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Simple vs. Optimal Mechanism Design

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Abstract

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Mechanism design has found various applications in today's economy, from ad auctions to online marketplaces. The goal of mechanism design is to design a mechanism or system such that a group of strategic agents are incentivized to choose actions that also help achieve the designer's objective. However, in many of the mechanism design problems, the theoretically optimal mechanisms are hard to characterize. They are often too complex to implement in practice.

The focus of this thesis is to resolve the discrepancy between theory and practice by studying the following questions: *Are the mechanisms used in practice close to optimal? Can we design simple mechanisms to approximate the optimal one?* In this thesis we focus on two important mechanism design settings: multi-item auctions and two-sided markets. We show that in both settings, there are indeed simple and approximately-optimal mechanisms.

Following Myerson's seminal result [Mye81], which provides a simple and revenue-optimal auction when a seller is selling a single item to multiple buyers, there has been extensive research effort on maximizing revenue in multi-item auctions. However, the revenue-optimal mechanism is known to be complex and randomized [HN13, DDT13]. We provide a unified framework to approximate the optimal revenue in general multi-item multi-bidder auctions. Our result substantially improves and extends existing results.

Another line of works in this thesis focuses on two-sided markets. The main difference between two-sided markets and auctions is that mechanism in two-sided markets is designed by a third party instead of the seller. The mechanism need to incentivize both buyers and sellers to report their private information truthfully. The impossibility result by Myerson and Satterthwaite [MS83] shows that even in the most basic bilateral trade setting (1 buyer, 1 seller, 1 item), the full efficiency is not achievable by a truthful mechanism that does not run a deficit. In this thesis we focus on approximating the efficiency in terms of the objective of gains from trade. We provide simple

mechanisms that approximate the optimal gains from trade, in bilateral trade and many other two-sided market settings.

Simple vs. Optimal Mechanism Design

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Chapter 1

Introduction

Mechanism Design is a fairly classical field that lies in the intersection of Economics, Game Theory and Optimization. Unlike most of existing directions in Economics and Game Theory, mechanism design is referred to as "reverse game theory": The core problem is how to design the right incentive structure, so that the group of strategic participants of a system are motivated to take actions that will help to realize the system designer's goals.

With the growth of online markets and platforms, mechanism design has become increasingly important and relevant as demonstrated by its abundant applications including auctioning ad slots to advertisers on websites (e.g. Google), buying and selling goods in online marketplaces (e.g. Amazon, eBay), routing internet traffic, scheduling tasks in the cloud, and more. In all these applications, designing a good mechanism in favor of the designer's goal is crucial, as it might cause a difference of billions of dollars. Besides its practical importance, the area also has great theoretical depth. Despite being a relatively young field, we have already witnessed an explosion of new technical tools that lead to the resolution of many longstanding open problems in the last 15 years.

The major challenge in mechanism design comes from the misalignment of incentive between the designer and participants. While the designer aims to implement an outcome in favor of her objective, the participants pick their actions according to their own preferences (e.g. maximize their utilities). To address this issue and make participants' actions predictable, the designed mechanism is often required to be *truthful*, which means that the participants are willing to honestly report her private preferences. As a seminal result of mechanism design, Vickrey [Vic61], Clarke [Cla71] and Groves [Gro73] proposed a mechanism that is truthful and maximizes the *social welfare* - the sum of the buyers' values for the chosen outcome - in general mechanism design settings.

Simple vs. Optimal

Clearly, the quality of the solution with respect to the designer's objective is crucial. However, one should also pay equal attention to another criterion of a mechanism, that is, its simplicity. Compared to complicated mechanisms, a simple mechanism has advantages for both the designer and participants. For the designer, the mechanism with a simple format can be implemented by people that are not only experts in mechanism design; For participants, when facing a complicated mechanism, they may be confused by the rules and thus unable to optimize their actions and react in unpredictable ways instead. This may lead to undesirable outcomes and poor performance of the mechanism.

The mechanism design community has yet to converge to a single definition of simplicity. Mechanisms designed in this thesis will satisfy the following two appealing properties: (i) The mechanisms are *deterministic*, i.e. the outcome of the mechanism is completely determined by its input. In a deterministic mechanism, participants can easily see their utilities without simulating the execution of the mechanism with random bits, and thus it's easier for them to optimize their actions. (ii) The mechanisms are *posted price* mechanisms and their variations, i.e., there is a separate price associated with each good on the market, and each participant can choose their favorite bundles by paying the according price.

In mechanism design, an ideal mechanism would be both optimal and simple. However, in many settings, the theoretically optimal mechanisms are complex and randomized. To move forward, one has to compromise – either settle with optimal but somewhat complex mechanisms or turn to simple but approximately optimal solutions. In this thesis, we focus on the latter approach. We discuss two well-motivated mechanism design problems whose optimal solutions are complex: revenue (or profit) maximization in multi-item auctions, and maximizing gains from trade in two-sided markets.

We design simple and approximately optimal mechanisms for both problems.

1.1 Revenue Maximization

One fundamental question in mechanism design is how to maximize the *revenue* of the designer. In particular, maximizing revenue in multi-item auctions, where a seller is selling multiple items to a set of buyers, has attracted lots of attention from the mechanism design community.

When there is a single item to sale, the seminal result by Myerson [Mye81] shows that there is a truthful mechanism of a simple format that maximizes the revenue. After Myerson's result, maximizing revenue in multi-item auctions has become one of the central problems over the past decades. However, no general characterization has been found for the revenue-optimal mechanism. Moreover, even in fairly basic settings, the revenue-optimal mechanism has been proved to be complex and suffer many undesirable properties including randomization, non-monotonicity, and others [RC98, Tha04, Pav11, HN13, HR12, BCKW10, DDT13, DDT14]. To facilitate this discussion, we focus on the case of two items and a single *additive* buyer and present some examples. A buyer is additive if her value for a bundle of items is equal to the sum of her value for each item.

Selling Items Separately is not Optimal: There is a single additive buyer whose value for each item is drawn independently from the uniform distribution on $\{1, 2\}$. Since the items are independent, the mechanism that sells both items separately becomes a natural candidate. The optimal revenue by doing so is 2. However, consider the mechanism that offers a take-it-or-leave-it price $3/2$ for both items together. The buyer will accept the price with probability $3/4$, which generates expected revenue $9/4 > 2$.

Hart and Nisan [HN13] then showed that when selling m independent items to a single additive buyer, the optimal revenue by selling items separately can only achieve an $O(\frac{1}{\log m})$-fraction of the revenue by selling them together at a take-it-or-leave-it price. When $m \to \infty$, the mechanism that sells items separately could be far from the optimal revenue.

Randomization: [DDT13] Consider a single additive buyer with 2 items. Her value for item 1 is drawn uniformly from $\{1, 2\}$ and her value for item 2 is drawn independently and uniformly from $\{1, 3\}$. Then the unique optimal mechanism allows the buyer to choose from the following three options: receive both items with price 4; or receive the first item and a lottery for the second item, meaning that the second item is given to buyer with probability 1/2, by paying 2.5; or receive nothing and pay nothing. Daskalakis, Deckelbaum, and Tzamos further provide an example with a single additive buyer and two items where the unique optimal mechanism offers a menu of uncountably many lotteries.

For the above examples, the revenue-optimal mechanism becomes significantly more complex even when there is a single buyer and two items. There is a line of works characterizing the revenue-optimal mechanisms for this special setting [DDT13, DDT14, TW17, GK14a, GK18, HH19]. But in general, a simple and optimal solution is unlikely to exist.

To move forward, there has been extensive research effort studying the performance of simple mechanisms in multi-item auctions through the lens of approximation. However, a constant factor approximation for the optimal revenue was proved only when buyers' valuations are *unit-demand*, i.e. the buyer is interested in purchasing at most one item, due to a line of work initiated by Chawla et al. [CHK07, CHMS10, CMS15, CDW16], or when buyers are additive due to a series of work initiated by Hart and Nisan [HN12, CH13, LY13, BILW14, Yao15, CDW16]. Chawla and Miller [CM16] considered a generalization of additive and unit-demand valuation, proving a constant factor approximation to the optimal revenue via a variation of posted price mechanism.

Nonetheless, all results above only apply to the settings where buyers have linear valuations, i.e. each buyer's value for a bundle of items (which may be subject to some constraint) equals to the sum of her value for each item in the bundle. On the other hand, buyers in real auctions and markets often have more complex non-linear valuations. For example, a telecom company has a certain number of target customers in each city, but also has a capacity on how many customers its service can support. When the number of customers exceeds the capacity, extending service to a new city will not increase the value for the company. However, when comes to non-linear valuations, a constant factor approximation is known only for a single buyer [RW15]. It is a major open problem to extend this result to multiple buyers.

Question 1. *Is there a simple and truthful mechanism that can approximate the optimal revenue in multi-item auctions, with multiple buyers who have non-linear valuations?*

In Chapter 3 of this thesis, we give an affirmative answer to this question:

Informal Theorem 1. *There exists a simple and truthful mechanism that achieves a constant fraction of the optimal revenue, even when there are multiple buyers with* fractionally-subadditive[1] *valuation functions. For a more general* subadditive[2] *valuations, our approximation ratio degrades to* $O(\log m)$*, where* m *is the number of items in the auction.*

We introduce a new class of mechanisms called *sequential posted price with entry fee* to approximate the optimal revenue. Informally, the procedure of the mechanism is shown as follows:

Sequential Posted Price with Entry Fee Mechanism (informal)

0: Before the mechanism starts, the seller designs a *posted price* for every buyer and every item in the auction.

1: Buyers will come to the auction and purchase items sequentially.

2: When a buyer comes, the seller shows her all remaining items as well as their posted price dedicated to this buyer.

3: The buyer is asked to pay an *entry fee*. If she agrees, then she can enter the mechanism and take any set of remaining items by paying the posted prices. If she refuses, she gets nothing and pays 0.

Our mechanism is deterministic and truthful. Moreover, it has a simple format by using posted prices for separate items and entry fees. In fact, mechanisms of this form have already found broad real-world applications, for example in Costco and Amazon Prime, where the entry fee can be viewed as a membership fee.

We prove our result via an extension of the duality framework proposed by Cai et al. [CDW16]. Their framework provided a unified treatment for approximating optimal revenue for additive and

1. Informally, it's the maximum over linear functions. See Chapter 2 for a formal definition.

2. A buyer has subadditive valuations if the sum of her value for two bundles is at least her value for their union. The result by Rubinstein and Weinberg [RW15] study this valuation but for a single buyer.

unit-demand valuations. Roughly speaking, the framework generates a useful benchmark of the optimal revenue and then designs mechanisms to approximate the benchmark. However, the original approach by Cai et al. [CDW16] is inadequate to provide an analyzable benchmark, when buyers have non-linear combinatorial valuations. In this paper, we show how to extend their duality framework to accommodate general subadditive valuations. Another major contribution of our work is a novel approach to analyze the benchmark. More specifically, we draw a connection between approximating our benchmark using the sequential posted price with entry fee mechanisms and approximating optimal welfare with posted price mechanisms.

Our results substantially improve the approximation ratios for many of the settings studied in the literature, and in the meantime generalize the results to broader cases. See Chapter 3 for a detailed comparison between the best ratios reported in the literature and the new ratios obtained in this thesis.

1.2 A General and Practical Model: Profit Maximization

Most results in mechanism design assume that the seller has no cost to produce or own the items. However, in real markets, the seller often has its own cost for each item. It may be a production cost, or an opportunity cost, e.g., there is an outside option to sell the item at a certain price if the item is unsold in this auction. Motivated by this phenomenon, we also study the multi-item auctions where the seller has costs for obtaining the items in this thesis. As these costs usually depend on private information that is only available to the seller, we assume that the costs are private to the seller but are drawn from a distribution also known to the buyers. The goal is to design a mechanism that maximizes the profit, that is, the total revenue minus the total cost. We refer to this problem as *Profit Maximization*. Revenue maximization in multi-item auctions, one of the most classical and widely studied problem in mechanism design, is clearly a special case of this problem, where the seller always has cost 0 for each item.

Despite being realistic and widely applicable, the profit maximization problem is not well-understood. To the best of our knowledge, the only case with non-zero and randomized costs that has been studied is in the context of ad auctions [BMS12, FJM$^+$12, DIR14, EFG$^+$14, DPT16]. In ad auctions, the auctioneer is selling an ad displaying slot to an advertiser and aims to maximize revenue. There are multiple types of viewers of the webpage. The advertiser has a different value for displaying his ad to a viewer of different type. In an ad auction, only the auctioneer observes the type of the viewer, and the advertiser only knows a prior distribution from which the viewer-type is drawn from. In fact the problem can be easily cast as a special case of the profit maximization problem by mapping each type of viewer to an item. We will give a more detailed comparison between profit maximization and ad auctions in Chapter 4.

In revenue maximization, the optimal mechanism is known to be randomized and complex in multi-item settings. As a more general model, it is clear that the profit-optimal mechanism also requires complex allocation rules and randomization. Thus following the major and successful research theme in designing simple and approximately revenue-optimal mechanisms, the thesis aims to design simple and approximately-optimal mechanisms in profit maximization.

Question 2. *In profit maximization, is there a simple and truthful mechanism that approximate the optimal profit?*

Before answering this question, it's worth to discuss a natural but crucial question raised by this general setting: *Is profit maximization substantially harder than revenue maximization?* To facilitate the discussion, we examine two natural but unsuccessful attempts to solve the profit maximization problem.

Two unsuccessful attempts: (i) Use a mechanism that (approximately) optimizes the revenue. This is a terrible solution as some of the items sold by the mechanism may have extremely high costs, and as a result the mechanism only generates low if any profit. (ii) After the seller sees the costs, reveal them to the buyer, then use the (approximately) optimal mechanism that is tailored to those particular costs. To see why this mechanism can be far away from optimal, let us consider the following example adapted from [HN17].

Example 1.1. A random variable X with support $[1, +\infty)$ follows the *equal revenue (ER) distribution* if and only if $\Pr[X \leq x] = 1 - \frac{1}{x}$. Consider a profit maximization problem with m

heterogeneous items and a single additive buyer. The buyer's value t_j for each item j is drawn from an ER distribution. Define δ_j to be the m-dimensional vector whose j-th entry is 0 and all the other entries are ∞. The seller has a private and random cost vector $\vec{c}$ as follows: $\vec{c} = (c_1, ..., c_m) = \delta_j$ with probability $1/n$ for each j.

The expected profit of the mechanism in *(ii)* is 1, since for every j, after revealing cost δ_j, the seller can only sell item j to the buyer, which can generate 1 expected profit.[3] However, consider an alternative mechanism which does not reveal the costs, but offers the buyer the following contract: if the buyer pays $\log m/2$ up front, the buyer can take any item that is available, e.g. has cost 0. The chance that the buyer accepts the contract is $\Pr\left[\frac{1}{m} \cdot \sum_{j\in[m]} t_j \geq \frac{\log m}{2}\right]$, and due to [HN17], it is at least 1/2. Hence, the mechanism has profit at least $\frac{1}{4} \cdot \log m$.

The two failed attempts highlight two major challenges of profit maximization compared to revenue maximization: (i) how to balance the revenue and cost; (ii) how to capture the informational rent of the buyer, leveraging the fact that the costs are private information to extract more revenue. In Chapter 4, we overcome these two challenges and give an affirmative answer for Question 2:

Informal Theorem 2. *In the profit maximization problem, there are simple and truthful mechanisms that achieve a constant fraction of the optimal profit. Our result holds for arbitrary distribution of the cost vector.*

To prove our result, we propose a novel mechanism called *permit-selling*. For the single-buyer case, an informal procedure of the mechanism is shown as follows:

Permit-Selling Mechanism (informal)

0: For each item, the mechanism creates a separate *permit* that allows the buyer to purchase the item at its cost.

1: The mechanism first sells the permits without revealing any information about the actual costs.

2: Then the mechanism reveals all the costs. The buyer can buy each item by only paying the cost, if the buyer has purchased the permit for this item in the first stage.

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3. For every j, the profit of selling item j is $\max_p p \cdot \Pr[t_j \geq p] = 1$, since the seller has 0 cost for this item.

The permit-selling mechanisms can in fact help addressing the two challenges of profit maximization as mentioned. Since in the second stage, the price for each item is the same as the seller's cost, the profit of the permit-selling mechanisms is exactly the revenue from the first stage. So any mechanism that achieves high revenue in the first stage also generates high profit. Moreover, the buyer needs to make a decision on what permits to purchase without learning the costs, therefore, the seller can extract the informational rent by pricing the permits appropriately. Indeed, we do not even need to use any complex pricing scheme in the first stage. We sell the permits separately or sell them as a grand bundle. To accommodate multiple buyers, we then generalize the permit-selling mechanisms by selling the permits sequentially.

Similar to revenue maximization, in the proof we first come up with a benchmark of the optimal profit using the Cai-Devanur-Weinberg duality framework [CDW16], which has become a standard tool for analyzing the performance of simple mechanisms. In most of the results based on this duality approach, a particular family of dual variables is used to provide a benchmark for the objective function. However, this set of dual variables does not provide an appropriate benchmark due to the existence of costs. We propose a new set of dual variables that is tailored to handle the costs. Indeed, these dual variables are so informative that they inspired us to introduce the permit-selling mechanisms.

1.3 Two-sided Markets

As discussed in previous sections, mechanism design for one-sided markets, where a single seller owns all the items and designs the mechanism, has been extensively studied in Economics and more recently investigated in Computer Science. In the past few years, there also has been increasing interest in understanding how to design mechanisms for two-sided markets. In two-sided markets, there are two distinct groups of selfish agents in a two-sided market – the *sellers* and *buyers*. Sellers own the items and have *costs* for parting with the items that they own; buyers have *values* for acquiring the items that are on the market. Both sides are assumed to act strategically in order to maximize their own utilities. The goal is to design a mechanism to facilitate trade between the two groups and optimize a certain objective, e.g., efficiency.

Two-sided markets are ubiquitous in today's economy: take for example the New York Stock Exchange, online ad exchange platforms (e.g., Google's Doubleclick, Microsoft's AdECN, etc.), where advertisers try to purchase ad slots from websites who wish to sell the slots, crowdsourcing platforms, FCC's spectrum auctions where the telecommunication companies try to purchase spectrum from television broadcasting companies, online market places (e.g., Amazon, eBay, etc.) where buyers and sellers trade on the large-scaled trading platforms, or sharing economy platforms such as Uber, Lyft, and Airbnb.

Mechanism design for two-sided markets, where both the buyer(s) and seller(s) are strategic, is known to be substantially harder than for one-sided markets. The additional challenges stem from the following requirements:

1. The mechanism must be truthful for *both* sides of the market;

2. The buyer and seller payments must satisfy *budget balance*, that is, the mechanism must not run a deficit.

The limitations of these constraints are best illustrated by the seminal impossibility result of Myerson and Satterthwaite [MS83]. They show that even in the most basic two-sided market—*bilateral trade*, when one seller is selling a single item to a buyer—no truthful and budget balanced mechanism can achieve full efficiency (the trade is made whenever the buyer's value is higher than the seller's cost). In the same paper, Myerson and Satterthwaite also provided the mechanism that maximizes the efficiency among all truthful and budget balanced mechanisms (known as the *second-best* mechanism). Unfortunately, even in bilateral trade, it's a non-trivial task to explicitly describe the second-best mechanism as it requires to solve a system of differential equations that depend on the buyer's and seller's distributions. Such a mechanism is difficult to implement in practice. Motivated by the aforementioned results, we aim to design simple and approximately-optimal mechanisms in two-sided markets in this thesis.

Gains from Trade: A More Challenging Objective

There are two ways to measure the efficiency of a mechanism in two-sided markets. One is the standard notion of *social welfare*, which is the sum of the agents' values/costs of the resulting allocation. An alternative objective is *gains from trade* (GFT), which is the social welfare of the final allocation minus the total cost of the sellers. In other words, the GFT captures how much more social welfare the mechanism generates. Clearly, the two measures are equivalent when they are maximized exactly. However, approximating the GFT is much more challenging than approximating the social welfare. For example, if the buyer's value is 10 and the seller's cost is 9, not trading the item is a 9/10-approximation to the optimal social welfare but an 0-approximation to the optimal GFT. Obviously, any good approximation to the optimal GFT immediately gives a good approximation to the optimal social welfare, but the opposite direction is rarely true.

Prior to our work, all results approximating the GFT study the bilateral trade setting, and rely on assumptions that make the value/cost distributions nice [McA08, BM16, CBGdK+17]. One natural question following their results is whether there are approximation results that work for arbitrary distributions:

Question 3. *In two-sided markets, is there a simple, truthful, and budget balanced mechanism that achieves an unconditional approximation guarantee for the optimal Gains from Trade?*

The last part of this thesis is devoted to approximating GFT in two-sided markets. Our result gives an affirmative answer to Question 3, for bilateral trade and more general two-sided markets with multiple agents and multiple items.

1.3.1 Result in Bilateral Trade and Its Generalization (Chapter 5)

For the bilateral trade setting, we consider the following two mechanisms and show that the better of the two achieves $\frac{1}{2}$ of the optimal GFT. The first mechanism is called *Seller-Offering (SO)*, proposed by Blumroson and Mizrahi [BM16]: The seller offers to sell the item at a take-it-or-leave-it price. The buyer receives the item if she pays the offered price to the seller. Otherwise, the seller keeps the item and no payment is transferred; The second mechanism *Buyer-Offering (BO)* is essentially the same mechanism but with the roles of buyer and seller exchanged: Now the buyer offers to buy the item at a take-it-or-leave-it price and the item is traded if the seller agrees to sell at this price. The result is achieved by bounding the GFT of any truthful and budget balanced mechanism by the buyer's utility plus the seller's utility of this mechanism, and then proving that both terms can be further upper-bounded by the GFT of the SO and BO mechanisms accordingly.

We extend our result to a more general setting with multiple buyers and sellers, which is known as the *double auction* in the literature. In a double auction, each seller is endowed with precisely one item, all items are identical, and each buyer is interested in buying one item. So, there is a single number associated with each buyer's value or seller's cost. We prove a 2-approximation to the optimal GFT via a generalization of the SO and BO mechanisms. In this thesis we only present our result for double auctions. But the result can in fact accommodate any two-sided market where agents' type can be described as a scalar.

Informal Theorem 3. *In bilateral trade and a more general double auction setting, there is a simple, truthful and budget balanced mechanism that achieves at least half of the optimal GFT.*

1.3.2 Approximating Gains from Trade with Asymptotically Efficient Mechanisms (Chapter 6)

The mechanism proposed in Chapter 5 achieves (in expectation) a constant approximation to the optimal GFT. However, one caveat of this mechanism is that its expected GFT does *not* asymptotically converge to the optimal GFT as the market grows large. In fact, even when the values (or costs) of all agents are sampled i.i.d. from the uniform distribution over $[0, 1]$, the mechanism will at most achieve a constant fraction (strictly smaller than 1) of the optimal GFT (see Section 6.4). This caveat makes the mechanism less attractive in real-world applications with large markets.

On the other hand, the famous *Trade Reduction* mechanism by McAfee [McA92] in double auctions does not suffer from the above drawback. It is *asymptotically efficient*, i.e. the mechanism gets close-to-optimal GFT when the market size goes large. Moreover, its asymptotically efficiency guarantee holds for any agents' profile, while our generalized SO and BO mechanisms in Chapter 5 only have GFT guarantee in expectation over agents' distributions. However, the trade reduction mechanism, while asymptotically efficient, fails to give any unconditional approximation to the GFT in expectation.

To achieve the best of both worlds, in Chapter 6, we aim to design mechanisms that are asymptotically efficient, and also achieve a constant factor approximation to the optimal GFT in expectation. We show that there is a truthful and budget balanced mechanism that achieves both guarantees in double auctions, and a more general *matching market* setting, where trade between some buyer-seller pairs are disallowed.

Informal Theorem 4. *There is a simple, truthful and budget balanced mechanism that are asymptotically efficient, and also achieves a constant fraction of the optimal GFT, in double auctions and matching markets.*

Our mechanism combines the generalized offering mechanism presented in Chapter 5 with the trade reduction mechanism. The major challenge here is truthfulness. Without the truthfulness restriction, one can simply compare the GFT generated by both mechanisms for every agents' input, and then choose to run the mechanism with a higher GFT. However, an agent may misreport her value (or cost) to make the designer choose the mechanism that gives her higher utility. In our result, we properly adjust the scenario that items are traded in the offering mechanism, as well as carefully design the payments to guarantee truthfulness.

1.3.3 Two-sided Markets with Heterogeneous Items (Chapter 7)

In the two-sided markets discussed in Chapter 5 and Chapter 6, all items are identical. In Chapter 7 we move on to the setting with heterogeneous items. We design a simple, truthful and budget balanced mechanism and prove that it approximates the optimal GFT, with an approximation ratio logarithmic in the number of items. To the best of our knowledge, this is the first result that achieves a worst-case approximation guarantee in two-sided markets with heterogeneous items.

Informal Theorem 5. *In two-sided markets with heterogeneous items, there is a simple, truthful and budget balanced mechanism that achieves unconditional approximation to the optimal GFT.*

More specifically, we focus on a setting with n heterogeneous items, where each item is owned by a different seller, and there is a constrained-additive[4] buyer. Under this setting, we prove an

4. It's a natural generalization of unit-demand, additive, and matroid-rank valuations. Informally, the buyer is

$O(\log^2(n))$-approximation using some simple mechanisms.[5]

To prove the result, we propose a new class of mechanisms called *seller adjusted posted price (SAPP)*. It is a variation of posted price mechanisms. Informally, the procedure of the mechanism is shown as follows:

Seller Adjusted Posted Price Mechanism (informal)

1: Each seller reports her private cost of her item to the mechanism.
2: The mechanism designs a posted price for each item according to the reported cost profile.
3: The mechanism shows to the buyer all the items, together with their posted prices.
4: The buyer is only allowed to purchase *at most one* item by paying the posted price for this item.

Comparing to a classic posted price mechanism where the prices are designed before the mechanism starts, the main advantage of using an SAPP mechanism is that it provides the flexibility to set prices based on the sellers' cost, which allows an SAPP mechanism to achieve a higher GFT. An astute reader may have already realized that the amount of money that sellers gain are not yet defined in the SAPP mechanism. In fact, it is the major technical challenge to carefully design the adjusted prices as well as the sellers' gains, to make the mechanism truthful and budget balanced. We postpone a more detailed discussion in Chapter 7.

1.4 Thesis Organization

In Chapter 2, we introduce all mechanism design concepts, definitions and technical background that are needed to read this thesis. Then we overview the related work on both lines of work: multi-item auctions and two-sided markets.

In Chapter 3, we present our result for revenue maximization in multi-item auctions. We first review the duality framework of [CDW16]. Then we derive an upper bound of the optimal

only interested in a set of items subject to some downward-closed constraint, and is additive over the items. See Chapter 2 for a formal definition.

5. Our approximation ratio is $O(\log(n))$ for a broad class of constraints (such as matroid, matching, knapsack, or the intersection of these).

revenue for subadditive buyers by combining the duality framework with our new techniques. To familiarize the readers with some basic ideas and techniques used to bound the benchmark, we first give our proof for the single buyer case. Then we show how to upper bound the optimal revenue for multiple fractionally-subadditive (or subadditive) buyers with our sequential posted price with entry fee mechanisms.

In Chapter 4, we present our results for profit maximization. We first introduce a benchmark of the optimal profit by extending the duality framework of [CDW16] to our profit setting. And then we show how to bound the benchmark with the permit-selling mechanisms. After that, we design simple mechanisms to approximate the benchmark for both the single-buyer and multiple-buyer case.

Starting from Chapter 5, we discuss approximating GFT in two-sided markets. In Chapter 5, we first prove a 2-approximation in the bilateral trade setting. And then we generalize this result to the double auction setting with an arbitrary trading constraint. In Chapter 6, we design asymptotically efficient mechanisms to approximate GFT. We first show by an example that the GFT of the mechanism we design in Chapter 5 does not converge to the optimal GFT when the market goes large. Then we design a truthful and budget balanced mechanism that are both approximately-optimal and asymptotically efficient in double auctions and a more general setting called matching market.

In Chapter 7, we present the results for two-sided markets with heterogeneous items. We start by proving a distribution-parameterized approximation to the optimal GFT. Then we prove an unconditional $O(\log^2 n)$-approximation by using the first result. At the end of this chapter, we draw a connection between a lower bound to our analysis and one of the major open problems in a special matching market.

In Chapter 8, we give a conclusion of this thesis and list some open questions following from our results.

1.5 Bibliographic Notes

The work presented in this thesis is contained in research papers with co-authors. Chapter 3 is based on joint work with Yang Cai [CZ17]. Chapter 4 is based on joint work with Yang Cai [CZ19]. Chapter 5 is based on joint work with Johannes Brustle, Yang Cai and Fa Wu [BCWZ17]. Chapter 6 is based on joint work with Moshe Babaioff, Yang Cai and Yannai A. Gonczarowski [BCGZ18]. Chapter 7 is based on joint work with Yang Cai, Kira Goldner and Steven Ma [CGMZ21].

Chapter 2

Background

In this chapter we introduce some basic definitions and notations, as well as useful tools in the literature that are needed to read this thesis. Then we overview the related work on both lines of works that this thesis focuses on: multi-item auctions and two-sided markets.

2.1 Basic Concepts in Mechanism Design

In this section we give the definition of basic concepts in mechanism design.

Agent's type: The *type* of an agent contains all of her private information in order to design mechanisms. A type *profile* is a collection of types of all agents.

Single-Dimensional vs. Multi-Dimensional: A mechanism design problem is called *single-dimensional* if the type of every agent can be described as a scalar. An example of the single-dimensional setting is single-item auction, where a seller is selling a single item. On the other hand, the problem is called *multi-dimensional* if the type is described as a vector of numbers.

Bayesian setting: Throughout this thesis, we consider mechanism design problems in a *Bayesian setting*, i.e. the type of each agent is drawn from some distribution that is known to the public.

(Direct) Mechanism: A mechanism takes inputs from all agents (called bid or report), decides an outcome (called *allocation rule*), and then charges each agent some payment (called *payment rule*).

Agent's value: An agent's *value* maps a tuple of her type and an outcome to a non-negative real number, which represents how much happiness the buyer has for this outcome. The agent is *risk-neutral*, meaning that her value for a randomized allocation equals to the expectation of her value over the randomness of the allocation.

Agent's utility: An agent's utility maps a tuple of her type, an outcome and a payment to a real number. An strategic agent aims to maximize her utility in any mechanism she joins. In this thesis, we assume that every agent has *quasi-linear* utility, which means that her utility equals to her value minus the payment.

Truthfulness/Incentive compatibility: A mechanism is *truthful* or *incentive compatible (IC)* if for every agent, reporting her type truthfully maximizes her utility. There are two IC concepts we consider in this thesis:

- *Bayesian Incentive Compatible (BIC)*: For every agent, reporting her type truthfully maximizes her expected utility if other agents also report truthfully. The expectation is taken over the distribution of other agents' types and the randomness of the mechanism.

- *Dominant Strategy Incentive Compatible (DSIC)*: For every agent, reporting her type truthfully maximizes her utility for any reported profile of other agents.

Individual Rationality (IR): A mechanism is *individual rational* (also called *interim individual rational* or interim IR) if for every agent, reporting her type truthfully gives her non-negative utility in expectation over other agents' types. A mechanism is ex-post IR if for every agent, reporting her type truthfully gives her non-negative utility for any reported profile of other agents.

Revelation Principle: A mechanism can have a more complicated format than a direct mechanism that simply asks the buyers to submit their types. Myerson introduced the *revelation principle* [Mye79], which states that every incomplete information game can be simulated by a direct mechanism. Thus throughout this thesis, we will, without loss of generality, focus on direct and truthful mechanisms.

2.2 Background of Multi-Item Auctions

In multi-item auctions, there are n buyers and m heterogenous items. For every buyer i, we denote her type t_i as $\langle t_{ij}\rangle_{j=1}^m$, where t_{ij} is buyer i's private information about item j. For each i, j, we assume t_{ij} is drawn independently from the distribution D_{ij}. Let $D_i = \times_{j=1}^m D_{ij}$ be the distribution of buyer i's type and $D = \times_{i=1}^n D_i$ be the distribution of the type profile. We use T_{ij} (or T_i, T) and f_{ij} (or f_i, f) to denote the support and density function of D_{ij} (or D_i, D). For notational convenience, we let t_{-i} to be the types of all buyers except i and $t_{<i}$ (or $t_{\leq i}$) to be the types of the first $i-1$ (or i) buyers. Similarly, we define D_{-i}, T_{-i} and f_{-i} for the corresponding distributions, support sets and density functions.

Throughout this thesis, we use the notation $[k] = \{1, 2, ..., k\}$ for any positive integer k.

Valuation Function: The buyer's *valuation* is a function that maps each type t and a subset of items $S \in [m]$ to a positive number, which represents how much she values the bundle S when her type is t. For every buyer i with type t_i, her valuation for a subset of items S is denoted by $v_i(t_i, S)$.

Independent System and Matroid: Given a finite ground set I, a *set system* $\mathcal{F} \subseteq 2^I$ is a family of subsets of I. $\mathcal{F}$ is called *downward-closed* if for every $S \in \mathcal{F}$, we have $S' \in \mathcal{F}, \forall S' \subseteq S$.

The pair $(I, \mathcal{F})$ is called a *matroid* if it satisfies all of the following properties:

1. $\emptyset \in \mathcal{F}$.

2. $\mathcal{F}$ is downward-closed.

3. $\mathcal{F}$ has *exchange property*: For every $S, S' \in \mathcal{F}$ and $|S'| > |S|$, there exists $e \in S' \backslash S$ such that $S \cup \{e\} \in \mathcal{F}$.

In the thesis we say $\mathcal{F}$ is a matroid constraint with respect to I if $(I, \mathcal{F})$ forms a matroid.

Hierarchy of the Valuation Functions: Throughout this thesis, every buyer in the auction has a valuation that is *subadditive over independent items*:

Definition 2.1. *[RW15] For every buyer, whose type t is drawn from a product distribution $\mathcal{D}$ with support T, her valuation function $v(t, \cdot)$ is subadditive over independent items if:*

- *$v(\cdot, \cdot)$ has no externalities, i.e., for each $t \in T$ and $S \subseteq [m]$, $v(t, S)$ only depends on $\langle t_j \rangle_{j \in S}$, formally, for any $t' \in T$ such that $t'_j = t_j$ for all $j \in S$, $v(t', S) = v(t, S)$.*
- *$v(\cdot, \cdot)$ is monotone, i.e., for all $t \in T$ and $U \subseteq V \subseteq [m]$, $v(t, U) \leq v(t, V)$.*
- *$v(\cdot, \cdot)$ is subadditive, i.e., for all $t \in T$ and $U, V \subseteq [m]$, $v(t, U \cup V) \leq v(t, U) + v(t, V)$.*

We give a formal definition of different classes of buyer's valuation covered in this thesis.

Definition 2.2. *Let t be the type and $v(t, S)$ be the value for bundle $S \in [m]$.*

- ***Additive**: $v(t, S) = \sum_{j \in S} v(t, \{j\})$.*
- ***Unit-demand**: $v(t, S) = \max_{j \in S} v(t, \{j\})$.*
- ***Constrained Additive**: $v(t, S) = \max_{R \subseteq S, R \in \mathcal{I}} \sum_{j \in R} v(t, \{j\})$, where $\mathcal{I} \subseteq 2^{[m]}$ is a downward-closed set system over the items specifying the feasible bundles (called* feasibility constraint*). The valuation function is called **matroid-rank** if $\mathcal{I}$ is a matroid. An equivalent way to represent any constrained additive valuations is to view the function as additive but the bidder is only allowed to receive bundles that are feasible, i.e., bundles in $\mathcal{I}$.*
- ***XOS/Fractionally Subadditive:** $v(t, S) = \max_{i \in [K]} v^{(i)}(t, S)$, where K is some finite number and $v^{(i)}(t, \cdot)$ is an additive function for any $i \in [K]$.*

- ***Subadditive:*** *$v(t, S_1 \cup S_2) \le v(t, S_1) + v(t, S_2)$ for any $S_1, S_2 \subseteq [m]$.*

Constrained additive valuation is a generalization of additive and unit-demand valuation by choosing $\mathcal{I} = 2^{[m]}$ and $\mathcal{I} = \{\{j\} \mid j \in [m]\}$ accordingly. We notice that since the valuation has no externalities, $v_i(t_i, \{j\})$ only depends on t_{ij} for all buyer i. To ease notations, for constrained additive valuations, we interpret t_i as an m-dimensional vector $(t_{i1}, t_{i2}, \cdots, t_{im})$ such that $t_{ij} = v_i(t_i, \{j\})$ is buyer i's value for item j.

Mechanisms: A mechanism M in multi-item auctions can be described as a tuple (x, p). For every type profile $\mathbf{t}$, buyer i and bundle $S \subseteq [m]$, $x_{iS}(\mathbf{t})$ is the probability of buyer i receiving the exact bundle S at profile $\mathbf{t}$, $p_i(\mathbf{t})$ is the payment for buyer i at the same type profile. To ease notations, for every buyer i and types t_i, we use $p_i(t_i) = \mathbb{E}_{t_{-i}}[p_i(t_i, t_{-i})]$ as the interim price paid by buyer i and $\sigma_{iS}(t_i) = \mathbb{E}_{t_{-i}}[x_{iS}(t_i, t_{-i})]$ as the interim probability of receiving the exact bundle S.

IC and IR constraints: A mechanism $M = (x, p)$ is BIC if:

$$\sum_{S \subseteq [m]} \sigma_{iS}(t_i) \cdot v_i(t_i, S) - p_i(t_i) \ge \sum_{S \subseteq [m]} \sigma_{iS}(t_i') \cdot v_i(t_i, S) - p_i(t_i'), \forall i, t_i, t_i' \in T_i.$$

The mechanism is DSIC if:

$$\sum_{S \subseteq [m]} x_{iS}(t_i, t_{-i}) \cdot v_i(t_i, S) - p_i(t_i, t_{-i}) \ge \sum_{S \subseteq [m]} x_{iS}(t_i', t_{-i}) \cdot v_i(t_i, S) - p_i(t_i', t_{-i}), \forall i, t_i, t_i' \in T_i, t_{-i} \in T_{-i}.$$

The mechanism is (interim) IR if:

$$\sum_{S \subseteq [m]} \sigma_{iS}(t_i) \cdot v_i(t_i, S) - p_i(t_i) \ge 0, \forall i, t_i \in T_i.$$

The mechanism is ex-post IR if:

$$\sum_{S \subseteq [m]} x_{iS}(t_i, t_{-i}) \cdot v_i(t_i, S) - p_i(t_i, t_{-i}) \ge 0, \forall i, t_i \in T_i, t_{-i} \in T_{-i}.$$

Designer's Objectives: Given a BIC and IR mechanism, the *social welfare* of this mechanism is the expected sum of buyers' value over the distributions of buyers. Formally, the social welfare of a mechanism $M = (x, p)$ is

$$\mathbb{E}_{\mathbf{t}\in T}\left[\sum_i \sum_{S\subseteq[m]} x_{iS}(\mathbf{t})\right] = \sum_i \mathbb{E}_{t_i\in T_i}\left[\sum_{S\subseteq[m]} \sigma_{iS}(t_i)\right]$$

The *revenue* of this mechanism is the expected sum of buyers' payment over the distributions of buyers. Formally,

$$\text{REV}(M) = \mathbb{E}_{\mathbf{t}\in T}\left[\sum_i p_i(\mathbf{t})\right] = \sum_i \mathbb{E}_{t_i\in T_i}\left[p_i(t_i)\right]$$

2.3 Myerson's Result

In this section we present the celebrated result by Myerson [Mye81] in single-item auctions. Note that the result also applies to any single-dimensional settings.

Mechanism in Single-Item Auction: Given any mechanism $M = (x, p)$. For every buyer i, $x_i(\mathbf{t})$ denote the probability that buyer i wins the item under type profile $\mathbf{t}$, and $p_i(\mathbf{t})$ denote her payment. Let $x_i(t_i) = \mathbb{E}_{t_{-i}}[x_i(\mathbf{t})]$ be the interim probability that buyer i wins the item and let $p_i(t_i) = p_i(\mathbf{t})$ be her interim payment.

Properties of Allocation Rule: An allocation rule x is *implementable* if there exists a payment rule p such that $M = (x, p)$ is DSIC and ex-post IR. An allocation rule x is *monotone* if for every buyer i and t_{-i}, $x_i(t_i, t_{-i})$ is non-decreasing on t_i.

Myerson's result [Mye81] gives an exact characterization of implementable allocation rule in single-item auction.

Lemma 2.1. *[Mye81] An allocation rule x is implementable if and only if x is monotone. Moreover, the payment rule p such that $M = (x, p)$ is DSIC and ex-post IR is defined as follows:*

$$p_i(t_i, t_{-i}) = t_i \cdot x_i(t_i, t_{-i}) - \int_0^{t_i} x_i(y, t_{-i})dy - \theta$$

for any $\theta \geq 0$. We refer to it as Myerson's payment identity.

Revenue vs. Virtual Welfare: For any BIC and IR mechanism $M = (x, p)$, the expected revenue equals to the expected "virtual welfare" for particular virtual value functions. Formally,

$$\sum_i \mathbb{E}_{t_i}[p_i(t_i)] = \sum_i \mathbb{E}_{t_i}[x_i(t_i) \cdot \varphi_i(t_i)].$$

Myerson's Virtual Value Function: $\varphi_i(t_i) = t_i - \frac{1-F_i(t_i)}{f_i(t_i)}$ is called Myerson's virtual value function with respect to distribution D_i.

According to Lemma 2.1, the revenue-optimal mechanism in single-item auction (Myerson's auction) is defined as follows:

Myerson's Mechanism in Single-item Auction

1: Each buyer i reports her value t_i to the mechanism.

2: The mechanism calculates the Myerson's virtual value $\varphi_i(t_i)$ for every buyer i.

3: The mechanism allocates the item to buyer i with the highest $\varphi_i(t_i)$ (break ties arbitrarily) if $\max_j \varphi_j(t_j) > 0$. Otherwise, the mechanism don't allocate the item.

4: The payment for each buyer follows from Lemma 2.1.

Given any distribution D_i, we say D_i is *regular* if $\varphi_i(t_i)$ is monotone non-decreasing. When D_i is regular for all i, the allocation rule defined above is indeed monotone. For general distributions, the allocation may not be monotone, and thus not implementable. Instead, Myerson performed an "ironing procedure" [Mye81] on the virtual value functions to get a monotone allocation rule.

Ironing: Consider a distribution D with cdf F and pdf f. Consider the *quantile space* for distribution D: For any $q \in [0, 1]$, let $R(q) = q \cdot F^{-1}(1 - q)$. By taking derivative, one can easily check that $R'(q) = \varphi(F^{-1}(1 - q))$ holds for every q. Thus $\varphi(\cdot)$ is non-decreasing if and only if $R(\cdot)$ is concave. Let $\tilde{R}(q) = \max_{q_1, q_2 \in [0,1]}(\delta \cdot R(q_1) + (1 - \delta) \cdot R(q_2))$, where $\delta \in [0, 1]$ is the unique value such that $q = \delta \cdot q_1 + (1 - \delta) \cdot q_2$. Now $\tilde{R}(\cdot)$ is a concave curve. The *Myerson's ironed virtual value* function is defined as $\tilde{\varphi}(b) = \tilde{R}'(1 - F(b))$.

2.4 Background of Two-sided Markets

Two-sided Markets: We use m to denote the number of sellers and n to denote the number of buyers. For each seller j (or buyer i), her type s_j (or type b_i), is drawn independently from her type distribution D_j^S (or D_i^B). Let $D^S = \times_{j=1}^m D_j^S$ and $D^B = \times_{i=1}^n D_i^B$ be the product distribution of sellers' and buyers' type profile respectively. For notational convenience, let D_{-j}^S (or D_{-i}^B) be the distribution of types of all sellers (or buyers) except j (or i). We use T_j^S (or T^S, T_i^B, T^B, T_{-j}^S, T_{-i}^B) and f_j^S (or f^S, f_i^B, f^B, f_{-j}^S, f_{-i}^B) to denote the support and density function of D_j^S (or $D^S, D_i^S, D^B, D_{-j}^S, D_{-i}^B$). Let $D = (D^S, D^B)$.

Double Auction: *Double auction* is a special case of two-sided markets, where sellers are unit-supply with identical items and buyers are unit-demand. In other words, each seller only owns one item, and each buyer only wishes to buy one item and treat all the items as the same. This is a single-dimensional setting as every buyer's type and each seller's type can be represented as a scalar. For each buyer i, b_i is the value of buyer i if she gets an item. For each seller j, s_j is her value for the item she owns. To distinguish from buyer's value, we refer to s_j as seller j's *cost*.

Let $V = \{(i, j) \mid i \in [n], j \in [m]\}$ be the set of all possible trading pairs between the sellers and buyers. We use $\mathcal{F} \subseteq 2^V$ to denote the *trading constraint*. More specifically, $\mathcal{F}$ is a set system that contains all of the feasible sets of seller-buyer pairs that can be traded simultaneously. We allow any $\mathcal{F}$ that satisfies the following two properties:

- Every $S \in \mathcal{F}$ is a matching, in other words, for every buyer i there is at most one seller $j \in [m]$ such that $(i, j) \in S$. Same for the sellers.
- $\mathcal{F}$ is downward closed, i.e., if $S \in \mathcal{F}$, and $S' \subseteq S$, then $S' \in \mathcal{F}$.

We refer to this setting as *double auction with trading constraint* $\mathcal{F}$. In the (classic) double auction, any matching is in $\mathcal{F}$, indicating that any buyer can trade with any seller.

Bilateral trade: It's the simplest setting in two-sided markets, with a single buyer, a single seller, and a single item. It's a special case of double auction when $n = m = 1$.

Mechanisms in Double Auctions: Any mechanism in double auctions (with trading constraint $\mathcal{F}$) can be specified as a tuple (A, p^B, p^S), where A represents the allocation rule, p^B and p^S are the payment rule for buyers and sellers. Given a type profile $\mathbf{s}$ and $\mathbf{b}$, $A(\mathbf{b}, \mathbf{s}) \in \mathcal{F}$ is a (random) matching that contains all the pairs of sellers and buyers who trade with each other under this type profile. In other words, for every pairs $(i, j) \in A(\mathbf{b}, \mathbf{s})$, buyer i receives the item from seller j and pays $p_i^B(\mathbf{b}, \mathbf{s})$ to the mechanism. $p_j^S(\mathbf{b}, \mathbf{s})$ is the amount of money seller j gains from the mechanism. For notational convenience, we slightly abuse notation to let $p_i^B(b_i) = \mathbb{E}_{b_{-i},\mathbf{s}}[p_i^B(b_i, b_{-i}, \mathbf{s})]$ be buyer i's expected payment when she reports type b_i, over the randomness of mechanism and other agents' types. Similarly, let $p_j^S(s_j)$ be seller j's expected gains when she reports type s_j.

In our analysis, we usually use an alternative representation of the allocation rule. For any type profile $\mathbf{b}$ and $\mathbf{s}$, for every $i \in [n]$, we use $x_i^B(\mathbf{b}, \mathbf{s})$ to denote the probability that buyer i gets an item under this type profile, i.e., the probability that $(i, j) \in A(\mathbf{b}, \mathbf{s})$ for some j. Similarly, for every $j \in [m]$, we use $x_j^S(\mathbf{b}, \mathbf{s})$ to denote the probability that seller j sells her item. Given a mechanism (x^B, x^S, p^B, p^S), for every type profile $(\mathbf{b}, \mathbf{s})$, buyer i's utility is her value minus payment, which is $b_i \cdot x_i^B(\mathbf{b}, \mathbf{s}) - p_i^B(\mathbf{b}, \mathbf{s})$, and seller j's utility is her gain minus cost, which is $p_j^S(\mathbf{b}, \mathbf{s}) - s_j \cdot x_j^S(\mathbf{b}, \mathbf{s})$.

We again slightly abuse notation to let $x_i^B(b_i) = \mathbb{E}_{b_{-i},\mathbf{s}}[x_i^B(b_i, b_{-i}, \mathbf{s})]$ be the interim probability that buyer i gets an item when she reports type b_i, over the randomness of mechanism and other agents' types. Similarly, let $x_j^S(s_j)$ be seller j's interim probability to sell her item when she reports type s_j.

An allocation rule $x = (x^B, x^S)$ is *monotone* if for every buyer $i \in [n]$, $x_i^B(b_i, b_{-i}, \mathbf{s})$ is non-decreasing in b_i for any fixed b_{-i} and $\mathbf{s}$, and for every seller $j \in [m]$, $x_j^S(\mathbf{b}, s_j, s_{-j})$ is non-increasing in s_j for any fixed $\mathbf{b}$ and s_{-j}.

Budget Balance Constraints: We give formal definitions of variants of the budget balance constraints. In Section 5.4, we discuss the connections between these variants.

- **Strong Budget Balance (SBB)**: Under any type profile, the sum of all buyers' (expected) payment is equal to the sum of all sellers' (expected) gains, over the randomness of mechanism.

Formally,

$$\sum_i p_i^B(\mathbf{b}, \mathbf{s}) = \sum_j p_j^B(\mathbf{b}, \mathbf{s}), \forall \mathbf{b}, \mathbf{s}$$

- **Weak Budget Balance (WBB)**: Under any type profile, the sum of all buyers' (expected) payment is at least the sum of all sellers' (expected) gains, over the randomness of mechanism. Formally,

$$\sum_i p_i^B(\mathbf{b}, \mathbf{s}) \geq \sum_j p_j^B(\mathbf{b}, \mathbf{s}), \forall \mathbf{b}, \mathbf{s}$$

- **Ex-ante Strong Budget Balance (Ex-ante SBB)**: The sum of all buyers' expected payment is equal to the sum of all sellers' expected gains, over the randomness of mechanism and the type profile of all agents. Formally,

$$\mathbb{E}_{\mathbf{b}\sim D^B, \mathbf{s}\sim D^S}\left[\sum_i p_i^B(\mathbf{b}, \mathbf{s}) - \sum_j p_j^B(\mathbf{b}, \mathbf{s})\right] = 0$$

- **Ex-ante Weak Budget Balance (Ex-ante WBB)**: The sum of all buyers' expected payment is at least the sum of all sellers' expected gains, over the randomness of mechanism and the type profile of all agents. Formally,

$$\mathbb{E}_{\mathbf{b}\sim D^B, \mathbf{s}\sim D^S}\left[\sum_i p_i^B(\mathbf{b}, \mathbf{s}) - \sum_j p_j^B(\mathbf{b}, \mathbf{s})\right] \geq 0$$

Gains from Trade (GFT): *Gains from trade* describes the gains of social welfare induced by the mechanism. Formally, given a mechanism $M = (A, p^B, p^S)$, the expected GFT for the mechanism is defined as

$$\mathrm{GFT}(M) = \mathbb{E}_{\mathbf{b}\sim D^B, \mathbf{s}\sim T^S}\left[\sum_{(i,j)\in A(\mathbf{b},\mathbf{s})} (b_i - s_j)\right] \tag{2.1}$$

or using the definition of x^B, x^S,

$$\mathrm{GFT}(M) = \mathbb{E}_{\mathbf{b}\sim D^B, \mathbf{s}\sim T^S}\left[\sum_{i=1}^n x_i^B(\mathbf{b}, \mathbf{s}) \cdot b_i - \sum_{j=1}^m x_j^S(\mathbf{b}, \mathbf{s}) \cdot s_j\right] \tag{2.2}$$

2.4.1 Impossibility Result by Myerson and Satterthwaite

In this section we overview the impossibility result by Myerson and Satterthwaite [MS83] in bilateral trade. We will use b and s to represent the buyer and seller's type accordingly. In any mechanism $M = (x, p^B, p^S)$ in bilateral trade, for every type profile (b, s), $x(b, s)$ is the probability that the item is traded. $p^B(b, s)$ and $p^S(b, s)$ are the payments. For simplicity, for every b let $x^B(b) = \mathbb{E}_s[x(b, s)], p^B(b) = \mathbb{E}_s[p^B(b, s)]$ be the buyer's interim allocation and payment. Similarly, let $x^S(s) = \mathbb{E}_b[x(b, s)], p^S(s) = \mathbb{E}_b[p^S(b, s)]$ for every s.

Given any allocation rule x, we say x is *implementable* if there exists a payment rule $p = p^B = p^S$, such that $M = (x, p)$ is BIC, IR and SBB.

Theorem 2.1. *[MS83] (Impossibility Result) In bilateral trade, consider the allocation rule x such that $x(b, s) = \mathbb{1}[b \geq s]$ for all b and s. Then x is not implementable.*

First-best vs. Second-best: The impossibility result distinguishes two maximum GFT in two-sided markets. The *first-best* GFT (denoted as FB-GFT) is defined as the maximum expected GFT without any constraint. In double auctions with trading constraint $\mathcal{F}$, we have

$$\text{FB-GFT} = \mathbb{E}_{\mathbf{b}\sim D^B, \mathbf{s}\sim T^S}\left[\max_{A\in\mathcal{F}} \sum_{(i,j)\in A}(b_i - s_j)\right]$$

On the other hand, the *second-best* GFT (denoted as SB-GFT) is defined as the maximum expected GFT obtainable by any IR, BIC, ex-ante WBB mechanism.

In the same paper, they also presented an exact characterization of implementable allocation rules.

Theorem 2.2. *[MS83] In bilateral trade, an allocation rule x is implementable if and only if*

- $x^B(b)$ *is non-decreasing on* b. $x^S(s)$ *is non-increasing on* s_j.
- $\mathbb{E}_{b\sim D^B, s\sim D^S}[x(b, s) \cdot (\varphi(b) - \tau(s))] \geq 0$.

Here $\varphi(\cdot)$ is the Myerson's virtual value function for D^B, i.e. $\varphi(b) = b - \frac{1-F^B(b)}{f^B(b)}, \forall b$. $\tau(\cdot)$ is *an analogous definition for the seller. We will refer to it as the Myerson's virtual cost function for D^S. For every s, $\tau(s) = s + \frac{F^S(s)}{f^S(s)}$.*

Theorem 2.2 can be proved using the following lemma, which is known as the Myerson's lemma for two-sided markets.

Lemma 2.2. *[MS83] For any BIC, IR mechanism $M = (x, p^B, p^S)$ in bilateral trade, we have*

1. *$x^B(b)$ is non-decreasing on b. $x^S(s)$ is non-increasing on s_j.*

2. *For every b,*
$$p^B(b) = b \cdot x^B(b) - \int_0^b x^B(t)dt - \theta, \tag{2.3}$$
where θ is some non-negative constant.

3. *For every s,*
$$p^S(s) = s \cdot x^S(s) + \int_s^\infty x^S(t)dt + \eta, \tag{2.4}$$
where η is some non-negative constant.

Furthermore, if (p^B, p^S) satisfies Equation (5.2) and (5.3), then

$$\begin{aligned} \mathbb{E}_b[p^B(b)] &= \mathbb{E}_b\left[x^B(b) \cdot \varphi(b)\right] - \theta \\ \mathbb{E}_s[p^S(s)] &= \mathbb{E}_s\left[x^S(s) \cdot \tau(s)\right] + \eta \end{aligned} \tag{2.5}$$

2.5 Related Work

2.5.1 Revenue Maximization in Multi-item Auctions

In recent years, we have witnessed several breakthroughs in designing (approximately) optimal mechanisms in multi-dimensional settings. The black-box reduction by Cai et al. [CDW12a, CDW12b, CDW13a, CDW13b] shows that we can reduce any Bayesian mechanism design problem to a similar algorithm design problem via convex optimization. Through their reduction, it is proved that all optimal mechanisms can be characterized as a distribution of virtual welfare maximizers, where the virtual valuations are computed by an LP. Although this characterization provides important insights about the structure of the optimal mechanism, the optimal allocation rule is unavoidably randomized and might still be complex as the virtual valuations are only a solution of an LP.

Another line of work considers the "Simple vs. Optimal" auction design problem. For instance, a sequence of results [CHK07, CHMS10, CMS10, CMS15] show that sequential posted price mechanism can achieve $\frac{1}{33.75}$ of the optimal revenue, whenever the buyers have unit-demand valuations over independent items. Another series of results [HN12, CH13, LY13, BILW14, Yao15] show that the better of selling the items separately and running the VCG mechanism with per bidder entry fee achieves $\frac{1}{69}$ of the optimal revenue, whenever the buyers' valuations are additive over independent items. Cai et al. [CDW16] unified the two lines of results and improved the approximation ratios to $\frac{1}{8}$ for the additive case and $\frac{1}{24}$ for the unit-demand case using their duality framework.

Works in the literature have shown that simple mechanisms can approximate the optimal revenue even when buyers have more sophisticated valuations. For instance, Chawla and Miller [CM16] showed that the sequential two-part tariff mechanism can approximate the optimal revenue when buyers have matroid rank valuation functions over independent items. Their mechanism requires every buyer to pay an entry fee up front, and then run a sequential posted price mechanism on buyers who have accepted the entry fee. Our sequential posted price with entry fee mechanism is similar to their mechanism, but with the following major difference: since buyers are asked to pay the entry fee before the seller visits them, the buyers have to make their decisions based on the expected utility (assuming every other buyer behaves truthfully) they can receive. Hence, the mechanism is only guaranteed to be BIC and interim IR. While in our mechanism, the buyers can see what items are still available before paying the entry fee, therefore the decision making is straightforward and our mechanism is DSIC and ex-post IR. For valuations beyond matroid rank functions, Rubinstein and Weinberg [RW15] showed that for a single buyer whose valuation is subadditive over independent items, either grand bundling or selling the items separately achieves at least $\frac{1}{338}$ of the optimal revenue.

Our results in Chapter 3 draw a connection between approximating our benchmark using the sequential posted price with entry fee mechanisms and approximating optimal welfare with posted price mechanisms. A great work by Feldman et al. [FGL15] proved an approximation ratio of 2 (or $O(\log m)$) to the optimal welfare via posted price mechanisms, for fractionally-subadditive (or subadditive) buyers. Their results help us to prove the desired approximation to the optimal revenue. A recent breakthrough by Dutting et al. [DKL20] improved the approximating ratio for welfare to $O(\log\log m)$, when the buyer is subadditive. In the same paper, they also proved an $O(\log\log m)$-approximation to the optimal revenue, following a similar approach to the conference version of our work [CZ17].

The Cai-Devanur-Weinberg duality framework [CDW16] has been applied to other intriguing Mechanism Design problems. For example, Eden et al. showed that the better of selling separately and bundling together gets an $O(d)$-approximation for a single bidder with "complementarity-d valuations over independent items" [EFF$^+$16b]. The same authors also proved a Bulow-Klemperer result for regular i.i.d. and constrained additive bidders [EFF$^+$16a]. Liu and Psomas provided a Bulow-Klemperer result for dynamic auctions [LP16]. Finally, Brustle et al. [BCWZ17] extended the duality framework to two-sided markets and used it to design simple mechanisms for approximating the Gains from Trade.

Strong duality frameworks have recently been developed for one additive buyer [DDT13,DDT15, Gia14, GK14b, GK15]. These frameworks show that the dual problem of revenue maximization can be viewed as an optimal transport/bipartite matching problem. Hartline and Haghpanah provided an alternative duality framework in [HH15]. They showed that if certain paths exist, these paths provide a witness of the optimality of a certain Myerson-type mechanism, but these paths are not guaranteed to exist in general. Similar to the Cai-Devanur-Weinberg framework, Carroll [Car15] independently made use of a partial Lagrangian over incentive constraints. These duality frameworks have been successfully provide conditions under which a certain type of mechanism is optimal when there is a single unit-demand or additive bidder. However, none of these frameworks succeeds in yielding any approximately optimal results in multi-buyer settings.

2.5.2 Profit Maximization

The ad auction problem has been extensively studied in the literature [BMS12, FJM$^+$12, DIR14, EFG$^+$14, DPT16]. Signaling mechanisms had been the focus. In a signaling mechanism, the seller first sends a signal to the buyer based on the type of the viewer and according to a signaling scheme known to the buyer. The buyer updates her posterior belief of the viewer type after observing the signal. The seller then uses a mechanism tailored to the buyer's updated posterior to sell the ad displaying slot. Many results have been obtained regarding the revenue-optimal signaling scheme. Overall, the optimal signaling scheme may be highly complex and hard to pin down. Interestingly, Daskalakis et al. showed that even if we can find the optimal signaling scheme the corresponding mechanism can still be bounded away from the optimum [DPT16]. They showed that the optimal mechanism is direct and does not involve any signaling. Motivated by their result, we focus on simple and direct mechanisms.

In [DPT16], they also showed how to use simple mechanisms to approximate the auctioneer's profit in an ad auction. They established the result by reducing the problem to revenue maximization in multi-item auctions with an additive buyer. However, their reduction is ad-hoc and heavily relies on a specific property of their cost distribution, that is, the cost is always one of the δ_is (see Example 1.1 for the definition). When the cost distribution is general, their reduction no longer holds, and thus is inapplicable to our problem.

Another result that is related to ours is [MSL15]. They consider the problem of maximizing profit for the single additive buyer case, when the seller has a fixed production cost for each item. They propose a mechanism called Pure Bundling with Disposal for Cost (PBDC), where after buying the bundle, the customer is allowed to return any subset of goods for their production cost. They prove that such a mechanism is almost optimal for a large number of independent goods, and the better of a PBDC mechanism and the mechanism that sells items separately is a constant factor approximation for the optimal profit. For the single buyer and fixed cost case, their PBDC mechanism is identical to our permit-bundling mechanism. In our paper, we focus on a more general setting with multiple buyers and non-fixed costs.

2.5.3 Two-sided Markets

Gains from Trade: The main related works are on worst-case GFT approximation. Blumrosen and Mizrahi [BM16] guarantee an e-approximation to the first-best GFT in the setting of bilateral trade—one buyer, one seller, one item—when the buyer's distribution satisfies the monotone hazard rate condition. Colini-Baldeschi et al. [CBGdK$^+$17] show that a simple fixed price mechanism obtains an $O(\frac{1}{r})$-approximation to GFT in the bilateral trade and double auction settings, but a more careful setting of the fixed price gives an $O(\log \frac{1}{r})$-approximation for bilateral trade. Our result in Chapter 7 generalizes the $O(\log \frac{1}{r})$-approximation of [CBGdK$^+$17] to multi-dimensional settings, while providing an unconditional $O(\log n)$-approximation.

Other lines of work provide (1) *asymptotic* approximation guarantees in the number of items optimally traded for settings as general as multi-unit buyers and sellers and k types of items [McA92, SHA18b, SHA18a], (2) dual asymptotic and worst-case guarantees for double auctions and matching markets [BCGZ18], and (3) Bulow-Klemperer-style guarantees of the number of additional buyers (or sellers) needed in double auctions in order for the GFT of the new setting running a simple mechanism to beat the first-best GFT of the original setting [BGG20].

Multi-Dimensional Revenue: In the setting where one seller owns all of the items, has no cost for the items, and is the mechanism designer, much more is known. However, even when selling to a single additive bidder (e.g. with no feasibility constraints), posted prices can achieve at best an $O(\log n)$-approximation [HN12, LY13]. In order to obtain a constant-factor approximation for an additive buyer, Babaioff et al. [BILW14] use the better of posted prices and posting a price on the grand bundle, and a variation works for a single subadditive (which includes constrained-additive) buyer as well [RW15]. However, in a two-sided market where items are owned by separate sellers, it is not clear how to implement bundling in an incentive-compatible way. The mechanisms used to obtain constant-approximations for multiple constrained-additive, XOS, or subadditive buyers [CM16, CZ17] are only more complex.

Welfare in Two-Sided Markets: Colini-Baldeschi et al. [CdKLT16] consider welfare maximization in the double auction setting with matroid feasibility constraints. They generalize sequential posted price mechanisms (SPMs) to the two-sided market setting, guaranteeing a constant-factor approximation to welfare. The mechanism posts prices for each buyer-seller combination (not just for each item), visits the buyers and sellers simultaneously in the given order, and advances on either side when the price is rejected. Trade occurs when both sides accept the trade. Follow up work of Colini-Baldeschi et al. [CBGK+20] generalizes the idea to the setting where buyers are XOS and sellers are additive. Here, there is a posted price for each item, but only "high welfare" items are considered. The buyers visit and pick out the bundles they want among the high welfare items. Then sellers are given the opportunity to sell their entire bundle of items demanded by the buyers (but not any subset), and they are skipped with some probability. Like the previous work, this mechanism is ex-post IR, DSIC, and strongly BB (buyer payments equal seller payments). As only "high welfare" items are considered, it is possible for their mechanism to not trade any item when the minimum trade probability r is a constant.

Blumrosen and Dobzinski [BD16] give an IR, BIC, and strongly BB mechanism for bilateral trade that obtains in expectation a constant-fraction of the optimal welfare. Dütting et al. [DRT14] study welfare maximization in the prior-free setting and present DSIC, IR, and weakly BB (buyer payments exceed seller payments) mechanisms for double auctions with feasibility constraints on either side.

2.6 Lagrangian Duality

In this section we give a brief introduction of the partial Lagrangian dual of a linear program. We will use these definitions and properties in Chapter 3 and Chapter 4.

We consider the following linear program (primal) with variable x.

$$\begin{aligned} \max \quad & f(x) \\ \text{s.t.} \quad & Ax \leq b \\ & x \in \mathcal{P} \end{aligned}$$

Denote x^* the optimal solution of the primal. We take the partial Lagrangian dual by using the Lagrangian multiplier λ_i for each constraint $(Ax)_i \leq b_i$:

Partial Lagrangian Dual: $\min_{\lambda \geq 0} \max_{x \in \mathcal{P}} \mathcal{L}(x, \lambda)$, where $\mathcal{L}(x, \lambda) = f(x) + \lambda^T(b - Ax)$.

For every feasible dual solution $\lambda \geq 0$, let $D(\lambda) = \max_{x \in \mathcal{P}} \mathcal{L}(x, \lambda)$. Denote λ^* the optimal dual solution: $\lambda^* \in \min_{\lambda \geq 0} D(\lambda)$.

Weak Duality: For any feasible dual solution λ, it holds that $f(x^*) \leq D(\lambda)$.

Proof.

$$D(\lambda) \geq \mathcal{L}(x^*, \lambda) = f(x^*) + \lambda^T(b - Ax^*) \geq f(x^*)$$

□

Strong Duality: The value of the primal equals to the value of the partial Lagrangian dual. In other words, $f(x^*) = D(\lambda^*) = \min_{\lambda \geq 0} \max_{x \in \mathcal{P}} \mathcal{L}(x, \lambda)$.

2.7 Online Contention Resolution Scheme

In this section, we introduce a useful technique in mechanism design called *online contention resolution scheme* (OCRS). It was first studied by Feldman et al. [FSZ16].

An OCRS is an algorithm defined for the following online selection problem: There is a ground set I, and the elements are revealed one by one, with item i *active* with probability x_i independent of the other items. The algorithm is only allowed to accept active elements and has to irrevocably make a decision whether to accept an element before the next one is revealed. Moreover, the algorithm can only accept a set of elements subject to a feasibility constraint $\mathcal{F}$. We use the vector x to denote active probabilities for the elements and $R(x)$ to denote the random set of active elements.

Definition 2.3 (relaxation)**.** *We say that a polytope $P \subseteq [0,1]^{|I|}$ is a relaxation of $P_{\mathcal{F}}$ if it contains the same $\{0,1\}$-points, i.e., $P \cap \{0,1\}^{|I|} = P_{\mathcal{F}} \cap \{0,1\}^{|I|}$.*

Definition 2.4. *An* Online Contention Resolution Scheme *(OCRS) for a polytope $P \subseteq [0,1]^{|I|}$ and feasibility constraint $\mathcal{F}$ is an online algorithm that selects a feasible and active set $S \subseteq R(x)$ and $S \in \mathcal{F}$ for any $x \in P$. A* ***greedy OCRS*** *π greedily decides whether or not to select an element in each iteration: given the vector $x \in P$, it first determines a sub-constraint $\mathcal{F}_{\pi,x} \subseteq \mathcal{F}$. When element i is revealed, it accepts the element if and only if i is active and $S \cup \{i\} \in \mathcal{F}_{\pi,x}$, where S is the set of elements accepted so far. In most cases, we choose P to be $P_{\mathcal{F}}$, the convex hull of all characteristic vectors of feasible sets in $\mathcal{F}$: $P_{\mathcal{F}} = conv(\mathbb{1}_S \mid S \in \mathcal{F})$.*

Definition 2.5 ((δ, η)-selectability [FSZ16])**.** *For any $\delta, \eta \in (0,1)$, a greedy OCRS π for P and $\mathcal{F}$ is (δ,η)-selectable if for every $x \in \delta \cdot P$ and $i \in I$,*

$$\Pr[S \cup \{i\} \in \mathcal{F}_{\pi,x}, \forall S \subseteq R(x), S \in \mathcal{F}_{\pi,x}] \geq \eta.$$

The probability is taken over the randomness of $R(x)$ and the subconstraint $\mathcal{F}_{\pi,x}$. We slightly abuse notation and say that $\mathcal{F}$ is (δ,η)-selectable if there exists a (δ,η)-selectable greedy OCRS for $P_{\mathcal{F}}$ and $\mathcal{F}$.

Feldman et al. [FSZ16] proved that a broad class of downward-closed feasibility constraints, such as matroids, matching constraints and knapsack constraints, are all (δ,η)-selectable for some constant $\delta, \eta \in (0,1)$. Moreover, they prove that (δ,η)-selectability has nice composability. In further chapters, we will see how these results help us to prove that our mechanism achieves a constant factor approximation.

Definition 2.6 (Matching, Knapsack Constraint)**.** *Given an undirected graph $G = (V, E)$. $\mathcal{F} \subseteq 2^E$ is a* matching constraint *with respect to the ground set E if $\mathcal{F} = \{M \subseteq E : M$ is a matching in $G\}$.*

A knapsack constraint $\mathcal{F}$ *with respect to the ground set* I *is defined as:* $\mathcal{F} = \{S \subseteq I : \sum_{i \in S} c_i \leq 1\}$. *Here* $c_i \in [0, 1]$ *is the weight of element* i.

Lemma 2.3 (Selectability of Natural Constraints)**.** *[FSZ16]*

- *For any matroid constraint* $\mathcal{F}$ *and any* $\delta \in (0, 1)$*, there exists a* $(\delta, 1-\delta)$*-selectable greedy OCRS for* $P_{\mathcal{F}}$*. Moreover, for any* $\epsilon \in (0, 1-\delta)$*, there exists a* $(\delta, 1-\delta-\epsilon)$*-selectable greedy OCRS* π *for* $P_{\mathcal{F}}$*, and the running time of* π *is polynomial on the input size and* $1/\epsilon$*.*

- *For any matching constraint* $\mathcal{F}$ *and any* $\delta \in (0, 1)$*, there exists an efficient* $(\delta, e^{-2\delta})$*-selectable greedy OCRS for* $P_{\mathcal{F}}$*.*

- *For any knapsack constraint* $\mathcal{F}$ *and any* $\delta \in (0, \frac{1}{2})$*, there exists an efficient* $(\delta, \frac{1-2\delta}{2-2\delta})$*-selectable greedy OCRS for* $P_{\mathcal{F}}$*.*

Lemma 2.4 (Composability of Selectability)**.** *[FSZ16] Given two downward-closed constraints* $\mathcal{F}_1$ *and* $\mathcal{F}_2$ *with respect to the same ground set* I*. Let* $\mathcal{F} = \mathcal{F}_1 \cap \mathcal{F}_2$*. Suppose there exist a* (δ, η_1)*-selectable greedy OCRS* π_1 *for* P_1 *and* $\mathcal{F}_1$*, and a* (δ, η_2)*-selectable greedy OCRS* π_2 *for* P_2 *and* $\mathcal{F}_2$*. Then there exists a* $(\delta, \eta_1 \cdot \eta_2)$*-selectable greedy OCRS* π *for* $P_1 \cap P_2$ *and* $\mathcal{F}$*. When* $P_1 = P_{\mathcal{F}_1}$ *and* $P_2 = P_{\mathcal{F}_2}$*, as* $P_{\mathcal{F}_1} \cap P_{\mathcal{F}_2} \subseteq P_{\mathcal{F}}$*,* π *is also* $(\delta, \eta_1 \cdot \eta_2)$*-selectable for* $P_{\mathcal{F}}$ *and* $\mathcal{F}$*. Moreover,* π *is efficient computable given* π_1 *and* π_2*.*

Chapter 3

Revenue Maximization in Multi-item Auctions

In this chapter we study the revenue maximization problem in multi-item auctions. We design simple and truthful mechanisms to approximate the optimal revenue for multiple subadditive buyers.

In Section 3.1 we give an overview of our results and techniques shown in this chapter. In Section 3.2 we introduce some additional definitions and notations need to read this chapter. In Section 3.3 and Section 3.4, we review the duality framework of [CDW16]. In Section 3.5, we derive an upper bound of the optimal revenue for subadditive buyers by combining the duality framework with our new techniques. In Section 3.6, we use the single buyer case to familiarize the readers with some basic ideas and techniques used to bound the benchmark. In Section 3.7, we show how to upper bound the optimal revenue for multiple XOS (or subadditive) buyers with our sequential posted price with entry fee mechanisms. A buyer in the auction can also be called as a bidder. We use both words interchangeably throughout this chapter.

3.1 Results and Techniques in This Chapter

In this chapter, we unify and strengthen all the results in the literature via an extension of the duality framework proposed by Cai et al. [CDW16]. Moreover, we show that even when there are

		Additive or Unit-demand	Matroid-Rank	Constrained Additive	XOS	Subadditive
Single Buyer	Previous	6 [BILW14] or 4 [CMS15]	31.1*	31.1 [CM16]	338*	338 [RW15]
	This Paper	-	11*	11	40*	40
Multiple Buyer	Previous	8 [CDW16] or 24 [CDW16]	133 [CM16]	?	?	?
	This Paper	-	70*	70	268	$O(\log m)^\dagger$

* The result is implied by another result for a more general setting.

† The state-of-the-art result is an $O(\log\log m)$-approximation by Dutting et al. [DKL20].

Table 3.1: Comparison of approximation ratios between previous and current work.

multiple buyers with XOS valuation functions, there exists a simple, deterministic and Dominant Strategy Incentive Compatible (DSIC) mechanism that achieves a constant fraction of the optimal Bayesian Incentive Compatible (BIC) revenue. For subadditive valuations, our approximation ratio degrades to $O(\log m)$.

Informal Theorem 6. *There exists a simple, deterministic and DSIC mechanism that achieves a constant fraction of the optimal BIC revenue in multi-item settings, when the buyers' valuation distributions are XOS over independent items. When the buyers' valuation distributions are subadditive over independent items, our mechanism achieves at least $\Omega(\frac{1}{\log m})$ of the optimal BIC revenue, where m is the number of items.*

The original paper by Cai et al. [CDW16] provided a unified treatment for additive and unit-demand valuations. However, it is inadequate to provide an analyzable benchmark for even a single subadditive bidder. In this paper, we show how to extend their duality framework to accommodate general subadditive valuations. Using this extended framework, we substantially improve the approximation ratios for many of the settings discussed above, and in the meantime generalize the results to broader cases. See Table 3.1 for the comparison between the best ratios reported in the literature and the new ratios obtained in this work.

Our mechanism is either a *rationed sequential posted price mechanism* (**RSPM**) or an *anony-*

mous sequential posted price with entry fee mechanism (**ASPE**). In an RSPM, there is a price p_{ij} for buyer i if she wants to buy item j, and she is allowed to purchase at most one item. We visit the buyers in some arbitrary order and the buyer takes her favorite item among the available items given the item prices for her. Here we allow personalized prices, that is, p_{ij} could be different from p_{kj} if $i \neq k$. In an ASPE, every buyer faces the same collection of item prices $\{p_j\}_{j\in[m]}$. Again, we visit the buyers in some arbitrary order. For each buyer, we show her the available items and the associated price for each item. Then we ask her to pay the entry fee to enter the mechanism, which may depend on what items are still available and the identity of the buyer. If the buyer accepts the entry fee, she can proceed to purchase any item at the given prices; if she rejects the entry fee, then she will leave the mechanism without receiving anything. Given the entry fee and item prices, the decision making for the buyer is straightforward, as she only accepts the entry fee when the surplus for winning her favorite bundle is larger than the entry fee. Therefore, both RSPM and ASPE are DSIC and ex-post Individually Rational (ex-post IR).

3.1.1 Our Contributions

To obtain the new generalizations, we provide important extensions to the duality framework in [CDW16], as well as novel analytic techniques and new simple mechanisms.

1. Accommodating subadditive valuations: the original duality framework in [CDW16] already unified the additive case and unit-demand case by providing an approximately tight upper bound for the optimal revenue using a single dual solution. A trivial upper bound for the revenue is the social welfare, which may be arbitrarily bad in the worst case. The duality based upper bound in [CDW16] improves this trivial upper bound, the social welfare, by substituting the value of each buyer's favorite item with the corresponding Myerson's virtual value. However, the substitution is viable only when the following condition holds – the buyer's marginal gain for adding an item solely depends on her value for that item (assuming it's feasible to add that item[1]), but not the set of items she has already received. This applies to valuations that are additive, unit-demand and more

1. WLOG, we can reduce any constrained additive valuation to an additive valuation with a feasibility constraint (see Definition 2.2)

generally constrained additive, but breaks under more general valuation functions, e.g., submodular, XOS or subadditive valuations. As a consequence, the original dual solution from [CDW16] fails to provide a nice upper bound for more general valuations. To overcome this difficulty, we take a different approach. Instead of directly studying the dual of the original problem, we first relax the valuations and argue that the optimal revenue of the relaxed valuation is comparable to the original one. Then, since we choose the relaxation in a particular way, by applying a dual solution similar to the one in [CDW16] to the relaxed valuation, we recover an upper bound of the optimal revenue for the relaxed valuation resembling the appealing format of the one in [CDW16]. Combining these two steps, we obtain an upper bound for subadditive valuations that is easy to analyze. Indeed, we use our new upper bound to improve the approximation ratio for a single subadditive buyer from 338 [RW15] to 40. See Section 3.5.1 for more details.

2. An adaptive dual: our second major change to the framework is that we choose the dual in an adaptive manner. In [CDW16], a dual solution λ is chosen up front inducing a virtual value function $\Phi(\cdot)$, then the corresponding optimal virtual welfare is used as a benchmark for the optimal revenue. Finally, it is shown that the revenue of some simple mechanism is within a constant factor of the optimal virtual welfare. Unfortunately, when the valuations are beyond additive and unit-demand, the optimal virtual welfare for this particular choice of virtual value function becomes extremely complex and hard to analyze. Indeed, it is already challenging to bound when the buyers' valuations are k-demand. In this paper, we take a more flexible approach. For any particular allocation rule σ, we tailor a special dual $\lambda^{(\sigma)}$ based on σ in a fashion that is inspired by Chawla and Miller's ex-ante relaxation [CM16]. Therefore, the induced virtual valuation $\Phi^{(\sigma)}$ also depends on σ. By duality, we can show that the optimal revenue obtainable by σ is still upper bounded by the virtual welfare with respect to $\Phi^{(\sigma)}$ under allocation rule σ. Since the virtual valuation is designed specifically for allocation σ, the induced virtual welfare is much easier to analyze. Indeed, we manage to prove that for any allocation σ the induced virtual welfare is within a constant factor of the revenue of some simple mechanism, when bidders have XOS valuations. See Section 3.5.2 and 3.5.3 for more details.

3. A novel analysis and new mechanism: with the two contributions above, we manage to derive an upper bound of the optimal revenue similar to the one in [CDW16] but for subadditive bidders. The third major contribution of this paper is a novel approach to analyzing this upper bound. The analysis in [CDW16] essentially breaks the upper bound into three different terms– Single, Tail and Core, and bound them separately. All three terms are relatively simple to bound for additive and unit-demand buyers, but for more general settings the Core becomes much more challenging to handle. Indeed, the analysis in [CDW16] was insufficient to tackle the Core even when the buyers have k-demand valuations[2]– a very special case of matroid rank valuations, which itself is a special case of XOS or subadditive valuations. Rubinstein and Weinberg [RW15] showed how to approximate the Core for a single subadditive bidder using grand bundling, but their approach does not apply to multiple bidders. Yao [Yao15] showed how to approximate the Core for multiple additive bidders using a VCG with per bidder entry fee mechanism, but again it is unclear how his approach can be extended to multiple k-demand bidders. A recent paper by Chawla and Miller [CM16] finally broke the barrier of analyzing the Core for multiple k-demand buyers. They showed how to bound the Core for matroid rank valuations using a sequential posted price mechanism by applying the *online contention resolution scheme (OCRS)* developed by Feldman et al. [FSZ16]. The connection with OCRS is an elegant observation, and one might hope the same technique applies to more general valuations. Unfortunately, OCRS is only known to exist for special cases of downward closed constraints, and as we show in Section 3.7.2, the approach by Chawla and Miller cannot yield any constant factor approximation for general constrained additive valuations.

We take an entirely different approach to bound the Core. Here we provide some intuition behind our mechanism and analysis. The Core is essentially the optimal social welfare induced by some truncated valuation v', and our goal is to design a mechanism that extracts a constant fraction of the welfare as revenue. Let M be any sequential posted price mechanism. A key observation is that when bidder i's valuation is subadditive over independent items, her utility in M, which is the largest surplus she can achieve from the unsold items, is also subadditive over independent

2. The class of k-demand valuations is a generalization of unit-demand valuations, where the buyer's value is additive up to k items.

items. If we can argue that her utility function is a-Lipschitz (Definition 3.4) with some small a, Talagrand's concentration inequality [Tal95, Sch03] allows us to set an entry fee for the bidder so that we can extract a constant fraction of her utility just through the entry fee. If we modify M by introducing an entry fee for every bidder, according to Talagrand's concentration inequality, the new mechanism M' should intuitively have revenue that is a constant fraction of the social welfare obtained by M [3]. Therefore, if there exists a sequential posted price mechanism M that achieves a constant fraction of the optimal social welfare under the truncated valuation v', the modified mechanism M' can obtain a constant fraction of Core as revenue. Surprisingly, when the bidders have XOS valuations, Feldman et al. [FGL15] showed that there exists an anonymous sequential posted price mechanism that always obtains at least half of the optimal social welfare. Hence, an anonymous sequential posted price with per bidder entry fee mechanism should approximate the Core well, and this is exactly the intuition behind our ASPE mechanism.

To turn the intuition into a theorem, there are two technical difficulties that we need to address: (i) the Lipschitz constants of the bidders' utility functions turn out to be too large (ii) we deliberately neglected the difference in bidders' behavior under M and M' in hope to keep our discussion in the previous paragraph intuitive. However, due to the entry fee, bidders may end up purchasing completely different items under M and M', so it is not straightforward to see how one can relate the revenue of M' to the welfare obtained by M. See Section 3.7.2 for a more detailed discussion on how we overcome these two difficulties.

3.2 Notation in This Chapter

In this chapter, we focus on revenue maximization in the combinatorial auction with n independent bidders and m heterogenous items. The valuation of each bidder is subadditive over independent items (see Definition 2.1). We denote bidder i's type t_i as $\langle t_{ij}\rangle_{j=1}^m$, where t_{ij} is bidder i's private information about item j. For each i, j, we assume t_{ij} is drawn independently from the distribution D_{ij}. Let $D_i = \times_{j=1}^m D_{ij}$ be the distribution of bidder i's type and $D = \times_{i=1}^n D_i$ be the distribution of

3. M's welfare is simply its revenue plus the sum of utilities of the bidders, and M' can extract some extra revenue from the entry fee, which is a constant fraction of the total utility from the bidders.

the type profile. We use T_{ij} (or T_i, T) and f_{ij} (or f_i, f) to denote the support and density function of D_{ij} (or D_i, D). For notational convenience, we let t_{-i} to be the types of all bidders except i and $t_{<i}$ (or $t_{\leq i}$) to be the types of the first $i-1$ (or i) bidders. Similarly, we define D_{-i}, T_{-i} and f_{-i} for the corresponding distributions, support sets and density functions. When bidder i's type is t_i, her valuation for a set of items S is denoted by $v_i(t_i, S)$. For every item j, we use $V_i(t_{ij})$ to denote $v_i(t_i, \{j\})$, as it only depends on t_{ij}.

Given D and $v = \{v_i(\cdot,\cdot)\}_{i\in[n]}$, we use $\textsc{Rev}(M, v, D)$ to denote the expected revenue of a BIC mechanism M. Throughout this chapter, we use the following notations for the simple mechanisms we consider.

Single-Bidder Mechanisms:

- SRev(v, D) denotes the optimal expected revenue achievable by any posted price mechanism that only allows the buyer to purchase at most one item, and we use SRev for short if there is no confusion[4].

- BRev(v, D) denotes the optimal expected revenue achievable by selling a grand bundle and we use BRev for short if there is no confusion.

Multi-Bidder Mechanisms:

- PostRev(v, D) denotes the optimal expected revenue achievable by selling the items via an RSPM to the bidders, and we use PostRev for short when there is no confusion.

- APostEnRev(v, D) denotes the optimal expected revenue achievable by selling the items via an ASPE to the bidders, and we use APostEnRev for short when there is no confusion.

Single-Dimensional Copies Setting: In the analysis for unit-demand bidders in [CHMS10, CDW16], the optimal revenue is upper bounded by the optimal revenue in the single-dimensional copies setting defined in [CHMS10]. We use the same technique. We construct nm agents, where agent (i, j) has value $V_i(t_{ij})$ of being served with $t_{ij} \sim D_{ij}$, and we are only allow to use matchings,

4. The mechanism is slightly different from selling separately, as we only allow the buyer to purchase at most one item.

that is, for each i at most one agent (i,k) is served and for each j at most one agent (k,j) is served[5]. Notice that this is a single-dimensional setting, as each agent's type is specified by a single number. Let $\text{OPT}^{\text{Copies-UD}}$ be the optimal BIC revenue in this copies setting.

Continuous vs. Discrete Distributions: We explicitly assume that the input distributions are discrete. Nevertheless, it is known that every D can be discretized into D^+ such that the optimal revenue for D and D^+ are within $(1 \pm \epsilon)$ of each other [CDW16]. So our results also apply to continuous distributions.

3.2.1 Our Mechanisms

In this section, we introduce a class of mechanisms called *Sequential Posted Price with Entry Fee.* For each bidder i, the mechanism first determines a posted price ξ_{ij} for each item j and an entry fee function $\delta_i(\cdot) : 2^{[m]} \to \mathbb{R}_{\geq 0}$ for each bidder i that maps the set of available items to a real value entry fee. The seller visits the bidders sequentially in some arbitrary order. For simplicity, we assume the bidders are visited in the lexicographical order. When bidder i is visited, let $S_i(t_{<i})$ be the set of items that are still available. Clearly, this set only depends on the types of bidders who are visited before i. The mechanism shows the set $S_i(t_{<i})$ to bidder i and asks her for an entry fee $\delta_i(S_i(t_{<i}))$. If she accepts the entry fee, she can enter the mechanism and take her favorite bundle S_i^* by paying $\sum_{j \in S_i^*} \xi_{ij}$.

If there exist multiple bundles with the same maximum surplus, the bidder can break ties arbitrarily. Sometimes, there is a feasibility constraint $\mathcal{F}$ on what items a buyer can purchase. In particular, if we say the mechanism is rationed, then $\mathcal{F} = \{\emptyset\} \cup \{\{j\} \mid j \in [m]\}$, i.e., a buyer can purchase at most one item. Formally, the favorite bundle S_i^* is defined as follows: $S_i^* = \text{argmax}_{S \subseteq S_i(t_{<i}) \wedge S \in \mathcal{F}} v_i(t_i, S) - \sum_{j \in S} \xi_{ij}$.

See Algorithm 3.1 for the formal specification of the above mechanism. Notice that before the bidder decides whether to pay the entry fee, she is aware of the set $S_i(t_{<i})$ which contains all

5. This is exactly the copies setting used in [CHMS10], if every bidder i is unit-demand and has value $V_i(t_{ij})$ with type t_i. Notice that this unit-demand multi-dimensional setting is equivalent as adding an extra constraint, each buyer can purchase at most one item, to the original setting with subadditive bidders.

Mechanism 3.1 Sequential Posted Price with Entry Fee Mechanism

Require: ξ_{ij} is the price for bidder i to purchase item j and $\delta_i(\cdot)$ is bidder i's entry fee function.

$S \leftarrow [m]$
for $i \in [n]$ **do**
    Show bidder i the set of available items S, and define entry fee as $\delta_i(S)$.
    if Bidder i pays the entry fee $\delta_i(S)$ **then**
        i receives her favorite bundle S_i^*, paying $\sum_{j \in S_i^*} \xi_{ij}$.
        $S \leftarrow S \backslash S_i^*$.
    else
        i gets nothing and pays 0.
    end if
end for

available items. Thus, she can compute her favorite bundle S_i^* and the corresponding utility if she chooses to enter the mechanism. She can then compare that utility with the entry fee and accept the entry fee if the former is greater than the latter. The mechanism described above is therefore deterministic and DSIC. Throughout this paper, we focus on the following two special cases of this class of mechanisms:

-Rationed Sequential Posted Price Mechanism (RSPM): Every buyer can purchase at most one item and the mechanism always charges 0 entry fee, i.e., $\mathcal{F} = \{\emptyset\} \cup \{\{j\} \mid j \in [m]\}$ and $\delta_i(S) = 0$ for all i and S.

