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### Essays in Industrial Organization

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# Abstract

Essays in Industrial Organization

Yuzhou Wang

2021

This dissertation studies a few topics in industrial organization. In the first two chapters, I study the education market from the perspective of industrial organization. Chapter 3 studies second-price auctions with participation costs.

Chapters 1 and 2 study the high school market of a large city in China that introduced a policy allowing public schools to offer both free and priced admission options within a centralized admission mechanism. Chapter 1 introduces the institutional background, such as details of the admission mechanism, the schools and their price levels, and the new policy. I then present descriptive analysis. I also estimate a high school value-added regression, where I regress exit exam scores on entrance exam scores and other variables. I interpret the value-added as the high school quality. Results show that quality increases after the implementation of the policy. A difference-in-differences regression further shows that top-tier schools are able to increase their value-added more. I use these results as both inputs and motivating facts for my study in Chapter 2.

Chapter 2 formally develops a demand and supply model. On the demand side, students consider both their preferences for a school and their probabilities of being admitted to that school to file a report of preferences. I develop an algorithm to quickly find students' optimal reports, and this algorithm helps to reduce the

computational burden in demand estimation. On the supply side, I model schools as maximizing a weighted average of profit and quality, so as to allow for the existence of excess demand for good schools. Demand estimation using students' strategic reports quantifies the extent to which students with higher entrance exam scores care more about quality relative to price. Supply side estimation shows that top-tier schools have lower marginal costs of quality and thus choose higher quality. Counterfactual analysis shows that introducing subsidies to low income students while keeping the current priced admission options would give students more equal access to good schools, while keeping the quality gain brought by market incentives. Another counterfactual analysis shows that the quality gain brought by market incentives is driven by an increase in funds to improve quality and schools' preference for quality.

Chapter 3 studies equilibria and efficiency in second-price auctions with public participation costs. This is joint work with José-Antonio Espín-Sánchez and Álvaro Parra. We generalize previous results by allowing arbitrary heterogeneity in bidders' distributions of valuations and in their participation costs. We develop a notion of bidder strength, based on the best response of a bidder when all of her opponents play the same strategy she does. We then show that a *herculean equilibrium*, in which stronger bidders have a lower participation threshold than weaker bidders, exists in general environments. In other words, the order of bidders given by their strength, which is a non-equilibrium concept and can be easily calculated for each bidder using only one equation, predicts the order of the participation thresholds in a certain equilibrium which exists in general. Combined with a sufficient condition for equilibrium uniqueness that we further provide, bidders' strength points out the direction for

finding and simplifies the formulation of the equilibrium. Furthermore, even though all equilibria are *ex-post* inefficient, an *ex-ante* efficient equilibrium always exists. Therefore, under the uniqueness condition, the *herculean equilibrium* is the unique equilibrium of the game and is *ex-ante* efficient.

Essays in Industrial Organization

A Dissertation  
Presented to the Faculty of the Graduate School  
of  
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Doctor of Philosophy

by  
Yuzhou Wang

Dissertation Director: Philip Haile, Steven Berry, José-Antonio  
Espín-Sánchez, and Yusuke Narita

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To my grandparents, parents, and brother

For their unconditional love and support

# Chapter 1

## Introducing Prices to Public School

### Assignment: Institutional

## Background and Empirical Evidence

### 1.1 Introduction

Market incentives have been introduced to many school systems around the world, with goals such as promoting school quality and equalizing student access. One of these policies, although not tried in the U.S. to my knowledge, is to allow public schools to offer paid admission options. Such a policy has been tried in China and is the subject of my empirical work.

From the supply perspective, allowing schools to charge tuition may create incentives for improvements in school quality (Betts and Tang (2011); Mizala and Torche

(2012)).<sup>1</sup> Such improvements in quality would not only benefit those who pay tuition to be admitted, but would also benefit those who do not pay. From the demand perspective, the availability of paid admission options may allow students to better express their preferences over price and quality (Harris *et al.* (2015)). However, such a policy may also undermine the goal of equal access to good schools by favoring those with greater ability to pay. Some existing theoretical and empirical literature has suggested similar concerns (Epple and Romano (2012); Hanushek *et al.* (2016); Manski (1992)). Most students prefer schools with higher value added, which allows such schools to charge higher prices. This, in turn, allows schools with higher value added to collect more funds and to increase school value added even further. As a result, the gap in school value-added may grow, with richer students better able to access better schools.

I study education markets from the perspective of industrial organization to understand the equilibrium effects on school quality and equal access of introducing prices to public-school systems.

Specifically, I capture two key features that differentiate education markets from other commonly studied industries. First and foremost, there are usually excess demand, typically for some top schools, in education markets. Although education services resemble other consumer goods in the dimension of being both vertical and horizontal differentiated, its vertical attribute, namely, the quality of a school, usually plays a more important role and a school with high quality can not be easily

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1. Some researchers have studied how school choices induced competition affects school outcome. See Hoxby (2003) and Rothstein (2006) for an example



substituted. While this brings good schools huge demand and local or even global market powers, capacity constraints usually means that not all students can be admitted to their most preferred school. This, in turn, induces the introduction of various centralized assignment or decentralized matching mechanisms, either by the government or school themselves. As a result, students' true preferences are usually entangled with their beliefs and strategies in most markets. Second, the existence of excess demand also implies that schools are not pure profit maximizers and they also care about other things, such as their reputations of high quality. The emphasize on quality by both students and schools also means that it is crucial to understand schools' marginal cost of producing quality.

The high school market of a large city in China that is characterized by these two features provides a representative environment to study the impact of introducing price to public-school systems. Several other attractive settings of this market further makes it suitable for the purpose of this study. All high schools, public or private, have to admit students through a centralized admission mechanism. Although private schools are not major players in this market, it nonetheless allows me to incorporate all schools into the model instead of treating private schools as outside options. Still, public schools are the major players in the market and are the focus of this study. The city introduced a policy allowing individual public schools to offer both free and paid admission options. These options are treated as different non-exclusive choices, and students can apply for both within the centralized admission mechanism.

Descriptive analysis shows that, after schools are allowed to charge tuition, many but not all schools choose to offer the paid admission options over the years, the

number of students admitted through such options increases over time, and the tuition increases too. Using a data set on high school entry and exit exam scores, a value-added regression shows that school value-added in test scores increases after the implementation of the policy.<sup>2</sup> Moreover, a difference-in-differences regression shows that top-tier schools are able to increase their value-added more, when compared to middle-tier and bottom-tier schools.<sup>3</sup> I use such results as both input to the demand model and motivating facts for the supply model, so as to explain why different schools choose different quality and to study the distributional implication of the policy.

I collect data on students' reports of school preferences. Students are allowed to rank a limited number of schools in a report that they must file before taking the entrance exam. Thus, students need to be strategic,<sup>4</sup> considering not only their true preferences but also their beliefs about admission probabilities. Students' exam scores are a crucial part in the centralized admission mechanism, which is a variation of serial dictatorship. This setting is different from the commonly studied setting of K-12 school admissions in the U.S. The students' ranking scores in such settings, which determines whom to be considered first, depends on their school district, status of their siblings, and their reports, instead of students' talents (Abdulkadiroğlu and Sönmez (2003)).<sup>5</sup>

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2. Other research also tries to compare performances of public schools and private schools. See Altonji *et al.* (2005) and Dobbie and Fryer Jr (2011) for an example.

3. Each high school is categorized into top-tier, middle-tier or bottom-tier by the government and such categorization didn't change in the time period of my study. More details about this in later sections.

4. Some researchers have shown that some students are strategic under other mechanisms too; see Abdulkadiroğlu *et al.* (2006) and Calsamiglia and Guell (2013) for an example.

5. See Abdulkadiroğlu *et al.* (2005) and Pathak and Sönmez (2013) for more details about ad-

The rest of this chapter is organized as follows: Section 1.2 introduces basic institutional backgrounds of this study, followed by a detailed explanation of the data I collected in Section 1.3. Section 1.4 describes some stylized facts from the data and estimates school valued-added in test scores, which is both an input to the demand model and an motivating fact to the supply model. Section 1.5 concludes this chapter.

## 1.2 Institutional Background

In this paper, I study the high school market in a northern city in China. Since 2006, public high schools in this city are allowed to offer both free and priced admission options. As a result, there are three possible channels for students to be admitted to a high school: free tuition, low tuition, and high tuition, the first two of which already exist before the new policy. Schools can choose whether to have the high tuition channel. In fact only schools good enough choose to have it. The price of high tuition varies between around 20,000 RMB and 50,000 RMB for three years. The price of low tuition varies between around 5,000 RMB and 20,000 RMB. The average individual income is around 40,000 RMB in this period. When schools choose to have the high tuition channel, they will also choose the number of possible seats of students to admit through this channel. However, this choice is highly regulated by the government. After students are admitted, their admission channels don't matter: each student has the same opportunity to be assigned to different small classes. Schools don't

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mission mechanisms such as Boston mechanism in the US. See Lise *et al.* (2004), Todd and Wolpin (2006), Bajari and Hortacsu (2005), and Pathak and Shi (2014) for use of data from random social experiments, lab experiments, or regime shifts.

systematically assign good teachers to students admitted through free tuition channels or high tuition channels. This means that the education services, teaching quality more specifically, are the same for different channels. Only the price is different for different channels within the same school.

Each high school is categorized by the government into top tier, middle tier, or bottom tier. Top-tier high schools are seen as the ones with the best quality, and all of them have all three possible channels (free, low, high tuition) to admit students. Most middle-tier schools have all three possible channels, but with lower prices for low and high tuition channels than those of top-tier schools. Bottom-tier schools only have free and low tuition channels to admit students. Each tier has about 10 high schools. Table 1.1 below summarizes price levels for different channels and school tiers.

Table 1.1: Price levels for school tiers and channels

RMB	Free	Low	High
Top tier	0	~ 18,000	~ 48,000
Middle tier	0	~ 15,000	~ 36,000
Bottom tier	0	~ 8,000	NA

Middle school graduates file a rank-ordered list of their preferences over school-channels before they take the High School Entrance Exam. Each school-channel is listed as one possible choice. The list of 16 ordered choices should be filled according to some specific rules that require students to choose from a certain subset of school channels for each spot in the list. Such subsets are exclusive and exhaustive and are based on school tiers and channels. For example, for the very first spot in the list, students can only choose from free tuition channels of top-tier high schools; for the

second and third spots, they can only choose from low tuition channels of top-tier schools; for the fourth and fifth spots, they can choose from free tuition channels of middle-tier schools. And for spots from sixth to tenth, they can choose from high tuition channels of top-tier schools, low tuition and high tuition channels of middle-tier schools, and free and low tuition channels of bottom-tier schools. See [Table 1.2](#) below for an illustration. Thus, such rank-ordered preference list is divided into sub-lists and truncated for each sub-list.

Table 1.2: An illustration of how school tiers and channels are allocated in the report

Slots	Free	Low	High
Top tier	1	2, 3	6 – 10
Middle tier	4, 5	6 – 10	6 – 10
Bottom tier	6 – 10	6 – 10	NA

For example, a school-channel that is from top-tier school and free should be reported in the 1st slot. A school-channel that is from a top-tier school and low price should be reported in the 2nd and 3rd slot.

After the exam scores are revealed, students are assigned to school channels by serial dictatorship mechanism: students are admitted one by one, from the highest score to the lowest score, to their most preferred school channel that has not been filled up yet when it is her turn. The mechanism is not a pure serial dictatorship due to the fact that the list is divided and truncated. Thus, such admission mechanism is not strategy-proof. When reporting preference lists, students consider not only their utilities from school-channels but also the probabilities that they will be admitted. The timeline is summarized in [Figure 1.1](#).

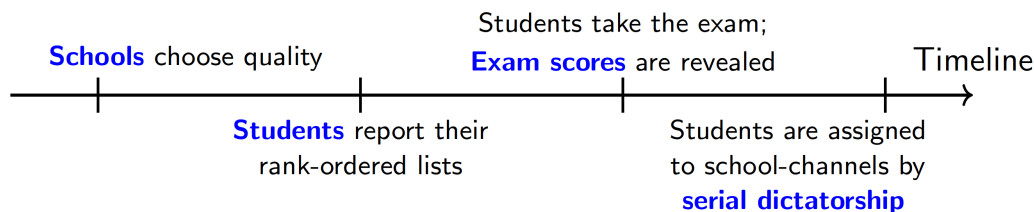


Figure 1.1: Timeline

### 1.3 Data

I collect three sets of data. The first set is high school entrance exam data from 2007 to 2016. For each year, the data set includes a random 60% sample from the full population. Each observation includes the student’s full reported rank-ordered list of preferred school channels, the high school entrance exam total scores together with scores for each of the seven subjects, the school channel admitted, the middle school attended, and personal information such as gender, race, and DOB. It also includes a personal identifier. For each year, there are approximately 15,000 observations.

The second set of data is high school exit exam data from 2007 to 2016 except 2011 and 2015. For each year, the data set includes the full population of those who graduate from high schools, corresponding to matriculating cohorts from 2004 to 2013. Only students from current cohort are included; those who spend another

year in high school after high school graduation are excluded. Such students spend another year in the same or a different high school to take the high school exit exam, a.k.a., the college entrance exam, more than once, with the hope of being admitted to a better college. I exclude such students since they study at possibly more than one high schools for more than three years. Each observation includes the high school exit exam, a.k.a., the college entrance exam, total scores together with scores for each of the four subjects. It also includes gender, race, DOB and a personal identifier. For each year, there are approximately 15,000 observations. I use the personal identifier to match the entrance exam data and the exit exam data, and it gives about 9,000 observations for each year.

The third set of data is high school-channel characteristics. This includes the price and number of seats for each school channel. I also calculate distance from each high school to each middle school. Since students are assigned to the middle school that is closest to her home, I use the distance between middle schools and high schools as a proxy for the distance from students' home to high schools.

I construct a data set of 5 years by matching the above three data sets. It covers students who enter high schools in 2007, 2009, 2010, 2011, and 2013.

## 1.4 School Quality

### 1.4.1 Basic facts

After high schools are allowed to offer paid admission options with high prices, top-tier schools increase their total number of seats in general and the number of seats in high tuition channels increases in general too. [Figure 1.2](#) shows that top-tier schools have a larger market share over the years, while both middle-tier and bottom-tier schools shrink. This means top-tier schools have much larger number of seats and are expanding, since each tier has about the same number of schools and the total number of students increases slightly over the time period.

[Figure 1.2](#) also shows that only top-tier schools can effectively attract a significant number of students to pay for their high tuition channels, and that the percentage of students admitted through high tuition channels is increasing over the years (the green bars).

[Figure 1.3](#) shows that the average price of top-tier schools (weighted by the number of students in each school-channel) increases significantly over the years while the average price of middle-tier and bottom-tier schools stays roughly the same.

### 1.4.2 Quality estimation

I estimate the teaching quality for each school-year and examine whether the teaching quality changes in each high school after they are allowed to offer paid admission options with high prices.

There is large literature on how to measure teaching quality, such as [Abdulka-](#)



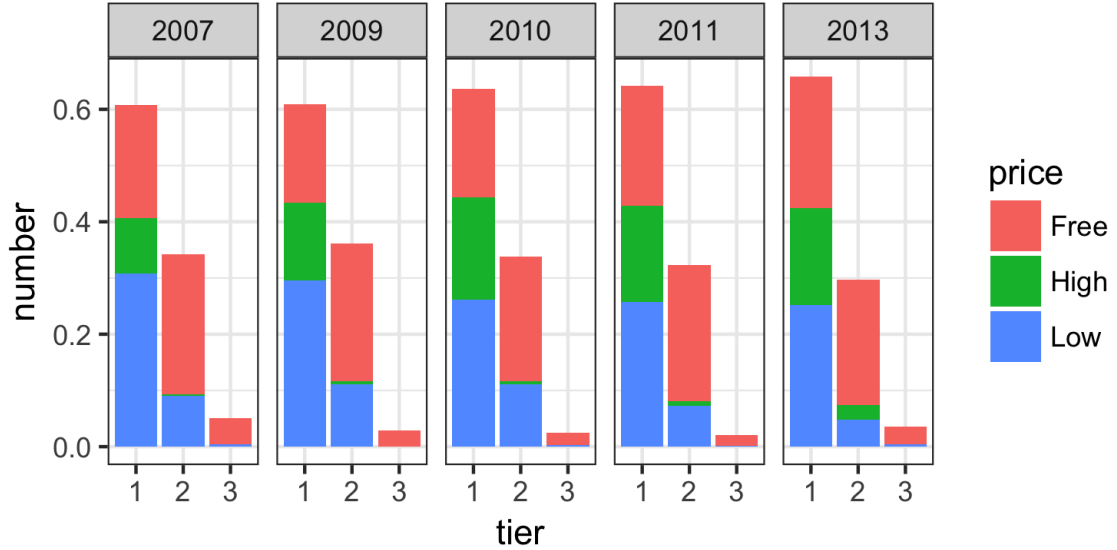


Figure 1.2: Change in number of seats for channels with different price

dirođlu *et al.* (2019) and Rockoff and Turner (2010). I use the matched data to estimate a value-added model and interpret the value-added as schools' quality.

$$\begin{aligned}
 ExitScore_{ijt} = & EntryScore_{ijt}\gamma_1 + X_{ijt}\gamma_2 + HighSchool_{jt} \\
 & + MiddleSchool_k + percentileAtEntry_{ijt}\gamma_3 + v_{ijt}
 \end{aligned}$$

*ExitScore* is student *i*'s total score of college entrance exam (high school exit exam). *EntryScore* is a vector of all seven subject scores of her high school entrance exam, which are Chinese, math, English, physics, chemistry, history, and political science. *X* are demographic variables which include gender and race of the student. *HighSchool* is a dummy for each high school-year. *MiddleSchool* is a dummy for each middle school. *percentileAtEntry* is the percentile of a student's total entrance exam score within the high school she is admitted to. I include this variable to account

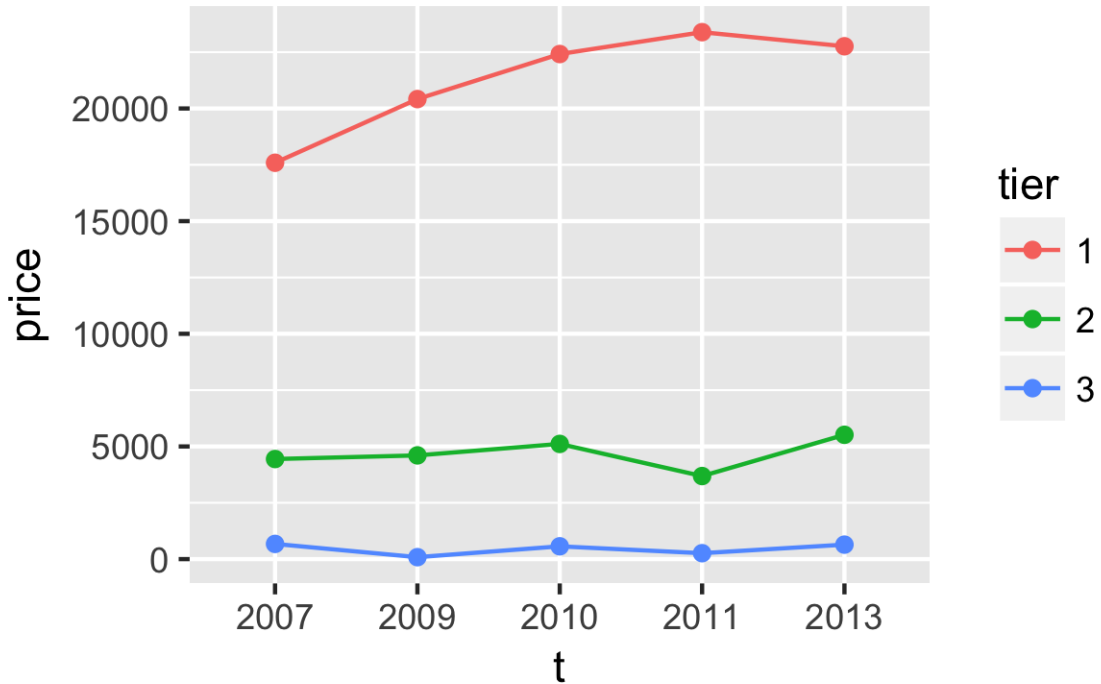


Figure 1.3: Change in average price of schools in different tiers

for the possible concern that students may benefit from an environment where they are the star and can be more confident. Similar ideas are also exploited in [Kapor \(2015\)](#).

*EntryScore* is normalized by each subject for each year based on the score distribution of the whole student body of the city. *ExitScore* is standardized for each year based on the score distribution of the whole province which the city is in. Since the difficulty of the exam varies year by year, it's better to standardize exam scores year by year rather than pooling several years together. Ideally, I would like to normalize *EntryScore* based on the distribution of the province too. But the high school entrance exam of this city is separate from the exam of the province. Students from outside this city don't take the exam of this city. Thus, there is no way to fairly compare scores across the whole province. The exit exam (college entrance exam),

however, is the same for every student from the province, and the exam scores are comparable. I choose to standardize the *ExitScore* by the score distribution of the province instead of the city. In this way, high school value-added is not a mere relative comparison between schools within a specific year. It can also be allowed to be compared across years.

I interpret the coefficient on *HighSchool<sub>jt</sub>* as the teaching quality of high school *j* at year *t*. There might be concerns of selection in this regression. Ideally, I would like to control for more student characteristics, for example, family income and parents' education level. However, I couldn't do this due to data limitations. I do control for the middle school a student attends. Since students are assigned to middle schools based on proximity of their home to the school, *MiddleSchool* not only controls for the value-added from the middle school, but also controls for, at least to some extent, the income level in that neighborhood. Another explanatory variable that is of our interest is peer effect or student composition. However, the observation of student composition is at the school level, and will lead to collinearity problems if I want to include it in the value-added regression.

For simplicity, I show a part of the results in [Table 1.3](#) to give a sense of what is happening. The benchmark is high school with code 7907 in year 2007. This is a bottom-tier high school.

High school *hs7701* *hs7703* are top-tier high schools, and *hs7701* is actually the very top school within the top-tier. It is the very first school to have the high tuition channel. *hs7705* *hs7901* are middle-tier schools and have high tuition channels at a rather late time, and charge lower prices than top-tier schools. *hs7003* *hs7709* are

Table 1.3: High school quality estimation

	2007	2009	2010	2011	2012	2013
<i>hs7701</i>	1.551	1.472	1.826	1.710	1.665	1.700
<i>hs7703</i>	1.157	1.123	1.151	1.166	1.205	1.374
<i>hs7705</i>	0.878	0.757	0.718	0.954	0.909	0.876
<i>hs7901</i>	0.608	0.613	0.668	0.658	0.813	0.862
<i>hs7003</i>	0.229	0.075	0.129	0.151	0.179	0.223
<i>hs7709</i>	0.187	-0.075	0.057	0.137	0.288	0.114

bottom-tier schools and never choose to have the high tuition channels.

