A Dihedral Acute Triangulation of the Cube

Evan VanderZee Anil N. Hirani Vadim Zharnitsky Damrong Guoy
Edgar Ramos

Abstract

It is shown that there exists a dihedral acute triangulation of the cube. The method of constructing the acute triangulation is described, and symmetries of the triangulation are discussed.

1 Introduction

Interest in acute triangulation of polyhedra dates back to the 1960's at least; when geometers were working on proving that abstract polyhedra could be realized geometrically, acute triangulations of polyhedra played a role in the solution [2]. The specific problem of obtaining dihedral acute triangulations of domains in high-dimensional spaces appears in some lecture notes of Pak [6]. He mentions that in $\mathbb{R}^5$ and in higher-dimensional spaces, there is no acute triangulation of space, leaving the proof to the reader. The problem is a combinatorial one, and a proof is given in the literature by Křížek [5]. The acute triangulation of $\mathbb{R}^3$ and of infinite slabs in $\mathbb{R}^3$ was solved by Eppstein, Sullivan, and Üngör [4], but they were unable to find an acute triangulation of the cube. Saraf, who recently made some contributions to the problem of acute triangulation of polyhedra, also mentions that finding an acute triangulation of the cube is an open problem [8].

This paper shows that the cube does have an acute triangulation. In fact, it has at least two acute triangulations. The primary acute triangulation, with 1370 tetrahedra, is described in Sec. 2, and some statistics are given that show the superior quality of the triangulation. The maximum dihedral angle is around 84.65°, well within the range of acute, and the minimum dihedral angle is a nice 35.89°. Section 3 describes the computer-assisted construction of the acute triangulation of the cube; a hand construction was combined with mesh optimization to build the mesh. The triangulation has some very symmetries, which are discussed in Sec 4. The symmetries greatly reduce the number of distinct tetrahedra from 1370 to 82, and can be used to generate the full set of 277 vertices from just 26 of them.

2 The Acute Triangulation

We present the first-known acute triangulation of the cube as a triangulation of a cube centered at the origin with corner vertices at $(\pm 1, \pm 1, \pm 1)$. For lack of a more elegant means of precisely describing the triangulation, we list its vertices in Table 1 in Appendix A. The coordinates given are the exact coordinates used for the statistics in this paper. The mesh connectivity is given by the Delaunay triangulation of the set of vertices. It has been shown that an acute triangulation in three dimensions is not necessarily a Delaunay triangulation [4]. This acute triangulation of the cube, however, is not only Delaunay, but also is one for which each tetrahedron properly contains its circumcenter[1], i.e., the triangulation is 3-well-centered [12].

[1]This property does not hold in general for acute triangulations. In fact, for a tetrahedral mesh in $\mathbb{R}^3$, if each tetrahedron contains its circumcenter, the mesh must be Delaunay [7] [9] [12], so any nonDelaunay acute triangulation, such as the one in [4], does not have this property.

Figure 1: Two views of a cutaway section of the first-known acute triangulation of the cube. The view at right is a 90° rotation about the z-axis from the view at left. On the left four of the triangles on the surface of the cube are visible. These triangles appear on the back face of the cube in the view on the right. This cutaway is a collection of one of each of the distinct 82 tetrahedra that are used in the acute triangulation. The tetrahedra fit together to cover a symmetry region as discussed in Sec. 4, and through rotations and reflections can generate the full acute triangulation of the cube.

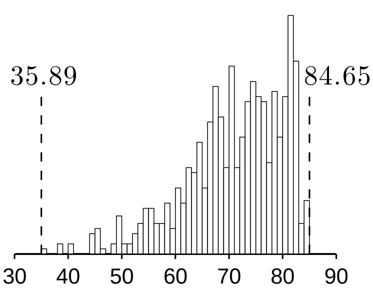

Figure 2: A histogram of the dihedral angles of the acute triangulation of the cube.