-Anonymous Sequential Posted Price with Entry Fee Mechanism (ASPE): The mechanism uses anonymous posted prices, i.e., $\xi_{ij} = \xi_{kj}$ for any item j and bidders $i \neq k$, but may charge positive and personalized entry fee. Also, any buyer can purchase any bundle available once she has paid the entry fee, i.e., $\mathcal{F} = 2^{[m]}$.

3.3 Duality Framework

The focus of [CDW16] was on additive and unit-demand valuations and their respective dual was derived from an LP that is only meaningful for constrained additive valuations. In order to tackle general valuations, we need to apply the duality framework to an LP that is meaningful for general valuations. Instead of using the "implicit forms" LP from [CDW13b, CDW16], we choose a slightly different and more intuitive LP formulation (see Figure 3.1). For all bidders i and types $t_i \in T_i$, we use $p_i(t_i)$ as the interim price paid by bidder i and $\sigma_{iS}(t_i)$ as the interim probability of receiving the exact bundle S. To ease the notation, we use a special type $\varnothing$ to represent the choice of not participating in the mechanism. More specifically, $\sigma_{iS}(\varnothing) = 0$ for any S and $p_i(\varnothing) = 0$. Now a Bayesian IR (BIR) constraint is simply another BIC constraint: for any type t_i, bidder i will not want to lie to type $\varnothing$. We let $T_i^+ = T_i \cup \{\varnothing\}$.

Following the recipe provided by [CDW16], we take the partial Lagrangian dual of the LP in Figure 3.1 by lagrangifying the BIC constraints. Let $\lambda_i(t_i, t_i')$ be the Lagrange multiplier associated with the BIC constraint that if bidder i's true type is t_i she will not prefer to lie to type t_i' (see Figure 3.2 and Definition 3.1). As shown in [CDW16], the dual solution has finite value if and only if the dual variables λ_i form a valid flow for every bidder i. The reason is that the payments $p_i(t_i)$ are unconstrained variables, therefore the corresponding coefficients must be 0 in order for the dual solution to have finite value. It turns out when all these coefficients are 0, the dual variables λ can be interpreted as a flow described in Lemma 3.1. We refer the readers to [CDW16] for a complete proof. From now on, we only consider λ that corresponds to a flow.

Variables:

- $p_i(t_i)$, for all bidders i and types $t_i \in T_i$, denoting the expected price paid by bidder i when reporting type t_i over the randomness of the mechanism and the other bidders' types.
- $\sigma_{iS}(t_i)$, for all bidders i, all bundles of items $S \subseteq [m]$, and types $t_i \in T_i$, denoting the probability that bidder i receives **exactly** the bundle S when reporting type t_i over the randomness of the mechanism and the other bidders' types.

Constraints:

- $\sum_{S \subseteq [m]} \sigma_{iS}(t_i) \cdot v_i(t_i, S) - p_i(t_i) \geq \sum_{S \subseteq [m]} \sigma_{iS}(t_i') \cdot v_i(t_i, S) - p_i(t_i')$, for all bidders i, and types $t_i \in T_i, t_i' \in T_i^+$, guaranteeing that the reduced form mechanism (σ, p) is BIC and Bayesian IR.
- $\sigma \in P(D)$, guaranteeing σ is feasible.

Objective:

- $\max \sum_{i=1}^{n} \sum_{t_i \in T_i} f_i(t_i) \cdot p_i(t_i)$, the expected revenue.

Figure 3.1: A Linear Program (LP) for Revenue Optimization.

Definition 3.1. *Let $\mathcal{L}(\lambda, \sigma, p)$ be the partial Lagrangian defined as follows:*

$$\begin{aligned}
&\mathcal{L}(\lambda, \sigma, p) \\
=&\sum_{i=1}^{n}\left(\sum_{t_i \in T_i} f_i(t_i)\cdot p_i(t_i) + \sum_{t_i\in T_i, t_i'\in T_i^+} \lambda_i(t_i, t_i') \cdot \left(\sum_{S\subseteq[m]} v_i(t_i, S)\cdot\left(\sigma_{iS}(t_i) - \sigma_{iS}(t_i')\right) - \left((p_i(t_i) - p_i(t_i')\right)\right)\right) \qquad (3.1)\\
=&\sum_{i=1}^{n}\left(\sum_{t_i\in T_i} p_i(t_i)\cdot\left(f_i(t_i) + \sum_{t_i'\in T_i}\lambda_i(t_i', t_i) - \sum_{t_i'\in T_i^+}\lambda_i(t_i, t_i')\right)\right)\\
&+\sum_{i=1}^{n}\left(\sum_{t_i\in T_i}\sum_{S\subseteq[m]}\sigma_{iS}(t_i)\cdot\left(v_i(t_i,S)\cdot\sum_{t_i'\in T_i^+}\lambda_i(t_i, t_i') - \sum_{t_i'\in T_i}\left(v_i(t_i', S)\cdot\lambda_i(t_i', t_i)\right)\right)\right) \quad (\sigma_i(\varnothing) = \boldsymbol{0},\ p_i(\varnothing) = 0) \qquad (3.2)
\end{aligned}$$

Variables:

- $\lambda_i(t_i, t_i')$ for all $i, t_i \in T_i, t_i' \in T_i^+$, the Lagrangian multipliers for Bayesian IC and IR constraints.

Constraints:

- $\lambda_i(t_i, t_i') \geq 0$ for all $i, t_i \in T_i, t_i' \in T_i^+$, guaranteeing that the Lagrangian multipliers are non-negative.

Objective:

- $\min_{\lambda} \max_{\sigma\in P(D), p} \mathcal{L}(\lambda, \sigma, p)$.

Figure 3.2: Partial Lagrangian of the Revenue Maximization LP.

Lemma 3.1 (Useful Dual Variables [CDW16])**.** *A set of feasible duals λ is* useful *if* $\max_{\sigma\in P(D),p}\mathcal{L}(\lambda, \sigma, p) < \infty$*. λ is useful iff for each bidder i, λ_i forms a valid flow, i.e., iff the following satisfies flow conservation at all nodes except the source and the sink:*

1. *Nodes: A super source s and a super sink $\varnothing$, along with a node t_i for every type $t_i \in T_i$.*

2. *An edge from s to t_i with flow $f_i(t_i)$, for all $t_i \in T_i$.*

3. *An edge from t_i to t_i' with flow $\lambda_i(t_i, t_i')$ for all $t_i \in T_i$, and $t_i' \in T_i^+$ (including the sink).*

Definition 3.2 (Virtual Value Function)**.** *For each flow λ, we define a corresponding virtual value* *function $\Phi(\cdot)$, such that for every bidder i, every type $t_i \in T_i$ and every set $S \subseteq [m]$,*

$$\Phi_i(t_i, S) = v_i(t_i, S) - \frac{1}{f_i(t_i)} \sum_{t_i' \in T_i} \lambda_i(t_i', t_i) \left(v_i(t_i', S) - v_i(t_i, S)\right).$$

The proof of Theorem 3.1 is essentially the same as in [CDW16]. We include it in Appendix A.1 for completeness.

Theorem 3.1 (Virtual Welfare $\geq$ Revenue [CDW16])**.** *For any flow λ and any BIC mechanism $M = (\sigma, p)$, the revenue of M is $\leq$ the virtual welfare of σ w.r.t. the virtual valuation $\Phi(\cdot)$ corresponding to λ.*

$$\sum_{i=1}^{n} \sum_{t_i \in T_i} f_i(t_i) \cdot p_i(t_i) \leq \sum_{i=1}^{n} \sum_{t_i \in T_i} f_i(t_i) \sum_{S \subseteq [m]} \sigma_{iS}(t_i) \cdot \Phi_i(t_i, S)$$

Let λ^* be the optimal dual variables and $M^* = (\sigma^*, p^*)$ be the revenue optimal BIC mechanism, then the expected virtual welfare with respect to Φ^* (induced by λ^*) under σ^* equals to the expected revenue of M^*.

3.4 Recap: Flow for Additive Valuations

In this section, we give a recap of the flow for additive valuations in [CDW16] and the appealing properties of the corresponding virtual valuation functions. When the valuations are additive, we simply view t_{ij} as bidder i's value for receiving item j. Although there are many possible ways to define a flow, we focus on a class of simple ones. Every flow in this class $\lambda^{(\beta)}$ is parametrized by a set of parameters $\beta = \{\beta_{ij}\}_{i \in [n], j \in [m]} \in \mathbb{R}^{nm}$. Based on $\beta_i = \{\beta_{ij}\}_{j \in [m]}$, we first partition the type space T_i for each bidder i into $m+1$ regions:

- $R_0^{(\beta_i)}$ contains all types t_i such that $t_{ij} < \beta_{ij}$ for all $j \in [m]$.
- $R_j^{(\beta_i)}$ contains all types t_i such that $t_{ij} - \beta_{ij} \geq 0$ and j is the smallest index in $\text{argmax}_k\{t_{ik} - \beta_{ik}\}$.

We use essentially the same flow as in [CDW16]. Here we provide a partial specification and state some desirable properties of the flow. See Figure 3.4 for an example with 2 items and [CDW16] for a complete description of the flow.

Partial Specification of the flow $\lambda^{(\beta)}$:

1. For every type t_i in region $R_0^{(\beta_i)}$, the flow goes directly to $\varnothing$ (the super sink).
2. For all $j > 0$, any flow entering $R_j^{(\beta_i)}$ is from s (the super source) and any flow leaving $R_j^{(\beta_i)}$ is to $\varnothing$.
3. For all t_i and t_i' in $R_j^{(\beta_i)}$ ($j > 0$), $\lambda_i^{(\beta)}(t_i, t_i') > 0$ only if t_i and t_i' only differ in the j-th coordinate.

Figure 3.3: Partial Specification of the flow $\lambda^{(\beta)}$.

Lemma 3.2 ([CDW16][6]). *For any β, there exists a flow $\lambda_i^{(\beta)}$ such that the corresponding virtual value function $\Phi_i(t_i, \cdot)$ satisfies the following properties:*

- *For any $t_i \in R_0^{(\beta_i)}$, $\Phi_i(t_i, S) = \sum_{k \in S} t_{ik}$.*
- *For any $j > 0$, $t_i \in R_j^{(\beta_i)}$,*

$$\Phi_i(t_i, S) \leq \sum_{k \in S \wedge k \neq j} t_{ik} + \tilde{\varphi}_{ij}(t_{ij}) \cdot \mathbb{1}[j \in S],$$

where $\tilde{\varphi}_{ij}(\cdot)$ is Myerson's ironed virtual value function for D_{ij}.

The properties above are crucial for showing the approximation results for simple mechanisms in [CDW16]. One of the key challenges in approximating the optimal revenue is how to provide a tight upper bound. A trivial upper bound is the social welfare, which may be arbitrarily bad in the worst case. By plugging the virtual value functions in Lemma 3.2 into the partial Lagrangian, we obtain a new upper bound that replaces the value of the buyer's favorite item with the corresponding Myerson's ironed virtual value. As demonstrated in [CDW16], this new upper bound is at most 8 times larger than the optimal revenue when the buyers are additive, and its appealing structure allows the authors to compare the revenue of simple mechanisms to it. In Section 3.5, we identify some difficulties in directly applying this flow to subadditive valuations. Then we show how to overcome these difficulties by relaxing the subadditive valuations and obtain a similar upper bound.

6. Note that this Lemma is a special case of Lemma 3 in [CDW16] when the valuations are additive.

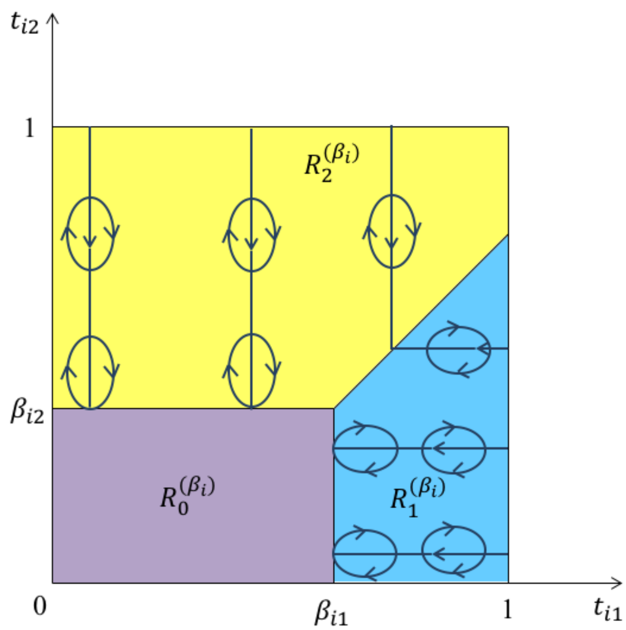


Figure 3.4: An example of $\lambda_i^{(\beta)}$ for additive bidders with two items.

3.5 Canonical Flow and Properties of the Virtual Valuations

In this section, we present a canonical way of setting the dual variables/flow that induces our benchmarks.

Although any flow can provide a finite upper bound of the optimal revenue, we focus on a particular class of flows, in which every flow $\lambda^{(\beta)}$ is parametrized by a set of parameters $\beta = \{\beta_{ij}\}_{i\in[n],j\in[m]} \in \mathbb{R}^{nm}_{\geq 0}$. Based on β, we partition the type set T_i of each buyer i into $m+1$ regions: **(i)** $R_0^{(\beta_i)}$ contains all types t_i such that $V_i(t_{ij}) < \beta_{ij}$ for all $j \in [m]$. **(ii)** $R_j^{(\beta_i)}$ contains all types t_i such that $V_i(t_{ij}) - \beta_{ij} \geq 0$ and j is the smallest index in $\text{argmax}_k\{V_i(t_{ik}) - \beta_{ik}\}$. Intuitively, if we view β_{ij} as the price of item j for bidder i, then $R_0^{(\beta_i)}$ contains all types in T_i that cannot afford any item, and any $R_j^{(\beta_i)}$ with $j > 0$ contains all types in T_i whose "favorite" item is j. We first provide a Partial Specification of the flow $\lambda^{(\beta)}$:

1. For every type t_i in region $R_0^{(\beta_i)}$, the flow goes directly to $\varnothing$ (the super sink).

2. For all $j > 0$, any flow entering $R_j^{(\beta_i)}$ is from s (the super source) and any flow leaving $R_j^{(\beta_i)}$ is to $\varnothing$.

3. For all t_i and t_i' in $R_j^{(\beta_i)}$ ($j > 0$), $\lambda_i^{(\beta)}(t_i, t_i') > 0$ only if t_i and t_i' only differ in the j-th coordinate.

For additive valuations and any type $t_i \in R_j^{(\beta_i)}$, the contribution to the virtual value function $\Phi(t_i, S)$ from any type $t_i' \in R_j^{(\beta_i)}$ is either 0 if $j \notin S$, or $\lambda_i^{(\beta)}(t_i', t_i)(v_i(t_i', S) - v_i(t_i, S)) = \lambda_i^{(\beta)}(t_i', t_i)(t_{ij}' - t_{ij})$ if t_i, t_i' only differs on the j-th coordinate and $j \in S$. In either case, the contribution does not depend on t_{ik} for any $k \neq j$. This is the key property that allows [CDW16] to choose a flow such that the value of the favorite item is replaced by the corresponding Myerson's ironed virtual value in the virtual value function $\Phi_i(t_i, \cdot)$. Unfortunately, this property no longer holds for subadditive valuations. When $j \in S$ and $\lambda_i^{(\beta)}(t_i', t_i) > 0$, the contribution $\lambda_i^{(\beta)}(t_i', t_i)(v_i(t_i', S) - v_i(t_i, S))$ heavily depends on t_{ik} of all the other item $k \in S$. All we can conclude is that the contribution lies in the range $[-\lambda_i^{(\beta)}(t_i', t_i) \cdot V_i(t_{ij}), \lambda_i^{(\beta)}(t_i', t_i) \cdot V_i(t_{ij}')]$[7], but this is not sufficient for us to convert the value of item j into the corresponding Myerson's ironed virtual value.

3.5.1 Valuation Relaxation

This is the first major barrier for extending the duality framework to accommodate subadditive valuations. We overcome it by considering a relaxation of the valuation functions. More specifically, for any β, we construct another function $v_i^{(\beta_i)}(\cdot, \cdot) : T_i \times 2^{[m]} \mapsto \mathbb{R}_{\geq 0}$ for every buyer i such that: (i) for any t_i, $v_i^{(\beta_i)}(t_i, \cdot)$ is subadditive and monotone, and for every bundle S the new value $v_i^{(\beta_i)}(t_i, S)$ is no smaller than the original value $v_i(t_i, S)$; (ii) for any BIC mechanism M with respect to the original valuations, there exists another mechanism $M^{(\beta)}$ that is BIC with respect to the new valuations and its revenue is comparable to the revenue of M; (iii) for the new valuations $v^{(\beta)}$, there exists a flow whose induced virtual value functions have properties similar to those in the additive case. Property (ii) implies that the optimal revenue with respect to $v^{(\beta)}$ can serve as a proxy for the original optimal revenue. Moreover, due to Theorem 3.1, the optimal revenue for $v^{(\beta)}$ is upper bounded by the partial Lagrangian dual with respect to $v^{(\beta)}$, which has an appealing format similar to the additive case by property (iii). Thus, we obtain a benchmark for subadditive bidders that resembles the benchmark for additive bidders in [CDW16].

7. $v_i(t, \cdot)$ is subadditive and monotone for every type $t \in T_i$, therefore $v_i(t_i, S) \in [v_i(t_i, S \backslash \{j\}), v_i(t_i, S \backslash \{j\}) + V_i(t_{ij})]$ and $v_i(t_i', S) \in [v_i(t_i', S \backslash \{j\}), v_i(t_i', S \backslash \{j\}) + V_i(t_{ij}')]$.

Definition 3.3 (Relaxed Valuation)**.** *Given β, for any buyer i, define $v_i^{(\beta_i)}(t_i, S) = v_i(t_i, S\backslash\{j\}) + V_i(t_{ij})$, if the "favorite" item is in S, i.e., $t_i \in R_j^{(\beta_i)}$ and $j \in S$. Otherwise, define $v_i^{(\beta_i)}(t_i, S) = v_i(t_i, S)$.*

In the next lemma, we show that for any BIC mechanism M for v, there exists a BIC mechanism $M^{(\beta)}$ for $v^{(\beta)}$ such that its revenue is comparable to the revenue of M (property (ii)). Moreover, the ex-ante probability for any buyer i to receive any item j in $M^{(\beta)}$ is no greater than in M (property (i)). We will see later that this is an important property for our analysis. The proof of Lemma 3.3 is similar to the ϵ-BIC to BIC reduction in [HKM11, BH11, DW12] and can be found in Appendix A.6.

Lemma 3.3. *For any β and any BIC mechanism M for subadditive valuation $\{v_i(t_i, \cdot)\}_{i\in[n]}$ with $t_i \sim D_i$ for all i, there exists a BIC mechanism $M^{(\beta)}$ for valuations $\{v_i^{(\beta_i)}(t_i, \cdot)\}_{i\in[n]}$ with $t_i \sim D_i$ for all i, such that*

(i) $\sum_{t_i\in T_i} f_i(t_i) \cdot \sum_{S:j\in S} \sigma_{iS}^{(\beta)}(t_i) \leq \sum_{t_i\in T_i} f_i(t_i) \cdot \sum_{S:j\in S} \sigma_{iS}(t_i)$, *for all i and j,*

(ii) $\textsc{Rev}(M, v, D) \leq 2\cdot\textsc{Rev}(M^{(\beta)}, v^{(\beta)}, D) + 2\cdot\sum_i \sum_{t_i\in T_i} \sum_{S\subseteq[m]} f_i(t_i)\cdot\sigma_{iS}^{(\beta)}(t_i)\cdot\left(v_i^{(\beta_i)}(t_i, S) - v_i(t_i, S)\right).$

$\textsc{Rev}(M, v, D)$ (or $\textsc{Rev}(M^{(\beta)}, v^{(\beta)}, D)$) is the revenue of the mechanism M (or $M^{(\beta)}$) while the buyers' types are drawn from D and buyer i's valuation is $v_i(t_i, \cdot)$ (or $v_i^{(\beta_i)}(t_i, \cdot)$). $\sigma_{iS}(t_i)$ (or $\sigma_{iS}^{(\beta)}(t_i)$) is the probability of buyer i receiving exactly bundle S when her reported type is t_i in mechanism M (or $M^{(\beta)}$).

3.5.2 Virtual Valuation for the Relaxed Valuation

For any β, based on the same partition of the type sets as in the beginning of Section 3.5, we construct a flow $\lambda^{(\beta)}$ that respects the partial specification, such that the corresponding virtual valuation function for $v^{(\beta)}$ has the same appealing properties as in the additive case. For the relaxed valuation, as $\lambda_i^{(\beta)}(t_i, t_i')$ is only positive for types t_i, $t_i' \in R_j^{(\beta_i)}$ that only differ in the j-th coordinate, the contribution from item j to the virtual valuation solely depends on t_{ij} and t_{ij}' but not t_{ik} for any other item $k \in S$. Notice that this property does not hold for the original valuation, and it is the main reason why we choose the relaxed valuation as in Definition 3.3. Moreover, we can choose $\lambda_i^{(\beta)}$ carefully so that the virtual valuation of $v^{(\beta)}$ has the following format:

Lemma 3.4. *Let F_{ij} be the distribution of $V_i(t_{ij})$ when t_{ij} is drawn from D_{ij}. For any β, there exists a flow $\lambda_i^{(\beta)}$ such that the corresponding virtual value function $\Phi_i^{(\beta_i)}(t_i,\cdot)$ of valuation $v_i^{(\beta_i)}(t_i,\cdot)$ satisfies the following properties:*

1. For any $t_i \in R_0^{(\beta_i)}$, $\Phi_i^{(\beta_i)}(t_i,S) = v_i(t_i,S)$.

2. For any $j>0$, $t_i \in R_j^{(\beta_i)}$, $\Phi_i^{(\beta_i)}(t_i,S) \le v_i(t_i,S)\cdot \mathbb{1}[j\notin S]+(v_i(t_i,S\backslash\{j\})+\tilde{\varphi}_{ij}(V_i(t_{ij})))\cdot\mathbb{1}[j\in S]$, where $\tilde{\varphi}_{ij}(V_i(t_{ij}))$ is the Myerson's ironed virtual value for $V_i(t_{ij})$ with respect to F_{ij}.

The proof of Lemma 3.4 is postponed to Appendix A.2. Next, we use the virtual welfare of the allocation $\sigma^{(\beta)}$ to bound the revenue of $M^{(\beta)}$.

Lemma 3.5. *For any β,*

$$
\begin{aligned}
&\textsc{Rev}(M^{(\beta)},v^{(\beta)},D) \le \sum_i \sum_{t_i\in T_i} f_i(t_i) \sum_{S\subseteq[m]} \sigma_{iS}^{(\beta)}(t_i)\cdot \Phi_i^{(\beta_i)}(t_i,S)\\
\le &\sum_i \sum_{t_i\in T_i} f_i(t_i)\cdot \mathbb{1}\left[t_i\in R_0^{(\beta_i)}\right]\cdot \sum_{S\subseteq[m]} \sigma_{iS}^{(\beta)}(t_i)\cdot v_i(t_i,S)\\
&+\sum_i \sum_{t_i\in T_i} f_i(t_i)\cdot \sum_{j\in[m]} \mathbb{1}\left[t_i\in R_j^{(\beta_i)}\right]\cdot\left(\sum_{S:j\in S}\sigma_{iS}^{(\beta)}(t_i)\cdot v_i(t_i,S\backslash\{j\})+\sum_{S:j\notin S}\sigma_{iS}^{(\beta)}(t_i)\cdot v_i(t_i,S)\right)\\
&+\sum_i \sum_{t_i\in T_i} f_i(t_i)\cdot \sum_{j\in[m]} \mathbb{1}\left[t_i\in R_j^{(\beta_i)}\right]\cdot \pi_{ij}^{(\beta)}(t_i)\cdot\tilde{\varphi}_{ij}(t_{ij}),
\end{aligned}
$$

where $\pi_{ij}^{(\beta)}(t_i)=\sum_{S:j\in S}\sigma_{iS}^{(\beta)}(t_i)$. $\textsc{Non-Favorite}(M,\beta)$ denotes the sum of the first two terms. $\textsc{Single}(M,\beta)$ denotes the last term.

Proof. The Lemma follows easily from the properties in Lemma 3.4 and Theorem 3.1. □

We obtain Theorem 3.2 by combining Lemma 3.3 and 3.5.

Theorem 3.2. *For any mechanism M and any β,*

$$\textsc{Rev}(M,v,D) \le 4\cdot \textsc{Non-Favorite}(M,\beta) + 2\cdot \textsc{Single}(M,\beta).$$

Proof of Theorem 3.2: First, let's look at the value of $v_i^{(\beta_i)}(t_i,S) - v_i(t_i,S)$. When $t_i \in R_j^{(\beta_i)}$ for some $j > 0$ and $j \in S$, $v_i^{(\beta_i)}(t_i,S) - v_i(t_i,S) = v_i(t_i,S\backslash\{j\}) + V_i(t_{ij}) - v_i(t_i,S) \le v_i(t_i,S\backslash\{j\})$, because $V_i(t_{ij}) \le v_i(t_i,S)$. For the other cases, $v_i^{(\beta_i)}(t_i,S) - v_i(t_i,S) = 0$. Therefore,

$$\begin{aligned}
&\sum_i \sum_{t_i \in T_i} \sum_{S \subseteq [m]} f_i(t_i) \cdot \sigma_{iS}^{(\beta)}(t_i) \cdot \left(v_i^{(\beta_i)}(t_i,S) - v_i(t_i,S)\right) \\
\le& \sum_i \sum_{t_i} f_i(t_i) \sum_j \mathbb{1}[t_i \in R_j^{(\beta_i)}] \sum_{S: j \in S} \sigma_{iS}^{(\beta)}(t_i) \cdot v_i(t_i,S\backslash\{j\}) \\
\le& \textsc{Non-Favorite}(M,\beta) \qquad \text{(Definition of } \textsc{Non-Favorite}(M,\beta))
\end{aligned}$$

Our statement follows from combining Lemma 3.3, Lemma 3.5 with the inequality above. □

3.5.3 Upper Bound for the Revenue of Subadditive Buyers

In Section 3.5.1, we have argued that for any β, there exists a mechanism $M^{(\beta)}$ such that its revenue with respect to the relaxed valuation $v^{(\beta)}$ is comparable to the revenue of M with respect to the original valuation. In Section 3.5.2, we have shown for any β how to choose a flow to obtain an upper bound for $\textsc{Rev}(M^{(\beta)}, v^{(\beta)}, D)$ and also an upper bound for $\textsc{Rev}(M, v, D)$. Now we specify our choice of β.

In [CDW16], the authors fixed a particular β, and shown that under any allocation rule, the corresponding benchmark can be bounded by the sum of the revenue of a few simple mechanisms. However, for valuations beyond additive and unit-demand, the benchmark becomes much more challenging to analyze[8]. We adopt an alternative and more flexible approach to obtain a new upper bound. Instead of fixing a single β for all mechanisms, we customize a different β for every different mechanism M. Next, we relax the valuation and design the flow based on the chosen β as specified in Section 3.5.1 and 3.5.2. Then we upper bound the revenue of M with the benchmark in Theorem 3.2 and argue that for any mechanism M, the corresponding benchmark can be upper bounded by the sum of the revenue of a few simple mechanisms. As we allow β, in other words the

8. Indeed, the difficulties already arise for valuations as simple as k-demand. A bidder's valuation is k-demand if her valuation is additive subject to a uniform matroid with rank k.

flow $\lambda^{(\beta)}$, to depend on the mechanism, our new approach may provide a better upper bound. As it turns out, our new upper bound is indeed easier to analyze.

Lemma 3.6 specifies the two properties of our β that play the most crucial roles in our analysis. We construct such a β in the proof of Lemma 3.6, however the construction is not necessarily unique and any β satisfying these two properties suffices. Note that our construction heavily relies on property **(i)** of Lemma 3.3.

Lemma 3.6. *For any constant $b \in (0,1)$ and any mechanism M, there exists a β such that: for the mechanism $M^{(\beta)}$ constructed in Lemma 3.3 according to β, any $i \in [n]$ and $j \in [m]$,*

(i) $\sum_{k\neq i} \Pr_{t_{kj}}\left[V_k(t_{kj}) \geq \beta_{kj}\right] \leq b$;

(ii) $\sum_{t_i \in T_i} f_i(t_i) \cdot \pi_{ij}^{(\beta)}(t_i) \leq \Pr_{t_{ij}}\left[V_i(t_{ij}) \geq \beta_{ij}\right]/b$, *where* $\pi_{ij}^{(\beta)}(t_i) = \sum_{S: j\in S} \sigma_{iS}^{(\beta)}(t_i)$.

Before proving Lemma 3.6, we provide some intuition behind the two required properties. Property **(i)** is used to guarantee that if item j's price for bidder i is higher than β_{ij} for all i and j in an RSPM, for any item j' and any bidder i', j' is still available with probability at least $(1-b)$ when i' is visited. As for any bidder $k \neq i'$ to purchase item j', $V_k(t_{kj'})$ must be greater than her price for item j'. By the union bound, the probability that there exists such a bidder is upper bounded by the LHS of property (i), and therefore is at most b. With this guarantee, we can easily show that the RSPM achieves good revenue (Lemma 3.19). Property **(ii)** states that the ex-ante probability for bidder i to receive an item j in $M^{(\beta)}$ is not much bigger than the probability that bidder i's value is larger than item j. This is crucial for proving our key Lemma 3.26, in which we argue that two different valuations provide comparable welfare under the same allocation rule $\sigma^{(\beta)}$. With Lemma 3.26, we can show that the ASPE obtains good revenue.

Proof of Lemma 3.6: When there is only one buyer, we can simply set every β_j to be 0 and both conditions are satisfied. When there are multiple players, we let

$$\beta_{ij} := \inf\{x \geq 0 : \Pr_{t_{ij}}\left[V_i(t_{ij}) \geq x\right] \leq b \cdot \sum_{t_i \in T_i} f_i(t_i) \cdot \pi_{ij}(t_i)\},$$

where $\pi_{ij}(t_i) = \sum_{S:j\in S} \sigma_{iS}(t_i)$. Clearly, when the distribution of $V_i(t_{ij})$ is continuous, then

$$\Pr_{t_{ij}}[V_i(t_{ij}) \geq \beta_{ij}] = b \cdot \sum_{t_i \in T_i} f_i(t_i) \cdot \pi_{ij}(t_i), \tag{3.3}$$

and therefore for any j,

$$\sum_i \Pr_{t_{ij}}[V_i(t_{ij}) \geq \beta_{ij}] = b \cdot \sum_i \sum_{t_i \in T_i} f_i(t_i) \cdot \pi_{ij}(t_i) \leq b.$$

So the first condition is satisfied. The second condition holds because by the first property in Lemma 3.3, $\sum_{t_i\in T_i} f_i(t_i) \cdot \pi_{ij}^{(\beta)}(t_i) \leq \sum_{t_i \in T_i} f_i(t_i) \cdot \pi_{ij}(t_i)$.

When the distribution for $V_i(t_{ij})$ is discrete, it is possible that Equation 3.3 does not hold, but this is essentially a tie breaking issue and not hard to fix. Let $\epsilon > 0$ be an extremely small constant that is smaller than $\left|V_i(t_{ij}) - V_i(t'_{ij})\right|$ for any $t_{ij}, t'_{ij} \in T_{ij}$, any i and any j. Let ζ_{ij} be a random variable uniformly distributed on $[0, \epsilon]$, and think of it as a random rebate that the seller gives to bidder i when she purchases item j. Now we modify the definition of β_{ij} as $\beta_{ij} := \inf\{x \geq 0 : \Pr_{t_{ij},\zeta_{ij}}[V_i(t_{ij}) + \zeta_{ij} \geq x] \leq b \cdot \sum_{t_i\in T_i} f_i(t_i) \cdot \pi_{ij}(t_i)\}$.

Both of the two properties in Lemma 3.6 hold if we replace $V_i(t_{ij})$ with $V_i(t_{ij}) + \zeta_{ij}$. The only change we need to make in the mechanism is to actually give the bidders ζ_{ij} as the corresponding rebate. Since we can choose ϵ to be arbitrarily small, the sum of the rebate is also arbitrarily small. For the simplicity of the presentation, we will omit ζ_{ij} and ϵ in the rest of the paper. The random rebate indeed makes our mechanism randomized(according to the random variable $\zeta_{ij} \sim [0, \epsilon]$). However, the randomized mechanism is a uniform distribution of deterministic DSIC mechanisms (after determining all ζ_{ij}), and the expected revenue of the randomized mechanism is simply the average revenue of all these deterministic mechanisms. Therefore, there must be one realization of the rebates such that the corresponding deterministic mechanism has revenue above the expectation, i.e., the expected revenue of the randomized one. Thus if the randomized mechanism is proved to achieve some approximation ratio, there must exist a deterministic one that achieves the same ratio. The deterministic mechanism will use a fixed value $z_{ij} \in [0, \epsilon]$ as the rebate.

Similarly, the same issue about discrete distributions arises when we define some other crucial parameters later, e.g., in the Definition of c, c_i and τ_i. We can resolve all of them together using the trick (adding a random rebate) described above, and we will not include a detailed proof for those cases. $\Box$

3.6 Warm Up: Single Buyer

To warm up, we first study the case where there is a single subadditive buyer and show how to improve the approximation ratio from 338 to 40. Since there is only one buyer, we will drop the subscript i in the notations. As specified in Section 3.5.3, we use a β that satisfies both properties in Lemma 3.6. For a single buyer, we can simply set β_j to be 0 for all j. We use $\textsc{Single}(M), \textsc{Non-Favorite}(M)$ in the following proof to denote the corresponding terms in Theorem 3.2 for $\beta = \mathbf{0}$. Notice $R_0^{(\mathbf{0})} = \emptyset$. Theorem 3.3 shows that the optimal revenue is within a constant factor of the better of selling separately and grand bundling.

Theorem 3.3. *For a single buyer whose valuation distribution is subadditive over independent items,*

$$\textsc{Rev}(M, v, D) \leq 24 \cdot \text{S}\textsc{Rev} + 16 \cdot \text{B}\textsc{Rev}$$

for any BIC mechanism M.

Recall that the revenue for mechanism M is upper bounded by $4 \cdot \textsc{Non-Favorite}(M) + 2 \cdot \textsc{Single}(M)$ (Theorem 3.2). We first upper bound $\textsc{Single}(M)$ by $\text{OPT}^{\textsc{Copies-UD}}$. Since $\sigma_S^{(\beta)}(t)$ is a feasible allocation in the original setting, $\mathbb{1}[t \in R_j^{(\beta)}] \cdot \pi_j^{(\beta)}(t)$ with $\pi_j^{(\beta)}(t) = \sum_{S:j\in S} \sigma_S^{(\beta)}(t)$ is a feasible allocation in the copies setting, and therefore $\textsc{Single}(M)$ is the Myerson Virtual Welfare of a certain allocation in the copies setting, which is upper bounded by $\text{OPT}^{\textsc{Copies-UD}}$. By [CHMS10], $\text{OPT}^{\textsc{Copies-UD}}$ is at most $2 \cdot \text{S}\textsc{Rev}$.

Lemma 3.7. *For any BIC mechanism M,* $\textsc{Single}(M) \leq \text{OPT}^{\textsc{Copies-UD}} \leq 2 \cdot \text{S}\textsc{Rev}$.

For $\textsc{Non-Favorite}(M)$, we will prove the following.

Lemma 3.8. *For any BIC mechanism M,* $\textsc{Non-Favorite}(M) \le 5 \cdot \textsc{SRev} + 4 \cdot \textsc{BRev}$.

We first bound it by the social welfare from all non-favorite items. Then we decompose the latter into two terms $\textsc{Core}(M)$ and $\textsc{Tail}(M)$, and bound them separately. For every $t \in T$, define $\mathcal{C}(t) = \{j : V(t_j) < c\}$, $\mathcal{T}(t) = [m] \backslash \mathcal{C}(t)$. Here the threshold c is chosen as

$$c := \inf \left\{ x \ge 0 : \ \sum_j \Pr_{t_j} [V(t_j) \ge x] \le 2 \right\}. \tag{3.4}$$

Since $v(t, \cdot)$ is subadditive for all $t \in T$, we have for every $S \subseteq [m]$, $v(t, S) \le v(t, S \cap \mathcal{C}(t)) + \sum_{j \in S \cap \mathcal{T}(t)} V(t_j)$. We decompose $\textsc{Non-Favorite}(M)$ based on the inequality above. Proof of Lemma 3.9 can be found in Appendix A.3.

Lemma 3.9.

$$\begin{aligned}
\textsc{Non-Favorite}(M) &\le \sum_{t \in T} f(t) \cdot \sum_j \mathbb{1}[t \in R_j^{(\beta)}] \cdot v(t, [m] \backslash \{j\}) \\
&\le \sum_{t \in T} f(t) \cdot v(t, \mathcal{C}(t)) \qquad\qquad (\textsc{Core}(M)) \\
&+ \sum_j \sum_{t_j : V(t_j) \ge c} f_j(t_j) \cdot V(t_j) \cdot \Pr_{t_{-j}} [\exists k \ne j, V(t_k) \ge V(t_j)] \qquad (\textsc{Tail}(M))
\end{aligned}$$

Using the definition of c and $\textsc{SRev}$, we can upper bound $\textsc{Tail}(M)$ with a similar argument as in [CDW16].

Lemma 3.10. *For any BIC mechanism M,* $\textsc{Tail}(M) \le 2 \cdot \textsc{SRev}$.

Proof. Since $\textsc{Tail}(M) = \sum_j \sum_{t_j : V(t_j) \ge c} f_j(t_j) \cdot V(t_j) \cdot \Pr_{t_{-j}} [\exists k \ne j, V(t_k) \ge V(t_j)]$, for each type $t_j \in T_j$ consider the mechanism that posts the same price $V(t_j)$ for each item but only allows the buyer to purchase at most one. Notice if there exists $k \ne j$ such that $V(t_k) \ge V(t_j)$, the mechanism is guaranteed to sell one item obtaining revenue $V(t_j)$. Thus, the revenue obtained by this mechanism is at least $V(t_j) \cdot \Pr_{t_{-j}} [\exists k \ne j, V(t_k) \ge V(t_j)]$. By definition, this is no more than $\textsc{SRev}$.

$$\textsc{Tail}(M) \leq \sum_j \sum_{t_j : V(t_j) \geq c} f_j(t_j) \cdot \text{SRev}{=}2 \cdot \text{SRev} \tag{3.5}$$

The last equality is because by the definition of c, $\sum_j \Pr_{t_j}[V(t_j) \geq c] = 2$.[9] □

The $\textsc{Core}(M)$ is upper bounded by $\mathbb{E}_t[v'(t,[m])]$ where $v'(t,S) = v(t, S \cap \mathcal{C}(t))$. We argue that $v'(t,\cdot)$ is drawn from a distribution that is subadditive over independent items and $v'(\cdot,\cdot)$ is c-Lipschitz (see Definition 3.4). Using a concentration bound by Schechtman [Sch03], we show $\mathbb{E}_t[v'(t,[m])]$ is upper bounded by the median of random variable $v'(t,[m])$ and c, which are upper bounded by BRev and SRev respectively.

Lemma 3.11. *For any BIC mechanism M, $\textsc{Core}(M) \leq 3 \cdot \text{SRev} + 4 \cdot \text{BRev}$.*

Recall that

$$\textsc{Core}(M) = \sum_{t \in T} f(t) \cdot v(t, \mathcal{C}(t)) \tag{3.6}$$

We will bound $\textsc{Core}(M)$ with a concentration inequality from [Sch03]. It requires the following definition:

Definition 3.4. *A function $v(\cdot,\cdot)$ is a-**Lipschitz** if for any type $t, t' \in T$, and set $X, Y \subseteq [m]$,*

$$\left|v(t,X) - v(t',Y)\right| \leq a \cdot \left(|X \Delta Y| + \left|\{j \in X \cap Y : t_j \neq t'_j\}\right|\right),$$

where $X \Delta Y = (X \backslash Y) \cup (Y \backslash X)$ is the symmetric difference between X and Y.

Define a new valuation function for the bidder as $v'(t,S) = v(t, S \cap \mathcal{C}(t))$, for all $t \in T$ and $S \subseteq [m]$. Then $v'(\cdot,\cdot)$ is $c-$ Lipschitz, and when t is drawn from the product distribution $D = \prod_j D_j$, $v'(t,\cdot)$ remains to be a valuation drawn from a distribution that is subadditive over independent items. See Appendix A.3 for the proof of Lemma 3.12 and Lemma 3.13.

Lemma 3.12. *For all $t \in T$, $v'(t,\cdot)$ satisfies monotonicity, subadditivity and no externalities defined in Definition 2.1.*

9. This clearly holds if $V(t_j)$ is drawn from a continuous distribution. When $V(t_j)$ is drawn from a discrete distribution, see the proof of Lemma 3.6 for a simple fix.

Lemma 3.13. *$v'(\cdot,\cdot)$ is $c-$Lipschitz.*

Next, we apply the following concentration inequality to derive Corollary 3.1, which is useful to analyze the Core(M).

Lemma 3.14. *[Sch03] Let $g(t,\cdot)$ with $t \sim D = \prod_j D_j$ be a function drawn from a distribution that is subadditive over independent items of ground set I. If $g(\cdot,\cdot)$ is c-Lipschitz, then for all $a > 0, k \in \{1, 2, ..., |I|\}, q \in \mathbb{N}$,*

$$\Pr_t[g(t,I) \geq (q+1)a + k \cdot c] \leq \Pr_t[g(t,I) \leq a]^{-q} q^{-k}.$$

Corollary 3.1. *Let $g(t,\cdot)$ with $t \sim D = \prod_j D_j$ be a function drawn from a distribution that is subadditive over independent items of ground set I. If $g(\cdot,\cdot)$ is c-Lipschitz, then if we let a be the median of the value of the grand bundle $g(t,I)$, i.e. $a = \inf\left\{x \geq 0 : \Pr_t[g(t,I) \leq x] \geq \frac{1}{2}\right\}$,*

$$\mathbb{E}_t[g(t,I)] \leq 2a + \frac{5c}{2}.$$

Proof. Let Y be $g(t,I)$. If we apply Lemma 3.14 to the case where a is the median and $q = 2$, we have

$$\begin{aligned}
\Pr_t[Y \geq 3a] \cdot \mathbb{E}_t[Y | Y \geq 3a] &= 3a \cdot \Pr_t[Y \geq 3a] + \int_{y=0}^{\infty} \Pr_t[Y \geq 3a + y] dy \\
&\leq 3a \cdot \Pr_t[Y \geq 3a] + c \cdot \sum_{k=0}^{|I|} \Pr_t[Y \geq 3a + k \cdot c] \quad (Y \leq |I| \cdot c) \\
&\leq 3a \cdot \Pr_t[Y \geq 3a] + c \cdot \sum_{k=0}^{2} \Pr_t[Y > a] + c \cdot \sum_{k=3}^{|I|} 4 \cdot 2^{-k} \quad \text{(Lemma 3.14)} \\
&\leq 3a \cdot \Pr_t[Y \geq 3a] + \frac{5}{2} c
\end{aligned}$$

With the inequality above, we can upper bound the expected value of Y.

$$\begin{aligned}
\mathbb{E}_t[Y] &\le a \cdot \Pr_t[Y \le a] + 3a \cdot \Pr_t[Y \in (a, 3a)] + \Pr_t[Y \ge 3a] \cdot \mathbb{E}_t[Y|Y \ge 3a] \\
&\le a \cdot \Pr_t[Y \le a] + 3a \cdot \Pr_t[Y \in (a, 3a)] + 3a \cdot \Pr_t[Y \ge 3a] + \frac{5}{2}c \\
&= a + 2a \cdot \Pr_t[Y > a] + \frac{5}{2}c \\
&\le 2a + \frac{5}{2}c
\end{aligned}$$

$\square$

Now, we are ready to prove Lemma 3.11.

Proof of Lemma 3.11: Let δ be the median of $v'(t, [m])$ when t is sampled from distribution D. Now consider the mechanism that sells the grand bundle with price δ. Notice that the bidder's valuation for the grand bundle is $v(t, [m]) \ge v'(t, [m])$. Thus with probability at least $\frac{1}{2}$, the bidder purchases the bundle. Thus, $\text{BRev} \ge \frac{1}{2}\delta$.

According to Corollary 3.1,

$$\text{Core}(M) = \mathbb{E}_{t\sim D}[v'(t, [m])] \le 2\delta + \frac{5c}{2} \tag{3.7}$$

It remains to argue that the Lipchitz constant c can be upper bounded using SRev. Notice that by AM-GM Inequality,

$$\begin{aligned}
\Pr_t\left[\exists j \in [m], V(t_j) \ge c\right] &= 1 - \prod_j \Pr_{t_j}[V(t_j) < c] \\
\ge& 1 - \left(\frac{\sum_j \Pr_{t_j}[V(t_j) < c]}{m}\right)^m = 1 - \left(1 - \frac{2}{m}\right)^m \ge 1 - e^{-2}
\end{aligned}$$

Consider the mechanism that posts price c for each item but only allow the buyer to purchase one item. Then with probability at least $1 - e^{-2}$, the mechanism sells one item obtaining expected revenue $(1 - e^{-2}) \cdot c$. Thus $c \le \frac{1}{1-e^{-2}} \cdot \text{SRev}$. Inequality (3.7) becomes

$$\textsc{Core}(M) \leq 2\delta + \frac{5c}{2} < 4 \cdot \text{BRev} + 3 \cdot \text{SRev} \tag{3.8}$$

□

Proof of Theorem 3.3: Since $\text{OPT}^{\textsc{Copies-UD}} \leq 2\text{SRev}$ (Lemma 3.7) and $\textsc{Non-Favorite}(M) \leq 5\text{SRev} + 4\text{BRev}$ (Lemma 3.10 and 3.11), $\textsc{Rev}(M, v, D) \leq 24 \cdot \text{SRev} + 16 \cdot \text{BRev}$ according to Theorem 3.2. □

3.7 Multiple Buyers

In this section, we prove our main result – simple mechanisms can approximate the optimal BIC revenue even when there are multiple XOS/subadditive bidders. First, we need the definition of supporting prices.

Definition 3.5 (Supporting Prices [DNS05])**.** *For any $\alpha \geq 1$, a type t and a subset $S \subseteq [m]$, prices $\{p_j\}_{j\in S}$ are α-supporting prices for $v(t, S)$ if (i) $v(t, S') \geq \sum_{j\in S'} p_j$ for all $S' \subseteq S$ and (ii) $\sum_{j\in S} p_j \geq \frac{v(t,S)}{\alpha}$.*

Theorem 3.4. *If for any buyer i, any type $t_i \in T_i$ and any bundle $S \in [m]$, $v_i(t_i, S)$ has a set of α-supporting prices $\{\theta_j^S(t_i)\}_{j\in S}$, then for any BIC mechanism M and any constant $b \in (0, 1)$,*

$$\textsc{Rev}(M, v, D) \leq 32\alpha \cdot \textsc{APostEnRev} + \left(12 + \frac{8}{1-b} + \alpha \cdot \left(\frac{16}{b(1-b)} + \frac{96}{1-b}\right)\right) \cdot \textsc{PostRev}$$

If $v_i(t_i, \cdot)$ is an XOS valuation for all i and $t_i \in T_i$, then $\alpha = 1$. By setting b to $\frac{1}{4}$, we have

$$\textsc{Rev}(M, v, D) \leq 236 \cdot \textsc{PostRev} + 32 \cdot \textsc{APostEnRev}.$$

For general subadditive valuations, $\alpha = O(\log(m))$ by [BR11], hence

$$\textsc{Rev}(M, v, D) \leq O(\log(m)) \cdot \max\{\textsc{PostRev}, \textsc{APostEnRev}\}.$$

Here is a sketch of the proof for Theorem 3.4. We show how to upper bound $\textsc{Single}(M, \beta)$ in Lemma 3.15. Then, we decompose $\textsc{Non-Favorite}(M, \beta)$ into $\textsc{Tail}(M, \beta)$ and $\textsc{Core}(M, \beta)$ in Lemma 3.16. We show how to construct a simple mechanism to approximate $\textsc{Tail}(M, \beta)$ in Section 3.7.1 and how to approximate $\textsc{Core}(M, \beta)$ in Section 3.7.2.

Analysis of Single(M, β)**:**

Lemma 3.15. *For any mechanism M,*

$$\textsc{Single}(M, \beta) \leq \text{OPT}^{\textsc{Copies-UD}} \leq 6 \cdot \textsc{PostRev}.$$

Proof. We construct a new mechanism M' in the copies setting based on $M^{(\beta)}$. Whenever $M^{(\beta)}$ allocates item j to buyer i and $t_i \in R_j^{(\beta)}$, M' serves the agent (i, j). Since there is at most one $R_j^{(\beta)}$ that t_i belongs to, M' serves at most one agent (i, j) for each of buyer i. Hence, M' is feasible in the copies setting, and $\textsc{Single}(M, \beta)$ is the expected Myerson's ironed virtual welfare of M'. Since every agent's value is drawn independently, the optimal revenue in the copies setting is the same as the maximum Myerson's ironed virtual welfare in the same setting. Therefore, $\text{OPT}^{\textsc{Copies-UD}}$ is no less than $\textsc{Single}(M, \beta)$.

As showed in [CHMS10,KW12], a simple posted-price mechanism with the constraint that every buyer can only purchase one item, i.e., an RSPM, achieves revenue at least $\text{OPT}^{\textsc{Copies-UD}}/6$ in the original setting. Hence, $\text{OPT}^{\textsc{Copies-UD}} \leq 6 \cdot \textsc{PostRev}$. $\square$

Core-Tail Decomposition of Non-Favorite(M, β)**:** we decompose $\textsc{Non-Favorite}(M, \beta)$ into two terms $\textsc{Tail}(M, \beta)$ and $\textsc{Core}(M, \beta)$[10]. First, we need the following definition.

Definition 3.6. *For every buyer i, let $c_i := \inf\left\{x \geq 0 : \sum_j \Pr_{t_{ij}}\left[V_i(t_{ij}) \geq \beta_{ij} + x\right] \leq \frac{1}{2}\right\}$. For every $t_i \in T_i$, let $\mathcal{T}_i(t_i) = \{j \mid V_i(t_{ij}) \geq \beta_{ij} + c_i\}$ and $\mathcal{C}_i(t_i) = [m] \backslash \mathcal{T}_i(t_i)$.*

Since $v_i(t_i, \cdot)$ is subadditive for all i and $t_i \in T_i$, we have $v_i(t_i, S) \leq v_i\left(t_i, S \cap \mathcal{C}_i(t_i)\right) + \sum_{j \in S \cap \mathcal{T}_i(t_i)} V_i(t_{ij})$. The term $\textsc{Non-Favorite}(M, \beta)$ can be decomposed into $\textsc{Tail}(M, \beta)$ and

10. In [CDW16], $\textsc{Non-Favorite}$ is decomposed into four different terms $\textsc{Under}$, $\textsc{Over}$, $\textsc{Core}$ and $\textsc{Tail}$. We essentially merge the first three terms into $\textsc{Core}(M, \beta)$ in our decomposition.

$\textsc{Core}(M,\beta)$ based on the inequality above. The complete proof of Lemma 3.16 can be found in Appendix A.4.

Lemma 3.16.

$$
\begin{aligned}
&\textsc{Non-Favorite}(M,\beta)\\
\leq&\sum_i\sum_{t_i} f_i(t_i)\sum_{S\subseteq[m]}\sigma_{iS}^{(\beta)}(t_i)\cdot v_i(t_i,S\cap\mathcal{C}_i(t_i)) \qquad\qquad (\textsc{Core}(M,\beta))\\
+&\sum_i\sum_j\sum_{t_{ij}:V_i(t_{ij})\geq\beta_{ij}+c_i} f_{ij}(t_{ij})\cdot V_i(t_{ij})\cdot\sum_{k\neq j}\Pr_{t_{ik}}\left[V_i(t_{ik})-\beta_{ik}\geq V_i(t_{ij})-\beta_{ij}\right] \qquad (\textsc{Tail}(M,\beta))
\end{aligned}
$$

3.7.1 Analyzing Tail(M,β) in the Multi-Bidder Case

In this section we show how to bound $\textsc{Tail}(M,\beta)$ with the revenue of an RSPM.

Lemma 3.17. *For any BIC mechanism M,* $\textsc{Tail}(M,\beta)\leq\frac{2}{1-b}\cdot\textsc{PostRev}$*.*

We first fix a few notations. Let

$$P_{ij}\in\text{argmax}_{x\geq c_i}(x+\beta_{ij})\cdot\Pr_{t_{ij}}[V_i(t_{ij})-\beta_{ij}\geq x],$$

$$r_{ij}=(P_{ij}+\beta_{ij})\cdot\Pr[V_i(t_{ij})-\beta_{ij}\geq P_{ij}]=\max_{x\geq c_i}(x+\beta_{ij})\cdot\Pr_{t_{ij}}[V_i(t_{ij})-\beta_{ij}\geq x],$$

$r_i=\sum_j r_{ij}$, and $r=\sum_i r_i$. We show in the following Lemma that r is an upper bound of $\textsc{Tail}(M,\beta)$.

Lemma 3.18. *For any BIC mechanism M,* $\textsc{Tail}(M,\beta)\leq r$*.*

Proof.

$$\begin{aligned}
\textsc{Tail}(M,\beta) \leq & \sum_i \sum_j \sum_{t_{ij}:V_i(t_{ij})\geq \beta_{ij}+c_i} f_{ij}(t_{ij}) \cdot (\beta_{ij}+c_i) \cdot \sum_{k\neq j} \Pr_{t_{ik}}\left[V_i(t_{ik})-\beta_{ik} \geq V_i(t_{ij})-\beta_{ij}\right] \\
& + \sum_i \sum_j \sum_{t_{ij}:V_i(t_{ij})\geq \beta_{ij}+c_i} f_{ij}(t_{ij}) \cdot (V_i(t_{ij})-\beta_{ij}) \cdot \sum_{k\neq j} \Pr_{t_{ik}}\left[V_i(t_{ik})-\beta_{ik} \geq V_i(t_{ij})-\beta_{ij}\right] \\
\leq & \frac{1}{2} \cdot \sum_i \sum_j \sum_{t_{ij}:V_i(t_{ij})\geq \beta_{ij}+c_i} f_{ij}(t_{ij}) \cdot (\beta_{ij}+c_i) \quad \text{(Definition of } c_i \text{ and } V_i(t_{ij}) \geq \beta_{ij}+c_i) \\
& + \sum_i \sum_j \sum_{t_{ij}:V_i(t_{ij})\geq \beta_{ij}+c_i} f_{ij}(t_{ij}) \cdot \sum_{k\neq j} r_{ik} \quad \text{(Definition of } r_{ik} \text{ and } V_i(t_{ij}) \geq \beta_{ij}+c_i) \\
\leq & \frac{1}{2} \cdot \sum_i \sum_j \Pr_{t_{ij}}[V_i(t_{ij}) \geq \beta_{ij}+c_i] \cdot (\beta_{ij}+c_i) + \sum_i r_i \cdot \sum_j \Pr_{t_{ij}}[V_i(t_{ij}) \geq \beta_{ij}+c_i] \\
\leq & \frac{1}{2} \cdot \sum_i \sum_j r_{ij} + \frac{1}{2} \cdot \sum_i r_i \quad \text{(Definition of } r_{ij} \text{ and } c_i) \\
= & r
\end{aligned}$$

In the second inequality, the first term is because $V_i(t_{ij}) - \beta_{ij} \geq c_i$, so

$$\sum_{k\neq j} \Pr_{t_{ik}}\left[V_i(t_{ik})-\beta_{ik} \geq V_i(t_{ij})-\beta_{ij}\right] \leq \sum_k \Pr_{t_{ik}}\left[V_i(t_{ik})-\beta_{ik} \geq c_i\right] \leq 1/2.$$

The second term is because for any t_{ij} such that $V_i(t_{ij}) \geq \beta_{ij}+c_i$,

$$(V_i(t_{ij})-\beta_{ij}) \cdot \Pr_{t_{ik}}\left[V_i(t_{ik})-\beta_{ik} \geq V_i(t_{ij})-\beta_{ij}\right] \leq (\beta_{ik}+V_i(t_{ij})-\beta_{ij}) \cdot \Pr_{t_{ik}}\left[V_i(t_{ik})-\beta_{ik} \geq V_i(t_{ij})-\beta_{ij}\right] \leq r_{ik}.$$

□

Next, we argue that r can be approximated by an RSPM. Indeed, we prove a stronger lemma, which is also useful for analyzing $\textsc{Core}(M,\beta)$.

Lemma 3.19. *Let $\{x_{ij}\}_{i\in[n],j\in[m]}$ be a collection of non-negative numbers, such that for any buyer i*

$$\sum_{j\in[m]} \Pr_{t_{ij}}\left[V_i(t_{ij}) \geq x_{ij}+\beta_{ij}\right] \leq 1/2,$$

then

$$\sum_i \sum_j (x_{ij} + \beta_{ij}) \cdot \Pr_{t_{ij}} [V_i(t_{ij}) \geq x_{ij} + \beta_{ij}] \leq \frac{2}{1-b} \cdot \textsc{PostRev}.$$

Proof. Consider a RSPM that sells item j to buyer i at price $\xi_{ij} = x_{ij} + \beta_{ij}$. The mechanism visits the buyers in some arbitrary order. Notice that when it is buyer i's turn, she purchases exactly item j and pays $x_{ij} + \beta_{ij}$ if all of the following three conditions hold: (i) j is still available, (ii) $V_i(t_{ij}) \geq x_{ij} + \beta_{ij}$ and (iii) $\forall k \neq j, V_i(t_{ik}) < x_{ik} + \beta_{ik}$. The second condition means buyer i can afford item j. The third condition means she cannot afford any other item $k \neq j$. Therefore, buyer i's purchases exactly item j.

Now let us compute the probability that all three conditions hold. Since every buyer's valuation is subadditive over the items, item j is purchased by someone else only if there exists a buyer $k \neq i$ who has $V_k(t_{kj}) \geq \xi_{kj}$. Because $x_{kj} \geq 0$ for all k, by the union bound, the event described above happens with probability at most $\sum_{k \neq i} \Pr_{t_{kj}} [V_k(t_{kj}) \geq \beta_{kj}]$, which is less than b by property (i) of Lemma 3.6. Therefore, condition (i) holds with probability at least $(1 - b)$. Clearly, condition (ii) holds with probability $\Pr_{t_{ij}} [V_i(t_{ij}) \geq x_{ij} + \beta_{ij}]$. Finally, condition (iii) holds with at least probability 1/2, because according to our assumption of the x_{ij}s, the probability that there exists any item $k \neq j$ such that $V_i(t_{ik}) \geq x_{ik} + \beta_{ik}$ is no more than 1/2. Since the three conditions are independent, buyer i purchases exactly item j with probability at least $\frac{(1-b)}{2} \cdot \Pr_{t_{ij}} [V_i(t_{ij}) \geq x_{ij} + \beta_{ij}]$. So the expected revenue of this mechanism is at least $\frac{(1-b)}{2} \cdot \sum_i \sum_j (\beta_{ij} + x_{ij}) \cdot \Pr_{t_{ij}} [V_i(t_{ij}) \geq x_{ij} + \beta_{ij}]$. $\square$

Proof of Lemma 3.17: Since $P_{ij} \geq c_i$, it satisfies the assumption in Lemma 3.19 due to the choice of c_i . Therefore,

$$r = \sum_{i,j} (\beta_{ij} + P_{ij}) \cdot \Pr_{t_{ij}} [V_i(t_{ij}) \geq P_{ij} + \beta_{ij}] \leq \frac{2}{1-b} \cdot \textsc{PostRev}. \tag{3.9}$$

Our statement follows from the above inequality and Lemma 3.18.$\square$

We have done the analysis for $\textsc{Tail}(M, \beta)$. Before starting the analysis for $\textsc{Core}(M, \beta)$, we show that r_i is within a constant factor of c_i. This Lemma is useful for bounding $\textsc{Core}(M, \beta)$.

Lemma 3.20. *For all $i \in [n]$, $r_i \geq \frac{1}{2} \cdot c_i$ and $\sum_i c_i/2 \leq \frac{2}{1-b} \cdot \textsc{PostRev}$.*

Proof. By the definition of P_{ij},

$$
\begin{aligned}
r_i &= \sum_j (\beta_{ij} + P_{ij}) \cdot \Pr[V_i(t_{ij}) - \beta_{ij} \geq P_{ij}] \geq \sum_j (\beta_{ij} + c_i) \cdot \Pr[V_i(t_{ij}) - \beta_{ij} \geq c_i] \\
&\geq \sum_j c_i \cdot \Pr[V_i(t_{ij}) - \beta_{ij} \geq c_i] \geq \frac{1}{2} \cdot c_i
\end{aligned}
$$

The last inequality is because when $c_i > 0$, $\sum_j \Pr_{t_{ij}}[V_i(t_{ij}) \geq \beta_{ij} + c_i]$ is at least $\frac{1}{2}$. As $\sum_i c_i/2 \leq r$, by Inequality (3.9), $\sum_i c_i/2 \leq \frac{2}{1-b} \cdot \textsc{PostRev}$. □

3.7.2 Analyzing Core(M, β) in the Multi-Bidder Case

In this section we upper bound $\textsc{Core}(M, \beta)$. Recall that

$$
\textsc{Core}(M, \beta) = \sum_i \sum_{t_i \in T_i} f_i(t_i) \cdot \sum_{S \subseteq [m]} \sigma_{iS}^{(\beta)}(t_i) \cdot v_i(t_i, S \cap \mathcal{C}_i(t_i))
$$

We can view it as the welfare of another valuation function v' under allocation $\sigma^{(\beta)}$ where $v'_i(t_i, S) = v_i(t_i, S \cap \mathcal{C}_i(t_i))$. In other words, we "truncate" the function at some threshold, i.e., only evaluate the items whose value on its own is less than that threshold. The new function still satisfies monotonicity, subadditivity and no externalities.

We first compare existing methods for analyzing the $\textsc{Core}$ with our approach before jumping into the proofs.

Comparison between the Existing Methods and Our Approach

As all results in the literature [CHMS10, Yao15, CDW16, CM16] only study special cases of constrained additive valuations, we restrict our attention to constrained additive valuations in the comparison, but our approach also applies to XOS and subadditive valuations.

We compare our approach to the state of the art result by Chawla and Miller [CM16]. They separate $\textsc{Core}(M, \beta)$ into two parts: (i) the welfare obtained from values below β, and (ii) the welfare obtained from values between β and $\beta + c$[11]. It is not hard to show that the latter can be

11. In particular, if bidder i is awarded a bundle S that is feasible for her, the contribution for the first part is

upper bounded by the revenue of a sequential posted price with per bidder entry fee mechanism. Due to their choice of β (similar to the second property of Lemma 3.6), the former is upper bounded by $\sum_{i,j} \beta_{ij} \cdot \Pr_{t_{ij}}\left[t_{ij} \geq \beta_{ij}\right]$. It turns out when every bidder's feasibility constraint is a matroid, one can use the OCRS from [FSZ16] to design a sequential posted price mechanism to approximate this expression. However, as we show in Example 3.1, $\sum_{i,j} \beta_{ij} \cdot \Pr_{t_{ij}}\left[t_{ij} \geq \beta_{ij}\right]$ could be $\Omega\left(\frac{\sqrt{m}}{\log m}\right)$ times larger than the optimal social welfare when the bidders have general downward closed feasibility constraints. Hence, such approach cannot yield any constant factor approximation for general constrained additive valuations.

As explained in the intro, we take an entirely different approach. We first construct the posted prices $\{Q_j\}_{j\in[m]}$ for our ASPE (Definition 3.7), Feldman et al. [FGL15] showed that the anonymous posted price mechanism with these prices achieves welfare $\Omega\left(\textsc{Core}(M,\beta)\right)$. If all bidders have valuations that are subadditive over independent items, for any bidder i and any set of available items S, i's surplus for S under valuation $v_i'(t_i,\cdot)$ ($max_{S'\subseteq S}\ v_i'(t_i,S') - \sum_{j\in S'} Q_j$) is also subadditive over independent items. According to Talagrand's concentration inequality, the surplus concentrates and its expectation is upper bounded by its median and its Lipschitz constant a. One can extract at least half of the median by setting the median of the surplus as the entry fee. How about the Lipschitz constant a? Unfortunately, a could be as large as $\frac{1}{2}\max_{j\in[m]}\{\beta_{ij}+c_i\}$, which is too large to be bounded.

Here is how we overcome this difficulty. Instead of considering v', we construct a new valuation $\hat{v}$ that is always dominated by the true valuation v. We consider the social welfare induced by $\sigma^{(\beta)}$ under $\hat{v}$ and define it as $\widehat{\textsc{Core}}(M,\beta)$. In Section 3.7.2, we show that $\widehat{\textsc{Core}}(M,\beta)$ is not too far away from $\textsc{Core}(M,\beta)$, so it suffices to approximate $\widehat{\textsc{Core}}(M,\beta)$ (Lemma 3.26). But why is $\widehat{\textsc{Core}}(M,\beta)$ easier to approximate? The reason is two-fold. **(i)** For any bidder i and any set of available items S, bidder i's surplus for S under $\hat{v}_i(t_i,\cdot)$ (defined as $\mu_i(t_i,S)$ in Definition 3.10, which is $max_{S'\subseteq S}\ \hat{v}_i(t_i,S') - \sum_{j\in S'} Q_j$), is not only subadditive over independent items, but also has a small Lipschitz constant τ_i (Lemma 3.27). Indeed, these Lipschitz constants are so small that $\sum_i \tau_i$ and can be upper bounded by $\textsc{PostRev}$ (Lemma 4.24). **(ii)** If we set the entry fee

$\sum_{j\in S} \min\left\{\beta_{ij}, t_{ij}\right\} \cdot \mathbb{1}\left[t_{ij} < \beta_{ij}+c_i\right]$ and the contribution to the second part is $\sum_{j\in S}\left(t_{ij}-\beta_{ij}\right)^+ \cdot \mathbb{1}\left[t_{ij} < \beta_{ij}+c_i\right]$

of our ASPE to be the median of $\mu_i(t_i, S)$ when t_i is drawn from D_i, using a proof inspired by Feldman et al. [FGL15], we can show that our ASPE's revenue collected from the posted prices plus the expected surplus of the bidders (over the randomness of all bidders' types) approximates $\widehat{\text{CORE}}(M, \beta)$ (implied by Lemma 3.28). Again by Talagrand's concentration inequality, we can bound bidder i's expected surplus by our entry fee and τ_i (Lemma 3.30). As $\hat{v}$ is always smaller than the true valuation v, thus for any type t_i of bidder i and any available items S, the surplus for S under $v_i(t_i, \cdot)$ must be larger than $\mu_i(t_i, S)$, and the entry fee is accepted with probability at least 1/2. Putting everything together, we demonstrate that we can approximate $\text{CORE}(M, \beta)$ with an ASPE or an RSPM (Lemma 3.31).

Construction of $\widehat{\textbf{Core}}(M, \beta)$

We first show that if for any i and $t_i \in T_i$ there is a set of α-supporting prices for $v_i(t_i, \cdot)$, then there is a set of α-supporting prices for $v'_i(t_i, \cdot)$.

Lemma 3.21. *If for any type t_i and any set S, there exists a set of α-supporting prices $\{\theta_j^S(t_i)\}_{j\in S}$ for $v_i(t_i, \cdot)$, then for any t_i and S there also exists a set of α-supporting prices $\{\gamma_j^S(t_i)\}_{j\in S}$ for $v'_i(t_i, \cdot)$. In particular, $\gamma_j^S(t_i) = \theta_j^{S\cap\mathcal{C}_i(t_i)}(t_i)$ if $j \in S \cap \mathcal{C}_i(t_i)$ and $\gamma_j^S(t_i) = 0$ otherwise. Moreover, $\gamma_j^S(t_i) \le V_i(t_{ij}) \cdot \mathbb{1}[V_i(t_{ij}) < \beta_{ij} + c_i]$ for all i, t_i, j and S.*

Proof. It suffices to verify that $\{\gamma_j^S(t_i)\}_{j\in S}$ satisfies the two properties of α-supporting prices. For any $S' \subseteq S$, $S' \cap \mathcal{C}_i(t_i) \subseteq S \cap \mathcal{C}_i(t_i)$. Therefore,

$$v'_i(t_i, S') = v_i(t_i, S' \cap \mathcal{C}_i(t_i)) \ge \sum_{j\in S'\cap\mathcal{C}_i(t_i)} \theta_j^{S\cap\mathcal{C}_i(t_i)}(t_i) = \sum_{j\in S'\cap\mathcal{C}_i(t_i)} \gamma_j^S(t_i) = \sum_{j\in S'} \gamma_j^S(t_i)$$

The last equality is because $\gamma_j^S(t_i) = 0$ for $j \in S\backslash\mathcal{C}_i(t_i)$. Also, we have

$$\sum_{j\in S} \gamma_j^S(t_i) = \sum_{j\in S\cap\mathcal{C}_i(t_i)} \theta_j^{S\cap\mathcal{C}_i(t_i)}(t_i) \ge \frac{v_i(t_i, S\cap\mathcal{C}_i(t_i))}{\alpha} = \frac{v'_i(t_i, S)}{\alpha}$$

Thus, $\{\gamma_j^S(t_i)\}_{j\in S}$ defined above is a set of α-supporting prices for $v'_i(t_i, \cdot)$. Next, we argue that $\gamma_j^S(t_i) \le V_i(t_{ij}) \cdot \mathbb{1}[V_i(t_{ij}) < \beta_{ij} + c_i]$ for all i, t_i, $j \in S$. If $V_i(t_{ij}) \ge \beta_{ij} + c_i$, $j \notin \mathcal{C}_i(t_i)$, by definition $\gamma_j^S(t_i) = 0$. Otherwise if $V_i(t_{ij}) < \beta_{ij} + c_i$, then $\{j\} \subseteq S \cap \mathcal{C}_i(t_i)$, by the first property of α-supporting prices, $\gamma_j^S(t_i) \leq v_i'(t_i, \{j\}) = V_i(t_{ij})$. □

Next, we define the prices of our ASPE.

Definition 3.7. *We define a price Q_j for each item j as follows,*

$$Q_j = \frac{1}{2} \cdot \sum_i \sum_{t_i \in T_i} f_i(t_i) \cdot \sum_{S: j \in S} \sigma_{iS}^{(\beta)}(t_i) \cdot \gamma_j^S(t_i),$$

where $\{\gamma_j^S(t_i)\}_{j \in S}$ are the α-supporting prices of $v_i'(t_i, \cdot)$ and set S for any bidder i and type $t_i \in T_i$.

$\textsc{Core}(M, \beta)$ can be upper bounded by $\sum_{j \in [m]} Q_j$. The proof follows from the definition of α-supporting prices (Definition 3.5) and the definition of Q_j (Definition 3.7).

Lemma 3.22. $2\alpha \cdot \sum_{j \in [m]} Q_j \geq \textsc{Core}(M, \beta)$.

Proof.

$$\begin{aligned}
\textsc{Core}(M, \beta) &= \sum_i \sum_{t_i \in T_i} f_i(t_i) \cdot \sum_{S \subseteq [m]} \sigma_{iS}^{(\beta)}(t_i) \cdot v_i'(t_i, S) \\
&\leq \alpha \cdot \sum_i \sum_{t_i \in T_i} f_i(t_i) \cdot \sum_{S \subseteq [m]} \sigma_{iS}^{(\beta)}(t_i) \cdot \sum_{j \in S} \gamma_j^S(t_i) \\
&= \alpha \cdot \sum_{j \in [m]} \sum_i \sum_{t_i \in T_i} f_i(t_i) \cdot \sum_{S: j \in S} \sigma_{iS}^{(\beta)}(t_i) \cdot \gamma_j^S(t_i) \\
&= 2\alpha \cdot \sum_{j \in [m]} Q_j
\end{aligned}$$

□

In the following definitions, we define $\widehat{\textsc{Core}}(M, \beta)$ which is the welfare of another function $\hat{v}$ under the same allocation $\sigma^{(\beta)}$.

Definition 3.8. *Let*

$$\tau_i := \inf\{x \geq 0 : \sum_j \Pr_{t_{ij}} [V_i(t_{ij}) \geq \max\{\beta_{ij}, Q_j + x\}] \leq \frac{1}{2}\}.$$

and define A_i to be $\{j \mid \beta_{ij} \leq Q_j + \tau_i\}$.

Definition 3.9. *For every buyer i and type $t_i \in T_i$, let $Y_i(t_i) = \{j \mid V_i(t_{ij}) < Q_j + \tau_i\}$,*

$$\hat{v}_i(t_i, S) = v_i\left(t_i, S \cap Y_i(t_i)\right)$$

and

$$\hat{\gamma}_j^S(t_i) = \gamma_j^S(t_i) \cdot \mathbb{1}[V_i(t_{ij}) < Q_j + \tau_i]$$

for any set $S \in [m]$. Moreover, let

$$\widehat{\textsc{Core}}(M, \beta) = \sum_i \sum_{t_i \in T_i} f_i(t_i) \cdot \sum_{S \subseteq [m]} \sigma_{iS}^{(\beta)}(t_i) \cdot \hat{v}_i(t_i, S).$$

In the next two Lemmas, we prove some useful properties of τ_i. In particular, we argue that $\sum_{i \in [n]} \tau_i$ can be upper bounded by $\frac{4}{1-b} \cdot \textsc{PostRev}$ (Lemma 4.24).

Lemma 3.23.

$$\sum_i \sum_j \max\left\{\beta_{ij}, Q_j + \tau_i\right\} \cdot \Pr_{t_{ij}}\left[V_i(t_{ij}) \geq \max\left\{\beta_{ij}, Q_j + \tau_i\right\}\right] \leq \frac{2}{1-b} \cdot \textsc{PostRev}$$

Proof. According to the definition of τ_i, for every buyer i, $\sum_j \Pr_{t_{ij}}\left[V_i(t_{ij}) \geq \max\{\beta_{ij}, Q_j + \tau_i\}\right] = \frac{1}{2}$, and $\max\{\beta_{ij}, Q_j + \tau_i\} \geq \beta_{ij}$. Our statement follows directly from Lemma 3.19. □

Lemma 3.24.

$$\sum_{i \in [n]} \tau_i \leq \frac{4}{1-b} \cdot \textsc{PostRev}.$$

Proof. Since Q_j is nonnegative,

$$\sum_i \sum_j \max\left\{\beta_{ij}, Q_j + \tau_i\right\} \cdot \Pr\left[V_i(t_{ij}) \geq \max\{\beta_{ij}, Q_j + \tau_i\}\right] \geq \sum_i \tau_i \cdot \sum_j \Pr\left[V_i(t_{ij}) \geq \max\{\beta_{ij}, Q_j + \tau_i\}\right].$$

According to the definition of τ_i, when $\tau_i > 0$,

$$\sum_j \Pr\left[V_i(t_{ij}) \geq \max\{\beta_{ij}, Q_j + \tau_i\}\right] = \frac{1}{2}.$$

Therefore, $\sum_{i\in[n]} \tau_i \leq \frac{4}{1-b} \cdot \textsc{PostRev}$ due to Lemma 3.23. $\square$

In the following two Lemmas, we compare $\widehat{\textsc{Core}}(M,\beta)$ with $\textsc{Core}(M,\beta)$. The proof of Lemma 3.25 is postponed to Appendix A.4.

Lemma 3.25. *For every buyer i, type $t_i \in T_i$, $\hat{v}_i(t_i,\cdot)$ satisfies monotonicity, subadditivity and no externalities. Furthermore, for every set $S \subseteq [m]$ and every subset S' of S, $\hat{v}_i(t_i,S') \geq \sum_{j\in S'} \hat{\gamma}_j^S(t_i)$.*

Lemma 3.26. *Let*

$$\hat{Q}_j = \frac{1}{2} \cdot \sum_i \sum_{t_i\in T_i} f_i(t_i) \cdot \sum_{S:j\in S} \sigma_{iS}^{(\beta)}(t_i) \cdot \hat{\gamma}_j^S(t_i).$$

Then,

$$Q_j \geq \hat{Q}_j, \text{ for all } j \in [m] \text{ and}$$

$$\sum_{j\in[m]} Q_j \leq \sum_{j\in[m]} \hat{Q}_j + \frac{(b+1)}{b\cdot(1-b)} \cdot \textsc{PostRev}.$$

Proof. From the definition of $\hat{Q}_j$, it is easy to see that $Q_j \geq \hat{Q}_j$ for every j. So we only need to argue that $\sum_{j\in[m]} Q_j \leq \sum_{j\in[m]} \hat{Q}_j + \frac{(b+1)}{b\cdot(1-b)} \cdot \textsc{PostRev}$.

$$\begin{aligned}
&\sum_j \left(Q_j - \hat{Q}_j\right) = \frac{1}{2} \cdot \sum_i \sum_j \sum_{t_i\in T_i} f_i(t_i) \cdot \sum_{S:j\in S} \sigma_{iS}^{(\beta)}(t_i) \cdot \left(\gamma_j^S(t_i) - \hat{\gamma}_j^S(t_i)\right)\\
\leq&\frac{1}{2} \cdot \sum_i \sum_j \sum_{t_i\in T_i} f_i(t_i) \cdot \sum_{S:j\in S} \sigma_{iS}^{(\beta)}(t_i) \cdot \left(\beta_{ij} \cdot \mathbb{1}\left[V_i(t_{ij}) \geq Q_j + \tau_i\right] + c_i \cdot \mathbb{1}\left[V_i(t_{ij}) \geq \max\{Q_j + \tau_i, \beta_{ij}\}\right]\right)\\
=&\frac{1}{2} \cdot \sum_i \sum_j \sum_{t_i\in T_i} f_i(t_i) \cdot \pi_{ij}^{(\beta)}(t_i) \cdot \left(\beta_{ij} \cdot \mathbb{1}\left[V_i(t_{ij}) \geq Q_j + \tau_i\right] + c_i \cdot \mathbb{1}\left[V_i(t_{ij}) \geq \max\{Q_j + \tau_i, \beta_{ij}\}\right]\right)
\end{aligned} \tag{3.10}$$

This first inequality is because $\gamma_j^S(t_i) - \hat{\gamma}_j^S(t_i)$ is non-zero only when $V_i(t_{ij}) \geq Q_j + \tau_i$, and the difference is upper bounded by β_{ij} when $V_i(t_{ij}) \leq \beta_{ij}$ and upper bounded by $\beta_{ij} + c_i$ when $V_i(t_{ij}) > \beta_{ij}$.

We first bound $\sum_i \sum_j \sum_{t_i \in T_i} f_i(t_i) \cdot \pi_{ij}^{(\beta)}(t_i) \cdot \beta_{ij} \cdot \mathbb{1}[V_i(t_{ij}) \geq Q_j + \tau_i]$.

$$\begin{aligned}
&\sum_i \sum_j \sum_{t_i \in T_i} f_i(t_i) \cdot \pi_{ij}^{(\beta)}(t_i) \cdot \beta_{ij} \cdot \mathbb{1}[V_i(t_{ij}) \geq Q_j + \tau_i] \\
\leq &\sum_i \sum_{j \in A_i} \beta_{ij} \cdot \sum_{t_i \in T_i} f_i(t_i) \cdot \mathbb{1}[V_i(t_{ij}) \geq Q_j + \tau_i] + \sum_i \sum_{j \notin A_i} \beta_{ij} \cdot \sum_{t_i \in T_i} f_i(t_i) \cdot \pi_{ij}^{(\beta)}(t_i) \\
\leq &\sum_i \sum_{j \in A_i} \beta_{ij} \cdot \Pr_{t_{ij}}[V_i(t_{ij}) \geq Q_j + \tau_i] + \sum_i \sum_{j \notin A_i} \beta_{ij} \cdot \Pr_{t_{ij}}[V_i(t_{ij}) \geq \beta_{ij}]/b \qquad (3.11) \\
\leq &(1/b) \cdot \sum_i \sum_j \max\{\beta_{ij}, Q_j + \tau_i\} \cdot \Pr_{t_{ij}}[V_i(t_{ij}) \geq \max\{\beta_{ij}, Q_j + \tau_i\}] \\
\leq &\frac{2}{b \cdot (1-b)} \cdot \textsc{PostRev}
\end{aligned}$$

The set A_i in the first inequality is defined in Definition 3.8. The second inequality is due to property (ii) in Lemma 3.6. The third inequality is due to Definition 3.8 and the last inequality is due to Lemma 3.23.

Next, we bound $\sum_i \sum_j \sum_{t_i \in T_i} f_i(t_i) \cdot \pi_{ij}^{(\beta)}(t_i) \cdot c_i \cdot \mathbb{1}[V_i(t_{ij}) \geq \max\{Q_j + \tau_i, \beta_{ij}\}]$.

$$\begin{aligned}
&\sum_i \sum_j \sum_{t_i \in T_i} f_i(t_i) \cdot \pi_{ij}^{(\beta)}(t_i) \cdot c_i \cdot \mathbb{1}[V_i(t_{ij}) \geq \max\{Q_j + \tau_i, \beta_{ij}\}] \\
\leq &\sum_i c_i \sum_j \sum_{t_i} f_i(t_i) \cdot \mathbb{1}[V_i(t_{ij}) \geq \max\{Q_j + \tau_i, \beta_{ij}\}] \\
\leq &\sum_i c_i \sum_j \Pr_{t_{ij}}[V_i(t_{ij}) \geq \max\{Q_j + \tau_i, \beta_{ij}\}] \qquad (3.12) \\
\leq &\sum_i c_i/2 \\
\leq &\frac{2}{(1-b)} \cdot \textsc{PostRev}
\end{aligned}$$

The last inequality is due to Lemma 3.20. Combining Inequality (3.10), (3.11) and (3.12), we have proved our claim. □

By Lemma 3.22, $\sum_{j \in [m]} Q_j \leq \textsc{Core}(M, \beta)/2\alpha$. Hence, Lemma 3.26 shows that to approximate $\textsc{Core}(M, \beta)$, it suffices to approximate $\widehat{\textsc{Core}}(M, \beta)$. Indeed, we will use $\sum_{j \in [m]} \hat{Q}_j$ as an proxy for $\textsc{Core}(M, \beta)$ in our analysis of the ASPE.

Design and Analysis of Our ASPE

Consider the sequential post-price mechanism with anonymous posted price Q_j for item j. We visit the buyers in the alphabetical order[12] and charge every bidder an entry fee. We define the entry fee here.

Definition 3.10 (Entry Fee)**.** *For any bidder i, any type $t_i \in T_i$ and any set S, let*

$$\mu_i(t_i, S) = \max_{S' \subseteq S} \big(\hat{v}_i(t_i, S') - \sum_{j \in S'} Q_j\big).$$

For any type profile $t \in T$ and any bidder i, let the entry fee for bidder i be

$$\delta_i(S_i(t_{<i})) = \textsc{Median}_{t_i}\left[\mu_i\left(t_i, S_i(t_{<i})\right)\right]^{13},$$

where $S_1(t_{<1}) = [m]$ and $S_i(t_{<i})$ is the set of items that are not purchased by the first $i-1$ buyers in the ASPE, when buyer ℓ's valuation is $v_\ell(t_\ell, \cdot)$ for all $\ell < i$. Notice that even though the seller does not know $t_{<i}$, she can compute the entry fee $\delta_i(S_i(t_{<i}))$, as she observes $S_i(t_{<i})$ after visiting the first $i-1$ bidders.

In Lemma 3.27, we show that τ_i is the Lipschitz constant for $\mu_i(\cdot, \cdot)$ and the proof is postponed to Appendix A.4. Moreover, $\sum_i \tau_i$ is upper bounded by $\frac{4}{1-b} \cdot \textsc{PostRev}$ due to Lemma 4.24.

Lemma 3.27. *For any i, the function $\mu_i(\cdot, \cdot)$ is τ_i-Lipschitz. Moreover, for any type $t_i \in T_i$, $\mu_i(t_i, \cdot)$ satisfies monotonicity, subadditivity and no externalities.*

The following Lemma is crucial for our proof. We show that in expectation over all type profiles, we can lower bound of the sum of $\mu_i(t_i, S_i(t_{<i}))$ for all bidders. In particular, this lower bound plus our ASPE's revenue from the posted prices already approximates $\widehat{\textsc{Core}}(M, \beta)$. The proof is inspired by Feldman et al. [FGL15]. Note that $\mu_i(t_i, S_i(t_{<i}))$ is a lower bound of the real surplus

12. We can visit the buyers in an arbitrary order. We use the the alphabetical order here just to ease the notations in the proof.

13. Here $\textsc{Median}_x[h(x)]$ denotes the median of a non-negative function $h(x)$ on random variable x, i.e. $\textsc{Median}_x[h(x)] = \inf\{a \geq 0 : \Pr_x[h(x) \leq a] \geq \frac{1}{2}\}$.

of buyer i for set $S_i(t_{<i})$. We choose to analyze the sum of $\mu_i(t_i, S_i(t_{<i}))$ because $\mu_i(\cdot,\cdot)$ has a small Lipschitz constant, which allows us to approximate $\mu_i(t_i, S_i(t_{<i}))$ with buyer i's entry fee $\mu_i(S_i(t_{<i}))$ and τ_i.

Lemma 3.28. *For all j, let Q_j (Definition 3.7) be the price for item j and every bidder's entry fee be described as in Definition 3.10. For every type profile $t \in T$, let $\text{SOLD}(t)$ be the set of items sold in the corresponding ASPE when buyer i's valuation is $v_i(t_i,\cdot)$. Then*

$$\begin{aligned}\mathbb{E}_t\left[\sum_{i\in[n]} \mu_i\left(t_i, S_i(t_{<i})\right)\right] &\geq \sum_j \Pr_t[j \notin \text{SOLD}(t)] \cdot (2\hat{Q}_j - Q_j) \\ &\geq \sum_{j\in[m]} \Pr_t\left[j \notin \text{SOLD}(t)\right] \cdot Q_j - \frac{(2b+2)}{b\cdot(1-b)} \cdot \textsc{PostRev}\end{aligned}$$

Proof.

$$
\begin{aligned}
&\mathbb{E}_t\left[\sum_i \mu_i\left(t_i, S_i(t_{<i})\right)\right] \\
\geq &\sum_i \mathbb{E}_{t_i,t_{-i},t'_{-i}}\left[\mu_i\left(t_i, S_i(t_{<i}) \cap M_i^{(\beta)}(t_i,t'_{-i})\right)\right] \\
\geq &\sum_i \mathbb{E}_{t_i,t_{-i},t'_{-i}}\left[\sum_{j\in M_i^{(\beta)}(t_i,t'_{-i})} \mathbb{1}\left[j\in S_i(t_{<i})\right]\cdot\left(\hat{\gamma}_j^{M_i^{(\beta)}(t_i,t'_{-i})}(t_i)-Q_j\right)^+\right] \\
= &\sum_i \mathbb{E}_{t_i,t'_{-i}}\left[\sum_{j\in M_i^{(\beta)}(t_i,t'_{-i})} \Pr_{t_{<i}}[j\in S_i(t_{<i})]\cdot\left(\hat{\gamma}_j^{M_i^{(\beta)}(t_i,t'_{-i})}(t_i)-Q_j\right)^+\right] \\
= &\sum_i \mathbb{E}_{t_i}\left[\sum_{S\subseteq[m]} \sigma_{iS}^{(\beta)}(t_i)\cdot\sum_{j\in S}\Pr_{t_{<i}}[j\in S_i(t_{<i})]\cdot\left(\hat{\gamma}_j^S(t_i)-Q_j\right)^+\right] \\
= &\sum_i \mathbb{E}_{t_i}\left[\sum_{j\in[m]} \Pr_{t_{<i}}[j\in S_i(t_{<i})]\cdot\sum_{S:j\in S}\sigma_{iS}^{(\beta)}(t_i)\cdot\left(\hat{\gamma}_j^S(t_i)-Q_j\right)^+\right] \\
= &\sum_i\sum_j \Pr_{t_{<i}}[j\in S_i(t_{<i})]\cdot\mathbb{E}_{t_i}\left[\sum_{S:j\in S}\sigma_{iS}^{(\beta)}(t_i)\cdot\left(\hat{\gamma}_j^S(t_i)-Q_j\right)^+\right] \\
\geq &\sum_i\sum_j \Pr_t[j\notin \text{SOLD}(t)]\cdot\mathbb{E}_{t_i}\left[\sum_{S:j\in S}\sigma_{iS}^{(\beta)}(t_i)\cdot\left(\hat{\gamma}_j^S(t_i)-Q_j\right)^+\right] \\
\geq &\sum_j \Pr_t[j\notin \text{SOLD}(t)]\sum_i\sum_{t_i} f_i(t_i)\cdot\sum_{S:j\in S}\sigma_{iS}^{(\beta)}(t_i)\cdot\left(\hat{\gamma}_j^S(t_i)-Q_j\right)
\end{aligned}
$$

t'_{-i} are fresh samples drawn from D_{-i}. The first inequality is because the $\mu_i(t_i, S)$ function is monotone in set S for any i and type $t_i \in T_i$. We use $\left(\hat{\gamma}_j^{M_i^{(\beta)}(t_i,t'_{-i})}(t_i)-Q_j\right)^+$ to denote $\max\left\{\hat{\gamma}_j^{M_i^{(\beta)}(t_i,t'_{-i})}(t_i)-Q_j, 0\right\}$. If we let S be the set of items that are in $S_i(t_{<i})\cap M_i^{(\beta)}(t_i,t'_{-i})$ and satisfy that $\hat{\gamma}_j^{M_i^{(\beta)}(t_i,t'_{-i})}(t_i)-Q_j \geq 0$, then $\mu_i\left(t_i, S_i(t_{<i})\cap M_i^{(\beta)}(t_i,t'_{-i})\right) \geq \hat{v}_i(t_i,S)-\sum_{j\in S}Q_j \geq \sum_{j\in S}\left(\hat{\gamma}_j^{M_i^{(\beta)}(t_i,t'_{-i})}(t_i)-Q_j\right)$ due to the definition of $\mu_i(t_i,\cdot)$ and Lemma 3.25. This inequality is exactly the second inequality above. The next equality is because $S_i(t_{<i})$ only depends on the types of bidders other than i. The second last inequality is because $\Pr_{t_{<i}}[j\in S_i(t_{<i})] \geq \Pr_t[j\notin \text{SOLD}(t)]$ for all j and i, as the LHS is the probability that the item is not sold after the seller has visited the

first $i-1$ bidders and the RHS is the probability that the item remains unsold till the end of the mechanism. Now, observe that $\sum_i \sum_{t_i} f_i(t_i) \cdot \sum_{S:j\in S} \sigma_{iS}^{(\beta)}(t_i) \cdot \hat{\gamma}_j^S(t_i) = 2\hat{Q}_j$ for any j according to the definition in Lemma 3.26. Therefore,

$$\begin{aligned}
&\sum_j \Pr_t[j \notin \text{SOLD}(t)] \sum_i \sum_{t_i} f_i(t_i) \cdot \sum_{S:j\in S} \sigma_{iS}^{(\beta)}(t_i) \cdot \left(\hat{\gamma}_j^S(t_i) - Q_j\right) \\
\geq &\sum_j \Pr_t[j \notin \text{SOLD}(t)] \cdot (2\hat{Q}_j - Q_j) \\
= &\sum_j \Pr_t[j \notin \text{SOLD}(t)] \cdot Q_j - \sum_j \Pr_t[j \notin \text{SOLD}(t)] \cdot 2(Q_j - \hat{Q}_j) \\
\geq &\sum_j \Pr_t[j \notin \text{SOLD}(t)] \cdot Q_j - \sum_j 2(Q_j - \hat{Q}_j) \qquad \text{(Due to Lemma 3.26, } Q_j - \hat{Q}_j \geq 0 \text{ for all } j) \\
\geq &\sum_j \Pr_t[j \notin \text{SOLD}(t)] \cdot Q_j - \frac{(2b+2)}{b\cdot(1-b)} \cdot \textsc{PostRev} \qquad \text{(Lemma 3.26)}
\end{aligned}$$

□

Since entry fee in the ASPE for every bidder as the median of her utility over the available items under $\hat{v}$. Clearly, bidders accept the entry fee with probability at least 1/2, as their true utilities (under v) are always higher than their utilities under $\hat{v}$. Combining the concentration property of the utility under $\hat{v}$ and Lemma 3.28, we can argue that the total revenue from our ASPE is comparable to $\widehat{\textsc{Core}}(M,\beta)$, and therefore is comparable to $\textsc{Core}(M,\beta)$.