The first column reflects the fact that top-tier schools have the highest value-added and bottom-tier schools have the lowest value-added. Such pattern doesn't change over the years.

For school *hs7701*, the value-added increases first and then decreases a little bit and then stays roughly the same. The value-added does increase after it starts to have a high tuition channel. The value-added does not increase further because it has already reached its capacity to collect tuition. It has already charged the highest price and admitted the maximum amount of students that the government allows.

For school *hs7703*, it starts its high tuition channel later than *hs7701* and it increases its price and number of admitted students over the years. As a result, its value-added keeps increasing.

For middle-tier school *hs7705* and *hs7901*, the value-added are increasing, but at a lower magnitude than those of top-tier schools, possibly due to the fact that they charge lower prices and admit fewer students through high tuition channels.

For bottom-tier school *hs7003* and *7709*, the value-added doesn't change much or even decreases, as expected, since they have no high tuition channels and can not

collect more money to increase their teaching quality.

### 1.4.3 Quality change across school tiers

To further look into the changes in school value-added across time and different school tiers, I run the following regression with  $q$  being the value-added estimated from above:

$$q_{jt} = \alpha_{2007,T} + \alpha_{2007,M} \text{MiddleTier}_j + \alpha_{2007,B} \text{BottomTier}_j + \sum_t \alpha_{t,T} \mathbf{D}_t + \sum_t \alpha_{t,M} \mathbf{D}_t \times \text{MiddleTier}_j + \sum_t \alpha_{t,B} \mathbf{D}_t \times \text{BottomTier}_j + \varepsilon_{jt}$$

*MiddleTier* is an indicator and equals 1 if the school is a middle-tier school. Similar for *BottomTier*.  $\mathbf{D}_t$  is an indicator for year  $t$ . The benchmark is then top-tier high schools in year 2007. Schools in different tiers have different abilities to collect money through paid admission options. The results are presented in Table 1.4. And the

Table 1.4: Quality change

$\alpha_{tk}$	Top	Middle	Bottom
2007	1.005***	-0.407**	-0.780***
2009	-0.090	-0.012	-0.065
2010	0.187	-0.334	-0.269
2011	0.230	-0.295	-0.238
2013	0.242	-0.240	-0.137

result is shown in Figure 1.4 too, where the value plotted is the value-added. For example, for bottom-tier year 2013, its  $\alpha_{2007,T} + \alpha_{2007,B} + \alpha_{2013,T} + \alpha_{2013,B}$

The results show a few things. First, comparing the first row of the table, we see value-added differences are quite significant across school tiers at year 2007. Second,

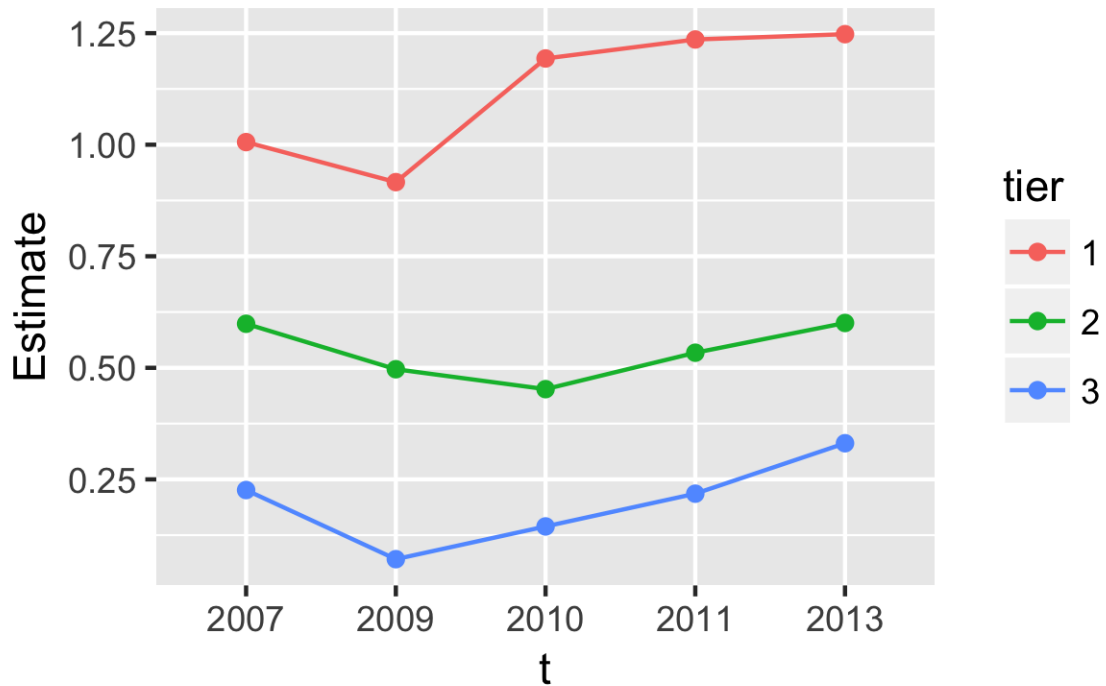


Figure 1.4: Change in teaching quality of schools in different tiers

the blue line in Figure 1.4 shows that the teaching quality for the bottom-tier schools increases a little in later years but barely changes when compared to 2007. Such schools do not have quality high enough to attract students so as to take full advantage of the paid admission options. Actually, their good teachers may even be recruited to other schools with better salaries. At last, the red line of Figure 1.4 shows that top-tier schools' value-added increases faster than that of middle-tier and bottom-tier schools, and the differences between them are spreading. This reflects the fact that top-tier schools have the strongest ability to attract students and to take the full advantage of the paid admission options to collect tuition which can then be used to increase teaching quality.

But notice that although the difference of value-added among schools are spreading, no category shows a significant reduction in quality. We also need to take into

account the fact that after schools are allowed to offer paid admission options with high prices, more students go to top-tier schools, leaving bottom-tier schools shrinking. This means that the total social gain from quality improvements may be even larger than the figure above suggests.

All results above, however, have not taken into account of family income, which is not in my data. In order to better understand the role of family income and its interaction with school demand, I need to turn to structural models which will allow me to simulate family income in the counterfactual analysis. More importantly, estimating a structural model will allow me better understand why different schools choose different levels of quality.

## 1.5 Conclusion

In this chapter, I provide institutional background for the high school market that is at the center of my study, and examine empirical evidence of the impact of allowing public high schools to offer both free and priced admission options. Descriptive evidence shows that after the implementation of the policy, seats from high price channels account for a larger share over the years. Using a data set on high school entry and exit exam scores that I collect, I estimate a value-added regression, and interpret the value-added as high school teaching quality. The results show that after the implementation of the policy, teaching quality measured by value-added increases over time.

Furthermore, a difference-in-differences regression shows that such increase in

teaching quality is not distributed equally among schools. Top-tier high schools with better ability to attract students and to collect tuition are more likely to have a larger increase in teaching quality.

To fully understanding of the impact of the policy on teaching quality and on equal access to good schools, I further develop and estimate a structural model in the next chapter.



# Chapter 2

## Introducing Prices to Public School

### Assignment: Structural Model and

### Counterfactual

#### 2.1 Introduction

In this chapter, I build and structurally estimate a model of school choice by students from the demand side and of quality choice by schools from the supply side. It allows me to understand students' heterogeneous demand and schools' behaviors that are in a different environment from a traditional consumer goods market. Furthermore, the counterfactual analysis provides insights into to why introducing price leads to quality change and what can be done to address the adverse impact on equal access.

Using the data set of students' strategic reports, as described in Chapter 1, I estimate students' true school preferences and beliefs about admission probabilities. I

allow students' beliefs to depend on a noisy signal of their test scores. I allow students with different characteristics to have different preferences over school characteristics such as quality and price. An important student characteristic is the signal of their exam scores. I allow students with different signals to have different preferences. Although the signal is not observable to the researcher, the true score is, making the preference coefficients random from the perspective of the econometrician. The spread of the characteristics between the first and the last reported schools on the list is useful to identify the variance of the noisy signal. Since I observe reports of each student, interactions between observed student and school characteristics help to identify important utility parameters.

I estimate the demand parameters using the method of simulated moments, together with an algorithm I propose to find optimal reports for individual students. This algorithm exploits two properties of optimal reports under a serial dictatorship mechanism. Specifically, I show that both cutoffs and utilities of the schools reported on the list should be decreasing with the order of the spots if the report is optimal. This algorithm also works for a range of mechanisms where the ranking score of an applicant does not depend on her exact report. My demand estimation shows that students with higher exam scores care more about quality relative to price.

From the supply side, I model schools to care about a weighted average between profit and its quality, so as to incorporate the facts that schools may want to keep excess demand in the market. To better understand schools choice of quality, I explicitly model schools' marginal costs of producing quality as being quadratic in quality. This not only captures the fact that producing quality is more and more costly with

higher level quality, but also allows for the problem of choosing optimal quality to be well-behaved. I combine such models of school competition with the demand side to estimate the marginal costs of different schools. My results show that top-tier schools have lower marginal costs of producing quality and will choose higher levels of teaching quality.

The estimated demand and supply models allow me to run counterfactual analysis and examine the impact of different policy changes. The first set of counterfactual is to study how introducing subsidies to students from low income families can impact their chances to be admitted to top-tier schools. Results show that 20% of students from the poorest 10% families are admitted to top-tier schools when there is no subsidy at all. As a comparison, 92% of students from the richest 10% families are admitted to top-tier schools in this scenario. When I introduce tuition subsidies to low income students of up to RMB 24,000, which can be used to cover tuition, 43% of students from the poorest 10% families are admitted to top-tier schools, and the percentage for students from the richest 10% families drops to 77%. When the tuition subsidy is up to RMB 50,000, the percentage for students from the poorest 10% families that are admitted to top-tier schools further rises to 57%, while the percentage for students from the richest 10% families further drops to 64%. This shows that introducing subsidies to low income students while keeping the current priced admission options would give students more equal access to better schools.

I run another counterfactual where I set the price of high tuition channel to the price of the low tuition channel of the same school. I then decompose the optimal quality and compare it with the decomposition of quality under current policy so

as to understand where the quality gain brought by introducing prices comes from. Using top-tier schools as an example, the results show that the increase in price of the high tuition channel leads to an increase in quality of 0.149 standard deviation (s.d.) of test scores. Out of this amount, the increase in price in a competitive setting brings a increase in quality of 1.042 s.d., driven by increase in price; the linear quality markdown brings a decrease in quality of 1.187 s.d., driven by the decrease of demand elasticity with respect to quality when the price increases; the quadratic markdown brings a decrease of 0.013 s.d., in a smaller scale when compared to previous two terms; and finally, the preference markup gives an increase in quality of 0.307 s.d., bringing the total change in quality to the positive territory.

Although this paper, to my best knowledge, is the first to study both demand and supply sides of education markets with excess demand and the effects of introducing to prices to public schools, it builds on previous research on related topics. The first strand of related research focuses on the topic of the effects of market incentives on education markets. Research by [Manski \(1992\)](#) and [Epple and Romano \(2008\)](#) pioneered the theoretical analysis of the effects of voucher programs. [Azevedo and Leshno \(2016\)](#) extend the discrete Gale-Shapley framework to an environment of a continuum of students to study the effects of school competitions. There is also research that use tools from industrial organization to study education markets, such as [Neilson \(2013\)](#) and [Allende \(2019\)](#). Both of them study the primary school markets in Chile with different focus from this paper. [Neilson \(2013\)](#) focus on the effectiveness of a voucher program while [Allende \(2019\)](#) focus on the social interaction among students.

The second strand of related literature studies the demand estimation from strategic reports. The two articles most closely related to this paper are Agarwal and Somaini (2018) and Calsamglia *et al.* (2018).<sup>1</sup> Agarwal and Somaini (2018) study the strategic reports of students under Boston Mechanism and interpret students' problem as finding the combination of schools that maximize expected utility. They rigorously formulate the problem, show the existence and uniqueness of equilibrium, the identification of the problem, and provide a roadmap of estimation. Calsamglia *et al.* (2018), on the other hand, exploit the two properties of the admission mechanisms, namely, sequentiality of spots in nature and reducible history, to simplify the problem. In terms of utility model setup, this paper follows the traditional IO literature to include interaction between student characteristics and school characteristics and allows for random coefficients of students' preferences. This extends Agarwal and Somaini (2018) and Calsamglia *et al.* (2018), both of which rely on fixed coefficients. This paper borrows and combines the ideas of Berry *et al.* (2004), Agarwal and Somaini (2018), Berry and Haile (2014) and Berry and Haile (2020) to identify the variance of the noisy signal and important utility parameters. While this paper adapts part of Agarwal and Somaini (2018)'s first part of estimation procedure to gain an estimation of admission probabilities conditional on an parameter to be estimated, I depart from them in the estimation of demand parameters. I exploit two properties of optimal reports under serial dictatorship to reduce the computational burden of finding the optimal reports, as briefly mentioned above.

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1. Other research about estimation of the equilibrium under mechanisms and settings less relevant to this paper includes, for example, De Haan *et al.* (2015), He (2015) and Hwang (2016).

The third strand of literature focuses on understanding supply side behavior in the education markets. Fu (2014) and Kapor (2015) both study college admission and the matching problem. Fu (2014) takes quality as given and explores how colleges choose admission policy and tuition levels in the market to maximize a weighted average between revenue and students' ability. This paper differs from this to model schools as maximizing a weighted average between profit and school quality, which makes more sense in the centralized admission mechanism where schools cannot choose students and their only way to attract students is through increasing desirable product characteristics such as quality. While Allende (2019) explicitly models marginal cost of quality, this paper further models the marginal cost to be quadratic in quality, a more realistic model which allows one to explain the existence of excess demand.

The rest of this chapter is organized as follows: Section 2.2 discusses the demand model and its estimation. Section 2.3 discusses the supply model and its estimation. Section 2.4 presents results, and Section 2.5 presents counterfactual analysis and Section 2.6 concludes this chapter.

## **2.2 Model of School Choice by Students**

### **2.2.1 Timeline of admission**

As discussed above, students first file their reports, and then take the high school entrance exam. After the exam scores are revealed, each student is assigned one by one, from the one with the highest score to the one with the lowest score, to their most

preferred school-channel that has not been filled up yet. Since the rank-ordered list is divided into sub-list, and each sub-list is truncated, students need to be strategic, and consider both their true utilities and their beliefs about admission probabilities of different school-channels. Both of these are potentially related to students' scores: students with higher exam scores are more likely to be admitted to better schools, and demand of students with higher signals of exam scores may be more sensitive to teaching quality, and less sensitive to price and distance. However, students don't know their true exam scores  $s_i$  when they file their preference lists. They only know a noisy signal  $a_i$  of the true score, which I model as:

$$a_i = s_i + e_i$$

where  $s_i$  is the total score of the high school entrance exam for student  $i$ , and the white noise  $e_i$  is independent of  $s$  and is normally distributed  $e_i \sim N(0, \sigma_e^2)$ .

### 2.2.2 Students' objective and choices

A student's objective is to maximize her expected utility by choosing which school-channels to fill in the rank-ordered preference list. The utility of student  $i$  from being admitted through school-channel  $j$  is

$$\begin{aligned} u_{ij} &= \beta_i X_{ij} + \epsilon_{ij} \\ &= (\beta_0 + a_i \beta_1 + z_i \beta_2) X_{ij} + \epsilon_{ij} \end{aligned}$$

where school-channel characteristics  $X_{ij}$  includes  $d_{ij}$ , the distance from student  $i$ 's home to high school  $j$ , measured by the distance between the middle school and high school she attends; and price-income term  $\log(\text{income}_i - p_j)$  of school-channel  $p_j$ . Notice that I do not observe individual income but I observe a distribution of income instead. I will simulate income in the demand estimation.  $X_{ij}$  also includes the teaching quality of school  $j$  in year  $t$   $q_{jt}$ , which is the value added I estimated from above. More importantly, I assume students with different scores may have different preferences over school-channel characteristics. Students' tastes over school-channel characteristics also depend on student characteristics, which include their race, gender, and the middle school the student attends.

I assume that the random term  $\epsilon_{ij}$  is i.i.d and follows  $N(0,1)$ . I normalize the variance to 1 to normalize the scale of the utility. I also assume  $\epsilon$  is orthogonal to other variables. The outside option of high school-channels is vocational high schools. In this paper, I always talk about academic high school-channels whenever I do not specifically point it out as vocational high schools. I normalize the utility of the outside option to zero.

Let  $L_{R_i, a_i}^j$  denote the probability student  $i$  will be admitted to school-channel  $j$  when her signal of score is  $a_i$  and her preference report is  $R_i$ . Let  $L_{R_i, a_i}$  be the vector of probability of admission for all possible school channels.

The student's objective is then to find a report  $R_i$  such that

$$u_i \cdot L_{R_i, a_i} \geq u_i \cdot L_{R'_i, a_i}, \forall R'_i \in \mathcal{R}_i \quad (2.1)$$



where  $u_i$  is the utility vector and  $\mathcal{R}_i$  is the set of all possible reports for  $i$ .

### 2.2.3 Beliefs about admission probabilities

Let  $\sigma$  denote the preference reporting strategies taken by students. Such strategies will affect what school-channels they end up reporting on their preference rank ordered list, and will in turn decide the cutoff scores of admission for each school-channel. Thus, when considering how to fill their preference lists before the exam, students will form their beliefs of assignment probabilities from their beliefs about the cutoffs of each school channel, which in turn requires students to have a belief about the reporting strategy of other students and have a belief of the distribution of scores of other students.

I assume students have rational expectations of other students' reporting strategy  $\sigma$  and the population distribution of signals  $a$ .

Let  $\Phi_{ij}^n((R_i, a_i), (R_{-i}, a_{-i}))$  denote the probability that, given the cutoffs, which are determined given the report of other students  $R_{-i}$  and their signals  $a_{-i}$ , student  $i$  will be admitted to school-channel  $j$  when her own signal is  $a_i$  and her own report is  $R_i$ .  $n$  denotes the total number of students. Then,

$$\Phi_{ik}^n = \begin{cases} 0 & \text{for } k \notin R_i \\ \int Pr(c_{k-1} > s_i > c_k | c, a_i) de_{-i} & \text{for } k \in R_i \end{cases}$$

where  $k - 1$  denotes the school-channel listed ahead of school-channel  $k$  on the report

list and  $c_k$  denotes the cutoff for school-channel  $k$ . Then, by definition,

$$L_{R_i, a_i}^{n, \sigma} = \mathbb{E}_{\sigma, a}[\Phi_i^n((R_i, a_i), (R_{-i}, a_{-i})) | R_i, a_i] \quad (2.2)$$

which takes into account a student's belief of other students' reporting strategy  $\sigma$  and belief of other students' signals of scores  $a$ .

## 2.2.4 Properties of optimal reports & finding optimal reports

While the admission mechanism in my setting is a variation of serial dictatorship mechanism, it doesn't change the key feature of it: the reported schools are considered one by one from the one in the first spot to the one in the last spot, and that the report does not change the ranking score which gives the sequence to assign students. Together with the rational expectation assumption, this gives two properties of the optimal reports in such mechanisms.

**Property.** Under the serial dictatorship or variations of it where schools are constrained to be reported in certain spots, if students have rational expectations of the distribution of other students' signal and reporting strategies, then the schools listed on the optimal report have

- (1) decreasing cutoffs from the first spot to the last spot and
- (2) decreasing utilities from the first spot to the last spot

*Proof.* The assumption of rational expectation means that students will have a rational expectation for the cutoffs of each school. When the number of students is large, then the distribution of the cutoffs will degenerate to the true cutoffs. Thus, students

with rational expectation see the cutoffs with no randomness and rationally expect the true cutoffs.

In the serial dictatorship or variations of it where schools are constrained to be reported in certain spots, schools are considered in order from the school in the first spot to the school in the last spot. I prove that schools listed have decreasing cutoffs using contradiction. Suppose there exists an optimal report where there exists two spots on the list,  $k$  and  $j$  with  $k < j$ , such that the school reported in the  $k$ -th spot has lower cutoff than the school reported in the  $j$ -th spot, denoted by  $cutoff_{(k)} < cutoff_{(j)}$ . Then, if the student is not admitted until the  $j$ -th spot, it means that the student's score  $s$  is lower than  $cutoff_{(k)}$ , which is smaller than  $cutoff_{(j)}$ . This means that the probability of being admitted to school ( $j$ ) is 0. To leave spot  $j$  empty will give the same expected utility to the student. This, in turn, is dominated by fill the  $j$ -th spot with schools with lower cutoff than that reported on the  $j - 1$  spot. This means that the original report cannot be optimal.

Similarly, I use contradiction to show that on the optimal report schools are listed with decreasing utilities. Suppose there exists an optimal report where there exists two spots on the list,  $k$  and  $k + 1$ , such that the school reported in the  $k$ -th spot has lower utilities for the student than the school reported in the  $k + 1$ -th spot, denoted by  $u_{i(k)} < u_{i(k+1)}$ . Since  $cutoff_{(k)} < cutoff_{(k+1)}$ , as I have established above, the

following will be true

$$\begin{aligned}
& u_{i(k)}\mathbb{P}(cutoff_{(k)} \leq s < cutoff_{(k-1)}) + u_{i(k+1)}\mathbb{P}(cutoff_{(k+1)} \leq s < cutoff_{(k)}) \\
& < u_{i(k+1)}\mathbb{P}(cutoff_{(k+1)} \leq s < cutoff_{(k-1)})
\end{aligned}$$

where the left hand side represents the report being considered, and the right hand side represents the report where the student leave the  $k$ -th spot empty. Since reports on all other spots are unchanged, this means that the original report cannot be optimal. Since each two consecutive spots have decreasing utilities in the optimal report, the full report also has decreasing utilities.  $\square$

I propose an algorithm where I exploit these two properties to find optimal reports for individual students. To give the general idea of the algorithm, I will first focus on the traditional serial dictatorship mechanism where there is one full-list with no sub-lists.

1. Only consider schools with positive utilities, and delete all schools with non-positive utilities
2. For all the schools left after step 1, list them from the one with the highest cutoff to the one with the lowest cutoff
3. Start with the last school on current list, i.e., the one with the lowest cutoff, denote it as school  $[n]$ . Compare the utilities of school  $[n]$  and school  $[n - 1]$ , the  $(n - 1)$ -th school on the current list, i.e., the one with the second lowest cutoff. If  $u_{i[n-1]} \leq u_{i[n]}$ , delete school  $[n - 1]$  from the current list; otherwise, do

nothing. Move on to school  $[n - 2]$ , if  $u_{i[n-2]} \leq u_{i[n]}$ , delete school  $[n - 2]$  from the current list; otherwise, do nothing. Continue until school  $[1]$  is compared with school  $[n]$ .