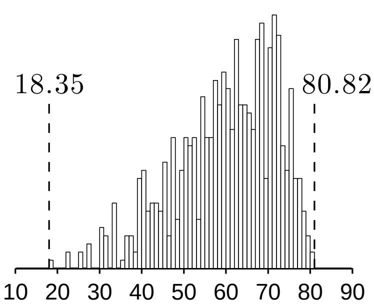

Figure 3: A histogram of the face angles of the acute triangulation of the cube.

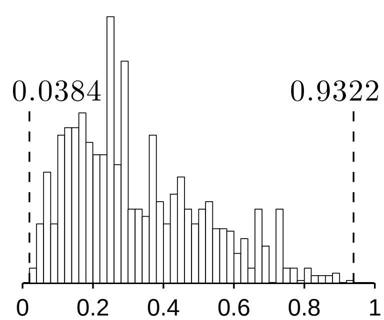

Figure 4: A histogram of the tetrahedron h/R values of the acute triangulation of the cube.

Figure 1, a cutaway view of the acute triangulation of the cube, visually shows the nice quality of the triangulation. Figures 2 through 4 give a quantitative measures of the quality of the triangulation. Each is a histogram of some quantitative measurement of the quality of tetrahedra. In each case, the histogram summarizes all of the values of the quantity in the mesh. For instance, the histogram of dihedral angles shows all of the dihedral angles, not just the maximum dihedral angle of each tetrahedron. The h/R values of Fig. 4, which may be less familiar to readers than the other measurements, is related to 3-well-centeredness. The range of the quantity over all tetrahedra is $(-1, 1)$, with 3-well-centered tetrahedra having all values in the range $(0, 1)$. The h/R values in a regular tetrahedron are all $1/3$. See [12] or [10] for more details.

Combinatorics plays an important role in acute triangulation, so we briefly mention some of the combinatorial statistics of the acute triangulation of the cube. There are 277 vertices, 1688 edges, and 1370 tetrahedra. Of the edges, 126 are boundary edges. Of the interior edges, 1506 have the minimum possible number of incident tetrahedra for an acute triangulation, i.e., 5 tetrahedra, and the remaining 56 each have 6 incident tetrahedra. For the vertices, 44 are on the boundary, and 233 are interior. The large majority of the interior vertices (200 of them) have icosahedral neighborhoods, thus they have 12 incident edges. There are 10 vertices with 14 incident edges, 18 vertices with 15 incident edges, 4 vertices with 16 incident edges, and 1 vertex—the central vertex located at $(0, 0, 0)$—with 22 incident edges.

The high degree central vertex can be replaced with a regular tetrahedron to give a combinatorially different acute and completely well-centered triangulation of the cube, one with 1387 tetrahedra. To obtain this triangulation, replace the vertex at the origin with the four vertices at $(-0.05, -0.05, 0.05)$, $(-0.05, 0.05, -0.05)$, $(0.05, -0.05, -0.05)$, and $(0.05, 0.05, 0.05)$. We see that there are at least two combinatorially distinct acute triangulations of the cube with the same surface triangulation.

3 Method of Construction

The basic methodology for the construction was one of an advancing front. It is absolutely necessary to have an acute surface triangulation, since an acute tetrahedron has acute facets [4], and more generally, all facets of an acute simplex are acute [1]. We began with a very nice acute surface triangulation of the cube; the midpoint of each edge of the cube was added, and each face was triangulated with a 14-triangle acute triangulation that conforms to this boundary and has a maximum face angle around $73.3°$.

Starting from the acute surface triangulation, we built inward, carefully adding vertices and tetrahedra to satisfy the combinatorial constraints. That is, each tetrahedron edge coinciding with an edge of the cube must have at least two incident tetrahedra, each tetrahedron edge lying in a facet of the cube must have at least three incident tetrahedra, and each interior edge must have at least five incident tetrahedra. The addition of vertices and tetrahedra was performed by hand with the frequent use of the Delaunay triangulation to help get the proper mesh connectivity.