Lemma 3.29. *For all i and $t_{<i}$, bidder i accepts $\delta_i(t_{<i})$ with probability at least 1/2 when t_i is drawn from D_i. Moreover,*

$$\textsc{APostEnRev} \geq \frac{1}{4} \cdot \sum_j Q_j - \left(\frac{5}{2(1-b)} + \frac{(b+1)}{2b\cdot(1-b)}\right) \cdot \textsc{PostRev}.$$

Proof. For any bidder i, type $t_i \in T_i$ and any set S, define bidder i's utility as $u_i(t_i,S) = \max_{S'\subseteq S}\left(v_i(t_i,S') - \sum_{j\in S'} Q_j\right)$. Clearly, $u_i(t_i,S) \geq \mu_i(t_i,S)$ for any type t_i and set S. For any $t_{<i}$, as long as $u_i(t_i, S_i(t_{<i})) \geq \delta_i(S_i(t_{<i}))$, buyer i accepts the entry fee. Since $\delta_i(S_i(t_{<i}))$ is the median

of $\mu_i(t_i, t_{<i})$, $u_i(t_i, S_i(t_{<i})) \geq \delta_i(S_i(t_{<i}))$ with probability at least 1/2 when t_i is drawn from D_i. So the revenue from entry fee is at least $\frac{1}{2} \cdot \sum_i \mathbb{E}_{t_{<i}}\left[\delta_i\left(S_i(t_{<i})\right)\right].$

For any i and $t_{<i}$, by Lemma 3.27 and Corollary 3.1, we are able to derive a lower bound for $\delta_i\left(S_i(t_{<i})\right)$, as shown in Lemma 3.30.

Lemma 3.30. *For all i and $t_{<i}$,*

$$2\delta_i\left(S_i(t_{<i})\right) + \frac{5\tau_i}{2} \geq \mathbb{E}_{t_i}\left[\mu_i\left(t_i, S_i(t_{<i})\right)\right].$$

Proof. It directly follows from Lemma 3.27 and Corollary 3.1. For any i and $t_{<i}$, let $S_i(t_{<i})$ be the ground set I. Therefore, $\mu_i(t_i, \cdot)$ with $t_i \sim D_i$ is a function drawn from a distribution that is subadditive over independent items. Since, $\mu_i(\cdot, \cdot)$ is τ_i-Lipschitz and $\delta_i(S_i(t_{<i})) = \textsc{Median}_{t_i}\left[\mu_i\left(t_i, S_i(t_{<i})\right)\right]$,

$$2\delta_i\left(S_i(t_{<i})\right) + \frac{5\tau_i}{2} \geq \mathbb{E}_{t_i}\left[\mu_i\left(t_i, S_i(t_{<i})\right)\right].$$

□

Back to the proof of Lemma 3.29. According to Lemma 3.30, the revenue from the entry fee is at least $\frac{1}{4} \cdot \sum_i \mathbb{E}_{t_{<i}, t_i}\left[\mu_i(t_i, S_i(t_{<i}))\right] - \frac{5}{8} \cdot \sum_i \tau_i$, which is equal to $\frac{1}{4} \cdot \sum_i \mathbb{E}_t\left[\mu_i(t_i, S_i(t_{<i}))\right] - \frac{5}{8} \cdot \sum_i \tau_i$. Combining Lemma 4.24 and Lemma 3.28, we can further show that the revenue from the entry fee is at least $\frac{1}{4} \sum_j \Pr_t[j \notin \text{SOLD}(t)] \cdot Q_j - (\frac{5}{2(1-b)} + \frac{(b+1)}{2b(1-b)})\textsc{PostRev}$. Since the revenue from the posted prices is exactly $\sum_j \Pr_t[j \in \text{SOLD}(t)] \cdot Q_j$, the total revenue of the ASPE is at least $\frac{1}{4} \cdot \sum_j Q_j - \left(\frac{5}{2(1-b)} + \frac{(b+1)}{2b \cdot (1-b)}\right) \cdot \textsc{PostRev}$. □

Combining everything together, we have the main result of Section 3.7.2.

Lemma 3.31. *For any BIC mechanism M,*

$$\textsc{Core}(M, \beta) \leq 8\alpha \cdot \textsc{APostEnRev} + 4\alpha\left(\frac{6}{1-b} + \frac{1}{b(1-b)}\right)\textsc{PostRev}.$$

Proof. It follows directly from Lemma 3.22 and 3.29. □

Now, we have upper bounded $\textsc{Single}(M,\beta)$, $\textsc{Tail}(M,\beta)$ and $\textsc{Core}(M,\beta)$ using the sum of the revenue of simple mechanisms (RSPM and ASPE). Combining these bounds, we complete the proof of Theorem 3.4.

Proof of Theorem 3.4: The proof follows from combining Theorem 3.2, Lemma 3.15, 3.16, 3.17 and 3.31. □

3.7.3 Bad Example for Chawla and Miller's Approach

Let bidders be constrained additive and $\mathcal{F}_i$ be bidder i feasibility constraint. We use $P_{\mathcal{F}_i} = conv(\{1^S | S \in \mathcal{F}_i\})$ to denote the feasibility polytope of bidder i. Let $\{q_{ij}\}_{i\in[n],j\in[m]}$ be a collection of probabilities that satisfy $\sum_i q_{ij} \leq 1/2$ for all item j and $\boldsymbol{q}_i = (q_{i1}, \ldots, q_{im}) \in b \cdot P_{\mathcal{F}_i}$. Let $\beta_{ij} = F_{ij}^{-1}(q_{ij})$. The analysis by Chawla and Miller [CM16] needs to upper bound $\sum_{i,j} \beta_{ij} \cdot q_{ij}$ using the revenue of some BIC mechanism. When $\mathcal{F}_i$ is a matroid for every bidder i, this expression can be upper bounded by the revenue of a sequential posted price mechanism constructed using OCRS from [FSZ16]. Here we show that if the bidders have general downward closed feasibility constraints, this expression is gigantic. More specifically, we prove that even when there is only one bidder, the expression could be $\Omega\left(\frac{\sqrt{m}}{\log m}\right)$ times larger than the optimal social welfare.

Consider the following example.

Example 3.1. *The seller is selling $m = k^2$ items to a single bidder. The bidder's value for each item is drawn i.i.d. from distribution F, which is the equal revenue distribution truncated at k, i.e.,*

$$F(x) = \begin{cases} 1 - \frac{1}{x}, & \text{if } x < k \\ 1, & \text{if } x = k \end{cases}$$

Items are divided into k disjoint sets $A_1, ..., A_k$, each with size k. The bidder is additive subject to feasibility constraint $\mathcal{F} = \{S \subseteq [m] | \exists i \in [k], S \subseteq A_i\}$.

Lemma 3.32. *Let $P_{\mathcal{F}} = conv(\{1^S | S \in \mathcal{F}\})$ be the feasibility polytope for the bidder in Example 3.1. Let SW be the optimal social welfare. Then for any constant $b > 0$, there exists $q \in b \cdot P_{\mathcal{F}}$ such* *that for sufficiently large k,*

$$\sum_{j\in[m]} q_j \cdot F^{-1}(1-q_j) = \Theta(\frac{k}{\log k}) \cdot SW$$

Proof. For any $b > 0$, consider the following feasible allocation rule: w.p. $(1-b)$, don't allocate anything, and w.p. b, give the buyer one of the sets A_i uniformly at random. The corresponding ex-ante probability vector q satisfies $q_j = \frac{b}{k}, \forall j \in [m]$. Thus $q \in b \cdot P_{\mathcal{F}}$. Since $q_j < \frac{1}{k}$, $F^{-1}(1-q_j) = k$ for all $j \in [m]$. We have $\sum_{j\in[m]} q_j \cdot F^{-1}(1-q_j) = k^2 \cdot \frac{b}{k} \cdot k = b \cdot k^2$. We use V_i to denote the random variable of the bidder's value for set A_i. It is not hard to see that $SW = \mathbb{E}[\max_{i\in[k]} V_i]$.

Lemma 3.33. *For any $i \in [k]$,*

$$\Pr\left[V_i > 3 \cdot k\log(k)\right] \leq k^{-3}$$

Proof. Let X be random variable with cdf F. Notice $E[X] = \log(k)$, $E[X^2] = 2k$, and $|X| \leq k$. For every i, by the Bernstein concentration inequality, for any $t > 0$,

$$\Pr\left[V_i - k\log(k) > t\right] \leq \exp\left(-\frac{\frac{1}{2}t^2}{2k^2 + \frac{1}{3}kt}\right)$$

Choose $t = 2k\log(k)$, we have

$$\Pr\left[V_i > 3k\log(k)\right] \leq \exp(-3\log(k)) = k^{-3}$$

□

By the union bound, $\Pr[\max_{i\in[k]} V_i > 3 \cdot k\log(k)] \leq k^{-2}$. Therefore, $\mathbb{E}[\max_{i\in[k]} V_i] \leq 3k\log k + k^2 \cdot k^{-2} \leq 4k\log k$. □

3.8 Improved Analysis for Constrained Additive Valuation

In this section, we show that for constrained additive bidders, we do not need to relax the valuations, as applying directly the flow in Section 3.5 already gives an upper bound with the right format. So we can take $M^{(\beta)}$ to simply be M. In particular, we can derive the following improved upper bound for $\textsc{Rev}(M, v, D)$ using essentially the same proof as in Section 3.5.

Theorem 3.5. *If for any bidder i any type $t_i \in T$, $v_i(t_i, \cdot)$ is a constrained additive valuation, then for any mechanism M and any $\beta = \{\beta_{ij}\}_{i\in[n],j\in[m]}$,*

$$\textsc{Rev}(M, v, D) \leq \textsc{Non-Favorite}(M, \beta) + \textsc{Single}(M, \beta).$$

Combining the same upper bounds we obtained for $\textsc{Non-Favorite}(M, \beta)$ and $\textsc{Single}(M, \beta)$ and the improved upper bound in Theorem 3.5, we can improve the approximation ratio when the bidder(s) have constrained additive valuations.

Theorem 3.6. *For a single buyer whose valuation is constrained additive,*

$$\textsc{Rev}(M, v, D) \leq 7 \cdot \text{SRev} + 4 \cdot \text{BRev},$$

for any BIC mechanism M.

Theorem 3.7. *For multiple buyers whose valuations are constrained additive,*

$$\begin{aligned} \textsc{Rev}(M, v, D) \leq\ & 8 \cdot \text{A}\textsc{Post}\text{E}\textsc{n}\text{R}\textsc{ev} \\ & + \left(6 + \frac{22}{1-b} + \frac{4(b+1)}{(1-b)b}\right) \cdot \textsc{PostRev} \end{aligned} \tag{3.13}$$

for any BIC mechanism M. In particular, if we set b to be $\frac{1}{4}$, then

$$\textsc{Rev}(M, v, D) \leq 8 \cdot \text{A}\textsc{Post}\text{E}\textsc{n}\text{R}\textsc{ev} + 62 \cdot \textsc{PostRev}.$$

Chapter 4

Profit Maximization

In this chapter we study the profit maximization problem in multi-item auctions, where the seller has private costs associated with her items. We design simple and truthful mechanisms and prove that they achieve a constant factor approximation to the optimal profit.

In Section 4.1 we give an overview of our results and techniques shown in this chapter. In Section 4.2 we introduce some additional definitions and notations need to read this chapter. In Section 4.3, we introduce a benchmark of the optimal profit using the CDW duality framework. We formulate the maximization problem as an LP, take the Lagrangian dual (Section 4.3.1), and then define a new set of dual variables (a flow) to derive our benchmark (Section 4.3.2). In Section 4.4, we prove our result for the single constraint-additive buyer case. In Section 4.5 we study the case with multiple buyers.

4.1 Results in This Chapter

To state our result, we will first focus on the single buyer case. In our model, there are m items for sale, and the seller has cost c_j for parting with item j. The costs $(c_1, \ldots, c_m)$ are drawn from a distribution $\mathcal{C}$ that is known to both the seller and the buyer. We allow the seller's costs to be correlated across items. Consider constrained-additive buyers, that is, the buyer has a downward-closed feasibility constraint $\mathcal{F} \subseteq 2^{[m]}$ that specifies what bundles of items are allowed. The buyer has value t_j for item j, and her value for a bundle S is defined as $\max_{A \in \mathcal{F}, A \subseteq S} \sum_{i \in A} t_j$. Similar to most results in the simple vs. optimal literature, we assume t_j to be drawn from $\mathcal{D}_j$ independently across items.

We propose a new class of mechanisms called *permit-selling*. These mechanisms have two stages. For each item j, we create a separate permit that allows the buyer to purchase the item at its cost. In the first stage, we sell the permits without revealing any information about the actual costs. In the second stage, the seller reveals all the costs, and the buyer can buy item j by only paying the cost c_j if the buyer has purchased the permit for item j in the first stage. How does the buyer make a decision in such a mechanism? In the first stage, the buyer needs to choose her favorite bundle of permits to purchase. Since she knows the distribution $\mathcal{C}$, she can compute her utility for each bundle of permits. In the second stage, the buyer simply picks her favorite set of items based on the permits she own, the costs of the items, and her valuation function. Why do the permit-selling mechanisms help addressing the two challenges? Note that the profit of the permit-selling mechanisms is exactly the revenue from the first stage, so any mechanism that achieves high revenue in the first stage also generates high profit. Moreover, the buyer needs to make a decision on what permits to purchase without learning the costs, therefore, the seller can extract the informational rent by pricing the permits appropriately.

Indeed, we do not even need to use any complex pricing scheme in the first stage. We sell the permits separately or sell them as a grand bundle. In our proof we need one more mechanism, which simply sells the items separately, and the prices change according to the seller's costs. The reason why this class of mechanism is required is more subtle and we only sketch the intuition here. In the permit-selling mechanism, for a fixed buyer type profile $\mathbf{t}$, the buyer purchases a set of permits P and thus the seller can only extract revenue from items in P, no matter what realized costs she has. However, for different cost vectors, the seller may have different items from which she can extract more revenue. By posting item prices that depend on her cost, the seller is able to target the profitable items based on her realized cost vector. This approach does not capture the informational rent but may generate high profit in certain cases.

Here are the mechanisms we use.

- **sell-items-separately (IS)**: for each possible cost vector $\mathbf{c} = (c_1, \cdots, c_m)$, sell the items separately, and the price $p_j(\mathbf{c})$ for item j depends on $\mathbf{c}$.
- **sell-permits-separately (PS)**: sell the permits separately, and the price p_j for the j-th permit is independent from the seller's costs.
- **permit-bundling (PB)**: sell all the permits as a grand bundle at a price p that is independent from the seller's costs.

Since in all these mechanisms, the seller does not even ask the buyer to report her valuation, the mechanism is clear incentive compatible (IC) and individually rational (IR). We show that the best mechanism among these three classes of mechanisms can already achieve a constant fraction of the optimal profit.

Theorem 4.1. *For any valuation distribution $\mathcal{D} = \mathcal{D}_1 \times \ldots \times \mathcal{D}_m$, cost distribution $\mathcal{C}$, and any downward-closed feasibility constraint $\mathcal{F}$, the best mechanism among all sell-items-separately, sell-permits-separately, and permit-bundling mechanisms is an* 11*-approximation to the optimal profit.*

When the buyer's valuation is additive, we can improve the approximation factor to 6.

Theorem 4.2. *If the buyer has additive valuation, for any valuation distribution $\mathcal{D} = \mathcal{D}_1 \times \ldots \times \mathcal{D}_m$ and cost distribution $\mathcal{C}$, the best mechanism among all sell-items-separately, sell-permits-separately, and permit-bundling mechanisms is a* 6*-approximation to the optimal profit.*

We then generalize the result to accommodate multiple buyers. With multiple buyers, we sell the permits with a sequential mechanism: buyers arrive in some arbitrary order. When a buyer arrives, we first offer the buyer the permits without revealing any information about the seller's cost or other buyers' types. Next, the buyer is given the remaining item set as well as an item price for each item. For single buyer case the item price is always chosen as the seller's cost for this item. Now we allow any prices that may depend on the cost vector.[1] Again the buyer can purchase any item from the remaining item set by paying the corresponding item price, if she has the permit

1. In the proof, the item prices is always chosen to be no less than the seller's cost.

for it. The mechanism is BIC and interim IR as the buyer needs to purchase the permit without knowing the seller's cost or what items are still available. In the proof, we use Sequential Permit Posted Price (SPP) mechanisms that sell permits separately and Sequential Permit Bundling (SPB) mechanisms that sell permits as a whole bundle. Similar to the single buyer case, we need another mechanism called Constrained Sequential Item Posted Price mechanism (CSIP). It resembles the Sequential Posted Price mechanism from [CHMS10]: the items are sold sequentially with posted prices that depend on the seller's cost vector. The mechanism also imposes a constraint on the set of items a buyer can purchase. See Section 4.2 for more details.

We prove that the best of three classes of mechanisms can already achieve a constant fraction of the optimal profit, if every buyer has matroid-rank valuation.

Theorem 4.3. *For any cost distribution and buyers' valuation distributions, if every buyer has a matroid-rank valuation, the best mechanism among all CSIP, restricted SPP*[2]*, and SPB mechanisms is a* 44*-approximation to the optimal profit.*

4.1.1 Proof Sketch and Techniques

Since the costs are private, it is a priori not clear that it is sufficient to consider only direct mechanisms. Indeed, signaling mechanisms, a class of indirect mechanisms, are widely studied in the ad auction setting [BMS12, FJM+12, DIR14, EFG+14, DPT16]. We first prove a revelation principle for our problem similar to the one proved in [DPT16] for ad auctions. Our revelation principle states that w.l.o.g. we can restrict our attention to direct, BIC, and interim IR mechanisms. Moreover, we can formulate the profit maximization problem as an LP. We next apply the Cai-Devanur-Weinberg duality framework [CDW16]. The framework has become a standard tool for analyzing the performance of simple mechanisms. In most of the results based on this duality approach, a particular family of dual variables, called the "canonical dual" [CDW16, CZ17], is used to provide a benchmark for the objective function. However, this set of dual variables does not provide an appropriate benchmark due to the existence of costs. We propose a new set of dual variables that is

2. It's closed to the Sequential Permit Posted Price mechanism, except that the mechanism may hide some items randomly, preventing the buyer from buying some item even she has permit. See Section 4.2 for more details.

tailored to handle the costs. Indeed, these dual variables are so informative that they inspired us to introduce the permit-selling mechanisms. In the multi-buyer case, the choice of the dual variables is also inspired by the ex-ante relaxation technique from [CM16]. A similar set of dual variables are used in [CZ17] to provided a benchmark for the optimal revenue.

The benchmark induced by our dual variables can be easily decomposed into three components – Single, Prophet and Non-Favorite. Single can be bounded by the profit of the CSIP mechanism using relatively standard analysis. For Prophet, we bound the term using the same class of mechanisms, with the help of the Online Contention Resolution Scheme [FSZ16].

For Non-Favorite, in order to establish a connection between profit maximization and revenue maximization, we provide a separate and clean proof for the single buyer case. Instead of directly analyzing the term, we construct an auxiliary revenue maximization problem for selling m items to help approximate Non-Favorite. Intuitively, each item in the auxiliary problem corresponds to a permit. We first show that any mechanism in the auxiliary problem can be turned into a permit-selling mechanism in the original problem, such that the revenue in the auxiliary problem is the same as the profit of the permit-selling mechanism. Next, we argue that the buyer has subadditive valuation in the auxiliary problem whenever the buyer has constrained additive valuation in the original problem. Note that the better of selling the items separately and grand bundling is a constant factor approximation of the optimal revenue when the buyer has subadditive valuation [RW15, CZ17]. Unfortunately, we cannot use this approximation as a black-box, as it is not yet clear how the revenue in the auxiliary problem relates to the Non-Favorite term. Luckily, Cai and Zhao obtain their result via the CDW duality framework, and in their analysis, they show that a term identical to Non-Favorite can be approximated by the revenue of selling the items separately or grand bundling. Putting everything together, we prove that the profit of a sell-permits-separately or permit-bundling mechanism approximates Non-Favorite, and that completes our proof. For general case, it's not straightforward build such a connection. We use the standard Core-Tail Decomposition technique [LY13, CDW16], dividing Non-Favorite further into two terms Tail and Core. Tail can be approximated using RSPP. For Core, it can be viewed as all buyers' truncated welfare with respect to a related fractionally-subadditive valuation and we can bound it using SPB and RSPP, by applying the Talagrand's concentration inequality [Sch03].

4.2 Additional Notations

We consider the auction where a seller is selling m heterogeneous items to n buyers. Each buyer has a constrained-additive valuation. Following the notations from Chapter 2, we denote buyer i's type t_i as $\langle t_{ij}\rangle_{j=1}^m$, where t_{ij} is buyer i's value for item j. For every buyer i, denote $\mathcal{F}_i$ the downward-closed feasibility constraint of buyer i's valuation. On the other hand, the seller has a private cost c_j for producing each item j. Denote $\mathbf{c}$ the cost vector and $\mathbf{c}$ is drawn from distribution $\mathcal{C}$. Let T^S be the support of $\mathcal{C}$. We allow correlated costs in our problem.

For any direct[3] mechanism M and any $\mathbf{t}, \mathbf{c}$, denote $x_{ij}(\mathbf{t}, \mathbf{c})$ the probability that buyer i is receiving item j, when the buyers has type profile $\mathbf{t}$ and seller has cost $\mathbf{c}$. Let $\pi_{ij}(t_i, \mathbf{c}) = \mathbb{E}_{t_{-i}}[x_{ij}(\mathbf{t}, \mathbf{c})]$ be the interim allocation probability. Similarly, use $p_i(\mathbf{t}, \mathbf{c})$ to denote the payment for buyer i. For any $\mathbf{t}$ and $\mathbf{c}$, buyer i's utility $u_i(\mathbf{t}, \mathbf{c}) = \mathbf{t}_i \cdot x_i(\mathbf{t}, \mathbf{c}) - p_i(\mathbf{t}, \mathbf{c})$. The seller has profit (revenue minus cost) $\sum_i(p_i(\mathbf{t}, \mathbf{c}) - \mathbf{c} \cdot x_i(\mathbf{t}, \mathbf{c}))$.

In our setting where the seller has a cost vector drawn from some distribution, the IC and IR concepts of the mechanism takes the expectation over the seller's cost vector. A formal definition can be found as follows:

- Bayesian Incentive Compatible (BIC): reporting the true value maximizes the buyer's expected utility $\mathbb{E}_{t_{-i},\mathbf{c}}[u_i(t_i, t_{-i}, \mathbf{c})]$.
- Dominant Strategy Incentive Compatible (DSIC): for every $\mathbf{c}$ and every t_{-i}, reporting the true value maximizes the buyer's utility $u_i(t_i, t_{-i}, \mathbf{c})$.
- interim Individual Rational (interim IR): reporting the true value induces non-negative expected utility. $\mathbb{E}_{t_{-i},\mathbf{c}}[u_i(t_i, t_{-i}, \mathbf{c})] \geq 0$.
- ex-post Individual Rational (ex-post IR): for every $\mathbf{c}$ and t_{-i}, reporting the true value induces

3. By Lemma 4.1, the revelation principle holds in the profit maximization problem. It suffices to consider direct, BIC, and interim IR mechanisms.

non-negative utility. $u_i(t_i, t_{-i}, \mathbf{c}) \geq 0$.

If the mechanism allocates set S to some buyer, and the buyer is only interested in a feasible subset of items $U \subset S$, the mechanism can simply allocate set U instead. This does not affect the truthfulness for all buyers and increases the seller's profit. In this paper, we will only consider mechanisms that always allocate a feasible set of items $U \in \mathcal{F}_i$ to each buyer i. Denote $\mathcal{P}(\{\mathcal{F}_i\}_{i=1}^n)$ the region for all feasible allocations x.

For every mechanism M, denote $\textsc{Profit}(\mathcal{D}, \mathcal{C}, \{\mathcal{F}_i\}_{i=1}^n, M)$ the seller's expected profit in M. We use $\textsc{Profit}(M)$ for short when $(\mathcal{D}, \mathcal{C}, \{\mathcal{F}_i\}_{i=1}^n)$ is clear and fixed.

$$\textsc{Profit}(M) = \sum_i \mathbb{E}_{\mathbf{t},\mathbf{c}}[p_i(\mathbf{t}, \mathbf{c}) - \mathbf{c} \cdot x_i(\mathbf{t}, \mathbf{c})]$$

As we will explain in Lemma 4.1, it is w.l.o.g. to only consider direct, BIC, and interim IR mechanisms. Let $\text{OPT}_{\textsc{Profit}}(\mathcal{D}, \mathcal{C}, \{\mathcal{F}_i\}_{i=1}^n)$ be the optimal profit among all BIC and interim IR mechanisms (use $\text{OPT}_{\textsc{Profit}}$ for short when $(\mathcal{D}, \mathcal{C}, \{\mathcal{F}_i\}_{i=1}^n)$ is clear and fixed). Our goal is to use a simple mechanism to approximate $\text{OPT}_{\textsc{Profit}}$.

4.3 Benchmark for the Maximum Profit

In this section, we construct a benchmark for the optimal profit using the Cai-Devanur-Weinberg duality framework. Before getting into the framework and benchmark, we first show that the revelation principle holds in the profit maximization problem. Therefore, it suffices to find a benchmark for the optimal profit attainable by any direct, BIC, and interim IR mechanisms. The proof is postponed to Appendix B.2.

Lemma 4.1. *Any ex-post implementable mechanism in the profit maximization problem can be implemented by a direct, BIC, and interim IR mechanism.*

4.3.1 Duality Framework

The framework is first developed in [CDW16] and is widely used in mechanism design. Here we apply the framework to our profit maximization problem. We obtain an upper bound of the optimal profit similar to the upper bound of the optimal revenue obtained in [CDW16]. More specifically, the profit of any BIC, interim IR mechanism is upper bounded by the sum of all buyers' virtual welfare minus the seller's total cost for the same allocation, with respect to some virtual value function. We will only show a sketch of the framework in the main body and refer the readers to Appendix B.1 for a complete description.

In the framework, we first formulate the profit maximization problem as an LP. Then take the partial Lagrangian dual of the LP by lagrangifying the BIC and interim IR constraints. Since the buyer's payment is unconstrained in the partial Lagrangian, one can argue that to obtain any finite benchmark, the corresponding dual variables must form a flow. The virtual value function in the benchmark is then defined according to the choice of the dual variables/flow.

Lemma 4.2. *For any dual solution λ that induces a finite benchmark of the optimal profit and any BIC, interim IR mechanism $M = (x, p)$,*

$$\textsc{Profit}(M) \le \mathbb{E}_{\boldsymbol{t},\boldsymbol{c}}\left[\sum_i \pi_i(t_i, \boldsymbol{c}) \cdot (\Phi_i^{(\lambda)}(t_i) - \boldsymbol{c})\right]$$

where

$$\Phi_i^{(\lambda)}(t_i) = t_i - \frac{1}{f_i(t_i)} \cdot \sum_{t_i' \in T_i} \lambda_i(t_i', t_i)(t_i' - t_i)$$

can be viewed as buyer i's virtual value function. Here $\pi_i(t_i, \boldsymbol{c}) = \mathbb{E}_{t_{-i}}[x_i(t_i, t_{-i}, \boldsymbol{c})]$ is the interim allocation. $\lambda(t_i', t_i)$ is the Lagrangian dual variable for the BIC/IR constraint that says when the buyer has true type t_i' she does not want to misreport t_i.

4.3.2 Our Flow

Now we choose the dual variables λ carefully to induce a useful benchmark. First, let us use the single buyer case to provide some intuition behind our flow.

Single Buyer

In [CDW16] and [CZ17], they cleverly choose the canonical flow in the revenue maximization setting. They divide the type space T into m regions $R_1, ..., R_m$ by finding the largest value t_j among all items (called "favorite" item). It is the item that contributes the most to the buyer's welfare. Then they let the flow go between two nodes $\mathbf{t}, \mathbf{t}' \in R_j$ only if they differs only on the j-th coordinate. However, the same flow does not give us a useful benchmark in our setting, as the way to divide the type space does not even depend on the information of the seller's costs (i.e. the realized cost $\mathbf{c}$ or the cost distribution $\mathcal{C}$). In [CDW16], they also analyze another flow that is considered as a distribution of several canonical flows. We could define our flow similar to theirs: first for any fixed cost vector $\mathbf{c}'$, divide the region by which item has the largest value $t_i - c_i'$ and use the above flow. Next, define our flow as a distribution of the flow for $\mathbf{c}'$, over the randomness of $\mathbf{c}'$. This attempt does take the cost distribution into account. Unfortunately, this flow does not work as the mechanism constructed based on the sampled cost $\mathbf{c}'$ will not represent the seller's true profit based on $\mathbf{c}$.

For single buyer, we introduce the following flow. For every $j \in [m]$, let $\bar{v}_j(t_j) = \mathbb{E}_{\mathbf{c}}[(t_j - c_j)^+]$. Define every R_j as follows: R_j contains all types $\mathbf{t} \in T$ such that j is the smallest index among $\text{argmax}_k \, \bar{v}_k(t_k)$. We route the flow in a similar manner, that is, there is a flow between two nodes $\mathbf{t}, \mathbf{t}' \in R_j$ if they only differ on the j-th coordinate (see Definition 4.3). Here is the intuition behind our division. Inspired by the canonical flow, we again want to identify the favorite item for the buyer and divide the regions accordingly. However, the favorite item now should be defined as the one that contributes the most to the buyer's utility instead of the overall welfare. Note that $\bar{v}_j(t_j) = \mathbb{E}_{\mathbf{c}}[(t_j - c_j)^+]$ is exactly the expected utility from item j when the item price is c_j, which is the lowest price that the seller is willing to sell the item. That is why we choose $\bar{v}_j(t_j)$ to represent the contribution of item j to the buyer's utility. Interestingly, the SPS mechanisms are inspired by our flow, because when there is only one buyer, $\bar{v}_j(t_j)$ can also be viewed as the buyer's "value" for the j-th permit when the item price $p_j(\mathbf{c}) = c_j$. If we can design a mechanism to extract high revenue from selling the permits, then we have a mechanism that generates high profit. We will make this intuitive connection more concrete in Section 4.4.3.

Multiple Buyers

Inspired by the single buyer case, we again aim to extract high revenue from selling the permits to make sure our mechanism generates high profit. When there are multiple buyers in the auction, we sell items sequentially to the buyers and our mechanism should satisfy the following two properties:

- The item price should be carefully chosen as the item can not be over-allocated. Usually in the sequential mechanism, the item price should be large enough, to make sure that the item is available to every buyer when she comes to the auction, with certain probability.
- The item price should be at least the seller's cost, to make sure the revenue extracted from selling the items is enough to cover the cost.

Intuitively, how the flow is chosen should also depend on the format of the mechanism we aim to use. To satisfy both properties, we combine our flow in Section 4.3.2 with the ex-ante relaxation technique purposed in [CM16]. [CZ17] uses the same technique to construct the flow. They divide the type space by comparing the difference between value and the quantile induced from ex-ante allocation probability. Here we involve different quantile thresholds for different cost realization. Furthermore, in order to satisfy the second property, we choose our threshold as the maximum between the quantile and seller's cost.

Definition 4.1. *(Ex-ante relaxation) Fix mechanism $M(\pi, p)$. For every $i \in [n], j \in [m]$ and $\boldsymbol{c} \in T^S$, define $q_{ij}(\boldsymbol{c}) = \frac{1}{2} \cdot \mathbb{E}_{\boldsymbol{t}}[\pi_{ij}(t_i, \boldsymbol{c})]$, and let*

$$\beta_{ij}(\boldsymbol{c}) = \inf\left\{a \geq 0 : \Pr[t_{ij} \geq \max\{a, c_j\}] \leq q_{ij}(\boldsymbol{c})\right\}$$

$\beta_{ij}(\mathbf{c}) = 0$ if $\Pr[t_{ij} \geq c_j] \leq q_{ij}(\mathbf{c})$. If not, for simplicity we assume that there exists $\beta_{ij}(\mathbf{c})$

such that $\Pr[t_{ij} \geq \beta_{ij}(\mathbf{c})] = q_{ij}(\mathbf{c})$. This is true for continuous distribution D_{ij}. For discrete distributions, our results will hold by dealing with a tie-breaking issue. We refer the readers to Section 5.3 of [CZ17] for more details. In the further proof we will focus on continuous distributions and a same fix will apply for discrete distributions.

We denote β the mappings from $\mathbf{c}$ to $\beta_{ij}(\mathbf{c})$ for all i, j. Before defining the flow, we need the following definition.

Definition 4.2. *Fix β. For every i, t_i and set $P \subseteq [m]$, define*

$$\bar{v}_i^{(\beta)}(t_i, P) = \mathbb{E}_{\boldsymbol{c}}\left[\max_{S \subseteq P, S \in \mathcal{F}_i} \sum_{j \in S}(t_j - \max\{\beta_{ij}(\boldsymbol{c}), c_j\})\right]$$

Remark: $\bar{v}_i^{(\beta)}(t_i, P)$ is equal to $\bar{u}_i^{p}(t_i, P)$ by choosing $p_{ij}(\mathbf{c}) = \max\{\beta_{ij}(\mathbf{c}), c_j\}$.

For notational convenience, let $\bar{v}_{ij}^{(\beta)}(t_{ij}) = \bar{v}_i^{(\beta)}(t_i, \{j\}) = \mathbb{E}_{\mathbf{c}}[(t_{ij} - \max\{\beta_{ij}(\mathbf{c}), c_j\})^+]$[4], which only depends on t_{ij}. It coincides with the definition in Section 4.3.2 with $\beta = 0$.

Now we are ready to define our flow for multiple buyer case.

Definition 4.3. *(Our flow) Fix β. For every $i \in [n], j \in [m]$, $R_{ij}^{(\beta)}$ contains all types $t_i \in T_i$ such that j is the smallest index among $\text{argmax}_k\, \bar{v}_{ik}^{(\beta)}(t_{ik})$. Define the flow as follows: Each node t_i receives flow of weight $f_i(t_i)$ from the source. For every node $t_i, t_i' \in R_{ij}^{(\beta)}$, $\lambda_i(t_i', t_i) > 0$ only if $t_{ik}' = t_{ik}$ for all $k \neq j$, and t_{ij}' is the predecessor type of t_{ij}*[5]*. For node $t_i = (t_{ij}, t_{i,-j}) \in R_{ij}^{(\beta)}$, if there does not exist a successor type t_{ij}' of t_{ij} such that $(t_{ij}', t_{i,-j}) \in R_{ij}^{(\beta)}$, all flow entering node t_i goes to the sink $\varnothing$. Figure 4.1 shows an example of our flow for some buyer i when $m = 2$. The curve in the graph contains all (t_{i1}, t_{i2}) such that $\bar{v}_{i1}^{(\beta)}(t_{i1}) = \bar{v}_{i2}^{(\beta)}(t_{i2})$.*

Since for all β, i, j, $\bar{v}_{ij}^{(\beta)}(\cdot)$ is non-decreasing, each region $R_{ij}^{(\beta)}$ is upward-closed: for every $t_i = (t_{ij}, t_{i,-j}) \in R_{ij}^{(\beta)}$ and $t_{ij}' > t_{ij}$, $(t_{ij}', t_{i,-j}) \in R_{ij}^{(\beta)}$. With this property, we have the following Lemma from [CDW16].

4. For any value x, denote $x^+ = \max\{x, 0\}$

5. In other words, t_{ij}' is the smallest value in the support set T_{ij} that is greater than t_{ij}.

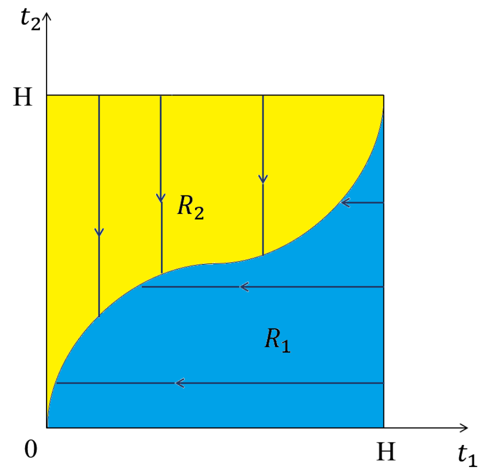


Figure 4.1: An example of our flow for two items.

Lemma 4.3. *[CDW16] Fix any β. There exists a flow λ such that for every $i \in [n]$, $t_i \in R_{ij}^{(\beta)}$,*

$$\Phi_{ik}^{(\lambda)}(t_i) = \begin{cases} t_{ik}, & \textit{if } k \neq j \\ \tilde{\varphi}_{ij}(t_{ij}), & \textit{if } k = j \end{cases}$$

where $\tilde{\varphi}_{ij}(\cdot)$ is the Myerson's ironed virtual value function w.r.t. D_{ij}.

With Lemma 4.2 and 4.3, we have obtained a benchmark for any BIC, interim IR mechanism and divide it into three terms. Note that the benchmark may differ for different mechanisms. The proof of Theorem 4.4 can be found in Appendix B.2.

Theorem 4.4. *For any BIC, interim IR mechanism M, let β be the mapping associated with M in Definition 4.1, then*

$$\begin{aligned} \textsc{Profit}(M) \leq & \sum_i \mathbb{E}_{t_i,\boldsymbol{c}}\big[\sum_j \mathbb{1}[t_i \in R_{ij}^{(\beta)}] \cdot \pi_{ij}(t_i,\boldsymbol{c}) \cdot (\tilde{\varphi}_{ij}(t_{ij}) - c_j)\big] \\ & + 2 \cdot \sum_i \sum_j \mathbb{E}_{\boldsymbol{c}}\big[q_{ij}(\boldsymbol{c}) \cdot (\max\{\beta_{ij}(\boldsymbol{c}), c_j\} - c_j)\big] \\ & + \sum_i \mathbb{E}_{t_i}\big[\sum_j \mathbb{1}[t_i \in R_{ij}^{(\beta)}] \cdot \bar{v}_i^{(\beta)}(t_i, [m]\backslash\{j\})\big] \end{aligned}$$

We use $\textsc{Single}(\beta)$, $\textsc{Prophet}(\beta)$ and $\textsc{Non-Favorite}(\beta)$ to denote the three terms accordingly. Note that all three terms depend on β. For the rest of the paper we show that these three terms can be bounded by the profit of simple mechanisms for any β induced by a BIC, interim IR mechanism M. In fact, to prove an approximation of the optimal profit, it is sufficient to consider one specific β induced by the optimal mechanism. In the proof, we fix β and omit it in the notation.

4.4 Warm-up: Single, Constrained-Additive Buyer

In this section, we bound the benchmark for single, constrained-additive buyer. In this case, all βs are set to 0 and thus $\textsc{Prophet} = 0$. We first introduce the class of mechanisms used in our proof.

4.4.1 Our Mechanisms for Single Buyer

We bound the optimal profit by the following three classes of mechanisms. The first mechanism is *sell-items-separately* (IS) mechanism. The mechanism is similar to the posted price mechanism in the revenue maximization problem, except that the seller may decide the posted price according to her cost vector. Before the auction starts, the seller will decide a posted price $p_i(\mathbf{c})$ for each i based on her cost vector $\mathbf{c}$. Then the posted prices are revealed to the buyer and she will choose her favorite bundle and pay the posted prices. Clearly the mechanism is ex-post IC and ex-post IR. We use IS-$\textsc{Profit}$ to denote the optimal seller's profit among all sell-items-separately mechanisms.

Next, we define the two permit-selling mechanisms that we need. We first need the following definition.

Definition 4.4. *For every $\boldsymbol{t}$ and set $P \subseteq [n]$, define*

$$\bar{v}(\boldsymbol{t}, P) = \mathbb{E}_{\boldsymbol{c}}\left[\max_{S\subseteq P, S\in\mathcal{F}} \sum_{i\in S}(t_i - c_i)\right]$$

This is the expected surplus of the buyer given a set of permits P in the second stage of a permit-selling mechanism. In the first stage of the permit-selling mechanism, the buyer is required to find her favorite set of permits by calculating $\bar{v}(\mathbf{t}, P)$ for all P, which is computationally hard. However, we show that as long as the buyer can make the right decision in scenarios where she can only derive positive utility from a single permit, our mechanism is already a constant factor approximation to the optimal profit. See Footnote 3 for more details.

The second class of mechanism is called *sell-permits-separately* (PS). There are two stages in the mechanism. In the first stage, instead of selling the items, the seller sells a *permit* for each item. She decides a price p_i for permit i independent from the seller's cost vector $\mathbf{c}$. The buyer is allowed to purchase any permit i by paying p_i. In the second stage, the seller reveals her cost vector $\mathbf{c}$ to the buyer, and the buyer can purchase any item i at a price of c_i if the buyer has permit i. The buyer is not allowed to purchase item i if she does not have the corresponding permit. The buyer chooses her favorite bundle among the items that she is allowed to purchase. Notice that in the second stage, the buyer with set of permits $P \subseteq [n]$ will choose the bundle $S^* = \text{argmax}_{S \subseteq P, S \in \mathcal{F}} \sum_{i \in S}(t_i - c_i)$. Thus, in the first stage, by knowing her valuation $\mathbf{t}$, all the permit prices p_js, as well as the cost distribution $\mathcal{C}$, the buyer is able to calculate her expected surplus in the second stage $\bar{v}(\mathbf{t}, P)$ (see Definition 4.4) for any $P \subseteq [n]$. She will hence choose the best set P^* that maximizes her expected utility in the whole auction and buy all the permits in set P^*[6]. Thus the mechanism is IC and IR. See Mechanism 4.4 for details.

Mechanism 4.1 Sell-permits-separately

Require: p_i, the price for permit i, for all $i \in [n]$.

1: Show all the permit prices to the buyer.
2: The buyer chooses a set of permits $P^* = \text{argmax}_{P \subseteq [n]} \bar{v}(\mathbf{t}, P)$ and pays $\sum_{i \in P^*} p_i$.
3: Reveal $\mathbf{c}$ to the buyer.
4: The buyer chooses a set of items $S^* = \text{argmax}_{S \subseteq P^*, S \in \mathcal{F}} \sum_{i \in S}(t_i - c_i)$ and pays $\sum_{i \in S^*} c_i$.

Here is a quick remark: the sell-items-separately mechanism may look similar to the sell-permits-separately mechanism. However, there is a major difference between the two mechanisms. In the IS mechanism, by posting a item price that depends on her cost vector, the seller is revealing information about her costs (the private information) to the buyer before the buyer makes any

6. The readers may wonder how can the buyer find the best set P^*. To explain this, we consider any type profile $\mathbf{t}$ where there exists i such that $\bar{v}(\mathbf{t}, \{i\}) > p_i$ and $\bar{v}(\mathbf{t}, \{k\}) < p_k, \forall k \neq i$. In this scenario buying the permit i only is the unique way to derive positive utility ($\bar{v}$ is subadditive by Lemma 4.6) and the buyer will buy permit i for sure. We point out that only counting the profit from these types suffices to to bound SINGLE. Please see footnote 8 for more details. Thus, our result holds even if the buyer cannot find the best set in all scenarios. As long as she can make the right decision in the above scenarios, enough profit is generated by our mechanism.

decision. That's why the mechanism is ex-post truthful. While in the PS mechanism, the buyer has no information about the costs in the first stage, and the mechanism is only truthful in expectation over the seller's costs. We use PS-PROFIT to denote the optimal profit among all PS mechanisms.

The third mechanism is *permit-bundling* (PB). The seller bundles all permits together and sell them as a grand bundle in the first stage. The seller decides a price p for the permit bundle independent from $\mathbf{c}$. The buyer refuses to pay p, then she get no permit and therefore cannot purchase anything in the second stage. If the buyer buys the permit bundle, the seller reveals her cost vector $\mathbf{c}$ to the buyer and asks for an item price c_i for item i. The buyer then chooses her favorite bundle and pays the item prices. The mechanism is also IC and IR due to a similar argument as for the PS mechanisms. We use PB-PROFIT to denote the optimal profit for the PB mechanisms. See Mechanism 4.5 for details.

Mechanism 4.2 Permit-bundling

Require: p, the price for the grand permit-bundle

```
Show p to the buyer.
if The buyer pays p then
    Reveal c to the buyer.
    The buyer chooses a set of items S* = argmax_{S⊆[n],S∈F} ∑_{i∈S}(t_i − c_i) and pays ∑_{i∈S*} c_i.
else
    The buyer pays nothing and receives nothing.
end if
```

In the rest of this section, we will prove the following theorem.

Theorem 4.5. *When $n = 1$, for any valuation distribution $\mathcal{D}$, cost distribution $\mathcal{C}$ and any downward-closed feasibility constraint $\mathcal{F}$,*

$$\text{OPT}_{\text{PROFIT}} \leq 2 \cdot \text{IS-PROFIT} + 5 \cdot \text{PS-PROFIT} + 4 \cdot \text{PB-PROFIT}$$

When the buyer's valuation is additive,

$$\text{OPT}_{\text{PROFIT}} \leq \text{IS-PROFIT} + 3 \cdot \text{PS-PROFIT} + 2 \cdot \text{PB-PROFIT}$$

Theorem 4.5 implies that a simple randomization among the three mechanisms achieves at least

$\frac{1}{11}$ the optimal profit for any downward-closed $\mathcal{F}$. And for additive valuations, a randomization among the three mechanisms is a 6-approximation to the optimal profit.

4.4.2 Bounding Single

To bound Single, we will consider the Copies Setting from [CHMS10], which is a single-dimensional setting in the revenue maximization problem. Here is a sketch of the proof. For any fixed cost vector $\mathbf{c}$, we first focus on a related revenue maximization problem with a single buyer and multiple items, by simply subtracting the fixed cost $\mathbf{c}$ from the buyer's value $\mathbf{t}$. Next, we show that the optimal revenue in the copies setting of the related revenue maximization problem is an upper bound of Single. According to [CHMS10], there exists a posted price mechanism in the multi-item setting whose revenue approximates the optimal revenue in its copies setting. Finally, we show an Item Posted Pricing mechanism whose expected profit is the same as to the expected revenue of the posted price mechanism.

For any fixed $\mathbf{c}$, we will first focus on the following revenue maximization problem with a single buyer and m items. Buyer has value $t_j - c_j$ for each item j, where t_j is drawn independently from $\mathcal{D}_j$. Since $\mathbf{c}$ is a fixed vector, the buyer's values are independent across items. The buyer is constraint-additive with respect to the feasibility constraint $\mathcal{F}$.

The Copies Setting of the above problem is as follows: there are m buyers in the auction and m copies to sell. Buyer j only interests in the j-th copy and has value $t_j - c_j$ for it, where t_j is drawn independently from $\mathcal{D}_j$. Since $\mathbf{c}$ is a fixed vector, all buyers' values are also independent. The seller has no cost for the copies but has a downward-closed constraint $\mathcal{F}$ that specifies which copies can simultaneously be sold. Denote $\text{OPT}^{\text{Copies-UD}}(\mathbf{c})$ the optimal revenue for the copies setting. Since it is a single dimensional setting, Myerson's auction achieves the optimal revenue, which equals to the maximum ironed virtual welfare

$$\text{OPT}^{\text{Copies-UD}}(\mathbf{c}) = \mathbb{E}_{\mathbf{t}}[\max_{S\in\mathcal{F}} \sum_{j\in S} (\tilde{\varphi}_j(t_j) - c_j)^+].^7$$

7. Notice that the ironed Myerson's virtual value for buyer j is $\tilde{\varphi}_j(t_j) - c_j$.

Moreover, let $\text{OPT-Rev}^{\text{Copies-UD}}(\mathbf{c})$ be the optimal revenue if we further restrict the seller to sell at most one copy. Similarly $\text{OPT-Rev}^{\text{Copies-UD}}(\mathbf{c}) = \mathbb{E}_{\mathbf{t}}[\max_j(\tilde{\varphi}_j(t_j) - c_j)^+]$. We have the following lemma.

Lemma 4.4. $\text{Single} \leq \mathbb{E}_{vc}[\text{OPT-Rev}^{\text{Copies-UD}}(\boldsymbol{c})] \leq 2 \cdot \text{IS-Profit}$. *When the buyer is additive, we further have* $\text{Single} \leq \text{IS-Profit}$. *Moreover, there exists an IP mechanism M where $p_j(\boldsymbol{c})$ only depends on c_j for every $j \in [m]$, such that* $\text{Single} \leq \text{Profit}(M)$.

Proof. We first prove the result for arbitrary downward-closed constraint $\mathcal{F}$. Notice that for every $\mathbf{t}$, the indicator $\mathbb{1}[\mathbf{t} \in R_j]$ is 1 for only one j. Since $\pi_j(\mathbf{t}, \mathbf{c}) \in [0, 1]$, we have

$$\begin{aligned}
\text{Single} &= \mathbb{E}_{\mathbf{t},\mathbf{c}}\big[\sum_j \mathbb{1}[\mathbf{t} \in R_j] \cdot \pi_j(\mathbf{t}, \mathbf{c}) \cdot (\tilde{\varphi}_j(t_j) - c_j)\big] \\
&\leq \mathbb{E}_{\mathbf{t},\mathbf{c}}\big[\max_j(\tilde{\varphi}_j(t_j) - c_j)^+\big] = \mathbb{E}_{\mathbf{c}}\big[\text{OPT-Rev}^{\text{Copies-UD}}(\mathbf{c})\big]
\end{aligned}$$

By [CHMS10], there exists a posted price mechanism $M(\mathbf{c})$ in the revenue maximization problem whose revenue is at least $\frac{1}{2}\text{OPT-Rev}^{\text{Copies-UD}}(\mathbf{c})$. Let $\hat{p}_j(\mathbf{c})$ be the posted price for item j.

Now we move back to our profit maximization setting and define the IP mechanism M' as follows: For every cost vector $\mathbf{c}$, define the posted price for item j as $\hat{p}_j(\mathbf{c}) + c_j$. Notice that for every $\mathbf{t}$ and $\mathbf{c}$, the buyer in M' will purchase the same bundle $B^*(\mathbf{t}, \mathbf{c})$ as the one in $M(\mathbf{c})$. Here $B^*(\mathbf{t}, \mathbf{c}) = \text{argmax}_{S \in \mathcal{F}} \sum_{j \in S}(t_j - c_j - \hat{p}_j(\mathbf{c}))$. Thus the seller's profit of M' is

$$\begin{aligned}
\mathbb{E}_{\mathbf{t},\mathbf{c}}\big[\sum_{j \in B^*(\mathbf{t},\mathbf{c})} (\hat{p}_j(\mathbf{c}) + c_j - c_j)\big] &= \mathbb{E}_{\mathbf{t},\mathbf{c}}\big[\sum_{j \in B^*(\mathbf{t},\mathbf{c})} \hat{p}_j(\mathbf{c})\big] \\
&\geq \frac{1}{2}\mathbb{E}_{\mathbf{c}}\big[\text{OPT-Rev}^{\text{Copies-UD}}(\mathbf{c})\big] \geq \frac{1}{2} \cdot \text{Single}
\end{aligned}$$

When the buyer is additive, for any fixed $\mathbf{c}$, it is not hard to realize that $\text{OPT}^{\text{Copies-UD}}(\mathbf{c})$ equals to the revenue of selling each item separately using the monopoly reserve in the revenue maximization problem. Let $\hat{p}_j(\mathbf{c})$ be the monopoly reserve for item j in the revenue maximization problem. Following the same proof as above, the IP mechanism M' with price $\hat{p}_j(\mathbf{c})+c_j$ achieves expected profit at least

$$\begin{aligned}\mathbb{E}_{\mathbf{c}}[\text{OPT}^{\text{Copies-UD}}(\mathbf{c})] &= \mathbb{E}_{\mathbf{t},\mathbf{c}}\big[\sum_j (\tilde{\varphi}_j(t_j)-c_j)^+\big] \\ &\geq \mathbb{E}_{\mathbf{t},\mathbf{c}}\big[\sum_j \mathbb{1}[\mathbf{t}\in R_j]\cdot \pi_j(\mathbf{t},\mathbf{c})\cdot(\tilde{\varphi}_j(t_j)-c_j)\big] = \textsc{Single}\end{aligned}$$

□

4.4.3 Bounding Non-Favorite

Before bounding Non-Favorite, we will first prove a crucial lemma of this section. Consider the revenue maximization problem with a single buyer and m items. The buyer's type $\mathbf{t}\sim\mathcal{D}$. She has valuation function $\bar{v}(\mathbf{t},\cdot)$ when her type is $\mathbf{t}$. For simplicity, we will call this revenue maximization problem *the revenue setting*, and the original profit maximization problem *the profit setting*. Recall that in the single buyer case,

$$\bar{v}(\mathbf{t},P)=\mathbb{E}_{\mathbf{c}}\big[\max_{S\subseteq P, S\in\mathcal{F}}\sum_{j\in S}(t_j-c_j)\big]$$

The following lemma converts any truthful mechanism in the revenue setting into a BIC and interim IR mechanism in the profit setting, without changing the value of the objective (revenue and profit accordingly). The intuition behind the lemma is as follows. For any mechanism in the profit setting that sells the permit before revealing her true cost, the buyer with type $\mathbf{t}$ has expected "value" $\bar{v}(\mathbf{t},P)$, that is, how much the buyer can make from the second stage if given a set of permits P, for all set of permits P. Thus, the mechanism can be viewed as a corresponding mechanism in the revenue setting where the permits are being sold and the buyer has valuation $\bar{v}$ over the permits.

Lemma 4.5. *For any truthful mechanism M in the revenue setting, there exists an IC and IR mechanism M' in the profit setting such that, the revenue of M equals to the seller's profit of M'.*

Proof. For any $\mathbf{t}$, let $X(\mathbf{t})$ be the (possibly random) set of items that the buyer is allocated in mechanism M, when the buyer reports $\mathbf{t}$. Let $p(\mathbf{t})$ be the payment for the buyer in M. Define M' as follows: in the first stage, the buyer reports her type $\mathbf{t}$ and the seller gives the set of permits $X(\mathbf{t})$ to the buyer and charge $p(\mathbf{t})$. In the second stage, the seller reveals the cost vector $\mathbf{c}$ and the buyer can buy any item that she has a permit by paying item price c_j. To prove M' is an IC and IR mechanism, it suffices to show that the buyer has no incentive to lie in the first stage. If the buyer with type $\mathbf{t}$ reports $\mathbf{t}'$ in M', she will receive the set of permits $X(\mathbf{t}')$ and purchase her favorite bundle of items under item prices $\mathbf{c}$. Her expected utility is

$$\mathbb{E}_{\mathbf{c},X(\mathbf{t}')}\big[\max_{S\in\mathcal{F},S\subseteq X(\mathbf{t}')}\sum_{j\in S}(t_j-c_j)\big]-p(\mathbf{t}')=\mathbb{E}_{X(\mathbf{t}')}[\bar{v}(\mathbf{t},X(\mathbf{t}'))]-p(\mathbf{t}')$$

Here the expectation is taken over the randomness of $X(\mathbf{t}')$. Since M is truthful, for any $\mathbf{t}\in T,\mathbf{t}'\in T^{+}$ [8], $\mathbb{E}_{X(\mathbf{t})}[\bar{v}(\mathbf{t},X(\mathbf{t}))]-p(\mathbf{t})\geq\mathbb{E}_{X(\mathbf{t}')}[\bar{v}(\mathbf{t},X(\mathbf{t}'))]-p(\mathbf{t}')$. It states that when the buyer has type $\mathbf{t}$, reporting $\mathbf{t}$ in the first stage maximizes her expected utility. Thus M' is IC and IR. Notice that in the second stage of M', the total item prices paid by the buyer is equal to the seller's total cost. Thus the seller's profit is exactly the payment in the first stage. Since M' use $p(\mathbf{t})$ as the payment rule, the seller's profit of M' equals to the revenue of M. □

Now we are ready to bound the term Non-Favorite. Recall that

$$\textsc{Non-Favorite}=\mathbb{E}_{\mathbf{t}}\big[\sum_{j}\mathbb{1}[\mathbf{t}\in R_j]\cdot\bar{v}(\mathbf{t},[n]\backslash\{j\})\big].$$

Consider the revenue setting where the buyer has valuation function $\bar{v}$ and let $\text{OPT}_{\text{Rev}}(\bar{v})$ be the optimal revenue among all truthful mechanisms. Here we omit $\mathcal{D}$ and $\mathcal{C}$ in the notation as they are fixed. Given Lemma 4.5, it is tempting to find a simple mechanism that approximates $\text{OPT}_{\text{Rev}}(\bar{v})$ and convert it into a permit-selling mechanism. However, since we do not know what class of valuation $\bar{v}$ belongs to, it not a priori clear any simple vs. optimal result applies here.

8. Recall that $T^{+}=T\cup\{\varnothing\}$ contains the choice of not attending the auction.

As the original valuation in the profit setting is constrained additive, it is natural to think that $\bar{v}$ is also constrained additive. Unfortunately, we are not able to prove such a claim as there is no clear feasibility that is associated with $\bar{v}$. The good news is that we are able to relax the class of valuations and show that $\bar{v}$ is indeed a *subadditive function*, which allows us to leverage the result in Chapter 3 (also in [RW15]).

To remind the readers, let us first review the results. In Chapter 3 we bound $\text{OPT}_{\textsc{Rev}}(\bar{v})$ when $\bar{v}$ is subadditive over independent items (see Definition 2.1). In the proof, we separate the benchmark of the optimal revenue into two terms (called "single" and "non-favorite"). We then bound the two terms by the optimal revenue of the Selling Separately mechanism ($\text{S}\textsc{Rev}(\bar{v})$) and Bundling mechanism ($\text{B}\textsc{Rev}(\bar{v})$) respectively. The second term "non-favorite" is defined as the expected welfare from all non-favorite items. Here for any fixed $\mathbf{t}$, the favorite item is defined as the j that maximizes $\bar{v}(\mathbf{t}, \{j\})$. Interestingly, this is how we divide the region into R_js and the term "non-favorite" is exactly the same as Non-Favorite. We will use $\textsc{Non-Fav}_{\textsc{Rev}}(\bar{v})$ to denote "non-favorite" here to emphasize that it is from the revenue setting.

In order to apply the result in the revenue setting, we first show that the function $\bar{v}(\cdot,\cdot)$ in Definition 4.2 is indeed subadditive over independent items (Definition 2.1). The proof of Lemma 4.6 is postponed to Appendix B.3.

Lemma 4.6. $\bar{v}(\cdot,\cdot)$ *is monotone, subadditive and has no externalities.*

Lemma 4.7. *(Restatement of Lemma 3.8) Suppose $\bar{v}$ is subadditive over independent items, then*

$$\textsc{Non-Fav}_{\textsc{Rev}}(\bar{v}) = \mathbb{E}_{\boldsymbol{t}}\Big[\sum_j \mathbb{1}[\boldsymbol{t} \in R_j] \cdot \bar{v}(\boldsymbol{t}, [n]\backslash\{j\})\Big] \leq 5 \cdot \text{S}\textsc{Rev}(\bar{v}) + 4 \cdot \text{B}\textsc{Rev}(\bar{v})$$

When the buyer is additive, [CDW16] has an improved bound for $\textsc{Non-Fav}_{\textsc{Rev}}(\bar{v})$ using $\text{S}\textsc{Rev}(\bar{v})$ and $\text{B}\textsc{Rev}(\bar{v})$.

Lemma 4.8. *[CDW16] If $\bar{v}$ is an additive function, then*

$$\textsc{Non-Fav}_{\textsc{Rev}}(\bar{v}) \leq 2 \cdot \text{S}\textsc{Rev}(\bar{v}) + 3 \cdot \text{B}\textsc{Rev}(\bar{v})$$

By Lemma 4.5, the Selling Separately mechanism in the revenue setting can be converted to the PP mechanism in the profit setting and has profit equals to $\text{SRev}(\bar{v})$. Also the Bundling mechanism can be converted to PB and obtains profit $\text{BRev}(\bar{v})$. Furthermore, when the buyer is additive, there is no constraint $\mathcal{F}$ and for every $\mathbf{t} \in T, P \subseteq [m]$,

$$\bar{v}(\mathbf{t}, P) = \mathbb{E}_{\mathbf{c}}\big[\max_{S \subseteq P} \sum_{j \in S} (t_j - c_j)\big] = \mathbb{E}_{\mathbf{c}}\big[\sum_{j \in P} (t_j - c_j)^+\big] = \sum_{j \in P} \bar{v}(\mathbf{t}, \{j\}).$$

Thus, $\bar{v}$ is an additive function. We have the following Corollary:

Corollary 4.1. $\text{Non-Favorite} \leq 5 \cdot \text{PS-Profit} + 4 \cdot \text{PB-Profit}$. *When the buyer is additive,* $\text{Non-Favorite} \leq 2 \cdot \text{PS-Profit} + 3 \cdot \text{PB-Profit}$.

Proof of Theorem 4.5: It follows from Theorem 4.4, Lemma 4.4 and Corollary 4.1. □

When the buyer is additive, according to [HN17], $\text{BRev}(\bar{v})$, the revenue of the optimal bundling mechanism, is bounded by $O(\log(m)) \cdot \text{SRev}(\bar{v})$. Thus Corollary 4.1 implies an $O(\log(m))$-approximation to the optimal profit with only IP and PP mechanisms. Both mechanisms sell the items separately. Thus the optimal profit for m items is bounded by $O(\log(m))$ times the sum of the optimal profit for every single item.

Theorem 4.6. *When the buyer is additive,*

$$\text{OPT}_{\text{Profit}} \leq \text{IS-Profit} + O(\log(m)) \cdot \text{PS-Profit}$$

Moreover,

$$\text{OPT}_{\text{Profit}} \leq O(\log(m)) \cdot \sum_{j \in [m]} \text{OPT}_{\text{Profit}}(\{j\}) = \log(m) \cdot \sum_{j \in [m]} \mathbb{E}_{t_j, c_j}[(\tilde{\varphi}_j(t_j) - c_j)^+]$$

where $\tilde{\varphi}_j(\cdot)$ is the Myerson's ironed virtual value function for D_j.

Proof. According to [HN17], $\text{BRev}(\bar{v}) \leq O(\log(m)) \cdot \text{SRev}(\bar{v})$. Combining this result with Lemma 4.4 and Corollary 4.1, we have the following: there exists an IP mechanism M where the posted price

$p_j(c_j)$ only depends on c_j for every $j \in [m]$, such that

$$\text{OPT}_{\textsc{Profit}} \leq \textsc{Profit}(M) + O(\log(m)) \cdot \text{PS-}\textsc{Profit}$$

Since the buyer is additive, $\textsc{Profit}(M)$ is equivalent to the sum (over all j) of the profit that sells a single item j with price $p_j(c_j)$. Note that for any PP mechanism with permit prices $\{\ell_j\}_{j\in[m]}$, the profit is equivalent to the sum (over all j) of the profit that sells a single item j with permit price ℓ_j. Thus

$$\text{OPT}_{\textsc{Profit}} \leq \textsc{Profit}(M) + O(\log(m)) \cdot \text{PS-}\textsc{Profit} \leq O(\log(m)) \cdot \sum_{j\in[m]} \text{OPT}_{\textsc{Profit}}(\{j\})$$

It remains to prove that for every j, the optimal profit when selling a single item j, is at most $\mathbb{E}_{t_j,c_j}[(\tilde{\varphi}_j(t_j) - c_j)^+]$. In the auction with a single item j, by Lemma 4.2, we have following for any dual variable λ:

$$\text{OPT}_{\textsc{Profit}}(\{j\}) \leq \max_{\pi} \mathbb{E}_{t_j,c_j}\left[\pi(t_j, c_j) \cdot (\Phi^{(\lambda)}(t_j) - c_j)\right]$$

where $\Phi^{(\lambda)}(t_j) = t_j - \frac{1}{f_j(t_j)} \cdot \sum_{t'_j \in T_j} \lambda(t'_j, t_j)(t'_j - t_j)$. Note that in the auction for selling a single item j, both the buyer's value and seller's cost are scalar. By Corollary 18 of [CDW19], when the optimal dual variable λ is chosen, $\Phi^{(\lambda)}(t_j) = \tilde{\varphi}_j(t_j)$. Thus

$$\text{OPT}_{\textsc{Profit}}(\{j\}) = \max_{\pi} \mathbb{E}_{t_j,c_j}\left[\pi(t_j, c_j) \cdot (\tilde{\varphi}_j(t_j) - c_j)\right] = \mathbb{E}_{t_j,c_j}[(\tilde{\varphi}_j(t_j) - c_j)^+],$$

where the last equality follows from $\pi(t_j, c_j) \in [0, 1], \forall t_j, c_j$. □

At the last of this section, we will prove the following lemma that connects the profit maximization problem to the revenue maximization problem. We show that any truthful mechanism in the revenue setting that is an α-approximation to the optimal revenue can be converted to a BIC and interim IR mechanism in the profit setting that is a $(9\alpha + 2)$-approximation to the optimal profit.

Lemma 4.9. *Recall that the revenue setting is the revenue maximization problem where the buyer* *has valuation $\bar{v}$. Then any truthful mechanism in the revenue setting that is an α-approximation to the optimal revenue $\text{OPT}_{\text{Rev}}(\bar{v})$ can be converted to an IC and IR mechanism in the profit setting that is a $(9\alpha + 2)$-approximation to the optimal profit $\text{OPT}_{\text{Profit}}$.*

Proof. By Lemma 3.8,

$$\text{Non-Fav}_{\text{Rev}}(\bar{v}) \leq 5 \cdot \text{SRev}(\bar{v}) + 4 \cdot \text{BRev}(\bar{v}) \leq 9 \cdot \text{OPT}_{\text{Rev}}(\bar{v})$$

Let M be the α-approximation mechanism in the revenue setting. Then the revenue of M satisfies:

$$\text{Rev}(M) \geq \frac{1}{\alpha} \cdot \text{OPT}_{\text{Rev}}(\bar{v}) \geq \frac{1}{9\alpha} \cdot \text{Non-Fav}_{\text{Rev}}(\bar{v})$$

By Lemma 4.5, there exists a BIC and interim IR mechanism M' in the profit setting such that $\text{Profit}(M') = \text{Rev}(M)$. Notice that $\text{Non-Favorite} = \text{Non-Fav}_{\text{Rev}}(\bar{v})$. By Theorem 4.4,

$$\text{OPT}_{\text{Profit}} \leq \text{Single} + \text{Non-Favorite} \leq 2 \cdot \text{IS-Profit} + 9\alpha \cdot \text{Profit}(M')$$

Thus a randomization between M' and the optimal Item Posted Pricing mechanism is a $(9\alpha+2)$-approximation. □

4.5 Multiple, Matroid-Rank Buyers

In this section, we will bound the benchmark in Theorem 4.4 for multiple, matroid-rank buyers. We remark that although in this thesis we only consider the setting where each buyer has a matroid-rank valuation, our result applies to any constrained additive buyer, where the feasibility constraint $\mathcal{F}_i$ is an intersection of constant number of matroid, polytope and knapsack constraints, by using the online contention resolution scheme (Section 2.7).

4.5.1 Our Mechanisms

We bound the optimal profit by the following three classes of mechanisms. The first mechanism is a variant of the Sequential Item Posted Price (SIP) mechanism, which is first purposed by [CHMS10] in the revenue maximization problem. Here we allow the seller to decide posted prices according to her cost vector. Before the auction starts, the seller decides a posted price $p_{ij}(\mathbf{c})$ for each buyer i and item j, based on her cost vector $\mathbf{c}$. Then buyers come one by one in an arbitrary order. Each buyer can choose her favorite bundle among all remaining items by paying the posted prices. The mechanism is DSIC and ex-post IR. We call the mechanism Constrained Sequential Item Posted Price (CSIP) if it further adds a sub-constraint on the set of items the buyer can purchase. The mechanism first decides a constraint $\mathcal{J}'(\mathbf{c})$ on the ground set of all buyer-item pairs $J = \{(i,j) \mid i \in [n], j \in [m]\}$, based on her true cost $\mathbf{c}$. A (possibly random) set $A \subseteq J$ represents a way of allocating the items. It's feasible if

- Each item is allocated to at most one buyer: $\forall j, O_j = \{i : (i,j) \in A\}, |O_j| \leq 1$.
- Each buyer is allocated a feasible set of items: $\forall i, P_i = \{j : (i,j) \in A\}, P_i \in \mathcal{F}_i$.

Let $\mathcal{J}$ be the family of all feasible sets. $\mathcal{J}'(\mathbf{c})$ must satisfy $\mathcal{J}'(\mathbf{c}) \subseteq \mathcal{J}$. When each buyer comes, she is only allowed to take the item that doesn't ruin the constraint $\mathcal{J}'(\mathbf{c})$. See Mechanism 4.3 for details.

Mechanism 4.3 Constrained Sequential Item Posted Price Mechanism

Require: $p_{ij}(\mathbf{c})$, the item price for $i \in [n], j \in [m]$; the constraint $\mathcal{J}' \subseteq \mathcal{J}$.

$A \leftarrow \emptyset$.
for $i \in [n]$ **do**
  Reveal item prices $\{p_{ij}(\mathbf{c})\}_{j=1}^{m}$ to the buyer.
  i can choose a bundle S_i such that $A \cup \{(i,j)\}_{j \in S_i} \in \mathcal{J}'$.
  i receives her favorite bundle S_i^*, paying $\sum_{j \in S_i^*} p_{ij}(\mathbf{c})$.
  $A \leftarrow A \cup \{(i,j)\}_{j \in S_i^*}$.
end for

For the CSIP used in our proof, the corresponding sub-constraint $\mathcal{J}'(\mathbf{c})$ can be computed efficiently. See Section 4.5.3 for more details. We use CSIP-PROFIT to denote the optimal seller's profit among all CSIP mechanisms.

Next, we define the two Sequential Permit Selling mechanisms used in the proof. The second class of mechanism is called Sequential Permit Posted Price(SPP). Before the auction starts, the seller decides a posted price $p_{ij}(\mathbf{c})$ for each buyer i and item j, based on her cost vector $\mathbf{c}$. Then buyers come one by one in an arbitrary order. For each buyer i there are two stages: the permit-purchasing stage and item-purchasing stage. In the permit-purchasing stage, instead of selling the items, the seller sells a *permit* for each item. She decides a price l_{ij} for permit j independent from the seller's cost vector $\mathbf{c}$ and buyer type profile $\mathbf{t}$. The buyer is allowed to purchase any permit $j \in [m]$ by paying l_{ij}. The decision must be made before she sees the remaining item set $S_i(t_{<i}, \mathbf{c})$. In the item-purchasing stage, the seller reveals $S_i(t_{<i}, \mathbf{c})$ and her cost vector $\mathbf{c}$ to the buyer, and the buyer can purchase any remaining item j at a price of $p_{ij}(\mathbf{c})$ if the buyer has permit j. The buyer is not allowed to purchase item j if she does not have the corresponding permit. The buyer chooses her favorite bundle among the items that she is allowed to purchase. Notice that in the second stage, the buyer with set of permits $P \subseteq [m]$ will choose the bundle $S^* = \text{argmax}_{S \subseteq P \cap S_i(t_{<i}, \mathbf{c}), S \in \mathcal{F}_i} \sum_{j \in S}(t_{ij} - p_{ij}(\mathbf{c}))$. Thus, in the first stage, by knowing her type t_i, all the permit prices l_{ij}s, as well as the cost distribution $\mathcal{C}$, the buyer is able to calculate her expected surplus in the second stage for any $P \subseteq S_i(t_{<i})$. She will hence choose the best set P^* that maximizes her expected utility in the whole auction and buy all the permits in set P^*. The mechanism is only BIC and Interim IR as buyers have to make decisions before getting any information about other buyers' types and the seller's costs. See Mechanism 4.4 for details.

Mechanism 4.4 Sequential Permit Posted Price

Require: $l_{ij}, p_{ij}(\mathbf{c})$, the permit and item price for $i \in [n], j \in [m]$.

$S \leftarrow [m]$
for $i \in [n]$ **do**
  Show buyer i the permit price l_{ij} for every j.
  i chooses a set of permits $P^* \subseteq [m]$ and pays $\sum_{j \in P^*} l_{ij}$.
  Reveal S and item prices $\{p_{ij}\}_{j=1}^{m}$ to the buyer.
  i receives her favorite bundle $S_i^* \subseteq S$, paying $\sum_{j \in S_i^*} p_{ij}(\mathbf{c})$.
  $S \leftarrow S \backslash S_i^*$.
end for

In our proof we will use restricted Sequential Permit Posted Price mechanisms (RSPP) by adding the following two changes to the mechanism: Firstly, the buyer is only allowed to purchase at most one permit on the permit-purchasing stage. Secondly, we will further allow the mechanism to hide some items from the buyer on the item-purchasing stage. Formally, the mechanism will choose a (possibly random) set $S_i'(t_{<i}, \mathbf{c}) \subseteq S_i(t_{<i}, \mathbf{c})$ and the buyer is only allowed to purchase item in $S_i'(t_{<i}, \mathbf{c})$. We will now briefly explain how our mechanism used in the proof chooses this set. Fix some parameter $b \in (0, 1)$ which is determined later. In the proof, the item price $p_{ij}(\mathbf{c})$ are chosen such that for every $i, j, \mathbf{c}$, $\Pr_{t_{<i}}[j \in S_i(t_{<i}, \mathbf{c})] \geq 1 - b$. We define the random set $S_i'(t_{<i}, \mathbf{c})$ as follows: for any $j \in S_i(t_{<i}, \mathbf{c})$, put j in $S_i'(t_{<i}, \mathbf{c})$ with probability $(1 - b)/\Pr_{t_{<i}}[j \in S_i(t_{<i}, \mathbf{c})]$, independently. Now we have $\Pr_{t_{<i}}[j \in S_i'(t_{<i}, \mathbf{c})] = 1 - b$. For every buyer i, she has to make the choice whether to purchase each permit j before knowing if the corresponding item is still available or not. The above equation guarantees that for every cost profile $\mathbf{c}$, in expectation over the type profile of buyers coming before her, each item j is available with probability exactly $1 - b$. That probability is independent of $\mathbf{c}$. Thus the expected value for purchasing every permit j is exactly $(1 - b)$ times the utility she could get from this item. This is a crucial property in our proof. See Lemma 4.17) for more details. In the rest of the paper, when we mention RSPP, we refer to the mechanism that hides the item as above. We denote RSPP-Profit the optimal profit of these mechanisms.

The third mechanism is Sequential Permit Bundling(SPB). When every buyer i comes, the seller bundles the permit of all items together and sell them as a grand bundle at some price δ_i in the first stage. δ_i is independent from $\mathbf{c}$. If the buyer refuses to pay the price, then she gets no permit and therefore cannot purchase anything in the second stage. If the buyer buys the permit bundle, the seller then reveals the remaining item set $S_i(t_{<i}, \mathbf{c})$ and the item prices $\{p_{ij}(\mathbf{c})\}_{j=1}^m$ to the buyer. The buyer then chooses her favorite bundle and pays the item prices. The mechanism is also BIC and interim IR due to a similar argument as for SPP. We use SPB-Profit to denote the optimal profit for all SPB mechanisms. See Mechanism 4.5 for details.

In the rest of this section, we prove the following theorem.

Theorem 4.7. *For any valuation distribution $\mathcal{D}$, cost distribution $\mathcal{C}$ and any matroid feasibility*

Mechanism 4.5 Sequential Permit Bundling

Require: $p_{ij}(\mathbf{c})$, item price for $i \in [n], j \in [m]$; δ_i, the price for the permit bundle.

```
S ← [m]
for i ∈ [n] do
    Show buyer i the permit bundle price δ_i.
    if buyer i pays price δ_i then
        Reveal S and item prices {p_ij}_{j=1}^m to the buyer.
        i receives her favorite bundle S_i^* ⊆ S, paying Σ_{j∈S_i^*} p_ij(c).
        S ← S\S_i^*.
    else
        The buyer pays nothing and receives nothing.
    end if
end for
```

constraints $\{\mathcal{F}_i\}_{i=1}^m$,

$$\text{OPT}_{\text{PROFIT}} \leq 14 \cdot \text{CSIP-PROFIT} + 22 \cdot \text{RSPP-PROFIT} + 8 \cdot \text{SPB-PROFIT}$$

Again a simple randomization among the three mechanisms achieves at least $\frac{1}{44}$ the optimal profit.

4.5.2 Bounding Single

Similar to the single buyer case, for every fixed vector $\mathbf{c}$, we consider the related revenue maximization problem where every buyer i has value $t_{ij} - c_j$ for item j. The corresponding Copies Setting is a revenue maximization problem with mn buyers and m items. Every buyer (i, j) only interests in item j and has value $t_{ij} - c_j$ on it. For every i, at most one (i, j) can be served in the mechanism. We denote $\text{OPT-REV}^{\text{COPIES-UD}}(\mathbf{c})$ the optimal revenue of this setting. In Lemma 4.10 we first bound SINGLE by $\mathbb{E}_{\mathbf{c}}[\text{OPT-REV}^{\text{COPIES-UD}}(\mathbf{c})]$. Then according to [CHMS10], $\text{OPT-REV}^{\text{COPIES-UD}}(\mathbf{c})$ can be approximated by the revenue of the optimal sequential posted price mechanism in the related revenue maximization setting. Assume the posted price for buyer i and item j is $\hat{p}_{ij}(\mathbf{c})$. We show that in our setting, an SIP mechanism with $p_{ij}(\mathbf{c}) = \hat{p}_{ij}(\mathbf{c}) + c_j$ has profit the same as the expected (over the randomness of $\mathbf{c}$) revenue of the above sequential posted price mechanism.

Lemma 4.10. $\text{SINGLE} \leq \mathbb{E}_{\boldsymbol{c}}[\text{OPT-REV}^{\text{COPIES-UD}}(\boldsymbol{c})] \leq 6 \cdot \text{CSIP-PROFIT}$.

Proof. Recall that

$$\textsc{Single} = \sum_i \mathbb{E}_{t_i,\mathbf{c}}\big[\sum_j \mathbb{1}[t_i \in R_{ij}^{(\beta)}] \cdot \pi_{ij}(t_i,\mathbf{c}) \cdot (\tilde{\varphi}_{ij}(t_{ij}) - c_j)\big]$$

For every BIC, interim IR mechanism $M = (\pi, p)$ and every $\mathbf{c}$, consider the following mechanism M' in the Copies Setting: M' serves agent (i,j) if and only if M allocates item j to buyer i and $t_i \in R_{ij}^{(\beta)}$. Since M is feasible, for every j there exist at most one i such that (i,j) is served in M'. Also since every t_i stays in one region, for every i there exists at most one j such that (i,j) is served in M'. Thus M' is feasible and the expected revenue equals to $\sum_i \mathbb{E}_{t_i}\big[\sum_j \mathbb{1}[t_i \in R_{ij}^{(\beta)}] \cdot \pi_{ij}(t_i,\mathbf{c}) \cdot (\tilde{\varphi}_{ij}(t_{ij}) - c_j)\big]$. Thus we have

$$\textsc{Single} \leq \mathbb{E}_{\mathbf{c}}[\text{OPT-}\textsc{Rev}^{\textsc{Copies-UD}}(\mathbf{c})]$$

By [CHMS10], for every $\mathbf{c}$ there exists a sequential posted price mechanism $M(\mathbf{c})$ in the related revenue maximization setting[9], where every buyer can purchase at most one item, such that its revenue is at least $\text{OPT-}\textsc{Rev}^{\textsc{Copies-UD}}(\mathbf{c})/6$. Suppose the posted price for buyer i and item j is $\hat{p}_{ij}(\mathbf{c})$. Now let's consider the Constrained Sequential Item Posted Price mechanism with $p_{ij}(\mathbf{c}) = \hat{p}_{ij}(\mathbf{c}) + c_j$ in our profit maximization setting, where every buyer is only allowed to purchase at most one item. For every $\mathbf{t}, \mathbf{c}$, let $A_i(t_{<i}, \mathbf{c})$ be the remaining item sets when buyer i comes to the auction. Then she will choose her favorite item $\text{argmax}_{j \in A_i(t_{<i},\mathbf{c})}(t_{ij} - c_j - \hat{p}_{ij}(\mathbf{c}))$ (or choose not to purchase anything). Notice that this is also buyer i's favorite item in $M(\mathbf{c})$ under the same scenario. Thus the allocation rule for CSIP under $\mathbf{c}$ is the same as the one for $M(\mathbf{c})$. Then profit of the constructed CSIP is equal to the expected revenue of $M(\mathbf{c})$ over the randomness of $\mathbf{c}$, as the extra item prices just cover the seller's costs. According to [CHMS10], the profit is at least $\mathbb{E}_{\mathbf{c}}[\text{OPT-}\textsc{Rev}^{\textsc{Copies-UD}}(\mathbf{c})]/6$. The proof is done. □

9. Recall that in this setting every buyer i has value $t_{ij} - c_j$ for item j.

4.5.3 Bounding Prophet

In this section we will bound Prophet with a Constrained Sequential Item Posted Price mechanism. The proof uses the Online Contention Resolution Scheme (OCRS) developed by Feldman et al. [FSZ16]. See Section 2.7 for details.

Lemma 4.11. *([FSZ16]) Consider the online selection setting. If there exists a (b, c)-selectable greedy OCRS Π for $P_{\mathcal{J}}$, then for every $y \in b \cdot P_{\mathcal{J}}$, consider the strategy that the agent takes elements greedily subject to the sub-constraint $\mathcal{J}_{\Pi,y}$. Then the agent will select each element e with probability at least $c \cdot y_e$. The result applies for any almighty adversary*[10].

Before getting to the proof, let's first discuss the connection between OCRS and bounding Prophet. Recall that

$$\text{PROPHET} = 2 \cdot \sum_i \sum_j \mathbb{E}_{\mathbf{c}}\left[q_{ij}(\mathbf{c}) \cdot (\max\{\beta_{ij}(\mathbf{c}), c_j\} - c_j)\right]$$

Fix $\mathbf{c}$. For every $(i, j) \in J$, let $y_{(i,j)} = q_{ij}(\mathbf{c})$. Since $q_{ij}(\mathbf{c})$ is half the ex-ante probability that a feasible mechanism M serves the pair (i, j) when the true cost is $\mathbf{c}$, thus $y = (y_{(i,j)})_{(i,j)\in J} \in \frac{1}{2} \cdot \mathcal{J}$. Now consider the CSIP with item prices $\max\{\beta_{ij}(\mathbf{c}), c_j\}$. Then by Definition 4.1, each buyer i can afford item j with probability $q_{ij}(\mathbf{c})$[11], i.e. the element (i, j) is active with probability $q_{ij}(\mathbf{c})$. In Lemma 4.12 we show that for one specific almighty adversary, the set of element (i, j) chosen by the agent following the greedy OCRS is exactly same as the set of buyer-item pair served in the mechanism, for every type profile. Then (b, c)-selectability guarantees that every buyer i will purchase every item j in the mechanism with probability at least c given the fact that she can afford this item. This gives a lower bound of CSIP-Profit.

Lemma 4.12. *Fix seller's cost vector $\mathbf{c}$. Suppose there exists a (b, c)-selectable greedy OCRS Π for polytope $P(\mathcal{J})$, for some constant $c \in (0, 1)$. For every l_{ij}s such that $l \in b \cdot P(\mathcal{J})$, consider the*

10. The adversary can determine the order of elements shown to the agent. An almighty adversary has all the information it needs to decide the order, including the agent's type and strategies, and the realization of all possible randomness. In other words, the adversary will choose the worst order for the agent.

11. It's true when $\beta_{ij}(\mathbf{c}) \geq c_j$. For those (i, j) such that $\beta_{ij}(\mathbf{c}) < c_j$, the corresponding term in Prophet is 0. We could simply never serve those pairs.

CSIP under the specific cost profile $\boldsymbol{c}$*, with posted price* $p_{ij}(\boldsymbol{c}) = F_{ij}^{-1}(1 - l_{ij})$ *and sub-constraint* $\mathcal{J}_{\Pi,l}$*. Then the mechanism will gain profit at least*

$$c \cdot \sum_{i,j} l_{ij} \cdot (p_{ij}(\boldsymbol{c}) - c_j)$$

under cost $\boldsymbol{c}$*.*

Proof. Under cost $\mathbf{c}$, consider the CSIP with posted price $p_{ij}(\mathbf{c})$, associated with the constraint $\mathcal{J}_{\Pi,y}$. When every buyer i comes, let A_i be the set of buyer-item pairs that have already been served. And let B_i^* be her favorite bundle among the remaining items, such that after taking those items, the sub-constraint $\mathcal{J}_{\Pi,y}$ is not violated. Now consider the online selection setting with the following almighty adversary: Ground set is J. The agent has value t_{ij} for each element (i, j). Each element (i, j) is active if $t_i \geq p_{ij}(\mathbf{c})$, i.e. is active with probability l_{ij}. The adversary divides the whole item-revealing process into n stages. For each stage i, let B_i be the set of elements that have been selected in the past. The adversary first reveals all (i, j)s where $j \in B_i^*$, one after another. Then it reveals the remaining (i, j)s.

Notice that by following the greedy OCRS Π, the agent will follow the constraint $\mathcal{J}_{\Pi,l}$ and choose all the element (i, j) where $j \in B_i^*$ on each stage, as taking those elements won't violate the constraint by the definition of B_i^*. It's equivalent to the buyer-item pair selection process in the CSIP. Thus under the above adversary, the set of element (i, j) chosen by the agent is exactly same as the set of buyer-item pair served in the mechanism. Since each element (i, j) is active with probability l_{ij} and $l \in b \cdot P(\mathcal{J})$, by Lemma 4.11, each element is chosen by the agent with probability at least $c \cdot l_{ij}$. In other words, in CSIP, each buyer i purchases item j with probability at least $c \cdot l_{ij}$, under cost $\mathbf{c}$. Thus the obtained profit is at least

$$c \cdot \sum_{i,j} l_{ij} \cdot (p_{ij} - c_j)$$

□

Now it's sufficient to show that there exists a $(\frac{1}{2}, c)$-selectable greedy OCRS Π for $P_{\mathcal{J}}$. Recall that every $A \in \mathcal{J}$ satisfies:

- Each item is allocated to at most one buyer: $\forall j, O_j = \{i : (i,j) \in A\}, |O_j| \leq 1$.
- Each buyer is allocated a feasible set of items: $\forall i, P_i = \{j : (i,j) \in A\}, P_i \in \mathcal{F}_i$.