4. For schools on the current list, denote the last one as school  $[m]$ , since after step 3, the total number of schools left on the list might change. But it is still the same school as school  $[n]$  from step 3. Start with the second but last school on the current list,  $[m - 1]$ , and compare the utilities of school  $[m - 1]$  and school  $[m - 2]$ . If  $u_{i[m-2]} \leq u_{i[m-1]}$ , delete school  $[m - 2]$  from the current list; otherwise, do nothing. Move on to school  $[m - 3]$ , if  $u_{i[m-3]} \leq u_{i[m-1]}$ , delete school  $[m - 3]$  from the current list; otherwise, do nothing. Continue until school  $[1]$  on the current list is compared with school  $[m - 1]$ .
5. Continue similar procedures until no more schools can be deleted. The list left is one with decreasing utilities.
6. If the length of the current list is no larger than the total number of spots on the report, then report the current list. The current list is optimal.
7. If the length of the current list is larger than the total number of spots on the report, consider all possible ways of deleting schools from the current list to make it fit the number of spots allowed on the report. Find the one with the highest expected utilities.<sup>2</sup>

The general idea of the algorithm is to first find a list, as long as possible, with

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2. To further simplify step 7, one may also try to exploit the two properties discovered by Cal-samglia *et al.* (2018).

decreasing cutoffs. Then use the property of decreasing utilities to eliminate schools from the list, so that I only need to consider a smaller set of possible combinations of schools to report. Usually, the longer the list after step 2, the more school channels we can eliminate in steps 3-5, and the smaller the set of possible combinations of schools will be in step 7. This will significantly reduce the computational burden.

The above algorithm can be further adapted to be applied to my current case with sub-lists. Under the admission mechanism in my study, school-channels are categorized into different sub-groups. The report is divided into several sub-reports. To fill the spots on the first sub-report, students have to choose school-channels from the first group. To fill the spots on the second sub-report, students have to choose school-channels from the second group, etc. The first report is listed before the second report, etc. To find the optimal report under such mechanism, the idea is still to exploit the two properties, namely, decreasing cutoffs and decreasing utilities. The difficulty, however, comes from the fact that I cannot list schools freely in step 2 in the above algorithm, since the order of sub-groups are given. I adapt step 2 as follows.

1. Same as above.
2. The goal is to find a list with decreasing cutoffs.
  - (a) Consider all school-channels in the first sub-group, list them from the one with the highest cutoff to the one with the lowest cutoffs. Do the same for all school-channels within each sub-group. For each sub-group, I have a sub-list with decreasing cutoffs.

- (b) To form one possible full list with decreasing cutoffs, list all school-channels from the first sub-list. Then, for the second sub-list, delete all school-channels with higher cutoffs than the cutoff of the last school-channel in the first sub-list; list the rest from the second sub-list after the first sub-list. For the third sub-list, delete all school-channels with higher cutoffs than the cutoff of the last school-channel in the second sub-list; list the rest from the third sub-list after the second sub-list. Continue until all sub-list are considered. Now, I get a full list with decreasing cutoffs.
- (c) To form another possible full list with with decreasing cutoffs, list all but the last school-channels from the first sub-list. Then, for the second sub-list, delete all school-channels with higher cutoffs than the cutoff of the second but last school-channel in the first sub-list, which is also the last school-channel on the current full list that I am filling; list the rest from the second sub-list after the current full list that I am filling. For the third sub-list, delete all school-channels with higher cutoffs than the cutoff of the last school-channel in the second sub-list; list the rest from the third sub-list after the second sub-list. Continue until all sub-list are considered. Now, I get another full list with decreasing cutoffs.
- (d) Go through similar procedures in step 2(c) to exploit all possible full lists with decreasing cutoffs.
3. For each possible full list I get after step 2(d), go through step 3-5 in the original algorithm to find a full list with decreasing utilities.

4. For each possible full list I get after step 3, if the length of each sub-list of this full list is no longer than the number of spots allowed in the sub-report, do nothing; the current full list is a candidate for the optimal report.
5. For each possible full list I get after step 3, if the length of some sub-list of this full list is longer than the number of spots allowed in the sub-report, consider all possible ways of deleting schools from the current sub-list to make it fit the number of spots allowed on the sub-report. Find the one with the highest expected utilities. This is another candidate for the optimal report.
6. For each possible full list I get after step 3, I now have a full report after step 4 and 5. These are candidates for the optimal report. Compare among such candidates and find the one with the highest expected utility. This is the optimal report.

The general idea of the adapted algorithm is still to first find a list, as long as possible, with decreasing cutoffs. Then use the property of decreasing utilities to eliminate schools from the list, so that I only need to consider a smaller set of possible combinations of schools to report. The additional work is to find all possible list with decreasing cutoffs in step 2, given the constraints on reporting rules. Usually, the fewer the number of sub-groups, the less possibilities to consider in step 2, the longer the list after step 2, the more school channels we can eliminate in steps 3.



### 2.2.5 Identification

The uniqueness of equilibrium follows from Agarwal and Somaini (2018). They show that the equilibrium is unique if schools are weak substitutes and are locally connected substitutes. These two conditions hold in my case, since there is no sub-group of school-channels where they only substitute within the sub-group. Thus, the equilibrium is unique.

What I observe is students' report  $R_i$  and true exam score  $s_i$ . I will use the method of simulated moments to match three sets of predicted moments to their data counterpart. Specifically, the first set of moments is the covariances between the observed first-choice school-channel characteristics,  $X_{ij}$ , and the observed students' attributes,  $s_i$  and  $z_i$ . This is useful to identify  $\beta_1$   $\beta_2$ , the coefficients on the interactions between school characteristics and student characteristics. The second set of moments is the covariances between the first-choice school-channel characteristics and the last-choice school-channel characteristics. Since each student knows  $a_i$  and guesses  $s_i|a_i$ , the larger  $\sigma_e^2 > 0$ , the more spread  $s_i|a_i$  will be, the more spread between the characteristics of the first and last school-channel reported. Thus, such moments will help to identify the variance  $\sigma_e^2 > 0$  of the noisy signal. The third set of moments is the "market share" of a school channel in a specific slot. Specifically, I use a unique feature of the reporting rule that for the last sub-list, possible choices cover all possible levels of prices and teaching quality. More conveniently, most slots of this last sub-list are not full. The very first slot within this sub-list can be thus regarded as a less constrained "first choice" within this sub-list, i.e., the perceived

probability plays a less important role here. I define the "market share of  $j$ " as the percentage of students that report  $j$  on the very first slot of this sub-list. This will be useful to identify the mean utility term  $\beta_0$  of a school channel  $j$ .

## 2.2.6 Demand estimation

### Estimation of perceived assignment probability conditional on $\sigma_e$

I can further simplify the expression in equation (2) above by using the fact that a student's score does not depend on how she reports her preferences. Define

$$\phi_{ij}^n(a_i, (R_{-i}, a_{-i})) := \int Pr(s_i > c_j | c, a_i) de_{-i}$$

Then

$$\Phi_{ik}^n = \begin{cases} 0 & \text{for } k \notin R_i \\ \phi_{ik}^n(a_i, (R_{-i}, a_{-i})) - \phi_{ik-1}^n(a_i, (R_{-i}, a_{-i})) & \text{for } k \in R_i \end{cases}$$

The benefit of such notation is that  $\phi$  no longer depends on a student's report  $R_i$  but only on her signal  $a_i$ .

Define

$$l_{a_i}^{n,\sigma} = \mathbb{E}_{\sigma,a}[\phi^n((a_i), (R_{-i}, a_{-i})) | a_i]$$

then it is easy to see that we can derive  $L_{R_i, a_i}^{n,\sigma}$  once we have  $l_{a_i}^{n,\sigma}$ .

To estimate perceived assignment probabilities, I actually only need to estimate  $l_{a_i}^{n,\sigma}$ . Since I assume students have rational expectations, the expectation over  $\sigma$  and

signals  $a$  in the expression of  $l_{a_i}$  will be the expectation over true reports and the true distribution of ability. Following the idea of Agarwal and Somaini (2018), I resample from the whole observation set of students true reports to estimate the expectation. Specifically, I estimate  $l_{a_i}^{n,\sigma}$  by

$$l_{a_i}^{n,\sigma} = \frac{1}{B} \sum \phi^n((a_i), (R_{-i}, a_{-i}))_b = \frac{1}{B} \sum \mathbb{P}(s_i > c_b | a_i)$$

where  $B$  is the size of resampling and  $b$  denotes each draw. To find cutoff  $c_b$  corresponding to a sample  $b$ , we only need to apply the admission mechanism to all individuals in this sample and assign them to school-channels, as it happens in reality. Further, since the number of students are large enough,  $c_b$  should not vary a lot in different samples, and I can use the true cutoff  $c$ . Thus, I have dealt with the randomness stemming from the distribution of students' reports  $R$ , the distribution of their scores  $c$ , and the white noises of students other than  $i$ . The probability should be simply estimated as follows now:

$$l_{a_i}^{n,\sigma} = \mathbb{P}(s_i > c | a_i)$$

Using the definition of  $a_i$ , I can estimate  $l_{a_i}^{n,\sigma}$  by

$$\begin{aligned}
l_{a_i}^{n,\sigma} &= \mathbb{P}(s_i > c|a_i) \\
&= \frac{\int_c^\infty f_e(a_i - s)dF_s(s)}{\int_{-\infty}^\infty f_e(a_i - s)dF_s(s)} \\
&= \frac{(1 - F_s(c))\frac{1}{B_1} \sum f_e(a_i - s)}{\frac{1}{B_2} \sum f_e(a_i - s)}
\end{aligned}$$

The first equality holds from the definition of  $a$ . In the second equality, I use the sampling procedure again to estimate the integrals, where  $B_1$  is a random draw from  $s > c$ ; and  $B_2$  is a random draw from all  $s$ .

After estimating  $l_{a_i}^{n,\sigma}$ , we can use this to derive the estimation for students' perceived assignment probability  $L_{R_i, a_i}$  for each report  $R_i$ .

### **Estimation of utility parameters and $\sigma_e$**

To get the predicted moments, I simulate signal  $a_i$  given  $\sigma_e$ , and then calculate the perceived assignment probabilities conditional on  $a_i$  as explained above. Then I draw income and  $\epsilon_{ij}$  to calculate utilities. I then find the optimal report for each student using the algorithm explained in section 2.2.4. Finally, I calculate the predicted moments.

## 2.3 Model of Quality Choice by Schools

### 2.3.1 School model and school behavior

I focus on the supply side in this subsection. I model high schools as choosing their teaching quality only and taking prices and the number of available seats for each channel as given by the government. High schools in the market can actually choose their quality, price levels and quantity levels. The latter two, however, are highly regulated by the government. In practice, most schools from the same tier choose similar if not the same price for the same channel. Although prices among different tiers are significantly different, price variation within tiers is small and changes only a little over time. This suggests that schools actually only choose from a specific set of possible prices, which is given by the government. The same pattern happens for the number of seats for each school-channel. Thus, I assume that prices and the number of available seats for each channel are given by the government and are not chosen by schools in the model. This allows me to focus on the most important characteristics of a school, its teaching quality.

#### **Schools' objective and first order conditions**

I assume that schools care about both their profits and their teaching quality. Unlike traditional firms, public high schools need to focus not only on their profitability, which represents how well the schools are operated, but also on their teaching quality, which is what schools are for and which is used for evaluation of schools by the government. This is further supported by what I observe from the data. The data

shows that there are always some very top schools whose most priced channels are over demanded. This means that schools are keeping a high level of quality even though they could increase their profit by lowering the quality without losing students paying the high price. Another way to understand teaching quality as part of the objective, is to interpret it as a proxy for schools' reputation. Schools would like to keep a good reputation over time and thus incorporate it into their objective functions.

Thus, I model schools' problem to be

$$\begin{aligned} \max_{q_j} \gamma \sum_c [p_{jc} - mc(q_j)] Q_{jc}(p, q) + (1 - \gamma) q_j \\ \text{s.t. } Q_{jc}(p, q) \leq \bar{Q}_{jc} \end{aligned} \quad (2.3)$$

where school  $j$  has admission channel  $cs$  and choose a quality  $q_j$  to maximize its objective function, which is a weighted average between its profit and teaching quality, with  $\gamma$  as the weight. Notice that from now on, I use  $j$  only to represent school, instead of school-channel as above, and a  $jc$  combination represents a school-channel. Schools take price of each channel  $p_{jc}$  and the number of available seats for each channel  $\bar{Q}_{jc}$  as given. Schools will incur a marginal cost of  $mc(q_j)$  to produce teaching quality of level  $q_j$  and I will specify the marginal cost function  $mc(\cdot)$  in more details later. The number of students admitted in each channel is  $Q_{jc}$ , which depends on the prices and quality of all schools.

Schools compete with each other by setting and committing to its quality before students file their reports of preferences. The trade-off for schools in choosing the optimal quality is the typical trade-off between demand and unit profit: an increase

quality will lead to more demand for the school, but will reduce the unit profit it can get. In addition, schools' preference for quality in the objective function also add a third factor into the trade-off.

Before I characterize their equilibrium strategies using each school's first order conditions, I can further simplify their objective functions. Data shows that schools usually have at most one channel whose capacity is not binding. Thus, I can further assume each school's choice of quality only affects the number of students admitted via one specific channel. Specifically, for bottom-tier schools, their free channel capacities are not binding, and no one pays to be admitted by the low price channels. Thus, I model them as solving

$$\max_{q_j} \gamma [p_F - mc(q_j)] Q_{jF}(p, q) + (1 - \gamma) q_j$$

Similarly, for middle-tier schools, free and low tuition channel capacities are usually binding, and their high tuition channel capacities are usually not binding. Thus, I model them as solving

$$\begin{aligned} \max_{q_j} \gamma \{ [p_F - mc(q_j)] \bar{Q}_{jF} + [p_{jL} - mc(q_j)] \bar{Q}_{jL} \\ + [p_{jH} - mc(q_j)] Q_{jH}(p, q) \} + (1 - \gamma) q_j \end{aligned}$$

It's slightly different for the top-tier schools. Some top-tier schools are like middle-tier schools: their free and low tuition channel capacities are usually binding, but their

high tuition channel capacities are usually not binding.

$$\begin{aligned} &max_{q_j} \gamma \{ [p_F - mc(q_j)] \bar{Q}_{jF} + [p_{jL} - mc(q_j)] \bar{Q}_{jL} \\ &+ [p_{jH} - mc(q_j)] Q_{jH}(p, q) \} + (1 - \gamma) q_j \end{aligned}$$

However, for some very top schools within the top tier, even their high price channel capacities are binding. I model them as solving

$$\begin{aligned} &max_{q_j} \gamma \{ [p_F - mc(q_j)] \bar{Q}_{jF} + [p_{jL} - mc(q_j)] \bar{Q}_{jL} \\ &+ [p_{jH} - mc(q_j)] \bar{Q}_{jH} \} + (1 - \gamma) q_j \end{aligned}$$

As a result, the first order conditions of schools from each tier can be written as follows. For bottom-tier schools:

$$\gamma \left\{ -\frac{\partial mc(q_j)}{\partial q_j} Q_{jF}(p, q) + (p_F - mc(q_j)) \frac{\partial Q_{jF}(p, q)}{\partial q_j} \right\} + (1 - \gamma) = 0$$

For middle-tier schools and top-tier schools with non-binding high tuition channel capacity:

$$\begin{aligned} &\gamma \left\{ -\frac{\partial mc(q_j)}{\partial q_j} [\bar{Q}_{jF} + \bar{Q}_{jL} + Q_{jH}(p, q)] + (p_{jH} - mc(q_j)) \frac{\partial Q_{jH}(p, q)}{\partial q_j} \right\} \\ &+ (1 - \gamma) = 0 \end{aligned}$$



And for top-tier schools with binding high tuition channel capacity:

$$\gamma \left\{ -\frac{\partial mc(q_j)}{\partial q_j} [\bar{Q}_{jF} + \bar{Q}_{jL} + \bar{Q}_{jH}] \right\} + (1 - \gamma) = 0$$

### Schools' marginal cost of producing quality

Motivated by observations and settings above, I assume the following to characterize the marginal cost functions of producing quality for each school:

$$mc(q_{j(k)t}) = (\alpha_k^0 + \xi_j + \epsilon_{j(k)t}^s) q_{j(k)t} + \alpha_k^1 q_{j(k)t}^2$$

which gives that

$$mc'(q_{j(k)t}) = 2\alpha_k^1 q_{j(k)t} + \alpha_k^0 + \xi_j + \epsilon_{j(k)t}^s$$

School  $j$  from tier  $k$  will choose quality  $q_{j(k)t}$  at time  $t$ , and the marginal cost of producing it is quadratic in  $q_{j(k)t}$  with the quadratic coefficient  $\alpha_k^1$  and linear coefficient  $\alpha_k^0 + \xi_j + \epsilon_{j(k)t}^s$ .  $\xi_j$  is an unobserved heterogeneity in marginal costs across different schools, which is essential in explaining why different schools within the same tier choose different quality. I set  $\xi$  of the highest quality schools within each tier to be 0 as the benchmark. I assume that  $\epsilon_{j(k)t}^s$  is independent of other variables and has 0 mean.

## Estimation

Plugging the marginal cost functions into the first order conditions of schools of different tiers will give us several equations. From each equation I can write the error term  $\epsilon_{j(k)t}^s$  as a function of parameters and variables whose values can be calculated from the demand side. I use the assumption that  $E(\epsilon_{j(k)t}^s) = 0$  to form moment conditions to estimate  $\alpha_k^0$ ,  $\alpha_k^1$ , and  $\gamma$ . Notice that there are two groups of different schools within the top tier. The split in the top-tier schools gives an additional equation which can be used to identify  $\gamma$ .

### 2.3.2 Optimal quality and its decomposition

In this subsection, I find optimal quality from schools first order conditions and further decompose it so as to analyze what are the sources that drive changes in quality. Without loss of generality, I take as an example the optimization problem of the top-tier schools with non-binding capacity for their high tuition channels. From such schools' first order condition, plug in the specification of marginal cost, rearrange and I have

$$q_{j(k)t}^* = \frac{p_{jH} - \alpha_T^1 q_{j(k)t}^{*2}}{\alpha_T^0 + \xi_j + \epsilon_{j(k)t}^s} - \frac{\bar{Q}_{jF} + \bar{Q}_{jL} + Q_{jH}(p, q_{j(k)t}^*)}{\frac{\partial Q_{jH}(p, q_{j(k)t}^*)}{\partial q}}$$

$$- \frac{2\alpha_T^1 q_{j(k)t}^* [\bar{Q}_{jF} + \bar{Q}_{jL} + Q_{jH}(p, q_{j(k)t}^*)]}{(\alpha_T^0 + \xi_j + \epsilon_{j(k)t}^s) \frac{\partial Q_{jH}(p, q_{j(k)t}^*)}{\partial q}} + \frac{\frac{1-\gamma}{\gamma}}{(\alpha_T^0 + \xi_j + \epsilon_{j(k)t}^s) \frac{\partial Q_{jH}(p, q_{j(k)t}^*)}{\partial q}}$$

where  $q_{j(k)t}^*$  denote the optimal quality.

There are four terms on the right hand side of the expression for  $q_{j(k)t}^*$ . The first term is an analog of the usual competitive market induced quality, where we set profit to zero. The term is always positive, and it represents the part of quality brought by investing all tuition  $p$  to produce quality. This is the first and main source of quality. The second term is the usual quality markdown due to market power of school  $j$ . The term  $\frac{\partial Q_{jH}(p, q_{j(k)t}^*)}{\partial q}$  on the denominator is the derivative of demand with respect to quality. The third term is also a quality markdown due to market power of school  $j$ , but this is due to the fact that marginal cost function  $mc(q)$  is quadratic, instead of linear, in  $q$ . Such additional cost gives schools additional incentives to reduce the quality level in equilibrium, to which extent the schools can markdown the quality depends on the derivative of demand with respect to  $q$ . The second and third term together is the total quality markdown due to the market power possessed by the school. The last term is a preference markup. It comes from schools preference (taste) for quality in their objective function. Again, this preference is weighted by the derivative of demand ( $\frac{\partial Q_{jH}(p, q_{j(k)t}^*)}{\partial q}$ ) and the linear coefficient of the marginal cost function. It means that the steeper the marginal cost of producing quality, the less the preference  $\frac{1-\gamma}{\gamma}$  is considered. Similarly, the more elastic the demand is with respect to quality, the less the preference is considered.

In summary, the optimal quality found through the FOC can be correspondingly decomposed into three sources: the competitive level (first term) which school would choose if it makes 0 profit, and it also maximizes demand; the quality markdown (second and third term), similar to the price markup in a traditional industry where firms choose price, but in an opposite direction. Such markdown comes from the

market power schools have and schools can thus keep a positive unit profit by marking down quality and thus cost. The third one is the preference markup (fourth term) since schools care about quality

## 2.4 Results

Estimation results for important demand parameters are presented in Table 2.1

Table 2.1: Important utility parameters

	Estimate	
<i>price</i>	-0.0311***	(0.0143)
<i>quality</i>	0.2830***	(0.1257)
<i>distance</i>	-0.0214***	(0.0078)
<i>price</i> × <i>a</i>	0.0225**	(0.0149)
<i>quality</i> × <i>a</i>	0.0297***	(0.0105)
<i>distance</i> × <i>a</i>	0.0582***	(0.0290)
<i>price</i> × <i>gender</i>	-0.0035**	(0.0019)
<i>quality</i> × <i>gender</i>	0.0021	(0.0027)
<i>distance</i> × <i>gender</i>	0.0014	(0.0012)
$\sigma_e$	0.1095**	(0.0631)

unit: *price*(10,000 rmb), *distance* (10km); for simplicity of presenting results, the coefficient on *price* in the table is the negative of the coefficient on  $\log(\text{income} - \text{price})$

First, students' utility will increase with teaching quality but will decrease with price and distance, all of which are as expected. This means that students prefer high quality but dislike price and distance. The coefficient of *price* × *a* means that when students' signal of their exam scores increases by one unit, which is one standard deviation of the normalized exam scores, the taste for price will increase by 0.0225. Notice that the mean taste for price is negative, -0.0311, then this means that students with higher signal will have less negative tastes for price than those with lower

signals. Students with higher signals care less about price and are less sensitive to it, as expected. Similar results hold for distance. The coefficient of  $distance \times a$  means that when students' signal of their scores increases by one unit, the taste for distance will increase by 0.0582. Notice that the mean taste for distance is negative,  $-0.0214$ , this means that students with higher signals will have less negative tastes for distance than those with lower signals. Students with higher signals care less about distance and are less sensitive to it, as expected. The implication for teaching quality, however, is different. Although the coefficient of  $quqality \times a$  is also positive, meaning that the taste for price will increase by 0.0297 when students' signal of their scores increases by one unit, the mean taste for quality is positive, 0.2830. This means that students with higher signals will have more positive tastes for quality than those with lower signals. Students with higher signals care more about quality and are more sensitive to it.