After each layer or partial layer was constructed by hand, the mesh was optimized to obtain a set of acute tetrahedra conforming to the boundary of the cube. The optimization did not explicitly seek a dihedral acute triangulation, but instead tried to make the meshes completely well-centered. This type of optimization was introduced for two dimensions in [11] and later generalized to higher dimensions in [12]. At each layer, a moderately aggressive version of the optimization yielded a mesh that was both completely well-centered and dihedral acute. In most cases, more aggressive optimization produced a mesh that was well-centered but not acute, and less aggressive optimization produced a mesh that was neither well-centered nor acute. When a dihedral acute and completely well-centered mesh was obtained from the optimization, a new layer consisting of more tetrahedra and vertices was added by hand.

Eventually this process reached a stage in which all of the edges on the internal boundary already had three incident tetrahedra. This and the rest of the combinatorics and geometry worked out so that adding a vertex at the center of the cube completed an acute, completely well-centered triangulation of the cube.

When the acute triangulation was obtained and optimized with the optimization based on well-centeredness, it had a maximum dihedral angle around 87.8°. Later some additional optimization was applied that directly optimized the dihedral angle as well as optimizing for well-centeredness, and the symmetry discussed in the next section was enforced exactly. The additional optimization allowed boundary vertices to move constrained to the surfaces of the cube. This optimization produced the final mesh presented in this paper, except that the vertices were rounded to the nearest .001 for ease of presentation.

4 Symmetries of the Triangulation

A quick examination of the list of vertex coordinates reveals that the same numbers keep reappearing, and in many cases two of the coordinates of a vertex are the same with perhaps a difference in sign. This is not accidental. The acute triangulation of the cube presented in this paper has an S_4 symmetry group. More precisely, it has all of the symmetries of a regular tetrahedron whose vertices are four pairwise nonadjacent corners of the cube. Consider, for instance, the regular tetrahedron with vertices $(-1,-1,1)$, $(-1,1,-1)$, $(1,-1,-1)$, and $(1,1,1)$. Each of the 24 symmetries of this regular tetrahedron—rotations or reflections in $\mathbb{R}^3$ that map the tetrahedron to itself—is a symmetry that maps this acute triangulation of the cube to itself.

In fact, it is possible to use these symmetries to construct the full set of 277 vertices from just 26 of them. There are multiple ways to do this, one of which is the following. Take the first 26 vertices in the vertex list (reading down the first column, across page breaks if necessary). The last vertex is the vertex at the origin. These 26 vertices are the vertices that lie in the 1/24th of the cube specified by the inequalities $x \geq y$, $x \leq z$, and $x \leq -z$.

We will transform this initial set of vertices using the orthogonal matrices

$$A_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \quad A_2 = \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad A_3 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad A_4 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

Each of these matrices is a symmetry of the aforementioned regular tetrahedron and a symmetry of the cube. Matrix A_1 is a 120° rotation about the main diagonal of the cube through $(-1,-1,1)$ and $(1,1,-1)$. Matrix A_2 is a reflection through the plane $x = -y$. Matrix A_3 is a 180° rotation about the z-axis. Finally, matrix A_4 is a 180° rotation about the y-axis combined with a reflection through the plane $x = z$. It could also be thought of as reflection through the y-axis.

Applying both A_1 and A_1^2 (rotation by 240°) to the initial vertex set yields a set of vertices for 1/8th of the cube. Applying A_2 to this new vertex set covers 1/4th of the cube. Applying A_3 to this vertex set and A_4 to the subsequent vertex set, we obtain a set of vertices that covers the full cube. A large number of vertices in this vertex set are duplicates of each other, but when all of the duplicates are removed, there are 277 vertices that remain.