Let $\mathcal{J}_1$(or $\mathcal{J}_2$) be the subfamily that contains all set A that satisfies the first(or second) bullet point. It's straightforward to see that $\mathcal{J}_1$ forms a partition matroid. We will show that $\mathcal{J}_2$ is also a matroid, given the fact that every $\mathcal{F}_i$ is a matroid.

Lemma 4.13. *$\mathcal{J}_2$ is a matroid.*

Proof. Consider any $A, A' \in \mathcal{J}_2$ such that $|A| > |A'|$. For every i, let $P_i = \{j : (i,j) \in A\}$ and $P_i' = \{j : (i,j) \in A'\}$. We have $P_i \in P_i' \in \mathcal{F}_i$. Notice that $|A| = \sum_i |P_i| > |A'| = \sum_i |P_i'|$, there must exist i_0 such that $|P_{i_0}| > |P_{i_0}'|$. Since $\mathcal{F}_{i_0}$ is a matroid, there exists some $j_0 \in P_{i_0} \backslash P_{i_0}'$ such that $P_{i_0}' \cup \{j_0\} \in \mathcal{F}_{i_0}$. By definition of $\mathcal{J}_2$, we also have $(i_0, j_0) \in A \backslash A'$ and $A' \cup (i_0, j_0) \in \mathcal{J}_2$. Thus $\mathcal{J}_2$ is a matroid. □

Note that $\mathcal{J} = \mathcal{J}_1 \cap \mathcal{J}_2$. $\mathcal{J}$ is an intersection of two matroids. We can show that there exists a $(\frac{1}{2}, \frac{1}{4})$-selectable greedy OCRS for $P(\mathcal{J})$ by Lemma 2.3 and Lemma 2.4.

Put everything together, we are able to bound PROPHET using CSIP-PROFIT.

Lemma 4.14. PROPHET $\leq 8 \cdot$ CSIP-PROFIT*.*

Proof. First for those (i,j) such that $\beta_{ij}(\mathbf{c}) < c_j$, the corresponding term in PROPHET is 0. We could simply never serve those pairs. Thus without loss of generality, we assume that $\beta_{ij}(\mathbf{c}) \geq c_j$ for every (i,j). For every $\mathbf{c}$, since $q_{ij}(\mathbf{c})$ is half the ex-ante probability that a feasible mechanism M serves the pair (i,j) when the true cost is $\mathbf{c}$, thus $q(\mathbf{c}) = (q_{ij}(\mathbf{c}))_{(i,j)\in J} \in \frac{1}{2} \cdot \mathcal{J}$. By Lemma 2.3 and 2.4, there exists a $(\frac{1}{2}, \frac{1}{4})$-selectable greedy OCRS Π for $P(\mathcal{J})$. We thus consider the CSIP with posted price $p_{ij}(\mathbf{c}) = \max\{\beta_{ij}(\mathbf{c}), c_j\}$, associated with the constraint $\mathcal{J}_{\Pi, q(\mathbf{c})}$. By Lemma 4.12, the profit of the mechanism is at least

$$\frac{1}{4} \cdot \sum_i \sum_j \mathbb{E}_{\mathbf{c}}\left[q_{ij}(\mathbf{c}) \cdot (\max\{\beta_{ij}(\mathbf{c}), c_j\} - c_j)\right]$$

□

4.5.4 Bounding Non-Favorite

In this section we will bound NON-FAVORITE. As discussed in Section 4.3, we will fix β and omit it in the notation. We first give an informal proof by reducing the multiple buyer problem to single buyer problems with a new valuation $\bar{v}_i$ in Definition 4.2. This shows a connection to the single buyer setting as well as the ex-ante relaxation by Chawla and Miller [CM16]. In their paper they solve the revenue maximization problem for multiple matroid-rank buyers, bounding the benchmark by the sum of optimal revenue for the single buyer problem under an ex-ante constraint.

Recall that

$$\text{NON-FAVORITE} = \sum_i \mathbb{E}_{t_i}\big[\sum_j \mathbb{1}[t_i \in R_{ij}] \cdot \bar{v}_i(t_i, [m]\backslash\{j\})\big]$$

This is the sum of all buyer's welfare contributed by those non-favorite "items"[12], under a new valuation $\bar{v}_i$. Consider the Sequential Permit Selling mechanism with posted price $p_{ij}(\mathbf{c}) = \max\{\beta_{ij}(\mathbf{c}), c_j\}$. Since $p_{ij}(\mathbf{c}) \geq c_j$ for every $\mathbf{c}$, the profit of the mechanism will be at least the revenue extracted from the permit copies.

Notice that for every $i, j, \mathbf{c}$, buyer i can afford the item price for j with probability $q_{ij}(\mathbf{c})$. Thus by union bound, each item j is still available when buyer i comes with probability at least $1 - \sum_j q_{ij}(\mathbf{c}) \geq \frac{1}{2}$. By Lemma 4.21, buyer i's expected utility in the second stage after purchasing a set of permit copies P, is at least $\frac{1}{2} \cdot \bar{v}_i(t_i, P)$. Now we have reduced the multiple buyer problem to n single buyer problems where buyer i has valuation $\bar{v}_i(\cdot, \cdot)$ for the set of permit copies. Thus from Section 4.4, the term $\mathbb{E}_{t_i}\big[\sum_j \mathbb{1}[t_i \in R_{ij}] \cdot \bar{v}_i(t_i, [m]\backslash\{j\})\big]$ can be extracted from selling the permit copies separately and as a whole bundle.

The above argument doesn't give a formal proof because the buyer's expected utility on the copies does not exactly equal to $\frac{1}{2} \cdot \bar{v}_i(t_i, P)$ and thus a reduction like Lemma 4.5 cannot be directly obtained. Now we provide a formal and separate proof, bounding NON-FAVORITE with RSPP and SPB mechanisms. First we decompose the term using a standard Core-Tail decomposition technique [LY13, CDW16], according to $\bar{v}_{ij}(t_{ij})$. For every $i \in [n]$, define $\tau_i = \inf\{a \geq 0 :$

12. Here we use quotations on the word 'item' as in the corresponding single buyer problem, the goods sold to the buyers are permits, not real items.

$\sum_j \Pr_{t_{ij}}[\bar{v}_{ij}(t_{ij}) \geq a] \leq \frac{1}{2}\}$. For every t_i, let $C_i(t_i) = \{j \in [m] : \bar{v}_{ij}(t_{ij}) \leq \tau_i\}$.

Lemma 4.15.

$$\textsc{Non-Favorite} \leq \sum_i \sum_j \mathbb{E}_{t_{ij}:\bar{v}_{ij}(t_{ij})>\tau_i}[\bar{v}_{ij}(t_{ij}) \cdot \Pr_{t_{i,-j}}[\exists k \neq j \ s.t.\ \bar{v}_{ik}(t_{ik}) \geq \bar{v}_{ij}(t_{ij})]] \quad (\textsc{Tail})$$
$$+ \sum_i \mathbb{E}_{t_i}[\bar{v}_i(t_i, C_i(t_i))] \quad (\textsc{Core})$$

Proof.

$$\begin{aligned}
\textsc{Non-Favorite} &= \sum_i \mathbb{E}_{t_i}\big[\sum_j \mathbb{1}[t_i \in R_{ij}] \cdot \bar{v}_i(t_i, [m]\backslash\{j\})\big] \\
&\leq \sum_i \mathbb{E}_{t_i}\big[\sum_j \mathbb{1}[t_i \in R_{ij}] \cdot (\bar{v}_i(t_i, C_i(t_i)\backslash\{j\}) + \bar{v}_i(t_i, [m]\backslash\{j\}\backslash C_i(t_i)))\big] \\
&\leq \sum_i \mathbb{E}_{t_i}[\bar{v}_i(t_i, C_i(t_i))] + \sum_i \mathbb{E}_{t_i}\big[\sum_j \mathbb{1}[t_i \in R_{ij}] \cdot \sum_{k \in [m]\backslash\{j\}\backslash C_i(t_i)} \bar{v}_{ik}(t_{ik})\big] \\
&= \sum_i \mathbb{E}_{t_i}[\bar{v}_i(t_i, C_i(t_i))] + \sum_i \sum_k \mathbb{E}_{t_{ik}:\bar{v}_{ik}(t_{ik})>\tau_i}[\bar{v}_{ik}(t_{ik}) \cdot \Pr_{t_{i,-k}}[(t_{ik}, t_{i,-k}) \notin R_{ij}]] \\
&= \textsc{Core} + \textsc{Tail}
\end{aligned}$$

□

Bounding Tail

We will bound Tail using RSPP mechanisms. For every i, j, let $r_{ij} = \max_{a \geq \tau_i} a \cdot \Pr_{t_{ij}}[\bar{v}_{ij}(t_{ij}) \geq a]$, which is the optimal revenue from selling permit j to buyer i. Let $r = \sum_i \sum_j r_{ij}$. We first show that Tail $\leq \frac{1}{2} \cdot r$ and then bound r using a RSPP.

Lemma 4.16. Tail $\leq \frac{1}{2} \cdot r$.

Proof.

$$\begin{aligned}
\textsc{Tail} &= \sum_i \sum_j \mathbb{E}_{t_{ij}:\bar{v}_{ij}(t_{ij})>\tau_i}[\bar{v}_{ij}(t_{ij}) \cdot \Pr_{t_{i,-j}}[\exists k \neq j \text{ s.t. } \bar{v}_{ik}(t_{ik}) \geq \bar{v}_{ij}(t_{ij})]] \\
&\leq \sum_i \sum_j \mathbb{E}_{t_{ij}:\bar{v}_{ij}(t_{ij})>\tau_i}\left[\bar{v}_{ij}(t_{ij}) \cdot \sum_{k\neq j} \Pr_{t_{ik}}[\bar{v}_{ik}(t_{ik}) \geq \bar{v}_{ij}(t_{ij})]\right] \\
&\leq \sum_i \sum_j \mathbb{E}_{t_{ij}:\bar{v}_{ij}(t_{ij})>\tau_i}\left[\sum_{k\neq j} r_{ik}\right] \\
&\leq \sum_i \sum_j \Pr_{t_{ij}}[\bar{v}_{ij}(t_{ij}) > \tau_i] \cdot r_i \\
&= \frac{1}{2} \cdot r
\end{aligned}$$

□

The following lemma bounds r using the RSPP.

Lemma 4.17. *For any positive $\{\xi_{ij}\}_{i,j}$ such that $\sum_j \Pr_{t_{ij}}[\bar{v}_{ij}(t_{ij}) \geq \xi_{ij}] \leq \frac{1}{2}$, we have*

$$\sum_i \sum_j \xi_{ij} \cdot \Pr_{t_{ij}}[\bar{v}_{ij}(t_{ij}) \geq \xi_{ij}] \leq 4 \cdot \text{RSPP-}\textsc{Profit}$$

Proof. Consider the RSPP mechanism with permit price $\frac{1}{2}\xi_{ij}$ and item price $\max\{\beta_{ij}(\mathbf{c}), c_j\}$. Notice that for every buyer i, her expected utility for purchasing each permit j is $\frac{1}{2} \cdot \bar{v}_{ij}(t_{ij})$. She will purchase every permit j for sure if both of the events happen:

1. She is willing to purchase permit j, i.e., $\bar{v}_{ij}(t_{ij}) \geq \xi_{ij}$.

2. She is not willing to purchase other permits, i.e., $\bar{v}_{ik}(t_{ik}) < \xi_{ik}, \forall k \neq j$.

(1) happens with probability $\Pr[\bar{v}_{ij}(t_{ij}) \geq \xi_{ij}]$; By union bound, (2) happens with probability at least $\frac{1}{2}$ as $\sum_j \Pr_{t_{ij}}[\bar{v}_{ij}(t_{ij}) \geq \xi_{ij}] \leq \frac{1}{2}$. Furthermore, both events are independent and thus buyer i will purchase permit j and pay the permit price with probability at least $\frac{1}{2} \cdot \Pr_{t_{ij}}[\bar{v}_{ij}(t_{ij}) \geq \xi_{ij}]$. □

We point out that in the above lemma, it's necessary to make every buyer i's expected utility for purchasing each permit j to be exactly $\frac{1}{2} \cdot \bar{v}_{ij}(t_{ij})$. This is the reason the RSPP mechanism needs to hide each item randomly to make each item available with probability exactly $\frac{1}{2}$ (See Section 4.5.1). If the mechanism doesn't hide the item, we only know that her expected utility for each permit is at least that much. We are not able to lower bound the probability that (2) happens using union bound.

Lemma 4.18. $\textsc{Tail} \leq 2 \cdot \text{RSPP-}\textsc{Profit}$.

Proof. It directly follows from Lemma 4.16 and 4.17 by applying $\text{argmax}_{a \geq \tau_i} a \cdot \Pr_{t_{ij}}[\bar{v}_{ij}(t_{ij}) \geq a]$ as ξ_{ij} (Notice that it satisfies the constraint in Lemma 4.17 by the definition of τ_i). ☐

Bounding Core

Now we bound $\textsc{Core}$ using RSPP and SPB.

Theorem 4.8. $\textsc{Core} \leq 8 \cdot \text{SPB-}\textsc{Profit} + 20 \cdot \text{RSPP-}\textsc{Profit}$.

Recall that $\textsc{Core} = \sum_i \mathbb{E}_{t_i}[\bar{v}_i(t_i, C_i(t_i))]$. In the proof we will consider the SPB mechanism with item prices $\max\{\beta_{ij}(\mathbf{c}), c_j\}$ and permit bundle price $\delta_i = \frac{1}{2} \cdot median_{t_i}(\bar{v}_i(t_i, C_i(t_i)))$. In order to show that each buyer will accept this bundle price with at least half probability, we will prove the expected utility for the item-purchasing stage is at least $\frac{1}{2} \cdot \bar{v}_i(t_i, [m])$. We need the following definition.

Definition 4.5. *Consider the above SPB mechanism. For every $i, t_i, \boldsymbol{c}$ and $P \subseteq [m]$, let*

$$u_i(t_i, \boldsymbol{c}, P) = \max_{S \subseteq P, S \in \mathcal{F}_i} \sum_{j \in S} (t_{ij} - \max\{\beta_{ij}(\boldsymbol{c}), c_j\})$$

By Definition 4.5, buyer i's expected utility for the item purchasing stage is $\mathbb{E}_{t_{<i}, \mathbf{c}}[u_i(t_i, \mathbf{c}, S_i(t_{<i}, \mathbf{c}))]$. Recall that $S_i(t_{<i}, \mathbf{c})$ is the set of available items in the above SPB mechanism. We notice that for every $i, j, \mathbf{c}$, buyer i can afford the item price for j with probability $q_{ij}(\mathbf{c})$. Thus by union bound, each item j is still available when buyer i comes with probability at least $1 - \sum_j q_{ij}(\mathbf{c}) \geq \frac{1}{2}$, i.e. $\Pr_{t_{<i}}[j \in S_i(t_{<i}, \mathbf{c})] \geq \frac{1}{2}$. Then by showing that all u_is are XOS valuations, we prove that every buyer has expected utility at least $\frac{1}{2} \cdot \bar{v}_i(t_i, [m])$ to enter the auction.

Lemma 4.19. *([DNS05]) A function $v(\cdot)$ is XOS if and only if for every $S \subseteq [m]$, there exist prices $\{p_j\}_{j\in S}$ (called supporting prices) such that*

- $v(S') \geq \sum_{j\in S'} p_j$ *for all* $S' \subseteq S$.
- $\sum_{j\in S} p_j \geq v(S)$.

Lemma 4.20. *For every $i, t_i, \mathbf{c}$, $u_i(t_i, \mathbf{c}, \cdot)$ is an XOS function.*

Proof. Fix $i, t_i, \mathbf{c}$. For every $P \subseteq [m]$, let $S^* = \text{argmax}_{S\subseteq P, S\in\mathcal{F}_i} \sum_{j\in S}(t_{ij} - \max\{\beta_{ij}(\mathbf{c}), c_j\})$. Define supporting prices for set P as follows: $p_j^P = (t_{ij} - \max\{\beta_{ij}(\mathbf{c}), c_j\}) \cdot \mathbb{1}[j \in S^*]$. It's easy to check that p_j^Ps satisfy both constraints in Lemma 4.19. Thus $u_i(t_i, \mathbf{c}, \cdot)$ is an XOS function. □

Lemma 4.21. *Consider the above SPB mechanism. For every i, buyer i will accept the bundle price δ_i with at least $\frac{1}{2}$ probability.*

Proof. As stated above, for every buyer i with type t_i, her expected utility on the item-purchasing stage is $\mathbb{E}_{t_{<i},\mathbf{c}}[u_i(t_i, \mathbf{c}, S_i(t_{<i}, \mathbf{c}))]$. For every $i, j, \mathbf{c}$, buyer i can afford the item price for j with probability $q_{ij}(\mathbf{c})$. Thus by union bound, $\Pr_{t_{<i}}[j \in S_i(t_{<i}, \mathbf{c})] \geq 1 - \sum_j q_{ij}(\mathbf{c}) \geq \frac{1}{2}$.

By Lemma 4.19 and 4.20, let $p_j^P(t_i, \mathbf{c})$ be the supporting price for $u_i(t_i, \mathbf{c}, \cdot)$ and set P. We have

$$
\begin{aligned}
\mathbb{E}_{t_{<i},\mathbf{c}}[u_i(t_i, \mathbf{c}, S_i(t_{<i}, \mathbf{c}))] &\geq \mathbb{E}_{t_{<i},\mathbf{c}}\Big[\sum_{j\in S_i(t_{<i},\mathbf{c})} p_j^{[m]}(t_i, \mathbf{c})\Big] \\
&= \mathbb{E}_{\mathbf{c}}\big[\sum_{j\in[m]} p_j^{[m]}(t_i, \mathbf{c}) \cdot \Pr_{t_{<i}}[j \in S_i(t_{<i}, \mathbf{c})]\big] \\
&\geq \frac{1}{2} \cdot \mathbb{E}_{\mathbf{c}}[u_i(t_i, \mathbf{c}, [m])] = \frac{1}{2}\bar{v}_i(t_i, [m]) \geq \frac{1}{2} \cdot \bar{v}_i(t_i, C_i(t_i))
\end{aligned}
$$

Thus buyer i will pay $\delta_i = \frac{1}{2} \cdot median_{t_i}(\bar{v}_i(t_i, C_i(t_i)))$ with probability at least $\frac{1}{2}$. □

Now it's sufficient to show that $\mathbb{E}_{t_i}[\bar{v}_i(t_i, C_i(t_i))]$ is comparable to δ_i for every i. This is obtained by applying the Talagrand's concentration inequality on $\bar{v}_i(t_i, C_i(t_i))$. Let $\mu_i(t_i, S) = \bar{v}_i(t_i, C_i(t_i) \cap S)$. We show that μ_i is subadditive and has small Lipschitz constant (Definition 3.4). The proof is Lemma 4.22 is postponed to Appendix B.4.

Lemma 4.22. *$\mu_i(t_i,\cdot)$ is monotone, subadditive, no exteralities and has Lipschitz constant τ_i.*

The following lemma is a restatement of Corollary 3.1. It shows that for every i, $\mathbb{E}_{t_i}[\bar{v}_i(t_i, C_i(t_i))]$ is bounded by δ_i and the Lipschitz constant τ_i.

Lemma 4.23. *(Restatement of Corollary 3.1)*

$$\mathbb{E}_{t_i}[\bar{v}_i(t_i, C_i(t_i))] = \mathbb{E}_{t_i}[\mu_i(t_i, [m]) \leq 4 \cdot \delta_i + \frac{5}{2} \cdot \tau_i$$

For the last step, $\sum_i \tau_i$ can be bounded using the RSPP.

Lemma 4.24. $\sum_i \tau_i \leq 8 \cdot \text{RSPP-PROFIT}$.

Proof. By definition, $\sum_j \Pr_{t_{ij}}[\bar{v}_{ij}(t_{ij}) \geq \tau_i] = \frac{1}{2}$ for every i. By Lemma 4.17,

$$\text{RSPP-PROFIT} \geq \frac{1}{4} \sum_i \tau_i \cdot \sum_j \Pr_{t_{ij}}[\bar{v}_{ij}(t_{ij}) \geq \tau_i] = \frac{1}{8} \cdot \sum_i \tau_i$$

□

Proof of Theorem 4.8: Consider the SPB mechanism with item prices $\max\{\beta_{ij}(\mathbf{c}), c_j\}$ and permit bundle price $\delta_i = \frac{1}{2} \cdot median_{t_i}(\bar{v}_i(t_i, C_i(t_i)))$. According to Lemma 4.21, Lemma 4.24 and Lemma 4.23,

$$\text{SPB-PROFIT} \geq \frac{1}{2} \cdot \sum_i \delta_i \geq \frac{1}{8} (\sum_i \mathbb{E}_{t_i}[\bar{v}_i(t_i, C_i(t_i))] - \frac{5}{2}\tau_i) \geq \frac{1}{8} \cdot (\text{CORE} - 20 \cdot \text{RSPP-PROFIT})$$

□

Chapter 5

Single-Dimensional Two-sided Markets

Starting from this chapter, we consider the problem of approximating gains from trade (GFT) in two-sided markets. In this chapter, we will focus on single-dimensional settings, or more specifically, bilateral trade and double auctions associated with any downward-closed trading constraints $\mathcal{F}$ (see Section 2.4). We will design a simple, truthful and budget balanced mechanism and prove that the GFT of this mechanism is at least half of the second-best GFT.

In Section 5.1, we characterize the set of allocation rules that are implementable by an IR, BIC, SBB mechanism in double auctions. In Section 5.2, we study the bilateral trade setting, proving a 2-approximation to the second-best GFT using some simple, IR, BIC, SBB mechanisms. In Section 5.3, we generalize the 2-approximation result to any double auction with arbitrary downward-closed feasibility constraint. In Section 5.4, we discuss the relationship of mechanisms with different budget balanced constraints.

5.1 Characterizing the Implementable Allocation Rules in Double Auctions

In this section, we characterize the set of allocation rules that are implementable by an IR, BIC, SBB mechanism in Theorem 5.1. It generalizes Myerson and Satterthwaite's result [Mye81] (see Section 2.4) to double auctions. In particular, their result is a special case of ours when $n = m = 1$.

For simplicity, in the proof of Theorem 5.1 we assume all agents have continuous distributions. In other words, for every buyer i (or seller j), $f_i^B(\cdot)$ (of $f_j^S(\cdot)$) is a continuous function and positive in its domain $[\underline{b_i}, \overline{b_i}]$ (or $[\underline{s_j}, \overline{s_j}]$). Both $\overline{b_i}$ and $\overline{s_j}$ can be ∞. For discrete distributions, the theorem follows from a similar argument.

Theorem 5.1. *Given an allocation rule $x = (x^B, x^S)$, there exists payment rule (p^B, p^S) such that the mechanism $M = (x^B, x^S, p^B, p^S)$ is IR, BIC and SBB if and only if*

- *For every i, $x_i^B(b_i)$ is non-decreasing on b_i. For every j, $x_j^S(s_j)$ is non-increasing on s_j.*
-
$$\mathbb{E}_{\boldsymbol{b},\boldsymbol{s}}\left[\sum_{i=1}^{n} x_i^B(\boldsymbol{b},\boldsymbol{s}) \cdot \varphi_i(b_i) - \sum_{j=1}^{m} x_j^S(\boldsymbol{b},\boldsymbol{s}) \cdot \tau_j(s_j)\right] \geq 0 \tag{5.1}$$

The following lemma characterizes the payments in an IR, BIC mechanism. The proof is similar to the analysis for bilateral trade in [MS83].

Lemma 5.1. *Suppose a mechanism $M = (x^B, x^S, p^B, p^S)$ is IR and BIC, then*

- *For every i, $x_i^B(b_i)$ is non-decreasing on b_i. For every j, $x_j^S(s_j)$ is non-increasing on s_j.*
- *For every buyer i and any of her type b_i,*

$$p_i^B(b_i) = b_i \cdot x_i^B(b_i) - \int_{\underline{b_i}}^{b_i} x_i^B(t)dt - \theta_i, \tag{5.2}$$

θ_i is some non-negative constant.

- *For every seller j and any of her type s_j,*

$$p_j^S(s_j) = s_j \cdot x_j^S(s_j) + \int_{s_j}^{\overline{s_j}} x_j^S(t)dt + \eta_j, \tag{5.3}$$

  η_j is some non-negative constant.

Furthermore, if (p^B, p^S) satisfies Equation (5.2) and (5.3), then

$$\sum_{i=1}^{n} \mathbb{E}_{b_i}[p_i^B(b_i)] = \mathbb{E}_{\boldsymbol{b},\boldsymbol{s}}\left[\sum_{i=1}^{n} x_i^B(\boldsymbol{b},\boldsymbol{s}) \cdot \varphi_i(b_i)\right] - \sum_{i=1}^{n} \theta_i \tag{5.4}$$

$$\sum_{j=1}^{m} \mathbb{E}_{s_j}[p_j^S(s_j)] = \mathbb{E}_{\boldsymbol{b},\boldsymbol{s}}\left[\sum_{j=1}^{m} x_j^S(\boldsymbol{b},\boldsymbol{s}) \cdot \tau_j(s_j)\right] - \sum_{j=1}^{m} \eta_j \tag{5.5}$$

Proof. For every i and b_i, let $U_i^B(b_i)$ be buyer i's expected utility when she reports her true type b_i. Since M is BIC, for every buyer i, and two types $b_i, b_i' \in [\underline{b_i}, \overline{b_i}]$,

$$U_i^B(b_i) = b_i \cdot x_i^B(b_i) - p_i^B(b_i) \geq b_i \cdot x_i^B(b_i') - p_i^B(b_i')$$

$$U_i^B(b_i') = b_i' \cdot x_i^B(b_i') - p_i^B(b_i') \geq b_i' \cdot x_i^B(b_i) - p_i^B(b_i)$$

The two inequalities imply

$$(b_i' - b_i) \cdot x_i^B(b_i) \leq U_i^B(b_i') - U_i^B(b_i) \leq (b_i' - b_i) \cdot x_i^B(b_i')$$

when $b_i' > b_i$, $x_i^B(b_i') \geq x_i^B(b_i)$. $x_i^B(b_i)$ is non-decreasing on b_i and thus Riemann integrable. Let $b_i' = b_i + \epsilon$,

$$\epsilon \cdot x_i^B(b_i) \leq U_i^B(b_i + \epsilon) - U_i^B(b_i) \leq \epsilon \cdot x_i^B(b_i + \epsilon) \tag{5.6}$$

For any value z, taking integral of b_i on $[\underline{b_i}, z]$ and let $\epsilon \to 0$, we have

$$U_i^B(z) = U_i^B(\underline{b_i}) + \int_{\underline{b_i}}^{z} x_i^B(b_i)db_i \tag{5.7}$$

$$p_i^B(z) = z \cdot x_i^B(z) - \int_{\underline{b_i}}^{z} x_i^B(b_i) db_i - U_i^B(\underline{b_i}) \tag{5.8}$$

Clearly, $U_i^B(\underline{b_i}) \geq 0$, as M is interim IR. Equation (5.2) follows from setting $\theta_i = U_i^B(\underline{b_i})$. Similarly, we can show that Equation (5.3) holds for every j.

Furthermore, for every i, by Equation (5.2),

$$\begin{aligned}
\mathbb{E}_{b_i}[p_i^B(b_i)] &= \int_{\underline{b_i}}^{\overline{b_i}} b_i x_i^B(b_i) f_i^B(b_i) db_i - \int_{\underline{b_i}}^{\overline{b_i}} \int_{\underline{b_i}}^{b_i} x_i^B(t) f_i^B(b_i) dt db_i - \theta_i \\
&= \int_{\underline{b_i}}^{\overline{b_i}} b_i x_i^B(b_i) f_i^B(b_i) db_i - \int_{\underline{b_i}}^{\overline{b_i}} x_i^B(t) \int_{t}^{\overline{b_i}} f_i^B(b_i) db_i dt - \theta_i \\
&= \int_{\underline{b_i}}^{\overline{b_i}} b_i x_i^B(b_i) f_i^B(b_i) db_i - \int_{\underline{b_i}}^{\overline{b_i}} x_i^B(b_i) \left(1 - F_i^B(b_i)\right) db_i - \theta_i \\
&= \int_{\underline{b_i}}^{\overline{b_i}} \varphi_i(b_i) x_i^B(b_i) f_i^B(b_i) db_i - \theta_i
\end{aligned} \tag{5.9}$$

Similarly, for every seller j, by Equation (5.3),

$$\begin{aligned}
\mathbb{E}_{s_j}[p_j^S(s_j)] &= \int_{\underline{s_j}}^{\overline{s_j}} s_j x_j^S(s_j) f_j^S(s_j) ds_j + \int_{\underline{s_j}}^{\overline{s_j}} \int_{s_j}^{\overline{s_j}} x_j^S(t) f_j^S(s_j) dt ds_j + \eta_j \\
&= \int_{\underline{s_j}}^{\overline{s_j}} s_j x_j^S(s_j) f_j^S(s_j) ds_j \int_{\underline{s_j}}^{\overline{s_j}} x_j^S(t) \int_{\underline{s_j}}^{t} f_j^S(s_j) ds_j dt + \eta_j \\
&= \int_{\underline{s_j}}^{\overline{s_j}} s_j x_j^S(s_j) f_j^S(s_j) ds_j \int_{\underline{s_j}}^{\overline{s_j}} x_j^S(s_j) F_j^S(s_j) ds_j + \eta_j \\
&= \int_{\underline{s_j}}^{\overline{s_j}} \tau_j(s_j) x_j^S(s_j) f_j^S(s_j) ds_j + \theta_j
\end{aligned} \tag{5.10}$$

Equations (5.4) and (5.5) directly follows from the two equations above.

□

Proof of Theorem 5.1: If there exists a payment rule (p^B, p^S) such that the mechanism $M = (x^B, x^S, p^B, p^S)$ is IR, BIC and SBB, by Lemma 5.1, x is monotone and there exists non-negative

θ_i's and η_j's such that Equations (5.4) and (5.5) hold. Since M is SBB,

$$\mathbb{E}_{\mathbf{b},\mathbf{s}}\left[\sum_{i=1}^{n} x_i^B(\mathbf{b},\mathbf{s})\cdot\varphi_i(b_i) - \sum_{j=1}^{m} x_j^S(\mathbf{b},\mathbf{s})\cdot\tau_j(s_j)\right] = \sum_{i=1}^{n}\theta_i + \sum_{j=1}^{m}\eta_j \geq 0 \tag{5.11}$$

If the given allocation rule x is monotone, and satisfies Inequality (5.1), define payment rule $p = (p^B, p^S)$ as follows:

$$p_i^B(\mathbf{b},\mathbf{s}) = b_i \cdot x_i^B(\mathbf{b},\mathbf{s}) - \int_{\underline{b_i}}^{b_i} x_i^B(t, b_{-i}, \mathbf{s})dt \tag{5.12}$$

$$p_j^S(\mathbf{b},\mathbf{s}) = s_j \cdot x_j^S(\mathbf{b},\mathbf{s}) + \int_{s_j}^{\overline{s_j}} x_j^S(\mathbf{b}, t, s_{-j})dt \tag{5.13}$$

This is a threshold payment and thus $M = (x, p)$ is DSIC and IR. According to Equations (5.4), (5.5) and Inequality (5.1), M is ex-ante WBB. By Lemma 5.11 and Theorem 5.12, there exists payment rule $(p^{B'}, p^{S'})$ such that $M' = (x^B, x^S, p^{B'}, p^{S'})$ is IR, BIC, and SBB. □

5.2 Bilateral Trade

To warm up, we first study the classic bilateral trade setting, when there is only one buyer, one seller and a single item. Throughout this section, we will use b and s to represent the buyer and seller's type accordingly. In any mechanism $M = (x, p^B, p^S)$, for every type profile (b, s), $x(b, s)$ is the probability that the item is traded. $p^B(b, s)$ and $p^S(b, s)$ are the payments. For simplicity, for every b let $x^B(b) = \mathbb{E}_s[x(b, s)], p^B(b) = \mathbb{E}_s[p^B(b, s)]$ be the buyer's interim allocation and payment. Similarly, let $x^S(s) = \mathbb{E}_b[x(b, s)], p^S(s) = \mathbb{E}_b[p^S(b, s)]$ for every s.

We will present two simple IR, BIC, SBB mechanisms and prove that the better one obtains at least half of the optimal GFT.

1. **Seller-Offering Mechanism (SO)**: The seller posts a take-it or leave-it price $q_B(s)$ for the item to the buyer. The item price depends on her true type s and the buyer's value distribution. The buyer has to pay $q_B(s)$ to the seller if she chooses to take the item.

2. **Buyer-Offering Mechanism (BO)**: The buyer posts a take-it or leave-it price $q_S(b)$ for the item to the seller. The item price depends on her true type b and the seller's value distribution. The seller can get $q_S(b)$ from the buyer if she chooses to sell the item.

Since the item is sold through a posted price, both mechanisms are clearly SBB. For SO, the seller has the information of her true type s and the buyer's type distribution D^B. She will choose $q_B(s)$ to maximize her expected utility and thus the mechanism is BIC and IR for the seller. The buyer sees the posted price and will buy if and only if her value b is greater than $q_B(s)$, no matter what s is. Hence the mechanism is DSIC and IR for the buyer. Similarly, BO is BIC for the buyer, DSIC for the seller, and IR for both agents.

Let GFT_{SO} and GFT_{BO} be the expected GFT for SO and BO. We will prove that the second-best GFT (SB-GFT) is bounded by GFT_{SO} plus GFT_{BO}. Hence a simple randomization between the two mechanisms achieves a 2-approximation.

Theorem 5.2. $\text{SB-GFT} \leq \text{GFT}_{\text{SO}} + \text{GFT}_{\text{BO}}$

To prove Theorem 5.2, we first focus on the left hand side of the inequality - the optimal GFT. In [MS83], the authors give an exact characterization of the optimal mechanism, yet its GFT is not easy to analyze. Thus it's necessary to provide an upper bound of the optimal GFT that has a simple form.

5.2.1 Upper Bound for OPT

Notice that the GFT of any two-sided market mechanism equals to the buyer's expected utility of this mechanism, plus the seller's expected utility, plus the difference between buyer's and seller's expected payment. Denote OPT_{BU} (or OPT_{SU}) the optimal buyer's (or seller's) utility attainable by any IR, BIC and ex-ante WBB mechanism. Note that in any ex-ante WBB mechanism, the buyer's expected payment is at least the seller's expected gains. Thus OPT is upper bounded by OPT_{BU} plus OPT_{SU}.

Now it remains to bound OPT_{SU} and OPT_{BU}. By Lemma 5.1, for any BIC and IR mechanism $M = (x, p)$, $x(b, s)$ is non-decreasing on b and non-increasing on s. The seller's expected utility for the mechanism is $\mathbb{E}_{b,s}[p^B(b,s)-x(b,s)\cdot s] = \mathbb{E}_{b,s}[x(b,s)\cdot(\varphi(b)-s)]$. When D^B is regular, it is at most $\mathbb{E}_{b,s}[(\varphi(b)-s)^+]$ when $x(b,s) = \mathbb{1}[\varphi(b) \geq s]$.[1] For irregular D^B, $\varphi(b)$ might not be non-decreasing and thus $\mathbb{1}[\varphi(b) \geq s]$ might not be a monotone allocation rule. To get an achievable bound, we will follow the ironing procedure on the virtual value function [Mye81] and bound OPT_{SU} by $\mathbb{E}_{b,s}[(\tilde{\varphi}(b)-s)^+]$, where $\tilde{\varphi}(\cdot)$ is the Myerson's ironed virtual value function.

Lemma 5.2. *[Har13] For any non-decreasing function $x^B(\cdot): T^B \to [0,1]$,*

$$\mathbb{E}_b[x^B(b)\cdot\varphi(b)] \leq \mathbb{E}_b[x^B(b)\cdot\tilde{\varphi}(b)].$$

The inequality holds with equality if $x^B(b) = x^B(b')$ for every two values b, b' in the same ironed interval (or formally, $\tilde{\varphi}(b) = \tilde{\varphi}(b')$).

For the seller's virtual cost function, we will perform a similar ironing procedure if D^S is irregular[2]. A formal definition of the ironing procedure can be found in Definition 5.1. Lemma 5.3 is analogous to Lemma 5.2, which states that the seller's expected virtual cost is at least the expected ironed virtual cost, under any monotone allocation rule.

Definition 5.1. *(ironing for seller's distribution) Let $G(\cdot), g(\cdot)$ be the cdf and pdf of D^S. Consider the quantile space for the distribution. For any $r \in [0,1]$, let $R(r) = r\cdot G^{-1}(r)$.[3] Let $\tilde{R}(r) = \min_{r_1,r_2\in[0,1]}(\delta\cdot R(r_1)+(1-\delta)\cdot R(r_2))$, where $\delta\in[0,1]$ is the unique value such that $r = \delta\cdot r_1 + (1-\delta)\cdot r_2$. Now $\tilde{R}(\cdot)$ is a convex curve. The ironed virtual cost function is defined as $\tilde{\tau}(s) = \tilde{R}'(G(s))$.*

Lemma 5.3. *For any non-increasing function $x^S(\cdot): T^S \to [0,1]$,*

$$\mathbb{E}_s[x^S(s)\cdot\tau(s)] \geq \mathbb{E}_s[x^S(s)\cdot\tilde{\tau}(s)].$$

The inequality holds with equality if $x^S(s) = x^S(s')$ for every two values s, s' in the same ironed

1. $x^+ = \max\{x, 0\}$

2. A seller's distribution D^S is regular if and only if $\tau(\cdot)$ is monotone non-decreasing.

3. By taking derivative, one can easily check that $R'(r) = \tau(G^{-1}(r))$ holds for every r. Thus $\tau(\cdot)$ is non-decreasing if and only if $R(\cdot)$ is convex.

interval (or formally, $\tilde{\tau}(s) = \tilde{\tau}(s')$).

Proof. Notice that $R'(F^S(s)) = f^S(s)\tau(s)$ for every s. By integration by parts, for monotone allocation rule $x^S(\cdot)$ we have,

$$\mathbb{E}_s[x^S(s) \cdot \tau(s)] = -\mathbb{E}_s[(x^S)'(s) \cdot R(F^S(s))]$$

Similarly,

$$\mathbb{E}_s[x^S(s) \cdot \tilde{\tau}(s)] = -\mathbb{E}_s[(x^S)'(s) \cdot \tilde{R}(F^S(s))]$$

By definition, for every $r \in [0, 1]$, $\tilde{R}(r) \leq R(r)$. Moreover, since $x^S(\cdot)$ is non-increasing, we have

$$\mathbb{E}_s[x^S(s) \cdot \tau(s)] = -\mathbb{E}_s[(x^S)'(s) \cdot R(F^S(s))] \geq -\mathbb{E}_s[(x^S)'(s) \cdot \tilde{R}(F^S(s))] = \mathbb{E}_s[x^S(s) \cdot \tilde{\tau}(s)]$$

The inequality holds with equality if $(x^S)'(s) = 0$ for every s such that $R(F^S(s)) > \tilde{R}(F^S(s))$. The proof is done by noticing that whenever $R(F^S(s)) > \tilde{R}(F^S(s))$, s must be in the interior of the some ironed interval. □

Lemma 5.4. $\mathrm{OPT}_{\mathrm{SU}} \leq \mathbb{E}_{b,s}[(\tilde{\varphi}(b) - s)^+], \mathrm{OPT}_{\mathrm{BU}} \leq \mathbb{E}_{b,s}[(b - \tilde{\tau}(s))^+]$.

Proof. By Lemma 5.1, for any BIC, IR, and ex-ante WBB mechanism $M = (x, p)$, $x(b, s)$ is non-decreasing on b and non-increasing on s. The seller's expected utility for the mechanism is

$$\begin{aligned}
\mathbb{E}_{b,s}[p^S(b, s) - x(b, s) \cdot s] \leq & \mathbb{E}_{b,s}[p^B(b, s) - x(b, s) \cdot s] \quad (M \text{ is ex-ante WBB}) \\
= & \mathbb{E}_{b,s}[x(b, s) \cdot (\varphi(b) - s)] - \theta \quad (\text{by Lemma 5.1}) \\
\leq & \mathbb{E}_{b,s}[x(b, s) \cdot (\tilde{\varphi}(b) - s)] - \theta \quad (\text{by Lemma 5.2}) \\
\leq & \mathbb{E}_{b,s}[(\tilde{\varphi}(b) - s)^+]
\end{aligned}$$

Similarly, for any mechanism that is BIC and IR for the seller, the buyer's expected utility for the mechanism is

$$\mathbb{E}_{b,s}[x(b,s)\cdot b - p^B(b,s)] \le \mathbb{E}_{b,s}[x(b,s)\cdot b - p^S(b,s)]$$
$$= \mathbb{E}_{b,s}[x(b,s)\cdot(b-\tau(s))] - \eta \le \mathbb{E}_{b,s}[x(b,s)\cdot(b-\tilde{\tau}(s))] - \eta \le \mathbb{E}_{b,s}[(b-\tilde{\tau}(s))^+]$$

where the second inequality follows from Lemma 5.3. The proof is done. □

We remark that according to Lemma 5.5 in Section 5.2.2, both of the inequalities in Lemma 5.4 hold with equality as the bound is achieved by SO (and BO). In the rest of this section we will first characterize the value of GFT_{SO} and GFT_{BO}, and then prove Theorem 5.2.

5.2.2 GFT of the SO and BO Mechanism

Lemma 5.5.

$$\text{GFT}_{\text{SO}} = \mathbb{E}_{b,s}[(b-s)\cdot \mathbb{1}[\tilde{\varphi}(b) \ge s]]$$

$$\text{GFT}_{\text{BO}} = \mathbb{E}_{b,s}[(b-s)\cdot \mathbb{1}[b \ge \tilde{\tau}(s)]]$$

Proof. First consider SO. For every seller's type s, if the seller uses q as the posted price in the SO, the seller's expected utility is $u_S(s,q) = (q-s)\cdot \Pr_{b\sim D^B}[b \ge q] = \mathbb{E}_b[(\varphi(b)-s)\cdot \mathbb{1}[b \ge q]]$ according to Myerson's lemma. By Lemma 5.2, this is at most $\mathbb{E}_b[(\tilde{\varphi}(b)-s)\cdot \mathbb{1}[b\ge q]]$. Since $\tilde{\varphi}(\cdot)$ is monotone, $b \ge q$ if and only if $\tilde{\varphi}(b) \ge \tilde{\varphi}(q)$. Thus choosing $q^* = \min\{q|\tilde{\varphi}(q) = s\}$[4] maximizes the term $\mathbb{E}_b[(\tilde{\varphi}(b)-s)\cdot \mathbb{1}[b \ge q]]$. Moreover, since q^* is not in the interior of any ironed interval, $u_S(s,q^*) = \mathbb{E}_b[(\tilde{\varphi}(b)-s)\cdot \mathbb{1}[b \ge q^*]]$. Thus q^* also maximizes the seller's utility. In SOM, the trade happens whenever $\tilde{\varphi}(b) \ge s$.

The proof for BO is analogous: In BO, the buyer with type b will choose $\max\{q|\tilde{\tau}(q) = b\}$ as the posted price, and the trade happens whenever $\tilde{\tau}(s) \le b$. □

Now we are ready to prove Theorem 5.2. If both of the distributions are regular, Theorem 5.2

4. In fact if there are multiple values of q that satisfy $\tilde{\varphi}(q) = s$, choosing $q^* = \max\{q|\tilde{\varphi}(q) = s\}$ derives the same maximum expected utility. In other words, the seller can choose either the start-point or the end-point of that ironed interval as her posted price. Here we pick the one that allows more trade.

directly follows from OPT $\leq$ $\text{OPT}_{\text{SU}} + \text{OPT}_{\text{BU}}$ and the fact that the buyer's virtual value is always less than the true value and the seller's virtual cost is always greater than her true cost. For general distributions, we will apply Lemma 5.2 to prove the theorem.

Proof of Theorem 5.2: By Lemma 5.4, we have

$$\text{SB-GFT} \leq \text{OPT}_{\text{BU}} + \text{OPT}_{\text{SU}} \leq \mathbb{E}_{b,s}[(\tilde{\varphi}(b) - s)^+] + \mathbb{E}_{b,s}[(b - \tilde{\tau}(s))^+]$$

For every s, note that $\mathbb{1}[\tilde{\varphi}(b) \geq s]$ is a monotone function and the jump from 0 to 1 happens at the minimum value of b such that $\tilde{\varphi}(b) = s$, which is not in the interior of any ironed interval. Thus

$$\mathbb{E}_{b,s}[(\tilde{\varphi}(b) - s)^+] = \mathbb{E}_{b,s}[(\varphi(b) - s) \cdot \mathbb{1}[\tilde{\varphi}(b) \geq s]] \leq \mathbb{E}_{b,s}[(b - s) \cdot \mathbb{1}[\tilde{\varphi}(b) \geq s]] = \text{GFT}_{\text{SO}}$$

Similarly we have $\mathbb{E}_{b,s}[(b - \tilde{\tau}(s))^+] \leq \text{GFT}_{\text{BO}}$. Thus $_{\leq}\text{GFT}_{\text{SO}} + \text{GFT}_{\text{BO}}$. □

5.3 Double Auction Setting

In this section, we consider the double auction setting. In Section 5.2, we proposed two simple mechanisms approximating the optimal GFT for bilateral trade. Both of the mechanisms are IR, BIC and SBB. However in double auctions, such a mechanism appears to be hard to design directly. One significant barrier is to find a payment rule that simultaneously satisfies all three conditions mentioned above. Indeed, given an allocation rule, even monotone, such a payment rule is not guaranteed to exist (Theorem 5.1).

However, even knowing there exists a payment rule that makes the mechanism IR, BIC and SBB, it is still not easy to explicitly describe these payments. We circumvent this difficulty by first proposing ex-ante WBB mechanisms whose GFT is at least $\frac{\text{OPT}}{2}$. Then the mechanism can be transformed into an SBB mechanism, while maintaining the same GFT. See Theorem 5.4 in Section 5.4 for more details about the transformation.

Theorem 5.3. *In any double auction, there exists an IR, DSIC and ex-ante WBB mechanism* *whose GFT is at least* $\frac{\text{OPT}}{2}$.

To prove Theorem 5.3, we first bound the optimal GFT using a similar argument as in Section 5.2. Denote OPT_{BU} (or OPT_{SU}) the maximum value for the sum of all buyers' (or sellers') utility attainable by any IR, BIC and ex-ante WBB mechanism. Lemma 5.6 gives an upper bound of the optimal GFT. Before stating the lemma, we need the following definition.

Definition 5.2. *Given type profile $(\boldsymbol{b}, \boldsymbol{s})$, build a complete bipartite graph between the buyers and the sellers with edge weight $w_{ij}(b_i, s_j)$ equal to $\tilde{\varphi}_i(b_i) - s_j$ for the edge between buyer i and seller j. Then find a matching $A(\boldsymbol{b}, \boldsymbol{s}) \in \mathcal{F}$ that maximizes the total weight $\sum_{(i,j)\in A(\boldsymbol{b},\boldsymbol{s})} w_{ij}(b_i, s_j)$. Denote $A^1(\boldsymbol{b}, \boldsymbol{s})$ the maximum weight matching. For the other side, denote $A^2(\boldsymbol{b}, \boldsymbol{s})$ the maximum weight matching when the edge weight $w_{ij}(b_i, s_j)$ equal to $b_i - \tilde{\tau}_j(s_j)$.*

Lemma 5.6.

$$\text{SB-GFT} \leq \text{OPT}_{\text{SU}} + \text{OPT}_{\text{BU}} \leq \mathbb{E}_{\boldsymbol{b},\boldsymbol{s}}\left[\sum_{(i,j)\in A^1(\boldsymbol{b},\boldsymbol{s})} (\tilde{\varphi}_i(b_i) - s_j)\right] + \mathbb{E}_{\boldsymbol{b},\boldsymbol{s}}\left[\sum_{(i,j)\in A^2(\boldsymbol{b},\boldsymbol{s})} (b_i - \tilde{\tau}_j(s_j))\right].$$

Proof. By Lemma 5.1, for any BIC, IR and ex-ante WBB mechanism $M = (x, p)$, for every $\mathbf{b}, \mathbf{s}$, $x_i^B(\mathbf{b}, \mathbf{s})$ is non-decreasing on b_i for every i. And for every j, $x_j^S(\mathbf{b}, \mathbf{s})$ is non-increasing on s_j. Denote $x_{ij}(\mathbf{b}, \mathbf{s})$ the probability that buyer i trades with seller j under profile $(\mathbf{b}, \mathbf{s})$. Clearly $x_i^B(\mathbf{b}, \mathbf{s}) = \sum_{j\in[m]} x_{ij}(\mathbf{b}, \mathbf{s})$, $x_j^S(\mathbf{b}, \mathbf{s}) = \sum_{i\in[n]} x_{ij}(\mathbf{b}, \mathbf{s})$.

The sum of all sellers' expected utility for the mechanism is

$$
\begin{aligned}
\sum_j \mathbb{E}_{\mathbf{b},\mathbf{s}}[p_j^S(\mathbf{b},\mathbf{s}) - x_j^S(\mathbf{b},\mathbf{s}) \cdot s_j] &\le \mathbb{E}_{\mathbf{b},\mathbf{s}}[\sum_i p_i^B(\mathbf{b},\mathbf{s}) - \sum_j x_j^S(\mathbf{b},\mathbf{s}) \cdot s_j] \quad (M \text{ is ex-ante WBB}) \\
&= \mathbb{E}_{\mathbf{b},\mathbf{s}}[\sum_i x_i^B(\mathbf{b},\mathbf{s}) \cdot \varphi_i(b_i) - \sum_j x_j^S(\mathbf{b},\mathbf{s}) \cdot s_j] - \sum_j \theta_j \quad (\text{Lemma 5.1}) \\
&\le \mathbb{E}_{\mathbf{b},\mathbf{s}}[\sum_i x_i^B(\mathbf{b},\mathbf{s}) \cdot \tilde{\varphi}_i(b_i) - \sum_j x_j^S(\mathbf{b},\mathbf{s}) \cdot s_j] - \sum_j \theta_j \quad (\text{Lemma 5.2}) \\
&= \mathbb{E}_{\mathbf{b},\mathbf{s}}[\sum_{i,j} x_{ij}(\mathbf{b},\mathbf{s}) \cdot (\tilde{\varphi}_i(b_i) - s_j)] - \sum_j \theta_j \\
&\le \mathbb{E}_{\mathbf{b},\mathbf{s}}\left[\sum_{(i,j)\in A^1(\mathbf{b},\mathbf{s})} (\tilde{\varphi}_i(b_i) - s_j)\right]
\end{aligned}
$$

where the first inequality follows from Lemma 5.2.

Similarly, for any mechanism that is BIC, IR and ex-ante WBB, the sum of all buyers' expected utility for the mechanism is

$$
\begin{aligned}
\sum_i \mathbb{E}_{\mathbf{b},\mathbf{s}}[x_i^B(\mathbf{b},\mathbf{s}) \cdot b_i - p_i^B(\mathbf{b},\mathbf{s})] &\le \mathbb{E}_{\mathbf{b},\mathbf{s}}[\sum_i x_i^B(\mathbf{b},\mathbf{s}) \cdot b_i - \sum_j p_i^S(\mathbf{b},\mathbf{s})] \quad (M \text{ is ex-ante WBB}) \\
&= \mathbb{E}_{\mathbf{b},\mathbf{s}}[\sum_i x_i^B(\mathbf{b},\mathbf{s}) \cdot b_i - \sum_j x_j^S(\mathbf{b},\mathbf{s}) \cdot \tau_j(s_j)] - \sum_i \eta_i \quad (\text{Lemma 5.1}) \\
&\le \mathbb{E}_{\mathbf{b},\mathbf{s}}[\sum_i x_i^B(\mathbf{b},\mathbf{s}) \cdot b_i - \sum_j x_j^S(\mathbf{b},\mathbf{s}) \cdot \tilde{\tau}_j(s_j)] - \sum_i \eta_i \quad (\text{Lemma 5.2}) \\
&= \mathbb{E}_{\mathbf{b},\mathbf{s}}[\sum_{i,j} x_{ij}(\mathbf{b},\mathbf{s}) \cdot (b_i - \tilde{\tau}_j(s_j))] - \sum_i \eta_i \\
&\le \mathbb{E}_{\mathbf{b},\mathbf{s}}\left[\sum_{(i,j)\in A^2(\mathbf{b},\mathbf{s})} (b_i - \tilde{\tau}_j(s_j))\right]
\end{aligned}
$$

□

5.3.1 Generalized BO and SO

We propose two mechanisms and show that the better one has GFT at least $\frac{1}{2}\text{OPT}$. The allocation and payment rule are defined as follows:

Definition 5.3. *We consider the following two mechanisms which are the generalizations of SO and BO in double auctions.*

- ***Generalized Seller-Offering Mechanism (GSO)****: Given type profile $\boldsymbol{b}, \boldsymbol{s}$, buyer i trades with seller j if and only if $(i, j) \in A^1(\boldsymbol{b}, \boldsymbol{s})$ (see Definition 5.2). According to Lemma 5.7, this allocation rule is monotone, and the buyer (or the seller) pays (or receives) the* threshold payment*. See below for more details about the payment rule.*

- ***Generalized Buyer-Offering Mechanism (GBO)****: Given type profile $\boldsymbol{b}, \boldsymbol{s}$, buyer i trades with seller j if and only if $(i, j) \in A^2(\boldsymbol{b}, \boldsymbol{s})$ (see Definition 5.2). According to Lemma 5.7, this allocation rule is monotone, and the buyer (or the seller) pays (or receives) the* threshold payment*.*

Lemma 5.7. *Suppose $w_{ij}(b_i, s_j)$ is non-decreasing in b_i and non-increasing in s_j for every buyer i and seller j. For any type profile $\boldsymbol{b}, \boldsymbol{s}$, if edge (i, j) is in the maximum weight matching $\mathcal{M}$, this edge is in the maximum weight matching under type profile $(b_i', b_{-i}, \boldsymbol{s})$ (or $(\boldsymbol{b}, s_j', s_{-j})$) for any $b_i' > b_i$ (or any $s_j' < s_j$).*

Proof. We prove that for any $b_i' > b_i$, $\mathcal{M}$ is still a maximum weight matching under type profile $(b_i', b_{-i}, \mathbf{s})$. For convenience, we use $w_{ij}(\mathbf{b}, \mathbf{s})$ to represent the weight of edge (i, j) under type profile $(\mathbf{b}, \mathbf{s})$. $w_{ij}(\mathbf{b}, \mathbf{s}) = w_{ij}(b_i, s_j)$. For every matching $\mathcal{M}' \in \mathcal{F}$, notice that $w_{i'j'}(\mathbf{b}, \mathbf{s}) = w_{i'j'}(b_i', b_{-i}, \mathbf{s})$ for all $i' \neq i$ and all $j' \in [m]$. Hence,

$$\begin{aligned}
\sum_{(i',j') \in \mathcal{M}} w_{i'j'}(b_i', b_{-i}, \mathbf{s}) &= \sum_{(i',j') \in \mathcal{M}} w_{i'j'}(\mathbf{b}, \mathbf{s}) + \big(w_{ij}(b_i', s_j) - w_{ij}(b_i, s_j)\big) \\
&\geq \sum_{(i',j') \in \mathcal{M}'} w_{i'j'}(\mathbf{b}, \mathbf{s}) + (w_{ij}(b_i', s_j) - w_{ij}(b_i, s_j)) \quad \text{(Optimality of } \mathcal{M}) \\
&\geq \sum_{(i',j') \in \mathcal{M}'} w_{i'j'}(b_i', b_{-i}, s)
\end{aligned} \tag{5.14}$$

The last inequality is an equality if $(i, j) \in \mathcal{M}'$. If $(i, j) \notin \mathcal{M}'$, the inequality is because $w_{ij}(b_i', s_j) - w_{ij}(b_i, s_j) \geq 0$ and $\sum_{(i',j') \in \mathcal{M}'} w_{i',j'}(\mathbf{b}, \mathbf{s}) = \sum_{(i',j') \in \mathcal{M}'} w_{i'j'}(b_i', b_{-i}, \mathbf{s})$. Thus, $\mathcal{M}$ is a maximum weight matching under type profile $(b_i', b_{-i}, \mathbf{s})$. Similarly, we can show that $\mathcal{M}$ is a maximum weight matching under type profile $(\mathbf{b}, s_j, s_{j'})$ for all $s_j' < s_j$. $\square$

By Lemma 5.7, the allocation rules for GSO and GBO are both monotone due to the monotonicity of functions $\tilde{\varphi}_i(\cdot)$ and $\tilde{\tau}_j(\cdot)$ for all i, j. We use the *threshold payment*, that is, given any type profile $\mathbf{b}, \mathbf{s}$, for every buyer i, if $x_i^B(\mathbf{b}, \mathbf{s}) = 0$, $p_i^B(\mathbf{b}, \mathbf{s})$ is also 0, and if $x_i^B(\mathbf{b}, \mathbf{s}) = 1$, $p_i^B(\mathbf{b}, \mathbf{s})$ equals the smallest b_i' such that $x_i^B(b_i', b_{-i}, \mathbf{s}) = 1$. Similarly, for every seller j, if $x_j^S(\mathbf{b}, \mathbf{s}) = 0$, $p_j^S(\mathbf{b}, \mathbf{s})$ is also 0, and if $x_j^S(\mathbf{b}, \mathbf{s}) = 1$, $p_j^S(\mathbf{b}, \mathbf{s})$ equals to the largest s_j' such that $x_j^S(\mathbf{b}, s_j', s_{-j}) = 1$. As the allocation rule is monotone and the threshold payment is used, the mechanism is IR and DSIC for every agent.

5.3.2 GSO and GBO in Bilateral Trade

To get a better understanding of the two mechanisms, we will first compare GSO with SO in the bilateral trade setting (the buyer side is analogous). For GSO, the pair is selected in the optimal matching if and only if the weight $\tilde{\varphi}(b) - s \geq 0$. Notice that this is exactly same as the allocation rule used in SO. Thus the two mechanisms have the same GFT, yet the different payment rule. Different from SO which is SBB simply by the definition of the mechanism, it's not straightforward that GSO satisfies any budget balance criteria: By applying the threshold payment on both agents, any monotone allocation rule can be used to construct a DSIC and IR mechanism. However, Myerson and Satterthwaite's impossibility result [MS83] implies that not all allocation rule can induce a budget balanced mechanism. In Lemma 5.8, we show that with the specific allocation rule in Definition 5.3, GSO is indeed ex-ante SBB.

Lemma 5.8. *In the Bilateral Trade setting, GSO is an IR, DSIC and ex-ante SBB mechanism.*

Proof. We only need to prove that the mechanism is ex-ante SBB in bilateral trade. By the definition of the mechanism, if $x(b, s) = 1$, $p^B(b, s) = b'(s)$ where $b'(s) = \min\{b_0 : \tilde{\varphi}(b_0) \geq s\}$, $p^S(b, s) = \tilde{\varphi}(b)$. For any s,

$$\mathbb{E}_b[p^B(b, s)] = b'(s) \cdot \Pr_b[\tilde{\varphi}(b) \geq s]$$

$$\mathbb{E}_b[p^S(b, s)] = \mathbb{E}_b[\tilde{\varphi}(b) \cdot \mathbb{1}[\tilde{\varphi}(b) \geq s]]$$

According to the definition of $b'(s)$, $b'(s)$ does not lie in the interior of any ironed interval. Thus by Lemma 5.2,

$$\mathbb{E}_b[\tilde{\varphi}(b) \cdot \mathbb{1}[\tilde{\varphi}(b) \geq s]] = \mathbb{E}_b[\varphi(b) \cdot \mathbb{1}[\tilde{\varphi}(b) \geq s]] = \int_{b'(s)}^{\bar{b}} \varphi(b) f^B(b) db = b'(s) \cdot \Pr_b[\tilde{\varphi}(b) \geq s]$$

Thus we have $\mathbb{E}_b[p^B(b,s)] = \mathbb{E}_b[p^S(b,s)]$. □

Similarly, we can prove that GBO is ex-ante SBB. The trade happens if and only if the weight $b - \tilde{\tau}(s) \geq 0$ in GBO, which is exactly the same allocation rule with BO. Using a similar argument as in Lemma 5.8, we can prove that GBO is also ex-ante SBB.

Lemma 5.9. *In the Bilateral Trade setting, GBO is an IR, DSIC and ex-ante SBB mechanism.*

5.3.3 Finishing the Proof

In this section, we consider the double auction setting and give the proof of Theorem 5.3. We first need to prove that both GSO and GBO are ex-ante WBB. The idea is to consider each pair (i, j) separately and show that the expected payment of buyer i for trading with seller j is greater than the expected gains of seller j for trading with buyer i.

Lemma 5.10. *Both GSO and GBO are IR, DSIC and ex-ante WBB mechanisms.*

Proof. We will give the proof for GSO and a similar argument applies to GBO. Let (x^B, x^S, p^B, p^S) be the allocation and payment rule for GSO. For every i, j and type profile $\mathbf{b}, \mathbf{s}$, let $x_{ij}(\mathbf{b}, \mathbf{s}) = \mathbb{1}[(i,j) \in A^1(\mathbf{b}, \mathbf{s})]$ representing whether buyer i is trading with seller j. Clearly $x_i^B(\mathbf{b}, \mathbf{s}) = \sum_{j \in [m]} x_{ij}(\mathbf{b}, \mathbf{s})$, $x_j^S(\mathbf{b}, \mathbf{s}) = \sum_{i \in [n]} x_{ij}(\mathbf{b}, \mathbf{s})$. We notice that with threshold payments, $p_i^B(\mathbf{b}, \mathbf{s})$ (or $p_j^S(\mathbf{b}, \mathbf{s})$) is non-zero only if $x_i^B(\mathbf{b}, \mathbf{s})$ (or $x_j^S(\mathbf{b}, \mathbf{s})$) is 1. Then the difference between all buyers' expected payments and sellers' expected gains can be written as

$$\begin{aligned}\mathbb{E}_{\mathbf{b},\mathbf{s}}\Big[\sum_{i=1}^{n} x_i^B(\mathbf{b},\mathbf{s})p_i^B(\mathbf{b},\mathbf{s}) - \sum_{j=1}^{m} x_j^S(\mathbf{b},\mathbf{s})p_j^S(\mathbf{b},\mathbf{s})\Big] \\ =&\mathbb{E}_{\mathbf{b},\mathbf{s}}\Big[\sum_{(i,j)\in A^1(\mathbf{b},\mathbf{s})} x_{ij}(\mathbf{b},\mathbf{s})\cdot\left(p_i^B(\mathbf{b},\mathbf{s})-p_j^S(\mathbf{b},\mathbf{s})\right)\Big] \\ =&\mathbb{E}_{\mathbf{b},\mathbf{s}}\Big[\sum_{i,j} x_{ij}(\mathbf{b},\mathbf{s})\cdot\left(p_i^B(\mathbf{b},\mathbf{s})-p_j^S(\mathbf{b},\mathbf{s})\right)\Big] \\ =&\sum_{i=1}^{n}\sum_{j=1}^{m}\mathbb{E}_{\mathbf{b},\mathbf{s}}\left[x_{ij}(\mathbf{b},\mathbf{s})\cdot\left(p_i^B(\mathbf{b},\mathbf{s})-p_j^S(\mathbf{b},\mathbf{s})\right)\right]\end{aligned} \tag{5.15}$$

Now we fix i, j, b_{-i} and s_{-j}. Lemma 5.7 states that if a pair (i, j) is in the max weight matching, then increasing the value of b_i or decreasing the value of s_j will not remove this pair from the maximum weight matching. In other words, $x_{ij}(b_i, b_{-i}, s_j, s_{-j})$ is non-decreasing in b_i and non-increasing in s_j. Next, we characterize the threshold payments of i and j. For every $b_i \in T_i^B$, define $s'_j(b_i)$ to be the largest value of s_j such that $x_{ij}(b_i, b_{-i}, s_j, s_{-j}) = 1$. Notice that when $(i, j) \in A^1(\mathbf{b},\mathbf{s})$, $\tilde{\varphi}_i(b_i) - s_j$ must be non-negative, as the feasibility constraint $\mathcal{F}$ is downward-closed, so removing a pair with negative weight gives a strictly better matching. Thus, $s'_j(b_i) \le \tilde{\varphi}_i(b_i)$.

Similarly, for every $s_j \in T_j^S$, define $b'_i(s_j)$ to be the smallest value of b_i such that $x_{ij}(b_i, b_{-i}, s_j, s_{-j}) = 1$. By the definition of threshold payments, given b_i, s_j, if $x_{ij}(\mathbf{b},\mathbf{s}) = 1$, $p_j^S(\mathbf{b},\mathbf{s}) = s'_j(b_i)$. As for the buyer, $p_i^B(\mathbf{b},\mathbf{s}) = b'_i(s_j)$. The reason is that when $s_j \ge s'_j(b_i)$ (or $b_i \le b'_i(s_j)$) then $x_j^S(\mathbf{b},\mathbf{s})$ (or $x_i^B(\mathbf{b},\mathbf{s})$) must be 0. Imagine this is not the case, and j (or i) is in the maximum matching with some other buyer i' (or seller j') under profile $(\mathbf{b},\mathbf{s})$, then clearly if we decrease the value of s_j (or increase the value of b_i), (i', j) (or (i, j')) should remain in the maximum matching according to Lemma 5.7. Contradiction.

Now fix s_j. $x_{ij}(\mathbf{b},\mathbf{s}) = 1$ if and only if $b_i \ge b'_i(s_j)$. We have

$$\mathbb{E}_{b_i}\left[x_{ij}(\mathbf{b},\mathbf{s})p_i^B(\mathbf{b},\mathbf{s})\right] = b'_i(s_j)\cdot\Pr[b_i \ge b'_i(s_j)] \tag{5.16}$$

$$\mathbb{E}_{b_i}\left[x_{ij}(\mathbf{b},\mathbf{s})p_i^S(\mathbf{b},\mathbf{s})\right] = \mathbb{E}_{b_i}[s'_j(b_i)\cdot\mathbb{1}[b_i \ge b'_i(s_j)]] \le \mathbb{E}_{b_i}[\tilde{\varphi}_i(b_i)\cdot\mathbb{1}[b_i \ge b'_i(s_j)]] \tag{5.17}$$

Again notice that $b'_i(s_j)$ does not lie in the interior of any ironed interval for all s_j. So

$$\mathbb{E}_{b_i}[\tilde{\varphi}_i(b_i) \cdot \mathbb{1}[b_i \geq b'_i(s_j)]] = \mathbb{E}_{b_i}[\varphi_i(b_i) \cdot \mathbb{1}[b_i \geq b'_i(s_j)]] = b'_i(s_j) \cdot \Pr[b_i \geq b'_i(s_j)]$$

Hence,

$$\mathbb{E}_{b_i}\left[x_{ij}(\mathbf{b},\mathbf{s})p_i^B(\mathbf{b},\mathbf{s})\right] \geq \mathbb{E}_{b_i}\left[x_{ij}(\mathbf{b},\mathbf{s})p_i^S(\mathbf{b},\mathbf{s})\right].$$

Take expectation over s_j, b_{-i}, s_{-j}, and sum over all i, j:

$$\sum_{i=1}^{n}\sum_{j=1}^{m}\mathbb{E}_{\mathbf{b},\mathbf{s}}\left[x_{ij}(\mathbf{b},\mathbf{s})(p_i^B(\mathbf{b},\mathbf{s}) - p_j^S(\mathbf{b},\mathbf{s}))\right] \geq 0 \tag{5.18}$$

Hence GSO is ex-ante WBB. □

Now we are ready to prove Theorem 5.3.

Proof of Theorem 5.3: We use GFT_{GSO} (or GFT_{GBO}) to denote the expected GFT of GSO (or GBO). According to Lemma 5.10, both GSO and GBO are IR, BIC and ex-ante WBB, so we only need to prove that $\text{GFT}_{\text{GSO}} + \text{GFT}_{\text{GBO}} \geq \text{SB-GFT}$. By Lemma 5.6,

$$\leq \mathbb{E}_{\mathbf{b},\mathbf{s}}\left[\sum_{(i,j)\in A^1(\mathbf{b},\mathbf{s})}(\tilde{\varphi}_i(b_i) - s_j)\right] + \mathbb{E}_{\mathbf{b},\mathbf{s}}\left[\sum_{(i,j)\in A^2(\mathbf{b},\mathbf{s})}(b_i - \tilde{\tau}_j(s_j))\right] \tag{5.19}$$

For every i, j, fix $b_{-i}, \mathbf{s}$. As in Lemma 5.10, we will continue to use $b'_i(s_j)$ to denote the smallest value of b_i such that $x_{ij}(b_i, b_{-i}, s_j, s_{-j}) = 1$ in GSO, and use $x_{ij}(\mathbf{b},\mathbf{s}) = \mathbb{1}[(i,j) \in A^1(\mathbf{b},\mathbf{s})]$ to denote whether buyer i trades with seller j in GSO. Since $b'_i(s_j)$ does not lie in the interior of any ironed interval, we have

$$\begin{aligned}
&\mathbb{E}_{b_i}[(\tilde{\varphi}_i(b_i) - s_j) \cdot x_{ij}(\mathbf{b},\mathbf{s})] \\
=&\mathbb{E}_{b_i}[\tilde{\varphi}_i(b_i) \cdot \mathbb{1}[b_i \geq b'_i(s_j)]] - s_j \cdot \mathbb{E}_{b_i}[x_{ij}(\mathbf{b},\mathbf{s})] \\
=&\mathbb{E}_{b_i}[\varphi_i(b_i) \cdot \mathbb{1}[b_i \geq b'_i(s_j)]] - s_j \cdot \Pr[b_i \geq b'_i(s_j)] \quad \text{(Lemma 5.2)} \\
\leq&\mathbb{E}_{b_i}\left[(b_i - s_j) \cdot \mathbb{1}[b_i \geq b'_i(s_j)]\right] \qquad (\varphi_i(b_i) < b_i) \\
=&\mathbb{E}_{b_i}\left[(b_i - s_j) \cdot x_{ij}(\mathbf{b},\mathbf{s})\right]
\end{aligned} \tag{5.20}$$

Take expectation on $b_{-i}, \mathbf{s}$, and then sum up over all i, j:

$$\begin{aligned}\mathbb{E}_{\mathbf{b},\mathbf{s}}\left[\sum_{(i,j)\in A^1(\mathbf{b},\mathbf{s})}(\tilde{\varphi}_i(b_i)-s_j)\right] &= \sum_{i,j}\mathbb{E}_{\mathbf{b},\mathbf{s}}\left[(\tilde{\varphi}(b_i)-s_j)\cdot x_{ij}(\mathbf{b},\mathbf{s})\right] \\ &\leq \sum_{i,j}\mathbb{E}_{\mathbf{b},\mathbf{s}}\left[(b_i-s_j)\cdot x_{ij}(\mathbf{b},\mathbf{s})\right] = \text{GFT}_{\text{GSO}}\end{aligned} \tag{5.21}$$

Similarly, we have $\mathbb{E}_{\mathbf{b},\mathbf{s}}\left[\sum_{(i,j)\in A^2(\mathbf{b},\mathbf{s})}(b_i - \tilde{\tau}_j(s_j))\right] \leq \text{GFT}_{\text{GBO}}$. Combining this with Inequality (5.19), we have SB-GFT $\leq \text{GFT}_{\text{GSO}} + \text{GFT}_{\text{GBO}}$. □

5.4 Transformation between Mechanisms with Different Budget-Balanced Constraints

In this section, we argue how to transform an IR, BIC and ex-ante WBB mechanism to an IR, BIC and SBB mechanism without changing the allocation rule. Clearly, the GFT remains the same after the transformation. The result applies to general two-sided markets and we will use the notations from Section 2.4 throughout this section.

Theorem 5.4. *Given an IR, BIC, ex-ante WBB mechanism $M = (x^B, x^S, p^B, p^S)$ with non-negative payment rule, there exists another non-negative payment rule $(p^{B'}, p^{S'})$ such that the mechanism $M' = (x^B, x^S, p^{B'}, p^{S'})$ is IR, BIC and SBB.*

As a SBB mechanism is clearly ex-ante WBB, thus with Theorem 5.4 we will have an equivalence among the ex-ante WBB constraint and the SBB constraint for IR and BIC mechanisms. The intuition behind Theorem 5.4 is that if our mechanism has positive surplus, we can simply divide the surplus to the agents evenly and independently from their reported types (Lemma 5.11). Now we have an ex-ante SBB mechanism. Next, we massage the payments so that all interim payments remain unchanged, while under every type profile the sum of buyers' payments equal the sum of sellers' gains. This is achieved via an interesting linear transformation on the payments (Lemma 5.12).

Lemma 5.11. *Given an IR, BIC, ex-ante WBB mechanism $M = (x^B, x^S, p^B, p^S)$ with non-negative payment rule, there exists another non-negative payment rule $(p^{B'}, p^{S'})$ such that mechanism $M' = (x^B, x^S, p^{B'}, p^{S'})$ is IR, BIC and ex-ante SBB.*

Proof. Let $\delta = \mathbb{E}_{\mathbf{b},\mathbf{s}}\left[\sum_{i=1}^n p_i^B(\mathbf{b},\mathbf{s}) - \sum_{j=1}^m p_j^S(\mathbf{b},\mathbf{s})\right] \geq 0$. Define $(p^{B'}, p^{S'})$ as follows: for every $\mathbf{b},\mathbf{s}$, $p^{B'}(\mathbf{b},\mathbf{s}) = p^B(\mathbf{b},\mathbf{s}) \geq 0$, $p^{S'}(\mathbf{b},\mathbf{s}) = p^S(\mathbf{b},\mathbf{s}) + \frac{\delta}{m} \geq 0$. Then

$$\mathbb{E}_{\mathbf{b},\mathbf{s}}\left[\sum_{i=1}^n p_i^{B'}(\mathbf{b},\mathbf{s}) - \sum_{j=1}^m p_j^{S'}(\mathbf{b},\mathbf{s})\right] = \mathbb{E}_{\mathbf{b},\mathbf{s}}\left[\sum_{i=1}^n p_i^{B}(\mathbf{b},\mathbf{s}) - \sum_{j=1}^m p_j^{S}(\mathbf{b},\mathbf{s})\right] - m \cdot \frac{\delta}{m} = 0 \tag{5.22}$$

M' is ex-ante SBB. In mechanism M', it first gives $\frac{\delta}{m}$ to each seller and then follows the allocation rule and payment of mechanism M. Since each seller receives a fixed amount of money at the beginning of the mechanism, M' will still be IR and BIC. □

The next lemma provides a transformation for turning an IR, BIC, ex-ante SBB mechanism to a SBB mechanism without modifying the allocation rule.

Lemma 5.12. *Given an IR, BIC and ex-ante SBB mechanism $M = (x^B, x^S, p^B, p^S)$ with non-negative payment rule, there exists another non-negative payment rule $(p^{B'}, p^{S'})$ such that the mechanism $M' = (x^B, x^S, p^{B'}, p^{S'})$ is IR, BIC and SBB.*

Proof. We will construct $p^{B'}$ such that for every buyer i and her type b_i, the expected payment for buyer i to report type b_i in M' is the same as her payment in M. Formally,

$$\mathbb{E}_{b_{-i},\mathbf{s}}\left[p_i^{B'}(b_i, b_{-i}, \mathbf{s})\right] = \mathbb{E}_{b_{-i},\mathbf{s}}\left[p_i^{B}(b_i, b_{-i}, \mathbf{s})\right].$$

Similarly, for each seller j and her type s_j, the expected gains for seller j to report type s_j in M' is the same as her gains in M, that is,

$$\mathbb{E}_{\mathbf{b},s_{-j}}\left[p_j^{S'}(\mathbf{b}, s_j, s_{-j})\right] = \mathbb{E}_{\mathbf{b},s_{-j}}\left[p_j^{S}(\mathbf{b}, s_j, s_{-j})\right].$$

This property guarantees that the expected utility for buyer i (or seller j) when reporting type b_i (or type s_j) stays unchanged. Since M is BIC and IR, M' is also BIC and IR.

Suppose we are given an IR, BIC and ex-ante SBB mechanism $M = (x^B, x^S, p^B, p^S)$. Define the payment rule $(p^{B'}, p^{S'})$ as follows. Let $\Omega^B = \left\{i \in [n] : \mathbb{E}_{\mathbf{b}',\mathbf{s}'}[p_i^B(\mathbf{b}', \mathbf{s}')] > 0\right\}$ and $\Omega_S = \{j \in [m] : \mathbb{E}_{\mathbf{b}',\mathbf{s}'}[p_j^S(\mathbf{b}', \mathbf{s}')] > 0\}$.

For $i \notin \Omega^B$, since all payments $p_i^B(\mathbf{b}', \mathbf{s}')$ are non-negative, we must have $p_i^B(\mathbf{b}', \mathbf{s}') = 0$ for all $\mathbf{b}', \mathbf{s}'$. We define $p_i^{B'}(\mathbf{b}, \mathbf{s}) = 0$ for all $i \notin \Omega^B$ and every type profile $(\mathbf{b}, \mathbf{s})$. Similarly, for all $j \notin \Omega^S$, let $p_j^{S'}(\mathbf{b}, \mathbf{s}) = 0$ for every type profile $(\mathbf{b}, \mathbf{s})$. For simplicity, we will slightly abuse the notation, using $p_i^B(b_i)$ (or $p_j^S(s_j)$) to denote $\mathbb{E}_{b_{-i},\mathbf{s}}\left[p_i^B(b_i, b_{-i}, \mathbf{s})\right]$ (or $\mathbb{E}_{s_{-j},\mathbf{b}}\left[p_j^S(\mathbf{b}, s_j, s_{-j})\right]$), and p_i^B (or p_j^S) to denote $\mathbb{E}_{\mathbf{b}',\mathbf{s}'}[p_i^B(\mathbf{b}', \mathbf{s}')]$ (or $\mathbb{E}_{\mathbf{b}',\mathbf{s}'}[p_j^S(\mathbf{b}', \mathbf{s}')]$).

For $i \in \Omega^B$ and $j \in \Omega^S$, define

$$p_i^{B'}(\mathbf{b}, \mathbf{s}) = \prod_{i' \in \Omega^B} \frac{p_{i'}^B(b_{i'})}{p_{i'}^B} \cdot \prod_{j' \in \Omega^S} \frac{p_{j'}^S(s_{j'})}{p_{j'}^S} \cdot p_i^B \tag{5.23}$$

$$p_j^{S'}(\mathbf{b}, \mathbf{s}) = \prod_{i' \in \Omega^B} \frac{p_{i'}^B(b_{i'})}{p_{i'}^B} \cdot \prod_{j' \in \Omega^S} \frac{p_{j'}^S(s_{j'})}{p_{j'}^S} \cdot p_j^S \tag{5.24}$$

For every $\mathbf{b}, \mathbf{s}$, since M is ex-ante SBB, $\sum_{i \in \Omega^B} p_i^B = \sum_{j \in \Omega^S} p_j^S$, which implies that $\sum_{i \in \Omega^B} p_i^{B'}(\mathbf{b}, \mathbf{s}) = \sum_{j \in \Omega^S} p_j^{S'}(\mathbf{b}, \mathbf{s})$. Since the payments of buyers (or sellers) that are not in Ω_B (or Ω_S) are 0, Mechanism M' is SBB. Moreover, for every $i \in \Omega^B$ and type b_i, if we take expectation of $p_i^{B'}(\mathbf{b}, \mathbf{s})$ over all $b_{-i}, \mathbf{s}$, we have

$$\begin{aligned}\mathbb{E}_{b_{-i},\mathbf{s}}[p_i^{B'}(\mathbf{b}, \mathbf{s})] &= \frac{p_i^B(b_i) \cdot \prod\limits_{i' \neq i, i' \in \Omega^B} \mathbb{E}_{b_{i'} \sim D_{i'}^B}[p_{i'}^B(b_{i'})] \cdot \prod\limits_{j' \in \Omega^S} \mathbb{E}_{s_{j'} \sim D_{j'}^S}[p_{j'}^S(s_{j'})] \cdot p_i^B}{\prod\limits_{i' \in \Omega^B} p_{i'}^B \cdot \prod\limits_{j' \in \Omega^S} p_{j'}^S} \\ &= p_i^B(b_i)\end{aligned} \tag{5.25}$$

If $i \notin \Omega^B$, $\mathbb{E}_{b_{-i},\mathbf{s}}[p_i^{B'}(\mathbf{b}, \mathbf{s})] = 0 = p_i^B(b_i)$. Similarly, for every seller j and any of her type s_j, $\mathbb{E}_{\mathbf{b},s_{-j}}[p_j^{S'}(\mathbf{b}, \mathbf{s})] = p_j^S(s_j)$. Thus, M' is an IR, BIC and SBB mechanism. □

Proof of Theorem 5.4: It directly follows from Lemma 5.11 and Lemma 5.12. □

With Lemma 5.11 and Lemma 5.12, one can show that under IR and BIC constraints, mecha-

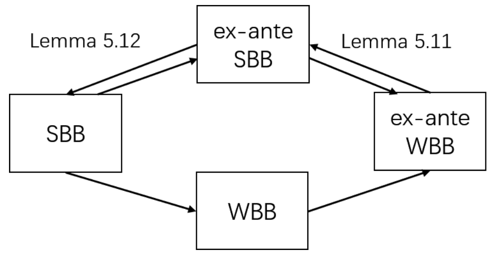


Figure 5.1: Transformation between Mechanisms with Different BB Constraints

nisms with all variants of the Budget-Balance constraint can be transformed to one another, without changing the allocation rule (and thus not affecting the GFT).

Figure 5.1 describes the transformation between mechanisms with different budget balance constraints. In the figure, all the simple arrows are directed from a stronger constraint to a weaker one.

Chapter 6

Approximating GFT with Asymptotically Efficient Mechanisms

The mechanism proposed in Chapter 5 achieves a constant approximation to the second-best GFT in bilateral trade and double auction with arbitrary downward-closed trading constraint. However, one caveat of this mechanism is that its expected GFT does not asymptotically converge to the optimal GFT as the market grows large (see Example 6.1). In this chapter, we aim to design mechanisms that are asymptotically efficient, and also achieve a constant factor approximation to the second-best GFT in expectation.

In Section 6.1, we give an introduction of the results and techniques covered in this chapter. In Section 6.2 and Section 6.3 we introduce our setting and go through mechanisms in the literature that is useful in this chapter. In Section 6.4 we present our results for double auctions. In Section 6.5 and Section 6.6, we present our results for a more general setting matching market. In Section 6.7 we give some discussion about our results in this chapter.

6.1 Results in This Chapter

In light of the seminal impossibility result of Myerson–Satterthwaite [MS83], follow-up work in the two-sided market literature has looked at IR, BIC, and BB mechanisms that are approximately efficient, rather than precisely efficient. The current state-of-the-art mechanisms in the literature can be categorized as giving one of two guarantees:

1. A constant ex-ante guarantee, measured with respect to the second-best GFT, that is, the (possibly very complex) mechanism obtaining the highest expected GFT of any IR and BIC mechanism that is weakly budget balanced, or

2. An asymptotically optimal ex-post guarantee, measured with respect to the first-best GFT, that is, the mechanism obtaining full efficiency (VCG).

In this chapter, we aim to construct simple mechanisms that simultaneously achieve both guarantees. We study settings in which each seller is endowed with precisely one item, all items are identical, and each buyer is interested in buying one item. In the *double-auction* setting, any seller can trade with any buyer, while in the more general *matching market* setting, trade between some buyer-seller pairs is disallowed. Before describing our results, we first survey the state-of-the-art mechanisms giving each guarantee in more depth.

Ex-Ante Guarantees [BCWZ17] (henceforth BCWZ, presented in Chapter 5) present a simple mechanism that is IR, BIC, and weakly BB, and obtains, in expectation, at least half of the expected GFT of the second-best GFT, in any double auction with arbitrary downward-closed trading constraint.

While the mechanism obtains at least half of the second-best GFT in expectation, we observe that it does not give any ex-post efficiency guarantees, and moreover, even its expected GFT does *not* asymptotically converge to the GFT of the second-best (let alone the first-best) mechanism as the market grows large. This holds even for the very simple double-auction market with n sellers, each selling an identical item, and n buyers, each interested in buying a single item, with the values (or costs) of the agents sampled i.i.d. from the uniform distribution over $[0, 1]$. Even when n is large, the mechanism of BCWZ will only give in expectation a constant fraction (strictly smaller than 1) of the second-best GFT, and no more than that (see Example 6.1 in Section 6.4). In particular, even in a large market, the efficiency of their mechanism does not converge to full efficiency.

Ex-Post Guarantees The Trade Reduction mechanism[1] of McAfee [McA92], which is defined for the double-auction setting, does not suffer from the above drawback and is asymptotically efficient. The mechanism circumvents the impossibility result [MS83] for bilateral trade, by providing an ex-post efficiency guarantee only when more than one trade is possible in the double-auction market. The mechanism works as follows: it first finds the efficient trade — the allocation that maximizes the GFT for the particular input profile. Denote the size (number of pairs) of this trade by q. It then removes the least efficient trade (one buyer-seller pair), and only allows for the remaining trades (the $q-1$ most efficient trades) to realize, charging the winning buyers the value of the removed buyer, and paying the winning sellers the cost of the removed seller. This creates an IR and ex-post IC mechanism. As the value of the removed buyer is at least the cost of the removed seller, each trade is weakly budget-balanced. The mechanism obtains at least a $1-1/q$ fraction[2] of the realized optimal (first-best) GFT. In the double-auction example above, as n grows q will also grow, and this fraction will tend to 1. Unfortunately, when $q=1$ this mechanism performs no trade and provides no guarantees at all. (Failing to provide an ex-post guarantee unconditionally is of course inevitable in light of the impossibility result of Myerson–Satterthwaite [MS83].[3]) We note that the Trade Reduction mechanism, while asymptotically efficient, fails to give any unconditional approximation to the GFT, even with respect to the GFT of the second-best mechanism (as the mechanism of BCWZ does give).

The Best of Both Worlds In this work we aim to design simple mechanisms that are IR, BIC, and weakly BB, and simultaneously provide both types of efficiency guarantees discussed above. First, in the spirit of the guarantee of BCWZ, we aim to guarantee for the expected GFT to be at least a constant fraction of the expected GFT of the second-best mechanism. Second, in the spirit of the guarantee of [McA92], we aim to guarantee for the ex-post GFT to be at least a realization-

1. McAfee's original mechanism is slightly more involved. We use a simplified version that provides the same worst-case guarantees.

2. Recall that q is a function of the valuation profile.

3. Ex-post approximation to the GFT requires the mechanism to trade whenever there is positive gain, but the impossibility result implies that for some of these profiles trade will not occur.

dependent fraction of the realized optimal GFT (first-best), such that this fraction tends to 1 "as the market grows large" and the efficient trade size grows[4] to infinity.

6.1.1 Our Results

We present results both for the double-auction setting and for the more involved matching-market setting.Providing a result for this more involved scenario is considerably more challenging than for the double-auction setting, and is the main result of this paper.

Double Auctions

We first present our result for double auctions.

Theorem 6.1. *For the double-auction setting, there exists a simple mechanism that is ex-post IR, BIC and ex-post weakly budget balanced, and satisfies both of the following.*

- *The expected GFT of this mechanism is at least* $1/4$ *of the expected GFT of the second-best mechanism.*
- *This mechanism guarantees at least* $1-1/q$ *of the realized optimal (first-best) GFT, where* q *is the size of the most efficient trade. Thus the mechanisms is asymptotically efficient (converges to full efficiency as the trade size* q *grows large).*

Note that the asymptotic efficiency that is obtained is with respect to the most demanding benchmark of the realized optimal GFT (the first-best and not only the second-best), providing the same guarantee as the one provided by the Trade Reduction mechanism [McA92]. The concurrent ex-ante guarantee is with respect to the second-best, similarly to the result of BCWZ.

Before examining the problem thoroughly, one might be tempted to think that it is trivial to come up with such a mechanism for double auctions. Here is a natural naïve candidate for such a mechanism: first, the mechanism computes the efficient trade size q. If $q > 1$, it runs McAfee's

4. The condition on the efficient trade size ensures that the growth in the market size does not, for example, come from adding agents that are "irrelevant," such as buyers with 0 value and sellers with very high costs, since in such a case it would not be possible to provide any guarantee that is better for large markets than for small ones (such as bilateral trade markets).

Trade Reduction mechanism. Otherwise, it runs the mechanism of BCWZ. This naïve approach turns out to fail miserably as the allocation is not even monotone: it may well be the case that the two agents that trade when $q = 1$ (i.e., those that trade according to the mechanism of BCWZ) are not the highest-value buyer and the lowest-cost seller, and so in certain scenarios an agent that is reduced in the $q > 1$ case (by the Trade Reduction mechanism) may be able to reduce her bid to move to the $q = 1$ case and trade (for more details see Section 6.4).