The estimation results for supply parameters are presented in [Table 2.2](#).

Table 2.2: Marginal costs parameters

$\gamma$	$\alpha_T^0$	$\alpha_T^1$	$\alpha_M^0$	$\alpha_M^1$	$\alpha_B^0$	$\alpha_B^1$
0.686	0.197	0.011	0.240	2.445	0.346	10.523
(0.231)	(0.104)	(0.008)	(0.132)	(0.859)	(0.227)	(2.815)

unit: Q(1,000 students)

The results show that the marginal cost of producing quality are lowest for top-tier schools, with the small numbers on  $\alpha_T$ s, and highest for bottom-tier schools with the large numbers on  $\alpha_B$ s. For top-tier schools, the linear term  $\alpha_T^0$  in marginal cost plays a more significant role than the quadratic term  $\alpha_T^1$ . For middle-tier and bottom-tier schools, however, the quadratic terms  $\alpha_M^1$  and  $\alpha_B^1$  are more important than the linear

terms  $\alpha_M^0$  and  $\alpha_B^0$ . Also, the difference in the quadratic terms  $\alpha^1$ s among different tiers are larger than the difference in the linear terms  $\alpha^0$ s. This means that the main difference in marginal cost among different tiers comes from different quadratic terms. Due to such differences in marginal cost of producing quality, top-tier schools would be able to produce teaching quality more efficiently at lower costs and will in turn choose higher quality.

## 2.5 Counterfactual Analysis

### 2.5.1 Distributional effects of the policy and possible remedy

The main benefit of the current policy of paid admission options is the gain in teaching quality. However, such quality gain is not equally distributed across students for two reasons as explained above. Namely, better schools benefit more from this policy and are able to increase their teaching quality more than other schools do, creating an inequality among schools. Also, this policy gives richer students more choices while leaving poorer students no more access to good schools than before. In order to address such restricted access to higher teaching quality, I propose a modification to the current policy that introduces subsidies to low income students.

Specifically, since I only observe a distribution of income instead of individual level income, I need to simulate the admission results under current policy with an assumed relationship between income distribution and students' exam scores. For simplicity, I assume income is independent of students' exam scores and then draw

an income for each individual. Then, I use the estimated demand and supply models to find the new equilibrium quality chosen by schools. This allows me to simulate students' reports and the admission results, from which I can calculate the percentage of students admitted to different tiers of schools from each income group.

Second, I use the same draw of income to find the new equilibrium teaching quality and simulate students reports and admission results under the modified policy. From the simulated admission results, I can calculate a new percentage of students admitted to different tiers of schools from each income group. Comparing these results will show us how access to good schools are affected by the modified policy. The modified policies give tuition subsidies up to a certain amount to low income students if they are admitted through a priced admission option. The simulation results are presented in [Table 2.3](#).

Table 2.3: Percentage of students from each income group who are admitted to top-tier schools under different levels of subsidies

	Current Policy	8000	24000	50000
90% – 100%	0.92	0.90	0.77	0.64
70% – 90%	0.80	0.80	0.72	0.62
50% – 70%	0.67	0.64	0.61	0.60
30% – 50%	0.58	0.59	0.59	0.60
10% – 30%	0.40	0.42	0.49	0.58
0% – 10%	0.20	0.23	0.43	0.57

The first column represents the current policy of priced admission options. For students from the top-10%-income families, 92% of them are admitted to top-tier schools, through either free, low or high tuition options. When family income decreases (going down the column), the percentage of students from the corresponding income group who are admitted to top-tier schools decreases quickly. For the bottom-

10%-income families, only 20% of them attend top-tier schools. This is consistent with the fact that approximately 20% of total seats of the market are provided by top-tier school free-channels. Remember from the graph above that about 60-65% of total seats of the market are provided by top-tier schools. This means that under the current policy, rich students are over represented while poor students are under represented.

In the second column, I introduce a subsidy up to RMB 8000 to poor students, who I define to students from the poorer half of the population. The results show that only a few more students from bottom-30%-income families go to top-tier schools. The small effect can be explained by the fact that RMB 8000 is less than half of the low price of top-tier schools, and thus makes little difference. But it does increase the percentage of such students who attend middle-tier schools more significantly.

The third column represents a subsidy up to RMB 24000 for poor students. We can see that the percentage of students from bottom-30%-income families that go to top-tier schools increases more. Actually, 24000 is enough for all low price channels of top-tier schools, and thus the increase mostly comes from the increase of admissions through low price channels for poor students. Since 24000 is only half of the prevailing RMB 50000 for high price admission channels, such subsidy has little effect on who are admitted through high price channels. The changes for students from top-income families are mostly driven by decreases in their admission through the low tuition channels.

Finally, in the last column, I introduce a subsidy of up to RMB 50000, which covers all high price channels of top-tier schools. The results show that many more



students from low income families go to top-tier schools, while far fewer students from rich families attend top-tier schools. In fact, the percentage of students who go to top-tier schools are roughly the same for each income group. This is not surprising since now income is rarely a constraint and I assume income is independent of exam scores. Actually, the distribution of students across different tier of schools under this policy is similar to that under no policy at all, since tuition is no longer a constraint. More importantly, if the government taxes all of the tuition richer students pay for the high-price channels to finance its subsidies to poorer students, the government can actually break-even, since I define low income as students from the poorer half of the population. Meanwhile, there is still gain in the teaching quality since the subsidies are collected by the schools as tuition eventually. In short, introducing subsidies to low income students on top of current policy of priced admission options will give students more equal access to good schools while keeping the quality gain.

### **2.5.2 Understanding the sources of quality change brought by prices**

Results above show that introducing prices to public schools leads to increase in teaching quality. However, such results are not rigorous, and the increase in teaching quality may come not only from introducing prices, but also from introducing new capacities for new channels. In this subsection, I separate these two forces by comparing the current policy with a counterfactual, where the capacities for each channel are kept fixed, and the price of the high tuition channel of each school is reduced to

the price of the low tuition channel of that school. I further decompose the optimal quality in these two scenarios to understand the sources of change of quality. As in section 2.3.2, without loss of generality, I use as an example the optimization problem of the top-tier schools with non-binding capacity for their high tuition channels. Table 2.4 below shows the decomposition of quality under current policy on the first row and the decomposition of quality under the counterfactual in the second row. For simplicity of explanation, the change ( $\Delta$ ) in the third row is the change from the counterfactual (second row) to current policy (first row).

Table 2.4: Decomposition of quality under current policy and reduced price for high tuition channels

	q	competitive level	markdown (linear)	markdown (quadratic)	preference markup
Current policy	0.950	2.232	-1.705	-0.018	0.441
$p_{jH}$ set to $p_{jL}$	0.801	1.189	-0.517	-0.005	0.134
$\Delta$	0.149	1.042	-1.187	-0.013	0.307
%	100	701	-798	-9	206

the change ( $\Delta$ ) in the third row is the change from the counterfactual (second row) to current policy (first row).

When the price of the high tuition channel is increased from a hypothetical reduced price to the current level, the increase in quality is 0.149 standard deviation (s.d.) of the test scores. Out of this change, changes from competitive level of quality and linear markdown are the main drivers, at similar magnitude but in different directions. The competitive level of quality brings an increase in quality of 1.042 s.d.. This is mainly driven by the huge increase in price. A linear markdown brings a decrease in quality of 1.187 s.d. The change in linear markdown, however, is mainly driven by the change in the demand derivative with respect to quality. A unit change in quality

will bring a smaller demand when the price (tuition) is higher, making the demand derivative with respect to quality to be decreasing in price. As a result, when the price goes up, the demand derivative with respect to quality decreases, giving schools more market power. This means that schools can markdown quality more when the price is higher. The quadratic markdown and its change are in a smaller scale when compared to the linear markdown. It brings a decrease in quality of 0.013 s.d. Finally, the preference markup brings an increase in quality of 0.307 s.d., and brings the total change of quality to an increase.

## 2.6 Conclusion

In this paper, I examine the impact of allowing public high schools to offer both free and priced admission options.

On the demand side, I estimate an empirical model of school choice by students. I collect data of students' reports of school preferences. Students are allowed to rank a limited number of schools of their choice in a report that they must file before taking the entrance exam. They are then admitted through a variation of serial dictatorship mechanism. Thus, students need to be strategic, and consider not only their true preferences, but also their beliefs about admission probabilities based on a noisy signal of their test scores. I use such reported preferences to estimate students' true school preferences and beliefs about admission probabilities. The spread of the characteristics between the first and last reported schools is useful to identify the variance of the noisy signal. The interaction between student and school characteristics helps

to identify important utility parameters. I estimate the parameters using simulated moments, together with an algorithm I propose to find optimal reports for individual students. Results show that students prefer quality and dislike price. More importantly, students with higher scores, when compared to students with lower scores, care more about quality and less about price.

On the supply side, I model schools to care about a weighted average between profit and its quality, so as to incorporate the facts that schools may want to keep excess demand in the market. To better understand schools' choice of quality, I explicitly model schools' marginal costs of producing quality as being quadratic in quality. I combine such models of school competition with the demand side to estimate the model. Results show that better schools have lower marginal cost of producing quality and will choose a higher level of teaching quality.

The counterfactual analysis shows that introducing subsidies to low income students while keeping the current priced admission options would give students more equal access to better schools, while keeping the quality gain brought by market incentives. Another counterfactual analysis shows that the quality gain brought by market incentives are driven by more funds to improve quality and schools' preference for quality.

# Chapter 3

## Second-Price Auctions with

## Participation Costs, (with

## José-Antonio Espín-Sánchez and

## Álvaro Parra)

### 3.1 Introduction

In this article, we study participation in a second-price auction with independent private values and public participation costs. Our framework expands existing models by accommodating rich forms of bidder heterogeneity, which enables a wide range of empirical applications facilitating policy analysis of auction markets. Our main contributions are a general characterization of the game's equilibria and identifica-

tion of sufficient conditions that guarantee both equilibrium uniqueness and efficient outcomes.

We develop a notion of relative competitiveness called *strength* to characterize the auction's set of equilibria. Using the publicly known characteristics of bidders, strength ranks potential bidders by their ability to endure competition. The strength of a bidder is computed by the unique symmetric strategy profile in which the bidder is best-responding; that is, strength is the hypothetical participation-cutoff value that would make a bidder indifferent to participate, conditional on all other bidders using the same strategy as the bidder. Although this strategy is generally not an equilibrium, it neatly captures a bidder's ability to endure competition. Relative to another bidder's strength, a lower strength value—i.e., a stronger bidder—indicates that the bidder is willing to participate in the auction at a lower valuation even while facing competitors who are also participating at lower valuations; i.e., the bidder is more willing to participate despite facing more competition.

We define a *herculean* equilibrium as an equilibrium in which stronger bidders are more willing to participate—i.e., the bidders' participation strategies are ordered by their relative strength. We show that, in general environments, a herculean equilibrium always exists. Despite the *ex-post* inefficiencies that are created by costly participation, we also show that an *ex-ante* efficient equilibrium exists. In addition, we show that when the distributions of valuations are concave, the game has a unique equilibrium. We also provide a sufficient condition for uniqueness when the distributions of valuations are not concave. Therefore, when any of these sufficiency conditions hold, the unique equilibrium is both herculean and efficient.

A bidder's strength can be computed in *any* game, allowing to rank every potential auction participant. Strength allows to generalize existing work on *quasi-symmetric* games to environments without restrictions on the distributions of valuations or on participation costs. Quasi-symmetric games are those in which: (i) bidders have identical participation costs and their valuations are ordered by first-order stochastic dominance (FOSD; [Tan and Yilankaya, 2006](#)); or (ii) bidders with symmetric distributions of valuations, ordered by their participation costs ([Cao and Tian, 2013](#)). Strength is specially suited for applied research, as it allows us to rank bidders, and consequently, characterize equilibria in games with *any* degree of bidder heterogeneity.

To illustrate the previous point, consider the US Forest Service timber auctions studied in [Roberts and Sweeting \(2013\)](#). They assume a quasi-symmetric environment in which large bidders' (mills) valuations FOSD those of small bidders (loggers) but have identical participation costs. Suppose that, after the model has been estimated, we want to evaluate a policy that recommends subsidizing the participation cost of loggers.<sup>1</sup> This counterfactual scenario is no longer quasi-symmetric, as the millers' advantage in drawing higher valuations might be offset by the loggers' lower cost of participating. Whether millers are still *stronger* than loggers depends on the size of the subsidy. Since bidders are not quasi-symmetric, existing models cannot predict which firms are more likely to participate, whether the game has a unique equilibrium and, consequently, whether counterfactual analyses are robust to the existence of other equilibria. This article characterizes which firms are more likely to participate in the

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1. As example of this type of policy, [Marion \(2007\)](#) and [Krasnokutskaya and Seim \(2011\)](#) evaluate entry fees and subsidies in the context of first price auctions.

new scenario and provides a sufficient condition for equilibrium uniqueness that can be easily checked.

This article contributes to the literature of auctions with participation costs. In this literature, there are two broad classes of models that describe bidders' own information about their valuations. [Levin and Smith \(1994\)](#) study auction participation in environments where participation decisions are made with no private information (see also [Jehiel and Lamy, 2015](#); [McAfee and McMillan, 1987](#); [Tan, 1992](#)). In this framework, participation becomes a coordination game, and generally leads to multiple equilibria. When signals are informative but public—i.e., observed by all bidders—environments also resemble coordination games, as in [Levin and Smith \(1994\)](#). By contrast, our framework builds upon [Samuelson \(1985\)](#), who studied a symmetric environment in which bidders learn their private information *prior* to the participation decision. Within this framework, [Campbell \(1998\)](#) studies coordinated entry, whereas [Tan and Yilankaya \(2007\)](#) examine collusive outcomes and [Menezes and Monteiro \(2000\)](#) study optimal auction design. Recent articles have allowed more general information structures in which bidders receive (private) *signals* about their valuations before participating in the auction (c.f. [Gentry and Li, 2014](#); [Roberts and Sweeting, 2016](#); [Sweeting and Bhattacharya, 2015](#)). We discuss such models in Section 3.6.

[Tan and Yilankaya \(2006\)](#) and [Cao and Tian \(2013\)](#) identify conditions for a unique equilibrium in the context of quasi-symmetric games. The restricted degree of bidder-heterogeneity in their frameworks, however, constrains bidders' behavior in meaningful ways, making quasi-symmetric environments inadequate for applied work. We show that, in quasi-symmetric games where bidders play an herculean



equilibrium, bidders have the same ranking in their probability of participating, equilibrium cutoffs, and expected revenues. Hence, when a unique equilibrium exists, quasi-symmetric environments cannot accommodate a bidder who is more likely to participate, but receives lower expected profits on average than a competitor. In line with [Maskin and Riley \(2000\)](#), we show that high-participation low-profit behavior can emerge in models with richer degrees of bidder heterogeneity. This article, therefore, provides a theoretical framework that better meets applied researchers' needs to accommodate behavior observed in data.

Our welfare analysis expands the early work of [Stegeman \(1996\)](#) (see also [Lu, 2009](#)). Although every equilibrium is *ex-post* inefficient, [Stegeman \(1996\)](#) shows that SPA with participation costs have one equilibrium that is *ex-ante* efficient. We provide a direct proof of Stegeman's result. Furthermore, we show that each equilibrium corresponds to a (possibly local) maximum or a saddle point of the social welfare function. Finally, by identifying the equilibrium that survives when the uniqueness condition holds, we partially characterize the efficient equilibria.

Finally, [Espín-Sánchez and Parra \(2019\)](#) generalizes the ideas of strength and herculean equilibrium developed here to characterize entry into oligopolistic markets. Despite the similarities in goals, there are key differences in terms of methodologies and results that make the contributions of this article distinctive. This article crucially relies on the linear-payoff structure of second-price auctions. In particular, welfare results, the sharper sufficient conditions for uniqueness, and more importantly the induction argument used in the characterization of the  $n$ -bidder scenario do not extend to their environment. In contrast, the techniques in [Espín-Sánchez and Parra \(2019\)](#)

rely on a post-entry strict payoff-monotonicity assumption that is not satisfied in second-price auctions.

The rest of the article is organized as follows. Section 3.2 presents the model. Section 3.3 characterizes all equilibria, establishes existence and discusses efficiency. Section 3.4 defines strength, herculean equilibria and presents the main results of the article. Section 3.5 discusses the importance of allowing models with more heterogeneity than quasi-symmetry. Section 3.6 extends the results to environments with a reserve price and environments with partially informed bidders. Section 3.7 concludes. All proofs are relegated to the Appendix.

## 3.2 Setup

Consider a sealed-bid second-price auction with no reservation price in an independent private values environment.<sup>2</sup> The auction consists of one seller,  $n$  potential bidders, and one indivisible good. Before making any participation decision, each bidder  $i$  observes her valuation for the object  $v_i$  which is drawn from an atomless distribution function  $F_i$  with full support on  $\mathbb{R}_+$ . We assume that each  $F_i$  is continuously differentiable and has a finite expectation.<sup>3</sup> Upon privately observing their own valuation, each bidder, independently and simultaneously, decides whether to participate in the auction. If bidder  $i$  decides to participate, she incurs a cost  $c_i > 0$ . The tuple  $(F_i, c_i)_{i=1}^n$ , which includes the number of potential bidders  $n$ , is commonly known by

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2. For results in a common value setting see Murto and Välimäki (2015).

3. Our results would still hold if the support of  $F_i$  were the interval  $[0, b_i]$  with  $b_i > 0$ . This, however, would complicate our exposition as we would have to consider corner solutions.

all the bidders.

**Definition** (Symmetric and quasi-symmetric games). A game is called *symmetric* if  $F_i = F$  and  $c_i = c$  for all  $i$ . A game is called quasi-symmetric if either: (i)  $F_i = F$  for all  $i$ , or (ii)  $c_i = c$  for all  $i$  and  $F_i$  are ordered by first-order stochastic dominance (FOSD).

After bidders make participation decisions, they observe other participating agents' identities. Afterwards, every participant submits their bid simultaneously. We simplify the bidding stage by assuming that each player bids their valuation; i.e., bidders play their weakly dominant strategy.<sup>4</sup> Therefore, we restrict attention to participation strategies. A participation strategy for bidder  $i$  is a mapping from bidder  $i$ 's valuation to a probability of participating in the auction  $\tau_i : \mathbb{R}_+ \rightarrow [0, 1]$ . We assume that bidder  $i$ 's strategy is an integrable function with respect to her own type  $v_i$ . We study the Bayesian Equilibrium of the participation game.

Given a strategy profile  $\tau = (\tau_1, \tau_2, \dots, \tau_n)$ , define

$$T_i(v) = F_i(v) + \int_v^\infty (1 - \tau_i(s)) dF_i(s)$$

to be the *ex-ante* probability that bidder  $i$  does not obtain the object when the highest bid among her opponents is  $v$ . Observe that  $T_i(v) > 0$  whenever  $v > 0$ . The expected utility of a bidder who participates in the auction with probability  $\tau_i(v)$ ,

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4. Tan and Yilankaya (2006) model non-participation as submission of a zero bid. Technically, their model is a one-stage game in which a bidder's dominant strategy is not to bid their valuation. By contrast, we explicitly model the sequential bid process. Both formulations are equivalent.

faces opponents playing  $\tau_{-i}$ , and values the good by  $v$  is:

$$u_i(\tau, v) = \tau_i(v) \left[ vG_i(v) - \int_0^v s dG_i(s) - c_i \right], \quad (3.1)$$

where  $G_i(v) = \prod_{k \neq i} T_k(v)$  is the probability that bidder  $i$  obtains the object when her valuation is  $v$ . In other words, conditional on participating, the expected utility of bidder  $i$  is the expected value of getting the good  $vG_i(v)$ , minus the participation costs  $c_i$ , minus the expected price paid, which distributes according to  $dG_i(v)$  and is equal to the second highest bid in the auction.

### 3.3 Preliminary Results

In this section, we provide a preliminary characterization of the equilibria and efficiency properties of the game. We establish the existence of an equilibrium and we show that, without loss of generality, we can restrict our attention to cutoff strategies. In addition, we prove that, although every equilibrium of the game is *ex-post* inefficient, an *ex-ante* efficient equilibrium always exists.

#### 3.3.1 Equilibrium existence

**Definition** (Cutoff strategy). A strategy  $\tau_i(v)$  is called *cutoff* if there exists  $x > 0$  such that

$$\tau_i(v) = \begin{cases} 1 & \text{if } v \geq x \\ 0 & \text{if } v < x \end{cases}.$$

A cutoff strategy specifies whether a bidder participates in the auction with certainty depending on her valuation being above some given threshold. [Lemma 1](#) below shows that, without loss of generality, we can restrict our attention to cutoff strategies.

**Lemma 1** (Cutoff are best responses). *For each profile of opponent's strategies  $\tau_{-i}$ , bidder  $i$  has a unique best response. Bidder  $i$ 's best response is a cutoff strategy given by the unique value of  $v$  that solves  $u_i(\tau_i = 1, \tau_{-i}, v) = 0$ .*

[Lemma 1](#) follows from showing that, conditional on participation, and regardless of their opponents' strategies, a bidder's (expected) utility is monotonically increasing with respect to their own valuation,  $v_i$ . Then, because a bidders' utility is linear in the participation probability, and since they want to participate whenever there is positive expected utility to do so, bidders best respond by playing a cutoff strategy. The cutoff is defined by the valuation that gives zero expected utility for participating in the auction. When a bidder's valuation is equal to its cutoff, the bidder is indifferent to whether or not to participate in the auction. We break this indifference by assuming that bidders participate. The main consequence of [Lemma 1](#) is that each equilibrium, if any exists, must be in cutoff strategies.

From now on, we abuse notation by denoting a cutoff strategy in terms of the cutoff itself. In addition, and without loss of generality, we order the bidders' identities according to their equilibrium cutoffs, with  $x_1$  being the bidder with the lowest cutoff and  $x_n$  the bidder with the highest. For a given vector of cutoff strategies  $\mathbf{x} = (x_1, x_2, \dots, x_n)$  define  $\mathbf{x}^i = (x_1, x_2, \dots, x_i)$  to be the vector of cutoffs up to bidder  $i$ . Let  $A_i^n = \prod_{j>i}^n F_j(x_j)$  be the probability that bidders playing cutoffs *above*  $x_i$  do

not participate in the auction; let  $B_i(v) = \prod_{j < i} F_j(v)$  be the probability that bidders playing cutoffs *below* bidder  $i$  obtain valuations lower than  $v$ , and; let

$$r_i(\mathbf{x}^i) = x_i B_i(x_i) - \sum_{j=1}^{i-1} \left( A_j^{i-1} \int_{x_j}^{x_{j+1}} s dB_{j+1}(s) \right), \quad (3.2)$$

be bidder  $i$ 's expected *revenue* when bidder  $i$  plays the highest participation cutoff in a game with  $n = i$  potential bidders and bidder  $i$ 's valuation is equal to its cutoff.<sup>5</sup>

The next lemma characterizes every equilibria in the participation game.