Because of these symmetries of the cube, there are only 82 distinct tetrahedra used in the 1370-tetrahedron acute triangulation of the cube. The views of the acute triangulation of the cube from Fig. 1 actually show just one of each of these 82 tetrahedra as they fit together to cover the generating 1/24th section of the cube. It is clear from Fig. 1 that many of the tetrahedra do not align with the boundaries of the generating 1/24th section of the cube, which is bounded by the planes $y = -1$, $x = y$, $x = z$, and $x = -z$. Tetrahedra that intersect the boundaries of the generating region are mapped onto themselves by one or more of the symmetries. In fact, there are only 38 tetrahedra that are interior to the generating section of the cube. There are, of course, 24 copies of each of these tetrahedra in the final result. As far as the other tetrahedra in the generateing set are concerned, 35 of them intersect one of the planes bounding the region, 8 of them intersect one of the main diagonals of the cube, and 1 intersects the y-axis. With some thought about the symmetries involved, one can see that in the full acute triangulation of the cube there are 12 copies of each tetrahedron that intersect a plane, 4 copies of each tetrahedron that intersect a main diagonal, and 6 copies of the tetrahedron that intersects the y-axis.

5 Conclusions

We have demonstrated that there exists an acute triangulation of the cube. The fact that the cube has an acute triangulation directly implies that many other regions of $\mathbb{R}^3$ also have acute triangulations. In particular, one can reflect the acute triangulation of the cube through one of its faces to get an acute triangulation of a square prism twice as long as a cube with vertices that match on the two square faces. One can identifying the matching vertices with each other to get an acute triangulation of a periodic domain. Alternatively, one can stack infinitely many of these objects together to obtain an acute triangulation of an infinitely long square prism. Using reflections and translations of this acute triangulation, one can easily obtain an acute triangulation of an infinite slab in $\mathbb{R}^3$, or of all of $\mathbb{R}^3$, as alternatives to the constructions in [4]. In fact, one can use translations and reflections of an initial acute triangulation of the cube to acutely triangulate any object in $\mathbb{R}^3$ that can be tiled with cubes.

But does every polyhedron have a dihedral acute triangulation? This remains an open question. Does every tetrahedron have a dihedral acute triangulation? This question, too, remains open, and so far there is still no nontrivial acute triangulation of the regular tetrahedron that is known to the authors. It is likely that a computer-assisted construction like the one discussed in this paper could be used to obtain such an acute triangulation, but there may be more direct methods. Even a directly constructive, perhaps simpler, acute triangulation of the cube would be nice.

Another interesting problem is that of finding the smallest possible acute triangulation of the cube, where size is measured in terms of the number of tetrahedra. It may be that the 1370 tetrahedra acute triangulation presented here is the smallest acute triangulation of the cube possible, but the authors suspect this is not the case. The analogous question in two dimensions—the smallest acute triangulation of the square—has been answered; an acute triangulation of the square requires at least eight triangles [3].

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A Vertex Coordinates

Table 1: List of coordinates of the vertices of an acute triangulation of the cube.

x	y	z
−1	−1	−1
−0.24	−1	−0.24
−1	−1	0
−0.347	−1	0.347
−1	−1	1
−0.517	−0.517	−0.23
−0.559	−0.559	0.357
−0.122	−0.624	0.122
−0.399	−0.598	0.052
−0.152	−0.472	−0.152
−0.27	−0.523	0.27
−0.254	−0.254	−0.254
−0.336	−0.336	0.336
−0.325	−0.325	−0.02
−0.376	−0.376	0.18
0	−0.388	0
−0.21	−0.398	0.099
−0.224	−0.316	0.224
−0.127	−0.269	0.127
−0.158	−0.263	−0.027
−0.2	−0.2	0.2
−0.258	−0.258	0.115
−0.115	−0.115	−0.115
0	−0.214	0
−0.15	−0.15	0.061
0	0	0
−1	−0.24	−0.24
−1	0	−1
−1	0.347	−0.347
−1	1	−1
−0.517	−0.23	−0.517
−0.559	0.357	−0.559
−0.624	0.122	−0.122
−0.598	0.052	−0.399