To present our mechanism, let us first very roughly review the behavior of the mechanism of BCWZ in the bilateral-trade case: in this special case, the mechanism flips a coin; with probability 50%, the seller offers a take-it-or-leave-it price to the buyer (calculated so as to maximize the expected utility of the seller), and with probability 50%, the buyer offers a take-it-or-leave-it price to the seller (calculated so as to maximize the expected utility of the buyer). In order to obtain the mechanism that we seek, we carefully make two main modifications to the naïve "compound" mechanism described above: first, in order to address the above-discussed source of non-monotonicity, instead of running the mechanism of BCWZ on the entire market, we run their bilateral-trade mechanism only on the (unique) pair in the efficient trade. To make the resulting mechanism truthful, we need to make an additional adjustment: in the seller-offering case (the adjustment to the buyer-offering case is analogous), we on the one hand force the seller to set a price that is at most the threshold bid that puts her in the efficient (first-best) trade, and on the other hand notify her of the values of all buyers except the one that she is facing, and calculate the price that she offers to maximize her expected utility conditioned upon the fact that the buyer that she is facing has value larger than all of these values. Both adjustments, and in particular the first one, make the proof of the ex-ante guarantee, as well as the proof that the mechanism is BIC, quite subtle.

The main challenge in obtaining the approximation guarantee for the case where $q = 1$ is to reconcile the fact that the pair that our mechanism attempts to trade on is determined by maximizing the realized GFT (first-best) and might not be the same as the pair that would have traded according to the mechanism of BCWZ. The main hurdle to obtaining the approximation guarantee for this case is therefore that for some valuation profiles, an offer between the unique pair in the efficient trade will be rejected, resulting in no trade in our hybrid mechanism, while in the mechanism of BCWZ an offer will be made — and accepted — between a different pair. To overcome this, we have to carefully charge such losses in GFT to gains in GFT by other parts of our hybrid mechanism.

Matching Markets

As stated above, the mechanism of BCWZ does not converge to the efficient outcome in large double-auction markets, and thus will clearly not do so in the more general matching market setting. Our goal is to present a mechanism for matching market that is IR, BIC and ex-post weakly BB, but also provide ex-ante guarantees for the GFT as well as ex-post guarantees that converges to full efficiency "as the market grows large". While in the double auction setting, every buyer can trade with every seller, this is no longer the case in a matching market. Our notion of a large matching market aims to generalize the fact that in a large market there are many agents that are "equivalent" up to their values. The sense of agents being equivalent in a matching market is that they can trade with exactly the same set of agents. So, we can naturally partition agents to equivalence classes, with every two agents of the same class being interchangeable in any matching (up to their valuations). We consider matching markets with a fixed set of such classes, and think about a large market as a market in which the number of agents of each class is growing large, yet the number of different classes that any agent can trade with remains bounded by some constant d.

Recall that the Trade Reduction mechanism [McA92] is defined for a double-auction setting. We first present a generalization for matching markets of the Trade Reduction mechanism (section 6.5.1) and prove that it is ex-post asymptotically efficient "as the market grows large" in the above sense. To our knowledge, this nontrivial generalization of the Trade Reduction mechanism, which may also be of separate interest, is novel. Similarly to the Trade Reduction mechanism [McA92] for double-auction settings, this mechanism does not give any ex-ante approximation guarantee.

As with the double-auction case, we cannot directly combine our Trade Reduction mechanism for matching markets with the mechanism of BCWZ while maintaining truthfulness. Therefore, we present a novel mechanism (section 6.5.2), which we call the *Offering Mechanism for Matching Markets*. Like the mechanism of BCWZ, this mechanism does not provide the ex-post guarantee we are after, but we manage to carefully define it in a way that allows us to combine it with the Trade Reduction mechanism for matching markets to obtain a truthful mechanism that provides both types of guarantees that we are after. The precise definition of the Offering Mechanism that allows for both the truthfulness and the efficiency guarantees of the hybrid mechanism has been quite elusive to pin down, and the proofs of truthfulness, and in particular of the ex-ante guarantee, are considerably more subtle than in the double-auction setting. To prove the ex-ante guarantee of the Offering Mechanism, we compare it to the mechanism of BCWZ, showing that it attains at least half of the GFT of their mechanism, resulting in an ex-ante guarantee of at least $1/4$ of the expected GFT of the second best mechanism. Proving the ex-ante guarantee of the Offering Mechanism is the most technically challenging part of our analysis. To prove this guarantee, we show that it is possible to carefully "charge" every edge of the matching of BCWZ to edges of the first-best matching that will be traded in our Offering Mechanism, proving that the expected GFT of our Offering Mechanism is at least half the expected GFT of the mechanism of BCWZ. The combination of the Offering Mechanism for matching markets with the Trade Reduction mechanism for matching markets creates the *Hybrid Mechanism for Matching Markets* (section 6.5.3), giving us our main result.

Theorem 6.2. *For the matching market setting, there exists a simple mechanism that is ex-post IR, BIC and ex-post weakly budget balanced, and satisfies both of the following.*

- *The expected GFT of this mechanism is at least $1/4$ of the expected GFT of the second-best mechanism.*
- *When $1 - d/q \geq 1/2$, this mechanism guarantees at least $1 - d/q$ of the realized optimal (first-best) GFT where d denotes the maximum number of classes that any agent can trade with, and q denotes the minimal positive number of trading agents of the same class in the welfare maximizing outcome. Thus the mechanism is asymptotically efficient in the sense that it converges to full efficiency as the number of trading agents in every class grows large, as long*

as d is fixed.

We remark that while our mechanism ex-ante guarantees a quantitatively smaller fraction of the second-best GFT than the 1/2 fraction guaranteed by the mechanism of BCWZ in Chapter 5, the mechanism presented in this chapter has two qualitative advantages over the other mechanism: first we obtain asymptotic efficiency and an ex-post guarantee, and moreover, our mechanism is *ex-post* weakly BB, while the mechanism in Chapter 5 is only *ex-ante* weakly BB.

6.2 Definitions and Notations

6.2.1 Model and Definitions

Agents and Utilities We focus on double auctions in this chapter. To remind the readers, in a double-auction market for identical goods, there is a finite set S of sellers with one good each, and a finite set B of unit-demand buyers, with $|S| \geq 2$ and $|B| \geq 2$. Each seller $j \in S$ has a cost $s_j > 0$ that she incurs if she sells her item, and each buyer $i \in B$ has a value $b_i > 0$ that she derives if she purchases an item. We assume that an agent who does not trade does not incur any cost or derive utility from this. Let $\mathbf{s}$ be the vector of sellers' costs and $\mathbf{b}$ be the vector of buyers' values. The costs and values are sampled from agent-specific (but not necessarily identical) distributions D_i^B for each buyer $i \in B$ and D_j^S for each seller $j \in S$, each independent of all other distributions. Agents have quasi-linear utilities and are risk neutral.

Trading Constraints In this chapter, we consider a special class of downward-closed trading constraint where the set of trading buyer-seller pairs must form a matching in the bipartite graph between buyers and sellers. Such a special case is called *matching market*. In a matching market setting, an undirected bipartite graph $G = (S, B, E)$ with the sellers on one side and the buyers on the other constraints transactions. A set of *trading agents* K is a set of buyers and of sellers that can be partitioned into pairs, each of one buyer and one seller that are neighbors in G (this is equivalent to a matching of the set K in G) — the set K corresponds to each seller selling her item, and each buyer buying one of the items sold from one of its neighbors in G. The *size of trade* of K is defined to be $|K \cap S| = |K \cap B|$.

Gains from Trade The *gains from trade (GFT)* when the set K (of trading agents) is trading is defined to be $\sum_{i \in K \cap B} b_i - \sum_{j \in K \cap S} s_j$. Given a valuation profile $(\mathbf{b}, \mathbf{s})$, a set of trading agents is *efficient* if it maximizes the gains from trade from $(\mathbf{b}, \mathbf{s})$ among all sets of trading agents.

Benchmarks Given a valuation profile $(\mathbf{b}, \mathbf{s})$, let $M(\mathbf{b}, \mathbf{s})$ be the *first-best* matching, or the maximum-weight matching in G, where ties between agents are broken by the "lexicographic order by IDs" formally defined in Definition C.1 in appendix C.5.2.[5] Slightly abusing notation, we use $M(\mathbf{b}, \mathbf{s})$ to also denote the set of agents in the matching $M(\mathbf{b}, \mathbf{s})$. Let FB-GFT$(\mathbf{b}, \mathbf{s})$ be the GFT of the "first-best" $M(\mathbf{b}, \mathbf{s})$, that is FB-GFT$(\mathbf{b}, \mathbf{s}) = \sum_{(i,j) \in M(\mathbf{b}, \mathbf{s})} (b_i - s_j)$. Note that the VCG mechanism (which is not budget balanced) attains a GFT of FB-GFT$(\mathbf{b}, \mathbf{s})$ on every valuation profile $(\mathbf{b}, \mathbf{s})$.

A mechanism is called *second-best* if it maximizes the expected gains from trade among all BIC, interim IR and ex-ante weakly budget balanced mechanisms.

Special Cases The case where G is the *complete* bipartite graph (i.e., any seller can trade with any buyer) is called the *double-auction* setting. In the double-auction setting, for every valuation profile $(\mathbf{b}, \mathbf{s})$ we denote the size of the efficient set of trading agents by $q(\mathbf{b}, \mathbf{s})$. The case where $|S| = |B| = 1$ and the buyer and the seller are connected by an edge in G (so this is also a special case of double-auction) is called the *bilateral-trade* setting.

6.2.2 The Trade Reduction Mechanism

In the double-auction setting, the *Trade Reduction (TR) mechanism* [McA92] is a mechanism that finds the most efficient trade of only $q(\mathbf{b}, \mathbf{s}) - 1$ items,[6] and charges each agent his critical value for winning. That is, the $q(\mathbf{b}, \mathbf{s}) - 1$ highest-value buyers trade and pay the bid of the reduced buyer

5. This tie breaking rule satisfies two properties we use extensively: 1) it does not depend on weights, and 2) it is *subset consistent* in the sense that when removing an edge (i, j) from some matching M and picking a matching on the remaining nodes $M \setminus \{i, j\}$, it will pick the matching of M on these nodes.

6. If $q(\mathbf{b}, \mathbf{s}) = 0$ there is no trade in the TR mechanism, and no payments are made.

(the $q(\mathbf{b}, \mathbf{s})$-highest buyer); they trade with the $q(\mathbf{b}, \mathbf{s}) - 1$ lowest-cost sellers, each seller getting paid the cost of the reduced seller (the $q(\mathbf{b}, \mathbf{s})$-lowest seller).

Theorem 6.3 ([McA92]). *In the double-auction setting, the TR mechanism is ex-post IC, ex-post IR, and ex-post (direct trade) weakly budget balanced. For every valuation profile* $(\boldsymbol{b}, \boldsymbol{s})$*, the gains from trade of this mechanism are at least an* $1 - \frac{1}{q(\boldsymbol{b},\boldsymbol{s})}$ *fraction of* $\text{FB-GFT}(\boldsymbol{b}, \boldsymbol{s})$.

Note that if $q(\mathbf{b}, \mathbf{s}) = 1$, then no ex-ante approximation to the GFT is achieved by the TR mechanism, while for $q(\mathbf{b}, \mathbf{s}) \geq 2$, Theorem 6.3 guarantees at least half the efficient GFT for $(\mathbf{b}, \mathbf{s})$, ex-post.

6.2.3 The Random Virtual-Welfare Maximizing Mechanism

Here we recap the mechanism in Chapter 5 (see Definition 5.3) for double auctions with downward-closed constraints (which subsume matching constraints), which we will refer to throughout this chapter as the *Random Virtual-Welfare Maximizing (RVWM)* mechanism.

Observation 6.1. *Let* $(\boldsymbol{b}, \boldsymbol{s})$ *be a valuation profile. If trade occurs with some positive probability on a given edge* (i, j) *in the RVWM mechanism, then trade would occur on the same edge with at least the same probability in the mechanism that runs one of the following, with probability* 50% *each:*

- *Seller* j *offers a price to buyer* i *that maximizes the utility of seller* j *in expectation over the distribution* D_i^B *from which buyer* i*'s valuation was drawn, and trade occurs if and only if this price is at most buyer* i*'s valuation* b_i*.*
- *Buyer* i *offers a price to seller* j *that maximizes the utility of buyer* i *in expectation over the distribution* D_j^S *from which seller* j*'s valuation was drawn, and trade occurs if and only if this price is at least seller* j*'s cost* s_j*.*

Theorem 6.4. *(Restatement of Theorem 5.3 and Lemma 5.10) The RVWM mechanism is BIC, ex-post IR, and ex-ante weakly budget balanced, and in expectation gets a* $1/2$*-fraction of the gains from trade of the second-best mechanism.*

Note that while the ex-ante guarantee of BCWZ is with respect to the *second-best* mechanism, the ex-post guarantee of McAfee [McA92] is with respect to the more demanding benchmark of the realized optimal *first-best* gains from trade.

6.3 The Seller-Offering, Buyer-Offering, and Randomized-Offerer Mechanisms

Before we turn to our main results, in this section we present a slightly modified version of the bilateral-trade construction of the SO and BO mechanism shown in Section 5.2, which we will use as a building block in the construction of our hybrid mechanisms, and prove several properties thereof.

Definition 6.1 (SO, BO, RO Mechanisms)**.** *Fix D_s and D_b to be nonnegative-valued distributions, and fix $\bar{s} \geq \sup \operatorname{Support} D_s$ and $\bar{b} \leq \inf \operatorname{Support} D_b$ s.t. $\bar{s} \geq \bar{b}$. We define three mechanisms for trade between a seller with cost $s \sim D_s$ and a buyer with value $b \sim D_b$.*

- *The* Seller-Offering (SO) mechanism with offer constraint $\bar{s}$ and target distribution D_b *is the mechanism in which a seller with cost s offers to the buyer the lowest price p among the prices that maximize the utility of the seller in expectation over $b \sim D_b$, under the constraint $p \leq \bar{s}$. That is, the offered price is $\min\bigl\{p \bigm| p \in \arg\max_{p \leq \bar{s}}(p - s) \cdot \bigl(1 - D_b(p)\bigr)\bigr\}$. The buyer accepts this price if it is no greater than the realized value b of the buyer. If the buyer accepts this price, then trade occurs at this price; otherwise, no trade occurs.*

- *The* Buyer-Offering (BO) mechanism with offer constraint $\bar{b}$ and target distribution D_s *is the mechanism in which a buyer with value b offers to the seller the highest price p among the prices that maximize the utility of the buyer in expectation over $s \sim D_s$, under the constraint $p \geq \bar{b}$. That is, the offered price is $\max\bigl\{p \bigm| p \in \arg\max_{p \geq \bar{b}}(b - p) \cdot \bigl(1 - D_s(p)\bigr)\bigr\}$. The seller accepts this price if it is no less than the realized cost s of the seller. If the seller accepts this price, then trade occurs at this price; otherwise, no trade occurs.*

- *The (Bilateral)* Randomized Offerer (RO) mechanism with SO parameters $\bar{s}$ and D_b and BO parameters $\bar{b}$ and D_s *is the mechanism that flips a coin, with probability* $1/2$ *it runs the SO mechanism with offer constraint* $\bar{s}$ *and target distribution* D_b*, and otherwise it runs the BO mechanism with offer constraint* $\bar{b}$ *and target distribution* D_s*.*

We slightly strengthen the special case of the incentive and budget guarantees of Theorem 6.4 for bilateral trade, and prove that they still hold even with offer constraints as in the RO mechanism.[7] We furthermore show that whenever trade occurs, the trading happens at a price that indeed lies between the seller's and the buyer's constraint.

Lemma 6.1. *Fix* D_s *and* D_b *to be nonnegative-valued distributions and fix* $\bar{s} \geq \sup \operatorname{Support} D_s$ *and* $\bar{b} \leq \inf \operatorname{Support} D_b$ *s.t.* $\bar{s} \geq \bar{b} \geq 0$*. Consider the RO mechanism with SO parameters* $\bar{s}$ *and* D_b *and BO parameters* $\bar{b}$ *and* D_s*.*

1. *When valuations are drawn from* $D_s \times D_b$*, this mechanism is a BIC, ex-post IR, and ex-post (direct trade) strongly budget balanced mechanism.*
2. *Whenever trade occurs in this mechanism, it holds that the price* p *that the seller pays the buyer satisfies* $\bar{b} \leq p \leq \bar{s}$*.*

The proof of lemma 6.1 is given in appendix C.2. To conclude this section, we will prove two more properties of the RO mechanisms that will allow us to lower-bound its GFT guarantee: the first will allow us to compare its GFT to that of the first-best, and the second will allow us to compare its GFT to that of the RVWM mechanism.

Lemma 6.2. *Let* D_s *and* D_b *be distributions and let* $\bar{s} \geq \bar{b} \geq 0$*. Let* $s \leq \bar{s}$ *be a cost for the seller and let* $b \geq \bar{b}$ *be a value for the buyer. Consider the RO mechanism with SO parameters* $\bar{s}$ *and* $D_b|_{\geq \bar{b}}$ *and BO parameters* $\bar{b}$ *and* $D_s|_{\leq \bar{s}}$*.*[8]

1. *If* $\bar{b} \geq s$ *or* $\bar{s} \leq b$*, then the probability that trade occurs in this mechanism is at least* $1/2$*.*

7. We note that each of the SO and BO mechanisms is a deterministic and ex-post monotone mechanism, and so can be made ex-post IC (and ex-post IR) by charging the threshold winning prices. The resulting modified mechanisms, however, are not ex-post (even weakly) budget balanced, but only ex-ante (strongly) budget balanced.

8. For a distribution D and a value c, we use $D|_{\leq c}$ to denote this distribution conditioned upon the drawn value being at most c, and use $D|_{\geq c}$ to denote this distribution conditioned upon the drawn value being at least c.

2. *If $\bar{b} \leq s$ and $\bar{s} \geq b$, then the probability that trade occurs in this mechanism is at least as high as the probability that trade occurs in the RO mechanism with SO parameters ∞ and D_b and BO parameters 0 and D_s.*

The proof of Lemma 6.2 is given in Appendix C.2. In a nutshell, part 1 holds since if, e.g., $\bar{b} \geq s$, then an offer by the buyer will always be accepted by the seller, and part 2 holds since under the given assumptions, if trade occurs in the unconstrained and unconditioned RO mechanism, then the price offered there also satisfies all of the extra restrictions of the constrained and conditioned RO mechanism, and therefore the same price will be offered in that mechanism as well, resulting in trade there as well.

6.4 Double Auctions

In this section, we present our results for the double-auction setting, in which there are no constraints on which seller can trade with which buyer (i.e., the graph G is the full bipartite graph). We first show that the RVWM mechanism is not asymptotically efficient, even ex-ante, and then present our hybrid mechanism for double auctions, which is an ex-post IR, BIC, ex-post weakly budget balanced mechanism, which ex-ante guarantees a constant fraction of the second-best, and is ex-post asymptotically efficient.

6.4.1 Asymptotic Inefficiency of the RVWM Mechanism

We first observe that the the RVWM mechanism is not asymptotically efficient for double auctions, even ex-ante and compared to the second-best.

Example 6.1. *Consider a double-auction market with n seller and n buyers, with agents' values and costs sampled i.i.d. from the uniform distribution over $[0,1]$. We claim that even when n is large, the RVWM mechanism will only give in expectation a constant fraction (strictly smaller than 1) of the expected GFT of the second-best mechanism. In particular, even in a large market, and even in expectation, the efficiency of the RVWM mechanism with respect to the second best (and thus also with respect to the first-best) does not converge to full efficiency.*

Proof sketch. We prove Example 6.1 in Appendix C.1, and here we give some intuition. When n is large, it is easy to observe that in an efficient trade roughly the $n/2$ lowest-cost sellers (essentially distributed uniformly in $[0, 1/2]$) will sell their items to roughly the $n/2$ highest-value buyers (essentially distributed uniformly in $[1/2, 1]$), increasing the welfare by about $1/2$ in expectation in each trade, resulting in the first-best having asymptotic expected GFT of about $n/4$. The second-best mechanism gets GFT that is in expectation at least the expected GFT of the Trade Reduction mechanism, so it has asymptotic expected GFT of about $n/4 - 1$, asymptotically the same as the first-best mechanism. On the other hand, when a buyer offers an optimized price facing a uniform distribution as in the RVWM mechanism, she offers only half of her value (and similarly, a seller offers a price that is half-way between her cost and 1). This results in only roughly the $n/3$ lowest-cost sellers (essentially distributed uniformly in $[0, 1/3]$) selling their items to roughly the $n/3$ highest-value buyers (essentially distributed uniformly in $[2/3, 1]$), increasing the welfare by about $2/3$ in expectation in each trade, resulting with asymptotic expected GFT of about $2n/9 < n/4$. □

6.4.2 A Hybrid Mechanism for Double Auctions

While the RVWM mechanism is not asymptotically efficient, the Trade Reduction (TR) mechanism [McA92] is asymptotically efficient as it guarantees, ex-post, an $1 - \frac{1}{q(\mathbf{b},\mathbf{s})}$ fraction of the first-best GFT, where $q(\mathbf{b}, \mathbf{s})$ is the size of the most efficient trade (Theorem 6.3).[9] As this mechanism gives no ex-ante guarantee (when $q(\mathbf{b}, \mathbf{s}) = 1$), we create a *hybrid mechanism* that runs the TR mechanism when $q(\mathbf{b}, \mathbf{s}) > 1$ and run the RO mechanism with some constraints and conditional distributions otherwise. We now present this mechanism.

Definition 6.2 (Hybrid Mechanism for Double Auctions)**.** *Our* hybrid mechanism *for double auctions is a direct revelation mechanism. Given the reports* $\boldsymbol{b}$ *and* $\boldsymbol{s}$ *(that are assumed to be truthful), we use* $b_{(1)}$ *to denote the buyer*[10] *with maximum value (when breaking ties lexicographically by IDs), i.e.,* $b_{(1)} \geq b_i$ *for every* $i \in B$*, and use* $b_{(2)}$ *to denote the buyer with maximum value after removing*

9. If there is a trade with GFT of 0, then there are efficient trades with different sizes. In this case trading according to the largest size will give full efficiency.

10. Somewhat abusing notation, we use $b_{(1)}$ to refer both to this buyer and to his value, and similarly for other agents.

buyer $b_{(1)}$. *Similarly, we use* $s_{(1)}$ *to denote the seller with minimal cost, and* $s_{(2)}$ *to denote the seller with the second-minimal cost.*[11] *The mechanism computes* $q(\boldsymbol{b}, \boldsymbol{s})$ *and runs as follows.*

- *If* $q(\boldsymbol{b}, \boldsymbol{s}) \leq 1$*,*[12] *the mechanism computes the set of trading agents and payments by running the RO mechanism with SO parameters* $\bar{s} = s_{(2)}$ *and* $D^B_{b_{(1)}}|_{\geq b_{(2)}}$ *and BO parameters* $\bar{b} = b_{(2)}$ *and* $D^S_{s_{(1)}}|_{\leq s_{(2)}}$*.*[13]
- *If* $q(\boldsymbol{b}, \boldsymbol{s}) \geq 2$*, the mechanism computes the set of trading agents and payments by running the TR mechanism on* $\boldsymbol{b}$ *and* $\boldsymbol{s}$*.*

We will now sketch the intuition behind our choice, in the case where $q(\mathbf{b}, \mathbf{s}) = 1$, of the constraints $\bar{s}, \bar{b}$ and the conditioned distributions $D^S_{s_{(1)}}|_{\leq s_{(2)}}$ and $D^B_{b_{(1)}}|_{\geq b_{(2)}}$ for which the offered prices are optimized. First, we would never want to allow $b_{(1)}$ to pay a price p such that if $b_{(1)}$ had valuation p then she would not be in the first-best. This is since such a possibility would create an incentive for her to manipulate her bid if her valuation really were slightly higher than p but still not high enough for her to be in the first-best: in this case, raising her bid would place her in the first-best, and she may end up paying p, which would give her positive utility. So, we have to make sure that $b_{(1)}$ never offers, nor is ever offered, such a p that is lower than $\bar{b} = b_{(2)}$. (In fact, the threshold bid of $b_{(1)}$ to be in the first-best is $\max\{b_{(2)}, s_{(1)}\}$, but by definition of the RO mechanism, she would never pay less than $s_{(1)}$ as this would result in negative GFT, so we only need to make sure that she never pays less than $\bar{b} = b_{(2)}$.) To make sure that $b_{(1)}$ never offers such a price p, we constrain her to offer at least $\bar{b}$ in the BO mechanism. To make sure that she is never offered such a price p in the SO mechanism, we have $s_{(1)}$ optimize her offer under the assumption that the value of $b_{(1)}$ is drawn from $D^B_{b_{(1)}}|_{\geq \bar{b}}$, which is equivalent to disclosing to $s_{(1)}$ that she has no point in offering a price lower than $\bar{b}$ since an offer of $\bar{b}$ will always be accepted. To see why the mechanism is truthful once we have set $\bar{b}$ (and $\bar{s}$) this way, consider the following hypothetical

11. Note that the maximal efficient set of trading agents is $\{s_{(1)}, \ldots, s_{q(\mathbf{b},\mathbf{s})}, b_{(1)}, \ldots, b_{q(\mathbf{b},\mathbf{s})}\}$.

12. Recall that in this case, if there is any trade with positive gains, then the maximal efficient set of trading agents is $\{s_{(1)}, b_{(1)}\}$.

13. We note that in this case since $q(\mathbf{b}, \mathbf{s}) = 1$, we have that $\bar{b} = b_{(2)} < s_{(2)} = \bar{s}$ and therefore indeed also $\bar{s} \geq \sup \operatorname{Support}(D^S_{s_{(1)}}|_{\leq s_{(2)}})$ and $\bar{b} \leq \inf \operatorname{Support}(D^B_{b_{(1)}}|_{\geq b_{(2)}})$.

scenario. Say that after calculating that $q(\mathbf{b}, \mathbf{s}) = 1$, the mechanism notifies $s_{(1)}$ and $b_{(1)}$ that they are the lowest-cost seller and highest-cost bidder, and furthermore notifies each of them of the values (and costs) of all other agents except the one that they are facing. In this case, the *posterior distribution* of $s_{(1)}$ regarding $b_{(1)}$ is $D^B_{b_{(1)}}|_{\geq \max\{b_{(2)}, s_{(1)}\}}$, so her best action is to optimize the price that she offers under this assumption, which is equivalent to optimizing the price that she offers for the distribution $D^B_{b_{(1)}}|_{\geq b_{(2)}}$ (but optimizing for the latter is easier to analyze, as it does not depend on the cost of $s_{(1)}$).

Theorem 6.5. *For the double auction setting the above simple hybrid mechanism for double auctions is ex-post individually rational, Bayesian incentive compatible, ex-post (direct trade) weakly budget balanced, and has both of the following efficiency guarantees:*

- *It gets at least a* $1/4$*-fraction of the efficient gains from trade ex-ante (second-best).*
- *It gets at least a* $\frac{q(\boldsymbol{b},\boldsymbol{s})-1}{q(\boldsymbol{b},\boldsymbol{s})}$*-fraction of the efficient gains from trade ex-post (first-best) . Note that the mechanism is asymptotically efficient: as the trade size* $q(\boldsymbol{b}, \boldsymbol{s})$ *goes to infinity, the fraction of the efficient gains from trade that it gets ex-post (first-best) goes to* 1*.*

6.4.3 Proof of Theorem 6.5

Proof of theorem 6.5. Recall that by Theorem 6.3 and Lemma 6.1, both the TR and the RO mechanisms are each ex-post IR, BIC, and ex-post (direct trade) weakly budget balanced.

Ex-post IR Ex-post IR holds since both the TR and the RO mechanisms are ex-post IR.

Bayesian IC We will show that our hybrid mechanism is BIC for any buyer.[14] A similar argument holds for truthfulness of the sellers.

We first claim that if a manipulation by a buyer does not change the choice of the mechanism that is run (TR or an instance of RO, where we consider each such instance to be a separate

14. In fact, when our hybrid mechanism runs TR, then it is ex-post IC for every agent, and when a price p is offered by an agent in the RO mechanism, then our hybrid mechanism is Bayesian IC for the agent making the offer, and ex-post IC for all other agents including the agent who receives the offer.

mechanism) by our hybrid mechanism, then it is nonbeneficial in expectation. For TR this follows since TR is ex-post IC. In Appendix C.3.1, we show that this holds also for each instance of the RO mechanism, by conditioning over the bids and identities of all agents except $s_{(1)}$ and $b_{(1)}$, and using both properties of Lemma 6.1.

We now claim that a buyer who is in the efficient trading set cannot change the efficient trading set while remaining in this set. Indeed, to see that this is the case, suppose a buyer in the efficient trading misreports by adding x (positive or negative) to his bid. The gains from trade from any trading set that includes this buyer therefore increase by x (while the gains from trade of any other trading set remains the same); therefore, since we break ties in the same manner without and with the deviation, no other trading set that includes this buyer other than the true efficient trading set can "become" (as a result of the misreport) the new efficient trading set.

Since (1) agents outside the efficient trading set never win, (2) a buyer in the efficient trading set cannot change the efficient trading set while remaining in this set, (3) the choice of the mechanism to run is completely determined by the efficient trading set and by the values/costs of the agents outside the efficient trading set, and (4) a manipulation that does not change the choice of the mechanism to run is nonbeneficial in expectation, we conclude that there are no strategic opportunities (in expectation) for any buyer who is in the efficient trading set.

To complete the proof that our hybrid mechanism is BIC, it is therefore enough to show that there is no beneficial manipulation by a buyer who is not in the efficient trading set. We will in fact show that the mechanism is ex-post IC for such agents; we do so by considering several cases.

- If $q(\mathbf{b},\mathbf{s}) \geq 2$, then a buyer who is not in the efficient trading set cannot cause a move to $q(\mathbf{b},\mathbf{s}) < 2$. Any manipulation by such a buyer is therefore nonbeneficial since the TR mechanism (which is run prior to, and following, the manipulation) is ex-post IC.

- If $q(\mathbf{b},\mathbf{s}) = 1$, then we consider two possible manipulations by some buyer $b_{(j)}$ who is not in the efficient trading set (and is therefore not the true $b_{(1)}$):

  - First, consider a manipulation by $b_{(j)}$ that causes a move to $q(\mathbf{b},\mathbf{s}) \geq 2$ and causes her to win. We claim that in this case, this buyer, who was previously not in the efficient trading set, must pay at least her true value whenever she wins. Indeed, by definition of TR and since truly $q(\mathbf{b}, \mathbf{s}) = 1$, since this buyer wins following the manipulation (and so is not reduced by the TR mechanism), she pays at least the original $b_{(1)}$, which is at least her true value. Therefore, she incurs non-positive utility.

  - We next consider a manipulation by $b_{(j)}$ that maintains $q(\mathbf{b}, \mathbf{s}) = 1$ and causes her to win (with some positive probability). We will show that whenever this buyer wins, she incurs non-positive utility. Since $q(\mathbf{b}, \mathbf{s}) = 1$ is maintained following the manipulation, we must have that $b_{(j)}$ raised her bid to be higher than the original $b_{(1)}$, who is now in the role of $b_{(2)}$. By Lemma 6.1, if the manipulating buyer wins, then she pays at least the new $b_{(2)}$, i.e., the original $b_{(1)}$, which is at least her true value, and so she incurs non-positive utility.

- Finally, consider the case $q(\mathbf{b}, \mathbf{s}) = 0$ and consider a manipulation by any buyer that causes her to win. Such a manipulation can only result in $q(\mathbf{b}, \mathbf{s}) = 1$, so the manipulator, if she wins, trades with $s_{(1)}$, and by definition of RO and since this mechanism is ex-post IR for this seller, this buyer pays at least $s_{(1)}$. Since $q(\mathbf{b}, \mathbf{s}) = 0$, we have that $s_{(1)}$ is larger than the true valuation of all buyers (including the manipulator), so the manipulator incurs negative utility whenever she wins.

Ex-post (direct trade) weak budget balance Our hybrid mechanism is ex-post (direct trade) weakly budget balanced since the two mechanisms TR and RO are both ex-post (direct trade) weakly budget balanced (the one is in fact ex-post (direct trade) strongly budget balanced).

Ex-post efficiency guarantee When $q(\mathbf{b}, \mathbf{s}) = 1$, then the guarantee vacuously holds, while when $q(\mathbf{b}, \mathbf{s}) \geq 2$, the guarantee follows from the same guarantee by the TR mechanism.

Ex-ante efficiency guarantee We will show that for each valuation profile $(\mathbf{b}, \mathbf{s})$, our hybrid mechanism achieves at least half of the gains from trade of the RVWM mechanism for the same valuation profile. Fix a valuation profile $(\mathbf{b}, \mathbf{s})$. We consider several cases.

- Consider the case where $s_{(2)} \leq b_{(2)}$. Note that this is precisely the case where $q(\mathbf{b},\mathbf{s}) \geq 2$. In this case, our hybrid mechanism runs the TR mechanism, which by theorem 6.3 guarantees at least a $\frac{q(\mathbf{b},\mathbf{s})-1}{q(\mathbf{b},\mathbf{s})} \geq 1/2$ fraction of the realized optimal gains from trade ex-post, and so at least a $1/2$ fraction of the gains from trade of the RVWM mechanism.

- Consider the case where $s_{(1)} > b_{(1)}$. Note that this is precisely the case where $q(\mathbf{b},\mathbf{s}) = 0$. In this case, it is efficient to have no trade for $(\mathbf{b},\mathbf{s})$, and this is what both our hybrid mechanism and the RVWM mechanism do, so our hybrid mechanism has the same gains from trade as the RVWM mechanism.

- Consider the case where $b_{(2)} < s_{(2)}$, and in addition either $b_{(2)} \geq s_{(1)}$ or $s_{(2)} \leq b_{(1)}$. In this case, since $q(\mathbf{b},\mathbf{s}) = 1$, we run the RO mechanism. By Lemma 6.2, in this case $s_{(1)}$ and $b_{(1)}$ trade with probability at least $1/2$, so our hybrid mechanism achieves at least a $1/2$ fraction of the realized optimal gains from trade, and so at least a $1/2$ fraction of the gains from trade of the RVWM mechanism.

- Finally, consider the case where $b_{(2)} < s_{(1)} \leq b_{(1)} < s_{(2)}$. In this case, the only possible trading pair with positive gains is of $s_{(1)}$ with $b_{(1)}$, so if the RVWM mechanism achieves positive gains from trade, then it trades this pair with positive probability. By observation 6.1, in this case the GFT of the RVWM mechanism are therefore at least those of the RO mechanism with SO parameters ∞ (no constraint) and $D^B_{b(1)}$ (unconditioned distribution) and BO parameters 0 (no constraint) and $D^S_{s_{(1)}}$ (unconditioned distribution) on that edge. Since $s_{(1)} > b_{(2)} = \bar{b}$ and $b_{(1)} < s_{(2)} = \bar{s}$, we have by Lemma 6.2 that the probability that trade occurs between $s_{(1)}$ and $b_{(1)}$ is at least as high in our hybrid mechanism (which runs the appropriate RO mechanism, constrained and conditioned) as it is in the unconstrained and unconditioned RO mechanism (that upper-bounds RVWM in this case). Therefore, in this case our hybrid mechanism achieves at least the gains from trade of the RVWM mechanism.

Combining all of the above, we have that the expected gains from trade of our hybrid mechanism are at least a $1/2$-fraction of those of the RVWM mechanism, and so by theorem 6.4 at least a $1/4$-fraction of the expected optimal gains from trade ex-ante (second-best). □

6.5 Main Results for Matching Markets

In this section we will generalize the results of Section 6.4 to matching markets. Recall that a matching market is given by an undirected bipartite graph $G = (S, B, E)$ with nodes on one side representing the sellers and nodes on the other side representing the buyers, with edges indicating possible trades. Recall that a profile $(\mathbf{b}, \mathbf{s})$ assigns a value b_i for each buyer $i \in B$ and a cost s_j for each seller $j \in S$.

6.5.1 A Trade Reduction Mechanism for Matching Markets

We first present a generalized Trade Reduction mechanism for matching markets. Like the Trade Reduction mechanism [McA92] for double-auctions, the *Trade Reduction Mechanism for matching markets* that we define below picks a subset of the "first-best" trade, and determines the payments based on the values and costs of the agents that it removed from the first-best. The details are, however, more subtle than in the double-auction setting.

Recall from Theorem 6.3 that for every valuation profile $(\mathbf{b}, \mathbf{s})$, the TR mechanism for double auctions attains GFT of at least a $1 - \frac{1}{q(\mathbf{b},\mathbf{s})}$ fraction of FB-GFT$(\mathbf{b}, \mathbf{s})$. We note that giving the same guarantee for matching markets, with $q(\mathbf{b}, \mathbf{s})$ remaining total the size of trade in the market, is not possible — just consider a matching market that consists of two connected components, each a double auction. So, to phrase our TR mechanism for matching markets, we will first have to define some notation that will eventually help us phrase its GFT guarantee (which will still generalize the $1 - \frac{1}{q(\mathbf{b},\mathbf{s})}$ of TR for double auctions, but in a slightly different way).

We say that the *classes of buyer i and i' are the same* if for any seller j it holds that $(i, j) \in E$ if and only if $(i', j) \in E$. Similarly, we define classes for sellers.[15] That is, two agents are of the same class if in any case that one of them can trade with some agent x, it also holds that the other agent can trade with agent x. Thus, nodes in the graph can be partitioned into equivalent classes, where each equivalent class consists of all agents of some fixed class. Each such class either includes only buyers, or only sellers, but never both. Let T_t denote the set of agents of class t. For each class t

15. Note that the classes that we define depend neither on the values of the buyers nor on the costs of the sellers (nor on the distributions from which these values/costs are drawn).

we denote by $q_t = q_t(M(\mathbf{b}, \mathbf{s}))$ the number of agents of class t that are matched in $M(\mathbf{b}, \mathbf{s})$, that is $q_t = |T_t \cap M(\mathbf{b}, \mathbf{s})|$. Additionally, we denote by $d_t = d_t(M(\mathbf{b}, \mathbf{s}))$ the number of distinct classes t' such that there is an edge in $M(\mathbf{b}, \mathbf{s})$ between an agent of class t and an agent of class t'.

Definition 6.3 (Trade Reduction Mechanism for Matching Markets)**.** *Fix a matching market $G = (S, B, E)$. The Trade Reduction mechanism for G gets as input a profile $(\boldsymbol{b}, \boldsymbol{s})$ and outputs an allocation and payments as follows.*

- *Given profile $(\boldsymbol{b}, \boldsymbol{s})$, let $M(\boldsymbol{b}, \boldsymbol{s})$ be the "first-best" matching. Any agent not in $M(\boldsymbol{b}, \boldsymbol{s})$ is marked as a loser and does not trade, paying 0.*
- *For each class t, recall that q_t is the number of agent of class t that are matched in $M(\boldsymbol{b}, \boldsymbol{s})$ and d_t is the number of different classes that trade with agents of class t in $M(\boldsymbol{b}, \boldsymbol{s})$.*
  - *For each buyer class t, the set of trading buyers will be the set of $q_t - d_t$ highest-value buyers of class t (breaking ties lexicographically by IDs).*[16] *We say that d_t buyers of class t were* reduced. *Each buyer of class t pays the highest value reported by any reduced buyer of class t.*
  - *For each seller class t, the set of trading sellers will be the set of $q_t - d_t$ lowest-cost sellers of class t (breaking ties lexicographically by IDs).*[17] *We say that d_t sellers of class t were* reduced. *Each buyer of class t is paid the lowest cost reported by any reduced seller of class t.*

We denote the set of agents that are trading under this mechanism by $TR(\boldsymbol{b}, \boldsymbol{s})$.

The following theorem presents the properties of the Trade Reduction Mechanism for matching markets. In particular, it shows that the mechanism provides some ex-post GFT guarantees which is a function of the maximum weight matching $M(\mathbf{b}, \mathbf{s})$. As with the TR mechanism for double auctions, this mechanism does not provide any ex-ante guarantees, though, even with respect to

16. Note that the number of trading buyers is non-negative, as for every class t it holds that $q_t \geq d_t$.

17. Note that the number of trading sellers is non-negative, as for every class t it holds that $q_t \geq d_t$.

the second-best mechanism. In particular, with a single trade in FB-GFT($\mathbf{b}, \mathbf{s}$), there will be no trade in this mechanism.

Theorem 6.6. *The Trade Reduction Mechanism for matching markets is ex-post IR, ex-post IC, ex-post (direct trade) weakly budget balanced, and for any* $(\boldsymbol{b}, \boldsymbol{s})$ *the fraction of the gains from trade of* FB-GFT$(\boldsymbol{b}, \boldsymbol{s})$ *that it attains is at least* $\min\{1 - \frac{d_t}{q_t} \mid \textit{class } t \textit{ s.t. } q_t > 0\}$.[18]

We note that guarantee from Theorem 6.6 of the TR mechanism attaining a fraction-of-FB-GFT($\mathbf{b}, \mathbf{s}$) of at least $\alpha(\mathbf{b}, \mathbf{s}) = \min\{1 - \frac{d_t}{q_t} \mid \text{class } t \text{ s.t. } q_t > 0\}$ coincides in the double-auction setting with the guarantee of at least $1 - \frac{1}{q(\mathbf{b},\mathbf{s})}$ from Theorem 6.3, and naturally generalizes it. Another generalization for matching markets of the fraction $1 - \frac{1}{q(\mathbf{b},\mathbf{s})}$ that one may find natural, which also coincides with it in the double-auction setting, is $\beta(\mathbf{b}, \mathbf{s}) = \min\{1 - \frac{1}{r_{t,t'}} \mid \text{classes } (t, t') \text{ s.t. } r_{t,t'} > 0\}$, where for any buyers' class t and sellers' class t', we use $r_{t,t'}$ to denote the number of buyers of class t that are matched with sellers of class t' in $M(\mathbf{b}, \mathbf{s})$. While this alternative generalization is conceptually interesting in its own right, we in fact show that for every valuation profile it holds that $\beta(\mathbf{b}, \mathbf{s}) \leq \alpha(\mathbf{b}, \mathbf{s})$, and so a GFT guarantee of $\beta(\mathbf{b}, \mathbf{s})$ follows from the GFT guarantee of $\alpha(\mathbf{b}, \mathbf{s})$ from theorem 6.6:

Corollary 6.1. *For any* $(\boldsymbol{b}, \boldsymbol{s})$*, the fraction of the GFT of* FB-GFT$(\boldsymbol{b}, \boldsymbol{s})$ *that the TR mechanism for matching markets attains is at least* $\min\{1 - \frac{1}{r_{t,t'}} \mid \textit{classes } (t, t') \textit{ s.t. } r_{t,t'} > 0\}$.

The proofs of Theorem 6.6 and corollary 6.1 are given in Appendix C.4.

6.5.2 Offering Mechanism for Matching Markets

Before defining our hybrid mechanism for matching markets, we first define an offering mechanism for matching markets, analogous to the specific instance of the RO mechanism (including the specific offer constraints and conditioned distributions) that our hybrid mechanism for double auctions runs whenever $q(\mathbf{b}, \mathbf{s}) = 1$ in that setting. In this mechanism, agents not in $M(\mathbf{b}, \mathbf{s})$ never trade, and

18. Note that d_t, q_t and $r_{t,t'}$ are all function of $M(\mathbf{b}, \mathbf{s})$, so they are functions of the profile $(\mathbf{b}, \mathbf{s})$. This is similar to $q(\mathbf{b}, \mathbf{s})$ being a function of $(\mathbf{b}, \mathbf{s})$ for Trade Reduction in double auctions.

agent in a pair $(i, j) \in M(\mathbf{b}, \mathbf{s})$ either trades in that pair or does not trade at all. This mechanism is defined as follows.

Definition 6.4 (Offering Mechanism for Matching Markets)**.** *The mechanism iterates over all edges* $(i, j) \in M(\boldsymbol{b}, \boldsymbol{s})$*, and for each such edge acts as follows.*

- *Let* $\bar{s} = \bar{s}_{(i,j)}(\boldsymbol{b}, \boldsymbol{s})$ *be the minimal bid of buyer* i *such that any higher bid causes* i *to be in the first-best in the market* $(S \setminus \{j\}, B)$*, i.e., the market without seller* j*. We set* $\bar{s} = \infty$ *if no such bid exists.*
- *Let* $\bar{b} = \bar{b}_{(i,j)}(\boldsymbol{b}, \boldsymbol{s})$ *be the maximal bid (reported cost) of seller* j *that causes* j *to be in the first-best in the market* $(S, B \setminus \{i\})$*, i.e., the market without buyer* i*. We set* $\bar{b} = 0$ *if no such bid exists.*

Now, to decide whether trade occurs between i *and* j *and at which price, run the RO mechanism on this edge with SO parameters* $\bar{s}$ *and* $D_i^B|_{\geq \bar{b}}$ *and BO parameters* $\bar{b}$ *and* $D_j^S|_{\leq \bar{s}}$*.*

We note that the above offer constraints $\bar{s}$ and $\bar{b}$ precisely generalize the offer constraints from our hybrid mechanism for double auctions from section 6.4. Indeed, in a double auction, the minimal bid of buyer $b_{(1)}$ that causes her to be in the first-best in the market $(S \setminus \{s_{(1)}\}, B)$ without $s_{(1)}$ is $\max\{b_{(2)}, s_{(2)}\}$, and when $q(s, b) = 1$ in the double-auctions setting (this is the case where we run the RO mechanism) it must be that $b_{(2)} < s_{(2)}$ and so $\max\{b_{(2)}, s_{(2)}\} = s_{(2)}$, which how we set the constraint $\bar{s}$ in that mechanism. The choice of $\bar{b}$ is similar. The careful definition of $\bar{s}$ and $\bar{b}$ above guarantees the two properties of these thresholds that our double-auction constraints readily satisfied: first, both $\bar{b}$ and $\bar{s}$ are completely independent of b_i and of s_j, and second, as we will see in our analysis, $\bar{b}$ coincides with the minimal winning bid of *buyer* i in the *original* market whenever this constraint is binding. (And similarly for $\bar{s}$ and seller j.)

To show that the Offering Mechanism is well-defined, we have to make sure that the SO and BO parameters that we specify for the RO mechanism meet the conditions imposed in the definition of that mechanism. The following lemma does precisely this.

Lemma 6.3. *For every* $(i, j) \in M(\boldsymbol{b}, \boldsymbol{s})$*, it holds that* (1) $\bar{s} \geq s_j$*,* (2) $\bar{b} \leq b_i$*, and* (3) $\bar{s} \geq \bar{b}$*.*

We next prove that the Offering Mechanism is truthful, budget balanced, and has an ex-ante guarantee.

Theorem 6.7. *The Offering Mechanism is BIC, ex-post IR, ex-post (direct trade) strongly budget balanced, and ex-ante guarantees at least a* $1/4$ *of the expected GFT of the second-best mechanism.*

The proofs of Lemma 6.3 and Theorem 6.7 is given in Appendix C.6. As noted in the introduction, proving the ex-ante guarantee of the Offering Mechanism for matching markets is the most technically challenging part of our analysis. The main ideas behind this proof are surveyed in Section 6.6.

6.5.3 Hybrid Mechanism for Matching Markets

We are now ready to define our hybrid mechanism for matching markets. It combines the TR mechanism and the Offering Mechanism in a proper way. We note that for double auctions, the mechanism defined below reduces precisely to our hybrid mechanism for double auctions from Section 6.4.

Definition 6.5 (Hybrid Mechanism for Matching Markets)**.** *Let* $G = (S, B, E)$ *be the constraints graph. Our* hybrid mechanism *is a direct revelation mechanism. Given the the reports* $(\boldsymbol{b}, \boldsymbol{s})$ *(which is assumed to be truthful), the mechanism computes* $M(\boldsymbol{b}, \boldsymbol{s})$ *and* $\alpha(\boldsymbol{b}, \boldsymbol{s}) = \min\left\{1 - \frac{d_t}{q_t} \;\middle|\; \text{class } t \text{ s.t. } q_t > 0\right\}$ *and runs as follows.*

- *If* $\alpha(\boldsymbol{b}, \boldsymbol{s}) \geq 1/2$*, the mechanism computes the set of trading agents and payments by running the Trade Reduction Mechanism for matching markets defined above.*
- *Otherwise, the mechanism computes the set of trading agents and payments by running the Offering Mechanism for matching markets defined above.*

We are now ready to formally state the main result of this paper.

Theorem 6.8. *The Hybrid Mechanism for matching markets is ex-post IR, BIC and ex-post (direct trade) weakly budget balanced, which satisfies both of the following.*

- *The expected GFT of this mechanism are at least* $1/4$ *of those of the second-best mechanism.*

- *For any* $(\boldsymbol{b}, \boldsymbol{s})$ *with* $\alpha(\boldsymbol{b}, \boldsymbol{s}) \geq 1/2$*, the fraction of the gains from trade of* FB-GFT$(\boldsymbol{b}, \boldsymbol{s})$ *that this mechanism attains is at least* $\alpha(\boldsymbol{b}, \boldsymbol{s}) \geq 1/2$.

The hybrid mechanism for matching markets inherits from the Trade Reduction mechanism for matching markets also the ex-post guarantee of Corollary 6.1:

Corollary 6.2. *Let* $\beta(\boldsymbol{b}, \boldsymbol{s}) = \min\{1 - \frac{1}{r_{t,t'}} \mid \text{classes } (t,t') \text{ s.t. } r_{t,t'} > 0\}$*. For any* $(\boldsymbol{b}, \boldsymbol{s})$ *with* $\beta(\boldsymbol{b}, \boldsymbol{s}) \geq 1/2$*, the fraction of the gains from trade of* FB-GFT$(\boldsymbol{b}, \boldsymbol{s})$ *that the Hybrid Mechanism for matching markets attains is at least* $\beta(\boldsymbol{b}, \boldsymbol{s}) \geq 1/2$.

The proofs of theorem 6.8 and corollary 6.2 are given in Appendix C.7.

6.6 Sketch of the Proof of Ex-Ante Guarantee of the Offering Mechanism for Matching Markets

In this section, we sketch the proof of the ex-ante guarantee of the Offering Mechanism, which has been stated in Theorem 6.7. The full proof is relegated to Appendix C.6. To prove that the Offering Mechanism ex-ante guarantees at least a $1/4$-fraction of the gains from trade of the second-best mechanism, we compare the Offering Mechanism to the RVWM mechanism. Due to Theorem 6.4, it suffices to show the following lemma.

Lemma 6.4. *For any valuation profile* $(\boldsymbol{b}, \boldsymbol{s})$*, the gains from trade of the Offering Mechanism for matching markets is at least half of the gains from trade of the RVWM mechanism for that profile.*

We first provide the intuition behind Lemma 6.4. Fix a valuation profile $(\mathbf{b}, \mathbf{s})$. Let $M_1^* = M_1^*(\mathbf{b}, \mathbf{s})$ be the maximum-weight matching of G when edge weight is $\tilde{\varphi}_i(b_i) - s_j$ for $(i, j) \in E$ and $M_2^* = M_2^*(\mathbf{b}, \mathbf{s})$ be the maximum-weight matching of G when edge weight is $b_i - \tilde{\tau}_j(s_j)$ for $(i, j) \in E$.[19] The RVWM mechanism runs the Generalized Seller Offering Mechanism (GSO) with probability $1/2$ and in that case obtains the GFT of the matching M_1^*, and it runs the Generalized

19. $\tilde{\varphi}_i$ and $\tilde{\tau}_j$ are the ironed virtual value functions of buyer i and seller j, respectively.

Buyer Offering Mechanism (GBO) with probability 1/2 and in that case obtains the GFT of the matching M_2^*. See Definition 5.3 for the formal definition of GSO and GBO. It suffices to show that the GFT of each of the two matchings M_1^* and M_2^* can be bounded by twice the GFT of the Offering Mechanism for the valuation profile $(\mathbf{b}, \mathbf{s})$. We will show how to bound the GFT of M_1^*. A similar argument can bound the GFT of M_2^*.

Consider the first-best matching $M = M(\mathbf{b}, \mathbf{s})$ together with the matching M_1^*. Each connected component of the union of the two matchings $M \cup M_1^*$ is either a maximal alternating path[20] or an alternating cycle[21]. We will show that all alternating cycles consist of two edges between the same seller and buyer (see Figure 6.1 (a)) due to our tie-breaking rules (proved in Corollary C.2) and that the GFT of the Offering Mechanism from that buyer-seller pair is at least the GFT of the RVWM mechanism from that pair. For an alternating path, we will consider the cases of an even or an odd number of edges of the alternating path separately, and show that in either case, the GFT of our Offering Mechanism from the path is at least half of the GFT of the matching M_1^* from that path. Given the fact that M and M_1^* are each a maximum-weight matching w.r.t. the edge weights $b_i - s_j$ and $\tilde{\varphi}_i(b_i) - s_j$ respectively, we prove that any maximal alternating path that is not a cycle, starts with a buyer and an edge from M. See Corollary C.3 for more details. Figure 6.1 illustrates the three different cases in our proof.

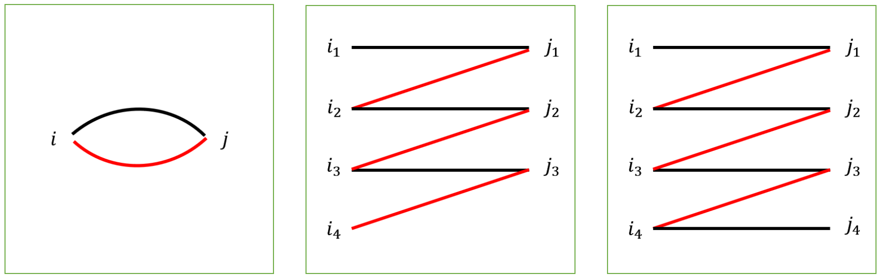
The following lemma plays a central role in our proof of Lemma 6.4. It provides a sufficient condition for a buyer-seller pair to trade in that mechanism.

Lemma 6.5. *Fix valuation profile $(\boldsymbol{b}, \boldsymbol{s})$. For every $(i, j) \in M(\boldsymbol{b}, \boldsymbol{s})$, if j is in $M_{-i}(\boldsymbol{b}, \boldsymbol{s})$ then buyer i will trade with seller j in the BO Mechanism, and if i is in $M_{-j}(\boldsymbol{b}, \boldsymbol{s})$ then buyer i will trade with seller j in the SO Mechanism. Thus, in either case i and j will trade with probability at least* 1/2 *in the Offering Mechanism.*

The next lemma shows that any seller who is not at the end of any alternating path, must still be in the first-best matching if we remove the buyer that is matched to her.

20. A path is called an *alternating path* if the edges of the path alternate between the two matchings. A path is *maximal* if it is not a subpath of any other path.

21. An *alternating cycle* is an alternating path whose two endpoints coincide.



(a) (b) (c)

Figure 6.1: Example of the three different cases considered in the proof of the ex-ante guarantee: (a) alternating cycle; (b) maximal alternating path with even number of edges; (c) maximal alternating path with odd number of edges.

Lemma 6.6. *Let A be an acyclic maximal alternating path of $M(\boldsymbol{b},\boldsymbol{s}) \cup M_1^*(\boldsymbol{b},\boldsymbol{s})$. For every seller $j \in A$ who is not at the end of the path, it holds that $j \in M_{-i}(\boldsymbol{b},\boldsymbol{s})$, where i is the buyer such that $(i,j) \in M(\boldsymbol{b},\boldsymbol{s})$.*

Combining Lemma 6.6 and Lemma 6.5, we show that all sellers in a maximal alternating path of even length will trade in the BO mechanism with the buyers that are matched to them in the first-best matching.

Next, we consider maximal alternating paths with odd length and present another useful characterization. We assume w.l.o.g. that any maximal alternating path of $M \cup M_1^*$ starts with a buyer and an edge in M (by Corollary C.3). Let $GFT_{M'}(U)$ be the GFT of all edges of M' that are contained in U.

Lemma 6.7. *For $K > 3$, let $A = (i_1 j_1 i_2 j_2 ... i_{L-1} j_{L-1} i_L j_L)$, be a maximal alternating path of odd number of edges of $M \cup M_1^*$ with i_l denoting buyers and j_l denoting sellers, and with $(i_1, j_1) \in M$. It holds that*

- *if $b_{i_L} > b_{i_1}$ then $i_L \in M_{-j_L}$.*
- *if $b_{i_L} \leq b_{i_1}$ then $GFT_M(A \setminus \{i_L, j_L\}) \geq GFT_{M_1^*}(A)$.*

Proof of Lemma 6.4. By Corollary C.2, any alternating cycle of M and M_1^* has only two (identical) undirected edges $(i,j) \in M \cap M_1^*$. If $j \in M_{-i}(\mathbf{b},\mathbf{s})$ or $i \in M_{-j}(\mathbf{b},\mathbf{s})$, by Lemma 6.5 buyer i will trade with seller j in the Offering Mechanism with probability at least $1/2$, which obtains at least half of the GFT that the RVWM mechanism obtains on $(i, j) \in M_1^*$ when the profile is $(\mathbf{b}, \mathbf{s})$. Otherwise, since $i \notin M_{-j}$ we have that $\bar{s} \geq b_i$, and since $j \notin M_{-i}$ we have that $\bar{b} \leq s_j$. Since trade occurs with positive probability on (i, j) in the RVWM mechanism, then similarly to the double-auction case, by Observation 6.1, our Offering Mechanism achieves at least the gains from trade of the RVWM mechanism on this edge (and therefore, on any alternating cycle).

Consider any maximal alternating path of even number of edges. By Lemma 6.5 and Lemma 6.6, every pair $(i, j) \in M$ trades in the BO mechanism, so whenever the BO mechanism runs, the maximal GFT (first-best) of the agents in the alternating path, which is at least the GFT of M_1^* from these agents, is obtained. The Offering Mechanism runs the BO mechanism is probability $1/2$, so in expectation it obtains at least $1/2$ the GFT of M_1^* from this path.

Now consider any maximal alternating path $(i_1 j_1 i_2 j_2 ... i_{L-1} j_{L-1} i_L j_L)$ of odd number of (at least 3)[22] edges, which starts with buyer i_1 and an edge from M. Here $L \geq 2$. By Lemma 6.5 and Lemma 6.6, for every $l = 1, 2, \ldots, L-1$, buyer i_l will trade with seller j_l in the BO mechanism. If $b_{i_1} \geq b_{i_L}$, the claim holds since by Lemma 6.7 the GFT of M_1^* from this path is at most the GFT of the first $L-1$ pairs in the first-best matching M, and all these $L-1$ pairs will be traded in the BO mechanism, which happens with probability $1/2$.

If $b_{i_1} < b_{i_L}$, by Lemma 6.5 and Lemma 6.7, buyer i_L will trade with seller j_L in the SO mechanism, which happens with probability $1/2$. Therefore, every pair $(i, j) \in M$ is traded with probability at least $1/2$. This obtains half the maximal GFT (first-best) of this path, which is at least half the GFT of M_1^* in this path.

Similarly, we can show that the Offering Mechanisms obtains at least $1/2$ of the GFT of M_2^*. Since the expected GFT of the RVWM mechanism is the average GFT of M_1^* and M_2^*, we conclude that the Offering Mechanisms obtains at least $1/2$ the GFT of the RVWM mechanism. □

22. If there is a single edge, then it is only in M. We only need to cover edges in M_1^*.

6.7 Discussion

One of the biggest pushbacks against constant-approximation mechanisms is that while they provide some worst-case guarantee, they often do not provide any guarantee for significantly better performance "when the instances are easy to handle". We believe that a mechanism that not only provides a worst-case guarantee, but also provides a guarantee of performing very well on "easy instances" is much more appealing and more likely to be used. In our setting, we implement this agenda by postulating that "nice instances" are large-market instances (for some formal sense of "large"), and we are able to achieve the best of both worlds: a guaranteed constant approximation on one hand, and asymptotic optimality when the markets are large on the other hand. We believe that presenting similar results in other settings is an interesting research direction.

Mechanisms with such "worse-case *and* best-case guarantees" are of particular appeal when the social planner needs to fix the mechanism well before the exact market characteristics are known, for example, when the mechanism is defined by some laws or regulations (e.g., FCC auctions) that are fixed well in advance.

Chapter 7

Multi-Dimensional Two-sided Markets

In this chapter we move on to multi-dimensional settings in two-sided markets. We focus on a special setting with multiple unit-supply sellers and a single constrained-additive buyer. We design simple, truthful and budget balanced mechanism to achieve an unconditional approximation to the second-best GFT.

In Section 7.1, we give an overview of the results and techniques covered in this chapter. In Section 7.2 we introduce the notations used in this chapter. In Section 7.3 we present a distribution-parameterized approximation to the first-best GFT. In Section 7.4, we prove an unconditional approximation to the second-best GFT, using results from Section 7.3. In Section 7.5 we draw a connection between a lower bound to our analysis and one of the major open problems in a special matching market. In Section 7.6 we present some examples toward the lower bound of our analysis.

7.1 Overview of Results and Techniques

We focus on a setting with n heterogeneous items, where each item is owned by a different seller i, and there is a constrained-additive buyer with feasibility constraint $\mathcal{F}$. The first main result is a distribution-parameterized approximation to the first-best GFT.

Result 1. *There is a* fixed posted price mechanism *whose GFT is an* $O(\frac{\log(1/r)}{\delta\eta})$*-approximation to the* first-best GFT *when the buyer's feasibility constraint* $\mathcal{F}$ *is* (δ, η)*-selectable (Definition 2.5), and an* $O(\log(n) \cdot \log(1/r))$*-approximation for a general constrained-additive buyer.* r *is a distributional parameter: the minimum* trade probability *over all items. We define the trade probability of item* i *as the probability that the buyer's value for* i *exceeds the seller's cost.*

As shown in Section 2.7, many familiar feasibility constraints such as matroid, matching, knapsack, and the compositions of each, are known to be (δ, η)-selectable with constant δ and η [FSZ16], so our result provides an $O(\log(1/r))$-approximation for all of these environments. The main takeaway from Result 1 is that there is an $O(\log(1/r))$-approximation to the first-best GFT for the feasibility constraints stated above. Next we introduce the class of *fixed posted price* mechanisms.

Fixed Posted Price (FPP): In a fixed posted price mechanism, there is a collection of fixed prices $\{(\theta_i^B, \theta_i^S)\}_{i\in[n]}$, where $\theta_i^B \geq \theta_i^S$ for each item i. Let R be the set of sellers that are willing to sell their item at price θ_i^S. The buyer can purchase any item i in R at price θ_i^B. Trade only occurs when the buyer wants to buy the item and the seller is willing to sell it.

Our result is a generalization of the result by Colini-Baldeschi et al. [CBGdK+17], where they provide the same approximation using a fixed posted price mechanism for bilateral trade. Importantly, our approximation ratio has the optimal dependence on r up to a constant factor. Example 7.1 (adapted from an example by Blumrosen and Dobzinski [BD16]) in Section 7.6 shows that, for any $r > 0$, there is an instance of our problem with minimum trade probability r such that no fixed posted price mechanism can achieve more than a $\frac{c}{\log(1/r)}$-fraction of even the second-best GFT for some absolute constant c. In our fixed posted price mechanism, we allow θ_i^B to be strictly greater than θ_i^S. This is crucial for our analysis, but makes the mechanism only ex-post weakly budget balanced. We leave it as an interesting open question as to whether our approximation ratio can be achieved by an ex-post strongly budget balanced fixed posted price mechanism.

When the trade probability of each item is not too low, our first result provides a good approximation to the first-best GFT using a simple fixed posted price mechanism. However, r can be arbitrarily small in the worst-case, making our approximation too large to be useful. Is it possible to produce an *unconditional worst-case approximation guarantee*? We provide an affirmative answer to this question with an unconditional $O(\log n)$-approximation to the second-best GFT.

Result 2. *There is a* dominant strategy incentive compatible *(DSIC), ex-post IR, and BB mechanism whose GFT is at least* $\Omega(\frac{\delta\eta}{\log n})$*-fraction of the* second-best GFT *when the buyer's feasibility constraint* $\mathcal{F}$ *is* (δ, η)*-selectable, and at least* $\Omega(\frac{1}{\log^2(n)})$*-fraction of the second-best GFT when the buyer is general constrained-additive.*

As we show in Example 7.1, no fixed posted price mechanism can provide such a guarantee. We develop two new mechanisms. The first one is a multi-dimensional extension of the Generalized Buyer Offering Mechanism (Definition 5.3). We provide a full description of the mechanism in Section 7.4.2. The second mechanism is a generalization of the fixed posted price mechanism that we call the *Seller Adjusted Posted Price Mechanism.*

Seller Adjusted Posted Price (SAPP): The sellers report their costs $\mathbf{s}$. The mechanism maps the cost profile to a collection of posted prices $\{\theta_i(\mathbf{s})\}_{i\in[n]}$ for the buyer. The buyer can purchase at most one item, and pays price $\theta_i(\mathbf{s})$ if she buys item i. An item trades if the buyer decides to purchase that item.

The main advantage of using a SAPP mechanism is that it provides the flexibility to set prices based on the sellers' costs, which allows a SAPP mechanism to achieve GFT that can be unboundedly higher than the GFT attainable by even the best fixed posted price mechanism (see Example 7.2). Example 7.3 in Section 7.6 shows that the class of SAPP mechanisms is necessary to obtain any finite approximation ratio to the second-best: both the best FPP mechanisms and the Generalized Buyer Offering Mechanism have an unbounded gap compared to the second-best GFT, even in the bilateral trade setting.

An astute reader may have already realized that the payments to the sellers are not yet defined in the SAPP mechanism. This is because the allocation rule of a SAPP mechanism is not necessarily monotone in the sellers' costs if the mappings $\{\theta_i(\cdot)\}_{i\in[n]}$ are not chosen carefully. Interestingly, we show that if the mappings $\{\theta_i(\cdot)\}_{i\in[n]}$ satisfy a strong type of monotonicity that we call *bi-monotonicity* (Definition 7.1), then the allocation rule is indeed monotone in each seller's reported cost. Since the sellers are single-dimensional, we can apply Myerson's payment identity to design an incentive compatible payment rule. The final property we need to establish is budget balance, which turns out to be the major technical challenge for us. We provide more details and intuition about our solution to this challenge in the discussion of the techniques.

In Section 7.5, we draw a connection between a lower bound to our analysis and one of the major open problems in single dimensional two-sided markets. We prove a reduction from approximating the first-best GFT in the *unit-demand* setting to bounding the gap between the first-best and second-best GFT in a related *single-dimensional* setting (Theorem 7.4). If in the latter market, the gap between first-best and second-best GFT is at most c, then our mechanism is a $2c$-approximation to the first-best GFT in the former market.

7.1.1 Our Approach and Techniques

1. $\log(1/r)$**-Approximation (Section 7.3):** Our starting point is similar to Colini-Baldeschi et al. [CBGdK+17]. We first argue that the probability space of each item i can be partitioned into $O(\log(1/r))$ events $\{E_{ij}\}_{j\in[\log(2/r)]}$, such that in each event E_{ij}, the median of the buyer's value b_i for item i dominates the median of the i-th seller's cost s_i. The first-best GFT is upper bounded by the sum of the contribution to GFT from each of these events. In bilateral trade, simply setting the posted price to be the median of the buyer's value is sufficient to obtain 1/2 of the optimal GFT from E_{ij} as shown by McAfee [McA08]. The $\log(1/r)$-approximation by Colini-Baldeschi et al. [CBGdK+17] essentially follows from this argument.

To illustrate the added difficulty from multiple items, it suffices to consider a unit-demand buyer. Setting the posted price on each item to be the median of the buyer's value does not provide a good approximation, because the buyer will purchase the item that gives her the highest surplus, which could be very different from the item that generates the most GFT. Similar scenarios are not uncommon in *multi-dimensional auction design*, and prophet inequalities [KW12, KS78] have been proven to be effective in addressing similar challenges. The main barrier for applying the prophet inequality to two-sided markets is choosing the appropriate random variable as the reward for the prophet/gambler. It is not obvious how to choose a random variable that will translate to a two-sided market mechanism, and in fact, for some choices, no translation between the thresholding policy for the gambler and a two-sided market mechanism is possible.[1] Our key insight is to replace event E_{ij} with a related but different event $\overline{E}_{ij}$ where there is a fixed number θ_{ij} such that s_i and b_i are always separated by θ_{ij} ($s_i \leq \theta_{ij} \leq b_i$). We further show that the GFT contribution from event $\overline{E}_{ij}$ is at least half of the GFT contribution from E_{ij}. Importantly, the GFT contributed by item i in event $\overline{E}_{ij}$: $(b_i - s_i)^+$ [2] $= (b_i - \theta_{ij})^+ \cdot \mathbb{1}[s_i \leq \theta_{ij}] + (\theta_{ij} - s_i)^+ \cdot \mathbb{1}[b_i \geq \theta_{ij}]$. Note that if we replace $\overline{E}_{ij}$ with E_{ij}, the LHS can exceed the RHS when $\theta_{ij} > b_i > s_i$. The decomposition of $(b_i - s_i)^+$ using θ_{ij} is critical for us to apply the prophet inequality. We can now choose the reward for the gambler to be $v_i = (\theta_{ij} - s_i)^+ \cdot \mathbb{1}[b_i \geq \theta_{ij}]$, and the thresholding policy with a threshold T can be implemented with a posted price mechanism where the price for the buyer is θ_{ij} and the price for the seller is $\theta_{ij} - T$.[3]

When the buyer's feasibility constraint is general downward-closed, the only known prophet inequalities are due to Rubinstein [Rub16] and are $O(\log n)$-competitive. Unfortunately, the prophet inequalities in [Rub16] are highly adaptive, and thus cannot translate into prices for a single buyer. Further, an almost matching lower bound of $O(\log n / \log \log n)$ is shown by Babaioff et al. [BIK07], precluding much possible improvement for this approach. Instead, we use a constrained fixed posted price mechanism that forces the buyer to buy at least h items (at their posted prices) if she wants to buy any; otherwise, she must leave with nothing. We divide the same variables v_i into $O(\log m)$ buckets based on their contribution to seller surplus. Within each bucket k, all variables v_i lie in $[L_k, 2L_k]$ for some L_k. We prove a concentration inequality for the *maximum size of a feasible and affordable set*. It guarantees that with constant probability, the buyer will be willing to purchase at least h items (for an appropriate choice of h), generating sufficient GFT.

1. For example, one can choose the GFT from the i^{th} item $(b_i - s_i)^+$ as the reward of the i^{th} round, but no fixed posted price mechanism corresponds to the policy that only accepts items whose GFT is above a certain threshold. Indeed, no BIC, IR, and BB mechanism can implement a thresholding policy with threshold 0 due to the impossibility result by Myerson and Satterthwaite [MS83].

2. $x^+ = \max\{x, 0\}$.

3. A similar fixed posted price mechanism can take care of $(b_i - \theta_{ij})^+ \cdot \mathbb{1}[s_i \leq \theta_{ij}]$.

2. Benchmark of the Second-Best GFT (Section 7.4.1): As our goal is to obtain a benchmark of the second-best GFT that is unconditional, the benchmark from the previous (distribution-parameterized) result cannot be used here. We derive a novel benchmark in two steps. Step (i): we create two imaginary one-sided markets: the *super seller auction* and the *super buyer procurement auction*. We show that the second-best GFT of the two-sided market is upper bounded by the *optimal profit* from the super seller auction and the *optimal buyer utility* from the super buyer procurement auction. Step (ii): we provide an extension of the marginal mechanism lemma [HN12, CH13] to the optimal profit. We show that the optimal profit for selling all items in $[m]$ is upper bounded by the *first-best* GFT from items in T and the optimal profit for selling items in $[n]\backslash T$, where T is an arbitrary subset of $[n]$. Our key insight is to choose T to be the "likely to trade" items, which are the ones with trade probability at least $1/n$, and apply the marginal mechanism lemma. This partition allows us to use our first result to provide an $O(\log n)$-approximation to the first-best GFT of the "likely to trade" items using a fixed posted price mechanism. Moreover, we prove that the optimal buyer utility from the super buyer procurement auction is upper bounded by the GFT of an extension of the "generalized buyer offering mechanism" [BCWZ17]. Finally, we provide an $O(\log n)$-approximation to the optimal profit for selling the "unlikely to trade" items using a SAPP mechanism. Note that the approximation crucially relies on the fact that in expectation at most one item can trade among the "unlikely to trade" items.

3. Budget Balance of Seller Adjusted Posted Price Mechanisms (Section 7.4.3): As mentioned earlier, we restrict our attention to bi-monotonic mappings from cost profiles to buyer prices $\{\theta_i(\cdot)\}_{i\in[n]}$ to guarantee incentive-compatibility. However, budget balance does not follow from bi-monotonic mappings. We extend the definition of bi-monotonicity to allocation rules and show that all bi-monotonic allocation rules can be transformed into a DSIC, IR, and BB SAPP mechanism. In our proof of the budget balance property, we identify an auxiliary allocation rule q, which may not be implementable by a BB mechanism. We then show that the allocation rule of our SAPP mechanism is "coupled" with q. In particular, our allocation probability is always between $q/4$ and $q/2$. The upper bound $q/2$ allows us to upper bound the payment to the seller, and the lower bound $q/4$ allows us to lower bound the payment we collect from the buyer. Surprisingly, we can prove that the upper bound of the payment to the seller is no more than the lower bound of the buyer's payment. We suspect this type of allocation coupling argument may also be useful in other problems.

7.2 Notations in This Chapter

Two-sided Markets. We focus on two-sided markets between a single buyer and n unit-supply sellers. Every seller i sells a heterogeneous item. For simplicity we denote the item sold by seller i as item i. Each seller i has cost s_i for producing item i, where s_i is drawn independently from distribution $\mathcal{D}_i^S$. The buyer has value b_i for every item i where b_i is drawn independently from distribution $\mathcal{D}_i^B$. $\mathcal{D}_i^S$ and $\mathcal{D}_i^B$ are public knowledge. Let $\mathcal{D}^B = \times_{i=1}^n \mathcal{D}_i^B$ be the distribution of the buyer's value profile and $\mathcal{D}^S = \times_{i=1}^n \mathcal{D}_i^S$ be the distribution of the cost profile for all sellers. Let $\mathbf{b} = (b_1, ..., b_n)$ and $\mathbf{s} = (s_1, ..., s_n)$ denote the value (or cost) profile for the buyer and all sellers. For notational convenience, for every i we denote b_{-i} (or s_{-i}) to be the value (or cost) profile without item i. For every i, F_i, f_i (or G_i, g_i) denote the cumulative distribution function and density function of $\mathcal{D}_i^B$ (or $\mathcal{D}_i^S$). Throughout this chapter we assume that all distributions are continuous over their support, and so the inverse cumulative functions F_i^{-1} and G_i^{-1} exist.[4] Throughout this chapter, we assume that the buyer has a constrained-additive valuation over the items (Definition 2.1). We denote $\mathcal{F}$ the downward-closed feasibility constraint of her valuation.

Mechanism and Constraints. Any mechanism in the two-sided market defined above is specified by the tuple (x, p^B, p^S) where x is the allocation rule of the mechanism and p^B, p^S are the payment rules. For every profile $(\mathbf{b}, \mathbf{s})$ and every i, $x_i(\mathbf{b}, \mathbf{s})$ is the probability that the buyer trades with seller i under profile $(\mathbf{b}, \mathbf{s})$. $p^B(\mathbf{b}, \mathbf{s})$ is the payment from the buyer and $p_i^S(\mathbf{b}, \mathbf{s})$ is the gains

4. Any discrete distribution can be made continuous by replacing each point mass a with a uniform distribution on $[a - \epsilon, a + \epsilon]$, for arbitrarily small ϵ. Thus our result applies to discrete distributions as well by losing arbitrarily small GFT.

for (or payment to) seller i. All agents in the market have linear utility functions.[5]

Gains from Trade. We aim to maximize the Gains from Trade (GFT), i.e. the gains of social welfare induced by the mechanism. Formally, given a mechanism $M = (x, p^B, p^S)$, the expected GFT of M is

$$\text{GFT}(M) = \mathbb{E}_{\mathbf{b}\sim\mathcal{D}^B,\mathbf{s}\sim\mathcal{D}^S}\left[\sum_{i=1}^n x_i(\mathbf{b},\mathbf{s})\cdot(b_i - s_i)\right].$$

We use SB-GFT to denote the optimal GFT attainable by any BIC, IR, ex-ante WBB mechanism (also known as the "second-best" mechanism). Let FB-GFT denote the maximum possible gains of social welfare among all feasible allocations (known as the "first-best"). Formally

$$\text{FB-GFT} = \mathbb{E}_{\mathbf{b}\sim\mathcal{D}^B,\mathbf{s}\sim\mathcal{D}^S}\left[\max_{S\in\mathcal{F}}\sum_{i\in S}(b_i - s_i)\right].$$

In Section 7.3, the distribution-parameterized approximation uses the parameter r, the minimum probability over all items i that the buyer's value for item i is at least seller i's cost. Formally, for every item $i \in [m]$, let $r_i = \Pr_{b_i\sim\mathcal{D}_i^B, s_i\sim\mathcal{D}_i^S}[b_i \geq s_i]$ denote the probability that the buyer's value for item i exceeds seller i's cost. Without loss of generality, assume that $r_i > 0$ for all $i \in [n]$.[6] Let $r = \min_{i\in[n]} r_i > 0$.

7.3 A Distribution-Parameterized Approximation

In this section, we present an $O(\frac{\log(1/r)}{\delta\eta})$-approximation to FB-GFT when the buyer's feasibility constraint $\mathcal{F}$ is (δ, η)-selectable, and an $O(\log(n)\cdot\log(\frac{1}{r}))$-approximation for a general constrained-additive buyer. In Section 7.3.1, we show that FB-GFT can be bounded by the sum of four separate terms. In Section 7.3.2 we show that two of the terms ("buyer surplus") are relatively

5. Without loss of generality we can assume that the mechanism will only allow the buyer to trade with a (possibly randomized) set S of sellers where $S \in \mathcal{F}$. For any trading set T, let S^* denote the utility-maximizing feasible subset, $S^* = \text{argmax}_{S\in\mathcal{F}, S\subseteq T}\sum_{i\in S} b_i$. If we only allow the buyer to trade with the sellers in S^* instead of all of T, the gains from trade of the mechanism will not decrease.

6. If $r_i = 0$ the mechanism should never trade between the buyer and seller i, and so it can remove seller i from the market. This will not decrease the GFT of the mechanism as $b_i < s_i$ with probability 1.

easy to bound using fixed posted price (FPP) mechanisms with the same prices posted on both sides. In Section 7.3.3, we consider the special case of a unit-demand buyer and bound the other two terms ("seller surplus") using FPP mechanisms combined with the prophet inequality. In Section 7.3.4, we bound the seller surplus for any selectable feasibility constraint by using a *constrained* FPP mechanism. In Section 7.3.5, we present our result for a general constrained-additive buyer.

7.3.1 Upper Bound of FB-GFT.

For every i, let $\overline{F_i} = 1 - F_i$ denote the complementary CDF of b_i. Let x_i and y_i be the $\frac{r_i}{2}$-quantile of the buyer's and seller's distribution for item i, respectively. Formally, $x_i = \overline{F_i}^{-1}(\frac{r_i}{2}), y_i = G_i^{-1}(\frac{r_i}{2})$. We first prove that $x_i \geq y_i$.

Lemma 7.1. *For every $i \in [n]$, $x_i \geq y_i$.*

Proof. Note that for every $i \in [n]$, $b_i < x_i \wedge s_i > x_i$ implies that $b_i < s_i$. We have

$$\begin{aligned}
1 - r_i = \Pr_{b_i \sim \mathcal{D}_i^B, s_i \sim \mathcal{D}_i^S}[b_i < s_i] &\geq \Pr_{b_i, s_i}[b_i < x_i \wedge s_i > x_i] \\
&= (1 - \frac{r_i}{2}) \cdot (1 - \Pr_{s_i}[s_i \leq x_i]).
\end{aligned}$$

Suppose $x_i < y_i$. Then $(1 - \frac{r_i}{2}) \cdot (1 - \Pr_{s_i}[s_i \leq x_i]) \geq (1 - \frac{r_i}{2})^2 > 1 - r_i$. This is a contradiction. Thus $x_i \geq y_i$. □

In the following upper bound, we will separate the probability space for each item i into $2\lceil \log(2/r) \rceil$ events, and then divide the GFT into buyer surplus and seller surplus terms according to the cutoff for each event. For every $\mathbf{b}, \mathbf{s}$, define the feasible set that maximizes the GFT as $S^*(\mathbf{b}, \mathbf{s}) = \text{argmax}_{S \in \mathcal{F}} \sum_{k \in S}(b_k - s_k)$, and break ties arbitrarily. Observe the following upper bound for the first-best GFT:

$$\begin{aligned}\text{FB-GFT} &= \mathbb{E}_{\mathbf{b},\mathbf{s}}[\max_{S\in\mathcal{F}}\sum_{i\in S}(b_i-s_i)^+] \\ &\le \mathbb{E}_{\mathbf{b},\mathbf{s}}\left[\sum_i (b_i-s_i)\cdot \mathbb{1}[i\in S^*(\mathbf{b},\mathbf{s})]\cdot \mathbb{1}[b_i\ge s_i \wedge s_i < x_i]\right] \\ &+ \mathbb{E}_{\mathbf{b},\mathbf{s}}\left[\sum_i (b_i-s_i)\cdot \mathbb{1}[i\in S^*(\mathbf{b},\mathbf{s})]\cdot \mathbb{1}[b_i\ge s_i \ge y_i]\right],\end{aligned}$$

where the inequality holds because $x_i \ge y_i$ for all i. We refer to the two terms of RHS as ① and ② accordingly. We first consider term ①. For every $i\in[n], j\in 1,2,\ldots,\lceil\log(\frac{2}{r})\rceil$, let $\theta_{ij}=\overline{F_i}^{-1}(\frac{1}{2^j})$. Let E_{ij} be the event that $\overline{F_i}^{-1}(\frac{1}{2^{j-1}})\le s_i\le \overline{F_i}^{-1}(\frac{1}{2^j})\wedge b_i\ge \overline{F_i}^{-1}(\frac{1}{2^{j-1}})$. Then we have

① $\le \sum_{j=1}^{\lceil\log(\frac{2}{r})\rceil} \mathbb{E}_{\mathbf{b},\mathbf{s}}\left[\sum_i (b_i-s_i)^+\cdot \mathbb{1}[i\in S^*(\mathbf{b},\mathbf{s})\wedge E_{ij}]\right]$.

As discussed in Section 7.1.1, in order to bound the benchmark with fixed posted price mechanisms, we will consider a more restrictive event $\overline{E}_{ij}$ and show that the GFT contribution from event $\overline{E}_{ij}$ is at least half of the GFT contribution from E_{ij}.

Lemma 7.2. *For every i,j, let $\overline{E}_{ij}$ be the event that $\overline{F_i}^{-1}(\frac{1}{2^{j-1}})\le s_i\le \overline{F_i}^{-1}(\frac{1}{2^j})\wedge b_i\ge \overline{F_i}^{-1}(\frac{1}{2^j})$. Then the following inequality holds for every $j=1,\ldots,\lceil\log(2/r)\rceil$:*

$$\begin{aligned}&\mathbb{E}_{\boldsymbol{b},\boldsymbol{s}}\left[\sum_i (b_i-s_i)^+\cdot \mathbb{1}[i\in S^*(\boldsymbol{b},\boldsymbol{s})\wedge E_{ij}]\right] \\ \le& 2\cdot \mathbb{E}_{\boldsymbol{b},\boldsymbol{s}}\left[\sum_i (b_i-s_i)^+\cdot \mathbb{1}[i\in S^*(\boldsymbol{b},\boldsymbol{s})\wedge \overline{E}_{ij}]\right].\end{aligned}$$

Moreover,

$$\begin{aligned}① \le 2\cdot &\sum_{j=1}^{\lceil\log(2/r)\rceil}\mathbb{E}_{\boldsymbol{b},\boldsymbol{s}}\left[\max_{S\in\mathcal{F}}\sum_{i\in S}\left\{(b_i-\theta_{ij})^+\cdot\mathbb{1}[s_i\le\theta_{ij}]\right\}\right] \\ +2\cdot &\sum_{j=1}^{\lceil\log(2/r)\rceil}\mathbb{E}_{\boldsymbol{b},\boldsymbol{s}}\left[\max_{S\in\mathcal{F}}\sum_{i\in S}\left\{(\theta_{ij}-s_i)^+\cdot\mathbb{1}[b_i\ge\theta_{ij}]\right\}\right].\end{aligned}$$

We will refer to the two terms of RHS as ③ and ④ accordingly.

Readers may notice that $\Pr[\overline{E}_{ij}]=\frac{1}{2}\cdot\Pr[E_{ij}]$. However, this alone does not prove the first statement of Lemma 7.2, since both the indicator $\mathbb{1}[E_{ij}]$ and the contributed GFT $(b_i - s_i)^+ \cdot \mathbb{1}[i \in S^*(\mathbf{b}, \mathbf{s})]$ depend on the realization of b_i, s_i. In Lemma 7.2 we show that the two random variables are *positively correlated* with respect to b_i, which allows us to prove the first statement. The second statement follows from the fact that $(b_i - s_i)^+ \leq (b_i - \theta_{ij})^+ + (\theta_{ij} - s_i)^+$ for every b_i, s_i, and that $S^*(\mathbf{b}, \mathbf{s}) \in \mathcal{F}$ for every $\mathbf{b}, \mathbf{s}$.

In Lemma 7.3, we bound term ② in a similar way. The proof of Lemmas 7.2 and 7.3 can be found in Appendix D.1.

Lemma 7.3. *For every $i \in [n]$ and $j = 1, \ldots, \lceil \log(2/r) \rceil$, let $\theta'_{ij} = G_i^{-1}(\frac{1}{2^j})$. Then*

$$\text{②} \leq 2 \cdot \sum_{j=1}^{\lceil \log(2/r) \rceil} \mathbb{E}_{\boldsymbol{b},\boldsymbol{s}}\left[\max_{S \in \mathcal{F}} \sum_{i \in S} \left\{(b_i - \theta'_{ij})^+ \cdot \mathbb{1}[s_i \leq \theta'_{ij}]\right\}\right] + 2 \cdot \sum_{j=1}^{\lceil \log(2/r) \rceil} \mathbb{E}_{\boldsymbol{b},\boldsymbol{s}}\left[\max_{S \in \mathcal{F}} \sum_{i \in S} \left\{(\theta'_{ij} - s_i)^+ \cdot \mathbb{1}[b_i \geq \theta'_{ij}]\right\}\right].$$

We will refer to the two terms of RHS as ⑤ and ⑥ accordingly.

We refer to terms ③ and ⑤ as buyer surplus, and ④ and ⑥ as seller surplus. In the rest of this section we will bound each term separately.

7.3.2 Bounding Buyer Surplus.

We bound terms ③ and ⑤ using fixed posted price mechanisms. Let GFT_{FPP} denote the optimal GFT among all fixed posted price mechanisms. Recall that our market is not symmetric: a single multi-dimensional buyer with a feasibility constraint faces multiple single-dimensional sellers. As a result, even for the general constrained-additive buyer, bounding buyer surplus is fairly straightforward using fixed price mechanisms that set $\theta_i^S = \theta_i^B = \theta_{ij}$ (or $\theta_i^S = \theta_i^B = \theta'_{ij}$) for each term.

Lemma 7.4. *For any $\{p_i\}_{i \in [n]} \in \mathbb{R}_+^n$,*

$$\mathbb{E}_{\boldsymbol{b},\boldsymbol{s}}\left[\max_{S \in \mathcal{F}} \sum_{i \in S} \{(b_i - p_i)^+ \cdot \mathbb{1}[s_i \leq p_i]\}\right] \leq \text{GFT}_{\text{FPP}}.$$

Thus both ③ and ⑤ are upper bounded by $O(\log(\frac{1}{r}))\cdot \text{GFT}_{\text{FPP}}$.

Proof. Consider the fixed posted price mechanism $\mathcal{M}$ with $\theta_i^S=\theta_i^B=p_i$. For every $\mathbf{s}$, let $A(\mathbf{s})=\{i\in[n]\mid s_i\leq p_i\}$ be the set of available items. Then the buyer will choose the best set $S\subseteq A(\mathbf{s}), S\in\mathcal{F}$ that maximizes $\sum_{i\in S}(b_i-p_i)^+$ (and not buy any item if $b_i-p_i\leq 0$ for all i). Thus the gains from trade $\sum_{i\in S}(b_i-s_i)$ is at least $\sum_{i\in S}(b_i-p_i)^+\geq 0$. We have

$$\begin{aligned}\text{GFT}(\mathcal{M})\geq&\mathbb{E}_{\mathbf{b},\mathbf{s}}\left[\max_{S\subseteq A(\mathbf{s}),S\in\mathcal{F}}\sum_{i\in S}(b_i-p_i)^+\right]\\=&\mathbb{E}_{\mathbf{b},\mathbf{s}}\left[\max_{S\in\mathcal{F}}\sum_{i\in S}\left\{(b_i-p_i)^+\cdot\mathbb{1}[s_i\leq p_i]\right\}\right].\end{aligned}$$

To bound terms ③ and ⑤, just apply the above inequality with $p_i=\theta_{ij}$ (or θ'_{ij}). □

7.3.3 Bounding Seller Surplus for One Unit-Demand Buyer.

In the remainder of this section, we will bound the seller surplus terms (④ and ⑥). As a warm-up, we first focus on the case where the buyer is unit-demand, i.e. the buyer is only interested in at most one item. Here, the prophet inequality suffices for our bound.