**Lemma 2** (Cutoff Equilibrium). *The vector  $\mathbf{x}$  of cutoff strategies constitutes an equilibrium if and only if the following condition holds for each bidder  $i$ :*

$$A_i^n r_i(\mathbf{x}^i) = c_i \quad (3.3)$$

To understand equation ((3.3)) recall that, in equilibrium, if a bidder's valuation is equal to its cutoff, they must be indifferent to participating in the auction. For any bidder  $i$ , if  $v_i = x_i$ , participation by any bidder with a higher cutoff would imply losing the object. This event occurs with probability  $1 - A_i^n$  and leaves bidder  $i$  with zero revenue. As a consequence, bidder  $i$  only makes revenue with probability  $A_i^n$ . In this scenario, bidder  $i$  is the participating bidder with the highest participation cutoff and receives revenue  $r_i(\mathbf{x}^i)$ . The expected revenue of bidder  $i$  is the expected revenue conditional on winning times the probability of winning. In equilibrium, when a bidder's valuation is equal to its participation cutoff, the expected revenue

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5. The following notation is being used throughout the article:  $\sum \emptyset = 0$ ,  $\prod \emptyset = 1$ , and  $x_0 = 0$ .

from participating is equal to the participation cost. This indifference condition must hold for each bidder.

[Lemma 2](#) characterizes all equilibria of the game but does not provide any information about whether equilibria exist or about which bidder plays which cutoff. [Section 3.4](#) links bidders' public characteristics to equilibrium cutoffs. The next proposition, which follows from Brouwer's fixed-point theorem, establishes equilibrium existence.

**Proposition 1** (Existence). *For any game  $(F_i, c_i)_{i=1}^n$  there exists an equilibrium.*

The intuition behind this result is simple. Because a bidder's valuation is unbounded above, and because the opponents' valuations have finite expected value, it is possible to find a cutoff for each bidder such that expected payoffs from participating in the auction are positive regardless of the opponents' behavior. With this upper bound, it is possible to compactify the set of feasible strategies. Due to the continuity of the best response functions, the existence result follows from Brouwer's Fixed-point theorem.

### 3.3.2 Welfare analysis

We now discuss efficiency. As [Stegeman \(1996\)](#) pointed out, when participation is costly, *ex-ante* and *ex-post* efficiency are not equivalent. Moreover, when participation is costly, the revelation principle no longer applies because, in the equivalent direct mechanism, each bidder incurs a cost  $c_i$  to send a message—i.e., submit a bid—([Myerson, 1981](#)). Thus, as there is no “cost free” way to elicit bidders' preferences. When more messages are solicited, any optimal mechanism trades off the direct cost

of *ex-ante* soliciting more messages with the potential benefits from a better *ex-post* allocation. Although such a mechanism may be *ex-ante* optimal, any mechanism that does not solicit messages from all bidders in general produces *ex-post* misallocation with positive probability.

To illustrate this point, consider Figure 3.1, which depicts an equilibrium with two potential bidders, each with equal participation costs ( $c_i = c$ ), but different cutoff equilibrium strategies ( $x_1 < x_2$ ). Note that for an allocation to be *ex-post* efficient, only the bidder with the highest valuation, which must be above the participation cost, should participate in the auction.

In general, three types of inefficiencies arise. (i) *Insufficient Participation* (dark-shaded area): represents realizations of  $(v_1, v_2)$  in which there is at least one bidder whose valuation is greater than participation costs, but bidders stay out of the auction. (ii) *Excessive Participation* (lightly-shaded area): represents situations in which both bidders enter the auction, paying excessive participation costs. (iii) *Misallocation* (dotted area): realizations in which exactly one bidder participates, but is the bidder with the lowest valuation for the good. It is worth noticing that, conditional on participation, the bidder with the highest valuation wins the auction independent of the number of participants. Therefore, inefficiencies only arise due to miscoordinated participation.

From an *ex-ante* perspective, however, there is an efficient equilibrium. Consider the problem that a planner faces when choosing a strategy for each bidder conditional on the bidder's private information; i.e., the planner chooses a set of functions  $\tau_i^* : \mathbb{R}_+ \rightarrow [0,1]$  determining the probability that bidder  $i$  participates given her valuation.



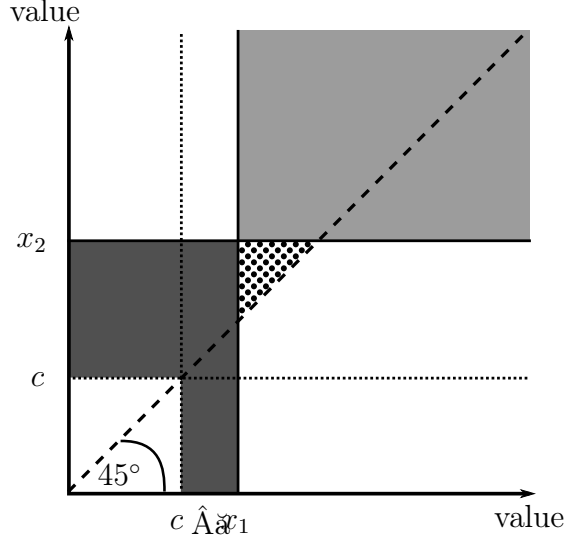


Figure 3.1: *Ex-post* inefficiency.

Using similar arguments to those in Lemma 1 it can be shown that the planner only considers cutoff functions.<sup>6</sup> Therefore, the planner chooses the vector of cutoffs  $\mathbf{x} = (x_1, x_2, \dots, x_n)$  that maximizes

$$W(\mathbf{x}) = \sum_{i=1}^n \left[ \int_{x_i}^{\infty} (v_i \Omega_i(v_i, x_{-i}) - c_i) dF_i(v_i) \right] \quad (3.4)$$

where  $\Omega_i(v_i, x_{-i}) = \prod_{k \neq i} F_k(\max\{v_i, x_k\})$  is the probability that bidder  $i$  obtains the object when her valuation is  $v_i$ . Notice that transfers between the winning bidder and the seller are irrelevant in terms of welfare. To explain (3.4) further, we focus on the planner's payoffs from bidder  $i$ . With probability  $dF_i(v_i)$ , bidder  $i$  draws the valuation  $v_i$  and participates in the auction whenever  $v_i \geq x_i$ , in which case she pays the participation cost  $c_i$  and wins the object with probability  $\Omega_i(v_i, x_{-i})$ . Total

6. Notice that in this case, the planner only solicits messages (bids) from bidders whose valuations are above the specified cutoff. That is, the planner does not solicit messages from all bidders with certainty.

welfare is simply the aggregation of all possible values for a given bidder (integral), aggregated across all bidders (summation).

**Proposition 2** (Welfare). *There exists an equilibrium that is ex-ante efficient. Every critical point of the welfare function corresponds to an equilibrium of the game. That is, each equilibrium is either a (possibly local) maximum, or saddle point of  $W(\mathbf{x})$ .*

The intuition behind the proposition is as follows: consider the social contribution of a marginal decrease in bidder  $i$ 's participation cutoff,  $x_i$ . By decreasing their cutoff, bidder  $i$  participates on a larger range of values, paying the participation cost  $c_i$  more often and, with probability  $\Omega_i$ , becoming the highest valuation bidder. This latter effect also decreases the opponents' probability of obtaining the good,  $\Omega_j$ . This decrease in probability occurs when an opponent who was winning the good is outbid by  $i$ . In these cases, bidder  $i$ 's social contribution is the gap between bidder  $i$ 's valuation and the second highest valuation. In a second-price auction, when the price paid is the valuation of the second-highest bidder, this gap is the same as bidder's  $i$  private gain. That is, the social trade-offs faced by the planner match the private trade-offs faced by a bidder. Because in an inflection point, the social (private) gain nets out from the entry cost  $c_i$ , every equilibrium matches an inflection point of the social welfare function. Notice, however, this equivalence may be broken if there is a reservation (minimum) bid.

This efficiency result is similar in spirit to [Levin and Smith \(1994\)](#), which shows that every participation equilibrium is *ex-ante* efficient when bidders are symmetric and have no private information at the moment of participation. Their findings do

not extend directly when private information exists. Privately informed bidders self-select according to their own characteristics, information, and expectations of other bidders' behavior. Different expectations may lead bidders to coordinate in inefficient equilibria, even if bidders are *ex-ante* symmetric.

To further illustrate the relation between uniqueness and efficiency, and to motivate the analysis that follows, we consider the case of  $n = 2$  potential bidders. The Hessian of the planner's problem, evaluated at a critical point, is equal to:

$$H(\mathbf{x}) = - \begin{pmatrix} f_1(x_1)F_2(x_2) & x_1f_1(x_1)f_2(x_2) \\ x_1f_1(x_1)f_2(x_2) & f_2(x_2)F_1(x_2) \end{pmatrix}.$$

Observe that, under concavity of  $F_i$ , the second order condition for a maximum is satisfied at *every* critical point.<sup>7</sup> Therefore, only one critical point exists and the game has a unique, efficient equilibrium. This finding suggests that some form of concavity of the CDF may be sufficient to guarantee both uniqueness and efficient outcomes. As we show below, this intuition extends to a large set of models that are relevant for applied analysis.

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7. Concavity of the CDF implies  $F_i(x) \geq xf_i(x)$  for every  $x > 0$ . Then, at every equilibrium  $x_1 < x_2$ , the first minor of  $H(\mathbf{x})$  is always negative and

$$\det(H(\mathbf{x})) = f_1(x_1)f_2(x_2) (F_1(x_1)F_2(x_2) - (x_1)^2f_1(x_1)f_2(x_2)) > 0.$$

### 3.4 Strength and Herculean Equilibrium

In this section, we present the main results of the article. In particular, we connect the bidders' public characteristics  $(F_i, c_i)_{i=1}^n$  to the game's equilibrium strategies. The definition below, which only uses the information given in the game fundamentals, ranks bidders in terms of their ability to endure competition. We use this notion to further describe bidders' participation strategies.

**Definition** (Strength). For a given game  $(F_i, c_i)_{i=1}^n$ , the strength of bidder  $i$  is the unique number  $s_i \in \mathbb{R}_+$  that solves:

$$s_i \prod_{k \neq i} F_k(s_i) = c_i. \quad (3.5)$$

We say that bidder  $i$  is stronger than  $j$  if  $s_i < s_j$ .

Observe that the left hand side of (3.5) is strictly increasing in  $s_i$ , takes the value of 0 when  $s_i = 0$ , and is unbounded above. Therefore, strength is well defined. Each bidder  $i$  has a unique scalar  $s_i$ . Thus, we can always use strength to rank all bidders in the game. Notice that the index  $s_i$  is inversely related to strength of a bidder. A lower  $s_i$  means a stronger bidder.

Strength uses all public information from the game to elicit bidders' ability to endure competition. The strength of bidder  $i$  is defined as the cutoff that bidder  $i$  plays in the unique symmetric strategy profile in which bidder  $i$  is best-responding (see equation (3.3) for the case of symmetric cutoffs). By computing this (symmetric) strategy in the context of asymmetric bidders, we can measure a bidder's willingness

to participate in the auction relative to their ability to endure competition. To see this, start by noticing that facing a lower participation cutoff from a competitor means that a bidder faces *more* competition. Because of symmetric behavior, a lower measure of strength  $s_i$  means that a bidder is willing to participate at a lower valuation, even when her competitors participate more often—i.e., a bidder is more willing to participate despite facing more competition.

In order to further understand strength, the next lemma relates it with the notions of bidder competitiveness developed by the previous literature.

**Lemma 3** (Strength in quasi-symmetric games).

1. *If bidders have the same participation costs, and if their distributions of valuations are ordered by FOSD, then bidders who stochastically dominate other bidders are stronger.*
2. *If bidders have the same distributions of valuations but different participation costs, then bidders who have lower participation costs are stronger.*

The order provided by strength coincides with existing notions of relative competitiveness among bidders, such as FOSD or participation-cost order. Strength, however, extends the order to scenarios in which relative competitiveness is not self-evident.<sup>8</sup> Take, for example, a bidder whose distribution of valuations first-order stochastically dominates that of another bidder, but has a higher participation cost. This scenario is likely to arise in practice when the auctioneer subsidizes participation costs of small

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8. Tan and Yilankaya (2006) calls the order induced by FOSD *intuitive*, whereas Cao and Tian (2013) calls the cost-order *monotone*. Because, in general, the order provided by strength might neither be intuitive nor monotone, we decided to avoid confusion and adopt the current nomenclature.

firms. In this case, although the former bidder may be stronger, as it is likely to draw a higher valuation, it may also be weaker than the latter bidder, who also has lower participation costs. Strength not only ranks bidders in this (or any other) scenario but also, as is shown below, provides meaningful information about equilibrium behavior.

**Definition** (Herculean Equilibrium). An equilibrium is called herculean if the equilibrium cutoffs are ordered by strength, with stronger bidders playing lower cutoffs. That is,  $x_i < x_j$  if and only if  $s_i < s_j$ .

Because stronger bidders are more able to endure competition, they should be more inclined than weaker bidders to participate in the auction. In terms of equilibrium behavior, stronger bidders should play lower participation cutoffs. As we show in the following sections, this intuition is correct: in most applications a herculean equilibrium will exist.

Finally, notice that in symmetric games all bidders are equally strong; thus, in a herculean equilibrium bidders must play symmetric strategies. Furthermore, the strength of each bidder coincides with their symmetric equilibrium cutoff; i.e.,  $x_i = s_i$ . Therefore, in symmetric games the notions of strength, symmetric equilibrium, and herculean equilibrium coincide. Because strength is well defined, this trivially implies that symmetric games have a unique symmetric equilibrium (of course, symmetric games may still have asymmetric equilibria.)

### 3.4.1 Herculean equilibrium under two bidders

In order to better illustrate our results, we start by presenting them in a two potential-bidder environment. From now on, unless otherwise noted, we order bidders' identities by their strength, with bidder 1 being the strongest bidder in the game. The following proposition is our main result in this context.

**Proposition 3** (Existence and uniqueness). *There always exists a herculean equilibrium. Every herculean equilibrium is characterized by cutoffs  $x_1 \leq x_2$  that jointly solve*

$$x_1 F_2(x_2) = c_1 \quad \text{and} \quad x_2 F_1(x_2) - \int_{x_1}^{x_2} v dF_1(v) = c_2. \quad (3.6)$$

*Moreover, a herculean equilibrium is the unique equilibrium of the game—and, therefore, ex-ante efficient—if the following condition holds for each bidder*

$$F_i(v) \geq v f_i(v) \quad \text{for all } v \geq c_j. \quad (3.7)$$

Proposition 3 generalizes existing results in the literature in two ways. By introducing the notion of strength, we associate bidders' public characteristics with equilibrium-cutoffs order in *any* game  $(F_i, c_i)_{i=1}^2$ —that is, without limiting our attention to specific distributions of valuations and without restrictions on participation costs. The proposition also confirms the intuition that an equilibrium in which the strong bidder plays a lower participation cutoffs should exist. Perhaps more importantly, the proposition provides a sufficient condition on the shape of the distributions

of valuations for the game to have a unique equilibrium. This result is particularly important for applied work, as it provides a testable condition that guarantees robust counterfactual analysis. Furthermore, as a consequence of Proposition 2, the sufficient condition for equilibrium uniqueness also guarantees efficient outcomes.<sup>9</sup>

In intuitive terms, condition (3.7) shapes the opponent’s best-response so that payoffs are monotone in a bidder’s strategy. It guarantees that bidder  $i$ ’s expected revenue is increasing in her cutoff  $x_i$ , even when bidder  $j$  best responds to the increase in  $x_i$  by decreasing  $x_j$  (increasing competition). This implies that only one cutoff makes bidder  $i$  indifferent to participate in the auction, leading to a unique equilibrium. From bidder  $i$ ’s perspective, bidder  $j$ ’s best response is a function of  $i$ ’s distribution (see equations in (3.6)). Because condition (3.7) regulates best-response behavior, the condition only needs to hold for valuations that are above the opponents’ entry costs, as no bidder would participate when her valuation is below her cost. The next lemma would help us to further characterize sufficient condition (3.7).

**Lemma 4.** *1) If  $(F_1, F_2)$  are concave, then (3.7) is satisfied and the equilibrium is unique. 2) If the distributions  $(F_1, F_2)$  become concave for high valuations, there exists a pair of entry costs  $(c_1, c_2)$  such that the game has a unique equilibrium.*<sup>10</sup>

Condition (3.7) is a weak form of concavity. In particular, auctions with concave distributions of valuations (e.g., Exponential, Generalized Pareto, or the standard Half-Normal distributions) always have a unique equilibrium. Many other distribu-

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9. Observe, however, that the proposition does not tell us that a herculean equilibrium is always *ex-ante* efficient. For instance, in symmetric games, when there are multiple equilibria, the symmetric equilibrium need not be the efficient equilibrium.

10. The proof of the lemma shows how to find the costs that guarantee equilibrium uniqueness.



tions, such as Beta, Gamma, or Weibull are concave for certain parameter specifications. Most distributions used in applications are concave for sufficiently high valuations. The Lemma also show that for these eventually-concave distributions there are sufficiently high participation costs guaranteeing equilibrium uniqueness. Example 3, below, illustrates this point.

**Examples.** To illustrate the usefulness of strength and herculean equilibria, and to illustrate the workings of the sufficient condition, we develop three examples. The first two examples make use of a Generalized Pareto distribution (GPD).<sup>11</sup> The choice of GPD yields a simple concave distribution with positive support that is flexible enough to change its mean and variance. Results and intuitions in the examples apply more generally. The third example corresponds to a log-normal distribution which is S-shaped.

1. **Second-order stochastic dominance.** Consider two asymmetric bidders whose distribution of valuations follows a GPD with shape parameter  $\kappa$  and scale parameter  $\sigma$ . Suppose both bidders have a symmetric participation cost  $c$ , but bidder 1 is characterized by  $(\kappa_1, \sigma_1) = (0, 1)$  and bidder 2 by  $(\kappa_2, \sigma_2) = (0.25, 0.75)$ . Both distributions have the same mean but the second distribution has twice the variance. That is, the second distribution is a mean-preserving spread of the first. Because the CDFs cross, distributions are *not* ordered by FOSD. This game is not quasi-symmetric and

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11. For  $\kappa \in \mathbb{R}$  and  $\sigma \in (0, \infty)$ , the Generalized Pareto CDF is defined over  $\mathbb{R}_+$  and given by

$$F(x|\kappa, \sigma) = \begin{cases} 1 - \left(1 + \frac{\kappa x}{\sigma}\right)^{-\frac{1}{\kappa}} & \kappa \neq 0 \\ 1 - e^{-\frac{x}{\sigma}} & \kappa = 0 \end{cases} .$$

The CDF is concave whenever  $\kappa > -1$ , its mean is well defined for  $\kappa < 1$  and given by  $\sigma/(1 - \kappa)$ , whereas its variance is defined for  $\kappa < 1/2$  and given by  $\sigma^2/(1 - \kappa)^2(1 - 2\kappa)$ .

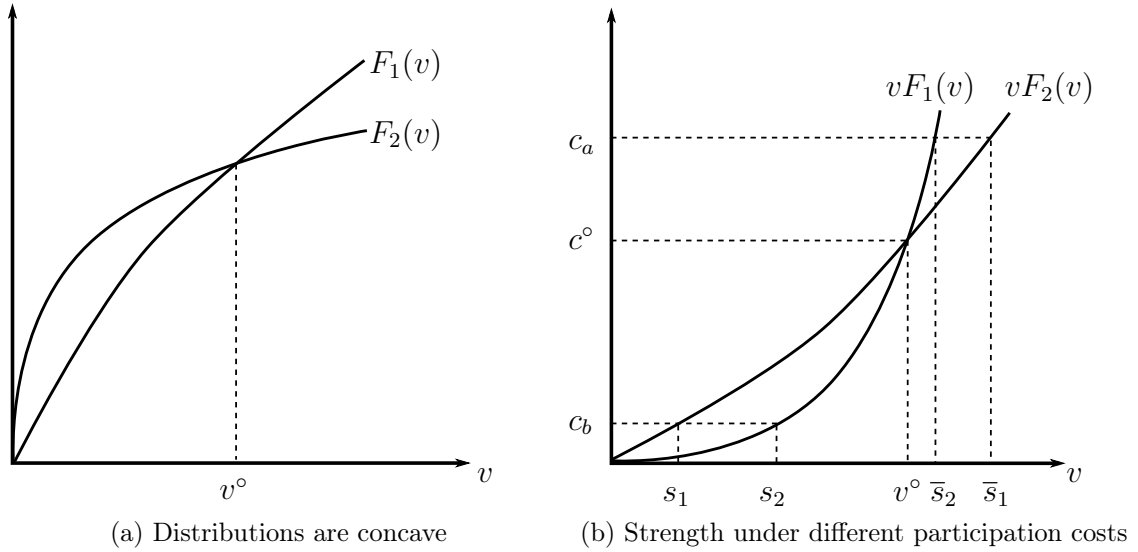


Figure 3.2: Strength under second-order stochastic dominance.

it is not self-evident which bidder is stronger. Consequently, existing tools in the literature cannot characterize equilibrium behavior, nor determine whether the game has a unique equilibrium.

Intuitively, the stronger bidder would be the one whose distribution of valuations has more mass to the right of the equilibrium cutoff strategies, as this implies the bidder is more likely to obtain higher valuations. If the equilibrium cutoff strategies are high, then bidder 2 would have more mass to the right of the cutoffs, and thus bidder 2 would be the stronger bidder. High equilibrium cutoff strategies are likely to occur when participation costs are high. Conversely, if the cutoff strategies are low, then bidder 1 would have more probability mass to the right of the cutoffs, and thus bidder 1 would be the stronger bidder. Low equilibrium cutoff strategies are likely to occur when participation costs are low.

This situation is illustrated in Figure 3.2. Panel (a) shows that both distributions are

concave, thus Lemma 4 implies that the participation game has a unique equilibrium for any participation costs  $c > 0$ . Panel (a) also shows that both distributions cross at  $v^\circ = 2.2007$ . Panel (b) depicts the bidders' strength. It shows that bidders are equally strong when  $c^\circ = 1.957$ . For participation costs above  $c^\circ$ , bidder 2 is stronger ( $s_2 < s_1$ ) and, in the unique equilibrium, bidder 2 plays a lower cutoff strategy ( $x_2 < x_1$ ). For instance, if  $c_a = 2 > c^\circ$ , then the vector of equilibrium cutoffs is  $\mathbf{x} = (2.241, 2.238)$ . Alternatively, when  $c < c^\circ$ , bidder 1 is stronger ( $s_1 < s_2$ ) and plays a lower equilibrium cutoff strategy ( $x_1 < x_2$ ). For example, if  $c_b = 1 < c^\circ$ , then the equilibrium is  $\mathbf{x} = (1.281, 1.383)$ .