x	y	z
0.214	0	0
0.15	0.15	0.061
0.24	1	−0.24
0	1	−1
−0.347	1	−0.347
0.23	0.517	−0.517
−0.357	0.559	−0.559
−0.122	0.624	−0.122
−0.052	0.598	−0.399
0.152	0.472	−0.152
−0.27	0.523	−0.27
0.02	0.325	−0.325
−0.18	0.376	−0.376
0	0.388	0
−0.099	0.398	−0.21
−0.224	0.316	−0.224
−0.127	0.269	−0.127
0.027	0.263	−0.158
−0.115	0.258	−0.258
0	0.214	0
−0.061	0.15	−0.15
0.24	0.24	−1
1	0	−1
0.517	0.23	−0.517
0.559	−0.357	−0.559
0.399	−0.052	−0.598
0.152	0.152	−0.472
0.325	0.02	−0.325
0.376	−0.18	−0.376
0.21	−0.099	−0.398
0.158	0.027	−0.263
0.258	−0.115	−0.258
0.15	−0.061	−0.15
−1	−0.347	0.347

x	y	z
1	0	1
0.517	−0.23	0.517
0.559	0.357	0.559
0.598	0.052	0.399
0.472	−0.152	0.152
0.325	−0.02	0.325
0.376	0.18	0.376
0.398	0.099	0.21
0.263	−0.027	0.158
0.258	0.115	0.258
0.15	0.061	0.15
0.24	−0.24	1
0	−1	1
−0.347	−0.347	1
0.23	−0.517	0.517
−0.357	−0.559	0.559
−0.122	−0.122	0.624
−0.052	−0.399	0.598
0.152	−0.152	0.472
−0.27	−0.27	0.523
0.02	−0.325	0.325
−0.18	−0.376	0.376
0	0	0.388
−0.099	−0.21	0.398
−0.224	−0.224	0.316
−0.127	−0.127	0.269
0.027	−0.158	0.263
−0.115	−0.258	0.258
0	0	0.214
−0.061	−0.15	0.15
−1	1	1
−1	0.24	0.24
−1	1	0
−0.517	0.517	0.23

Table 1: (continued)

x	y	z
−0.472	−0.152	−0.152
−0.523	0.27	−0.27
−0.336	0.336	−0.336
−0.325	−0.02	−0.325
−0.376	0.18	−0.376
−0.388	0	0
−0.398	0.099	−0.21
−0.316	0.224	−0.224
−0.269	0.127	−0.127
−0.263	−0.027	−0.158
−0.2	0.2	−0.2
−0.258	0.115	−0.258
−0.214	0	0
−0.15	0.061	−0.15
−0.24	−0.24	−1
0	−1	−1
0.347	−0.347	−1
1	−1	−1
−0.23	−0.517	−0.517
0.357	−0.559	−0.559
0.122	−0.122	−0.624
0.052	−0.399	−0.598
−0.152	−0.152	−0.472
0.27	−0.27	−0.523
0.336	−0.336	−0.336
−0.02	−0.325	−0.325
0.18	−0.376	−0.376
0	0	−0.388
0.099	−0.21	−0.398
0.224	−0.224	−0.316
0.127	−0.127	−0.269
−0.027	−0.158	−0.263
0.2	−0.2	−0.2
0.115	−0.258	−0.258
0	0	−0.214
0.061	−0.15	−0.15
1	1	−1
1	0.24	−0.24
1	1	0
1	0.347	0.347
1	1	1
0.517	0.517	−0.23
0.559	0.559	0.357
0.624	0.122	0.122
0.598	0.399	0.052
0.472	0.152	−0.152
0.523	0.27	0.27
0.254	0.254	−0.254
0.336	0.336	0.336
0.325	0.325	−0.02
0.376	0.376	0.18