Lemma 7.5. *When the buyer is unit-demand, for any $\{p_i\}_{i\in[n]}\in\mathbb{R}_+^n$, we have*

$$\mathbb{E}_{\boldsymbol{b},\boldsymbol{s}}\left[\max_i\{(p_i-s_i)^+\cdot\mathbb{1}[b_i\geq p_i]\}\right]\leq 2\cdot\text{GFT}_{\text{FPP}}.$$

Hence terms ④ and ⑥ are both upper-bounded by $O(\log(\frac{1}{r}))\cdot\text{GFT}_{\text{FPP}}$.

Proof. For every i, let $v_i=(p_i-s_i)^+\cdot\mathbb{1}[b_i\geq p_i]$ be a random variable that depends on b_i and s_i. Let $\mathbf{v}=\{v_i\}_{i\in[n]}$. Let V_i be the distribution of v_i where $b_i\sim\mathcal{D}_i^B, s_i\sim\mathcal{D}_i^S$, and $V=\times_{i=1}^n V_i$ be the distribution of $\mathbf{v}$. Then the LHS of the inequality in the Lemma statement is equal to $\mathbb{E}_{\mathbf{v}\sim V}[\max_i v_i]$.

Consider any threshold $\xi>0$. Observe that $v_i\geq\xi$ if and only if $b_i\geq p_i\wedge p_i-s_i\geq\xi$. Consider the fixed posted price mechanism $\mathcal{M}$ with $\theta_i^B=p_i$ and $\theta_i^S=p_i-\xi$ for every $i\in[n]$. Whenever the buyer purchases some item i, we must have $b_i\geq p_i$ (the buyer buys) and $s_i\leq p_i-\xi$ (the seller sells), and the contributed GFT satisfies $b_i - s_i \geq p_i - s_i \geq \xi$. In addition, the buyer will purchase *some* item if and only if there exists some i such that $v_i \geq \xi$. Therefore we can apply the prophet inequality [KW12, KS78, SC$^+$84] with threshold $\xi = \frac{1}{2} \cdot \mathbb{E}_{\mathbf{v}\sim V}[\max_i v_i]$ to ensure that the GFT of mechanism $\mathcal{M}$ is at least $\frac{1}{2}\mathbb{E}_{\mathbf{v}\sim V}[\max_i v_i]$. □

7.3.4 Bounding Seller Surplus with Selectability.

In this subsection we bound terms ④ and ⑥ for a more general class of constraints $\mathcal{F}$ using a variant of a fixed posted price (FPP) mechanism which we call *constrained* FPP. In the variant, the mechanism determines a (randomized) subconstraint $\mathcal{F}' \subseteq \mathcal{F}$ upfront. Then the buyer is only allowed to take a feasible set in $\mathcal{F}'$ (among all items that the sellers agree to sell at prices $\{\theta_i^S\}_{i\in[n]}$) and pays the price θ_i^B for each item she takes.[7] Let GFT_{CFPP} denote the the optimal GFT among all constrained FPP mechanisms.[8] Since all of the posted prices as well as the subconstraint are independent from the agents' reported profiles, the mechanism is DSIC and ex-post IR. The mechanism is also ex-post WBB since $\theta_i^B \geq \theta_i^S$ for all $i \in [n]$.

To prove our result, we prove the following lemma that is adapted from [FSZ16] and connects (δ, η)-selectability to constrained FPP mechanisms. Once again, the OCRS gives us both a GFT guarantee and a mechanism: variables v_i correspond to the bound on seller surplus, buyer item prices are $\{p_i\}_{i\in[n]}$, seller prices are $\{p_i - \xi_i\}_{i\in[n]}$, and the subconstraint is suggested by the OCRS.

Lemma 7.6. *Suppose there exists a (δ, η)-selectable greedy OCRS π for the polytope $P_{\mathcal{F}}$, for some $\delta, \eta \in (0, 1)$. Fix any $\{p_i\}_{i\in[n]} \in \mathbb{R}_+^n$. For every $i \in [n]$, let $v_i = (p_i - s_i)^+ \cdot \mathbb{1}[b_i \geq p_i]$. For any $\boldsymbol{q} \in P_{\mathcal{F}}$ that satisfies $q_i \leq \Pr_{b_i,s_i}[b_i \geq p_i > s_i]$ $\forall i$, let $\xi_i = p_i - G_i^{-1}(q_i/\Pr[b_i \geq p_i])$.[9] We have*

$$\textstyle\sum_i \mathbb{E}_{b_i,s_i}\left[v_i \cdot \mathbb{1}[v_i \geq \xi_i]\right] \leq \frac{1}{\delta\eta} \cdot \text{GFT}_{\text{CFPP}}.$$

7. Throughout this paper, we assume for simplicity that the buyer will purchase item i when $b_i = \theta_i^B$ as long as the bundle remains feasible after including i. Without this tie-breaking rule, one can simply decrease the posted price for each item by an arbitrarily small value ϵ, and the loss of GFT will be arbitrarily small.

8. Note that FPP is a subclass of constrained FPP, and therefore $\text{GFT}_{\text{FPP}} \leq \text{GFT}_{\text{CFPP}}$.

9. When $q_i \leq \Pr_{b_i,s_i}[b_i \geq p_i > s_i]$, $q_i/\Pr[b_i \geq p_i] \leq 1$. Thus ξ_i is well-defined.

Moreover, there exists a choice of $\boldsymbol{q}$ *such that*

$$\mathbb{E}_{\boldsymbol{b},\boldsymbol{s}}\left[\max_{S\in\mathcal{F}}\sum_{i\in S}\left\{(p_i-s_i)^+\cdot\mathbb{1}[b_i\geq p_i]\right\}\right]$$
$$\leq \sum_i \mathbb{E}_{b_i,s_i}\left[v_i\cdot\mathbb{1}[v_i\geq\xi_i]\right]\leq\frac{1}{\delta\eta}\cdot\mathrm{GFT}_{\mathrm{CFPP}}.$$

Proof of Lemma 7.6: Let V_i denote the distribution of v_i when $b_i\sim\mathcal{D}_i^B, s_i\sim\mathcal{D}_i^S$. Let $\mathbf{v}=\{v_i\}_{i\in[n]}$. Let $\hat{\mathbf{q}}$ be a scaled-down vector of $\mathbf{q}$ such that $\hat{q}_i=\delta\cdot q_i$ for every $i\in[n]$ and $\hat{\xi}_i=p_i-G_i^{-1}(\hat{q}_i/\Pr[b_i\geq p_i])$. This is also well-defined since $\hat{q}_i<q_i\leq\Pr[b_i\geq p_i]$. As $q\in P_{\mathcal{F}}$, then $\hat{q}\in\delta\cdot P_{\mathcal{F}}$. Consider the constrained FPP mechanism $\mathcal{M}$ with buyer posted prices $\{p_i\}_{i\in[n]}$, seller posted prices $\{p_i-\hat{\xi}_i\}_{i\in[n]}$, and subconstraint $\mathcal{F}_{\pi,\hat{q}}\in\mathcal{F}$ stated in Definition 2.5.

Fix any item $i\in[n]$. We say item i as active if $v_i\geq\hat{\xi}_i$. Similarly to Section 7.3.3, $v_i\geq\hat{\xi}_i$ if and only if $b_i\geq p_i\wedge s_i\leq p_i-\hat{\xi}_i$. That is, i is active if and only if item i is on the market and the buyer can afford it, which by choice of $\hat{\xi}_i$ happens independently across all i with probability $\Pr_{v_i}[v_i\geq\hat{\xi}_i]=\Pr_{b_i,s_i}[p_i-s_i\geq\hat{\xi}_i\wedge b_i\geq p_i]=\hat{q}_i$.

Then for any $\mathbf{v}$, the set of active items is $R(\mathbf{v})=\{j\in[n]:\ v_j\geq\hat{\xi}_j\}$. By (δ,η)-selectability (Definition 2.5) and the fact that $\hat{q}\in\delta\cdot P_{\mathcal{F}}$, we have

$$\Pr_{\pi,\mathbf{v}}[S\cup\{i\}\in\mathcal{F}_{\pi,\hat{q}},\forall S\subseteq R(\mathbf{v}),S\in\mathcal{F}_{\pi,\hat{q}}]\geq\eta. \tag{7.1}$$

Note that for the sets $S\in\mathcal{F}_{\pi,\hat{q}}$ that have $i\in S$, then $S\cup\{i\}\in\mathcal{F}_{\pi,\hat{q}}$ with probability 1. Thus, if we require $S\subseteq R(\mathbf{v})\backslash\{i\}$ instead, it can not be that $i\in S$, and so the following LHS occurs with equal probability, allowing us to rewrite inequality (7.1) as follows:

$$\Pr_{\pi,\mathbf{v}}[S\cup\{i\}\in\mathcal{F}_{\pi,\hat{q}},\forall S\subseteq R(\mathbf{v})\backslash\{i\},S\in\mathcal{F}_{\pi,\hat{q}}]\geq\eta. \tag{7.2}$$

For any $\mathbf{v}_{-i}$, let $R_i(\mathbf{v}_{-i})=\{j\neq i:\ v_j\geq\hat{\xi}_j\}$. Then inequality (7.2) is equivalent to

$$\Pr_{\pi,v_{-i}}[S\cup\{i\}\in\mathcal{F}_{\pi,\hat{q}},\forall S\subseteq R_i(\mathbf{v}_{-i}),S\in\mathcal{F}_{\pi,\hat{q}}]\geq\eta.$$

Define event $A_i=\{\mathbf{v}_{-i}:\ S\cup\{i\}\in\mathcal{F}_{\pi,\hat{q}},\forall S\subseteq R_i(\mathbf{v}_{-i}),S\in\mathcal{F}_{\pi,\hat{q}}\}$. We will argue that item i

must be in the buyer's favorite bundle S^* when both of the following conditions are satisfied: (i) $v_i \geq \hat{\xi}_i$, and (ii) event A_i happens. Note that in $\mathcal{M}$, the set of items in the market is $T = \{j : s_j \leq p_j - \hat{\xi}_j\}$, thus $S^* = \text{argmax}_{S\subseteq T, S\in\mathcal{F}_{\pi,\hat{q}}} \sum_{j\in S}(b_j - p_j)$. Suppose by way of contradiction that both conditions are satisfied but $i \notin S^*$. Clearly, for every $j \in S^*$, we have $b_j \geq p_j$, otherwise removing j from S^* will give the buyer greater utility. In addition, we have $s_j \leq p_j - \hat{\xi}_j$, so $S^* \subseteq R(\mathbf{v})$. By definition, S^* must lie in $\mathcal{F}_{\pi,\hat{q}}$. Since event A_i occurs, then $S^* \cup \{i\} \in \mathcal{F}_{\pi,\hat{q}}$. As $v_i \geq \hat{\xi}_i$, this implies that $b_i \geq p_i$. Thus adding i to S^* keeps the set feasible and does not decrease the buyer's utility $\sum_{j\in S^*}(b_j - p_j)$. Thus $i \in S^*$ (see footnote 5). This is a contradiction.

Note that condition (i) and (ii) are independent. Thus for every b_i and s_i such that $b_i \geq p_i \wedge s_i \leq p_i - \hat{\xi}_i$ (or equivalently $v_i \geq \hat{\xi}_i$), the expected GFT of item i over b_{-i}, s_{-i} is at least

$$\Pr[A_i] \cdot (b_i - s_i) \geq \eta \cdot (p_i - s_i) = \eta \cdot v_i.$$

Thus

$$\text{GFT}(\mathcal{M}) \geq \eta \cdot \sum_i \mathbb{E}_{v_i\sim V_i}[v_i \cdot \mathbb{1}[v_i \geq \hat{\xi}_i]] \geq \delta\eta \cdot \sum_i \mathbb{E}_{v_i\sim V_i}[v_i \cdot \mathbb{1}[v_i \geq \xi_i]],$$

where the last inequality is because for every i, we have $\mathbb{E}[v_i | v_i \geq \hat{\xi}_i] \geq \mathbb{E}[v_i | v_i \geq \xi_i]$ and $\Pr[v_i \geq \hat{\xi}_i] = \hat{q}_i = \delta \cdot \Pr[v_i \geq \xi_i]$.

For the second inequality stated in the lemma, note that

$$\mathbb{E}_{\mathbf{b},\mathbf{s}}\left[\max_{S\in\mathcal{F}} \sum_{i\in S} \left\{(p_i - s_i)^+ \cdot \mathbb{1}[b_i \geq p_i]\right\}\right] = \mathbb{E}_{\mathbf{v}}\left[\max_{S\in\mathcal{F}} \sum_{i\in S} v_i\right].$$

For every $\mathbf{v}$, let $\hat{S}(\mathbf{v}) = \text{argmax}_{S\in\mathcal{F}} \sum_{i\in S} v_i$, and break ties in favor of the set with smaller size. For every i, let $q_i = \Pr_{\mathbf{v}}[i \in \hat{S}(\mathbf{v})]$ be the probability that i is in the maximum weight feasible set. We have that $\mathbf{q} = \{q_i\}_{i\in[n]} \in P_{\mathcal{F}}$. Also for every i, $q_i = \Pr_{\mathbf{v}}[i \in \hat{S}(\mathbf{v})] \leq \Pr[v_i > 0] = \Pr[b_i \geq p_i > s_i]$. Moreover,

$$\mathbb{E}_{\mathbf{v}}\left[\max_{S\in\mathcal{F}}\sum_{i\in S} v_i\right] = \sum_{i\in[n]}\mathbb{E}_{\mathbf{v}}\left[v_i\cdot\mathbb{1}[i\in\hat{S}(\mathbf{v})]\right] \le \sum_{i\in[n]}\mathbb{E}_{v_i\sim V_i}\left[v_i\cdot\mathbb{1}[v_i\ge\xi_i]\right]$$

The inequality follows from the fact that for every i, both sides integrate the random variable v_i with a total probability mass q_i, while the right hand side integrates v_i at the top q_i-quantile. □

For each j in the summation, choose p_i from Lemma 7.6 to be θ_{ij} (or θ'_{ij}). Then both terms ④ and ⑥ are bounded by $\frac{\log(1/r)}{\delta\eta}\cdot \text{GFT}_{\text{CFPP}}$. Theorem 7.1 then follows directly from Lemmas 7.2, 7.3, 7.4, and 7.6.

Theorem 7.1. *Suppose the buyer's feasibility constraint $\mathcal{F}$ is (δ,η)-selectable for some $\delta,\eta\in(0,1)$. Then* FB-GFT $\le O(\frac{\log(1/r)}{\delta\eta})\cdot\text{GFT}_{\text{CFPP}}$.

Feldman et al. [FSZ16] show that many natural constraints—including matroids, matchings, knapsack, and their compositions—are (δ,η)-selectable for some constants δ and η (see Section 2.7). For all of these, Theorem 7.1 implies that GFT_{CFPP} is an $O(\log(1/r))$-approximation to FB-GFT.

Corollary 7.1. *Suppose the buyer's feasibility constraint is $\mathcal{F}=\bigcap_{t=1}^{d}\mathcal{F}_t$ for some constant d, where each $\mathcal{F}_t$ is a matroid, matching constraint, or knapsack constraint. Then* FB-GFT $\le O(\log(\frac{1}{r}))\cdot \text{GFT}_{\text{CFPP}}$.

Proof of Corollary 7.1: Pick any constant $\delta\in(0,\frac{1}{2})$. By Lemma 2.3 and 2.4, there exists some constant $\eta\in(0,1)$ such that there exists an efficient, (δ,η)-selectable greedy OCRS for $P_{\mathcal{F}}$. Then the result follows from Theorem 7.1.□

7.3.5 General Constrained-Additive Buyer.

In this section, we consider the case of a general constrained-additive buyer, and prove an $O(\log(n)\cdot\log(1/r))-$approximation to FB-GFT using constrained FPP mechanisms. Note that Lemmas 7.2, 7.3, and 7.4 still hold in this setting. It is sufficient to bound the seller surplus term with GFT_{CFPP}.

Throughout this section, we will use the following variant of FPP mechanisms: Other than posted prices, the mechanism also determines an integer $h>0$ upfront. The buyer can purchase any set of items of size at least h by paying the posted prices for each item in the set; otherwise, she leaves with nothing. This is a subclass of constrained FPP, with subconstraint $\mathcal{F}' = \{S \mid S \in \mathcal{F} \wedge |S| \geq h\} \subseteq \mathcal{F}$.[10]

Lemma 7.7. *For any* $\{p_i\}_{i\in[n]} \in \mathbb{R}^n_+$,

$$\begin{aligned}\mathcal{A} =& \mathbb{E}_{\boldsymbol{b},\boldsymbol{s}}\left[\max_{T\in\mathcal{F}} \textstyle\sum_{i\in T}\{(p_i - s_i)^+ \cdot \mathbb{1}[b_i \geq p_i]\}\right] \\ \leq& O(\log(n)) \cdot \mathrm{GFT}_{\mathrm{CFPP}}.\end{aligned}$$

Hence terms ④ *and* ⑥ *are both upper-bounded by* $O(\log(n) \cdot \log(\frac{1}{r})) \cdot \mathrm{GFT}_{\mathrm{CFPP}}$.

For every $i \in [n]$, again construct random variables $v_i = (p_i - s_i)^+ \cdot \mathbb{1}[b_i \geq p_i]$. The main issue here is that in an FPP mechanism, say with posted prices $\theta_i^B = \theta_i^S = p_i$, the buyer will pick the maximum weight feasible set (among all items that sellers are willing to sell) according to weight $b_i - p_i$ (her utility). However, this might be far from the set used in the benchmark, i.e. the maximum weight feasible set according to weight $p_i - s_i$. In the previous section (when the constraint $\mathcal{F}$ had selectability), by setting different prices for both sides and adding a more restrictive constraint, we guaranteed that if both the buyer and seller accept the posted prices for some item, then the buyer would purchase this item with at least constant probability. For general downward-closed $\mathcal{F}$, it is unclear how to achieve this property with a constrained FPP mechanism.

For every $\mathbf{b}, \mathbf{s}$, let $T^*(\mathbf{b}, \mathbf{s}) = \operatorname{argmax}_{T\in\mathcal{F}} \sum_{i\in T} v_i$ be the optimal set used in the benchmark. We divide $\mathcal{A}$ into three terms according to the value of v_i when i is in this optimal set: $v_i < \mathcal{A}/2n$, $v_i \in [\mathcal{A}/2n, 2n\mathcal{A}]$ and $v_i > 2n\mathcal{A}$. Denote the three terms $\mathcal{A}_S, \mathcal{A}_M, \mathcal{A}_L$ accordingly. First we notice that the contribution of $\mathcal{A}_S$ is at most a constant fraction of $\mathcal{A}$, as $\mathcal{A}_S = \mathbb{E}_{\mathbf{b},\mathbf{s}}\left[\sum_i v_i \cdot \mathbb{1}[i \in T^*(\mathbf{b}, \mathbf{s}) \wedge v_i < \frac{\mathcal{A}}{2n}]\right] < \mathbb{E}_{\mathbf{b},\mathbf{s}}\left[\frac{\mathcal{A}}{2n} \cdot n\right] = \frac{\mathcal{A}}{2}$.

For $\mathcal{A}_L$, in Lemma 7.9, we first prove that $\Pr_{b_i,s_i}[b_i \geq p_i \wedge p_i - s_i \geq 2n\mathcal{A}] \leq \frac{1}{2n}$ holds for every i. This implies that in a standard FPP mechanism (where $h = 1$) with $\theta_i^B = p_i, \theta_i^S = (p_i - 2n\mathcal{A})^+$ for all i, the buyer purchases each item i with probability at least $\frac{1}{2}$ if both the buyer and seller i accept the posted prices. First, we will need Lemma 7.8.

10. If $h = 1$, the mechanism becomes a standard FPP mechanism without any subconstraint $\mathcal{F}'$.

Lemma 7.8. *Given any constrained FPP mechanism $\mathcal{M}$ with posted prices $\{\theta_i^B\}_{i\in[n]}$, $\{\theta_i^S\}_{i\in[n]}$ and $h = 1$, suppose $\sum_i \Pr[b_i \geq \theta_i^B \wedge s_i \leq \theta_i^S] \leq \frac{1}{2}$. Then*

$$\mathrm{GFT}(\mathcal{M}) \geq \tfrac{1}{2} \textstyle\sum_i \mathbb{E}_{b_i,s_i}\left[(b_i - s_i) \cdot \mathbb{1}[b_i \geq \theta_i^B \wedge s_i \leq \theta_i^S]\right].$$

Proof. For any item i, the buyer will purchase item i if both of the following events happen:

1. $b_i \geq \theta_i^B$ and $s_i \leq \theta_i^S$;
2. For all items $k \neq i$, either $s_k > \theta_k^S$ or $b_k < \theta_k^B$.

By the union bound, the second event happens with probability at least $1 - \sum_{k\neq i} \Pr[b_i \geq \theta_i^B \wedge s_i \leq \theta_i^S] \geq \frac{1}{2}$. Since both events are independent, we have

$$\mathrm{GFT}(\mathcal{M}) \geq \frac{1}{2} \sum_i \mathbb{E}_{b_i,s_i}\left[(b_i - s_i) \cdot \mathbb{1}[b_i \geq \theta_i^B \wedge s_i \leq \theta_i^S]\right].$$

□

Lemma 7.9.

$$\mathcal{A}_L = \mathbb{E}_{\boldsymbol{b},\boldsymbol{s}}\left[\sum_i v_i \mathbb{1}[i \in T^*(\boldsymbol{b},\boldsymbol{s}) \wedge v_i > 2n\mathcal{A}]\right] \leq 2 \cdot \mathrm{GFT}_{\mathrm{FPP}}.$$

Proof. Consider the FPP mechanism with $\theta_i^B = p_i, \theta_i^S = (p_i - 2n\mathcal{A})^+$ for all i (and $h = 1$).

Note that for every $i \in [n]$, it must hold that $\Pr_{b_i,s_i}[b_i \geq p_i \wedge p_i - s_i \geq 2n\mathcal{A}] \leq \frac{1}{2n}$. In fact,

$$\mathcal{A} \geq \mathbb{E}_{b_i,s_i}[(p_i - s_i)^+ \cdot \mathbb{1}[b_i \geq p_i]] \geq 2n\mathcal{A} \cdot \Pr_{b_i,s_i}[b_i \geq p_i \wedge p_i - s_i \geq 2n\mathcal{A}].$$

Thus by Lemma 7.8 and the fact that $b_i - s_i \geq p_i - s_i$ when $b_i \geq p_i$, we have

$$\mathrm{GFT}_{\mathrm{FPP}} \geq \frac{1}{2} \sum_i \mathbb{E}\left[(p_i - s_i) \cdot \mathbb{1}[b_i \geq p_i \wedge p_i - s_i \geq 2n\mathcal{A}]\right] \geq \frac{1}{2} \cdot \mathcal{A}_L.$$

□

In Lemma 7.10 we bound $\mathcal{A}_M$, which is the primary challenge for this approximation.

Lemma 7.10. $\mathcal{A}_M \le O(\log(n)) \cdot \text{GFT}_{\text{CFPP}}$.

Proof. We further divide the interval $[\mathcal{A}/2n, 2n\mathcal{A}]$ into $O(\log(n))$ buckets, where in each bucket k, v_i falls in the range $[L_k, 2L_k]$ for some L_k. Formally, for any $k \in \{1, 2, ..., \lceil 2\log(n) + 2\rceil\}$, let $L_k = 2^k \cdot \frac{\mathcal{A}}{4n}$. We have

$$\mathcal{A}_M \le \sum_{k=1}^{\lceil 2\log(n)+2\rceil} \mathbb{E}_{\mathbf{b},\mathbf{s}}\left[\sum_i v_i \mathbb{1}[i \in T^*(\mathbf{b},\mathbf{s}) \wedge v_i \in [L_k, 2L_k]]\right].$$

In the rest of the proof, we will show that for any k, there exists some constant $c > 0$ such that

$$\mathbb{E}_{\mathbf{b},\mathbf{s}}\left[\sum_i v_i \mathbb{1}[i \in T^*(\mathbf{b},\mathbf{s}) \wedge v_i \in [L_k, 2L_k]]\right] \le c \cdot \text{GFT}_{\text{CFPP}}.$$

Fix any k. For every $i \in [n]$, let $t_i^{(k)} = \frac{v_i}{2L_k} \cdot \mathbb{1}[L_k \le v_i \le 2L_k]$. This is a random variable in $[\frac{1}{2}, 1]$. Note that all random variables $t = \{t_i^{(k)}\}_{i\in[n]}$ are independent. Let $Z(t) = \max_{T\in\mathcal{F}} \sum_{i\in T} t_i^{(k)}$. Then the contribution to $\mathcal{A}_L$ from values in this range is bounded by the expectation of the random variable $Z(t)$:

$$\mathbb{E}_{\mathbf{b},\mathbf{s}}\left[\sum_i v_i \cdot \mathbb{1}[i \in T^*(\mathbf{b},\mathbf{s}) \wedge v_i \in [L_k, 2L_k]]\right] \le \mathbb{E}_{\mathbf{b},\mathbf{s}}\left[\max_{S\in\mathcal{F}} \sum_{i\in S} v_i \cdot \mathbb{1}[L_k \le v_i \le 2L_k]\right] = 2L_k \mathbb{E}_t[Z(t)].$$

Now consider the constrained FPP mechanism with $\theta_i^B = p_i$ and $\theta_i^S = (p_i - L_k)^+$ for every i (the threshold h is determined later). Then in the mechanism, whenever the buyer purchases an item, the contributed GFT is at least L_k. Thus it is sufficient to show that the expected size of the purchasing set is at least a constant factor of $\mathbb{E}[Z(t)]$. Note that $Z(t)$ is a random variable on t, which is the maximum weight feasible set over n independent random variables in $[0, 1]$. In Lemma 7.11, we prove that $Z(t)$ concentrates near its mean. The proof is postponed to Section 7.3.6.

Lemma 7.11. *For any* $c \in (0, 1)$,

$$\Pr_t[Z(t) \ge c \cdot \mathbb{E}[Z(t)]] \ge \frac{(1-c)^2}{1 + 1/\mathbb{E}[Z(t)]}.$$

We first suppose that $\mathbb{E}[Z(t)] \geq \frac{1}{4}$. By applying Lemma 7.11 with $c = \frac{1}{2}$, we get

$$\Pr_t\left[Z(t) \geq \frac{\mathbb{E}[Z(t)]}{2}\right] \geq \frac{1}{20}.$$

Let $h = \max\left\{\lfloor\frac{\mathbb{E}[Z(t)]}{2}\rfloor, 1\right\}$. In mechanism $\mathcal{M}_k$, note that for every i, $t_i^{(k)} > 0$ implies that item i is on the market and that the buyer can afford it. With probability at least $\frac{1}{20}$, $Z(t) \geq h$, which implies that the item set $\{i \mid i \in \text{argmax}_{S\in\mathcal{F}} \sum_{i\in S} t_i^{(k)} \wedge t_i^{(k)} > 0\}$ is a feasible set of size at least h. (Recall that all $t_i^{(k)}$ are in $[\frac{1}{2}, 1]$). In this scenario, the buyer will purchase a set of items of size at least h. For every item i she purchases, the contributed GFT is $b_i - s_i \geq \theta_i^B - \theta_i^S = L_k$. Thus, $\text{GFT}(\mathcal{M}_k) \geq \frac{1}{20} \cdot h \cdot L_k$. Readers who are familiar with mechanism design may notice that the role of the size threshold h is similar to an "entry fee" in the posted price mechanism in auctions [BILW14, CDW16, CZ17, CM16, RW15, Yao15], though the buyer doesn't have to pay extra money to attend the auction. It guarantees that the buyer will purchase at least h items when she can afford it, as otherwise she gets no utility.

When $\mathbb{E}[Z(t)] \geq \frac{1}{4}$, we have $h \geq \frac{\mathbb{E}[Z(t)]}{4}$. Thus

$$\mathbb{E}_{\mathbf{b},\mathbf{s}}\left[\sum_i v_i \mathbb{1}[i \in T^*(\mathbf{b},\mathbf{s}) \wedge v_i \in [L_k, 2L_k]]\right] \leq 160 \cdot \text{GFT}_{\text{CFPP}}.$$

Now we consider the case where $\mathbb{E}[Z(t)] < \frac{1}{4}$. For every i, let $q_i = \Pr[t_i^{(k)} > 0] = \Pr_{b_i,s_i}[b_i \geq p_i \wedge p_i - s_i \in [L_k, 2L_k]]$. Then it holds that $\Pr[\forall i, t_i^{(k)} = 0] = \prod_{i=1}^n (1 - q_i) > \frac{1}{2}$. This is because if there exists i such that $t_i^{(k)} > 0$, then $Z(t) = \max_{T\in\mathcal{F}} \sum_{i\in T} t_i^{(k)} \geq \frac{1}{2}$ as $t_i^{(k)} \in [\frac{1}{2}, 1]$ for every i. Thus if $\Pr[\forall i, t_i^{(k)} = 0] \leq \frac{1}{2}$, then $\mathbb{E}[Z(t)] \geq \frac{1}{4}$, which leads to a contradiction.

Consider the constrained FPP mechanism $\mathcal{M}$ with $\theta_i^B = p_i$, $\theta_i^S = (p_i - L_k)^+$, and $h = 1$. For every i, define event $E_i = \{t \mid t_i^{(k)} > 0 \wedge t_j^{(k)} = 0, \forall j \neq i\}$. Note that $t_i^{(k)} > 0$ implies that seller i accepts price θ_i^S and the buyer can afford item i. Under event E_i, there is at least one item on the market that the buyer can afford, i.e. item i. Thus the buyer must purchase *some* item j on the market that she can afford (possibly item i). For this item j, we have $b_j \geq \theta_j^B$ and $s_j \leq \theta_j^S$. Thus the contributed GFT is at least $b_j - s_j \geq p_j - s_j \geq L_k$. Since all E_is are disjoint events, we have

$\text{GFT}(\mathcal{M}) \geq \sum_i \Pr[E_i] \cdot L_k = L_k \cdot \sum_i q_i \cdot \prod_{j \neq i}(1-q_j) \geq L_k \cdot \sum_i q_i \cdot \prod_j (1-q_j) > \frac{1}{2} L_k \cdot \sum_i q_i$, where the equality uses the fact that all $t_i^{(k)}$s are independent. On the other hand, since $t_i^{(k)} \leq 1$ for any i, $\mathbb{E}[Z(t)] \leq \mathbb{E}\left[\sum_i t_i^{(k)} \cdot \mathbb{1}[t_i^{(k)} > 0]\right] \leq \sum_i q_i$. Thus

$$\mathbb{E}_{\mathbf{b},\mathbf{s}}\left[\sum_i v_i \cdot \mathbb{1}[i \in T^*(\mathbf{b},\mathbf{s}) \wedge v_i \in [L_k, 2L_k]]\right] \leq 2L_k \cdot \mathbb{E}[Z(t)] \leq 4 \cdot \text{GFT}_{\text{CFPP}}.$$

Summing the inequality over all k finishes the proof. □

Proof of Lemma 7.7: By Lemmas 7.9, 7.10, and the fact that $\mathcal{A}_S \leq \frac{\mathcal{A}}{2}$, we have that

$$\mathcal{A} \leq 2(\mathcal{A}_M + \mathcal{A}_L) \leq O(\log(n)) \cdot \text{GFT}_{\text{CFPP}}.$$

□

Theorem 7.2 summarizes our result for a general constrained-additive buyer. It directly follows from Lemmas 7.2, 7.3, 7.4, and 7.7.

Theorem 7.2. *For any downward-closed constraint $\mathcal{F}$,* FB-GFT $\leq O(\log(n) \cdot \log(\frac{1}{r})) \cdot \text{GFT}_{\text{CFPP}}$.

7.3.6 Proof of Lemma 7.11

We recall the statement of Lemma 7.11: *For any $c \in (0,1)$,*

$$\Pr_t[Z(t) \geq c \cdot \mathbb{E}[Z(t)]] \geq \frac{(1-c)^2}{1 + 1/\mathbb{E}[Z(t)]}.$$

Recall that $Z(t) = \max_{T \in \mathcal{F}} \sum_{i \in T} t_i^{(k)}$. In the proof we will omit the superscript k as it is fixed. The random seed t is also omitted if clear from context.

Lemma 7.12. *(Paley-Zygmund Inequality [PZ32]) For any random variable $Z \geq 0$ with finite variance, for any $c \in [0,1]$,*

$$\Pr[Z \geq c \cdot \mathbb{E}[Z]] \geq (1-c)^2 \cdot \frac{\mathbb{E}[Z]^2}{\text{Var}[Z] + \mathbb{E}[Z]^2}.$$

To use Lemma 7.12, we only need to show an upper bound on $\mathrm{Var}[Z(t)]$.

Lemma 7.13. $\mathrm{Var}[Z(t)] \le \mathbb{E}[Z(t)]$.

Proof. By the Efron-Stein Inequality [ES81],

$$\mathrm{Var}[Z(t)] \le \frac{1}{2}\sum_i \mathbb{E}_{t_i,t_i',t_{-i}}[(Z(t_i,t_{-i}) - Z(t_i',t_{-i}))^2] = \sum_i \mathbb{E}_{t_{-i}}[\mathrm{Var}[Z(t)|t_{-i}]].$$

Here t_i' shares the same distribution with t_i (a fresh sample). Note that for every fixed t_{-i}, $\mathrm{Var}_{t_i}[Z(t_i,t_{-i})] \le \mathbb{E}_{t_i}[(Z(t_i,t_{-i}) - a)^2]$ for any real constant a. For every i, let

$$Z_i(t_{-i}) = \max_{T\in\mathcal{F}, i\neq T}\sum_{j\in T} t_j$$

, which only depends on t_{-i}. We have

$$\mathrm{Var}[Z(t)] \le \sum_i \mathbb{E}_{t_{-i}}[\mathrm{Var}[Z(t)|t_{-i}]] \le \sum_i \mathbb{E}[(Z(t) - Z_i(t_{-i}))^2] \le \sum_i \mathbb{E}[Z(t) - Z_i(t_{-i})],$$

where the last inequality follows from the fact that $Z_i(t_{-i}) \le Z(t) \le Z_i(t_{-i}) + 1$, as every random variable $t_j \in [\frac{1}{2}, 1]$.

Now fix any t. Let $T^* = \mathrm{argmax}_{T\in\mathcal{F}}\sum_{j\in T} t_j$. Then for every i, by the definition of Z_i, $\sum_{j\in T^*\backslash\{i\}} t_j \le Z_i(t_{-i})$. Thus $\sum_i Z_i(t_{-i}) \ge \sum_i \sum_{j\in T^*\backslash\{i\}} t_j = (n-1)\cdot\sum_{j\in T^*} t_j = (n-1)\cdot Z(t)$. Hence,

$$\mathrm{Var}[Z(t)] \le \sum_i \mathbb{E}[Z(t) - Z_i(t_{-i})] \le \mathbb{E}[Z(t)].$$

□

7.4 An Unconditional Approximation for a Single Constrained-Additive Buyer

In this section, we prove Theorem 7.3, an unconditional $O(\log n)$-approximation when the buyer's feasibility constraint is selectable, and an unconditional $O(\log^2(n))$-approximation for a general constrained-additive buyer—without dependence on distributional parameters. The result combines the $\log(1/r)$-approximation and a novel mechanism—the *seller adjusted posted price mechanism.*

Theorem 7.3. *Suppose the buyer's feasibility constraint $\mathcal{F}$ is (δ, η)-selectable for some $\delta, \eta \in (0, 1)$. Then there exists a DSIC, ex-post IR, ex-ante WBB mechanism $\mathcal{M}$ such that* $\text{OPT} \leq O(\frac{\log n}{\delta \cdot \eta}) \cdot \text{GFT}(\mathcal{M})$. *Moreover, for a general constrained-additive buyer, there exists a DSIC, ex-post IR, ex-ante WBB mechanism $\mathcal{M}$ such that* $\text{OPT} \leq O(\log^2(n)) \cdot \text{GFT}(\mathcal{M})$.

7.4.1 An Upper Bound of the Second-Best GFT.

Formally, we use OPT to denote the optimal GFT attainable by any BIC, IR, ex-ante WBB mechanism. Notice that the GFT of any two-sided market mechanism can be broken down into the buyer's expected utility of this mechanism, plus the sum of all sellers' expected utilities (or profit), plus the difference between buyer's and sellers' expected payment. We show that the OPT is upper bounded by the sum of the designers' utilities in two related **one-sided markets**: the *super seller auction* and the *super buyer procurement auction.*

Super Seller Auction. Consider a one-sided market, where the designer is the super seller who owns all the items, and replaces all the original sellers. The buyer is the same as in our two-sided market setting. The super seller designs a mechanism to sell the items to the buyer. The main difference between the super seller auction and the original two-sided market is that the mechanism only needs to be BIC and IR for the buyer and does not have any incentive compatibility constraints for the super seller. We use OPT-S to denote the maximum profit (revenue minus her cost) achievable by any BIC and IR mechanism in the super seller auction.

To avoid ambiguity in further proofs, for every subset $T \subseteq [n]$ and downward-closed feasibility constraint $\mathcal{J}$ with respect to T, we let $\text{OPT-S}(T, \mathcal{J})$ denote the optimal profit in the following super seller auction: the super seller owns the set of items in T and has cost $s_i \sim \mathcal{D}_i^S$ for every item $i \in T$. The buyer has value $b_i \sim \mathcal{D}_i^B$ for every item $i \in T$ and is additive subject to constraint $\mathcal{J}$. We slightly abuse notation and write $\text{OPT-S}(T, \text{ADD})$ if the buyer is additive ($\mathcal{J} = 2^T$) and $\text{OPT-S}(T, \text{UD})$ if the buyer is unit-demand ($\mathcal{J} = \{\{i\} : i \in T\}$). Clearly, OPT-S =

OPT-S($[n], \mathcal{F}$).

Super Buyer Procurement Auction. Similarly, let the *super buyer procurement auction* be the one-sided market where the super buyer (same as the real buyer) designs the mechanism to procure items from the sellers. Here the mechanism only needs to be BIC and IR for all of the sellers, but not the buyer. We use OPT-B to denote the maximum utility (value minus payment) of the super buyer attainable by any BIC and IR mechanism in the super buyer procurement auction.

First, we prove that the GFT of any IR, BIC, ex-ante WBB mechanism $\mathcal{M} = (x, p^B, p^S)$ is upper bounded by OPT-S + OPT-B.

Lemma 7.14. OPT $\leq$ OPT-S + OPT-B.

Proof. Take any BIC, IR, ex-ante WBB mechanism $\mathcal{M} = (x, p^B, p^S)$. Since every seller i is BIC and IR, we have for any s_i, s_i',

$$\mathbb{E}_{\mathbf{b},s_{-i}}\left[p_i^S(\mathbf{b},\mathbf{s}) - s_i \cdot x_i(\mathbf{b},\mathbf{s})\right] \geq \max\left\{\mathbb{E}_{\mathbf{b},s_{-i}}[p_i^S(\mathbf{b},s_i',s_{-i})] - s_i \cdot x_i(\mathbf{b},s_i',s_{-i}), 0\right\}.$$

Observe that $\mathcal{M}' = (x, p^S)$ is a valid super buyer procurement auction. The above inequalities are exactly the BIC and IR constraints for seller i. Thus $\mathcal{M}'$ is BIC and IR. Similarly, $\mathcal{M}'' = (x, p^B)$ is BIC and IR, so it is a valid super seller auction. Since $\mathcal{M}$ is ex-ante WBB, $\mathbb{E}_{\mathbf{b},\mathbf{s}}[p^B(\mathbf{b},\mathbf{s}) - \sum_i p^S(\mathbf{b},\mathbf{s})] \geq 0$. Thus we have

$$\begin{aligned}
\text{GFT}(\mathcal{M}) &= \mathbb{E}_{\mathbf{b},\mathbf{s}}[\sum_{i\in[n]} x_i(\mathbf{b},\mathbf{s})(b_i - s_i)] \\
&\leq \mathbb{E}_{\mathbf{b},\mathbf{s}}[p^B(\mathbf{b},\mathbf{s}) - \sum_i x_i(\mathbf{b},\mathbf{s}) \cdot s_i] + \mathbb{E}_{\mathbf{b},\mathbf{s}}[\sum_i (x_i(\mathbf{b},\mathbf{s}) \cdot b_i - p_i^S(\mathbf{b},\mathbf{s}))] \\
&\leq \text{OPT-S} + \text{OPT-B}.
\end{aligned}$$

Taking $\mathcal{M}$ to be the GFT-maximizing mechanism completes the proof. $\square$

Next we prove an analog of the "Marginal Mechanism Lemma" [CH13, HN12] for the optimal profit. Namely, let (T, R) be a partition of the items in $[n]$. Then the optimal profit in a super seller auction with items in $[n]$ is upper bounded by the first-best GFT for items in T plus the optimal profit in a super seller auction with items in R.

Lemma 7.15 (Marginal Mechanism for Profit)**.** *For any subset $T \in [n]$, we let $\mathcal{F}\big|_T = \{S \subseteq T : S \in \mathcal{F}\}$ denote the restriction of $\mathcal{F}$ to T. We use* FB-GFT$(T, \mathcal{F}\big|_T)$ *to denote the first-best GFT obtainable between sellers in T and the $\mathcal{F}\big|_T$-constrained additive buyer, that is,*

$$\text{FB-GFT}(T, \mathcal{F}\big|_T) = \mathbb{E}_{\boldsymbol{b}_T, \boldsymbol{s}_T}[\max_{S \in \mathcal{F}|_T} \textstyle\sum_{i \in S}(b_i - s_i)^+],$$

where $\boldsymbol{b}_T = \{b_i\}_{i \in T}$, $\boldsymbol{s}_T = \{s_i\}_{i \in T}$. Let (R, T) be any partition of the items in $[n]$. Then

$$\text{OPT-S}([n], \mathcal{F}) \le \text{OPT-S}(R, \mathcal{F}\big|_R) + \text{FB-GFT}(T, \mathcal{F}\big|_T).$$

Proof. Consider the optimal BIC and IR mechanism $\mathcal{M} = (x, p)$ in the super seller auction with item set $[n]$. We will construct a BIC and IR mechanism $\mathcal{M}' = (x', p')$ in the super seller auction with item set R as follows. The mechanism only sells items in R using the same allocation x. The payment for the buyer is defined as the payment p in $\mathcal{M}$ minus the buyer's expected total value for all items in T. Formally, for every $\mathbf{b}_R = \{b_j\}_{j \in R}$, $\mathbf{s}_R = \{s_j\}_{j \in R}$ and $i \in R$, let

$$x_i'(\mathbf{b}_R, \mathbf{s}_R) = \mathbb{E}_{\mathbf{b}_T, \mathbf{s}_T}[x_i(\mathbf{b}, \mathbf{s})]$$

$$p'(\mathbf{b}_R, \mathbf{s}_R) = \mathbb{E}_{\mathbf{b}_T, \mathbf{s}_T}\left[p(\mathbf{b}, \mathbf{s}) - \textstyle\sum_{j \in T} b_j \cdot x_j(\mathbf{b}, \mathbf{s})\right].$$

Notice that in $\mathcal{M}'$, the expected utility of the buyer with type $\mathbf{b}_R$ when reporting $\mathbf{b}_R'$ is

$$\mathbb{E}_{\mathbf{s}_R}[\sum_{i \in T} b_i \cdot x_i'(\mathbf{b}_R', \mathbf{s}_R) - p'(\mathbf{b}_R', \mathbf{s}_R)] = \mathbb{E}_{\mathbf{b}_T, \mathbf{s}}[\sum_{i \in [n]} b_i \cdot x_i(\mathbf{b}_R', \mathbf{b}_T, \mathbf{s}) - p(\mathbf{b}_R', \mathbf{b}_T, \mathbf{s})],$$

Since $\mathcal{M}$ is BIC and IR, $\mathcal{M}'$ is also BIC and IR. Thus

$$\begin{aligned}\text{OPT-S}([n],\mathcal{F}) &= \mathbb{E}_{\mathbf{b},\mathbf{s}}[p(\mathbf{b},\mathbf{s}) - \sum_{i\in[n]} s_i \cdot x_i(\mathbf{b},\mathbf{s})] \\ &= \mathbb{E}_{\mathbf{b}_R,\mathbf{s}_R}[p'(\mathbf{b}_R,\mathbf{s}_R) - \sum_{i\in R} s_i \cdot x_i'(b_R,s_R)] + \mathbb{E}_{\mathbf{b},\mathbf{s}}[\sum_{i\in T}(b_i - s_i)\cdot x_i(\mathbf{b},\mathbf{s})] \\ &\leq \text{OPT-S}(R,\mathcal{F}\big|_R) + \text{FB-GFT}(T,\mathcal{F}\big|_T).\end{aligned}$$

□

We partition the items into the set of "likely to trade" items, that is, items with trade probability $r_i = \Pr_{b_i,s_i}[b_i \geq s_i] \geq 1/n$, and the "unlikely to trade" items. We can bound the OPT-S by the first-best GFT of the "likely to trade" items and the optimal profit of the super seller auction with the "unlikely to trade" items. We can further replace the first-best GFT of the "likely to trade" by $O(\log n)\cdot \text{GFT}_{\text{CFPP}}$ according to Theorem 7.1 or by $O(\log^2(n))\cdot\text{GFT}_{\text{CFPP}}$ according to Theorem 7.2 depending on the buyer's feasibility constraint. Formally,

Lemma 7.16. *Define $H = \{i \in [n] : r_i \geq \frac{1}{n}\}$ and $L = [n]\backslash H = \{i\in[n] : r_i < \frac{1}{n}\}$. Suppose the buyer's feasibility constraint $\mathcal{F}$ is (δ,η)-selectable for some $\delta,\eta\in(0,1)$. Then* OPT *is upper bounded by*

$$\begin{aligned}&\text{OPT-B} + \text{OPT-S}(L,\mathcal{F}\big|_L) + \text{FB-GFT}(H,\mathcal{F}\big|_H)\\ \leq&\text{OPT-B} + \text{OPT-S}(L,\mathcal{F}\big|_L) + O\left(\frac{\log n}{\delta\cdot\eta}\right)\cdot\text{GFT}_{\text{CFPP}}.\end{aligned}$$

For a general constrained-additive buyer, OPT *is upper bounded by*

$$\text{OPT-B} + \text{OPT-S}(L,\mathcal{F}\big|_L) + O\left(\log^2(n)\right)\cdot\text{GFT}_{\text{CFPP}}.$$

Proof. The first inequality follows from Lemma 7.14 and 7.15. Since $\mathcal{F}$ is (δ,η)-selectable, $\mathcal{F}\big|_H$ is also (δ,η)-selectable. We derive the second inequality by applying Theorem 7.1 on the items in H. For a general constrained-additive buyer, we derive the inequality by applying Theorem 7.2 on the items in H. $\square$

It is well known that in multi-item auctions, the revenue of selling the items separately is an $O(\log n)$-approximation to the optimal revenue when there is a single additive buyer [LY13]. Cai and Zhao [CZ19] provide an extension of this $O(\log n)$-approximation to profit maximization (Chapter 4). We build on this in Section 7.4.4 to upper bound the OPT-S$(L, \mathcal{F}\big|_L)$ term, where with $|L|$ items, we get a $\log(|L|)$ factor (Lemma 7.23).

All together, this gives the following upper bound on the second-best GFT.

Lemma 7.17 (Upper Bound on Second-Best GFT)**.** *Define $H = \{i \in [n] : r_i \geq \frac{1}{n}\}$ and $L = [n] \backslash H = \{i \in [n] : r_i < \frac{1}{n}\}$. Suppose the buyer's feasibility constraint $\mathcal{F}$ is (δ, η)-selectable for some $\delta, \eta \in (0,1)$. Then*

$$\begin{aligned}\text{OPT} \leq & \text{OPT-B} + O\left(\frac{\log n}{\delta \cdot \eta}\right) \cdot \text{GFT}_{\text{CFPP}} \\ & + O\left(\log(|L|) \cdot \sum_{i \in L} \mathbb{E}_{b_i, s_i}\left[(\tilde{\varphi}_i(b_i) - s_i)^+\right]\right).\end{aligned}$$

For a general constrained-additive buyer, the $O\left(\frac{\log n}{\delta \cdot \eta}\right)$ factor above becomes $O\left(\log^2(n)\right)$.

Next, Section 7.4.2 gives details on constructing a mechanism for a two-sided market whose GFT is at least OPT-B. In Section 7.4.3, we show how to use a generalization of posted price mechanisms to approximate the second term in the upper bound by the GFT of the Seller Adjusted Posted Price mechanism. The approximation heavily relies on the fact that in expectation, only one item can trade, so it is crucial that L only contains the "unlikely to trade" items.

7.4.2 Bounding the Optimal Buyer Utility in the Super Buyer Procurement Auction.

In this section, we construct a two-sided market to bound OPT-B for any constrained additive buyer.

Lemma 7.18. *Consider the mechanism $\mathcal{M}^* = (x, p^B, p^S)$ where for every item i, buyer profile $\boldsymbol{b}$, and seller profile $\boldsymbol{s}$, $x_i(\boldsymbol{b}, \boldsymbol{s}) = \mathbb{1}[b_i - \tilde{\tau}_i(s_i) \geq 0 \wedge i \in \text{argmax}_{S \in \mathcal{F}} \sum_{i \in S}(b_i - \widetilde{\tau}_i(s_i))^+]$. Here*

$\widetilde{\tau}_i(s_i)$ is Myerson's ironed virtual value function[11] *for seller i's distribution $\mathcal{D}_i^S$. For every seller i, since $\tilde{\tau}_i(s_i)$ is non-decreasing in s_i, $x_i(\boldsymbol{b}, \boldsymbol{s})$ is non-increasing in s_i. Define $p_i^S(\boldsymbol{b}, \boldsymbol{s})$ as the threshold payment for seller i, i.e., the largest cost s_i such that $x_i(\boldsymbol{b}, s_i, s_{-i}) = 1$. Define the buyer's payment $p^B(\boldsymbol{b}, \boldsymbol{s}) = \sum_i x_i(\boldsymbol{b}, \boldsymbol{s}) \cdot \tilde{\tau}_i(s_i)$. $\mathcal{M}^*$ is DSIC, ex-post IR, ex-ante SBB* [12] *and*

$$\mathrm{GFT}(\mathcal{M}^*) \geq \text{OPT-B} = \mathbb{E}_{\boldsymbol{b},\boldsymbol{s}}[\max_{S \in \mathcal{F}} \sum_{i \in S} (b_i - \widetilde{\tau}_j(s_j))^+].$$

Proof. Since the seller's allocation rule is monotone and we use the threshold payment, $\mathcal{M}^*$ is DSIC and ex-post IR for each seller.

Note that for any seller profile $\mathbf{s}$, when the buyer has true type $\mathbf{b}$, her expected utility from reporting $\mathbf{b}'$ is $\sum_i x_i(\mathbf{b}', \mathbf{s}) \cdot (b_i - \widetilde{\tau}_i(s_i))$. According to the definition of x, the buyer's utility is maximized when $\mathbf{b}' = \mathbf{b}$. Hence, $\mathcal{M}$ is DSIC for the buyer. Moreover we have ex-post IR, as the buyer's expected utility when reporting truthfully is $\max_{S \in \mathcal{F}} \sum_{i \in S} (b_i - \widetilde{\tau}_i(s_i))^+ \geq 0$.

It only remains to prove that the mechanism is ex-ante SBB and to lower bound its GFT. By Myerson's lemma, for every $\mathbf{b}$ we have

$$\mathbb{E}_{\mathbf{s}}\left[\sum_i p_i^S(\mathbf{b}, \mathbf{s})\right] = \mathbb{E}_{\mathbf{s}}\left[\sum_i x_i(\mathbf{b}, \mathbf{s}) \cdot \widetilde{\tau}_i(s_i)\right] = \mathbb{E}_{\mathbf{s}}[p^B(\mathbf{b}, \mathbf{s})].$$

Thus the mechanism is ex-ante SBB.

Why is $\text{OPT-B} = \mathbb{E}_{\mathbf{b},\mathbf{s}}[\max_{S \in \mathcal{F}} \sum_{i \in S} (b_i - \widetilde{\tau}_j(s_j))^+]$? Notice that only the sellers are strategic in a super buyer procurement auction, and their types are all single-dimensional. One can apply the standard Myersonian analysis to the super buyer procurement auction and show that the optimal buyer utility is exactly $\mathbb{E}_{\mathbf{b},\mathbf{s}}[\max_{S \in \mathcal{F}} \sum_{i \in S} (b_i - \widetilde{\tau}_j(s_j))^+]$.

Note that the buyer's expected utility in $\mathcal{M}^*$ is exactly OPT-B. As $\mathcal{M}^*$ is an ex-ante SBB mechanism, the expected GFT of $\mathcal{M}^*$ is equal to the buyer's expected utility plus the sum of all

11. The seller's unironed virtual value function is $\tau_i(s_i) = s_i + \frac{G_i(s_i)}{g_i(s_i)}$.

12. One can make the mechanism IR and ex-post SBB by defining $p^B(\mathbf{b}, \mathbf{s}) = \sum_i p_i^S(\mathbf{b}, \mathbf{s})$. The mechanism is still DSIC for all sellers. It is only BIC for the buyer, as the sellers' gains only equal the virtual welfare when taking expectation over sellers' profile.

sellers' expected utility, and the latter is non-negative since $\mathcal{M}^*$ is ex-post IR for every seller. □

7.4.3 The Seller Adjusted Posted Price Mechanism.

In this section, we introduce a new mechanism—the *Seller Adjusted Posted Price* (SAPP) Mechanism. We define an adjusted price mechanism to first elicit each seller's cost s_i, and then produce posted prices $\{\theta_i(\mathbf{s})\}_{i\in[n]}$ as a function of the reported profile $\mathbf{s}$; thus the mechanism is a collection of posted prices depending on the reported seller cost profile. The items are offered to the buyer at each posted price $\theta_i(\mathbf{s})$, with the buyer only allowed to purchase at most one item by paying the posted price. See Mechanism 7.1 for a complete description of the SAPP mechanism. We show that for a properly selected mapping $\{\theta_i(\cdot)\}_{i\in[n]}$, the SAPP mechanism is DSIC, ex-post IR, and ex-ante WBB. Moreover, its GFT is at least $\Theta\left(\sum_{i\in L}\mathbb{E}_{b_i,s_i}\left[(\tilde{\varphi}_i(b_i)-s_i)^+\right]\right)$.

Since the posted prices depend on the reported seller cost profile, we need to be careful to ensure that there is no incentive for any seller to misreport the cost. We identify a sufficient condition for the posted prices, called *bi-monotonicity*, to make sure the corresponding mechanism is DSIC and ex-post IR.

Definition 7.1 (Bi-monotonic Prices)**.** *We say the posted prices $\{\theta_i(\boldsymbol{s})\}_{i\in[n]}$ are bi-monotonic, if (i) $\theta_i(\boldsymbol{s})\geq s_i$ for all seller profile $\boldsymbol{s}$ and seller i; (ii) $\theta_i(\boldsymbol{s})$ is non-decreasing in s_i and non-increasing in s_j for all $j\neq i$.*

In Lemma 7.19, we prove that bi-monotonic posted prices induce a monotone allocation rule for every seller, enabling *threshold payments* [Mye81, MS83]. Formally, for every seller i let $\hat{x}_i(\mathbf{b},\mathbf{s})$ denote the probability that the buyer trades with seller i under profile $(\mathbf{b},\mathbf{s})$. This is either 0 or 1 since all $\theta_i(\mathbf{s})$ are fixed values when $\mathbf{s}$ is fixed. If $\hat{x}_i(\mathbf{b},\mathbf{s})=1$, $p_i^S(\mathbf{b},\mathbf{s})$ is defined as the maximum value s_i' such that $\hat{x}_i(\mathbf{b},s_i',s_{-i})=1$. Otherwise $p_i^S(\mathbf{b},\mathbf{s})=0$. This makes the SAPP mechanism DSIC and ex-post IR.

Lemma 7.19. *Let $\mathcal{M}$ be an* SAPP *mechanism with bi-monotonic posted prices $\{\theta_i(\boldsymbol{s})\}_{i\in[n]}$. Then the allocation of the mechanism $\hat{x}_i(\boldsymbol{b},\boldsymbol{s})$ is non-increasing in s_i for all sellers i, and $\mathcal{M}$ is DSIC and ex-post IR for the buyer and the sellers.*

Proof. Notice that for every type $\mathbf{b}$, the buyer chooses the item that maximizes $b_i - \theta_i(\mathbf{s})$ (and does not choose any item if she cannot afford any of the items). For every i, by bi-monotonicity, when s_i decreases, $b_i - \theta_i(\mathbf{s})$ does not decrease while $b_j - \theta_j(\mathbf{s})$ does not increase for all $j \neq i$. Thus if the buyer chooses item i under the original s_i, she must continue to choose item i for smaller reports s_i'. Thus $\hat{x}_i(\mathbf{b}, \mathbf{s})$ is non-increasing in s_i. Since every seller receives the threshold payment, she is DSIC and ex-post IR. As the buyer simply faces a posted price mechanism, the mechanism is DSIC and ex-post IR for the buyer. □

Mechanism 7.1 Seller Adjusted Posted Price Mechanism

Require: $\forall i \in [n]$, function $\theta_i(\cdot)$ that maps each seller cost profile to a price for item i. Input $(\mathbf{b}, \mathbf{s})$.

1: Given the sellers' reported cost profile $\mathbf{s}$, offer each item i to the buyer at price $\theta_i(\mathbf{s})$.
2: The buyer is allowed to purchase at most one item by paying the corresponding posted price.
3: If no item is picked, then no trade happens and payment is 0 for every agent. Otherwise, if the buyer chooses item i, she receives item i and pays $\theta_i(\mathbf{s})$. Seller i sells her item and receives threshold payment.

The main challenge we face here is establishing the budget balance condition. Unfortunately, having bi-monotonic posted prices is not sufficient. Consider the $n = 1$ case: the posted price $p(s) = s$ is trivially bi-monotonic. Clearly, the corresponding SAPP mechanism achieves FB-GFT. However, due to the impossibility result by Myerson and Satterthwaite [MS83], no BIC, IR, and ex-ante WBB mechanism can always achieve FB-GFT, so the SAPP mechanism must sometimes violate the budget balance constraint. In Lemma 7.20, we show that even though bi-monotonic posted prices do not imply budget balance, there is indeed a wide range of bi-monotonic posted prices that induce budget balanced SAPP mechanisms. Our budget balance proof heavily relies on an allocation coupling argument (Lemma 7.21) that simultaneously provides a lower bound on the buyer's payment, as well as an upper bound on the payment to the seller.

Lemma 7.20. *Let $x = \{x_i(\boldsymbol{b}, \boldsymbol{s})\}_{i\in[n]}$ be an arbitrary allocation rule that satisfies (i) the buyer never purchases more than one item in expectation under each profile $(\boldsymbol{b}, \boldsymbol{s})$, i.e., $\sum_{i\in[n]} x_i(\boldsymbol{b}, \boldsymbol{s}) \leq 1$, and (ii) for every buyer type $\boldsymbol{b}$ and seller i, $x_i(\boldsymbol{b}, \boldsymbol{s})$ is non-increasing in s_i, and non-decreasing in s_j for all $j \neq i$. We define $q_i(\boldsymbol{s}) = \mathbb{E}_{\boldsymbol{b}}[x_i(\boldsymbol{b}, \boldsymbol{s}) \cdot \mathbb{1}[\tilde{\varphi}_i(b_i) \geq s_i]]$, where $\tilde{\varphi}_i(b_i)$ is Myerson's ironed* *virtual value for* $\mathcal{D}_i^B$*, and* $\theta_i(\boldsymbol{s}) = F_i^{-1}(1 - \frac{q_i(\boldsymbol{s})}{2})$*. The posted prices* $\{\theta_i(\boldsymbol{s})\}_{i\in[n]}$ *are bi-monotonic, and the corresponding SAPP mechanism* $\mathcal{M}$ *is DSIC, ex-post IR, and ex-ante WBB. Moreover,* $\text{GFT}(\mathcal{M}) \geq \frac{1}{4}\mathbb{E}_{\boldsymbol{b},\boldsymbol{s}}\left[\sum_i (\widetilde{\varphi}_i(b_i) - s_i) \cdot x_i(\boldsymbol{b},\boldsymbol{s})\right]$.

Proof. It is not hard to verify that $\{\theta_i(\mathbf{s}) = F_i^{-1}(1 - \frac{q_i(\mathbf{s})}{2})\}_{i\in[n]}$ is bi-monotonic. Now we proceed to prove that the SAPP mechanism $\mathcal{M}$ is ex-ante WBB. We require the following lemma.

Lemma 7.21. *For every seller* i *and every seller profile* $\boldsymbol{s}$*,* $\hat{x}_i(\boldsymbol{s}) \in \left[\frac{q_i(\boldsymbol{s})+q_i(\boldsymbol{s})^2}{4}, \frac{q_i(\boldsymbol{s})}{2}\right]$.

Proof. Note that the buyer will purchase item i if both of the following conditions are satisfied:

1. The buyer can afford item i, i.e., $b_i \geq \theta_i(\mathbf{s})$.

2. The buyer cannot afford any other items, i.e., $b_j < \theta_j(\mathbf{s}), \forall j \neq i$.

By choice of $\theta_i(\mathbf{s})$, the first event happens with probability $\Pr[b_i \geq \theta_i(\mathbf{s})] = q_i(\mathbf{s})/2$.

Note that $\sum_{i\in[n]} q_i(\mathbf{s}) \leq \mathbb{E}_{\mathbf{b}}[\sum_{i\in[n]} x_i(\mathbf{b},\mathbf{s})] \leq 1$. For each $j \neq i$, $\Pr[b_j < \theta_j(\mathbf{s})] = 1 - \frac{q_j(\mathbf{s})}{2}$. Thus $\sum_{j\neq i}\left(1 - \frac{q_j(\mathbf{s})}{2}\right) \geq n - \frac{3}{2} + \frac{q_i(\mathbf{s})}{2}$. The second event happens with probability $\prod_{j\neq i}\left(1 - \frac{q_j(\mathbf{s})}{2}\right) \geq \frac{1}{2} + \frac{q_i(\mathbf{s})}{2}$. The equality holds when one out of the $n-1$ $q_j(\mathbf{s})$'s equals $1 - q_i(\mathbf{s})$ and the rest are all equal to 0. Notice that the two events are independent, so we have the upper and lower bound on $\hat{x}_i(\mathbf{s})$. □

We return to the proof of Lemma 7.20. For easy reference, we list our notation again:

- $x = \{x_i(\mathbf{b},\mathbf{s})\}_{i\in[n]}$ is an arbitrary allocation.

- $\hat{x}_i(\mathbf{b},\mathbf{s})$ is the probability that item i trades in $\mathcal{M}$ under profile $(\mathbf{b},\mathbf{s})$.

- $\hat{x}_i(\mathbf{s}) = \mathbb{E}_{\mathbf{b}}[\hat{x}_i(\mathbf{b},\mathbf{s})]$ is the probability that item i trades over the randomness of buyer valuations, i.e. the interim trade probability.

- $q_i(\mathbf{s}) = \mathbb{E}_{\mathbf{b}}[x_i(\mathbf{b},\mathbf{s}) \cdot \mathbb{1}[\tilde{\varphi}_i(b_i) \geq s_i]]$ is the probability that item i trades in allocation x *and* the buyer's ironed virtual value for item i is above the seller's cost.

- $\theta_i(\mathbf{s}) = F_i^{-1}(1 - \frac{q_i(\mathbf{s})}{2})$ is the buyer's posted price set such that $\Pr[b_i \geq \theta_i(\mathbf{s})] = q_i(\mathbf{s})/2$.

Fix any seller profile $\mathbf{s}$. For simplicity, we slightly abuse notation and use $\hat{x}_i(z)$ and $q_i(z)$ to denote $\hat{x}_i(z, s_{-i})$ and $q_i(z, s_{-i})$. The expected payment from the buyer under cost profile $\mathbf{s}$ is $\sum_{i\in[n]} \hat{x}_i(s_i) \cdot \theta_i(\mathbf{s})$. For every seller i, denote $p_i^S(\mathbf{s}) = \mathbb{E}_{\mathbf{b}}[p_i^S(\mathbf{b}, \mathbf{s})]$ as her expected payment under cost profile $\mathbf{s}$.

Note that for every $\mathbf{b}, \mathbf{s}$, the threshold payment $p_i^S(\mathbf{b}, \mathbf{s})$ can be rewritten as the quantity $\int_{s_i}^{\infty} \hat{x}_i(\mathbf{b}, t, s_{-i})dt + s_i \cdot \hat{x}_i(\mathbf{b}, s_i, s_{-i})$: When $\hat{x}_i(\mathbf{b}, \mathbf{s}) = 0$, then $\hat{x}_i(\mathbf{b}, t, s_{-i})$ for all $t \geq s_i$ since $\hat{x}_i(\mathbf{b}, \mathbf{s})$ is non-increasing in s_i. Thus the above quantity is 0. When $\hat{x}_i(\mathbf{b}, \mathbf{s}) = 1$, let s_i' be the maximum value such that $\hat{x}_i(\mathbf{b}, s_i', s_i) = 1$. Then the above quantity is equal to $\int_{s_i}^{s_i'} 1dt + s_i = s_i' = p_i^S(\mathbf{b}, \mathbf{s})$. Thus $p_i^S(\mathbf{s}) = \mathbb{E}_{\mathbf{b}}[p_i^S(\mathbf{b}, \mathbf{s})] = \int_{s_i}^{\infty} \hat{x}_i(z, s_{-i})dz + s_i \cdot \hat{x}_i(s_i, s_{-i})$. We will show that $p_i^S(\mathbf{s}) \leq \hat{x}_i(s_i) \cdot \theta_i(\mathbf{s})$. By definition,

$$\begin{aligned}
p_i^S(\mathbf{s}) &= \int_{s_i}^{\infty} \hat{x}_i(z)dz + s_i \cdot \hat{x}_i(s_i) \\
&= \int_{s_i}^{\infty} \int_0^{\infty} \mathbb{1}[\hat{x}_i(z) \geq t]dtdz + s_i \cdot \hat{x}_i(s_i) \\
&= \int_{s_i}^{\infty} \int_0^{\hat{x}_i(s_i)} \mathbb{1}[\hat{x}_i(z) \geq t]dtdz + s_i \cdot \hat{x}_i(s_i) \\
&= \int_0^{\hat{x}_i(s_i)} \int_{s_i}^{\infty} \mathbb{1}[\hat{x}_i(z) \geq t]dzdt + s_i \cdot \hat{x}_i(s_i).
\end{aligned}$$

The second inequality follows from $\hat{x}_i(z) = \int_0^{\infty} \mathbb{1}[\hat{x}_i(z) \geq t]dt, \forall z$. The third inequality is due to $\mathbb{1}[\hat{x}_i(z) \geq t] = 0, \forall z \geq s_i$ and $t > \hat{x}_i(s_i)$. The last equality follows from Fubini's Theorem, as the integral is finite due to the monotonicity of $\hat{x}_i(\cdot)$.

Moreover, since $\hat{x}_i(\cdot)$ is non-increasing, for every $z \leq s_i, t \leq \hat{x}_i(s_i)$, we have $\hat{x}_i(z) \geq \hat{x}_i(s_i) \geq t$. Thus $\int_0^{\hat{x}_i(s_i)} \int_0^{s_i} \mathbb{1}[\hat{x}_i(z) \geq t]dzdt = \int_0^{\hat{x}_i(s_i)} \int_0^{s_i} 1dzdt = s_i \cdot \hat{x}_i(s_i)$.

Combining the two equations, we have

$$
\begin{aligned}
p_i^S(\mathbf{s}) &= \int_0^{\hat{x}_i(s_i)} \int_0^{\infty} \mathbb{1}[\hat{x}_i(z) \geq t] dz dt \\
&\leq \int_0^{\hat{x}_i(s_i)} \int_0^{\infty} \mathbb{1}[q_i(z) \geq 2t] dz dt \\
&\leq \int_0^{\hat{x}_i(s_i)} \int_0^{\infty} \mathbb{1}\left[\Pr_{b_i}[\widetilde{\varphi}_i(b_i) \geq z] \geq 2t\right] dz dt
\end{aligned}
$$

The first inequality follows from Lemma 7.21 and the second inequality follows from the definition of $q_i(\cdot)$. For every t, we prove that $\int_0^\infty \mathbb{1}\left[\Pr_{b_i}[\widetilde{\varphi}_i(b_i) \geq z] \geq 2t\right] dz \leq \widetilde{\varphi}_i(F_i^{-1}(1-2t+\epsilon))$ for any $\epsilon > 0$. In fact, let $z^* = \widetilde{\varphi}_i(F_i^{-1}(1-2t+\epsilon))$. For every $z > z^*$, $\Pr[\widetilde{\varphi}_i(b_i) \geq z] \leq \Pr[\widetilde{\varphi}_i(b_i) > z^*] = \Pr[b_i > F_i^{-1}(1-2t+\epsilon)] \leq 2t - \epsilon$. So $\mathbb{1}\left[\Pr[\widetilde{\varphi}_i(b_i) \geq z] \geq 2t\right] = 0$ for all $z > z^*$.

Therefore, for any $\epsilon > 0$, we have the following. In the second line, we change the variable by denoting $y = F_i^{-1}(1-2t+\epsilon)$.

$$
\begin{aligned}
p_i^S(\mathbf{s}) \leq & \int_0^{\hat{x}_i(s_i)} \widetilde{\varphi}_i(F_i^{-1}(1-2t+\epsilon)) dt \\
= & \int_\infty^{F_i^{-1}(1-2\hat{x}_i(s_i)+\epsilon)} \widetilde{\varphi}_i(y) d\frac{1+\epsilon-F_i(y)}{2} \\
= & -\frac{1}{2}\int_\infty^{F_i^{-1}(1-2\hat{x}_i(s_i)+\epsilon)} \widetilde{\varphi}_i(y) f_i(y) dy \\
= & \frac{1}{2}\int_{F_i^{-1}(1-2\hat{x}_i(s_i)+\epsilon)}^{\infty} \widetilde{\varphi}_i(y) f_i(y) dy \\
= & \frac{1}{2} F_i^{-1}(1-2\hat{x}_i(s_i)+\epsilon) \cdot [1 - F_i(F_i^{-1}(1-2\hat{x}_i(s_i)+\epsilon))] \\
= & F_i^{-1}(1-2\hat{x}_i(s_i)+\epsilon) \cdot (\hat{x}_i(s_i) - \epsilon/2) \\
\leq & \hat{x}_i(s_i) \cdot F_i^{-1}(1-2\hat{x}_i(s_i)+\epsilon)
\end{aligned}
$$

If $q_i(s_i) = 0$, then $\hat{x}_i(s_i) \cdot F_i^{-1}(1-2\hat{x}_i(s_i)+\epsilon) = 0 = \hat{x}_i(s_i) \cdot \theta_i(\mathbf{s})$. Otherwise, choose ϵ to be any number in $(0, \frac{q_i(s_i)^2}{4})$. Then, according to Lemma 7.21 and our choice of ϵ,

$$1 - 2\hat{x}_i(s_i) + \epsilon \leq 1 - \frac{q_i(s_i)}{2} - \frac{q_i(s_i)^2}{4} < 1 - \frac{q_i(s_i)}{2}.$$

Hence, $F_i^{-1}(1 - 2\hat{x}_i(s_i) + \epsilon) < \theta_i(\mathbf{s})$. Thus $p_i^S(\mathbf{s}) \leq \hat{x}_i(\mathbf{s}) \cdot \theta_i(\mathbf{s})$ for every i and $\mathbf{s}$, which implies that $\mathbb{E}_{\mathbf{s}}\left[\sum_i \theta_i(\mathbf{s}) \cdot \hat{x}_i(\mathbf{s})\right] \geq \mathbb{E}_{\mathbf{s}}\left[\sum_i p_i^S(s_i, s_{-i})\right]$. Hence $\mathcal{M}$ is ex-ante WBB.

We now need to lower bound the GFT from mechanism $\mathcal{M}$.

$$\begin{aligned}
\text{GFT}(\mathcal{M}) =& \mathbb{E}_{\mathbf{b},\mathbf{s}}\left[\sum_i (b_i - s_i) \cdot \hat{x}_i(\mathbf{b}, \mathbf{s})\right] \\
\geq& \mathbb{E}_{\mathbf{s}}\left[\sum_i (\theta_i(\mathbf{s}) - s_i) \cdot \hat{x}_i(\mathbf{s})\right] \\
\geq& \frac{1}{2}\mathbb{E}_{\mathbf{s}}\left[\sum_i \left(F_i^{-1}\left(1 - \frac{q_i(\mathbf{s})}{2}\right) - s_i\right) \cdot \frac{q_i(\mathbf{s})}{2}\right] \\
=& \frac{1}{2}\mathbb{E}_{\mathbf{b},\mathbf{s}}\left[\sum_i (\widetilde{\varphi}_i(b_i) - s_i) \cdot \mathbb{1}\left[b_i \geq F_i^{-1}\left(1 - \frac{q_i(\mathbf{s})}{2}\right)\right]\right] \\
\geq& \frac{1}{4}\mathbb{E}_{\mathbf{b},\mathbf{s}}\left[\sum_i (\widetilde{\varphi}_i(b_i) - s_i) \cdot x_i(\mathbf{b}, \mathbf{s}) \cdot \mathbb{1}[\tilde{\varphi}_i(b_i) \geq s_i]]\right] \\
\geq& \frac{1}{4}\mathbb{E}_{\mathbf{b},\mathbf{s}}\left[\sum_i (\widetilde{\varphi}_i(b_i) - s_i) \cdot x_i(\mathbf{b}, \mathbf{s})]\right]
\end{aligned}$$

Here the second inequality uses the definition of $\theta_i(\mathbf{s})$, $q_i(\mathbf{s})$ and Lemma 7.21. The third inequality follows from Myerson's lemma. The second-to-last inequality uses the fact that

$$\mathbb{E}_{b_i}\left[\widetilde{\varphi}_i(b_i) \cdot \mathbb{1}[b_i \geq F_i^{-1}\left(1 - \frac{q_i(\mathbf{s})}{2}\right)\right] \geq \frac{1}{2} \cdot \mathbb{E}_{\mathbf{b}}\left[\widetilde{\varphi}_i(b_i) \cdot x_i(\mathbf{b}, \mathbf{s}) \cdot \mathbb{1}[\tilde{\varphi}_i(b_i) \geq s_i]\right]$$

holds for every $\mathbf{s}$ and i.

This is because the right hand side

$$\frac{1}{2} \cdot \mathbb{E}_{\mathbf{b}}[\widetilde{\varphi}_i(b_i) \cdot x_i(\mathbf{b}, \mathbf{s}) \cdot \mathbb{1}[\tilde{\varphi}_i(b_i) \geq s_i]] = \mathbb{E}_{b_i}[\widetilde{\varphi}_i(b_i) \cdot \frac{1}{2}\mathbb{E}_{b_{-i}}[x_i(\mathbf{b}, \mathbf{s}) \cdot \mathbb{1}[\tilde{\varphi}_i(b_i) \geq s_i]]]$$

can be viewed as the expectation of $\widetilde{\varphi}_i(b_i)$ on an event of b_i with a total probability mass $\mathbb{E}_{b_i}[\frac{1}{2}\mathbb{E}_{b_{-i}}[x_i(\mathbf{b},\mathbf{s})\cdot \mathbb{1}[\tilde{\varphi}_i(b_i) \geq s_i]]] = \frac{q_i(\mathbf{s})}{2}$, while the left hand side is the maximum expectation of $\widetilde{\varphi}_i(b_i)$ on any event of b_i with total probability mass $\frac{q_i(\mathbf{s})}{2}$, as $\widetilde{\varphi}_i(b_i)$ is non-decreasing on b_i. □

Lemma 7.22 shows how to choose an allocation rule x so that the induced SAPP mechanism (using Lemma 7.20) has GFT at least $\Omega\left(\sum_{i\in L}\mathbb{E}_{b_i,s_i}\left[(\tilde{\varphi}_i(b_i)-s_i)^+\right]\right)$. Note that the existence of such an x heavily relies on the fact that in expectation there is only one item that can trade among the "unlikely to trade" items.

Lemma 7.22. *We let* $\mathrm{GFT}_{\mathrm{SAPP}}(S)$ *denote the optimal GFT attainable by any DSIC, ex-post IR, and ex-ante WBB* SAPP *mechanism over items in* S *for any subset* $S \subseteq [n]$. $\mathrm{GFT}_{\mathrm{SAPP}}(L) \geq \frac{1}{4e}\cdot\sum_{i\in L}\mathbb{E}_{b_i,s_i}[(\widetilde{\varphi}_i(b_i)-s_i)^+]$.

Proof. Let $\mathbf{b}_L = \{b_i\}_{i\in L}$ and $\mathbf{s}_L = \{s_i\}_{i\in L}$. For every $i \in L$, define the event that only i is tradeable:

$$A_i = \{(\mathbf{b}_L, \mathbf{s}_L) : b_i \geq s_i \wedge b_j < s_j, \forall j \in L\backslash\{i\}\}\,.$$

We consider the following allocation rule:

$$x_i(\mathbf{b}_L,\mathbf{s}_L) = \begin{cases} \mathbb{1}[\widetilde{\varphi}_i(b_i) \geq s_i] & \text{, if } (\mathbf{b},\mathbf{s}) \in A_i \\ 0 & \text{, otherwise} \end{cases}$$

Notice that $(\mathbf{b}_L,\mathbf{s}_L) \in A_i$ implies that $(\mathbf{b}_L, s_i', \mathbf{s}_{L\backslash\{i\}}) \in A_i$ for any $s_i' \leq s_i$. Thus, $x_i(\mathbf{b}_L,\mathbf{s}_L)$ is non-increasing in s_i. Similarly, it is easy to verify that $x_i(\mathbf{b}_L,\mathbf{s}_L)$ is non-decreasing in all s_j where $j \in L\backslash\{i\}$. Furthermore, $\sum_{i\in L} x_i(\mathbf{b}_L,\mathbf{s}_L) \leq 1$ for all $\mathbf{b}_L,\mathbf{s}_L$. If we choose the posted prices according to Lemma 7.20, then the corresponding mechanism has GFT that is at least $\frac{1}{4}\mathbb{E}_{\mathbf{b},\mathbf{s}}\left[\sum_i(\widetilde{\varphi}_i(b_i)-s_i)\cdot x_i(\mathbf{b},\mathbf{s})\right]$.

Moreover, by the definition of $x_i(\mathbf{b},\mathbf{s})$,

$$\begin{aligned}\mathbb{E}_{\mathbf{b},\mathbf{s}}\left[\sum_{i\in L}(\widetilde{\varphi}_i(b_i)-s_i)\cdot x_i(\mathbf{b},\mathbf{s})\right] =& \sum_{i\in L}\mathbb{E}_{b_i,s_i}[(\widetilde{\varphi}_i(b_i)-s_i)^+)]\cdot\prod_{j\in L\backslash\{i\}}\Pr_{b_j,s_j}[b_j<s_j]\\ \geq& \sum_{i\in L}\mathbb{E}_{b_i,s_i}[(\widetilde{\varphi}_i(b_i)-s_i)^+)]\cdot(1-\frac{1}{n})^{|L|}\\ \geq& \frac{1}{e}\cdot\sum_{i\in L}\mathbb{E}_{b_i,s_i}[(\widetilde{\varphi}_i(b_i)-s_i)^+)]\end{aligned}$$

The first inequality holds because for each item $j\in L$, $\Pr_{b_j,s_j}[b_j<s_j]\geq 1-1/n$. Hence,

$$\text{GFT}_{\text{SAPP}}(L)\geq\frac{1}{4e}\cdot\sum_{i\in L}\mathbb{E}_{b_i,s_i}[(\widetilde{\varphi}_i(b_i)-s_i)^+].$$

□

7.4.4 Bounding the Optimal Profit from the Unlikely to Trade Items.

In this section, we provide an upper bound for the optimal super seller profit from items in L. It is well known that in multi-item auctions the revenue of selling the items separately is a $O(\log n)$-approximation to the optimal revenue when there is a single additive buyer [LY13]. In Chapter 4 we have provided a extension of this $O(\log n)$-approximation to profit maximization. Combining this approximation with some basic observations based on the Cai-Devanur-Weinberg duality framework [CDW16], we derive the following upper bound of $\text{OPT-S}(L,\mathcal{F}\big|_L)$.

Lemma 7.23. $\text{OPT-S}(L,\mathcal{F}\big|_L)\leq O\left(\log(|L|)\cdot\sum_{i\in L}\mathbb{E}_{b_i,s_i}\left[(\tilde{\varphi}_i(b_i)-s_i)^+\right]\right)$. *Here $\widetilde{\varphi}_i(b_i)$ is Myerson's ironed virtual value function*[13] *for the buyer's distribution for item i, $\mathcal{D}_i^B$.*

We need the following lemma proved in Chapter 4.

Lemma 7.24. *(Restatement of Lemma 4.2) For any $T\subseteq[n]$ and feasibility constraint $\mathcal{J}$ with respect to T, consider the super seller auction with item set T and a $\mathcal{J}$-constrained buyer. Any flow λ_T induces a finite benchmark for the optimal profit, that is,*

13. The buyer's unironed virtual value function is $\varphi_i(b_i)=b_i-\frac{1-F_i(b_i)}{f_i(b_i)}$. These values are averaged to make the function monotonic in quantile space, which creates $\widetilde{\varphi}_i(b_i)$.

$$\text{OPT-S}(T, \mathcal{J}) \leq \max_{x \in P_{\mathcal{J}}} \mathbb{E}_{\boldsymbol{b},\boldsymbol{s}} \left[\sum_{i \in T} x_i(\boldsymbol{b}, \boldsymbol{s}) \cdot (\Phi_i^T(\boldsymbol{b}) - s_i) \right]$$

where $\Phi_i^T(\boldsymbol{b}) = b_i - \frac{1}{f_i(b_i)} \sum_{\boldsymbol{b}'} \lambda_T(\boldsymbol{b}', \boldsymbol{b}) \cdot (b_i' - b_i)$ *can be viewed as buyer* i*'s virtual value function, and* $P_{\mathcal{J}}$ *is the set of all feasible allocation rules. More specifically,* $\lambda_T(\boldsymbol{b}', \boldsymbol{b})$ *is the Lagrangian multiplier for the BIC/IR constraint that states that when the buyer has true type* $\boldsymbol{b}$*, she does not want to misreport* $\boldsymbol{b}'$*. The equality sign is achieved when the optimal dual* λ_T^* *is chosen.*

Next, we show that $\text{OPT-S}(L, \mathcal{F}\big|_L)$ is no more than $\text{OPT-S}(L, \text{ADD})$ using Lemma 7.24.

Lemma 7.25. $\text{OPT-S}(L, \mathcal{F}\big|_L) \leq \text{OPT-S}(L, ADD)$.

Proof. Let $\hat{\lambda}_L$ be the optimal dual in Lemma 7.24 when the buyer is additive without any feasibility constraint, and $\hat{\Phi}_i^L(\cdot)$ be the induced virtual value function. We have that

$$\begin{aligned}
\text{OPT-S}(L, \mathcal{F}\big|_L) &\leq \max_{x \in P_{\mathcal{F}|_L}} \mathbb{E}_{\mathbf{b},\mathbf{s}} \left[\sum_{i \in L} x_i(\mathbf{b}, \mathbf{s}) \cdot (\hat{\Phi}_i^L(\mathbf{b}) - s_i) \right] \\
&\leq \max_{x_i(\mathbf{b},\mathbf{s}) \in [0,1]} \mathbb{E}_{\mathbf{b},\mathbf{s}} \left[\sum_{i} x_i(\mathbf{b}, \mathbf{s}) (\hat{\Phi}_i^L(\mathbf{b}) - s_i) \right] \\
&= \text{OPT-S}(L, \text{ADD}).
\end{aligned}$$

□

Theorem 4.6 in Chapter 4 gives a logarithmic upper bound of the optimal profit for a single additive buyer, using the sum of optimal profit for each individual item.

Lemma 7.26. *(Restatement of Theorem 4.6)*

$$\text{OPT-S}(L, ADD) \leq \log(|L|) \cdot \sum_{i \in L} \text{OPT-S}(\{i\}) = \log(|L|) \cdot \sum_{i \in L} \mathbb{E}_{b_i, s_i}[(\varphi_i(b_i) - s_i)^+].$$

Together, Lemmas 7.25 and 7.26 conclude the proof of Lemma 7.23:

$$\text{OPT-S}(L, \mathcal{F}\big|_L) \leq O\left(\log(|L|) \cdot \sum_{i \in L} \mathbb{E}_{b_i, s_i} \left[(\tilde{\varphi}_i(b_i) - s_i)^+ \right] \right).$$

Proof of Lemma 7.17: It directly follows from Lemmas 7.16 and 7.23. □

Proof of Theorem 7.3: The theorem follows directly from Lemmas 7.17, 7.18, 7.20, and 7.22. □

7.5 Lower Bounds and the First-Best–Second-Best Gap

In the unconditional approximation results stated in Section 7.4, we compare the GFT of our mechanism to OPT. Readers may be interested in whether our mechanism is also an approximation to FB-GFT. In fact, this question is related to one of the major open problems in two-sided markets: *How large is the gap between the second-best and the first-best GFT?* In this section, we consider a unit-demand buyer and present a reduction from achieving a FB-GFT approximation in our multi-dimensional setting to the open problem regarding the gap in single-dimensional two-sided markets.

Matching Markets. This setting has a two-sided market with n buyers, n sellers, and n identical items. Each seller owns one item and each buyer is interested in buying at most one item. Thus the value (or cost) for every agent is a scalar. Here we consider a special case where for every $i \in [n]$, buyer i and seller i can only trade with each other, and at most one pair of agents in the market can trade. This is bilateral trade when $n = 1$.

Theorem 7.4. *Suppose the buyer is unit-demand in the multi-dimensional setting, and define* $\text{FB-GFT}, \text{OPT-B}$*,* GFT_{SAPP} *as in the previous section. Also consider the following matching market with n buyers and n sellers: for every $i \in [n]$, buyer i has value drawn from $\mathcal{D}_i^B$ and seller i has cost drawn from $\mathcal{D}_i^S$. Let* $\text{FB-GFT}^{\text{SD}} = \mathbb{E}_{\boldsymbol{b},\boldsymbol{s}}[\max_i(b_i - s_i)]$ *be the first-best GFT of the matching market defined above (which is the same as* FB-GFT *in the multi-dimensional unit-demand setting) and* $\text{SB-GFT}^{\text{SD}}$ *be the second-best GFT. For any $c > 1$, suppose* $\text{SB-GFT}^{\text{SD}} \geq 1/c \cdot \text{FB-GFT}^{\text{SD}}$*, then*

$$\max\{\text{OPT-B}, \text{GFT}_{\text{SAPP}}\} \geq \frac{1}{2c} \cdot \text{FB-GFT}.$$

Proof. We construct the following allocation rule $x = \{x_i(\mathbf{b}, \mathbf{s})\}_{i\in[n]}$. For every i and $\mathbf{b}, \mathbf{s}$, let

$$x_i(\mathbf{b}, \mathbf{s}) = \mathbb{1}\left[i = \text{argmax}_k(\tilde{\varphi}_k(b_k) - s_k) \wedge \tilde{\varphi}_i(b_i) \geq s_i\right].$$

Then x satisfies both properties in Lemma 7.20. By Lemma 7.20,

$$\text{GFT}_{\text{SAPP}} \geq \mathbb{E}_{\mathbf{b},\mathbf{s}}\left[\sum_i (\tilde{\varphi}_i(b_i) - s_i) \cdot x_i(\mathbf{b}, \mathbf{s})\right] = \mathbb{E}_{\mathbf{b},\mathbf{s}}\left[\max_i (\tilde{\varphi}_i(b_i) - s_i)^+\right].$$

Moreover, by Lemma 5.4, we have that

$$\text{SB-GFT}^{\text{SD}} \leq \mathbb{E}_{\mathbf{b},\mathbf{s}}\left[\max_i (\tilde{\varphi}_i(b_i) - s_i)^+\right] + \mathbb{E}_{\mathbf{b},\mathbf{s}}\left[\max_i (b_i - \tilde{\tau}_i(s_i))^+\right].$$

Thus by Lemma 7.18, we have

$$\begin{aligned}\max\{\text{OPT-B}, \text{GFT}_{\text{SAPP}}\} \geq & \frac{1}{2} \cdot \left(\mathbb{E}_{\mathbf{b},\mathbf{s}}\left[\max_i (\tilde{\varphi}_i(b_i) - s_i)^+\right] + \mathbb{E}_{\mathbf{b},\mathbf{s}}\left[\max_i (b_i - \tilde{\tau}_i(s_i))^+\right]\right) \\ \geq & \frac{1}{2} \cdot \text{SB-GFT}^{\text{SD}} \geq \frac{1}{2c} \cdot \text{FB-GFT}^{\text{SD}}.\end{aligned}$$

□

The main takeaway of Theorem 7.4 is that, if the largest gap between $\text{FB-GFT}^{\text{SD}}$ and $\text{SB-GFT}^{\text{SD}}$ is at most (i.e. a constant) c for matching markets, then our mechanism is a $2c$-approximation to FB-GFT. Note that if the buyer is additive, such a reduction clearly exists: In the additive case, items can be treated separately without impacting the IC constraint. Then performing a Buyer (or Seller) Offering mechanism for every item separately obtains GFT at least $\text{SB-GFT}^{\text{SD}}$, thus approximating FB-GFT by the assumption. Theorem 7.4 shows that for a unit-demand buyer, a similar reduction also exists using the SAPP mechanism.