The example above illustrates a simple but important point. In games that are not quasi-symmetric, the relative bidders' strength is not self-evident. Two games that only differ in the (symmetric) participation cost can generate different strength rankings and different predictions on which bidder would participate more often.

**2. Subsidized participation.** Suppose two asymmetric bidders whose valuations follow a GPD with shape parameter  $\kappa = 0$ . Bidder 1 is characterized by  $(\sigma_1, c_1) = (1, 1)$  and bidder 2 by  $(\sigma_2, c_2) = (2, 2)$ . That is, bidder 2's valuation FOSD bidder 1's valuation, but bidder 1 has a lower participation cost. Situations like this may arise in a procurement auction when bidder 2 is a large firm and bidder 1 is a small (local) firm with subsidized participation. As before, the model is not quasi-symmetric and existing results do not apply.

We can use the notion of strength to characterize the equilibria in this game. In this example, bidder 1 is stronger than bidder 2, as  $s_1 = 1.73 < 2.24 = s_2$ . Consequently,

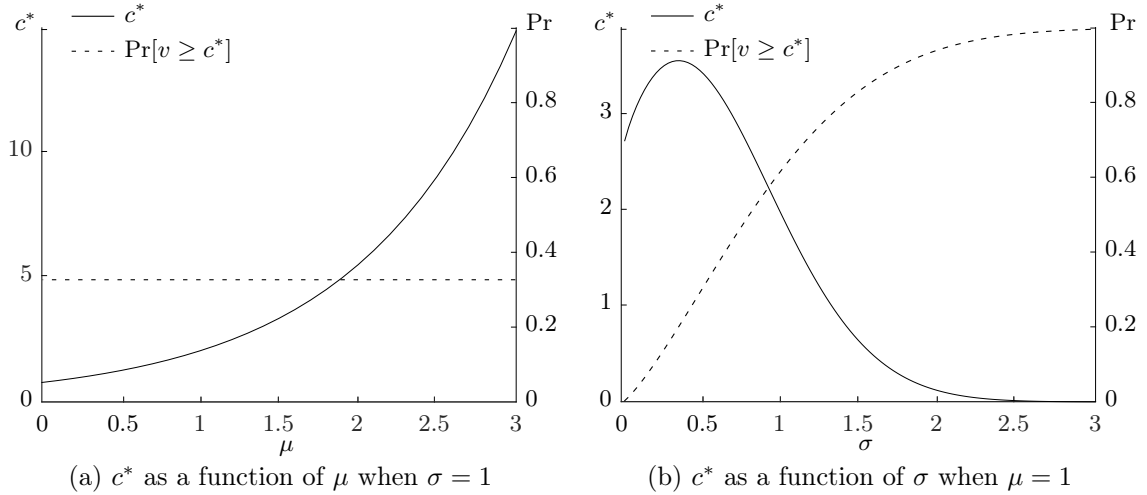


Figure 3.3: Sufficiency with Log-Normal distribution

in a herculean equilibrium, bidder 1 plays a lower cutoff. In this case,  $\mathbf{x} = (x_1, x_2) = (1.398, 2.511)$ . Because both distributions are concave, this is the unique equilibrium of the game. Notice that  $x_1 < s_1 < s_2 < x_2$ . That is, the equilibrium cutoff strategies are “farther apart” than the strength values. This feature not only holds in this example, but in any herculean equilibrium with two types of bidders. This property can be useful when estimating auction models, as it can reduce the computing power necessary to find the equilibrium cutoffs. In particular, the strength of the weak bidder provides a lower bound for its cutoff and the strength of the strong bidder provides an upper bound.

**3. Uniqueness under a log-normal distribution.** To illustrate the intuition behind the sufficient condition for uniqueness (3.7), consider the case with two symmetric bidders under Log-normal valuations with parameters  $(\mu, \sigma)$ .<sup>12</sup> This distribution is not concave, therefore existing results in the literature do not apply. By Lemma

12. A Log-normal distribution with parameters  $(\mu, \sigma)$  has mean  $\exp(\mu + \frac{\sigma^2}{2})$  and median  $\exp(\mu)$ .

4, however, we can find a participation cost  $c^*$  that is sufficiently high, so that the sufficient condition (3.7) holds.

Figure 3.3 depicts the threshold  $c^*$  and the mass of valuations below  $c^*$ , as a function of  $\mu$  and  $\sigma$ . Panel (a) shows that  $c^*$  increases in  $\mu$ . This means that for distributions with higher medians, the minimum participation cost that guarantees uniqueness is higher. Notice, however, that the proportion of valuations below the entry costs is independent of  $\mu$ . This observation implies that, under Log-normality, whether a game has a unique equilibrium only depends on the standard error of the distribution and the participation cost. To further understand the previous point, we show the relation between  $c^*$  and  $\sigma$  when  $\mu = 1$ . Panel (b) shows that the relation between  $c^*$  and  $\sigma$  is non-monotonic. In particular,  $c^*$  is maximal at 3.6493 when  $\sigma = .3507$ . This implies that any game with  $c^* > 3.6493$  has a unique equilibrium, for any value of  $\sigma$ . When  $\sigma > .3507$ ,  $c^*$  decreases with  $\sigma$ . Notice that the proportion of valuations below the entry costs is not independent of  $\sigma$ . The larger the variance of the distribution, the less demanding the condition for uniqueness becomes. In contrast, as  $\sigma \rightarrow 0$ , the mass of valuations above  $c^*$  converges to zero. That is, as the game converges to a complete information game—where equilibrium multiplicity is known to exist (c.f., Levin and Smith, 1994)—the sufficient condition for uniqueness is never met.

### 3.4.2 Herculean equilibrium for two groups of bidders

We now extend our previous results to environments with more than two bidders. Suppose first that there are two groups of bidders, say groups 1 and 2. Each group

$g$  consists of  $m_g$  bidders characterized by pairs  $(F_g, c_g)$ , for  $g = 1, 2$ . Without loss of generality, assume that bidders in group 1 are stronger than those in group 2 ( $s_1 \leq s_2$ ). Although bidders are symmetric within each group, the degree of asymmetry of the distribution of valuation or participation costs *across* groups is unrestricted. The two-group model is especially useful in applied work when bidders are divided by exogenous factors into two groups, such as incumbency (incumbent vs entrant) or size (small vs large). Examples of papers studying participation—not necessarily in the context of second price auctions—in environments with two groups of players include Athey *et al.* (2011), Krasnokutskaya and Seim (2011), Roberts and Sweeting (2013) among many others.

**Proposition 4** (Two-groups equilibria). *There always exists a herculean equilibrium. Every herculean equilibrium is characterized by the cutoffs  $x_1 \leq x_2$  that jointly solve*

$$x_1 F_1(x_1)^{m_1-1} F_2(x_2)^{m_2} = c_1 \quad (3.8)$$

$$F_2(x_2)^{m_2-1} \left[ x_2 F_1(x_2)^{m_1} - \int_{x_1}^{x_2} v d(F_1(v)^{m_1}) \right] = c_2. \quad (3.9)$$

*Moreover, the herculean equilibrium is the unique equilibrium of the game and, thus, efficient if for each bidder  $i$*

$$F_i(v) \geq v f_i(v) \text{ for every } v \geq \min\{c_k\}_{k \neq i}. \quad (3.10)$$

Proposition 4 generalizes Proposition 3 to the case in which there is more than one bidder in each group of bidders. The main difference between conditions (3.10)

and (3.7) is that, with more than one opponent, the range of valuations under which the condition has to hold needs to include the participation cost of every opponent, including opponents within the same group.

Condition (3.10) for bidder  $i$  in the two-groups scenario differs from (3.7) (in the two-bidders scenario) when there is more than one bidder in  $i$ 's group. In this case, the condition also has to hold for valuations above  $c_i$ , which might be lower than the cost of the other group. This is because the uniqueness proof has two steps. The first step shows that symmetric bidders—that is, bidders belonging to the same group—play symmetric strategies in equilibrium, so we can restrict our attention to group-symmetric strategies. This step only involves best-responses of bidders in  $i$ 's group, making use of condition (3.10) starting at  $c_i$ . The second step shows that among the group-symmetric class of strategies, the only equilibrium is the herculean one. This step makes use of condition (3.10), for values higher than the other group entry cost. Finally, it is worth noting that Lemma 4 extends to this environment without modification.

### 3.4.3 Robust strength order among bidders

The existence of herculean equilibrium in games with three or more groups of bidders is linked to the robustness of the ranking provided by strength. In particular, existence depends on whether the strength order between two bidders depends on the behavior of other bidders.

Consider a scenario with three bidders such that  $s_1 < s_2 < s_3$ . The construction

of strength assumes symmetric behavior among bidders. Suppose instead that bidder 3 is constrained to participate at some given cutoff  $\hat{x}_3$ . Given this restriction, we can recalculate the strength of bidders 1 and 2 and find a reversal in their strength order. This reversal is associated with non-existence of herculean equilibrium. To illustrate this, suppose we constructed best-response cutoffs for bidders 1 and 2, as a function of bidder 3's cutoff. Because  $s_1 < s_2$ , the initially constructed cutoffs satisfy  $x_1(s_3) < x_2(s_3)$ . For different values of  $x_3$ , for instance at  $x_3 = \hat{x}_3$ , the strength order between bidders 1 and 2 reverses, reversing their best responses; i.e.,  $x_1(\hat{x}_3) > x_2(\hat{x}_3)$ . In order to establish our existence and uniqueness results, we need to impose further structure to guarantee the robustness of strength. The next definition and lemma are instrumental to that purpose.

**Definition** (Cutoff upper bound). Let  $\bar{v}_i$  be the unique scalar that solves

$$\bar{v}_i G_i(\bar{v}_i) - \int_0^{\bar{v}_i} y dG_i(y) = c_i \quad (3.11)$$

where  $G_i(v) = \prod_{k \neq i} F_k(\max\{v, c_k\})$  is the probability that bidder  $i$  obtains the object when her valuation is  $v$  and other bidders participate in the auction for valuations above their participation cost.

The value of  $\bar{v}_i$  is well defined.<sup>13</sup> It provides an upper bound to bidder  $i$ 's set of feasible best responses. The cutoff  $\bar{v}_i$  corresponds to bidder  $i$ 's best response assuming that the other bidders always enter whenever their valuations are above their entry

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13. It is well defined, as the left hand side of (3.11) is strictly increasing in  $\bar{v}_i$ , starts from zero when  $\bar{v}_i = 0$ , and is unbounded above due to the finite expectation assumption.



costs; i.e.,  $\bar{v}_i$  is  $i$ 's participation cutoff under the highest level of feasible competition.

**Lemma 5** (Robust strength order). *Let  $\underline{c} = \min\{c_i\}_{i=1}^n$  and  $\bar{v} = \max\{\bar{v}_i\}_{i=1}^n$ . Suppose that for any two bidders  $i$  and  $j$ , with  $i < j$ , the following condition holds:*

$$F_i(v)c_i \leq F_j(v)c_j \text{ for all } v \in [\underline{c}, \bar{v}]. \quad (3.12)$$

*Then, bidders are ordered by strength with bidder 1 being the strongest bidder.*

The set of models satisfying condition (3.12) includes quasi-symmetric games as particular cases. Condition (3.12) further extends the existing literature in participation in quasi-symmetric environments in two ways. First, it allows distribution functions that cross and allows for cost orders that do not coincide with distribution orders. Second, the condition does not restrict bidders to belong to one of two groups. In particular, if condition (3.12) holds with equality for some bidders, the condition allows for an arbitrary number of (strictly ordered) groups of bidders, with each group having an arbitrary number of members.

**Example.** To illustrate that models satisfying condition (3.12) might not be quasi-symmetric, consider a scenario in which bidders valuations belong to the Exponentiated distribution family; i.e.,  $F_i(x) = F(x)^{\theta_i}$  for any  $F$  satisfying our assumptions and  $\theta_i > 0$ . Observe that bidder  $i$  FOSD  $j$  if and only if  $\theta_i > \theta_j$ .<sup>14</sup> Suppose  $\theta_i > \theta_j$ , then using  $\bar{v}$  we find that every  $c_i \leq c_j F(\bar{v})^{\theta_j - \theta_i}$  satisfies condition (3.12). In particular, the game is not quasi-symmetric whenever  $c_i \in (c_j, c_j F(\bar{v})^{\theta_j - \theta_i}]$ , as firm  $i$  first order

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14. This example includes quasi-symmetric environments when two bidders  $i < j$  satisfy  $c_i < c_j$  but  $\theta_i = \theta_j$  or when  $c_i = c_j$  but  $\theta_i > \theta_j$ .

stochastically dominates  $j$  but has a higher participation cost.

**Proposition 5** ( $n$  potential bidders). *Under condition (3.12), a herculean equilibrium always exists. Furthermore, the herculean equilibrium is the unique equilibrium of the game and, therefore, ex-ante efficient if*

$$F_i(v) \geq v f_i(v) \text{ for all } v \in [\underline{c}, \bar{v}]. \quad (3.13)$$

Proposition 5 is neither a particular case, nor a generalization of our previous results. On the one hand, the proposition extends existence and uniqueness of herculean equilibrium to the case with  $n$  potential bidders. On the other hand, the proposition requires condition (3.12) to hold whereas previous propositions do not. The proposition generalizes the existence-of-equilibrium result in Miralles (2008), who studied ‘intuitive’ equilibria in a scenario with  $n$ -bidders ordered by FOSD and symmetric participation costs. More importantly, it extends those findings to a larger set of models and show, as in the previous scenarios, that our weak form of concavity is sufficient to guarantee equilibrium uniqueness and efficient outcomes. As before, Lemma 4 applies without modification.

### 3.5 On the Importance of Bidder Heterogeneity

In previous sections we emphasized that our results apply to environments that allow for more bidder heterogeneity than quasi-symmetric models. Here, we highlight the importance of allowing this type of heterogeneity. In particular, we show that

quasi-symmetric models restrict the relation between observable outcomes and restrict bidder behavior under (exogenous) changes in competition.

**Equilibrium, Revenues, and Entry probability** In this section, we show that quasi-symmetric models limit the relation between the bidders' behavior—i.e., their participation cutoffs—and observed outcomes, such as the bidders' profitability and participation probability. For ease of exposition, we present the results with two bidders, but they could easily be extended to an arbitrary number of bidders.

In precise terms, we study the relationship between: (i) the cutoff strategies,  $x_i$ ; (ii) the *ex-ante* probability of participating in the auction,  $1 - F_i(x_i)$ ; and (iii) the *ex-ante* expected payoff of each bidder; which, for a given vector of cutoffs strategies  $\mathbf{x} = (x_1, x_2)$ , is equal to:

$$U_i(\mathbf{x}) = \int_{x_i}^{\infty} \left( vF_j(\max\{v, x_j\}) - \int_{x_j}^{\max\{v, x_j\}} s dF_j(s) - c_i \right) dF_i(v). \quad (3.14)$$

That is, for each valuation  $v_i$  under which bidder  $i$  participates (i.e., for each  $v_i > x_i$ ), the expected payoff of participating in the auction, weighted by the probability that  $v_i$  occurs.

The following proposition characterizes the relationship between these three objects as a function of the game's degree of bidder heterogeneity.

**Proposition 6** (Cutoffs and revenue ranking).

1. *In a symmetric game, a bidder playing a lower cutoff obtains higher (expected) payoffs and is more likely to participate.*

2. *In quasi-symmetric games where bidders play herculean equilibria, a bidder playing a lower cutoff obtains higher payoffs and is more likely to participate.*
3. *With general forms of bidder heterogeneity, cutoff, participation-probability, and payoff rankings may not coincide in a herculean equilibria.*

Consider a situation in which the data shows that distributions are concave, so that a unique equilibrium exists, and one bidder participates in an auction more often than another, but overall receives lower expected payoffs. [Proposition 6](#) shows that symmetric and quasi-symmetric models cannot account for this type of behavior. The behavior, however, can be accommodated if more degree of bidder heterogeneity is allowed.

In quasi-symmetric games, when bidders play a herculean equilibrium, payoffs, cutoffs and participation probabilities are always ordered in the same way; i.e., bidders with lower cutoffs are more likely to participate and receive higher expected profits. To see the intuition behind this result, consider an environment in which bidders are quasi-symmetric in costs. In a herculean equilibrium, the low-cost bidder plays the lower cutoff, which implies—due to bidders having symmetric distributions of valuations—that she participates with higher probability. Suppose, for the sake of argument, that both bidders play the *same* cutoff strategy. Because the stronger bidder has a lower participation costs and bidders have symmetric distribution of valuations, the stronger bidder would receive a higher expected payoff. This payoff order gets reinforced in equilibrium. Bidders participate whenever they have positive expected payoffs and the strong (low-cost) bidder participates at a larger range of

valuations, obtaining even higher profits than the weaker bidder.<sup>15</sup> This construction strongly relies on quasi-symmetry. Once we allow for general forms of bidder asymmetries, the relation breaks even within the herculean equilibrium class, as shown in the examples below.

**Example.** Recall example 1 from section 3.4.1. There, bidders are not ordered by FOSD as bidder 2's CDF is a mean preserving spread of bidder 1's. When the participation cost is equal to  $c^\circ$ , bidders are equally strong ( $s_i = v^\circ$ ). Because the CDFs are concave, the unique equilibrium is given by the symmetric cutoffs equal to the bidders' strength ( $x_i = v^\circ$ ). The expected payoff of bidder 2, however, is greater than the expected payoff of bidder 1. Using equation (3.14), we obtain  $(U_1, U_2) = (0.103, 0.185)$ . This means that although bidders' cutoffs are not ranked, their expected profits are. The intuition in this scenario follows from  $F_2(v) < F_1(v)$  for every  $v > v^\circ$ . Relative to bidder 1, bidder 2's valuations (distributed according to  $F_2(v)$ ) are skewed to the right tail of the distribution, whereas their expected payment price (distributed according to  $F_1(v)$ ) is skewed towards the left (see Figure 3.2.(a)). In other words, for valuations greater than  $v^\circ$ , bidder 2's conditional distribution of valuations FOSD the bidder 1's conditional distribution.

Beginning from the previous example, we construct an equilibrium in which bidder 1 receives a lower expected payoff than bidder 2, despite playing a lower participation cutoff and having a higher participation probability. By decreasing bidder 1's partic-

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15. Similar reasoning applies to quasi-symmetric games with FOSD distributions. If both bidders were to play the *same* participation cutoff, the stronger bidder would participate more often and receive a higher expected payoff due to FOSD. In a herculean equilibrium, the stronger bidder plays a lower cutoff, participating more often and obtaining even higher expected payoffs.

icipation cost, bidder 1 becomes stronger than bidder 2 and will play a lower cutoff in the unique equilibrium of the game. By continuity, if the decrease in bidder 1's cost is small, we can construct an equilibrium with said characteristics. Take for example  $(c_1, c_2) = (1.9, c^\circ)$ , then bidder 1 is stronger and plays a lower cutoff—in this case  $\mathbf{x} = (2.1327, 2.2196)$ —but also receives lower expected payoffs  $(U_1, U_2) = (1.11, 1.83)$ . At a cutoff equal to  $v^\circ$ , both bidders are equally likely to enter. Thus,  $x_1 < v^\circ < x_2$  implies that bidder 1 is simultaneously more likely to participate and receive a lower expected payoff.

Finally, to show that cutoff order need not coincide with entry-probability order, modify the participation costs to  $(c_1, c_2) = (1.1, 1)$ . In this scenario, bidder 1 plays a higher entry cutoff  $x_1 = 1.434 > 1.313 = x_2$  while also participating more frequently  $1 - F_1(x_1) = .238 > .234 = 1 - F_2(x_2)$ .

**Number of Competitors and Equilibrium Behavior** We now discuss how an (exogenous) increase in the number of competitors generate different equilibrium predictions in quasi-symmetric and non-quasi-symmetric models. In particular, [Lemma 6](#) below shows that, in non-quasi-symmetric models, increasing the number of competitors may change the relative strength position among existing bidders. Whereas, in quasi-symmetric models, this reversal cannot occur.

**Lemma 6.** *Adding a potential bidder to the game does not affect the existing strength-order among quasi-symmetric bidders but might change the order if bidders are not quasi-symmetric.*

Quasi-symmetric models restrict bidder behavior when faced with increased com-

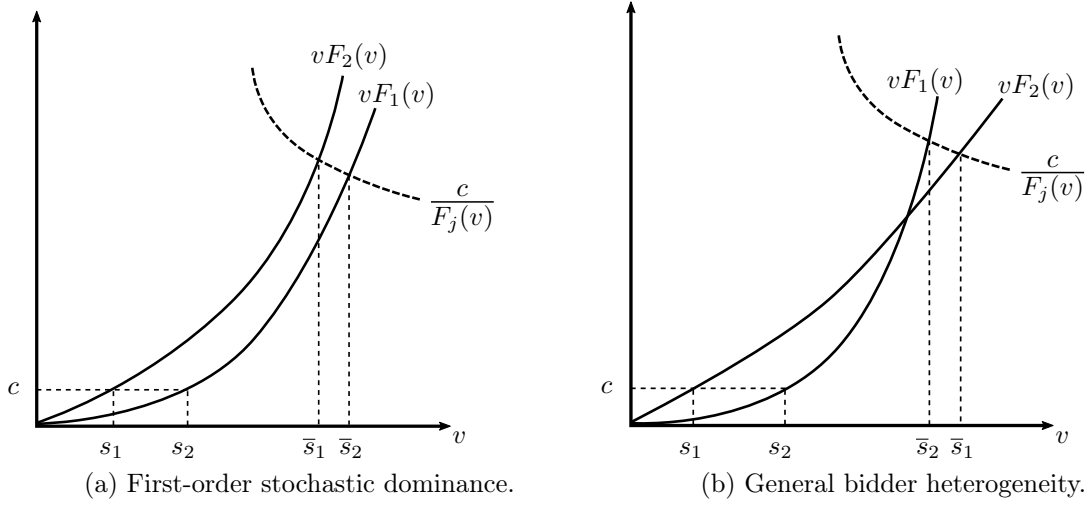


Figure 3.4: Competition and Strength.

petition. Consider a baseline scenario with two asymmetric bidders with symmetric participation costs  $c$ . Suppose that bidder 1 is stronger than bidder 2 and, for simplicity, that  $F_1(x)$  and  $F_2(x)$  are concave so that a unique equilibrium exists. Because bidder 1 is stronger, it plays a lower participation cutoff than bidder 2. Using the definition of strength in equation (3.5), under symmetric participation costs, the strength of bidder 1 in a game with two bidders is given by  $s_1 F_2(s_1) = c$ . In Figure 3.4.(a),  $s_1$  is solved by the intersection of the curve  $vF_2(v)$  with the horizontal line  $c$  (and analogously for bidder 2). Suppose a new potential bidder  $j$  joins the game. When the third bidder is added to the game, the strength of bidder 1 is determined by  $\bar{s}_1 F_2(\bar{s}_1) F_j(\bar{s}_1) = c$ . To simplify comparison with the case of two bidders, we rearrange the previous equations that define the strength of bidder 1 as  $\bar{s}_1 F_2(\bar{s}_1) = c / F_j(\bar{s}_1)$ . In Figure 3.4.(a),  $\bar{s}_1$  can be computed by the intersection of the curve  $vF_2(v)$  and the curve  $c / F_j(v)$  (and analogously for bidder 2).