x	y	z
−0.624	−0.122	0.122
−0.598	−0.399	0.052
−0.523	−0.27	0.27
−0.398	−0.21	0.099
−0.316	−0.224	0.224
−0.269	−0.127	0.127
−0.263	−0.158	−0.027
0.347	−1	−0.347
0.122	−0.624	−0.122
0.052	−0.598	−0.399
0.27	−0.523	−0.27
0.099	−0.398	−0.21
0.224	−0.316	−0.224
0.127	−0.269	−0.127
−0.027	−0.263	−0.158
−0.347	0.347	−1
−0.122	0.122	−0.624
−0.399	0.052	−0.598
−0.27	0.27	−0.523
−0.21	0.099	−0.398
−0.224	0.224	−0.316
−0.127	0.127	−0.269
−0.158	−0.027	−0.263
0.347	1	0.347
0.122	0.624	0.122
0.399	0.598	0.052
0.27	0.523	0.27
0.21	0.398	0.099
0.224	0.316	0.224
0.127	0.269	0.127
0.158	0.263	−0.027
1	−0.347	−0.347
0.624	−0.122	−0.122
0.598	−0.052	−0.399
0.523	−0.27	−0.27
0.398	−0.099	−0.21
0.316	−0.224	−0.224
0.269	−0.127	−0.127
0.263	0.027	−0.158
−0.052	0.399	−0.598
−0.099	0.21	−0.398
0.027	0.158	−0.263
1	−1	1
0.24	−1	0.24
1	−1	0
0.517	−0.517	0.23
0.559	−0.559	−0.357
0.399	−0.598	−0.052
0.152	−0.472	0.152
0.254	−0.254	0.254
0.325	−0.325	0.02

x	y	z
−0.559	0.559	−0.357
−0.598	0.399	−0.052
−0.472	0.152	0.152
−0.254	0.254	0.254
−0.325	0.325	0.02
−0.376	0.376	−0.18
−0.398	0.21	−0.099
−0.263	0.158	0.027
−0.258	0.258	−0.115
−0.115	0.115	0.115
−0.15	0.15	−0.061
−0.24	1	0.24
0	1	1
−0.23	0.517	0.517
0.357	0.559	0.559
0.052	0.598	0.399
−0.152	0.472	0.152
−0.02	0.325	0.325
0.18	0.376	0.376
0.099	0.398	0.21
−0.027	0.263	0.158
0.115	0.258	0.258
0.061	0.15	0.15
−0.24	0.24	1
−1	0	1
−0.517	0.23	0.517
−0.559	−0.357	0.559
−0.399	−0.052	0.598
−0.152	0.152	0.472
−0.325	0.02	0.325
−0.376	−0.18	0.376
−0.21	−0.099	0.398
−0.158	0.027	0.263
−0.258	−0.115	0.258
−0.15	−0.061	0.15
0.598	−0.399	−0.052
0.398	−0.21	−0.099
0.263	−0.158	0.027
−0.052	−0.598	0.399
−0.099	−0.398	0.21
0.027	−0.263	0.158
0.347	0.347	1
0.122	0.122	0.624
0.399	0.052	0.598
0.27	0.27	0.523
0.21	0.099	0.398
0.224	0.224	0.316
0.127	0.127	0.269
0.158	−0.027	0.263
−0.399	0.598	−0.052
−0.21	0.398	−0.099

Table 1: (continued)

x	y	z
0.388	0	0
0.398	0.21	0.099
0.316	0.224	0.224
0.269	0.127	0.127
0.263	0.158	−0.027
0.2	0.2	0.2
0.258	0.258	0.115
0.115	0.115	−0.115

x	y	z
0.376	−0.376	−0.18
0.21	−0.398	−0.099
0.158	−0.263	0.027
0.258	−0.258	−0.115
0.115	−0.115	0.115
0.15	−0.15	−0.061
1	−0.24	0.24

x	y	z
−0.158	0.263	0.027
−0.598	−0.052	0.399
−0.398	−0.099	0.21
−0.263	0.027	0.158
0.052	0.399	0.598
0.099	0.21	0.398
−0.027	0.158	0.263