On the other hand, finding a lower bound for our result (compared to OPT) is at least as hard as finding a lower bound for the approximation ratio w.r.t. FB-GFT, and thus is *at least as hard* as finding an instance in the matching market that separates $\text{FB-GFT}^{\text{SD}}$ from $\text{SB-GFT}^{\text{SD}}$—a problem that has long remained open. Indeed, even in bilateral trade, deciding whether the gap is finite or not is still open.

7.6 Some Lower-Bound Examples

Tight Example of the $\log\left(\frac{1}{r}\right)$-Approximation. Consider the case when $n = 1$ (bilateral trade). We introduce an example provided by Blumrosen and Dobzinski [BD16]. They prove that in this example, no fixed posted price mechanisms can achieve an approximation ratio better than $\Omega(\log(1/r))$ compared to the first-best GFT. In addition, we will verify that the statement also holds for the second-best GFT for the same example. It implies that our $\log(\frac{1}{r})$-approximation is tight even compared to the second-best GFT.

Example 7.1 (Example in Bilateral Trading [BD16])**.** *For any $t > 0$, consider a buyer and a seller with values on the support $[0, t]$. Let $\lambda = \frac{1}{1-e^{-t}}$. Let $F(b) = \lambda(1 - e^{-b})$ with $f(b) = \lambda e^{-b}$ and $G(s) = \lambda(e^{s-t} - e^{-t})$ with $g(s) = \lambda e^{s-t}$. Then*

$$r = \Pr[b \geq s] = \int_0^t \int_0^b \lambda e^{-b} \cdot \lambda e^{s-t} ds db = \lambda^2 \cdot e^{-t}(t - 1 + e^{-t}) = \frac{t-1}{e^t - 1} + \frac{t}{(e^t-1)^2}$$

$$\text{FB-GFT} = \int_0^t \int_0^b (b-s)\lambda e^{-b} \cdot \lambda e^{s-t} ds db = \lambda^2 \cdot (\frac{t-2}{e^t} + \frac{t+2}{e^{2t}})$$

In any fixed posted price mechanism, note that the mechanism always achieves a larger GFT by choosing the same price for both agents. The gains from trade from posting at price p:

$$\text{GFT}(p) = \int_0^p \int_p^v (b-s)\lambda e^{-b} \cdot \lambda e^{s-t} db ds = \lambda^2 (\frac{t+2}{e^{2t}} + \frac{2}{e^t} - \frac{p+2}{e^{p+t}} - \frac{e^p(t+2-p)}{e^{2t}}) < \lambda^2 (\frac{t+2}{e^{2t}} + \frac{2}{e^t})$$

When t is sufficiently large, FB-GFT *is about $\lambda^2 \cdot \frac{t-2}{e^t}$ while* $\text{GFT}(p)$ *is at most $\lambda^2 \cdot \frac{2}{e^t}$, as $\frac{t+2}{e^{2t}}$ is negligible. Thus $\text{GFT}(p) = O(1/t) \cdot$ FB-GFT. On the other hand, $r = \Theta(\frac{t}{e^t})$ for large t, $\log(\frac{1}{r}) = \Theta(t)$. Thus $\text{GFT}(p) = O(1/\log(\frac{1}{r})) \cdot$ FB-GFT.*

We now verify that $\text{GFT}(p) = O(1/\log(\frac{1}{r})) \cdot \text{OPT}$ for any $p \in [0, t]$ and sufficiently large t. By [BCWZ17],

$$\text{OPT} \geq \mathbb{E}_{b,s}[(b-s) \cdot \mathbb{1}[\tilde{\varphi}(b) - s \geq 0]]$$

For the above distribution, $\varphi(b) = b - \frac{1-F(b)}{f(b)} = b - 1 + e^{b-t}$ *is monotonic increasing in* b*. Thus* $\tilde{\varphi}(b) = \varphi(b)$*.*

$$\begin{aligned}
\frac{1}{\lambda^2}\cdot \mathbb{E}_{b,s}[(b-s)\cdot \mathbb{1}[\tilde{\varphi}(b)-s\geq 0]] &= \int_0^t\int_0^{b-1+e^{b-t}}(b-s)\cdot e^{s-b-t}dsdb \\
&\geq \int_0^t\int_0^{b-1}(b-s)\cdot e^{s-b-t}dsdb \\
&= e^{-t}\cdot\int_0^t\int_1^b k\cdot e^{-k}dkdb \qquad\qquad (k=b-s) \\
&= e^{-t}\cdot\int_0^t(-e^{-k}(k+1)\big|_1^b)ds \\
&= e^{-t}\cdot\int_0^t\left[\frac{2}{e}-e^{-b}(b+1)\right]ds \\
&= e^{-t}\cdot(\frac{2t}{e}+\frac{t+2}{e^t}-2)
\end{aligned}$$

Thus when t *is sufficiently large,* $\mathrm{OPT} = \Omega(\lambda^2\cdot\frac{t}{e^t})$ *and we have* $\mathrm{GFT}(p) = O(1/t)\cdot\mathrm{OPT} = O(1/\log(\frac{1}{r}))\cdot\mathrm{OPT}$*.*

Example 7.2 ($\mathrm{GFT}_{\mathrm{SAPP}}$ is unboundedly higher than $\mathrm{GFT}_{\mathrm{FPP}}$)**.** *For any fixed* n*, consider the following instance for an additive buyer.* $\mathcal{D}_1^B$ *and* $\mathcal{D}_1^S$ *are distributions in Example 7.1 some sufficiently large* t*. Pick any* $C > 0$*. For every* $i = 2, ldots, n$*,* $\mathcal{D}_i^B$ *is a degenerate distribution at* C*, i.e. the value is* C *with probability 1. Distribution* $\mathcal{D}_i^S$ *takes value* $C+\epsilon$ *with probability* $1-\frac{1}{2n}$ *and* C *with probability* $\frac{1}{2n}$*, for some small* $\epsilon > 0$*. As shown in Example 7.2, when* t *is large,* $r_1 = \Theta(\frac{t}{e^t}) < \frac{1}{n}$*. For* $i \geq 2$*,* $r_i = \frac{1}{2n}$*. Thus all items are "unlikely to trade" items (*$r_i < \frac{1}{n}$*).*

Note that for $i \geq 2$*,* b_i *is always no more than* s_i*. By Lemma 7.22,*

$$\mathrm{GFT}_{\mathrm{SAPP}} = \Omega\left(\sum_i \mathbb{E}_{b_i,s_i}[(\tilde{\varphi}_i(b_i)-s_i)^+]\right) = \Omega(\mathbb{E}_{b_1,s_1}[(\tilde{\varphi}_1(b_1)-s_1)^+])$$

In Example 7.1, when t *is sufficiently large,* $\mathbb{E}_{b_1,s_1}[(\tilde{\varphi}_1(b_1)-s_1)^+] = \Omega(\lambda^2\cdot\frac{t}{e^t})$*. On the other hand, any fixed price mechanism can only gain positive GFT from item 1. Thus* $\mathrm{GFT}_{\mathrm{FPP}} = O(\lambda^2\cdot\frac{2}{e^t})$*, which can be arbitrarily far from* $\mathrm{GFT}_{\mathrm{SAPP}}$ *as* t *goes to infinity.*

Dependence on r is Necessary. We show that the dependence on $r = \min_i r_i$ is necessary for the approximation result of fixed posted price mechanisms. More formally, suppose fixed posted price mechanisms achieves an approximation ratio of $f(r_1, \ldots, r_n)$, for some n-ary function f. We will show that $f(r_1, \ldots, r_n) = \Omega(\log(1/r))$. Consider the instance shown in Example 7.2. Clearly FB-GFT $= \mathbb{E}\left[(b_1 - s_1)^+\right]$. Since all items other than item 1 always contribute 0 gains from trade, no fixed posted price mechanism can achieve better than $\Omega(\log(1/r_1))$-approximation to the first-best. Thus $f(r_1, \ldots, r_n) = \Omega(\log(1/r_1))$. Similarly we have $f(r_1, \ldots, r_n) = \Omega(\log(1/r_i))$ for all $i = 1, \ldots, n$. Thus $f(r_1, \ldots, r_n) = \Omega(\log(1/r))$.

SAPP Mechanism is Necessary. We provide the following example (Example 7.3) to show that the class of SAPP mechanisms defined in Mechanism 7.4.3 is necessary to obtain any finite approximation ratio to OPT. More specifically, we show that in bilateral trading, the best FPP mechanism and the mechanism used in Lemma 7.18 can both be arbitrarily far from OPT.

Example 7.3. *For every positive integer $m \geq 2$, consider the bilateral trading instance where the seller's and buyer's (discrete) distributions are shown in the following tables. In the table, $g(s)$ (or $f(b)$) represents the density of the corresponding value in the support.*

s	0	$2^m - 2^{m-1}$	$\ldots$	$2^m - 2^k$	$\ldots$	$2^m - 1$
$g(s)$	$\frac{1}{2^m}$	$\frac{1}{2^m}$	$\ldots$	$\frac{1}{2^{k+1}}$	$\ldots$	$\frac{1}{2}$
$\tau(s)$	0	2^m	$\ldots$	2^m	$\ldots$	2^m

Table 7.1: Seller's Distribution

b	$2^m - 2^L$	$\ldots$	$2^m - 2^k$	$\ldots$	$2^m - 1$
$f(b)$	p_L	$\ldots$	p_k	$\ldots$	p_0

Table 7.2: Buyer's Distribution

For the seller's distribution, one can verify that the virtual value $\tau(s)$ is 0 if $s = 0$ and 2^m elsewhere.[14]

14. For discrete distributions, the virtual value for the seller's distribution is defined as $\tau(s) = s + \frac{\sum_{t<s} g(t)\cdot(s-s')}{g(s)}$, where s' is the largest type in the support that is smaller than s.

For the buyer's distribution, choose $L = \lfloor m - \log(m) \rfloor$*. Then define the sequence* $\{p_k\}_{k=0}^{L}$ *as follows: Construct the sequence* $\{q_k\}_{k=0}^{L}$ *with* $q_0 = 1$, $q_1 = \frac{1}{m-1}$, *and for every* $k = 2, \ldots, L$, $q_k = \frac{m-k+2}{m-k} \cdot q_{k-1}$*. Then for every* $k = 0, \ldots, L$ *define* $p_k = q_k / \sum_{j=0}^{L} q_j$ *for every* k*.*

By induction, we have $\sum_{j=0}^{k} q_j = q_{k+1} \cdot (m-k-1)$*. Thus*

$$\sum_{j=0}^{k} p_j = p_{k+1} \cdot (m-k-1). \tag{7.3}$$

Lemma 7.27. *For any sufficiently large integer* m*, let* $\mathcal{M}$ *be the mechanism used in Lemma 7.18.*[15] *Then in Example 7.3 we have*

$$\max\{\text{GFT}_{\text{FPP}}, \text{GFT}(\mathcal{M})\} \leq O(\frac{1}{\log(m)}) \cdot \text{OPT}.$$

Proof. In mechanism $\mathcal{M}$ (see Lemma 7.18), the buyer only trades with the seller when $b \geq \tau(s)$. Thus in Example 7.3, the item trades only when $s = 0$.

$$\text{GFT}(\mathcal{M}) = \sum_{k=0}^{L} (2^m - 2^k) \cdot p_k \cdot \frac{1}{2^m} \leq \sum_{k=0}^{L} p_k.$$

For FB-GFT, we have

$$\begin{aligned} \text{FB-GFT} &= \sum_{k=0}^{L} \left(\sum_{j=k+1}^{m-1} (2^j - 2^k) \cdot p_k \cdot \frac{1}{2^{j+1}} + \frac{(2^m - 2^k) p_k}{2^m} \right) \\ &\geq \sum_{k=0}^{L} p_k \cdot \left(\sum_{j=k+1}^{m-1} \frac{2^{j-1}}{2^{j+1}} + \frac{2^{m-1}}{2^m} \right) \geq \frac{1}{4} \cdot \sum_{k=0}^{L} p_k \cdot (m-k), \end{aligned}$$

where the first inequality follows from the fact that $2^j - 2^k \geq 2^{j-1}$ for any $j > k$.

Now consider any fixed posted price mechanism. Clearly the largest GFT is achieved when the posted prices are same for both the buyer and the seller. Without loss of generality we can assume the posted price p lies in the support of distributions, i.e., $p = 2^m - 2^k$ for $k = 0, \ldots, L$. For any k,

15. In bilateral trading, the mechanism is essentially the Buyer Offering mechanism [BCWZ17]: The buyer picks a take-it or leave-it price according to her value and the seller can choose whether to sell at that price.

the mechanism with posted price $p = 2^m - 2^k$ achieves GFT

$$\sum_{j=0}^{k}\left(\sum_{i=k+1}^{m-1}(2^i - 2^j)\cdot p_j \cdot \frac{1}{2^{i+1}} + \frac{(2^m-2^j)p_j}{2^m}\right) \le \sum_{j=0}^{k} p_j\left(\sum_{i=k+1}^{m-1}\frac{2^i}{2^{i+1}}+1\right) \le \sum_{j=0}^{k} p_j\cdot(m-k).$$

Let $Q_k = \sum_{j=0}^{k} p_j \cdot (m-k)$. Note that by the choice of sequence $\{p_k\}_{k=0}^{L}$ in Example 7.3, we have for any $k = 0, \dots, L-1$ that

$$Q_{k+1} - Q_k = \sum_{j=0}^{k+1} p_j\cdot(m-k-1) - \sum_{j=0}^{k} p_j\cdot(m-k) = p_{k+1}\cdot(m-k-1) - \sum_{j=0}^{k} p_j = 0.$$

Thus each Q_k is the same value. Let this value be Q. Then $Q = Q_L = \sum_{j=0}^{L} p_j\cdot(m-L) = m-L$. Moreover, $\text{GFT}_{\text{FPP}} \le \max_k Q_k = Q$. $\text{GFT}(\mathcal{M}) \le \sum_{k=0}^{L} p_k = \frac{Q_L}{m-L} \le \frac{1}{\log(m)}\cdot Q$. On the other hand,

$$\begin{aligned}\text{FB-GFT} &\ge \frac{1}{4}\cdot\sum_{k=0}^{L} p_k\cdot(m-k) = \frac{1}{4}\cdot\sum_{k=0}^{L}\sum_{j=0}^{k-1} p_j \qquad \text{(Equation 7.3)}\\ &= \frac{1}{4}\sum_{k=0}^{L}\frac{Q_{k-1}}{m-k+1} \ge Q\cdot\int_{m-L+1}^{m+1}\frac{1}{x}dx = \frac{Q}{4}\cdot\log\left(\frac{m+1}{m-L+1}\right).\end{aligned}$$

When m is sufficiently large, we have $\text{FB-GFT} \ge \frac{Q}{5}\cdot\log(m)$. Thus

$$\frac{\log(m)}{5}\cdot\max\{\text{GFT}_{\text{FPP}}, \text{GFT}(\mathcal{M})\} \le \text{FB-GFT}.$$

It remains to verify that OPT is a constant factor of FB-GFT. By Lemma 5.5,

$$\text{OPT} \ge \mathbb{E}_{b,s}[(b-s)\cdot \mathbb{1}[\tilde{\varphi}(b) - s \ge 0]].$$

Now we calculate the buyer's virtual value.[16] We have that $\varphi(2^m-1) = 2^m-1$, and for every

16. For discrete distributions, the buyer's virtual value is defined as $\varphi(b) = b - \frac{\sum_{t>b} f(t)\cdot(b'-b)}{f(b)}$, where b' is the smallest type in the support that is larger than b.

$k = 1, \dots, L$,

$$\varphi(2^m - 2^k) = (2^m - 2^k) - \frac{\sum_{j=0}^{k-1} p_j \cdot 2^{k-1}}{p_k} = 2^m - 2^k - 2^{k-1}(m-k) \geq 2^m - 2^k \cdot m,$$

thus $\varphi(\cdot)$ is monotone increasing and $\tilde{\varphi}(b) = \varphi(b)$ for every b. Note that when $b = 2^m - 2^k$ and $s \leq 2^m - 2^{k+\log(m)}$, it holds that $\varphi(b) \geq s$. We have

$$\begin{aligned}
\text{OPT} &\geq \sum_{k=0}^{L} \left(\sum_{j=k+\lceil \log(m) \rceil}^{m-1} (2^j - 2^k) \cdot p_k \cdot \frac{1}{2^{j+1}} + \frac{(2^m - 2^k) p_k}{2^m} \right) \\
&= \text{FB-GFT} - \sum_{k=0}^{L} \sum_{j=k+1}^{k+\lceil \log(m) \rceil - 1} (2^j - 2^k) \cdot p_k \cdot \frac{1}{2^{j+1}} \\
&\geq \text{FB-GFT} - \sum_{k=0}^{L} \sum_{j=k+1}^{k+\lceil \log(m) \rceil - 1} p_k \cdot \frac{2^j}{2^{j+1}} \\
&\geq \text{FB-GFT} - \frac{\log(m)}{2} \sum_{k=0}^{L} p_k = \text{FB-GFT} - \frac{\log(m)}{2}.
\end{aligned}$$

Note that $\text{FB-GFT} \geq \frac{Q}{5} \cdot \log(m) = \frac{m-L}{5} \cdot \log(m) \geq \frac{1}{5} \cdot \log^2(m)$. Thus when $m \to \infty$, $\frac{\text{OPT}}{\text{FB-GFT}} \to 1$. We finish the proof. $\square$

Chapter 8

Conclusion and Open Problems

In many of the mechanism design problems, there is a discrepancy between theory and practice: The mechanism that achieves the theoretically optimal objective and mechanisms used in practice, are often quite different. Motivated by the existence of this discrepancy, in this thesis we addressed the following question: **Can we design simple mechanisms to approximate the optimal mechanism?**

In this thesis, we addressed three central mechanism design problems. The first follows from Myerson's seminal result on the revenue-optimal auction in the single-item case, as well as the existence of many undesirable properties of the revenue-optimal auction in multi-item case. There has been extensive research effort on designing simple and approximately-optimal mechanisms, but no result has been found beyond a single buyer, or multiple buyers with linear valuations. Our work presents a unified approach to approximate the optimal revenue in multi-item auctions, with multiple buyers with non-linear (subadditive) valuations. We improve the approximate ratio for many of the results in the literature, and generalize the result to broader case, when the buyer is XOS or subadditive. The main open question following our results is:

Open Question 1. *Can we design simple mechanisms that obtain a constant fraction of the optimal revenue, for multiple buyers with valuations that are subadditive over independent items?*

A large fraction of the proof in this work already applies to subadditive valuations. More specifically, our upper bound for the optimal revenue from Theorem 3.2 holds for all subadditive valuations, and we have used it to obtain a constant factor approximation for a single subadditive buyer and a $O(\log m)$-approximation for multiple subadditive buyers. Our analysis for the term Single and Tail also applies to subadditive valuations.

The only component that does not extend to subadditive valuations is the analysis of the Core. Our proof is inspired by Feldman et al. [FGL15], who showed that there exists a sequential posted-price mechanism that is an $O(1)$-approximation to the optimal social welfare for bidders with XOS valuations. Their proof makes heavy use of the supporting prices for XOS valuations, and the approximation ratio degrades to $O(\log m)$ for subadditive valuations.

A recent breakthrough by Dutting et al. [DKL20] managed to improved the approximation ratio of Feldman et al. for subadditive valuations. They proved an $O(\log\log m)$-approximation to the optimal social welfare for buyers with subadditive valuations, using sequential posted-price mechanisms. In the same paper, they also uses this result to achieve the same approximation ratio to the optimal revenue, following a similar approach to our work. Thus to resolve the open question above, it is worthwhile to further improve the approximation ratio proved in [DKL20]. In particular,

Open Question 2. *Can sequential posted-price mechanisms obtain a constant fraction of the optimal social welfare when buyers have subadditive valuations?*

An astute reader may have noticed that our approximation results are only existential. Luckily, the only nonconstructive part of our argument is finding the right β, which is essentially the same as finding the ex-ante allocation probabilities of the optimal or an approximately optimal mechanism. In Appendix A.5, we show how to find the right β when the buyers are symmetric, but the asymmetric case remains open.

Open Question 3. *Can we design a polynomial time algorithm to compute these simple and approximately revenue-optimal mechanisms for constrained additive and XOS valuations?*

As the second mechanism design problem, we studied profit maximization in multi-item auctions, which is considered as a more general model of revenue maximization. We proposed a novel permit-selling mechanism, and showed a constant factor approximation to the optimal profit. Our result applies to a single constrained-additive buyer and multiple buyers with selectable feasibility constraints (such as Matroid). The main open question following this result is whether we can generalize the result to more general settings.

Open Question 4. *In the profit maximization problem, can we design simple mechanisms that obtain a constant fraction of the optimal profit, for a single subadditive buyer, or multiple constrained-additive buyers?*

The last but not least mechanism design problem discussed in this thesis is approximating gains from trade in two-sided markets. While the impossibility result by Myerson and Satterthwaite [MS83] shows that no truthful and budget balanced mechanisms can achieve full GFT and the optimal mechanism is rather complex even in the simplist bilateral trade setting, our work provide simple mechanisms that approximate the optimal GFT, in bilateral trade and more general two-sided markets.

The main open question in two-sided markets follows from the Myerson-Satterthwaite impossibility result. It shows that even in bilateral trade, the unconstrained first-best GFT is strictly larger than the second-best GFT, the maximum GFT attainable by truthful and budget balanced mechanisms. However, it remains open whether the two has a finite gap.

Open Question 5. *How big is the gap between the first-best GFT and second-best GFT in two-sided markets?*

We remark that our unconditional approximation proved in this thesis are all compared to the second-best GFT, while all other results in the literature compares to the first-best GFT. If the gap between the first-best GFT and second-best GFT is shown to be infinite, then our proposed framework to approximate the second-best GFT becomes an essential step of this problem.

In the thesis we made a first step on studying GFT approximation in multi-dimensional two-sided markets. We considered a special setting with a single buyer and multiple unit-supply sellers and proved an unconditional $O(\log m)$-approximation. The open questions following this result is whether the result can be improved or generalized.

Open Question 6. *Can we design a truthful and budget balanced mechanism that approximates the second-best GFT, in two-sided markets with multiple buyers or multi-dimensional sellers?*

Appendix A

Missing Details from Chapter 3

A.1 Proof of Theorem 3.1

Proof of Theorem 3.1: When λ is useful, we can simplify function $\mathcal{L}(\lambda, \sigma, p)$ by removing the term associated with p and replacing $\sum_{t_i' \in T_i^+} \lambda(t_i, t_i')$ with $f_i(t_i) + \sum_{t_i' \in T_i} \lambda(t_i', t_i)$. After the simplification, we have

$$\begin{aligned}\mathcal{L}(\lambda, \sigma, p) = \sum_{i=1}^{n} \sum_{t_i \in T_i} f_i(t_i) \cdot \sum_{S \subseteq [m]} \sigma_{iS}(t_i) \cdot \\ \left(v_i(t_i, S) - \frac{1}{f_i(t_i)} \sum_{t_i' \in T_i} \lambda_i(t_i', t_i) \left(v_i(t_i', S) - v_i(t_i, S) \right) \right) \\ = \sum_{i=1}^{n} \sum_{t_i \in T_i} f_i(t_i) \cdot \sum_{S \subseteq [m]} \sigma_{iS}(t_i) \cdot \Phi_i(t_i, S),\end{aligned}$$

which is exactly the virtual welfare of σ with respect to λ. Now, we only need to prove that $\mathcal{L}(\lambda, \sigma, p)$ is greater than the revenue of M. Let us think of $\mathcal{L}(\lambda, \sigma, p)$ using Expression (3.1). Since M is a BIC mechanism,

$$\sum_{S \subseteq [m]} v_i(t_i, S) \cdot \left(\sigma_{iS}(t_i) - \sigma_{iS}(t_i') \right) - \left((p_i(t_i) - p_i(t_i') \right) \geq 0$$

for any i, $t_i \in T_i$ and $t'_i \in T_i^+$. Also, all the dual variables λ are nonnegative. Therefore, it is clear that $\mathcal{L}(\lambda, \sigma, p)$ is at least as large as the revenue of M.

When λ^* is the optimal dual variable, by strong duality, we know $\max_{\sigma \in P(D), p} \mathcal{L}(\lambda^*, \sigma, p)$ equals to the revenue of $M^* = (\sigma^*, p^*)$. But we also know that $\mathcal{L}(\lambda^*, \sigma^*, p^*)$ is at least as large as the revenue of M^*, therefore σ^* maximizes the virtual welfare. □

A.2 Proof of Lemma 3.4

Lemma A.1. *For any flow $\lambda_i^{(\beta)}$ that respects the partial specification in Figure 3.3, the corresponding virtual valuation function $\Phi_i^{(\beta_i)}$ of $v_i^{(\beta_i)}$ for any buyer i is:*

- $v_i(t_i, S\backslash\{j\}) + V_i(t_{ij}) - \frac{1}{f_i(t_i)} \sum_{t'_i \in T_i} \lambda(t'_i, t_i) \cdot \Big(V_i(t'_{ij}) - V_i(t_{ij})\Big)$, *if $t_i \in R_j^{(\beta_i)}$ and $j \in S$.*
- $v_i(t_i, S)$, *otherwise.*

Proof of Lemma A.1: The proof follows the definitions of the virtual valuation function (Definition 3.2) and relaxed valuation (Definition 3.3). We use $t_{i,-j} = \langle t_{ij'} \rangle_{j' \neq j}$ to denote bidder i's information for all items except item j. If $t_i \in R_j^{(\beta_i)}$ and $j \in S$, $v_i^{(\beta_i)}(t_i, S) = v_i(t_i, S\backslash\{j\}) + V_i(t_{ij})$. Since $\lambda(t_i, t'_i) > 0$ only when $t_{i,-j} = t'_{i,-j}$ and $t'_i \in R_j^{(\beta_i)}$, $v_i^{(\beta_i)}(t'_i, S) = v_i(t'_i, S\backslash\{j\}) + V_i(t'_{ij}) = v_i(t_i, S\backslash\{j\}) + V_i(t'_{ij})$. Therefore,

$$\Phi_i^{(\beta_i)}(t_i, S) = v_i(t_i, S\backslash\{j\}) + V_i(t_{ij}) - \frac{1}{f_i(t_i)} \sum_{t'_i \in T_i} \lambda(t'_i, t_i) \cdot \big(V_i(t'_{ij}) - V_i(t_{ij})\big)$$

If $t_i \in R_j^{(\beta_i)}$ and $j \notin S$ or $t_i \in R_0^{(\beta_i)}$, then $v_i^{(\beta_i)}(t_i, S) = v_i(t_i, S)$. If $t_i \in R_0^{(\beta_i)}$, there is no flow entering t_i except from the source, so clearly $\Phi_i^{(\beta_i)}(t_i, S) = v_i(t_i, S)$. If $t_i \in R_j^{(\beta_i)}$, then for any t'_i that only differs from t_i in the j-th coordinate, we have $v_i(t'_i, S) = v_i(t_i, S)$, because $j \notin S$. Hence, $\Phi_i^{(\beta_i)}(t_i, S) = v_i(t_i, S)$. □

Proof of Lemma 3.4:

Let $\Psi_{ij}^{(\beta_i)}(t_i) = V_i(t_{ij}) - \frac{1}{f_i(t_i)} \sum_{t'_i \in T_i} \lambda(t'_i, t_i) \cdot \Big(V_i(t'_{ij}) - V_i(t_{ij})\Big)$. According to Lemma A.1, it suffices to prove that for any $j > 0$, any $t_i \in R_j^{(\beta_i)}$, $\Psi_{ij}^{(\beta_i)}(t_i) \leq \tilde{\varphi}_{ij}(V_i(t_{ij}))$.

Claim A.1. *For any type* $t_i \in R_j^{(\beta_i)}$*, if we only allow flow from type* t_i' *to* t_i*, where* $t_{ik} = t_{ik}'$ *for all* $k \neq j$ *and* $t_{ij}' \in \text{argmin}_{s \in T_{ij} \wedge V_i(s) > V_i(t_{ij})} V_i(s)$*, and the flow* $\lambda(t_i', t_i)$ *equals* $\frac{f_{ij}(t_{ij})}{\Pr_{v \sim F_{ij}}[v = V_i(t_{ij})]}$ *fraction of the total in flow to* t_i'*, then there exists a flow* λ *such that*

$$\Psi_{ij}^{(\beta_i)}(t_i) = \varphi_{ij}(V_i(t_{ij})) = V_i(t_{ij}) - \frac{\Big(V_i(t_{ij}') - V_i(t_{ij})\Big) \cdot \Pr_{v \sim F_{ij}}[v > V_i(t_{ij})]}{\Pr_{v \sim F_{ij}}[v = V_i(t_{ij})]},$$

where $\varphi_{ij}(V_i(t_{ij}))$ *is the Myerson virtual value for* $V_i(t_{ij})$ *with respect to* F_{ij}*.*

Proof. As the flow only goes from t_i' and t_i, where t_i' and t_i only differs in the j-th coordinate, and $t_{ij} \in \text{argmax}_{s \in T_{ij} \wedge V_i(s) < V_i(t_{ij}')} V_i(s)$. If t_{ij} is a type with the largest $V_i(t_{ij})$ value in T_{ij}, then there is no flow coming into it except the one from the source, so $\Psi_{ij}^{(\beta_i)}(t_i) = V_i(t_{ij})$. For every other value of t_{ij}, the in flow is exactly

$$\frac{f_{ij}(t_{ij})}{\Pr_{v \sim F_{ij}}[v = V_i(t_{ij})]} \prod_{k \neq j} f_{ik}(t_{ik}) \cdot \sum_{x \in T_{ij}: V_i(x) > V_i(t_{ij})} f_{ij}(x) = \prod_{k} f_{ik}(t_{ik}) \cdot \frac{\Pr_{v \sim F_{ij}}[v > V_i(t_{ij})]}{\Pr_{v \sim F_{ij}}[v = V_i(t_{ij})]}.$$

This is because each type of the form $(x, t_{i,-j})$ with $V_i(x) > V_i(t_{ij})$ is also in $R_j^{(\beta_i)}$. So $\frac{f_{ij}(t_{ij})}{\Pr_{v \sim F_{ij}}[v = V_i(t_{ij})]}$ of all flow that enters these types will be passed down to t_i (and possibly further, before going to the sink), and the total amount of flow entering all of these types from the source is exactly $\prod_{k \neq j} f_{ik}(t_{ik}) \cdot \sum_{x \in T_{ij}: V_i(x) > V_i(t_{ij})} f_{ij}(x)$. Therefore, $\Psi_{ij}^{(\beta_i)}(t_i) = \varphi_{ij}(V_i(t_{ij}))$. Whenever there is no more type $t_i \in R_j^{(\beta_i)}$ with smaller $V_i(t_{ij})$ value, we push all the flow to the sink. □

If F_{ij} is regular, this completes our proof. When F_{ij} is not regular, we can iron the virtual value function in the same way as in [CDW16]. Basically, for two types $t_i, t_i' \in R_j^{(\beta_i)}$ that only differ in the j-th coordinate, if $\Psi_{ij}^{(\beta_i)}(t_i) > \Psi_{ij}^{(\beta_i)}(t_i')$ but $V_i(t_{ij}) < V_i(t_{ij}')$, add a loop between t_i and t_i' with a proper weight to make $\Psi_{ij}^{(\beta_i)}(t_i) = \Psi_{ij}^{(\beta_i)}(t_i')$.

Lemma A.2. *[CDW16] For any* β *and* i*, there exists a flow* $\lambda_i(\beta)$ *such that for any* $t_i \in R_j^{(\beta_i)}$*,* $\Psi_{ij}^{(\beta_i)}(t_i) \leq \tilde{\varphi}_{ij}(V_i(t_{ij}))$*.*

□

A.3 Analysis for the Single-Bidder Case

Proof of lemma 3.9: In NON-FAVORITE, since $R_0^\beta = \emptyset$, the corresponding term is simply 0. Notice $v(t,\cdot)$ is a monotone valuation for every $t \in T$,

$$\begin{aligned}\textsc{Non-Favorite}(M) =&\sum_{t\in T} f(t)\cdot \sum_{j\in[m]} \mathbb{1}\left[t\in R_j^{(\beta)}\right]\cdot\left(\sum_{S:j\in S}\sigma_S^{(\beta)}(t)\cdot v(t,S\backslash\{j\})+\sum_{S:j\notin S}\sigma_S^{(\beta)}(t)\cdot v(t,S)\right)\\ \leq&\sum_{t\in T} f(t)\cdot \sum_{j\in[m]} \mathbb{1}\left[t\in R_j^{(\beta)}\right]\sum_{S}\sigma_S^{(\beta)}(t)\cdot v(t,[m]\backslash\{j\})\\ \leq&\sum_{t\in T} f(t)\cdot \sum_{j\in[m]} \mathbb{1}\left[t\in R_j^{(\beta)}\right]\cdot v(t,[m]\backslash\{j\}) \qquad (\sum_S \sigma_S^{(\beta)}(t)\leq 1)\end{aligned}$$

Recall that for all $t \in T$ and $S \subseteq [m]$, $v(t,S) \leq v\left(t, S\cap \mathcal{C}(t)\right)+\sum_{j\in S\cap\mathcal{T}(t)} V(t_j)$. We will replace $v(t,[m]\backslash\{j\})$ above with $v\left(t,([m]\backslash\{j\})\cap\mathcal{C}(t)\right)+\sum_{k\in([m]\backslash\{j\})\cap\mathcal{T}(t)} V(t_k)$. First, the contribution from $v\left(t,([m]\backslash\{j\})\cap\mathcal{C}(t)\right)$ is upper bounded by the CORE(M).

$$\begin{aligned}&\sum_{t\in T} f(t)\cdot \sum_{j\in[m]} \mathbb{1}\left[t\in R_j^{(\beta)}\right]\cdot v\left(t,([m]\backslash\{j\})\cap\mathcal{C}(t)\right)\\ \leq&\sum_{t\in T} f(t)\cdot \sum_{j\in[m]} \mathbb{1}\left[t\in R_j^{(\beta)}\right]\cdot v\left(t,\mathcal{C}(t)\right)=\sum_{t\in T} f(t)\cdot v\left(t,\mathcal{C}(t)\right) \quad (\textsc{Core}(M))\end{aligned}$$

The inequality comes from the monotonicity of $v(t,\cdot)$ and the fact that for every t only stays in one region $R_j^{(\beta)}$.

Next, we upper bound the contribution from $\sum_{k\in([m]\backslash\{j\})\cap\mathcal{T}(t)} V(t_k)$ by the $\textsc{Tail}(M)$.

$$\begin{aligned}
&\sum_{t\in T} f(t)\cdot\sum_{j\in[m]}\mathbb{1}\left[t\in R_j^{(\beta)}\right]\cdot\sum_{k\in([m]\backslash\{j\})\cap\mathcal{T}(t)} V(t_k)\\
=&\sum_{t\in T} f(t)\cdot\sum_{j\in\mathcal{T}(t)} V(t_j)\cdot\mathbb{1}\left[t\notin R_j^{(\beta)}\right]\\
\leq&\sum_{t\in T} f(t)\cdot\sum_{j\in\mathcal{T}(t)} V(t_j)\cdot\mathbb{1}\left[\exists k\neq j, V(t_k)\geq V(t_j)\right] \quad \text{(Definition of } R_j^{(\beta)}\text{)}\\
=&\sum_{j}\sum_{t_j:V(t_j)\geq c} f_j(t_j)\cdot V(t_j)\cdot\Pr_{t_{-j}}\left[\exists k\neq j, V(t_k)\geq V(t_j)\right] \quad (\textsc{Tail}(M))
\end{aligned}$$

□

Proof of Lemma 3.12: We argue the three properties one by one.

- *Monotonicity:* For all $t\in T$ and $U\subseteq V\subseteq[m]$, $U\cap\mathcal{C}(t)\subseteq V\cap\mathcal{C}(t)$. Since $v(t,\cdot)$ is monotone,

  $$v'(t,U)=v\left(t,U\cap\mathcal{C}(t)\right)\leq v\left(t,V\cap\mathcal{C}(t)\right)=v'(t,V).$$

  Thus, $v'(t,\cdot)$ is monotone.

- *Subadditivity:* For all $t\in T$ and $U,V\subseteq[m]$, notice $(U\cup V)\cap\mathcal{C}(t)=(U\cap\mathcal{C}(t))\cup(V\cap\mathcal{C}(t))$, we have

  $$v'(t,U\cup V)=v\left((t,(U\cap\mathcal{C}(t))\cup(V\cap\mathcal{C}(t))\right)\leq v\left(t,U\cap\mathcal{C}(t)\right)+v\left(t,V\cap\mathcal{C}(t)\right)=v'(t,U)+v'(t,V).$$

- *No externalities:* For any $t\in T$, $S\subseteq[m]$, and any $t'\in T$ such that $t_j=t'_j$ for all $j\in S$, to prove $v'(t,S)=v'(t',S)$, it is enough to show $S\cap\mathcal{C}(t)=S\cap\mathcal{C}(t')$. Since $V(t_j)=V(t'_j)$ for any $j\in S$, $j\in S\cap\mathcal{C}(t)$ if and only if $j\in S\cap\mathcal{C}(t')$.

□

Proof of Lemma 3.13: For any $t,t'\in T$, and set $X,Y\subseteq[m]$, define set $H=\left\{j\in X\cap Y: t_j=t'_j\right\}$.

Since $v'(\cdot,\cdot)$ has no externalities, $v'(t',H) = v'(t,H)$. Therefore,

$$\begin{aligned}
|v'(t,X) - v'(t',Y)| &= \max\left\{v'(t,X) - v'(t',Y), v'(t',Y) - v'(t,X)\right\} \\
&\leq \max\left\{v'(t,X) - v'(t',H), v'(t',Y) - v'(t,H)\right\} \quad \text{(Monotonicity)} \\
&\leq \max\left\{v'(t,X\backslash H), v'(t',Y\backslash H)\right\} \quad \text{(Subadditivity)} \\
&= \max\left\{v\left(t,(X\backslash H)\cap\mathcal{C}(t)\right), v\left(t',(Y\backslash H)\cap\mathcal{C}(t)\right)\right\} \quad \text{(Definition of } v'(\cdot,\cdot)\text{)} \\
&\leq c\cdot\max\left\{|X\backslash H|, |Y\backslash H|\right\} \\
&\leq c\cdot\left(|X\Delta Y| + |X\cap Y| - |H|\right)
\end{aligned}$$

The second last inequality is because both $v(t,\cdot)$ and $v(t',\cdot)$ are subadditive and for any item $j\in\mathcal{C}(t)$ $(\mathcal{C}(t'))$ the single-item valuation $V(t_j)$ $(V(t'_j))$ is less than c. □

A.4 Missing Proofs for the Multi-Bidder Case

Proof of Lemma 3.16: We replace every $v_i(t_i,S)$ in NON-FAVORITE(M,β) with $v_i\left(t_i,S\cap\mathcal{C}_i(t_i)\right) + \sum_{j\in S\cap\mathcal{T}_i(t_i)} V_i(t_{ij})$. Let the contribution from $v_i\left(t_i,S\cap\mathcal{C}_i(t_i)\right)$ be the first term and the contribution from $\sum_{j\in S\cap\mathcal{T}_i(t_i)} V_i(t_{ij})$ be the second term.

$$\begin{aligned}
&\sum_i\sum_{t_i\in T_i} f_i(t_i)\cdot\mathbb{1}\left[t_i\in R_0^{(\beta_i)}\right]\cdot\sum_{S\subseteq[m]}\sigma_{iS}^{(\beta)}(t_i)\cdot v_i(t_i,S\cap\mathcal{C}_i(t_i)) + \\
&\sum_i\sum_{t_i\in T_i} f_i(t_i)\cdot\sum_{j\in[m]}\mathbb{1}\left[t_i\in R_j^{(\beta_i)}\right]\cdot \\
&\qquad\left(\sum_{S:j\in S}\sigma_{iS}^{(\beta)}(t_i)\cdot v_i\left(t_i,(S\backslash\{j\})\cap\mathcal{C}_i(t_i)\right) + \sum_{S:j\notin S}\sigma_{iS}^{(\beta)}(t_i)\cdot v_i\left(t_i,S\cap\mathcal{C}_i(t_i)\right)\right) \\
\leq&\sum_i\sum_{t_i\in T_i} f_i(t_i)\cdot\sum_{S\subseteq[m]}\sigma_{iS}^{(\beta)}(t_i)\cdot v_i(t_i,S\cap\mathcal{C}_i(t_i)) \quad \text{(CORE}(M,\beta)\text{)}
\end{aligned}$$

The inequality comes from the Monotonicity of $v_i(t_i,\cdot)$ by replacing $v_i\left(t_i,(S\backslash\{j\})\cap\mathcal{C}_i(t_i)\right)$ with $v_i(t_i,S\cap\mathcal{C}_i(t_i))$.

For the second term, notice that when $t_i \in R_0^{(\beta_i)}$, $\mathcal{T}_i(t_i) = \emptyset$. It can be rewritten as:

$$\sum_i \sum_{t_i \in T_i} f_i(t_i) \cdot \sum_{j \in [m]} \mathbb{1}\left[t_i \in R_j^{(\beta_i)}\right] \cdot \left(\sum_{S: j \in S} \sigma_{iS}^{(\beta)}(t_i) \cdot \sum_{k \in (S \backslash \{j\}) \cap \mathcal{T}_i(t_i)} V_i(t_{ik}) + \sum_{S: j \notin S} \sigma_{iS}^{(\beta)}(t_i) \cdot \sum_{k \in S \cap \mathcal{T}_i(t_i)} V_i(t_{ik}) \right)$$

$$= \sum_i \sum_{t_i \in T_i} f_i(t_i) \cdot \sum_{j \in \mathcal{T}_i(t_i)} V_i(t_{ij}) \cdot \mathbb{1}\left[t_i \notin R_j^{(\beta_i)}\right] \cdot \pi_{ij}^{(\beta)}(t_i) \qquad (\text{Recall } \pi_{ij}^{(\beta)}(t_i) = \sum_{S: j \in S} \sigma_{iS}^{(\beta)}(t_i))$$

$$\leq \sum_i \sum_{t_i \in T_i} f_i(t_i) \cdot \sum_{j \in \mathcal{T}_i(t_i)} V_i(t_{ij}) \cdot \mathbb{1}\left[t_i \notin R_j^{(\beta_i)}\right] \qquad (\pi_{ij}^{\beta}(t_i) \leq 1)$$

$$= \sum_i \sum_{t_i \in T_i} f_i(t_i) \cdot \sum_{j \in \mathcal{T}_i(t_i)} V_i(t_{ij}) \cdot \sum_{k \neq j} \mathbb{1}\left[t_i \in R_k^{(\beta_i)}\right] \qquad (t_i \notin R_0^{(\beta_i)})$$

$$\leq \sum_i \sum_j \sum_{t_{ij}: V_i(t_{ij}) \geq \beta_{ij} + c_i} f_{ij}(t_{ij}) \cdot V_i(t_{ij}) \cdot \sum_{k \neq j} \Pr_{t_{ik}}\left[V_i(t_{ik}) - \beta_{ik} \geq V_i(t_{ij}) - \beta_{ij}\right] \quad (\textsc{Tail}(M, \beta))$$

□

Lemma A.3. *Let $\{x_{ij}\}_{i \in [n], j \in [m]}$ be a set of nonnegative numbers. For any buyer i, any type $t_i \in T_i$, let $X_i(t_i) = \{j \mid V_i(t_{ij}) < x_{ij}\}$, and let*

$$\bar{v}_i(t_i, S) = v_i(t_i, S \cap X_i(t_i)),$$

for any set $S \subseteq [m]$. Then for any bidder i, any type $t_i \in T_i$, $\bar{v}_i(t_i, \cdot)$, satisfies monotonicity, subadditivity and no externalities.

Proof of Lemma A.3: We will argue these three properties one by one.

- *Monotonicity:* For all $t_i \in T_i$ and $U \subseteq V \subseteq [m]$, since $v_i(t_i, \cdot)$ is monotone,

$$\bar{v}_i(t_i, U) = v_i(t_i, U \cap X_i(t_i)) \leq v_i(t_i, V \cap X_i(t_i)) = \bar{v}(t_i, V)$$

  Thus $\bar{v}_i(t_i, \cdot)$ is monotone.

- *Subadditivity:* For all $t_i \in T_i$ and $U, V \subseteq [m]$, $(U \cup V) \cap X_i(t_i) = (U \cap X_i(t_i)) \cup (V \cap X_i(t_i))$.

Since $v_i(t_i, \cdot)$ is subadditive, we have

$$\begin{aligned}\bar{v}_i(t_i, U \cup V) &= v_i(t_i, (U \cap X_i(t_i)) \cup (V \cap X_i(t_i))) \\ &\leq v_i(t_i, U \cap X_i(t_i)) + v_i(t_i, V \cap X_i(t_i)) = \bar{v}_i(t_i, U) + \bar{v}_i(t_i, V).\end{aligned}$$

- *No externalities:* For any $t_i \in T_i$, $S \subseteq [m]$, and any $t_i' \in T_i$ such that $t_{ij} = t_{ij}'$ for all $j \in S$, to prove $\bar{v}_i(t_i, S) = \bar{v}_i(t_i', S)$, it suffices to show $S \cap X_i(t_i) = S \cap X_i(t_i')$. Since $V_i(t_{ij}) = V_i(t_{ij}')$, for any item $j \in S$, $j \in S \cap X_i(t_i)$ if and only if $j \in S \cap X_i(t_i')$.

□

Proof of Lemma 3.25: By Lemma A.3 and Definition 3.9, $\hat{v}_i(t_i, \cdot)$ satisfies monotonicity, subadditivity and no externalities.

$$\hat{v}_i(t_i, S') = v_i\left(t_i, S' \cap Y_i(t_i)\right) \geq v_i\left(t_i, \left(S' \cap Y_i(t_i)\right) \cap \mathcal{C}_i(t_i)\right) = v_i'\left(t_i, S' \cap Y_i(t_i)\right).$$

Since $S' \cap Y_i(t_i) \subseteq S$,

$$v_i'\left(t_i, S' \cap Y_i(t_i)\right) \geq \sum_{j \in S' \cap Y_i(t_i)} \gamma_j^S(t_i) = \sum_{j \in S'} \hat{\gamma}_j^S(t_i).$$

for all $j \in S$, we have $\min\{t_{ij}^{(k)}, Q_j + \tau_i\} \geq \hat{\gamma}_j^S(t_i)$. Therefore, $\hat{v}_i(t_i, S) \geq \sum_{j \in S} \hat{\gamma}_j^S(t_i)$. □

Proof of Lemma 3.27: We first prove that $\mu_i(\cdot, \cdot)$ is τ_i-Lipschitz. For any $t_i, t_i' \in T_i$ and set $X, Y \in [m]$, let $X^* \in \text{argmax}_{S \subseteq X}\left(\hat{v}_i(t_i, S) - \sum_{j \in S} Q_j\right), Y^* \in \text{argmax}_{S \subseteq Y}\left(\hat{v}_i(t_i', S) - \sum_{j \in S} Q_j\right)$. Recall that $\hat{v}_i(t_i, X^*) = v_i\left(t_i, \{j \mid (j \in X^*) \wedge (V_i(t_{ij}) < Q_j + \tau_i)\}\right)$. This means that for every $k \in X^*$, $V_i(t_{ik})$ must be less than $Q_k + \tau_i$, because otherwise $\mu_i(t_i, X^*) < \mu_i(t_i, X^* \backslash \{k\})$. Therefore, $\hat{v}_i(t_i, S) = v_i(t_i, S)$ for all $S \subseteq X^*$. Since $v_i(t_i, \cdot)$ is subadditive, $v_i(t_i, X^*) \leq v_i(t_i, X^* \backslash \{k\}) + V_i(t_{ik})$. So by the optimality of X^*, it must be that $V_i(t_{ik}) \geq Q_k$ for all $k \in X^*$. Similarly, we can show that for every $k \in Y^*$, $V_i(t_{ik}') \in [Q_k, Q_k + \tau_i]$.

Now let set $H = \{j \mid j \in X \cap Y \wedge t_{ij} = t_{ij}'\}$, if $\mu_i(t_i, X) > \mu_i(t_i', Y)$.

$$\left|\mu_i(t_i,X)-\mu_i(t_i',Y)\right| = \left(\hat{v}_i(t_i,X^*)-\sum_{j\in X^*}Q_j\right)-\left(\hat{v}_i(t_i',Y^*)-\sum_{j\in Y^*}Q_j\right)$$

$$\leq \left(\hat{v}_i(t_i,X^*)-\sum_{j\in X^*}Q_j\right)-\left(\hat{v}_i(t_i',X^*\cap H)-\sum_{j\in X^*\cap H}Q_j\right) \quad \text{(Optimality of } Y^* \text{ and } X^*\cap H\subseteq Y\text{)}$$

$$\leq \hat{v}_i(t_i,X^*)-\hat{v}_i(t_i,X^*\cap H)-\sum_{j\in X^*\backslash H}Q_j \qquad \text{(No externalities of } \hat{v}_i(t_i,\cdot)\text{)}$$

$$\leq \hat{v}_i(t_i,X^*\backslash H)-\sum_{j\in X^*\backslash H}Q_j \qquad \text{(Subadditivity of } \hat{v}_i(t_i,\cdot)\text{)}$$

$$\leq \tau_i\cdot|X^*\backslash H| \qquad (V_i(t_{ij})\in[Q_j,Q_j+\tau_i] \text{ for all } j\in X^*)$$

$$\leq \tau_i\cdot|X\backslash H|$$

Similarly, if $\mu_i(t_i,X)\leq\mu_i(t_i',Y)$, $|\mu_i(t_i,X)-\mu_i(t_i',Y)|\leq\tau_i\cdot|Y\backslash H|$. Thus, $\mu_i(\cdot,\cdot)$ is τ_i-Lipschitz as

$$\left|\mu_i(t_i,X)-\mu_i(t_i',Y)\right|\leq\tau_i\cdot\max\{|X\backslash H|,|Y\backslash H|\}\leq\tau_i\cdot(|X\Delta Y|+|X\cap Y|-|H|).$$

Monotonicity follows directly from the definition of $\mu_i(t_i,\cdot)$. Next, we argue subadditivity. For all $U,V\subseteq[m]$, let $S^*\in\text{argmax}_{S\subseteq U\cup V}\left(\hat{v}_i(t_i,S)-\sum_{j\in S}Q_j\right)$, $X=S^*\cap U\subseteq U$, $Y=S^*\backslash X\subseteq V$. Since $\hat{v}_i(t_i,\cdot)$ is a subadditive valuation,

$$\mu_i(t_i,U\cup V)=\hat{v}_i(t_i,S^*)-\sum_{j\in S^*}Q_j\leq\left(\hat{v}_i(t_i,X)-\sum_{j\in X}Q_j\right)+\left(\hat{v}_i(t_i,Y)-\sum_{j\in Y}Q_j\right)\leq\mu_i(t_i,U)+\mu_i(t_i,V)$$

Finally, we argue that $\mu_i(t_i,\cdot)$ has no externalities. Consider a set S, and types $t_i,t_i'\in T_i$ such that $t_{ij}'=t_{ij}$ for all $j\in S$. For any $S'\subseteq S$, since $\hat{v}_i(t_i,\cdot)$ has no externalities, $\hat{v}_i(t_i,S')-\sum_{j\in S'}Q_j=\hat{v}_i(t_i',S')-\sum_{j\in S'}Q_j$. Thus, $\mu_i(t_i,S)=\mu_i(t_i',S)$. □

A.5 Efficient Approximation for Symmetric Bidders

In this section, we sketch how to compute the RSPM and ASPE to approximate the optimal revenue in polynomial time for symmetric bidders[1]. For any given BIC mechanism M, one can follow our proof to construct in polynomial time an RSPM and an ASPE such that the better of the two achieves a constant fraction of M's revenue. We will describe the construction of the RSPM and the ASPE separately in this section. The difficulty of applying the method described above to construct the desired simple mechanisms is that we need to know an (approximately) revenue-maximizing mechanism M^*. We will show how to circumvent this difficulty when the bidders are symmetric.

Indeed, we can directly construct an RSPM that approximates the PostRev. As we have restricted the buyers to purchase at most one item in an RSPM, the PostRev is upper bounded by the optimal revenue of the unit-demand setting where buyer i has value $V_i(t_{ij})$ for item j when her type is t_i. By [CDW16], we know that the optimal revenue in this unit-demand setting is upper bounded by $4\text{OPT}^{\text{Copies-UD}}$, so one can simply use the RSPM constructed in [CHMS10] to extract revenue at least $\frac{\text{PostRev}}{24}$. Note that the construction is independent of M.

Unlike the RSPM, our construction for the ASPE heavily relies on β which depends on M (Lemma 3.6). Given β, we first compute c_is according to Definition 3.6. Next, we compute the Q_js (Definition 3.7). Finally, we compute the τ_is (Defintion 3.8) and use them to compute the entry fee (Definition 3.10). A few steps of the algorithm above requires sampling from the type distributions, but it is not hard to argue that a polynomial number of samples suffices. The main reason that the information about M is necessary is because our construction crucially relies on the choice of β. Next, we argue that for symmetric bidders, we can essentially choose a β that satisfies all requirements in Lemma 3.6 for all mechanisms.

When bidders are symmetric, the important observation is that the optimal mechanism must also be symmetric, and for any symmetric mechanism we can directly construct a β that satisfies all the requirements in Lemma 3.6. For every $i \in [n], j \in [m]$, choose β_{ij} such that $\Pr_{t_{ij}}\left[V_i(t_{ij}) \geq \beta_{ij}\right] =$

1. Bidders are symmetric if for any two bidders i and i', we have $v_i(\cdot,\cdot) = v_{i'}(\cdot,\cdot)$ and $D_{ij} = D_{i'j}$ for all j.

$\frac{b}{n}$. Clearly, this choice satisfies property (i) in Lemma 3.6. Furthermore, the ex-ante probability for any bidder i to win item j is the same in any symmetric mechanism, and therefore is no more than $1/n$. Hence, property (ii) in Lemma 3.6 is also satisfied. Given this β, we can essentially follow the algorithm mentioned above to construct the ASPE. The only difference is that we no longer know the σ, which is required when computing the Q_js. This can be resolved by considering the welfare maximizing mechanism M' with respect to v'. We compute the prices Q_j using the allocation rule of M' and construct our ASPE. As M' is also symmetric, our β satisfies all requirements in Lemma 3.6 with respect to M'. Therefore, Lemma 3.31 implies that either this ASPE or the RSPM constructed above has at least a constant fraction of $\textsc{Core}(M', \beta)$ as revenue. Since M' is welfare maximizing, $\textsc{Core}(M', \beta) \geq \textsc{Core}(M^*, \beta)$, where M^* is the revenue optimal mechanism. Therefore, we construct in polynomial time a simple mechanism whose revenue is a constant fraction of the optimal BIC revenue.

A.6 Proof of Lemma 3.3

We first prove some properties of $v^{(\beta)}$, which will be useful for proving Lemma 3.3.

Lemma A.4. *For any β_i, $t_i \in T_i$ and $S \in [m]$, $v_i^{(\beta_i)}(t_i, S) \geq v_i(t_i, S)$.*

Proof. This follows from the fact that $v_i(t_i, \cdot)$ is a subadditive function over bundles of items for all t_i. □

Lemma A.5. *For any β_i and $t_i \in T_i$, $v_i^{(\beta_i)}(t_i, \cdot)$ is a monotone, subadditive function over the items.*

Proof. Monotonicity follows directly from the monotonicity of $v_i(t_i, \cdot)$. We only argue subadditivity here. If t_i belongs to $R_0^{(\beta_i)}$, $v_i^{(\beta_i)}(t_i, \cdot) = v_i(t_i, \cdot)$. So it is clearly a subadditive function. If t_i belongs to $R_j^{(\beta_i)}$ for some $j > 0$ and j is not in either U or V, then clearly $v_i^{(\beta_i)}(t_i, U \cup V) \leq v_i^{(\beta_i)}(t_i, U) + v_i^{(\beta_i)}(t_i, V)$. If j is in one of the two sets, without loss of generality let's assume it is in U. Then $v_i^{(\beta_i)}(t_i, U) + v_i^{(\beta_i)}(t_i, V) = v_i(t_i, U \backslash \{j\}) + V_i(t_{ij}) + v_i(t_i, V) \geq v_i(t_i, V \cup (U \backslash \{j\})) + V_i(t_{ij}) = v_i^{(\beta_i)}(t_i, U \cup V)$. □

Here we prove a stronger version of Lemma 3.3.

Lemma A.6. *For any β, any absolute constant $\eta \in (0,1)$ and any BIC mechanism M for sub-additive valuations $\{v_i(t_i,\cdot)\}_{i\in[n]}$ with $t_i \sim D_i$ for all i, there exists a BIC mechanism $M^{(\beta)}$ for valuations $\{v_i^{(\beta_i)}(t_i,\cdot)\}_{i\in[n]}$ with $t_i \sim D_i$ for all i, such that*

1. $\sum_{t_i\in T_i} f_i(t_i)\cdot\sum_{S:j\in S}\sigma_{iS}^{(\beta)}(t_i) \le \sum_{t_i\in T_i} f_i(t_i)\cdot\sum_{S:j\in S}\sigma_{iS}(t_i)$, *for all i and j,*

2. $\textsc{Rev}(M,v,D) \le$
$$\frac{1}{1-\eta}\cdot\textsc{Rev}(M^{(\beta)},v^{(\beta)},D)+\frac{1}{\eta}\cdot\sum_i\sum_{t_i\in T_i}\sum_{S\subseteq[m]} f_i(t_i)\cdot\sigma_{iS}^{(\beta)}(t_i)\cdot\Big(v_i^{(\beta_i)}(t_i,S)-v_i(t_i,S)\Big).$$

$\textsc{Rev}(M,v,D)$ (or $\textsc{Rev}(M^{(\beta)},v^{(\beta)},D)$) is the revenue of the mechanism M (or $M^{(\beta)}$) while the buyers' types are drawn from D and buyer i's valuation is $v_i(t_i,\cdot)$ (or $v_i^{(\beta_i)}(t_i,\cdot)$). $\sigma_{iS}(t_i)$ (or $\sigma_{iS}^{(\beta)}(t_i)$) is the probability of buyer i receiving exactly bundle S when her reported type is t_i in mechanism M (or $M^{(\beta)}$).

Proof of lemma A.6: Readers who are familiar with the ϵ-BIC to BIC reduction [HKM11, BH11, DW12] might have already realized that the problem here is quite similar. Our proof will follow essentially the same approach.

First, we construct mechanism $M^{(\beta)}$, which has two phases:

Phase 1: Surrogate Sale

1. For each buyer i, create $\ell-1$ *replicas* and ℓ *surrogates* sampled i.i.d. from D_i. The value of ℓ will be specified later.

2. Ask each buyer to report her type t_i.

3. For each buyer i, create a weighted bipartite graph with the replicas and the buyer i on the left and the surrogates on the right. The edge weight between a replica (or buyer i) with type r_i and a surrogate with type s_i is the expected value for a bidder with valuation $v_i^{(\beta_i)}(r_i,\cdot)$ to receive buyer i's interim allocation in M when she reported s_i as her type subtract the interim payment of buyer i multiplied by $(1-\eta)$. Formally, the weight is $\sum_S \sigma_{iS}(s_i)\cdot v_i^{(\beta_i)}(r_i,S)-(1-\eta)p_i(s_i)$, where $p_i(s_i)$ is the interim payment for buyer i if she reported s_i.

4. Compute the VCG matching and prices on the bipartite graph created for each buyer i. If a replica (or bidder i) is unmatched in the VCG matching, match her to a random unmatched surrogate. The surrogate selected for buyer i is whoever she is matched to.

Phase 2: Surrogate Competition

1. Apply mechanism M on the type profiles of the selected surrogates $\vec{s}$. Let $M_i(\vec{s})$ and $P_i(\vec{s})$ be the corresponding allocated bundle and payment of buyer i.

2. If buyer i is matched to her surrogate in the VCG matching, give her bundle $M_i(\vec{s})$ and charge her $(1-\eta)\cdot P_i(\vec{s})$ plus the VCG price. If buyer i is not matched in the VCG matching, award them nothing and charge them nothing.

Lemma A.7 ([HKM11])**.** *If all buyers play $M^{(\beta)}$ truthfully, then the distribution of types of the surrogate chosen by buyer i is exactly D_i.*

Proof. In the mechanism, first the buyer i's type is sampled from the distribution, then we sampled $\ell - 1$ replicas and ℓ surrogates i.i.d. from the same distribution. Now, imagine a different order of sampling. We first sample the ℓ replicas and ℓ surrogates, then we pick one replica to be buyer i uniformly at random. The two different orders above provide exactly the same joint distribution over the replicas, surrogates and buyer i. So we only need to argue that in the second order of sampling, the distribution of types of the surrogate chosen by buyer i is exactly D_i. Note that the perfect matching (VCG matching plus the uniform random matching with the leftover replicas/surrogates) only depends on the types but not the identity of the node (replica or buyer i). So we can decide who is buyer i after we have decided the perfect matching. Since buyer i is chosen uniformly at random among the replicas, the chosen surrogate is also uniformly at random. Clearly, the distribution of the types of a surrogate chosen uniformly at random is also D_i. The assumption that buyer i is reporting truthfully is crucial, because otherwise the distribution of buyer i's reported type will be different from the type of a replica, and in that case, we cannot use the second sampling order. □

Lemma A.8. *$M^{(\beta)}$ is a BIC mechanism with respect to valuation $v^{(\beta)}$.*

Proof. We need to argue that for every buyer i reporting truthfully is a best response, if every other buyer is truthful. In the VCG mechanism, buyer i faces a competition with the replicas to win a surrogate. If buyer i has type t_i, then her value for winning a surrogate with type s_i in the VCG mechanism is $\sum_S \sigma_{iS}(s_i) \cdot v_i^{(\beta_i)}(t_i, S) - (1-\eta)p_i(s_i)$ due to Lemma A.7. Clearly, if buyer i reports truthfully, the weights on the edges between her and all the surrogates will be exactly her value for winning those surrogates. Since buyer i is in a VCG mechanism, reporting the true edge weights is a dominant strategy for her, therefore reporting truthfully is also a best response for her assuming the other buyers are truthful. It is critical that the other buyers are reporting truthfully, otherwise we cannot invoke Lemma A.7 and buyer i's value for winning a surrogate with type s_i may be different from the weight on the corresponding edge. □

Lemma A.9. *For any i and j, $\sum_{t_i \in T_i} f_i(t_i) \cdot \sum_{S:j\in S} \sigma_{iS}^{(\beta)}(t_i) \le \sum_{t_i \in T_i} f_i(t_i) \cdot \sum_{S:j\in S} \sigma_{iS}(t_i)$.*

Proof. The LHS is the ex-ante probability for buyer i to win item j in $M^{(\beta)}$, and the RHS is the corresponding probability in M. By Lemma A.7, we know the surrogate selected by buyer i is participating in M against all other surrogates whose types are drawn from D_{-i}. Therefore, the ex-ante probability for the surrogate chosen by buyer i to win item j is the same as RHS. Clearly, the chosen surrogate's ex-ante probability for winning any item should be at least as large as the ex-ante probability for buyer i to win the item in $M^{(\beta)}$. □

Next, we want to compare $\text{Rev}(M^{(\beta)}, v^{(\beta)}, D)$ with $\text{Rev}(M, v, D)$. The following simple Lemma relates both quantities to the expected prices charged to the surrogates by mechanism M. As in the proof of Lemma A.7, we change the order of the sampling. We first sample ℓ replicas and ℓ surrogates then select a replica uniformly at random to be buyer i. Let s_i^k and r_i^k be the type of the k-th surrogate and replica, $\mathbf{s} = (s_i^1, \ldots, s_i^\ell)$, $\mathbf{r_i} = (r_i^1, \ldots, r_i^\ell)$ and $V(\mathbf{s}, \mathbf{r_i})$ be the VCG matching between surrogates and replicas with types $\mathbf{s}$ and $\mathbf{r_i}$. We will slightly abuse notation by using s_i^k (or r_i^j) $\in V(\mathbf{s}, \mathbf{r_i})$ to denote that s_i^k (or r_i^j) is matched in the VCG matching $V(\mathbf{s}, \mathbf{r_i})$.

Lemma A.10. *For every buyer i, her expected payments in $M^{(\beta)}$ is at least*

$$(1-\eta)\cdot \mathbb{E}_{\boldsymbol{s},\mathbf{r_i}}\left[\sum_{s_i^k\in V(\boldsymbol{s},\mathbf{r_i})} \frac{p_i(s_i^k)}{\ell}\right],$$

and her expected payments in M is

$$\mathbb{E}_{\boldsymbol{s}}\left[\sum_{k\in[\ell]} \frac{p_i(s_i^k)}{\ell}\right].$$

Proof. The revenue of $M^{(\beta)}$ contains two parts – the prices paid by the chosen surrogates and the revenue of the VCG mechanism. Let's compute the first part. For buyer i and each realization of $\mathbf{r_i}$ and $\mathbf{s}$ only when the buyer i's chosen surrogate is in $V(\mathbf{s},\mathbf{r_i})$, she pays the surrogate price. Since each surrogate is selected with probability $1/\ell$, the expected surrogate price paid by buyer i is exactly $(1-\eta)\cdot\mathbb{E}_{\mathbf{s},\mathbf{r_i}}\left[\sum_{s_i^k\in V(\mathbf{s},\mathbf{r_i})}\frac{p_i(s_i^k)}{\ell}\right]$. Since the VCG payments are nonnegative, we have proved our first statement.

The expected payment from buyer i in M is $\mathbb{E}_{t_i\sim D_i}\left[p_i(t_i)\right]$. Since all s_i^k is drawn from D_i, this is exactly the same as $\mathbb{E}_{\mathbf{s}}\left[\sum_{k\in[\ell]}\frac{p_i(s_i^k)}{\ell}\right]$. □

If the VCG matching is always perfect, then Lemma A.10 already shows that the revenue of $M^{(\beta)}$ is at least $(1-\eta)$ fraction of the revenue of M. But since the VCG matching may not be perfect, we need to show that the total expected price from surrogates who are not in the VCG matching is small. We prove this in two steps. First, we consider another matching $X(\mathbf{s},\mathbf{r_i})$ – a maximal matching that only matches replicas and surrogates that have the same type, and show that the expected cardinality of $X(\mathbf{s},\mathbf{r_i})$ is close to ℓ. Then we argue that for any realization $\mathbf{r_i}$ and $\mathbf{s}$ the total payments from surrogates that are in $X(\mathbf{s},\mathbf{r_i})$ but not in $V(\mathbf{s},\mathbf{r_i})$ is small.

Lemma A.11 ([HKM11]). *For every buyer i, the expected cardinality of a maximal matching that only matches replicas and surrogates with the same type is at least $\ell-\sqrt{|T_i|\cdot\ell}$.*

The proof can be found in Hartline et al. [HKM11].

Corollary A.1. *Let* $\mathcal{R} = \max_{i,t_i \in T_i} \max_{S \in [m]} v_i(t_i, S)$*, then*

$$\mathbb{E}_{\boldsymbol{s},\mathbf{r_i}}\left[\sum_{s_i^k \in X(\boldsymbol{s},\mathbf{r_i})} \frac{p_i(s_i^k)}{\ell}\right] \geq \mathbb{E}_{\boldsymbol{s}}\left[\sum_{k \in [\ell]} \frac{p_i(s_i^k)}{\ell}\right] - \sqrt{\frac{|T_i|}{\ell}} \cdot \mathcal{R}.$$

Proof. Since M is a IR mechanism when the buyers' valuations are v, $\mathcal{R} \geq p_i(t_i)$ for any buyer i and any type t_i of i. Our claim follows from Lemma A.11. □

Now we implement the second step of our argument. The plan is to show the total prices from surrogates that are unmatched by going from $X(\mathbf{s}, \mathbf{r_i})$ to $V(\mathbf{s}, \mathbf{r_i})$. For any $\mathbf{s}, \mathbf{r_i}$, $V(\mathbf{s}, \mathbf{r_i}) \cup X(\mathbf{s}, \mathbf{r_i})$ can be decompose into a disjoint collection augmenting paths and cycles. If a surrogate is matched in $X(\mathbf{s}, \mathbf{r_i})$ but not in $V(\mathbf{s}, \mathbf{r_i})$, then it must be the starting point of an augmenting path. The following Lemma upper bounds the price of this surrogate.

Lemma A.12 (Adapted from [DW12])**.** *For any buyer i and any realization of $\boldsymbol{s}$ and $\mathbf{r_i}$, let P be an augmenting path that starts with a surrogate that is matched in $X(\boldsymbol{s}, \mathbf{r_i})$ but not in $V(\boldsymbol{s}, \mathbf{r_i})$. It has the form of either (a) $\left(s_i^{\rho(1)}, r_i^{\theta(1)}, s_i^{\rho(2)}, r_i^{\theta(2)}, \ldots, s_i^{\rho(k)}\right)$ when the path ends with a surrogate, or*
(b) $\left(s_i^{\rho(1)}, r_i^{\theta(1)}, s_i^{\rho(2)}, r_i^{\theta(2)}, \ldots, s_i^{\rho(k)}, r_i^{\theta(k)}\right)$ when the path ends with a replica, where $r_i^{\theta(j)}$ is matched to $s_i^{\rho(j)}$ in $X(\boldsymbol{s}, \mathbf{r_i})$ and matched to $s_i^{\rho(j+1)}$ in $V(\boldsymbol{s}, \mathbf{r_i})$ (whenever $s_i^{\rho(j+1)}$ exists) for any j.

$$\sum_{s_i^{\rho(j)} \in P \cap X(\boldsymbol{s},\mathbf{r_i})} p_i\left(s_i^{\rho(j)}\right) - \sum_{s_i^{\rho(j)} \in P \cap V(\boldsymbol{s},\mathbf{r_i})} p_i\left(s_i^{\rho(j)}\right) \leq$$
$$\frac{1}{\eta} \cdot \sum_{j=1}^{k-1} \sum_S \sigma_{iS}\left(s_i^{\rho(j+1)}\right) \cdot \left(v_i^{(\beta_i)}(r_i^{\theta(j)}, S) - v_i(r_i^{\theta(j)}, S)\right).$$

Proof. Since $r_i^{\theta(j)}$ is matched to $s_i^{\rho(j)}$ in $X(\mathbf{s}, \mathbf{r_i})$, $r_i^{\theta(j)}$ must be equal to $s_i^{\rho(j)}$. M is a BIC mechanism when buyers valuations are v, therefore the expected utility for reporting the true type is better than lying. Hence, the following holds for all j:

$$\sum_S \sigma_{iS}\left(s_i^{\rho(j)}\right) \cdot v_i\left(r_i^{\theta(j)}, S\right) - p_i\left(s_i^{\rho(j)}\right) \geq \sum_S \sigma_{iS}\left(s_i^{\rho(j+1)}\right) \cdot v_i\left(r_i^{\theta(j)}, S\right) - p_i\left(s_i^{\rho(j+1)}\right) \quad \text{(A.1)}$$

The VCG matching finds the maximum weight matching, so the total edge weights in path $P \cap V(\mathbf{s}, \mathbf{r_i})$ is at least as large as the total edge weights in path $P \cap X(\mathbf{s}, \mathbf{r_i})$. Mathematically, it is the following inequalities.

- If P has format (a):

$$\sum_{j=1}^{k-1}\left(\sum_{S}\sigma_{iS}\left(s_i^{\rho(j+1)}\right)\cdot v_i^{(\beta_i)}\left(r_i^{\theta(j)},S\right)-(1-\eta)\cdot p_i\left(s_i^{\rho(j+1)}\right)\right)\geq \sum_{j=1}^{k-1}\left(\sum_{S}\sigma_{iS}\left(s_i^{\rho(j)}\right)\cdot v_i^{(\beta_i)}\left(r_i^{\theta(j)},S\right)-(1-\eta)\cdot p_i\left(s_i^{\rho(j)}\right)\right) \tag{A.2}$$

- If P has format (b):

$$\sum_{j=1}^{k-1}\left(\sum_{S}\sigma_{iS}\left(s_i^{\rho(j+1)}\right)\cdot v_i^{(\beta_i)}\left(r_i^{\theta(j)},S\right)-(1-\eta)\cdot p_i\left(s_i^{\rho(j+1)}\right)\right)\geq \sum_{j=1}^{k}\left(\sum_{S}\sigma_{iS}\left(s_i^{\rho(j)}\right)\cdot v_i^{(\beta_i)}\left(r_i^{\theta(j)},S\right)-(1-\eta)\cdot p_i\left(s_i^{\rho(j)}\right)\right) \tag{A.3}$$

Next, we further relax the RHS of inequality (A.2) using inequality (A.1).

$$\begin{aligned}
&\text{RHS of inequality (A.2)}\\
\geq&\sum_{j=1}^{k-1}\left(\sum_{S}\sigma_{iS}\left(s_i^{\rho(j)}\right)\cdot v_i\left(r_i^{\theta(j)},S\right)-p_i\left(s_i^{\rho(j)}\right)\right)+\eta\cdot\sum_{j=1}^{k-1}p_i\left(s_i^{\rho(j)}\right) \quad \text{(Lemma A.4)}\\
\geq&\sum_{j=1}^{k-1}\left(\sum_{S}\sigma_{iS}\left(s_i^{\rho(j+1)}\right)\cdot v_i\left(r_i^{\theta(j)},S\right)-p_i\left(s_i^{\rho(j+1)}\right)\right)+\eta\cdot\sum_{j=1}^{k-1}p_i\left(s_i^{\rho(j)}\right) \quad \text{(Inequality A.1)}
\end{aligned}$$

We can obtain the following inequality by combining the relaxation above with the LHS of inequality (A.2) and rearrange the terms.

$$\frac{1}{\eta}\cdot\sum_{j=1}^{k-1}\sum_{S}\sigma_{iS}\left(s_i^{\rho(j+1)}\right)\cdot\left(v_i^{(\beta_i)}\left(r_i^{\theta(j)},S\right)-v_i\left(r_i^{\theta(j)},S\right)\right)\geq p_i\left(s_i^{\rho(1)}\right)-p_i\left(s_i^{\rho(k)}\right).$$

The inequality above is exactly the inequality in the statement of this Lemma when P has format (a).

Similarly, we have the following relaxation when P has format (b):

$$
\begin{aligned}
&\text{RHS of inequality (A.3)} \\
\geq &\sum_{j=1}^{k}\left(\sum_{S}\sigma_{iS}\left(s_i^{\rho(j)}\right)\cdot v_i\left(r_i^{\theta(j)},S\right)-p_i\left(s_i^{\rho(j)}\right)\right)+\eta\cdot\sum_{j=1}^{k}p_i\left(s_i^{\rho(j)}\right) \quad \text{(Lemma A.4)} \\
\geq &\sum_{j=1}^{k-1}\left(\sum_{S}\sigma_{iS}\left(s_i^{\rho(j+1)}\right)\cdot v_i\left(r_i^{\theta(j)},S\right)-p_i\left(s_i^{\rho(j+1)}\right)\right)+\eta\cdot\sum_{j=1}^{k}p_i\left(s_i^{\rho(j)}\right) \quad \text{(Inequality A.1 and } M \text{ is IR)}
\end{aligned}
$$

Again, by combining the relaxation with the LHS of inequality (A.3), we can prove our claim when P has format (b).

$$
\frac{1}{\eta}\cdot\sum_{j=1}^{k-1}\sum_{S}\sigma_{iS}\left(s_i^{\rho(j+1)}\right)\cdot\left(v_i^{(\beta_i)}\left(r_i^{\theta(j)},S\right)-v_i\left(r_i^{\theta(j)},S\right)\right)\geq p_i\left(s_i^{\rho(1)}\right).
$$

□

Lemma A.13. *For any β,*

$$
\begin{aligned}
&\mathbb{E}_{\boldsymbol{s},\mathbf{r_i}}\left[\sum_{s_i^k\in X(\boldsymbol{s},\mathbf{r_i})}\frac{p_i(s_i^k)}{\ell}\right]\leq \\
&\qquad \mathbb{E}_{\boldsymbol{s},\mathbf{r_i}}\left[\sum_{s_i^k\in V(\boldsymbol{s},\mathbf{r_i})}\frac{p_i(s_i^k)}{\ell}\right]+\frac{1}{\eta}\cdot\sum_{t_i\in T_i}\sum_{S\subseteq[m]}f_i(t_i)\cdot\sigma_{iS}^{(\beta)}(t_i)\cdot\left(v_i^{(\beta_i)}(t_i,S)-v_i(t_i,S)\right).
\end{aligned}
$$

Proof. Due to Lemma A.12, for any buyer i and any realization of $\mathbf{r_i}$ and $\mathbf{s}$, we have

$$
\sum_{s_i^k\in X(\mathbf{s},\mathbf{r_i})}\frac{p_i(s_i^k)}{\ell}-\sum_{s_i^k\in V(\mathbf{s},\mathbf{r_i})}\frac{p_i(s_i^k)}{\ell}\leq\frac{1}{\eta\cdot\ell}\cdot\sum_{s_i^k\in V(\mathbf{s},\mathbf{r_i})}\sum_{S}\sigma_{iS}\left(s_i^k\right)\cdot\left(v_i^{(\beta_i)}(r_i^{\omega(k)},S)-v_i(r_i^{\omega(k)},S)\right),
$$

where $r_i^{\omega(k)}$ is the replica that is matched to s_i^k in $V(\mathbf{s},\mathbf{r_i})$. If we take expectation over $\mathbf{r_i}$ and $\mathbf{s}$ on the RHS, the expectation means whenever mechanism $M^{(\beta)}$ awards buyer i (with type t_i) bundle S, $\frac{1}{\eta} \cdot \left(v_i^{(\beta_i)}(t_i, S) - v_i(t_i, S)\right)$ is contributed to the expectation. Therefore, the expectation of the RHS is the same as

$$\frac{1}{\eta} \cdot \left(\sum_{t_i \in T_i} \sum_{S \subseteq [m]} f_i(t_i) \cdot \sigma_{iS}^{(\beta)}(t_i) \cdot \left(v_i^{(\beta_i)}(t_i, S) - v_i(t_i, S)\right) \right).$$

This completes the proof of the Lemma. □

Now, we are ready to prove Lemma A.6.

$$\begin{aligned}
&\textsc{Rev}(M, v, D) \\
=&\sum_i \mathbb{E}_{\mathbf{s}} \left[\sum_{k \in [\ell]} \frac{p_i(s_i^k)}{\ell} \right] \quad \text{(Lemma A.10)} \\
\leq&\sum_i \left(\mathbb{E}_{\mathbf{s}, \mathbf{r_i}} \left[\sum_{s_i^k \in X(\mathbf{s}, \mathbf{r_i})} \frac{p_i(s_i^k)}{\ell} \right] + \sqrt{\frac{|T_i|}{\ell}} \cdot \mathcal{R} \right) \quad \text{(Corollary A.1)} \\
\leq&\sum_i \mathbb{E}_{\mathbf{s}, \mathbf{r_i}} \left[\sum_{s_i^k \in V(\mathbf{s}, \mathbf{r_i})} \frac{p_i(s_i^k)}{\ell} \right] \\
&\qquad + \frac{1}{\eta} \cdot \sum_i \sum_{t_i \in T_i} \sum_{S \subseteq [m]} f_i(t_i) \cdot \sigma_{iS}^{(\beta)}(t_i) \cdot \left(v_i^{(\beta_i)}(t_i, S) - v_i(t_i, S)\right) + \sum_i \sqrt{\frac{|T_i|}{\ell}} \cdot \mathcal{R} \quad \text{(Lemma A.13)} \\
\leq&\frac{1}{1-\eta} \cdot \textsc{Rev}(M^{(\beta)}, v^{(\beta)}, D) \\
&\qquad + \frac{1}{\eta} \cdot \sum_i \sum_{t_i \in T_i} \sum_{S \subseteq [m]} f_i(t_i) \cdot \sigma_{iS}^{(\beta)}(t_i) \cdot \left(v_i^{(\beta_i)}(t_i, S) - v_i(t_i, S)\right) + \sum_i \sqrt{\frac{|T_i|}{\ell}} \cdot \mathcal{R} \quad \text{(Lemma A.10)}
\end{aligned}$$

Since $|T_i|$ and $\mathcal{R}$ are finite numbers, we can take ℓ to be sufficiently large, so that $\sum_i \sqrt{\frac{|T_i|}{\ell}} \cdot \mathcal{R} < \epsilon$ for any ϵ. Let $P^{(\beta)}$ be the set of all BIC mechanisms that satisfy the first condition in Lemma A.6. Clearly, $P^{(\beta)}$ is a compact set and contains all $M^{(\beta)}$ we constructed (by choosing different values for ℓ). Notice that both $\textsc{Rev}(M^{(\beta)}, v^{(\beta)}, D)$ and $\sum_i \sum_{t_i \in T_i} \sum_{S \subseteq [m]} f_i(t_i) \cdot \sigma_{iS}^{(\beta)}(t_i) \cdot \left(v_i^{(\beta_i)}(t_i, S) - v_i(t_i, S)\right)$ are linear functions over the allocation/price rules of mechanism $M^{(\beta)}$.

Therefore,

$$
\begin{aligned}
& \textsc{Rev}(M, v, D) \\
\leq & \max_{M^{(\beta)} \in P^{(\beta)}} \left(\frac{1}{1-\eta} \cdot \textsc{Rev}(M^{(\beta)}, v^{(\beta)}, D) + \frac{1}{\eta} \cdot \sum_i \sum_{t_i \in T_i} \sum_{S \subseteq [m]} f_i(t_i) \cdot \sigma_{iS}^{(\beta)}(t_i) \cdot \left(v_i^{(\beta_i)}(t_i, S) - v_i(t_i, S) \right) \right).
\end{aligned}
$$

This completes the proof of Lemma A.6. □

Appendix B

Missing Details from Chapter 4

B.1 Duality Framework

The seller aims to maximize her profit among all direct, BIC, and interim IR mechanisms. This maximization problem can be captured by the following LP (see Figure B.1). Here we use type $\varnothing$ to represent the choice of not participating in the mechanism. Now the IR constraint can be described as another BIC constraint that the buyer won't report type $\varnothing$. Let $T_i^+ = T_i \cup \{\varnothing\}$.

Variables:

- $\pi_i(t_i, \mathbf{c})$, for all $i \in [n]$, $t_i \in T_i, \mathbf{c} \in T^S$, denotes the interim probability vector that buyer i with type t_i receives each item, when the seller has cost $\mathbf{c}$.
- $p_i(t_i, \mathbf{c})$, for all $i \in [n]$, $t_i \in T_i, \mathbf{c} \in T^S$, denoting the buyer i's interim payment when she has type t_i and the seller has cost $\mathbf{c}$.

Constraints:

- $\mathbb{E}_{\mathbf{c}}[t_i \cdot \pi_i(t_i, \mathbf{c}) - p_i(t_i, \mathbf{c})] \geq \mathbb{E}_{\mathbf{c}}[t_i \cdot \pi_i(t_i', \mathbf{c}) - p_i(t_i', \mathbf{c})]$, for all $i \in [n], t_i \in T_i, t_i' \in T_i^+$, guaranteeing that the mechanism is BIC and interim IR.
- $\pi \in P(\{\mathcal{F}_i\}_{i=1}^n)$, guaranteeing the allocation is implementable.

Objective:

- $\max \sum_i \mathbb{E}_{t_i, \mathbf{c}}[p_i(t_i, \mathbf{c}) - \mathbf{c} \cdot \pi_i(t_i, \mathbf{c})]$, the expected seller's profit.

Figure B.1: A Linear Program (LP) for Maximizing Profit.

We then take the partial Lagrangian dual of the LP in Figure B.1 by lagrangifying the BIC and interim IR constraints. Let $\lambda_i(\mathbf{t}, \mathbf{t}')$ be the Lagrangian multiplier. The dual problem is described in Figure B.2.

Variables:

- $\pi_i(t_i, \mathbf{c})$ and $p_i(t_i, \mathbf{c})$.
- $\lambda_i(\mathbf{t}, \mathbf{t}')$ for all $i \in [n], t_i \in T_i, t_i' \in T_i^+$, the Lagrangian multiplier for buyer i's BIC and interim IR constraints.

Constraints:

- $\lambda_i(\mathbf{t}, \mathbf{t}') \geq 0$ for all $i \in [n], t_i \in T_i, t_i' \in T_i^+$.
- $\pi \in P(\{\mathcal{F}_i\}_{i=1}^n)$.

Objective:

- $\min_\lambda \max_{\pi,p} \mathcal{L}(\lambda, \pi, p)$.

Figure B.2: Partial Lagrangian of the LP for Maximizing Profit.

$$\begin{aligned}
&\mathcal{L}(\lambda, \pi, p) \\
&= \sum_i \mathbb{E}_{t_i,\mathbf{c}}[p_i(t_i, \mathbf{c}) - \mathbf{c} \cdot \pi_i(t_i, \mathbf{c})] \\
&\qquad + \sum_i \sum_{t_i, t_i'} \lambda_i(t_i, t_i') \cdot \mathbb{E}_{\mathbf{c}}[(t_i \cdot \pi_i(t_i, \mathbf{c}) - p_i(t_i, \mathbf{c})) - (t_i \cdot \pi_i(t_i', \mathbf{c}) - p_i(t_i', \mathbf{c}))] \\
&= \sum_i \sum_{t_i} \mathbb{E}_{\mathbf{c}}[p_i(t_i, \mathbf{c})] \cdot \left(f_i(t_i) + \sum_{t_i' \in T_i} \lambda_i(t_i', t_i) - \sum_{t_i' \in T_i^+} \lambda_i(t_i, t_i') \right) \\
&\qquad + \sum_i \sum_{t_i} \mathbb{E}_{\mathbf{c}} \left[\pi_i(t_i, \mathbf{c}) \cdot \left(\sum_{t_i' \in T_i^+} t_i \cdot \lambda_i(t_i, t_i') - \sum_{t_i' \in T_i} t_i' \cdot \lambda_i(t_i', t_i) - f_i(t_i) \cdot \mathbf{c} \right) \right]
\end{aligned} \tag{B.1}$$

Definition B.1. *A feasible dual solution* λ *is* useful *if* $\max_{\pi \in P(\{\mathcal{F}_i\}_{i=1}^n), p} \mathcal{L}(\lambda, \pi, p) < \infty$.

Similar to [CDW16], we show that every useful dual solution forms a flow.

Lemma B.1. *A dual solution* λ *is useful if and only if it forms the following flow:*

- *Nodes: For every $i \in [n]$ and $t_i \in T_i$ a node t_i. A source s and a sink $\varnothing$.*
- *For every $i \in [n]$ and $t_i \in T_i$, a flow of weight $f_i(t_i)$ from s to t_i.*
- *For every $i \in [n]$ and $t_i \in T_i, t_i' \in T_i^+$, a flow of weight $\lambda_i(t_i, t_i')$ from t_i to t_i'.*

Proof. Suppose there exists $i \in [n], t_i \in T_i, t_i' \in T_i^+$ such that

$$f_i(t_i) + \sum_{t_i' \in T_i} \lambda_i(t_i', t_i) - \sum_{t_i' \in T_i^+} \lambda_i(t_i, t_i') \neq 0$$

Without loss of generality, suppose it's positive. Notice that $p_i(t_i, \mathbf{c})$ is unconstrained. Thus when $\mathbb{E}_{\mathbf{c}}[p_i(t_i, \mathbf{c})] \to +\infty$, the Lagrangian also goes to $+\infty$ (see Equation B.1). Hence for every $i \in [n], t_i \in T_i, t_i' \in T_i^+$,

$$f_i(t_i) + \sum_{t_i' \in T_i} \lambda_i(t_i', t_i) - \sum_{t_i' \in T_i^+} \lambda_i(t_i, t_i') = 0$$

It's essentially the flow conservation equation for node t_i. Thus λ forms a flow. On the other hand, if λ forms a flow, the Lagrangian only depends on π and thus bounded since π is bounded. $\square$

For any useful dual solution λ, by Lemma B.1, we can replace $\sum_{t_i' \in T_i^+} \lambda_i(t_i, t_i')$ by $f_i(t_i) + \sum_{t_i' \in T_i} \lambda_i(t_i', t_i)$ in Equation (B.1) and simplify $\mathcal{L}(\lambda, \pi, p)$. For any BIC and interim IR mechanism $M = (\pi, p)$, both $\lambda_i(t_i, t_i')$ and $\mathbb{E}_{\mathbf{c}}[(t_i \cdot \pi_i(t_i, \mathbf{c}) - p_i(t_i, \mathbf{c})) - (t_i \cdot \pi_i(t_i', \mathbf{c}) - p_i(t_i', \mathbf{c}))]$ are non-negative for all $i \in [n], t_i \in T_i, t_i' \in T_i^+$. Thus by Equation (B.1), $\mathcal{L}(\lambda, \pi, p) \geq \textsc{Profit}(M)$. We have the following lemma.