Figure 3.4 shows how strength varies in two different scenarios. Panel (a) depicts a

situation in which bidders are ordered by FOSD (a quasi-symmetric game). Panel (b) shows a scenario without this restriction. Notice how in Panel (a) the two functions  $vF_i(v)$  never cross. However, without the FOSD restriction the two functions may cross in general. Intuitively, this means that adding a competitor in quasi-symmetric games does not alter the relative strength of existing bidders. In Panel (a), bidder 1 is always stronger than bidder 2. Whereas, in non-quasi-symmetric games, increased competition may affect the relative strength of bidders. As shown by Panel (b), bidder 1 is stronger in a two bidder scenario. But bidder 2 becomes stronger when a new potential bidder is added to the game, reversing equilibrium behavior among existing bidders.

## 3.6 Extensions and Discussion

In this section, we briefly discuss some assumptions in the model and extend our analysis in two important directions: scenarios with a reservation price and in which bidders are only partially informed about their valuation before making their participation decisions. The proofs in this section are relegated to the online Appendix.

**Participation Costs** In the model, participation costs could represent participation fees charged by the auctioneer, the cost of preparing and submitting a bid, the opportunity cost of attending the auction or, travel costs to the auction site. In all these cases, our results are a starting point for auction design with costly participation and heterogeneous agents (c.f. Celik and Yilankaya, 2009; Menezes and Monteiro,



2000; Moreno and Wooders, 2011, who study the case of symmetric agents).

**Reserve Prices** For ease in exposition we assumed throughout the paper that there is no reserve price. The notion of strength and our uniqueness result, however, can be easily extended to this setting. For simplicity we present results in a two-bidders context. Following similar steps, however, results can be generalized to environments with more than two bidders.

Assume that the auction has a reservation price of  $r \geq 0$ . We start by adapting the notion of strength to reflect the existence of the reserve price. In a scenario with two potential bidders, the strength of bidder  $i$  is the unique number that solves:

$$(s_i - r)F_j(s_i) = c_i. \quad (3.15)$$

As before, the strength of bidder  $i$  is defined as the unique symmetric strategy profile in which bidder  $i$  is best responding. The main difference with respect to the previous definition in equation (3.5) is that now the symmetric strategy profile takes into account the reserve price. Letting bidder 1 being the strong bidder of the game, our main results in the context of a reserve price is:

**Proposition 7** (Existence and uniqueness with reserve price). *There always exists a herculean equilibrium. Every herculean equilibria is characterized by cutoffs  $x_1 \leq x_2$  that jointly solve*

$$(x_1 - r)F_2(x_2) = c_1 \quad \text{and} \quad x_2F_1(x_2) - rF_1(x_1) - \int_{x_1}^{x_2} v dF_1(v) = c_2. \quad (3.16)$$

A herculean equilibrium is the unique equilibrium of the game if for both bidders

$$F_i(v) \geq v f_i(v) \text{ for all } v > r + c_j. \quad (3.17)$$

We can see from the proposition above that the existence of a reserve price weakens our sufficient condition for uniqueness. The lower bound for participating in the auction is now  $r + c_i$ . A higher reservation price acts as an increase in the entry costs and, by Lemma 4, it becomes more likely that condition (3.17) is satisfied.

**Partially Informed Bidders** Thus far, we have studied environments in which bidders are *perfectly* informed about their valuations before making participation decisions. More generally, however, bidders could be *partially* informed about their valuations and only learn their true valuation after paying the participation cost. This model, also called the *selective entry* model, has been used in empirical applications by Gentry and Li (2014), Sweeting and Bhattacharya (2015), and Roberts and Sweeting (2016). We now show how our methodologies and results can be extended to this framework.

Consider the two bidder scenario of Section 3.4.1. Suppose that the valuation of bidder  $i$  is now given by  $V_i = v_i \varepsilon_i$ , where the *signal*  $v_i$  is observed before the participation decision and the *noise*  $\varepsilon_i$  is observed *after* paying the participation cost but *before* submitting a bid. We maintain our distributional assumptions over  $v_i$  and assume that  $\varepsilon_i$  is independent from  $v_i$ , distributed according to  $\Phi_i$ , an atomless distribution with full support over  $\mathbb{R}^+$ , and  $\mathbb{E}(\varepsilon_i) = 1$ . At one extreme of *selective entry*

models, is Levin and Smith (1994) where bidders are only informed after participating; i.e.,  $v_i$  is degenerate at some point, conveying no private information. At the other extreme is Samuelson (1985) and our previous framework, in which bidders are fully and privately informed about their type before participating into the auction; i.e.,  $\varepsilon_i$  is degenerate at 1.

For a given realization of the signal  $v_i$ , when bidder  $i$  is the sole participant in the auction, its *interim* expected payoff—that is, the expected payoff of bidder  $i$  right after the participation decisions have been (simultaneously) made, but before bidders receive their second signal  $\varepsilon_i$  and submit a bid—is equal to  $v_i \int_0^\infty \varepsilon_i d\Phi_i(\varepsilon_i) = v_i$ . Similarly, for a given realization of signals  $\mathbf{v} = (v_1, v_2)$ , the *interim* expected payoff of a bidder that faces competition from the other bidder is:

$$\pi_i(\mathbf{v}) = \int_0^\infty \left( \int_0^{v_i \varepsilon_i} (v_i \varepsilon_i - s) d\Phi_j \left( \frac{s}{v_j} \right) \right) d\Phi_i(\varepsilon_i) \quad (3.18)$$

where, after a change in variables,  $\Phi_j(s/v_j)$  is the distribution of bidder  $j$ 's bid, conditional on observing the signal  $v_j$ .

We can use the objects above to define the bidder's strength. Notice that strength is an object defined *ex-ante*. Therefore, strength is defined using the structure of the first signal, taking into account expectations over the second signal. For a given game  $(F_i, \Phi_i, c_i)_{i=1}^2$ , the strength of bidder  $i$  is the unique number  $s_i \in \mathbb{R}_+$  that solves:

$$s_i F_j(s_i) + \int_{s_i}^\infty \pi_i(s_i, v_j) dF_j(v_j) = c_i. \quad (3.19)$$

The equation above finds the signal  $s_i$  that makes bidder  $i$  indifferent to participate in the auction when their opponent plays a cutoff strategy  $s_i$ .<sup>16</sup> The first term is bidder  $i$ 's payoff when she is the sole entrant. In this case, bidder  $i$ 's expected payoff matches her first signal,  $s_i$ , times the probability that bidder  $j$  does not participate when bidder  $j$  plays a cutoff equal to  $s_i$ ,  $F_j(s_i)$ . The second term includes the cases where bidder  $j$  participates; i.e.,  $v_j > s_i$ . For each realization of  $j$ 's signal, bidder  $i$ 's expected payoff is given by equation (3.18). The expression integrates over every possible realization of  $v_j$ .

We say that bidder  $i$  is stronger than  $j$  if  $s_i < s_j$ . It can be readily verified that the left-hand side of (3.19) is strictly increasing and unbounded above; i.e., strength is well defined. As before, strength elicits a bidder's ability to endure competition by computing the symmetric strategy that makes each bidder indifferent between participating and not. A bidder with a lower value of strength  $s_i$  is willing to participate at a lower valuation. Without loss of generality, let bidder 1 be the strongest bidder of the game. The following proposition characterizes equilibrium in general selective entry models.

**Proposition 8** (Partially informed bidders). *There always exists an herculean equilibrium, which is characterized by cutoffs  $x_1 \leq x_2$  that for bidder  $i$  solves*

$$x_i F_j(x_j) + \int_{x_j}^{\infty} \pi_i(x_i, y) dF_j(y) = c_i.$$

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16. Recall that in the partially-informed-bidder model  $v_i$  is a signal and  $V_i = v_i \varepsilon_i$  is the valuation.

A herculean equilibrium is the unique equilibrium of the game if for both bidders

$$F_i(v) \geq v f_i(v) \text{ for all } v > \min\{c_1, c_2\}. \quad (3.20)$$

In this case, the weak concavity of the signal  $v_i$  distribution is also a sufficient condition for equilibrium uniqueness. Although the distributions of noise  $(\Phi_i)$  do not show up in our sufficient condition, they may still affect the existence of multiplicity of equilibria in scenarios in which the distribution of signals  $(F_i)$  is *not* concave. Finally, observe that condition (3.20) is a bit stronger than (3.7). This is due to the non-linear relation that now exists between  $v_i$  and  $F_j$  when the other bidder participates.

### 3.7 Concluding Remarks

In this article we generalized existing results about second-price auctions with participation costs by allowing heterogeneity both in distributions of valuations and in participation costs. We developed the concept of strength, which uses bidders' public characteristics—here, distributions of valuations and participation costs—to rank bidders according to their ability to endure competition. We showed that an equilibrium with cutoffs ordered by strength—called herculean equilibrium—exists in situations of applied interest. Moreover, when the distributions of valuations are concave, we showed that the herculean equilibrium is the unique equilibrium of the game. Because there is always an *ex-ante* efficient equilibrium, when the conditions for uniqueness hold, the herculean equilibrium is both *ex-ante* efficient and the unique equilibrium

of the game.

We believe that the methodology developed here can be extended to study second-price auctions with more general environments such as interdependent or affiliated values. Our methodology can also be extended to auction settings in which the auction designer creates endogenous heterogeneity, including when a bid handicap is imposed on a subset of bidders during the bidding stage (*e.g.*, bid preference programs for entrants). The tools developed in this article can be applied to estimate optimal participation fees and to compare revenue from participation fees and reserve prices. We regard such models as promising avenues for future research.

### 3.A Appendix: Omitted Proofs

This section presents the proofs omitted from the main text.

**Proof of Lemma 1.** Pick any  $\tau_{-i}$ . Because  $i$ 's utility is linear in  $\tau_i$ , it is a best response to participate with probability 1 whenever there is a positive payoff of doing so. Hence, it is sufficient to show that, conditional on bidder  $i$  participating in the auction ( $\tau_i(v) = 1$ ),  $i$ 's utility crosses zero at a singleton point and from below. Differentiating  $u_i(\tau_i = 1, \tau_{-i}, v) = [vG_i(v) - \int_0^v x dG_i(x) - c_i]$  with respect to  $v$  we obtain that  $du_i/dv = G_i(v) > 0$  for all  $v > 0$ , which implies that  $i$ 's utility is strictly increasing in  $v$ . By the finite expectation assumption on  $F_i$ ,  $u_i$  is unbounded above in  $v$ . Therefore, because  $u_i(\tau_i = 1, \tau_{-i}, 0) < 0$ , there exist a unique best response which is given by the unique value of  $v$  that solves  $u_i(\tau_i = 1, \tau_{-i}, v) = 0$ . ■

**Proof of Lemma 2.** Let  $\mathbf{x} = (x_1, x_2, \dots, x_n)$  be a profile of cutoff strategies. Denote bidder  $i$ 's expected utility of participating in the auction when her valuation is  $v$ , and the opponents play the cutoffs  $\mathbf{x}_{-i}$  by  $u_i(0, \mathbf{x}_{-i}, v)$ . Lemma 1 shows that bidder  $i$ 's best response to  $\mathbf{x}_{-i}$  is given by the unique valuation  $x_i$  satisfying  $u_i(0, \mathbf{x}_{-i}, x_i) = 0$ . In particular, using equation (3.1) when every opponent uses a cutoff strategy,  $u_i(0, \mathbf{x}_{-i}, x_i) = 0$  is equivalent to equation (3.3) which proves the lemma. ■

**Proof of Proposition 1.** We establish that the conditions of Brouwer fixed-point theorem are met. Let  $\mathbf{x} = (x_1, x_2, \dots, x_n)$  be a collection of cutoffs. By Lemma 1, bidder  $i$ 's best response to the profile of strategies  $\mathbf{x}_{-i}$  is given by the unique valuation  $v$  that solves  $u_i(0, \mathbf{x}_{-i}, v) = 0$ . Since  $F_i$  is atomless and has full support, bidder  $i$ 's best response is continuous in each of the opponent cutoffs. Moreover, since  $u_i(0, \mathbf{x}_{-i}, v)$  is increasing in the opponents' cutoffs, the lowest utility for bidder  $i$  is achieved when each opponent participates with certainty (i.e.,  $\mathbf{x}_{-i} = \mathbf{0}_{-i}$ ). Let  $K_i$  be valuation of bidder  $i$  that satisfies  $u_i(\mathbf{0}, K_i) = 0$ . Hence, the vector of best responses is a continuous mapping from the compact and convex set  $\times_{i=1}^n [0, K_i]$  to itself and all conditions of Brouwer Fixed Point Theorem are met, proving existence of equilibrium. ■

**Proof of Proposition 2.** Let  $\mathbf{x} = (x_1, x_2, \dots, x_n)$  where, without loss of generality, we order the bidders identities from the lowest cutoff chosen by the planner,  $x_1$ , to the highest,  $x_n$ . Differentiating (3.4) with respect to  $x_i$  we obtain

$$W_{x_i}(\mathbf{x}) = f_i(x_i)(c_i - x_i \Omega_i(\mathbf{x})) + \sum_{k \neq i} \int_{x_k}^{\infty} s \left( \frac{d\Omega_k(s, \mathbf{x}_{-k})}{dx_i} \right) dF_k(s).$$

Observing that  $d\Omega_k(v, \mathbf{x}_{-k})/dx_i = f_i(x_i) \prod_{\ell \neq k, i} F_\ell(\max\{v, x_\ell\})$  if  $v \leq x_i$  and zero otherwise, we can write

$$W_{x_i}(\mathbf{x}) = -f_i(x_i) \left( x_i \Omega_i(\mathbf{x}) - \sum_{k=1}^{i-1} \left\{ \int_{x_k}^{x_i} s \prod_{\ell \neq k, i} F_\ell(\max\{s, x_\ell\}) dF_k(s) \right\} - c_i \right). \quad (3.21)$$

Corner solutions are not welfare maximizing as, when we take  $x_i = 0$ ,  $W_{x_i}(0, \mathbf{x}_{-i}) > 0$  for all  $\mathbf{x}_{-i}$ ; and  $\lim_{x_i \rightarrow \infty} W_{x_i}(x_i, \mathbf{x}_{-i}) < 0$  due to the unboundedness of  $x_i \Omega_i(\mathbf{x})$ . Therefore, an interior maximum exists, which is characterized by a value of  $x_i$  satisfying  $W_{x_i}(\mathbf{x}) = 0$ . The term inside the parenthesis in equation (3.21) is equal to zero whenever condition (3.3) holds.<sup>17</sup> Therefore, we conclude that there exists a cutoff equilibrium that is *ex-ante* efficient. Moreover, since every equilibrium satisfies  $W_{x_i} = 0$ , they are a critical point of  $W$ . Finally, because  $W_{x_i, x_i}(\mathbf{x}) = -f_i(x_i) \Omega_i(\mathbf{x}) < 0$ , the critical point cannot be a minimum.<sup>18</sup> Thus, every equilibria is either a local maximal or a saddle point. ■

**Proof of Lemma 3.** Both situations are particular cases of the proof of Lemma 5. ■

**Proof of Proposition 3.** See Proposition 4 in the case  $m_1 = m_2 = 1$ . ■

17. To see this observe that  $x_i \Omega_i(\mathbf{x}) = x_i B_i(x_i) A_i^n$ , that, for a given  $k$ ,

$$\int_{x_k}^{x_i} s \prod_{\ell \neq k, i} F_\ell(\max\{s, x_\ell\}) dF_k(s) = A_i^n A_k^{i-1} \int_{x_k}^{x_i} s \prod_{\ell < k} F_\ell(s) dF_k(s)$$

and  $dB_{k+1}(s) = \sum_{j=1}^k dF_j(s) \prod_{\ell=1, \ell \neq j}^k F_\ell(s)$ . Then, re-arrange the summation in (3.21) so that the limits of the integral are consecutive cutoffs (i.e., from  $x_k$  to  $x_{k+1}$ ) instead of  $x_k$  to  $x_i$ .

18. We would like to thank an anonymous referee for pointing this out.



**Proof of Lemma 4.** The proof of both statements make use that a concave differentiable function is bounded above by its first-order Taylor approximation; i.e., for every  $x$  and  $y$  such that  $x > y$

$$F(x) - F(y) \geq (x - y)f(x). \quad (3.22)$$

The first claim follows, from taking  $y = 0$  and using  $F(0) = 0$ . For the second statement, let  $y$  in (3.22) be inflection point under which  $F_i(v)$  becomes concave. Because of concavity,  $f_i(x)$  is non increasing for every  $x \geq y$ . Because  $F_i$  is bounded above (by 1),  $f_i(x)$  converges to zero as  $x$  goes to infinity. Thus, the cost  $c_j > y$  such that condition (3.7) holds is implicitly given by  $F_i(y) = f_i(c_j)y$ . Then for every  $x \geq c_j$  we have:  $F_i(x) \geq xf_i(x) + F_i(y) - yf_i(x) \geq xf_i(x)$ . ■

**Proof of Proposition 4.** Begin by observing that equations (3.8) and (3.9) define an equilibrium as they correspond to equation (3.3) for the case in which bidders play symmetrically within group.

*Existence.* By construction. If  $s_1 = s_2 = s$  there is a herculean equilibrium with cutoffs  $x_1 = x_2 = s$ . Assume  $s_1 < s_2$ , let  $g(x)$  the function implicitly defined by

$$g(x)F_1(g(x))^{m_1-1}F_2(x)^{m_2} = c_1.$$

The function  $g(x) > c_1$  and represents the cutoff that bidders in group 1 have to play so that condition (3.8) is satisfied when everyone in group 2 plays the cutoff  $x_2 = x$ . Observe that  $g(x)$  is strictly decreasing in  $x$  and satisfies  $g(s_1) = s_1$ . Define

the function  $h : [s_1, \infty) \rightarrow \mathbb{R}$  by

$$h(x) = F_2(x)^{m_2-1} \left[ xF_1(x)^{m_1} - \int_{g(x)}^x yd(F_1(y)^{m_1}) \right] - c_2.$$

The function  $h(x)$  is continuous and corresponds to the payoffs that a member of group 2 obtains by playing the cutoff  $x_2 = x$  when all other members of group 2 play  $x$  and all members of group 1 respond by playing  $x_1 = g(x)$ . A herculean equilibrium exists if there is  $x^*$  such that  $h(x^*) = 0$  and  $x^* > g(x^*)$ . The next two claims prove the result.

**Claim 1.**  $x^* \in (s_1, \infty)$  is necessary and sufficient for  $x_1 < x_2$ .

*Proof.* Because  $g(x)$  is weakly decreasing in  $x$  and  $g(s_1) = s_1$ ,  $x_1 = g(x^*) < x^* = x_2$  if and only if  $x^* \in (s_1, \infty)$ . □

**Claim 2.**  $h(s_1) < 0$  and  $h(x)$  is unbounded above.

*Proof.* Group 2 being weak (i.e.,  $s_1 < s_2$ ) implies

$$h(s_1) = s_1 F_1(s_1)^{m_1} F_2(s_1)^{m_2-1} - c_2 < s_2 F_1(s_2)^{m_1} F_2(s_2)^{m_2-1} - c_2 = 0.$$

On the other hand,  $h(x)$  is unbounded above as  $xF_1(x)^{m_1}F_2(x)^{m_2-1}$  is unbounded and the finite expectation assumption. □

By the intermediate value theorem, Claim 2 plus continuity imply that there exists  $x^* \in (s_1, \infty)$  such that  $h(x^*) = 0$ . On the other hand,  $h(x^*) = 0$  holds if and only if equations (3.8) and (3.9) are satisfied. Therefore, by Claim 1, we have a herculean equilibrium with  $x_1 = g(x^*)$  and  $x_2 = x^*$ .

*Uniqueness.* From Lemma 9 in the Auxiliary Results section of the appendix we know that, under condition (3.10), symmetric bidders must play symmetric cutoffs. We need to show that there is no other herculean equilibrium, and that no non-herculean equilibria exists.

**Claim 3.** There exists a unique herculean equilibrium.

*Proof.* In a herculean equilibrium bidders are ordered by strength, thus we have to show there is no other equilibrium such that  $x_1 < x_2$  and equations (3.8) and (3.9) hold; i.e., there exists a unique  $x^* > s_1$  such that  $h(x^*) = 0$ . It is sufficient to show that  $h'(x) > 0$  for all  $x \geq s_1$ , so that  $h(x)$  single-crosses zero from below.

Differentiating

$$h'(x) = F_2(x)^{m_2-1} \left\{ (m_2 - 1) \frac{f_2(x)}{F_2(x)} \left[ xF_1(x)^{m_1} - \int_{g(x)}^x y d(F_1(y)^{m_1}) \right] + F_1(x)^{m_1} + m_1 g'(x) g(x) f_1(g(x)) F_1(g(x))^{m_1-1} \right\}.$$

Because  $F_2(x)^{m_2-1} > 0$ , it is sufficient to show that the term in braces is non-negative for all  $x \geq s_1$ . Implicitly differentiating  $g(x)$

$$g'(x) = - \frac{m_2 g(x) F_1(g(x))}{F_1(g(x)) + (m_1 - 1) g(x) f_1(g(x))} \frac{f_2(x)}{F_2(x)}$$

replacing into the expression in braces delivers

$$(m_2 - 1) \frac{f_2(x)}{F_2(x)} \left[ xF_1(x)^{m_1} - \int_{g(x)}^x y d(F_1(y)^{m_1}) \right] + \left[ F_1(x)^{m_1} - \frac{m_1 m_2 g(x)^2 f_1(g(x)) F_1(g(x))^{m_1}}{F_1(g(x)) + (m_1 - 1)g(x)f_1(g(x))} \frac{f_2(x)}{F_2(x)} \right]. \quad (3.23)$$

It is shown that a lower bound for the expression above is always positive. Maximize the subtracting term in the first square brackets by taking the upper bound  $x \int_{g(x)}^x dF_1(y)^{m_1}$  in the integral. Using condition (3.10), maximize the subtracting term in the second square brackets by substituting  $F_1(g(x))$  for  $g(x)f_1(g(x))$  in the denominator (recall that  $g(x) > c_1$ ). Then, equation (3.23) becomes

$$F_1(x)^{m_1} + [(m_2 - 1)x - m_2 g(x)] F_1(g(x))^{m_1} \frac{f_2(x)}{F_2(x)} \geq F_1(x)^{m_1} \left( 1 - \frac{g(x)}{x} \right)$$

where  $x \geq g(x)$  for  $x \geq s_1$ , and  $f_2(x)/F_2(x) \leq x^{-1}$  were used to obtain the inequality. Hence the lower bound of (3.23) is non-negative if and only if  $x \geq g(x)$ , which is true as  $x \geq s_1$ . □

**Claim 4.** There is no equilibrium in which strong bidders play a higher cutoff than weak bidders.

*Proof.* To prove that the only equilibrium is the herculean, suppose we have a non-herculean equilibrium. By Lemma 9, in the Auxiliary Results section, symmetric bidders must play symmetric cutoffs under condition (3.10). Thus, the only possibility

is to have  $x_1 > x_2$  but  $s_1 < s_2$ . Define  $\bar{g}(x)$  to be the function that satisfies

$$\bar{g}(x)F_2(\bar{g}(x))^{m_2-1}F_1(x)^{m_1} = c_2.$$

Similarly, define

$$\bar{h}(x) = F_1(x)^{m_1-1} \left[ xF_2(x)^{m_2} - \int_{\bar{g}(x)}^x yd(F_2(y)^{m_2}) \right] - c_1.$$

The function  $\bar{g}(x)$  is decreasing in  $x$ , satisfies  $\bar{g}(s_2) = s_2$ , and represents the cutoff that group 2 has to play so that condition (3.8) is satisfied when everyone in group 1 plays the cutoff  $x_1 = x$ . The continuous function  $\bar{h}(x)$  corresponds to the payoffs that a member of group 1 obtains by playing the cutoff  $x_1 = x$  when all other members of group 1 play  $x$  and all members of group 2 respond by playing  $x_2 = \bar{g}(x)$ . We show that there is no  $x$  such that  $x_1 = x > \bar{g}(x) = x_2$  and  $\bar{h}(x) = 0$ , which implies that condition (3.9) does not hold and no non-herculean equilibrium exists.