Lemma B.2. *(Restatement of Lemma 4.2) For any useful dual solution λ and any BIC, interim IR mechanism $M = (x, p)$,*

$$\textsc{Profit}(M) \leq \mathbb{E}_{\boldsymbol{t}, \boldsymbol{c}}\left[\sum_i \pi_i(t_i, \boldsymbol{c}) \cdot (\Phi_i^{(\lambda)}(t_i) - \boldsymbol{c})\right]$$

where

$$\Phi_i^{(\lambda)}(t_i) = t_i - \frac{1}{f_i(t_i)} \cdot \sum_{t_i' \in T_i} \lambda(t_i', t_i)(t_i' - t_i)$$

can be viewed as buyer i's virtual value function.

B.2 Missing Proofs from Section 4.3

Proof of Lemma 4.1: Consider a mechanism M that is ex-post implementable. For every i, t_i, let $A_i(t_i)$ be buyer i's (possibly randomized) equilibrium strategy, when her type is t_i. It specifies all the actions that the buyer takes in mechanism M. For every $\mathbf{c}$, let $X_i(\vec{A}, \mathbf{c})$ be the vector of (possibly randomized) indicator variables that indicate whether buyer i gets each item j when buyers choose strategies $\vec{A} = (A_1, ..., A_n)$ and the seller's realized cost vector is $\mathbf{c}$; let $P_i(\vec{A}, \mathbf{c})$ be the payment for the buyer. For every i and t_{-i}, denote $A_{-i}(t_{-i}) = (A_1(t_1), ..., A_{i-1}(t_{i-1}), A_{i+1}(t_{i+1}), ..., A_n(t_n))$.

Since A is an equilibrium strategy for the buyers, for every $i, t_i, t_i' \in T_i$, acting as $A_i(t_i)$ induces more utility than $A_i(t_i)$, when the buyer's type is t_i and other buyers follow strategy A_{-i}. We have

$$\begin{aligned} &\mathbb{E}_{\mathbf{c},t_{-i}}[t_i \cdot X_i(A_i(t_i), A_{-i}(t_{-i}), \mathbf{c}) - P_i(A_i(t_i), A_{-i}(t_{-i}), \mathbf{c})] \geq \\ &\qquad \mathbb{E}_{\mathbf{c},t_{-i}}[t_i \cdot X_i(A_i(t_i'), A_{-i}(t_{-i}), \mathbf{c}) - P_i(A_i(t_i'), A_{-i}(t_{-i}), \mathbf{c})] \end{aligned} \tag{B.2}$$

We now define the direct mechanism $M' = (x, p)$ as follows: for every profile $(\mathbf{t}, \mathbf{c})$, let $x_i(\mathbf{t}, \mathbf{c}) = X_i(A(\mathbf{t}), \mathbf{c})$ and $p_i(\mathbf{t}, \mathbf{c}) = P_i(A(\mathbf{t}), \mathbf{c})$ for all i. It's the allocation and payment rule when the reported type profile is $\mathbf{t}$ and the seller's realized cost vector is $\mathbf{c}$. Then Inequality (B.2) is equivalent to: for every $i, t_i, t_i' \in T_i$

$$\mathbb{E}_{\mathbf{c},t_{-i}}[t_i \cdot x_i(t_i, t_{-i}, \mathbf{c}) - p_i(t_i, t_{-i}, \mathbf{c})] \geq \mathbb{E}_{\mathbf{c},t_{-i}}[t_i \cdot x_i(t_i', t_{-i}, \mathbf{c}) - p_i(t_i', t_{-i}, \mathbf{c})]$$

It's exactly the BIC constraint for M'. Thus, M' is BIC.

Moreover, each buyer can choose not to participate in M, so $\mathbb{E}_{\mathbf{c},t_{-i}}[t_i \cdot X_i(A_i(t_i), A_{-i}(t_{-i}), \mathbf{c}) - P_i(A_i(t_i), A_{-i}(t_{-i}), \mathbf{c})] \geq 0$, which implies that $\mathbb{E}_{\mathbf{c},t_{-i}}[t_i \cdot x_i(t_i, t_{-i}, \mathbf{c}) - p_i(t_i, t_{-i}, \mathbf{c})] \geq 0$. Hence, M'

is also interim IR. □

Proof of Theorem 4.4: Let $\Phi^{(\lambda)}(\cdot)$ be the virtual value function induced by the above canonical flow λ. By Lemma 4.2 and Lemma 4.3,

$$\begin{aligned}
\textsc{Profit}(M) \leq & \sum_i \mathbb{E}_{t_i,\mathbf{c}}\left[\sum_j \pi_{ij}(t_i,\mathbf{c}) \cdot (\Phi_{ij}^{(\lambda)}(t_{ij}) - c_j)\right] \\
= & \sum_i \mathbb{E}_{t_i,\mathbf{c}}\left[\sum_j \mathbb{1}[t_i \in R_{ij}^{(\beta)}] \cdot \pi_{ij}(t_i,\mathbf{c}) \cdot (\tilde{\varphi}_{ij}(t_{ij}) - c_j)\right] \\
& + \sum_i \mathbb{E}_{t_i,\mathbf{c}}\left[\sum_j \mathbb{1}[t_i \notin R_{ij}^{(\beta)}] \cdot \pi_{ij}(t_i,\mathbf{c}) \cdot (\max\{\beta_{ij}(\mathbf{c}), c_j\} - c_j)\right] \\
& + \sum_i \mathbb{E}_{t_i,\mathbf{c}}\left[\sum_j \mathbb{1}[t_i \notin R_{ij}^{(\beta)}] \cdot \pi_{ij}(t_i,\mathbf{c}) \cdot (t_{ij} - \max\{\beta_{ij}(\mathbf{c}), c_j\})\right] \\
\leq & \sum_i \mathbb{E}_{t_i,\mathbf{c}}\left[\sum_j \mathbb{1}[t_i \in R_{ij}^{(\beta)}] \cdot \pi_{ij}(t_i,\mathbf{c}) \cdot (\tilde{\varphi}_{ij}(t_{ij}) - c_j)\right] \qquad (\textsc{Single}) \\
& + 2 \cdot \sum_i \sum_j \mathbb{E}_{\mathbf{c}}\left[q_{ij}(\mathbf{c}) \cdot (\max\{\beta_{ij}(\mathbf{c}), c_j\} - c_j)\right] \qquad (\textsc{Prophet}) \\
& + \sum_i \mathbb{E}_{t_i}\left[\sum_j \mathbb{1}[t_i \in R_{ij}^{(\beta)}] \cdot \bar{v}_i^{(\beta)}(t_i, [m]\backslash\{j\})\right] \qquad (\textsc{Non-Favorite})
\end{aligned}$$

The first inequality is due to Lemma 4.2, and the first equality is due to Lemma 4.3. The second inequality is because: For the second term, notice that $\max\{\beta_{ij}(\mathbf{c}), c_j\} - c_j \geq 0$, we bound the indicator by 1 and use the fact that $\mathbb{E}_{t_i}[\pi_{ij}(t_i,\mathbf{c})] = 2 \cdot q_{ij}(\mathbf{c})$ for every $\mathbf{c}$; For the third term, notice that for every $i, t_i \in R_{ij}^{(\beta)}$ and $\mathbf{c}$, since π is feasible, we have

$$\sum_{k \neq j} \pi_{ij}(t_i,\mathbf{c}) \cdot (t_{ik} - \max\{\beta_{ij}(\mathbf{c}), c_j\}) \leq \max_{S \in \mathcal{F}_i, j \notin S} \sum_{k \in S} (t_{ik} - \max\{\beta_{ij}(\mathbf{c}), c_j\}).$$

Taking expectation over $\mathbf{c}$, the RHS equals to $\bar{v}_i^{(\beta)}(t_i, [m]\backslash\{j\})$. Thus the inequality holds. □

B.3 Missing Proofs from Section 4.4

Proof of Lemma 4.6:

Monotonicity: Fix any $\mathbf{t} \in T$, $U \subseteq V \subseteq [m]$. For all $\mathbf{c}$ we have $\max_{S\subseteq U, S\in\mathcal{F}} \sum_{j\in S}(t_j - c_j) \leq \max_{S\subseteq V, S\in\mathcal{F}} \sum_{j\in S}(t_j - c_j)$. Taking expectation over $\mathbf{c}$ on both sides proves the monotonicity.

Subadditivity:

Fix any $\mathbf{t} \in T$ and $U, V \subseteq [m]$. For every $\mathbf{c}$, let $S^*(\mathbf{c}) = \text{argmax}_{S\subseteq U\cup V, S\in\mathcal{F}} \sum_{j\in S}(t_j - c_j)$. Clearly, $t_j - c_j \geq 0$ for all $j \in S^*(\mathbf{c})$. Notice that $S^*(\mathbf{c}) \cap U \subseteq U$ and $S^*(\mathbf{c}) \cap U \in \mathcal{F}$; also $S^*(\mathbf{c}) \cap V \subseteq V$ and $S^*(\mathbf{c}) \cap V \in \mathcal{F}$. We have

$$\begin{aligned}
\max_{S\subseteq U\cup V, S\in\mathcal{F}} \sum_{j\in S}(t_j - c_j) &\leq \sum_{j\in S^*(\mathbf{c})\cap U}(t_j - c_j) + \sum_{j\in S^*(\mathbf{c})\cap V}(t_j - c_j)\\
&\leq \max_{S\subseteq U, S\in\mathcal{F}} \sum_{j\in S}(t_j - c_j) + \max_{S\subseteq V, S\in\mathcal{F}} \sum_{j\in S}(t_j - c_j)
\end{aligned}$$

Taking expectation over $\mathbf{c}$ on both sides, we have $\bar{v}(\mathbf{t}, U \cup V) \leq \bar{v}(\mathbf{t}, U) + \bar{v}(\mathbf{t}, V)$.

No externalities: fix any $\mathbf{t} \in T$, $S \subseteq [m]$ and any $\mathbf{t}' \in T$ such that $t'_j = t_j$ for all $j \in S$. To prove $\bar{v}(\mathbf{t}', S) = \bar{v}(\mathbf{t}, S)$, it suffices to show that for any $\mathbf{c}$,

$$\max_{U\subseteq S, U\in\mathcal{F}} \sum_{j\in U}(t_j - c_j) = \max_{U\subseteq S, U\in\mathcal{F}} \sum_{j\in U}(t'_j - c_j)$$

It follows directly from the fact that $t'_j = t_j$ for all $j \in S$. □

B.4 Missing Proofs from Section 4.5

Proof of Lemma 4.22:

Monotonicity: Fix any t_i, $U \subseteq V \subseteq [m]$. For all $\mathbf{c}$ we have $\max_{S\subseteq U\cap C_i(t_i), S\in\mathcal{F}_i} \sum_{j\in S}(t_{ij} - \max\{\beta_{ij}(\mathbf{c}), c_j\}) \leq \max_{S\subseteq V\cap C_i(t_i), S\in\mathcal{F}_i} \sum_{j\in S}(t_{ij} - \max\{\beta_{ij}(\mathbf{c}), c_j\})$. Taking expectation over $\mathbf{c}$ on both sides proves the monotonicity.

Subadditivity:

Fix any t_i and $U, V \subseteq [m]$. For every $\mathbf{c}$, let $S^*(\mathbf{c}) = \text{argmax}_{S\subseteq(U\cup V)\cap C_i(t_i), S\in\mathcal{F}_i} \sum_{j\in S}(t_{ij} - \max\{\beta_{ij}(\mathbf{c}), c_j\})$. Clearly, $t_{ij} - \max\{\beta_{ij}(\mathbf{c}), c_j\} \geq 0$ for all $j \in S^*(\mathbf{c})$. Notice that $S^*(\mathbf{c}) \cap U \subseteq U \cap C_i(t_i)$ and $S^*(\mathbf{c}) \cap U \in \mathcal{F}_i$; also $S^*(\mathbf{c}) \cap V \subseteq V \cap C_i(t_i)$ and $S^*(\mathbf{c}) \cap V \in \mathcal{F}_i$. We have

$$\begin{aligned}
&\max_{S\subseteq(U\cup V)\cap C_i(t_i), S\in\mathcal{F}_i} \sum_{j\in S}(t_{ij} - \max\{\beta_{ij}(\mathbf{c}), c_j\}) \\
\leq &\sum_{j\in S^*(\mathbf{c})\cap U}(t_{ij} - \max\{\beta_{ij}(\mathbf{c}), c_j\}) + \sum_{j\in S^*(\mathbf{c})\cap V}(t_{ij} - \max\{\beta_{ij}(\mathbf{c}), c_j\}) \\
\leq &\max_{S\subseteq U\cap C_i(t_i), S\in\mathcal{F}_i} \sum_{j\in S}(t_{ij} - \max\{\beta_{ij}(\mathbf{c}), c_j\}) + \max_{S\subseteq V\cap C_i(t_i), S\in\mathcal{F}_i} \sum_{j\in S}(t_{ij} - \max\{\beta_{ij}(\mathbf{c}), c_j\})
\end{aligned}$$

Taking expectation over $\mathbf{c}$ on both sides, we have $\mu_i(t_i, U \cup V) \leq \mu_i(t_i, U) + \mu_i(t_i, V)$.

No externalities: fix any t_i, $S \subseteq [m]$ and any t_i such that $t'_{ij} = t_{ij}$ for all $j \in S$. To prove $\mu_i(t'_i, S) = \mu_i(t_i, S)$, it suffices to show that for any $\mathbf{c}$,

$$\max_{U\subseteq S\cap C_i(t_i), U\in\mathcal{F}_i} \sum_{j\in U}(t_{ij} - \max\{\beta_{ij}(\mathbf{c}), c_j\}) = \max_{U\subseteq S\cap C_i(t_i), U\in\mathcal{F}_i} \sum_{j\in U}(t'_{ij} - \max\{\beta_{ij}(\mathbf{c}), c_j\})$$

It follows directly from the fact that $t'_{ij} = t_{ij}$ for all $j \in S$.

Now we prove that $\mu_i(t_i, \cdot)$ has Lipschitz constant τ_i. For any t_i, t'_i, and set $X, Y \subseteq [m]$, define set $H = \left\{j \in X \cap Y : t_{ij} = t'_{ij}\right\}$. Since $\mu_i(\cdot, \cdot)$ has no externalities, $\mu_i(t_i, H) = \mu_i(t'_i, H)$. Therefore,

$$\begin{aligned}
|\mu_i(t_i, X) - \mu_i(t'_i, Y)| &= \max\left\{\mu_i(t_i, X) - \mu_i(t'_i, Y), \mu_i(t'_i, Y) - \mu_i(t_i, X)\right\} \\
&\leq \max\left\{\mu_i(t_i, X) - \mu_i(t'_i, H), \mu_i(t'_i, Y) - \mu_i(t_i, H)\right\} \quad \text{(Monotonicity)} \\
&\leq \max\left\{\mu_i(t_i, X\backslash H), \mu_i(t'_i, Y\backslash H)\right\} \quad \text{(Subadditivity)} \\
&\leq \tau_i \cdot \max\{|X\backslash H|, |Y\backslash H|\} \\
&\leq \tau_i \cdot (|X\Delta Y| + |X \cap Y| - |H|)
\end{aligned}$$

The second last inequality is because $\mu_i(t_i, \cdot)$ is subadditive and for any item $j \in \mathcal{C}_{\rangle}(t_i)$ ($\mathcal{C}_{\rangle}(t'_i)$) the single-item valuation $\bar{v}_{ij}(t_{ij})$ ($\bar{v}_{ij}(t'_{ij})$) is less than τ_i.

$\square$

Appendix C

Missing Details from Chapter 6

C.1 Proof of Example 6.1

Proof of Example 6.1. First consider the first-best mechanism. Fix buyer i. We'll prove that if her value $b_i > 1/2 + \epsilon$ for some small ϵ that will be determined later, she will trade in the first-best mechanism with high probability. Let $\mathcal{S}$ be the number of sellers whose cost is smaller than $1/2 + \epsilon$, and let $\mathcal{B}$ be the number of buyers whose value is larger than $1/2 + \epsilon$. Notice that there are at most $\mathcal{B}$ buyers with value larger than i. Thus if $\mathcal{B} < \mathcal{S}$, all buyers with value $> 1/2 + \epsilon$, including i, must have traded with a seller whose cost is smaller than $1/2 + \epsilon$ in the first-best mechanism.

Both $\mathcal{S}$ and $\mathcal{B}$ are the sum of independent Bernoulli random variables, with expectation $(1/2+\epsilon)n$ and $(1/2 - \epsilon)(n-1)$ respectively. By Chernoff bound,

$$\Pr\left[\mathcal{S} < (1-\epsilon)\left(\frac{1}{2}+\epsilon\right)n\right] \leq exp\left(-\frac{\epsilon^2}{2}\cdot\left(\frac{1}{2}+\epsilon\right)n\right) \leq exp\left(-\frac{1}{12}\epsilon^2 n\right),$$

$$\Pr\left[\mathcal{B} > (1+\epsilon)\left(\frac{1}{2}-\epsilon\right)(n-1)\right] \leq exp\left(-\frac{\epsilon^2}{3}\cdot\left(\frac{1}{2}-\epsilon\right)(n-1)\right) \leq exp\left(-\frac{1}{12}\epsilon^2 n\right).$$

With probability at least $\left(1 - exp\left(-\frac{1}{12}\epsilon^2 n\right)\right)^2$,

$$\mathcal{S} \geq (1-\epsilon)\left(\frac{1}{2}+\epsilon\right)n > (1+\epsilon)\left(\frac{1}{2}-\epsilon\right)(n-1) \geq \mathcal{B}.$$

In other words, if $b_i > {}^1\!/_2 + \epsilon$, buyer i will trade in the first-best mechanism with probability at least $\left(1 - exp\left(-\frac{1}{12}\epsilon^2 n\right)\right)^2$, taking expectation over all other agents' types.

Choose $\epsilon = n^{-1/3}$. The expected GFT contributed by buyer i is at least

$$\int_{\frac{1}{2}+n^{-\frac{1}{3}}}^{1} b_i db_i \cdot \left(1 - exp\left(-\frac{1}{12}n^{\frac{1}{3}}\right)\right)^2 = \frac{3}{8} + o(1).$$

By linearity of expectation and the fact that all buyers are i.i.d., the expected GFT contributed by all buyers is at most $3n/8 + o(n)$.

Similarly for every seller j, if her cost $s_j > {}^1\!/_2 + \epsilon$ for some small ϵ, she can only trade in the first-best mechanism with small probability. The expected GFT contributed by all sellers (a negative term) is at least $-n/8 + o(n)$. Thus the first-best mechanism obtains GFT at least $n/4 + o(n)$. The second-best mechanism gets GFT that is in expectation at least the expected GFT of the Trade Reduction mechanism, and as the GFT of the TR mechanism is at least the GFT of the first best minus 1, the second-best mechanism obtains GFT at least $n/4 + o(n)$ as well.

In the mechanism of BCWZ, with probability a half, the mechanism implements GSOM that finds all the efficient trade based on buyers' virtual value and sellers' cost. And probability a half, it implements GBOM that finds all the efficient trade based on buyers' value and seller' virtual cost. We will only give the proof that the expected GFT of GSOM is at most $2n/9 + o(n)$. An analogous proof shows that the expected GFT of GBOM is at most $2n/9 + o(n)$.

For each buyer whose value is drawn from uniform distribution $[0, 1]$, her virtual value follows the uniform distribution $[-1, 1]$. Fix buyer i, we'll show that if her value $b_i < {}^2\!/_3 - \epsilon$ (which means her virtual value $\varphi_i(b_i) < {}^1\!/_3 - 2\epsilon$) for some small ϵ she can only trade in the GSOM with exponentially small probability. Notice that buyer i can only trade in GSOM with a seller whose cost is smaller than ${}^1\!/_3 - 2\epsilon$. Let $\mathcal{S}$ be the number of sellers whose cost is smaller than ${}^1\!/_3 - 2\epsilon$, and let $\mathcal{B}$ be the number of buyers whose value is at least ${}^2\!/_3 - \epsilon$. Then if buyer i trades in GSOM, $\mathcal{B} \leq \mathcal{S}$. This is because there are at least $\mathcal{B}$ buyers with value larger than i (and thus have a larger virtual value since since all buyers are i.i.d.). If $\mathcal{B} > \mathcal{S}$, those buyers will take away all the sellers with cost $< {}^1\!/_3 - 2\epsilon$. It contradicts with the fact that buyer i must trade with a seller with cost

$< 1/3 - 2\epsilon$ in GSOM. Notice that again $\mathcal{S}$ and $\mathcal{B}$ are the sum of independent Bernoulli random variables. And the expectation of $\mathcal{S}$ $(1/3 - 2\epsilon)n$ is smaller than the expectation of $\mathcal{B}$, $(1/3 + \epsilon)(n-1)$ when $\epsilon = n^{-\frac{1}{3}}$ and n sufficiently large. By Chernoff bound (and a similar calculation as above), $\mathcal{B} \leq \mathcal{S}$ happens with exponentially small probability. The expected GFT contributed by each buyer i is at most

$$\int_{\frac{2}{3}-n^{-\frac{1}{3}}}^{1} b_i db_i + o(1) = \frac{5}{18} + o(1).$$

Similarly, for each seller j whose value $s_j < 1/3 - \epsilon$ for some small ϵ, she will trade in GSOM with high probability. The expected GFT contributed by each seller (a negative term) is at most $-1/18 + o(1)$. Thus GSOM obtains GFT at most $2n/9 + o(n)$.

We conclude that the expected GFT the mechanism of BCWZ obtains is at most $2n/9 + o(n)$, which is only a constant fraction of the second-best mechanism. □

C.2 Proofs of Lemmas 6.1 and 6.2

Proof of lemma 6.1. We start by proving part 1. We will show all of these properties for the SO mechanism with the given parameters. Similar arguments show them for the BO mechanism, and therefore for the RO mechanism. That the SO mechanism with the given parameters is BIC for the seller is immediate since the the seller chooses a price that maximizes her expected utility over the distribution from which the buyer is drawn. That SO is ex-post IC and ex-post IR for the buyer is immediate from the buyer choosing whether to accept or reject the offer in a way that maximizes her utility. Strong budget balance is also immediate from definition. Finally, to show that that the SO mechanism with the given parameters is ex-post IR for the seller, we note that since the seller's cost is drawn from D_s, and since $\bar{s} \geq \sup \text{Support}\, D_s$, we have that $\bar{s}$ is at least the seller's cost. Therefore, seller j can always ask for a price equal to his cost (without violating the constraint $\bar{s}$), which will result in zero utility for her (regardless of whether the buyer accepts or rejects this price), and in particular guarantees nonnegative utility for her.

We move on to prove part 2. We will show that any offered price is at least $\bar{b}$. An analogous proof shows it to be at most $\bar{s}$. In case of the BO mechanism, this holds by definition since any offer by the buyer is constrained to be at least $\bar{b}$. We note that any offer by the seller will also be at least $\bar{b}$, since this seller knows that the buyer will buy at this price with probability 1 since $\bar{b} \leq \inf \text{Support}\, D_b$ (and since this price is at least $\bar{s}$, offering it does not violate this seller's constraint), and therefore the seller will never make a lower offer, and the proof is complete. $\square$

Proof of lemma 6.2. We start by proving part 1. If $\bar{s} \leq b$, then any price offered by the seller in the SO part of the RO mechanism with the given parameters will be accepted by the buyer (as it will be at most $\bar{s}$ and therefore at most b). Similarly, if $\bar{b} \geq s$, then any price offered by the buyer in the BO part of the RO mechanism with the given parameters will be accepted by the seller (as it will be at least $\bar{b}$ and therefore at least s). Either way, trade will occur in the RO mechanism with probability at least $1/2$.

We move on to prove part 2. It is enough to show that if trade occurs in the SO part of the latter mechanism, then trade occurs also in the SO part of the former mechanism. (The BO part is handled analogously.) If trade occurs in the SO part of the latter mechanism (unconstrained and unconditioned), then it means that in that mechanism, the price p that maximizes the revenue of the seller from D_b (unconstrained and unconditioned), which is the price that was offered, is at most b (since the offered price is accepted) and at least s (since the mechanism is ex-post IR). Therefore, since $b \leq \bar{s}$ and $s \geq \bar{b}$, we have $\bar{s} \geq b \geq p \geq s \geq \bar{b}$, and so p is also the price that maximizes the revenue of the seller from $D_b|_{\geq \bar{b}}$ constrained upon the price being at most $\bar{s}$, and so this is also (at most, in case of multiple utility-maximizing prices) the price offered in the SO mechanism with parameters $\bar{s}$ and $D_b|_{\geq \bar{b}}$, and so the price offered by the seller is accepted in this mechanism as well. $\square$

C.3 Details Omitted from the Proof of Theorem 6.5 and Commentary

C.3.1 Details Omitted from the Proof of Theorem 6.5

We will show that that the region of the space of valuation/cost profiles where our hybrid mechanism for double auctions runs each instance of the RO mechanism can be partitioned into disjoint subsets where our hybrid mechanism is BIC on each such subset under the profile distribution conditioned upon being in that subset.

Fix a choice of the identity (but not the cost) of seller $s_{(1)}$ and the identity (but not the value) of bidder $b_{(1)}$, and fix a profile of costs and valuations for all other sellers and buyers (so in particular the cost $s_{(2)}$ and value $b_{(2)}$ are fixed). We first claim that either our hybrid mechanism runs the same instance of RO on all possible profiles b, s that agree with these fixed choices, or does not run any instance of RO on any of these profiles. Indeed, if $s_{(2)} \leq b_{(s)}$ (a conditioned fully determined by these fixed choices) then TR is run on all such profiles, and otherwise the RO mechanism with SO parameters $\bar{s} = s_{(2)}$ and $D^B_{b_{(1)}}|_{\geq b_{(2)}}$ and BO parameters $\bar{b} = b_{(2)}$ and $D^S_{s_{(1)}}|_{\leq s_{(2)}}$ (note that all of these parameters are fully determined by the above fixed choices and do not depend on the cost $s_{(1)}$ or the value $b_{(1)}$) is run on all such profiles.

We will next show that our hybrid mechanism is BIC on the subset of all profiles that agree with these fixed choices. Note that when conditioning the distribution of all profiles to those that agree with such fixed choices, the cost of the seller $s_{(1)}$ (conditioned to agree with these fixed choices) is distributed precisely according to $s_{(1)} \sim D^S_{s_{(1)}}|_{\leq s_{(2)}}$ and the value of the buyer $b_{(1)}$ (conditioned to agree with these fixed choices) is distributed precisely according to $b_{(1)} \sim D^B_{b_{(1)}}|_{\geq b_{(2)}}$. By Lemma 6.1, we therefore have that our hybrid mechanism is BIC for the offering agent (and ex-post IC for any other agent) over all profiles that agree with these choices. We have therefore shown that if a manipulation by a buyer does not change the choice of the mechanism that is run by our hybrid mechanism, then it is nonbeneficial in expectation.

C.3.2 Why The Proof of the Ex-Ante Guarantee Gives a Factor of 1/4 and Not 1/2

Having read the proof of the ex-ante guarantee of theorem 6.5, we note that at first glance, one may be tempted to consider the following naïve adaptation of this proof into a "proof" of an ex-ante guarantee of $1/2$ (rather than $1/4$) of the second-best:

> In each case analyzed above, the hybrid mechanism attains either at least the GFT of the RVWM mechanism, or at least half of the GFT of the first-best, which in turn is at least half of the GFT of the second-best. Since the GFT of the RVWM mechanism is in turn also at least half of the GFT of the second-best, we get that in either case our hybrid mechanism attains half of the GFT of the second-best.

The problem with this "proof" is that it mixes ex-ante and ex-post guarantees. While indeed the hybrid mechanism attains, on each profile $(\mathbf{b}, \mathbf{s})$, either at least the GFT of the RVWM mechanism or at least half of the GFT of the (first-best and therefore of the) second-best, it is wrong to assume that on each profile $(\mathbf{b}, \mathbf{s})$ the GFT of the RVWM mechanism is at least half of the GFT of the second-best, as we only know that the expected GFT of the RVWM mechanism, over all profiles, is at least half of the expected GFT, over all profiles, of the second-best. In other words, it may hypothetically be that the RVWM mechanism performs poorly on the profiles on which our hybrid mechanism attains at least the GFT of the RVWM mechanism, and that the RVWM mechanism performs very well, surpassing half of the GFT of the second-best, and even half of the GFT of the first-best, on the profiles on which our hybrid mechanism attains at least half of the GFT of the first-best (so on average, the RVWM mechanism would indeed attain its guarantee), and in such a case, the above "proof" obviously fails.

C.4 The Trade Reduction Mechanism for Matching Markets: Proofs

Proof of theorem 6.6. We first observe that the allocation is indeed feasible, that is, that we can perfectly match all winning buyers and sellers. Indeed, the set of winners can be obtained by taking the matching $M(\mathbf{b}, \mathbf{s})$ and then removing the agents that correspond to one edge between any two classes t and t' that are trading (have $r_{t,t'} > 0$), and then switching agents of the same class (removing every agent in the leftover matching that has value lower than a removed agent of the same class, and adding the removed agent in her stead), maintaining a perfect matching of $TR(\mathbf{b}, \mathbf{s})$. Thus, there is a matching of the winners $TR(\mathbf{b}, \mathbf{s})$ such that there are exactly $r_{t,t'} - 1$ trades of agents of classes t and t' whenever $r_{t,t'} > 0$. The reduced agents can be perfectly matched with exactly a single edge between agents of classes t and t' whenever $r_{t,t'} > 0$.

Now, the theorem will directly follow from the following sequence of claims.

Claim C.1. *The Trade Reduction Mechanism for matching markets is ex-post IR and ex-post IC.*

Proof. We first observe that the TR mechanism for matching markets is monotone. It is enough to show this for the buyers. Assume that the value of a winning buyer i increases by δ. We show that she still wins after the increase. Every matching that includes this buyer improves by the same amount δ, while the value of any other matching did not change, so the same matching $M(\mathbf{b}, \mathbf{s})$ will be picked after the value increase (ties are broken the same way, independent of values). Finally, the reduction will also not change as for any class t, the values of q_t and d_t did not change, and if buyer i is of class t, she will still not be in the set of d_t lowest-value buyers after her value has increased, so she will not be reduced.

To complete the proof that the mechanism is ex-post IR and ex-post IC, we need to show that payments are by critical values. Indeed, assume that for a winning buyer i of class t, the value of the highest reduced buyer of class t (if i bids truthfully) is x. If buyer i of class t changes her bid but keeps it above x, she wins. Now assume that she drops her bid below x. If she is not in $M(\mathbf{b}, \mathbf{s})$ she loses. If she is in $M(\mathbf{b}, \mathbf{s})$ and bids below x, then the matching $M(\mathbf{b}, \mathbf{s})$ will contain the exact same set of agents, and will be the same (due to the tie breaking rule which is independent of the actual bids), so she will lose while the agent with bid x will win instead of her, as both q_t and d_t are the same but now i is no longer one of the $q_t - d_t$ highest-bidding agents of class t. Thus, bidding above x implies winning for i, while bidding below x implies losing, so x is indeed the critical value for buyer i to win. Similar arguments prove the Claim for sellers. □

Claim C.2. *The Trade Reduction Mechanism for matching markets is ex-post (direct trade) weakly budget balanced.*

Proof. To see that the mechanism is ex-post (direct trade) weakly BB we prove that every trade is ex-post weakly BB. Indeed, consider a trade between two agents of respective classes t and t' in the matching of $TR(\mathbf{b}, \mathbf{s})$ that contains exactly $r_{t,t'} - 1$ trades of agents of classes t and t' whenever $r_{t,t'} > 0$, and consider the reduced edge between these two classes. The buyer pays at least the value of the reduced buyer of the same class on the reduced edge between classes t and t', while the seller receives at most the cost of the reduced seller on that edge. As that reduced edge has non-negative gain (otherwise removing it will increase the welfare of the first-best matching $M(\mathbf{b}, \mathbf{s})$), the trade is ex-post weakly BB. □

Claim C.3. *For any profile $(\boldsymbol{b}, \boldsymbol{s})$, the fraction of the realized gains from trade (first-best) that the Trade Reduction Mechanism for matching markets obtains ex-post is at least* $\min\{1 - \frac{d_t}{q_t} \mid \text{class } t \text{ s.t. } q_t > 0\}$.

Proof. Recall that T_t is the set of agents of class t. Let $\alpha(\mathbf{b}, \mathbf{s}) = \min\{1 - \frac{d_t}{q_t} \mid \text{class } t \text{ s.t. } q_t > 0\}$. To prove the claim that the mechanism guarantees an α fraction of the welfare of FB-GFT$(\mathbf{b}, \mathbf{s})$, we let v_k be the value of agent k ($v_k = b_i$ for a buyer $k = i$, and $v_k = -s_j$ for seller $k = j$), and assuming W is the set of agents in $M(\mathbf{b}, \mathbf{s})$ we observe that

$$\text{FB-GFT}(\mathbf{b}, \mathbf{s}) = \sum_{(i,j) \in M(\mathbf{b}, \mathbf{s})} (b_i - s_j) = \sum_{t: q_t > 0} \quad \sum_{k \in W \cap T_t} v_k$$

.

As for each class t with $q_t > 0$ we remove d_t agents each with value at most the value of any winner, we obtain at least a $\frac{q_t - d_t}{q_t}$ fraction of the value of agent of class t. Thus,

$$\alpha(\mathbf{b}, \mathbf{s}) \cdot OPT(\mathbf{b}, \mathbf{s}) \leq \sum_{t: q_t > 0} \frac{q_t - d_t}{q_t} \quad \sum_{k \in W \cap T_t} v_k \leq TR(\mathbf{b}, \mathbf{s}),$$

as needed. □

Theorem 6.6 follows from Claim C.1, Claim C.2 and Claim C.3. □

Proof of corollary 6.1. Let $\beta(\mathbf{b},\mathbf{s}) = \min\{1 - \frac{1}{r_{t,t'}} \mid \text{classes } (t,t') \text{ s.t. } r_{t,t'} > 0\}$. By the guarantee of theorem 6.6 with respect to $\alpha(\mathbf{b},\mathbf{s})$, it is enough to show that $\beta(\mathbf{b},\mathbf{s}) \le \alpha(\mathbf{b},\mathbf{s})$. Indeed, for every class t such that $q_t > 0$, since $q_t = \sum_{t'} r_{t,t'}$:

$$\frac{q_t - d_t}{q_t} = \sum_{t':r_{t,t'}>0} \frac{r_{t,t'} - 1}{q_t} = \sum_{t':r_{t,t'}>0} \frac{r_{t,t'} - 1}{r_{t,t'}} \cdot \frac{r_{t,t'}}{q_t} \ge \min_{t':r_{t,t'}>0} \frac{r_{t,t'} - 1}{r_{t,t'}},$$

where the inequality is since a weighted average of values is always at least the minimal value. Taking the minimum of both sides of the obtained inequality over all classes t s.t. $q_t > 0$, we obtain that $\alpha(\mathbf{b},\mathbf{s}) \ge \beta(\mathbf{b},\mathbf{s})$, as required. □

C.5 Additional Preliminaries for Appendices C.6 and C.7

C.5.1 Notation

First we give some notations specialized in this setting. Given profile $(\mathbf{b},\mathbf{s})$, let $M(\mathbf{b},\mathbf{s})$ be the first-best matching, or the maximum weight matching[1], under graph G with edge weight $b_i - s_j$ on each edge $(i,j) \in E$. For each agent a, denote $M_{-a}(\mathbf{b},\mathbf{s})$ the maximum weight matching[2] after removing a and its related edges. For each buyer i such that $(i,j) \in M(\mathbf{b},\mathbf{s})$, let $P_i(\mathbf{b},\mathbf{s})$ be the VCG payment of buyer i. Formally,

$$P_i(\mathbf{b},\mathbf{s}) = \sum_{(i',j')\in M_{-i}(\mathbf{b},\mathbf{s})} (b_{i'} - s_{j'}) - \sum_{(i',j')\in M(\mathbf{b},\mathbf{s})} (b_{i'} - s_{j'}) + b_i$$

Similarly, let $P_j(\mathbf{b},\mathbf{s})$ be the VCG payment received by seller j:

$$P_j(\mathbf{b},\mathbf{s}) = \sum_{(i',j')\in M(\mathbf{b},\mathbf{s})} (b_{i'} - s_{j'}) - \sum_{(i',j')\in M_{-j}(\mathbf{b},\mathbf{s})} (b_{i'} - s_{j'}) + s_j$$

For simplicity, when the valuation profile $(\mathbf{b},\mathbf{s})$ is fixed, we will abuse the notation and use M

1. We break ties lexicographically by IDs.

2. Follow the same breaking tie rules as the first-best matching.

(or M_{-a}, P_i, P_j) instead in the proof, without writing the valuation profile.

C.5.2 Lexicographic Tie-Breaking by ID

In this section, we define the tie-breaking rule that we use whenever we have to choose between multiple maximum weight matchings when picking a matching with maximum weight anywhere throughout this paper. We first define a strict total order over matchings, which we call the *Lexicographic order by IDs.*

Definition C.1 (Lexicographic order by IDs)**.** *Fix a bipartite graph, and let M' and M'' be two matchings in this graph. The* Lexicographic order by IDs *decides which of M' and M'' is ranked higher as follows. It first sorts the edges of the matching by the index of the buyer. For each k, let (i'_k, j'_k) and (i''_k, j''_k) be the k^{th} sorted edges (according to the index of the buyer) in M' and in M'', respectively. Let k be the lowest index such that it is not the case that the two edges (i'_k, j'_k) and (i''_k, j''_k) are both defined and are the same edge.*

- *If one matching has a k^{th} edge while the other does not, then the matching with more edges is ranked higher.*
- *Otherwise, the matching with the lower buyer index in the k^{th} edge is ranked higher.*
- *Otherwise, the matching with the lower seller index in the k^{th} edge is ranked higher.*

As noted above, throughout this paper when two matchings have the same weight, we use the Lexicographic by IDs order to break ties when choosing a maximum weight matching, so we in fact choose the lexicographically-by-IDs-highest matching among those with maximum weight. We will refer to this practice as using the *Lexicographic by IDs tie-breaking rule.* We will now formalize the two properties of this tie-breaking rule, which we will use in our analysis:[3]

Lemma C.1. *The Lexicographic by IDs tie-breaking rule satisfies the following two properties:*

- *The tie-breaking is* weight independent*: ties between maximum weight matchings are broken independently of any weight function. That is, if W and W' are two weight functions, if $\mathcal{M}$*

3. Indeed, our results would still hold for any other tie-breaking rule that satisfies these two properties.

and $\mathcal{M}'$ are the respective corresponding sets of maximum weight matchings, and if M and M' are the respective corresponding chosen matchings, then if $\mathcal{M} \subseteq \mathcal{M}'$ and $M' \in \mathcal{M}$, then $M = M'$. So, the set of matched nodes that this tie-breaking rule picks (among all possible maximum weight options), as well as the matching that this rule picks within that set, does not depend on the weight function.

- *The choice function is* subset consistent*: if the chosen maximum weight matching among all matchings of the vertices U is the matching M, then for any $(i, j) \in M$, the chosen maximum weight matching among all matchings of the vertices $U \setminus \{i, j\}$ is the matching $M \setminus \{(i, j)\}$.*

Proof. Weight-independence is by definition of the Lexicographic order by IDs. For subset consistency, let $M' \neq M \setminus \{(i, j)\}$ be another maximum weight matching of $U \setminus \{(i, j)\}$, and note that when adding the edge (i, j) to M', one obtains a maximum weight matching of U. By definition of M, it is ranked higher than $M' \cup \{(i, j)\}$ by the Lexicographic by IDs order, and since the shared edge makes no difference in the tie breaking, we have that after its removal $M \setminus \{(i, j)\}$ is (still) ranked higher than M' by the Lexicographic by IDs order (so the tie is be broken in the same way). □

C.6 The Offering Mechanism for Matching Markets: Proofs

We will now prove theorem 6.7, which states that the Offering Mechanism for matching markets is BIC, ex-post IR, ex-post (direct trade) strongly budget balanced, and ex-ante guarantees at least a 1/4-fraction of the optimal GFT (second-best). The Offering Mechanism is ex-post strongly (direct trade) budget balanced as the RO mechanism is ex-post (direct trade) strongly budget balanced. To show the remaining properties, we first develop some machinery.

C.6.1 Supporting Machinery

Lemma C.2. *In the offering mechanism when run on a profile $(\boldsymbol{b}, \boldsymbol{s})$, for every $(i, j) \in M(\boldsymbol{b}, \boldsymbol{s})$ the following hold:*

- *If $i \in M_{-j}(\boldsymbol{b}, \boldsymbol{s})$, then $\bar{s} = P_j(\boldsymbol{b}, \boldsymbol{s})$.*

- *If $j \in M_{-i}(\boldsymbol{b}, \boldsymbol{s})$, then $\bar{b} = P_i(\boldsymbol{b}, \boldsymbol{s})$.*

Proof. We will prove the first statement (the second is analogous). Since $i \in M_{-j}(\mathbf{b}, \mathbf{s})$, the VCG price of buyer i in the market without seller j is the minimal bid that causes her to be in the first-best in that market, and so $\bar{s} = P_i(\mathbf{b}, \mathbf{s}_{-j})$. Now observe that:

$$P_j(\mathbf{b}, \mathbf{s}) = \sum_{(i',j') \in M} (b_{i'} - s_{j'}) - \sum_{(i',j') \in M_{-j}} (b_{i'} - s_{j'}) + s_j =$$

$$\sum_{(i',j') \in M \setminus \{(i,j)\}} (b_{i'} - s_{j'}) - \sum_{(i',j') \in M_{-j}} (b_{i'} - s_{j'}) + b_i = P_i(\mathbf{b}, \mathbf{s}_{-j}) = \bar{s} \qquad \square$$

We are now ready to prove the first two parts of lemma 6.3:

Claim C.4. *For every $(i, j) \in M(\boldsymbol{b}, \boldsymbol{s})$, it holds that $\bar{s} \geq s_j$ and $\bar{b} \leq b_i$.*

Proof. We will show the former; the latter is analogous. If $\bar{s} = \infty$ the the claim immediately holds, so we assume that $\bar{s} < \infty$. Consider the profile $((\mathbf{b}_{-i}, b'_i), \mathbf{s})$ for $b'_i = \max\{\bar{s} + 1, b_i\}$. Since we have only increased the bid of i, we still have that $(i, j) \in M((\mathbf{b}_{-i}, b'_i), \mathbf{s})$. By definition, $\bar{s}$ is the same for the profile $((\mathbf{b}_{-i}, b'_i), \mathbf{s})$ as it is for $(\mathbf{b}, \mathbf{s})$. By definition of $\bar{s}$, we have by $b'_i > \bar{s}$ that $i \in M_{-j}((\mathbf{b}_{-i}, b'_i), \mathbf{s})$. Therefore, by lemma C.2, $\bar{s} = P_j((\mathbf{b}_{-i}, b'_i), \mathbf{s})$. Since $(i, j) \in M((\mathbf{b}_{-i}, b'_i), \mathbf{s})$, we have that $s_j \leq P_j((\mathbf{b}_{-i}, b'_i), \mathbf{s})$, and so $s_j \leq \bar{s}$, as required. $\square$

Claim C.5. *For every $(\boldsymbol{b}, \boldsymbol{s})$ and $(i, j) \in M(\boldsymbol{b}, \boldsymbol{s})$, it is the case that $P_j(\boldsymbol{b}, \boldsymbol{s}) \geq P_i(\boldsymbol{b}, \boldsymbol{s})$.*

Proof. by the Second Welfare Theorem, there exist prices $p = (p_{j'})_{j' \in S}$ (where $p_{j'}$ denotes a price for the good of seller j) such that $(M(\mathbf{b}, \mathbf{s}); p)$ is a Walrasian equilibrium. Therefore, p_j is a price received by s_j and paid by b_i in some Walrasian equilibrium. By Theorem 8 of [GS99], we therefore have that $P_j(\mathbf{b}, \mathbf{s}) \geq p_j \geq P_i(\mathbf{b}, \mathbf{s})$, completing the proof. $\square$

We are now ready to prove the third and final part of lemma 6.3:

Claim C.6. *For every $(i, j) \in M(\boldsymbol{b}, \boldsymbol{s})$, it holds that $\bar{s} \geq \bar{b}$.*

Proof. Assume for contradiction that there exists a profile $(\mathbf{b}, \mathbf{s})$ and a pair $(i, j) \in M(\mathbf{b}, \mathbf{s})$ such that $\bar{s} < \bar{b}$. By claim C.4 we have that $b_i \geq \bar{b} > \bar{s}$ and $s_j \leq \bar{s} < \bar{b}$. Therefore, we have by definition of $\bar{s}$ and $\bar{b}$ that both $i \in M_{-j}(\mathbf{b}, \mathbf{s})$ and $j \in M_{-i}(\mathbf{b}, \mathbf{s})$. Therefore, by lemma C.2, $\bar{s} = P_j(\mathbf{b}, \mathbf{s})$ and $\bar{b} = P_i(\mathbf{b}, \mathbf{s})$, and so $P_j(\mathbf{b}, \mathbf{s}) < P_i(\mathbf{b}, \mathbf{s})$ — a contradiction to claim C.5. □

Proof of Lemma 6.3. Follows from Claim C.4 and Claim C.6. □

Lemma C.3. *Fix valuation profile $(\boldsymbol{b}, \boldsymbol{s})$ and a pair $(i, j) \in M$. If $j \notin M_{-i}$, then $P_i(\boldsymbol{b}, \boldsymbol{s}) = s_j$. Similarly if $i \notin M_{-j}$, then $P_j(\boldsymbol{b}, \boldsymbol{s}) = b_i$.*

Proof. We only give the proof for $P_i = P_i(\mathbf{b}, \mathbf{s})$ and similar argument holds for seller's VCG payment $P_j = P_j(\mathbf{b}, \mathbf{s})$. If $j \notin M_{-i}$, it holds that $M_{-i} = M \setminus \{(i, j)\}$ by subset consistency of the tie breaking rule. Now, by definition of P_i, we have $P_i = s_j$. □

C.6.2 Incentive Guarantees

Claim C.7. *The Offering Mechanism is ex-post IR.*

Proof. That the Offering Mechanism is ex-post IR follows from claim C.4 and lemma 6.1. □

Claim C.8. *The Offering Mechanism is BIC.*

Proof. We will prove that the Offering Mechanism is BIC for the seller. A similar argument holds for the buyer. For each seller j with cost s_j, suppose she misreports her cost to be $s'_j \neq s_j$. We will show that taking expectation over other agents' valuation profile $\mathbf{b}, s_{-j}$, the expected utility of s_j when reporting truthfully is at least the expected utility of seller j with true cost s_j when reporting s'_j.

We first consider $\mathbf{b}, s_{-j}$ such that j is not in the first-best $M(\mathbf{b}, s_{-j}, s_j)$. It is sufficient to consider manipulations s'_j that cause j to become part of the first-best $M(\mathbf{b}, s_{-j}, s'_j)$. Let s'_j be such a manipulation, and note that in this case, $s_j \geq P_j(\mathbf{b}, s_{-j}, s'_j) \geq s'_j$ (since $P_j(\mathbf{b}, s_{-j}, s'_j)$ is the threshold bid of seller j to become part of the first-best). Let i be the agent such that $(i, j) \in M(\mathbf{b}, s_{-j}, s'_j)$. We will complete the proof of this case by considering two cases. First, if $i \in M_{-j}(\mathbf{b}, s_{-j}, s'_j)$, then $\bar{s} = P_j(\mathbf{b}, s_{-j}, s'_j)$ by lemma C.2. Therefore, by lemma 6.1, seller j with reported cost s'_j can only trade with i in the RO mechanism at a price $p \leq \bar{s} = P_j(\mathbf{b}, s_{-j}, s'_j) \leq s_j$, which derives non-positive utility for seller j. Second, if $i \notin M_j(\mathbf{b}, s_{-j}, s'_j)$, then by lemma C.3,

$P_j(\mathbf{b}, s_{-j}, s'_j) = b_i$. Since the mechanism is ex-post IR for buyer i, seller j can only trade with i (who has value b_i) in the RO mechanism at a price at most $b_i = P_j(\mathbf{b}, s_{-j}, s_j) \leq s_j$, which again derives non-positive utility for seller j.

Now for every buyer i, consider those $\mathbf{b}, s_{-j}$ such that $(i, j) \in M(\mathbf{b}, s_{-j}, s_j)$. In this case, we note that if seller j misreports to s'_j, then either the first-best is unchanged (and so j participates in the same RO mechanism with the same buyer i) or seller j is no longer in the first-best, receiving utility 0. Either way, she cannot change the RO mechanism that is run, or the buyer that she is facing. When the BO mechanism is processed, if seller j is in the first-best, she will be asked to accept a price. This is ex-post truthful.

When the SO mechanism is processed, then it is enough to show that for every i and fixed b_{-i}, s_{-j}, s_j, in expectation over all b_i such that $(i, j) \in M(b_{-i}, b_i, s_{-j}, s_j)$, the utility of j when she reports $s'_j \neq s_j$ (denoted as $u_j(s'_j)$) is at most the utility of j when she reports s_j (denoted as $u_j(s_j)$) truthfully. Notice that either i can never connect to j in the first-best $M(b_{-i}, b_i, s_{-j}, s_j)$, or $P_i(\mathbf{b}, s_{-j}, s_j)$ (which does not depend on b_i) is the threshold bid of buyer i to connect to j in the first-best. Thus it is enough to prove the claim that when there exists a bid for i such that i connects to j in the first-best, in expectation over $b_i \geq P_i(\mathbf{b}, s_{-j}, s_j)$, the utility of j when she reports s'_j is at most the utility of j when she reports s_j.

Note that fixed b_{-i}, s_{-j}, s_j, when $b_i \geq P_i(\mathbf{b}, s_{-j}, s_j)$, the outcome of the Offering Mechanism for j is as if the SO mechanism with parameters $\bar{s}$ and $D_i^B|_{\geq \bar{b}}$ had been run between j and i. Notice that the parameters $\bar{s}$ and $\bar{b}$ do not depend on s_j or on b_i. By lemma 6.1, the SO mechanism with parameters $\bar{s}$ and $D_i^B|_{\geq \bar{b}}$ is BIC when the buyer's valuation is drawn from $D_i^B|_{\geq \bar{b}}$. In other words,

$$\mathbb{E}_{b_i \geq \bar{b}}[u_j(s'_j)] \leq \mathbb{E}_{b_i \geq \bar{b}}[u_j(s_j)]. \tag{C.1}$$

Note that $P_i(\mathbf{b}, s_{-j}, s_j) \geq \bar{b}$ by claim C.4. If $P_i(\mathbf{b}, s_{-j}, s_j) = \bar{b}$, then the above claim trivially holds. Otherwise we have to reason about the case $P_i(\mathbf{b}, b_{-j}, s_j) > b_i \geq \bar{b}$, which is included in the expectation in eq. (C.1) but not in the expectation in the above claim.

In this case, since $P_i(\mathbf{b}, s_{-j}, s_j) > \bar{b}$, then lemmas C.2 and C.3, $P_i(\mathbf{b}, s_{-j}, s_j) = s_j$. We notice that when $s_j > b_i \geq \bar{b}$, seller j won't trade with buyer i in the Offering Mechanism when j reports s_j, as (i, j) can't be in the first-best $M(\mathbf{b}, \mathbf{s})$. The utility of j (contributed by buyer i) is thus 0 in this case. When $s_j > b_i \geq \bar{b}$ and j reports s'_j, since the payment goes directly from buyer i to seller j when they trade, they do so at price at most $b_j < s_j$ (since we assume that buyer i reports truthfully, and since the mechanism is ex-post IR for her), so seller j's utility (contributed by buyer i) in this case is negative if they trade, and 0 otherwise, so it is nonpositive. Combined this with eq. (C.1), we obtain

$$\mathbb{E}_{b_i \geq s_j}[u_j(s'_j)] \leq \mathbb{E}_{b_i \geq s_j}[u_j(s_j)]$$

which finishes the proof as $s_j = P_i(\mathbf{b}, s_{-j}, s_j)$. □

C.6.3 Efficiency Guarantee

Given a bipartite graph (B, S, E) and two matchings M and M' over the graph, a path is called an *alternating path* of $M \cup M'$ if the edges on the path alternate between edges of M and M'. If $a_K = a_1$, we call it an *alternating cycle*. A path is *maximal* if it is not a sub-path of any other path.

It is well-known that the union of two matchings in a bipartite graph can be divided into disjoint maximal alternating paths and cycles.

Observation C.1. *Given any set of nodes V and two undirected graphs $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$ such that in both graphs the degree of any node is at most 1 (i.e., each is a matching), it holds that in $G_{1,2} = (V, E_1 \cup E_2)$ every node has degree at most 2 and thus $G_{1,2}$ is a disjoint union of maximal alternating paths and maximal alternating cycles.*

Given a bipartite graph (V_1, V_2, E), a set $U \subseteq V_1 \cup V_2$ of nodes is *matchable* if it is possible to find a perfect matching of all of the nodes in U using edges in E. Note that if U is matchable then $|U \cap V_1| = |U \cap V_2|$. A *node weight function* is a function W that assigns a weight $W(i)$ to any node $i \in V_1 \cup V_2$. A *node-based* weighted matching problem is a matching problem in which for some node weight function W, the weight of every edge $(i, j) \in E$ is the sum of the weights of the two nodes incident on the edge, that is, $W(i, j) = W(i) + W(j)$. The weight of a matchable set of nodes U is $W(U) = \sum_{u \in U} W(u)$. For a weighted matching problem, a *weight-maximizing set* is a matchable set of nodes that has maximum weight, over all matchable sets. Clearly, for any node-based weighted matching problem, the weight of any matching over the same matchable set of nodes U is the same. Moreover, if U is a weight maximizing set, then any perfect matching of it does not include any edge of negative weight. For our mechanisms to work, we will need to carefully define the tie-breaking rule that will be used to choose the weight-maximizing set, as well as the perfect matching of its elements.

Observation C.2. *Fix a bipartite graph. Let W_V and W'_V be two node-based weight functions for the graph, and let M and M' be the two maximum weight matchings picked by the tie-breaking rule for these two weight functions, respectively. If the sets of matched nodes of M and M' are the same, then M and M' must be the same.*

We also observe that if an agent is in the first best, by changing his bid he cannot influence the picked matching while staying in the first best.

Observation C.3. *Fix a bipartite graph. Assume that with node-based weight function W, the maximum weight matching M is picked by the tie-breaking rule. Fix any i and let W' be a node-based weight function such that $W'(k) = W(k)$ for any $k \neq i$. Let M' be the the maximum weight matching picked for W'. Then if $i \in M'$ it holds that $M = M'$.*

We prove the following two lemmas about VCG prices which are both useful in our proofs.

Corollary C.1. *Consider the VCG mechanism with lexicographic by IDs tie-breaking rule.*

If $(i,j) \in M(\boldsymbol{b},\boldsymbol{s})$ for some $(\boldsymbol{b},\boldsymbol{s})$ then for any b'_i such that i trades when the bids are $((\boldsymbol{b}_{-i}, b'_i), \boldsymbol{s})$, it holds that buyer i trades with j and pays $P_i(\boldsymbol{b},\boldsymbol{s})$. Moreover, for any such b'_i it holds that $b'_i \geq P_i(\boldsymbol{b},\boldsymbol{s}) \geq s_j$.

Similarly, if $(i,j) \in M(\boldsymbol{b},\boldsymbol{s})$ for some $(\boldsymbol{b},\boldsymbol{s})$ then for any s'_j such that seller j trades when the bids are $(\boldsymbol{b}, (\boldsymbol{s}_{-j}, s_j))$, it holds that j trades with i and pays $P_j(\boldsymbol{b},\boldsymbol{s})$. Moreover, for any such s'_j it holds that $s'_j \leq P_j(\boldsymbol{b},\boldsymbol{s}) \leq b_i$.

Proof. The inequality $b'_i \geq P_i(\mathbf{b},\mathbf{s})$ holds by VCG being ex-post IR. It holds that $P_i(\mathbf{b},\mathbf{s}) \geq s_j$ as otherwise, if $P_i(\mathbf{b}, \mathbf{s}) < s_j$ then for b_i s.t. $P_i(\mathbf{b}, \mathbf{s}) < b_i < s_j$ there is an inefficient trade in M, a contradiction. Similar arguments prove imply the claim for seller j. □

Observe that $M = M(\mathbf{b}, \mathbf{s})$, $M_1^* = M_1^*(\mathbf{b}, \mathbf{s})$ and $M_2^* = M_2^*(\mathbf{b}, \mathbf{s})$ are each a maximum weighted matching for some node based weight function, all defined over the same undirected bipartite graph $G = (S, B, E)$ and chosen using the same tie-breaking rule. $M(\mathbf{b}, \mathbf{s})$ is derived from the node-based function W that assigns weight b_i to any node $i \in B$ and weight $-s_j$ to any node $j \in S$. Similarly, $M_1^*(\mathbf{b}, \mathbf{s})$ is derived from the node-based function W_1 that assigns weight $\tilde{\varphi}_i(b_i)$ to any node $i \in B$ and weight $-s_j$ to any node $j \in S$, where $\tilde{\varphi}_i(b_i)$ is the ironed virtual value of i when his value is b_i. Finally, $M_2^*(\mathbf{b}, \mathbf{s})$ is derived from the node-based function W_2 that assigns weight b_i to any node $i \in B$ and weight $-\tilde{\tau}_j(s_j)$ to any node $j \in S$, where $\tilde{\tau}_j(s_j)$ is the ironed virtual cost of j when his cost is s_j.

A direct corollary of Observation C.2, is that any alternating cycle in $M \cup M_1^*$ cannot include more than two distinct nodes, as any such alternating cycle is actually two different matchings over the same matchable set of nodes. We state the claim for M_1^*; the same claim holds also for M_2^*.

Corollary C.2. *Let $(a_1 a_2 ... a_K)$, be an alternating cycle of $M \cup M_1^*$. Then $K = 3$. In other words, $a_1 = a_3$ and the undirected edge (a_1, a_2) is in both M and in M_1^*.*

For a bipartite graph (V_1, V_2, E), we say that the node-based weight function W is a V_1-*weak-improvement of* W' if for any node $i \in V_1$ it holds that $W(i) \geq W'(i)$ and for any $j \in V_2$ it holds that $W(j) = W'(j)$. By definition of $\tilde{\varphi}$ and $\tilde{\tau}$, we have that the node-based weight function W used to derive M is a B-weak improvement to the node-based weight function W_1 used to derive M_1^* (and similarly, W is an S-weak improvement to the node-based weight function W_2 used to derive M_2^*).

Lemma C.4. *Fix a bipartite graph (B, S, E), and assume that the node-based weight function W is a B-weak-improvement of W'. Let M and M' be the maximum weight matchings that are picked by the tie-breaking rule for W and W' respectively. Consider a maximal alternating path of $M \cup M'$ that is not a cycle. It holds that path cannot both start and end with an edge from M'. Moreover, it holds that the path (or its inverse) starts with a node in B and that the first edge belongs to M.*

Proof. Let $A = (a_1 a_2 ... a_K)$ be a maximal alternating path in $M \cup M'$ that is not a cycle, and let U be the set of nodes in the path A. W.l.o.g., if there is a a node in B on any end of the path, it is the first in the path. Assume that the path starts and ends with an edge from M'. In this case the path must have an odd number of edges (as any edge from M' is followed by an edge from M), and it starts with a node $a_1 \in B$ and ends with a node $a_k \in S$. If $W'(a_1) + W'(a_k) < 0$, then $U' = U \setminus \{a_1, a_k\}$ is matchable and has higher weight than U for W', a contradiction to the maximality of M'. If $W'(a_1) + W'(a_k) \geq 0$ then since W is a V_1-weak-improvement of W' it holds that $W(a_1) + W(a_k) \geq W'(a_1) + W'(a_k) \geq 0$. If $W(a_1) + W(a_k) > 0$ then the matchable set U' has higher weight than U with respect to W, contradicting the maximality of M. If on the other hand $W(a_1) + W(a_k) = W'(a_1) + W'(a_k) = 0$ both U and U' are matchable sets of the same weight with respect to both W and W', so by set consistency of the tie breaking, both M and M' must have matched the same set, a contradiction.

Now, if the path starts and ends with an edge in M, it has an odd number of edges, so it has a node in B on one end and a node in S on the other, and as we can assume w.l.o.g. that the node in B is first, this completes the proof.

We are left with the case that the path has an edge from M on one end, and an edge from M' on the other. In this case it has an even number of edges and thus either both ends are in B, or both are in S. We prove that both are in B, completing the proof of the claim. Assume by way of contradiction that both a_1 and a_k are in S. Since W is a B-weak-improvement of W', for any node $j \in S$ we have $W(j) = W'(j)$ and thus $W(a_1) = W'(a_1)$ and $W(a_k) = W'(a_k)$. If $W(a_1) < W(a_k)$ then $U \setminus \{a_k\}$ is a matchable set with higher weight than the set $U \setminus \{a_1\}$ with respect to W, contradicting the optimality of M. Similarly if $W'(a_1) = W(a_1) > W(a_k) = W'(a_k)$ then $U \setminus \{a_1\}$ is a matchable set with higher weight than the set $U \setminus \{a_k\}$ with respect to W', contradicting the optimality of M'. Thus it must be the case that $W'(a_1) = W'(a_k)$. So both matchable sets $U \setminus \{a_1\}$ and $U \setminus \{a_k\}$ have exactly the same weight with respect to both W and W', and as the tie breaking is weight-independent, both should have picked the same set, a contradiction. □

From Lemma C.4 and Corollary C.2 we immediately get the following corollary.

Corollary C.3. *Let $(a_1a_2...a_K)$, be a maximal alternating path of $M \cup M_1^*$. Precisely one of the following holds:*

- *$K = 3$, $a_1 = a_3$ and the undirected edge $(a_1, a_2) \in M \cap M_1^*$, or*
- *(w.l.o.g.) the path starts with a buyer and an edge from M.*

lemma 6.5, which plays a central role in our proof of the ex-ante guarantee of the Offering Mechanism, provides a sufficient condition for a buyer-seller pair to trade in that mechanism. Here we restate and prove the lemma.

Lemma C.5 (Restatement of lemma 6.5)**.** *Fix valuation profile $(\boldsymbol{b}, \boldsymbol{s})$. For every $(i, j) \in M(\boldsymbol{b}, \boldsymbol{s})$, if j is in $M_{-i}(\boldsymbol{b}, \boldsymbol{s})$ then buyer i will trade with seller j in the BO Mechanism, and if i is in $M_{-j}(\boldsymbol{b}, \boldsymbol{s})$ then buyer i will trade with seller j in the SO Mechanism. Thus, in each such case the edge (i, j) will be traded with probability at least $1/2$ in the Offering Mechanism.*

Proof. For every pair $(i, j) \in M(\mathbf{b}, \mathbf{s})$, if $j \in M_{-i}(\mathbf{b}, \mathbf{s})$, we have by lemma C.2 that $\bar{b} = P_i(\mathbf{b}, \mathbf{s}) \geq s_j$, where the inequality is by lemma C.3. Similarly, if $i \in M_{-j}(\mathbf{b}, \mathbf{s})$, then $\bar{s} = P_j(\mathbf{b}, \mathbf{s}) \leq b_i$. Therefore, in either case, by lemma 6.2 the edge (i, j) will be traded with probability at least $1/2$ in the Offering Mechanism. □

Consider a maximal alternating path that is not a cycle. lemma 6.6 shows that for every seller $j \in A$ that is not at one of the ends such a path, it holds that $j \in M_{-i}$, where i is the buyer that is matched to j in M. Here we restate and give the proof of the lemma.

Lemma C.6 (Restatement of lemma 6.6)**.** *Let A be a maximal alternating path of $M \cup M_1^*$ that is not a cycle. For every seller $j \in A$ who is not at one of the ends of the path, it holds that $j \in M_{-i}$, where i is the buyer such that $(i, j) \in M$.*

Proof. By corollary C.3 we can assume w.l.o.g. that A starts with a buyer and an edge in M. So, if the path has an even number of edges, then $A = (i_1j_1i_2j_2...i_{L-1}j_{L-1}i_L)$ and if it is odd then $A = (i_1j_1i_2j_2...i_{L-1}j_{L-1}i_Lj_L)$, where in either case each agent i_l denotes a buyer and each agent j_l denotes a seller, such that for every $l \in \{1, 2, ..., L-1\}$ it holds that $(i_l, j_l) \in M$. If the path is odd, it furthermore holds that $(i_L, j_L) \in M$.

We need to show that for every $l \in \{1, 2, ..., L-1\}$ it holds that $j_l \in M_{-i_l}$. Assume for contradiction that $j_l \notin M_{-i_l}$ for some $l \in \{1, 2, ..., L-1\}$. Then $M_{-i_l} = M \setminus \{(i_l, j_l)\}$ by subset consistency of the tie-breaking rule, and in A the matching M_{-i_l} matches the set of agents[4] $A' = M \cap (A \setminus \{i_l, j_l\})$.

If the path has an even number of edges, then $i_L \notin A'$. To derive a contradiction we observe that the set $A'' = A' \cup \{j_l, i_L\} = A \setminus \{i_l\}$ is matchable (using the edges of M on the path A up to j_{l-1}, and the edges of M_1^* on the path A starting from j_l), and moreover, has weight with respect to W that is at least the weight of A'. This holds as M_1^* matched $A'' \cap R$ and not $A' \cap R$ for $R = \{j_l, i_{l+1}, j_{l+1}, \ldots, i_L\}$, and the weight of i_L is not lower in W than in W_1 (and the weight of j_l is the same in both). So we get an contradiction as either A'' is matchable and with a higher weight than A' with respect to W, or they have the same weight with regard to W and the same weight with regard to W_1, and ties were broken differently.

Next we consider the case that the path has an odd number of edges, in which case also $(i_L, j_L) \in M$. It must hold that $s_{j_l} \leq s_{j_L}$ as M_1^* matches $A \setminus \{i_1, j_L\}$ and not the matchable set $A \setminus \{i_1, j_l\}$. Recall that since $j_l \notin M_{-i_l}$, then $M_{-i_l} = M \setminus \{(i_l, j_l)\}$, but the matchable set $A'' = A \setminus \{i_l, j_L\} \subseteq A \setminus \{i_l\}$ has at least the weight of $A' = A \setminus \{i_l, j_l\}$ with respect to W (since sellers have the same weight in W and W_1), so we get an contradiction as either A'' is matchable and with a higher weight than A' with respect to W, or they have the same weight with regard to W and the same weight with regard to W_1, and ties were broken differently in M_{-i_l} and in M_1^*. □

lemma 6.7 considers such paths with an odd number of edges and present some additional characterization that will help us in bounding the GFT of our mechanism. By Corollary C.3 we can assume w.l.o.g. that any maximal alternating path of $M \cup M_1^*$ starts with a buyer and an edge in M. Let $GFT_{M'}(U)$ be the GFT of all edges of M' that are contained in U. We now restate and prove this lemma.

Lemma C.7 (Restatement of lemma 6.7). *For $K > 3$, let $A = (i_1 j_1 i_2 j_2 ... i_{L-1} j_{L-1} i_L j_L)$, be a maximal alternating path of odd number of edges of $M \cup M_1^*$ with any agent i_l denoting a buyer*

4. We slightly abuse notation by using A to also denote the set of agents in the path A.

and any agent j_l denoting a seller, and the first edge in M ($(i_1, j_1) \in M$). It holds that

- *if $b_{i_L} > b_{i_1}$ then $i_L \in M_{-j_L}$.*
- *if $b_{i_L} \le b_{i_1}$ then $GFT_M(A \setminus \{i_L, j_L\}) \ge GFT_{M_1^*}(A)$.*

Proof. We prove that if $b_{i_L} > b_{i_1}$ then $i_L \in M_{-j_L}$. Assume for contradiction that $i_L \notin M_{-j_L}$. Since the tie breaking is subset consistent, the matching picked on $A \setminus \{i_L, j_L\}$ will be the same as the one in M. Yet, the set $A \setminus \{i_1, j_L\}$ is matchable (by the edges of M_1^*) and has higher weight than the weight that M gets on $A \setminus \{i_L, j_L\}$, a contradiction.

We next consider the case that $b_{i_L} \le b_{i_1}$. Let $w = GFT_M(A) = \sum_{l=1}^{L}(b_{i_l} - s_{i_l})$. Notice that:

$$GFT_M(A \setminus \{i_L, j_L\}) = w - (b_{i_L} - s_{i_L}) \ge w - (b_{i_1} - s_{i_L}) = GFT_{M_1^*}(A \setminus \{i_1, j_L\}) = GFT_{M_1^*}(A). \quad \square$$

We are now ready to complete the proof of lemma 6.4, which states that for any valuation profile $(\mathbf{b}, \mathbf{s})$, the gains from trade of the Offering Mechanism for matching markets is at least half of the from trade of the RVWM mechanism for that profile.

Fix a valuation profile $(\mathbf{b}, \mathbf{s})$. To prove the claim we consider the connected components of $M(\mathbf{b}, \mathbf{s}) \cup M_1^*(\mathbf{b}, \mathbf{s})$ and show that in each connected component separately the GFT of the Offering Mechanism in expectation (over the randomness of the mechanism), is at least half the GFT of the RVWM mechanism on $(\mathbf{b}, \mathbf{s})$.

By observation C.1 each connected component is either a maximal alternating path or a cycle. By corollary C.2 any cycle has only two (identical) undirected edges, denote it by $(i, j) \in M(\mathbf{b}, \mathbf{s}) \cap M_1^*(\mathbf{b}, \mathbf{s})$. That is, the unique edge (i, j) of $M_1^* = M_1^*(\mathbf{b}, \mathbf{s})$ in this cycle is the same as the unique edge (i, j) of $M = M(\mathbf{b}, \mathbf{s})$ in that cycle. If $j \in M_{-i}(\mathbf{b}, \mathbf{s})$ or $i \in M_{-j}(\mathbf{b}, \mathbf{s})$, by lemma 6.5 buyer i will trade with seller j in the Offering Mechanism with probability at least $1/2$, which obtains at least half the GFT that the RVWM mechanism obtains on $(i, j) \in M_1^*(\mathbf{b}, \mathbf{s})$ when the profile is $(\mathbf{b}, \mathbf{s})$. Otherwise, since $i \notin M_{-j}$ we have that $\bar{s} \ge b_i$, and since $j \notin M_{-i}$ we have that $\bar{b} \le s_j$. Since trade occurs with positive probability on (i, j) in the RVWM mechanism, then by observation 6.1, in this case the GFT of the RVWM mechanism on this edge are therefore at least those of the RO mechanism with SO parameters ∞ (no constraint) and $D_{b_i}^B$ (unconditioned distribution) and

BO parameters 0 (no constraint) and $D^S_{s_j}$ (unconditioned distribution) on (i,j). Since $\bar{s} \geq b_i$ and $\bar{b} \leq s_j$, we have by lemma 6.2 that the probability that trade occurs between i and j is at least as high in our Offering Mechanism (which runs the appropriate RO mechanism, constrained and conditioned) as it is in the unconstrained and unconditioned RO mechanism (that upper-bounds the GFT of RVWM on this edge). Therefore, in this case our Offering Mechanism achieves at least the gains from trade of the RVWM mechanism on this edge (and therefore, on any alternating cycle).

By corollary C.3, any other maximal alternating path of is not a cycle, and is a path starts or ends with a buyer and an edge from M. We will assume w.l.o.g. that it start with a buyer and an edge from M, and we consider such paths of even and odd numbers of edges separately. Note that Corollary C.3 implies that there is no connected component that does not include at least one edge from M, so by going over all connected components with at least two edges, we cover all the edges of M_1^*.

If the number of edges in the path is even, by lemma 6.6, M matches every seller j in the path to some buyer i, and for any such pair (i,j) it holds that $j \in M_{-i}$. By lemma 6.5 buyer i will trade with seller j in the BO Mechanism, so whenever the BO mechanism runs, the maximal GFT (first best) of that connected component, which is at least the GFT of the M_1^* mechanism for that connected component, will be obtained. The Offering Mechanism runs the BO mechanism is probability $1/2$, so in expectation it obtains at least $1/2$ the GFT of M_1^* for this path.

We next consider the case that the number of edges in the path is odd and at least 3.[5] Let the path be $A = (i_1 j_1 i_2 j_2 ... i_{L-1} j_{L-1} i_L j_L)$ for some $L \geq 2$. By lemma 6.6, for any $l \in \{1, 2, ..., L-1\}$ it holds that $j_l \in M_{-i_l}$. Next, we use lemma 6.5 again. We consider two cases, using lemma 6.7.

- If $b_{i_L} > b_{i_1}$ then $i_L \in M_{-j_L}$. In this case all edges of M will each be traded with probability at least $1/2$ in the Offering Mechanisms, so in expectation it obtains at least $1/2$ the GFT of M in this path and thus also at least $1/2$ the GFT of M_1^* in this path A.

- If on the other hand $b_{i_L} \leq b_{i_1}$ then $GFT_M(A \setminus \{i_L, j_L\}) \geq GFT_{M_1^*}(A)$. Therefore, since every

5. As noted, if there is a single edge, it is only in M. We only need to cover edges in M_1^*.

edge (i_l, j_l) for $l \in \{1, 2, ..., L-1\}$ is traded with probability $1/2$ in the Offering Mechanisms, we have that in expectation the Offering Mechanism obtains in this path A at least $1/2$ of the GFT of M_1^* in this path A.

We conclude that the Offering Mechanisms obtains at least $1/2$ the GFT that the RVWM mechanism gets on M_1^*. Similar arguments show that the Offering Mechanisms obtains at least $1/2$ the GFT of the RVWM mechanism gets on M_2^*. Thus the Offering Mechanisms, obtains at least $1/4$ the total GFT of M_1^* and M_2^*. The expected GFT of the RVWM mechanism is the average GFT of M_1^* and of M_2^*. We conclude that the Offering Mechanisms obtains at least $1/2$ the GFT of the RVWM mechanism.

C.7 The Hybrid Mechanism for Matching Markets: Proofs

In this section, we prove theorem 6.8.

Ex-post IR, ex-post (direct trade) weakly budget balanced They directly come from the fact that both the TR mechanism and Offering Mechanism are ex-post IR and ex-post (direct trade) weakly budget balanced.

Bayesian IC Lemma C.8 proves that after combining the two mechanisms, the hybrid mechanism is still a BIC mechanism.

Lemma C.8. *The hybrid mechanism for matching markets is BIC.*

Proof. We will prove that the Hybrid Mechanism is BIC for the seller. A similar argument holds for the buyer. For each seller j with cost s_j, suppose she misreports her cost to be $s_j' \neq s_j$. We will show that taking expectation over other agents' valuation profile $\mathbf{b}, s_{-j}$, the expected utility of s_j when reporting truthfully is at least the expected utility of seller j with true cost s_j when reporting s_j'. We consider three cases:

- First, consider the case where when seller j reports s_j, then she is in the first-best and the TR Mechanism is run. In this case, we note that if seller j misreports to s_j', then either the first-best is unchanged (and so the TR is still run) or j is no longer in the first-best. In the former case, seller j does not profit since by theorem 6.6 the TR Mechanism is ex-post IC, and in the latter case seller j does not profit as she gets utility 0.

- Now, consider the case where when seller j reports s_j, then she is in the first-best and the Offering Mechanism is run with an offer on the edge (i, j). Similarly to above, we note that if j misreports to s'_j, then either the first-best is unchanged (and so the Offering Mechanism is still run) or j is no longer in the first-best, and has 0 utility. In particular, j cannot cause the TR mechanism to run without getting 0 utility. Also note that for the same reason, for every report of buyer i that keeps (i, j) in the first best, the Offering Mechanism is still run, and so it is enough to show truthfulness of j in expectation over all such reports of buyer i, and we have shown precisely that in claim C.8 using lemma 6.1. So, we have that when j has cost s_j such that there exists $\mathbf{b}, s_{-j}$ such that the Offering Mechanism is run with an offer on the edge (i, j), then in expectation over all such $\mathbf{b}, s_{-j}$, it is the case that s_j cannot gain from misreporting.

- Finally, consider the case where when seller j reports s_j, then she is not in the first-best. In this case, regardless of the mechanism that is actually run when j reports s_j, her outcome reporting s_j would have been the same under both mechanisms. So, since we have shown that when j is not in the first-best, truthtelling is ex-post IC in both mechanisms, we have that this implies that truthfulness is ex-post IC for seller j in the hybrid mechanism in this case. □

Ex-post efficiency guarantee Whenever $\alpha(\mathbf{b}, \mathbf{s}) \geq 1/2$, the hybrid mechanism run TR mechanism. The ex-post guarantee directly comes from Claim C.3.

Ex-ante efficiency guarantee Let $(\mathbf{b}, \mathbf{s})$ be a profile. If $\alpha(\mathbf{b}, \mathbf{s}) \geq \frac{1}{2}$, the hybrid mechanism runs the Trading Reduction mechanism, which achieves at least $1/2$-fraction of the first-best gains from trade. This is at least $1/2$-fraction the gains from trade of the RVWM mechanism. If $\alpha(\mathbf{b}, \mathbf{s}) < \frac{1}{2}$, the hybrid mechanism runs the Offering Mechanism, which by lemma 6.4 achieves at least a $1/2$-

fraction of the GFT of the RVWM mechanism for this profile. So, for any profile the hybrid mechanism achieves at least a $1/2$-fraction of the GFT of the RVWM mechanism for this profile, and so by theorem 6.4, it achieves at least a $1/4$-fraction of the GFT of the second-best mechanism, as required.

Proof of corollary 6.2. Since $\beta(\mathbf{b}, \mathbf{s}) \leq \alpha(\mathbf{b}, \mathbf{s})$ (see the proof of corollary 6.1), we have that in this case also $\alpha(\mathbf{b}, \mathbf{s}) \geq 1/2$, and so the hybrid mechanism runs the TR mechanism for matching markets, and so the claim following via corollary 6.1. □

Appendix D

Missing Details from Chapter 7

D.1 Missing Details from Section 7.3

Proof of Lemma 7.2:

For every $i, \mathbf{s}, b_i$, define

$$q_i(b_i, \mathbf{s}) = (b_i - s_i)^+ \cdot \Pr_{b_{-i}}[i \in S^*(\mathbf{b}, \mathbf{s})] \cdot \mathbb{1}\left[\overline{F_i}^{-1}(\frac{1}{2^{j-1}}) \leq s_i \leq \overline{F_i}^{-1}(\frac{1}{2^j})\right].$$

Then we have that $q_i(b_i, \mathbf{s}) \geq 0$ is non-decreasing in b_i, as both $b_i - s_i$ and the probability $\Pr_{b_{-i}}[i \in S^*(\mathbf{b}, \mathbf{s})]$ is non-decreasing in b_i.

Since $\theta_{ij} = \overline{F_i}^{-1}(\frac{1}{2^j})$ and $\Pr_{b_i}\left[b_i \geq \overline{F_i}^{-1}(\frac{1}{2^j})\right] = \frac{1}{2}\Pr_{b_i}\left[b_i \geq \overline{F_i}^{-1}(\frac{1}{2^{j-1}})\right]$, we have

$$\begin{aligned}\mathbb{E}_{b_i}\left[q_i(b_i, \mathbf{s}) \cdot \mathbb{1}\left[b_i \geq \theta_{ij}\right]\right] \geq & q_i(\theta_{ij}, \mathbf{s}) \cdot \Pr_{b_i}\left[b_i \geq \theta_{ij}\right] \\ = & q_i(\theta_{ij}, \mathbf{s}) \cdot \Pr_{b_i}\left[\overline{F_i}^{-1}(\frac{1}{2^{j-1}}) \leq b_i < \theta_{ij}\right] \\ \geq & \mathbb{E}_{b_i}\left[q_i(b_i, \mathbf{s}) \cdot \mathbb{1}\left[\overline{F_i}^{-1}(\frac{1}{2^{j-1}}) \leq b_i < \theta_{ij}\right]\right].\end{aligned} \tag{D.1}$$

Thus we have

$$
\begin{aligned}
&\mathbb{E}_{\mathbf{b},\mathbf{s}}\left[\sum_i (b_i - s_i)^+ \cdot \mathbb{1}[i \in S^*(\mathbf{b},\mathbf{s}) \wedge E_{ij}]\right] \\
=&\sum_i \mathbb{E}_{b_i,\mathbf{s}}\left[q_i(b_i,\mathbf{s}) \cdot \mathbb{1}[b_i \geq \overline{F_i}^{-1}(\frac{1}{2^{j-1}})]\right] \\
\leq& 2 \cdot \sum_i \mathbb{E}_{b_i,\mathbf{s}}\left[q_i(b_i,\mathbf{s}) \cdot \mathbb{1}[b_i \geq \theta_{ij}]\right] \qquad \text{(Inequality (D.1))} \\
=& 2 \cdot \mathbb{E}_{\mathbf{b},\mathbf{s}}\left[\sum_i (b_i - s_i)^+ \cdot \mathbb{1}[i \in S^*(\mathbf{b},\mathbf{s}) \wedge \overline{E}_{ij}]\right] \\
\leq& 2 \cdot \mathbb{E}_{\mathbf{b},\mathbf{s}}\left[\sum_i (b_i - \theta_{ij})^+ \cdot \mathbb{1}[i \in S^*(\mathbf{b},\mathbf{s}) \wedge \overline{E}_{ij}]\right] + 2 \cdot \mathbb{E}_{\mathbf{b},\mathbf{s}}\left[\sum_i (\theta_{ij} - s_i)^+ \cdot \mathbb{1}[i \in S^*(\mathbf{b},\mathbf{s}) \wedge \overline{E}_{ij}]\right]
\end{aligned}
$$

Moreover, we have

$$
\begin{aligned}
&\mathbb{E}_{\mathbf{b},\mathbf{s}}\left[\sum_i (b_i - \theta_{ij})^+ \cdot \mathbb{1}[i \in S^*(\mathbf{b},\mathbf{s})] \cdot \mathbb{1}[\overline{E}_{ij}]\right] \\
\leq& \mathbb{E}_{\mathbf{b},\mathbf{s}}\left[\sum_i (b_i - \theta_{ij})^+ \cdot \mathbb{1}[s_i \leq \theta_{ij}] \cdot \mathbb{1}[i \in S^*(\mathbf{b},\mathbf{s})]\right] \\
\leq& \mathbb{E}_{\mathbf{b},\mathbf{s}}\left[\max_{S \in \mathcal{F}} \sum_{i \in S} \left\{(b_i - \theta_{ij})^+ \cdot \mathbb{1}[s_i \leq \theta_{ij}]\right\}\right]
\end{aligned} \tag{D.2}
$$

Similarly,

$$
\mathbb{E}_{\mathbf{b},\mathbf{s}}\left[\sum_i (\theta_{ij} - s_i)^+ \cdot \mathbb{1}[i \in S^*(\mathbf{b},\mathbf{s}) \wedge \overline{E}_{ij}]\right] \leq \mathbb{E}_{\mathbf{b},\mathbf{s}}\left[\max_{S \in \mathcal{F}} \sum_{i \in S} \left\{(\theta_{ij} - s_i)^+ \cdot \mathbb{1}[b_i \geq \theta_{ij}]\right\}\right]
$$

□

Proof of Lemma 7.3: For every $i \in [n]$ and $j = 1, ..., \lceil \log(2/r) \rceil$, let E'_{ij} be the event that $G_i^{-1}(\frac{1}{2^j}) \leq b_i \leq G_i^{-1}(\frac{1}{2^{j-1}}) \wedge s_i \leq G_i^{-1}(\frac{1}{2^{j-1}})$ and $\overline{E}'_{ij}$ be the event that $G_i^{-1}(\frac{1}{2^j}) \leq b_i \leq G_i^{-1}(\frac{1}{2^{j-1}}) \wedge s_i \leq G_i^{-1}(\frac{1}{2^j})$. We have

$$
② \leq \sum_{j=1}^{\lceil \log(\frac{2}{r}) \rceil} \mathbb{E}_{\mathbf{b},\mathbf{s}}\left[\sum_i (b_i - s_i)^+ \cdot \mathbb{1}[i \in S^*(\mathbf{b},\mathbf{s}) \wedge E'_{ij}]\right].
$$

Fix any j. For every $i, \mathbf{b}, s_i$, define

$$q_i(\mathbf{b}, s_i) = (b_i - s_i)^+ \cdot \Pr_{s_{-i}}[i \in S^*(\mathbf{b}, \mathbf{s})] \cdot \mathbb{1}\left[G_i^{-1}(\frac{1}{2^j}) \leq b_i \leq G_i^{-1}(\frac{1}{2^{j-1}})\right].$$

Then we have that $q_i(\mathbf{b}, s_i) > 0$ is non-increasing in s_i. Since $\theta'_{ij} = G_i^{-1}(\frac{1}{2^j})$ and $\Pr_{s_i}\left[s_i \leq G_i^{-1}(\frac{1}{2^j})\right] = \frac{1}{2}\Pr_{s_i}\left[s_i \leq G_i^{-1}(\frac{1}{2^{j-1}})\right]$, we have

$$\begin{aligned}\mathbb{E}_{s_i}\left[q_i(\mathbf{b}, s_i) \cdot \mathbb{1}\left[s_i \leq \theta'_{ij}\right]\right] \geq & q_i(\mathbf{b}, \theta'_{ij}) \cdot \Pr_{s_i}\left[s_i \leq \theta'_{ij}\right] \\ = & \frac{1}{2} q_i(\mathbf{b}, \theta'_{ij}) \cdot \Pr_{s_i}\left[s_i \leq G_i^{-1}(\frac{1}{2^{j-1}})\right] \\ \geq & \frac{1}{2}\mathbb{E}_{s_i}\left[q_i(\mathbf{b}, s_i) \cdot \mathbb{1}\left[s_i \leq G_i^{-1}(\frac{1}{2^{j-1}})\right]\right].\end{aligned} \tag{D.3}$$

Thus we have

$$\begin{aligned} & \mathbb{E}_{\mathbf{b},\mathbf{s}}\left[\sum_i (b_i - s_i)^+ \cdot \mathbb{1}[i \in S^*(\mathbf{b}, \mathbf{s}) \wedge E'_{ij}]\right] \\ = & \sum_i \mathbb{E}_{\mathbf{b}, s_i}\left[q_i(\mathbf{b}, s_i) \cdot \mathbb{1}[s_i \leq G_i^{-1}(\frac{1}{2^{j-1}})]\right] \\ \leq & 2 \cdot \sum_i \mathbb{E}_{\mathbf{b}, s_i}\left[q_i(\mathbf{b}, s_i) \cdot \mathbb{1}[s_i \leq \theta'_{ij}]\right] \qquad \text{(Inequality D.3)} \\ = & 2 \cdot \mathbb{E}_{\mathbf{b},\mathbf{s}}\left[\sum_i (b_i - s_i)^+ \cdot \mathbb{1}[i \in S^*(\mathbf{b}, \mathbf{s}) \wedge \overline{E}'_{ij}]\right] \\ \leq & 2 \cdot \mathbb{E}_{\mathbf{b},\mathbf{s}}\left[\sum_i (b_i - \theta'_{ij})^+ \cdot \mathbb{1}[i \in S^*(\mathbf{b}, \mathbf{s}) \wedge \overline{E}'_{ij}]\right] + 2 \cdot \mathbb{E}_{\mathbf{b},\mathbf{s}}\left[\sum_i (\theta'_{ij} - s_i)^+ \cdot \mathbb{1}[i \in S^*(\mathbf{b}, \mathbf{s}) \wedge \overline{E}'_{ij}]\right]\end{aligned}$$

Moreover, we have

$$\begin{aligned} & \mathbb{E}_{\mathbf{b},\mathbf{s}}\left[\sum_i (b_i - \theta'_{ij})^+ \cdot \mathbb{1}[i \in S^*(\mathbf{b}, \mathbf{s})] \cdot \mathbb{1}[\overline{E}'_{ij}]\right] \\ \leq & \mathbb{E}_{\mathbf{b},\mathbf{s}}\left[\sum_i (b_i - \theta'_{ij})^+ \cdot \mathbb{1}[s_i \leq \theta'_{ij}] \cdot \mathbb{1}[i \in S^*(\mathbf{b}, \mathbf{s})]\right] \\ \leq & \mathbb{E}_{\mathbf{b},\mathbf{s}}\left[\max_{S \in \mathcal{F}} \sum_{i \in S} \left\{(b_i - \theta'_{ij})^+ \cdot \mathbb{1}[s_i \leq \theta'_{ij}]\right\}\right]\end{aligned} \tag{D.4}$$

Similarly,

$$\mathbb{E}_{\mathbf{b},\mathbf{s}}\left[\sum_i (\theta'_{ij} - s_i)^+ \cdot \mathbb{1}[i \in S^*(\mathbf{b},\mathbf{s}) \wedge \overline{E}'_{ij}]\right] \leq \mathbb{E}_{\mathbf{b},\mathbf{s}}\left[\max_{S \in \mathcal{F}} \sum_{i \in S} \left\{(\theta'_{ij} - s_i)^+ \cdot \mathbb{1}[b_i \geq \theta'_{ij}]\right\}\right]$$

□

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