Observe that  $x > \bar{g}(x)$  if and only if  $x \in (s_2, \infty)$  and that  $s_1 < s_2$  implies that

$$\bar{h}(s_2) = s_2F_1(s_2)^{m_1-1}F_2(s_2)^{m_2} - c_1 > s_1F_1(s_1)^{m_1-1}F_2(s_1)^{m_2} - c_1 = 0.$$

By an analogous argument given in Claim 3, condition (3.10) implies  $\bar{h}'(x) > 0$ , and  $\bar{h}(x) > 0$  for all  $x \in (s_2, \infty)$ . Therefore, there is no  $x > s_2$  such that  $\bar{h}(x) = 0$  and, by Lemma 2, no non-herculean equilibrium exists. □ ■

**Proof of Lemma 5.** By definition of  $i$ 's strength  $s_i \prod_{j \neq i} F_j(s_i) = c_i$ . Equation (3.12) implies  $c_{i+1} F_{i+1}(s_i) / F_i(s_i) > c_i$ . Substituting for  $c_i$  on the RHS of  $i$ 's strength and rearranging:  $s_i \prod_{j \neq i+1} F_j(s_i) < c_{i+1}$ . Since the LHS is increasing in  $s$ ,  $s_{i+1} > s_i$ . ■

**Proof of Proposition 5.** *Existence.* For a given vector  $\mathbf{v} = (v_1, \dots, v_n)$ , and following equation (3.2), define the family of functions  $h_i^n(\mathbf{v}) = A_i^n r_i(\mathbf{v}^i) - c_i$ .<sup>19</sup> This family of functions will be used in the proof of existence and uniqueness. Start by ordering bidders by strength, with bidder 1 being the strongest and  $n$  the weakest. By Lemma 2 a herculean equilibrium  $\mathbf{x} = (x_1, \dots, x_n)$  exists if and only if  $h_i^n(\mathbf{v}) = 0$  for all  $i$ . We construct  $\mathbf{x}$  recursively. Let  $\tilde{\mathbf{v}}^i = (v_i, \dots, v_n)$  represent the elements of  $\mathbf{v}$  in the  $i$ th and higher positions.

Start constructing  $x_1$ . For any vector  $\tilde{\mathbf{v}}^2$ , define  $x_1(\tilde{\mathbf{v}}^2)$  to be the value of  $v_1$  that solves  $h_1^n(v_1, \tilde{\mathbf{v}}^2) = 0$ ; i.e.,  $x_1(\tilde{\mathbf{v}}^2) = c_1 / A_1^n$ . We now construct  $x_2$  recursively, by using the constructed  $x_1(\tilde{\mathbf{v}}^2)$ . By substituting  $x_1(\tilde{\mathbf{v}}^2)$  in for the values of  $v_1$  in  $h_2^n(\mathbf{v})$ , we can write  $h_2^n(\mathbf{v})$  as a function of  $\tilde{\mathbf{v}}^2$  only. That is,  $h_2^n(\tilde{\mathbf{v}}^2) = A_2^n r_2(\tilde{\mathbf{v}}^2) - c_2$  where

$$r_2(\tilde{\mathbf{v}}^2) \equiv r_2(x_1(\tilde{\mathbf{v}}^2), v_2) = v_2 F_1(v_2) - \int_{c_1/A_1^n}^{v_2} x dF_1(x)$$

is the revenue function  $r_2(\mathbf{v}^2)$  after replacing the function  $x_1(\tilde{\mathbf{v}}^2)$  for the value of  $v_1$ .

The finite expectation assumption implies that  $h_2^n(v_2, \tilde{\mathbf{v}}^3)$  is unbounded above in  $v_2$ .

Define  $\hat{v}_2$  to be the *largest* value of  $v_2$  that satisfies  $\hat{v}_2 = x_1(\hat{v}_2, \tilde{\mathbf{v}}^3)$ . Observe that  $\hat{v}_2$

always exists, as  $v_2 \in \mathbb{R}_+$  and  $x_1(v_2, \tilde{\mathbf{v}}^3)$  is a continuous function of  $v_2$  with range in

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19. Recall that, for a given  $\mathbf{v}$ ,  $\mathbf{v}^i = (v_1, v_2, \dots, v_i)$ ,  $A_i^n = \prod_{j>i} F_j(v_j)$  and  $r_i(\mathbf{v}^i)$  is given by (3.2).

$(c_1, \bar{v}_1)$ . Also, for every  $v_2 > \hat{v}_2$ ,  $v_2 > x_1(v_2, \tilde{\mathbf{v}}^3)$ . Otherwise,  $v_2$  and  $x_1(v_2, \tilde{\mathbf{v}}^3)$  would cross again and  $\hat{v}_2$  wasn't the largest point in which they cross.

Using  $\hat{v}_2 = x_1(\hat{v}_2, \tilde{\mathbf{v}}^3) = c_1/(F_2(\hat{v}_2)A_2^n)$ , we find  $h_2^n(\hat{v}_2, \tilde{\mathbf{v}}^3) = c_1F_1(\hat{v}_2)/F_2(\hat{v}_2) - c_2$ . If the bidders are equally strong; i.e., condition (3.12) holds with equality,  $h_2^n(\hat{v}_2, \tilde{\mathbf{v}}^3) = 0$ . Then, we can define  $x_2(\tilde{\mathbf{v}}^3) = \hat{v}_2$ . If bidder 2 is strictly weaker, condition (3.12) implies,  $h_2^n(\hat{v}_2, \tilde{\mathbf{v}}^3) < 0$ . Thus, by the intermediate value theorem, there exists  $x_2(\tilde{\mathbf{v}}^3) > \hat{v}_2$  such that  $h_2^n(x_2(\tilde{\mathbf{v}}^3), \tilde{\mathbf{v}}^3) = 0$ . Because  $x_2(\tilde{\mathbf{v}}^3) > \hat{v}_2$ , we have  $x_2(\tilde{\mathbf{v}}^3) > x_1(x_2(\tilde{\mathbf{v}}^3), \tilde{\mathbf{v}}^3)$ . Observe that by replacing  $v_2 = x_2(\tilde{\mathbf{v}}^3)$  into  $x_1(\tilde{\mathbf{v}}^2)$ , we have written both  $x_1$  and  $x_2$  as functions of  $\tilde{\mathbf{v}}^3$ . Finally, by construction, the order between  $x_1(\tilde{\mathbf{v}}^3)$  and  $x_2(\tilde{\mathbf{v}}^3)$  is robust to any values of  $\tilde{\mathbf{v}}^3$ , implying that the order will not reverse when constructing cutoffs for weaker firms (though, the actual values of  $x_1$  and  $x_2$  do change when we change  $\tilde{\mathbf{v}}^3$ ).

Suppose we have shown that, for any vector  $\tilde{\mathbf{v}}^i$ ,  $x_1(\tilde{\mathbf{v}}^i) \leq x_2(\tilde{\mathbf{v}}^i) \leq \dots \leq x_{i-1}(\tilde{\mathbf{v}}^i)$  (strict whenever  $s_{k-1} < s_k$ ). For each  $k \leq i$ ,  $x_k(\tilde{\mathbf{v}}^i)$  has been recursively constructed by finding a value  $v_k$  solving  $h_k^n(v_k, \tilde{\mathbf{v}}^{k+1}) = 0$ —which gives us  $x_k(\tilde{\mathbf{v}}^{k+1})$ —and, for every  $j \in \{k+1, \dots, i-1\}$ , by replacing the solution of higher cutoffs  $x_j(\tilde{\mathbf{v}}^{j+1})$  into  $x_k(\tilde{\mathbf{v}}^{k+1})$ . We show that there exists  $x_i(\tilde{\mathbf{v}}^{i+1}) \geq x_{i-1}(x_i(\tilde{\mathbf{v}}^{i+1}), \tilde{\mathbf{v}}^{i+1})$  (strict if  $s_{i-1} < s_i$ ) solving  $h_i(x_i(\tilde{\mathbf{v}}^{i+1}), \tilde{\mathbf{v}}^{i+1}) = 0$ . Notice that  $h_{i-1}^n(x_{i-1}, \tilde{\mathbf{v}}^i) = 0$  implies  $r_{i-1}(x_{i-1}, \tilde{\mathbf{v}}^i) = c_{i-1}/A_{i-1}^n$ . Substituting the vector of solutions  $x^{i-1}$  we can write  $h_i^n(\mathbf{v})$  as  $h_i^n(\tilde{\mathbf{v}}^i) = A_i^n r_i(\tilde{\mathbf{v}}^i) - c_i$ . Because of the finite expectation assumption,  $h_i^n(\tilde{\mathbf{v}}^i)$  is unbounded above in  $v_i$ . Take  $\hat{v}_i$  to be the largest value of  $v_i$  that satisfies  $\hat{v}_i = x_{i-1}(\hat{v}_i, \tilde{\mathbf{v}}^{i+1})$ . This value exists by the same argument given to find  $\hat{v}_2$  and it also satisfies  $v_i > x_{i-1}(v_i, \tilde{\mathbf{v}}^{i+1})$  for  $v_i > \hat{v}_i$ . Using  $\hat{v}_i = x_{i-1}(\hat{v}_i, \tilde{\mathbf{v}}^{i+1})$  and Lemma 8.2 (see

the Auxiliary Result section) we know<sup>20</sup>

$$r_i(\hat{v}_i, \tilde{\mathbf{v}}^{i+1}) = F_{i-1}(x_{i-1}(\hat{v}_i, \tilde{\mathbf{v}}^{i+1}))r_{i-1}(x_{i-1}(\hat{v}_i, \tilde{\mathbf{v}}^{i+1}), \hat{v}_i, \tilde{\mathbf{v}}^{i+1}).$$

Then, using the property  $r_{i-1}(x_{i-1}(\tilde{\mathbf{v}}^i), \tilde{\mathbf{v}}^i) = c_{i-1}/A_{i-1}^n$  and  $\hat{v}_i = x_{i-1}(\hat{v}_i, \tilde{\mathbf{v}}^{i+1})$ , we can write  $h_i^n(\hat{v}_i, \tilde{\mathbf{v}}^{i+1}) = c_{i-1}F_{i-1}(\hat{v}_i)/F_i(\hat{v}_i) - c_i$ . If bidders  $i-1$  and  $i$  are equally strong,  $h_i^n(\hat{v}_i, \tilde{\mathbf{v}}^{i+1}) = 0$  by condition (3.12) and we can define  $x_i(\tilde{\mathbf{v}}^{i+1}) = \hat{v}_i$ . If bidder  $i$  is strictly weaker than  $i-1$ , condition (3.12) implies  $h_i^n(\hat{v}_i, \tilde{\mathbf{v}}^{i+1}) < 0$ . Then, by the intermediate value theorem, there exists  $x_i(\tilde{\mathbf{v}}^{i+1}) > \hat{v}_i$  such that  $h_i^n(x_i(\tilde{\mathbf{v}}^{i+1}), \tilde{\mathbf{v}}^{i+1}) = 0$ . Finally, because  $x_i(\tilde{\mathbf{v}}^{i+1}) > \hat{v}_i$ , we have  $x_i(\tilde{\mathbf{v}}^{i+1}) > x_{i-1}(x_i(\tilde{\mathbf{v}}^{i+1}), \tilde{\mathbf{v}}^{i+1})$ . Once again, the order between the cutoffs will be independent of the construction of cutoffs above.  $\square$

*Uniqueness: Preliminaries.* This proof uses induction. We begin by outlining the main argument. We order bidders from strongest to weakest. Define  $H_k^n : \mathbb{R}^n \rightarrow \mathbb{R}^k$  to be the function equal to  $h_i^n(\mathbf{v})$  (defined in the existence proof above) in the  $i^{\text{th}} \leq k$  dimension. Fix a value  $k$ , by the existence proof we know there exists recursively defined functions  $\mathbf{x}^k : \mathbb{R}^{n-k} \rightarrow \mathbb{R}^k$  satisfying  $H_k^n(\mathbf{x}^k(\tilde{\mathbf{v}}^{k+1}), \tilde{\mathbf{v}}^{k+1}) = 0$ . For any  $i \leq k$ ,

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20. The equation above uses the recursion notation. The formulation from the lemma is

$$r_i(x^{i-2}, x_{i-1}(\hat{v}_i, \tilde{\mathbf{v}}^{i+1}), \hat{v}_i) = F_{i-1}(x_{i-1}(\hat{v}_i, \tilde{\mathbf{v}}^{i+1}))r_{i-1}(x^{i-2}, x_{i-1}(\hat{v}_i, \tilde{\mathbf{v}}^{i+1})).$$



the total differential of  $h_i^n(x^k(\tilde{\mathbf{v}}^{k+1}), \tilde{\mathbf{v}}^{k+1})$  with respect to  $v_j, j > k$ , is:

$$A_i^n \left[ \sum_{s=1}^{i-1} A_s^{i-1} r_s(\mathbf{x}^s) f_s(x_s) \frac{dx_s}{dv_j} + B_i(x_i) \frac{dx_i}{dv_j} + r_i(\mathbf{x}^i) \left( \sum_{s>i}^k \frac{f_s(x_s)}{F_s(x_s)} \frac{dx_s}{dv_j} + \frac{f_j(v_j)}{F_j(v_j)} \right) \right]. \quad (3.24)$$

Using this equation and the implicit function theorem, we can write the vector of derivatives  $d\mathbf{x}^k(\tilde{\mathbf{v}}^{k+1})/dv_{k+1}$  as the solution to the following system of linear equations:

$$A_i^n \left[ M_k D_k + R_k \frac{f_{k+1}(v_{k+1})}{F_{k+1}(v_{k+1})} \right] = 0, \quad (3.25)$$

where ( $T$  denotes transpose):

$$D_k = \left( \frac{dx_1}{dx_{k+1}}, \frac{dx_2}{dx_{k+1}}, \dots, \frac{dx_k}{dx_{k+1}} \right)^T, \quad R_k = (r_1(x_1), r_2(\mathbf{x}^2), \dots, r_k(\mathbf{x}^k))^T$$

and

$$M_k = \begin{pmatrix} B_1(x_1) & r_1(x_1) \frac{f_2(x_2)}{F_2(x_2)} & r_1(x_1) \frac{f_3(x_3)}{F_3(x_3)} & \cdots & r_1(x_1) \frac{f_k(x_k)}{F_k(x_k)} \\ A_1^1 r_1(x_1) f_1(x_1) & B_2(x_2) & r_2(\mathbf{x}^2) \frac{f_3(x_3)}{F_3(x_3)} & \cdots & r_2(\mathbf{x}^2) \frac{f_k(x_k)}{F_k(x_k)} \\ A_1^2 r_1(x_1) f_1(x_1) & A_2^2 r_2(\mathbf{x}^2) f_2(x_2) & B_3(x_3) & \cdots & r_3(\mathbf{x}^3) \frac{f_k(x_k)}{F_k(x_k)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ A_1^{k-1} r_1(x_1) f_1(x_1) & A_2^{k-1} r_2(\mathbf{x}^2) f_2(x_2) & A_3^{k-1} r_3(\mathbf{x}^3) f_3(x_3) & \cdots & B_k(x_k) \end{pmatrix}.$$

If  $M_k$  is invertible, the the solution to (3.25) is given by:

$$D_k = -M_k^{-1} R_k f_{k+1}(v_{k+1}) / F_{k+1}(v_{k+1}). \quad (3.26)$$

We will show that  $M_k$  is invertible. Then, using (3.26), we show that  $dh_k^k(v_k)/dv_k > 0$ . This implies that  $h_k^k(v_k)$  single crosses zero, and  $x_k$  is uniquely defined. In words, in a game with  $n = k$  bidders, in which every cutoff  $\mathbf{x}^{k-1}(v_k)$  has been recursively constructed as a function of  $v_k$ , bidder  $k$  has a unique best response. Furthermore, we will show that  $dh_k^k(v_k)/dv_k > 0$  also implies  $dh_k^n(v_k, \tilde{\mathbf{v}}^{k+1})/dv_k > 0$  for any  $n > k$  and any vector  $\tilde{\mathbf{v}}^{k+1}$ . This implies that in every game with  $n > k$  bidders, for any  $\tilde{\mathbf{v}}^{k+1}$ , bidder  $k$  has a unique best response.<sup>21</sup> Then, by the induction argument, each step of the construction  $x_k(\tilde{\mathbf{v}}^k)$  is uniquely defined and the herculean equilibrium is unique.

**Claim 5.** There exists a unique herculean equilibrium.

*Proof.* Fix a step  $k$  and assume there is  $n \geq k + 1$  bidders. For that given  $n$ , let  $\mathbf{x}^k(\tilde{\mathbf{v}}^{k+1})$  to be the vector of functions constructed until step  $k$  in the recursion in the existence proof above. For any positive vector  $\tilde{\mathbf{v}}^{k+2}$ , we need to show that there is a unique value of  $x_{k+1}$  that solves  $h_{k+1}^n(x_{k+1}, \tilde{\mathbf{v}}^{k+2}) = 0$ . In particular, we show  $dh_{k+1}^n(v_{k+1}, \tilde{\mathbf{v}}^{k+2})/dv_{k+1} > 0$ , so that  $h_{k+1}^n(v_{k+1}, \tilde{\mathbf{v}}^{k+2})$  single crosses zero from below.

Using (3.24),

$$\frac{dh_{k+1}^n(v_{k+1}, \tilde{\mathbf{v}}^{k+2})}{dv_{k+1}} = A_{k+1}^n(d_k D_k + B_{k+1}(v_{k+1}))$$

where  $d_k = (A_1^k r_1(x_1) f_1(x_1), A_2^k r_2(\mathbf{x}^2) f_2(x_2), \dots, A_k^k r_k(\mathbf{x}^k) f_k(x_k))$ . Using (3.26), if

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21. Notice that this does not imply that for each  $n > k$ , bidder  $k$ 's best response function is the same across different  $n$ .

$M_k$  is invertible we can write  $D_k = -M_k^{-1}R_k f_{k+1}(v_{k+1})/F_{k+1}(v_{k+1})$  and

$$\frac{dh_{k+1}^n(v_{k+1}, \tilde{\mathbf{v}}^{k+2})}{dv_{k+1}} = A_{k+1}^n \left( B_{k+1}(v_{k+1}) - q_k \frac{f_{k+1}(v_{k+1})}{F_{k+1}(v_{k+1})} \right)$$

where  $q_k = d_k M_k^{-1} R_k$ . Because  $A_{k+1}^n > 0$  for all  $n \geq k+1$ , it is sufficient to show that the parenthesis (which corresponds to  $dh_{k+1}^{k+1}(v_{k+1})/dv_{k+1}$ ) is positive for all relevant values of  $v_{k+1}$ . We show the previous statement and the invertibility of  $M_k$  by induction.

Observe  $h_1^n(\mathbf{v}) = A_1^n v_1$ , thus  $dh_1^n(\mathbf{v})/dv_1 > 0$  and bidder 1 has a unique best response for any  $n \geq 1$  (given by  $x_1(\tilde{\mathbf{v}}^2) = c_1/A_1^n$ ). For bidder 2, observe  $M_1 = B_1(x_1) = 1$  is invertible and  $q_1 = (x_1)^2 f_1(x_1)$  is well defined. Then,  $B_2(v_2) - q_1 f_2(v_2)/F_2(v_2) = F_1(v_2) - (x_1)^2 f_1(x_1) f_2(v_2)/F_2(v_2)$ . Using condition (3.13) twice,  $x_1 F_1(x_1)/v_2$  is an upper bound for the subtracting term. Since, by construction, we are interested in  $v_2 \geq x_1$ ,  $B_2(v_2) - q_1 f_2(v_2)/F_2(v_2) > 0$ .

Suppose we have shown that  $M_{j-1}$  is invertible and  $B_j(x_j) - q_{j-1} f_j(x_j)/F_j(x_j) > 0$  for all  $j \leq k$ . Let  $l_k = (B_k(x_k) - q_{k-1} f_k(x_k)/F_k(x_k))^{-1}$  and observe that  $l_k > 0$  by induction hypothesis; then, by the definition of  $M_k$  and using blockwise inversion,

$$M_k = \begin{pmatrix} M_{k-1} & R_{k-1} \frac{f_k(x_k)}{F_k(x_k)} \\ d_{k-1} & B_k(x_k) \end{pmatrix} \text{ and } M_k^{-1} = \begin{pmatrix} O & -\frac{f_k(x_k)}{F_k(x_k)} P_k (M_{k-1}^{-1} R_{k-1}) \\ -l_k (d_{k-1} M_{k-1}^{-1}) & l_k \end{pmatrix}$$

where  $O = M_{k-1}^{-1} + \frac{f_k(x_k)}{F_k(x_k)} l_k (M_{k-1}^{-1} R_{k-1} d_{k-1} M_{k-1}^{-1})$ , and the inverse of  $M_k$  is well defined.

We need to show  $B_{k+1}(v_{k+1}) - q_k f_{k+1}(v_{k+1})/F_{k+1}(v_{k+1}) > 0$ . Observing that  $R_k = (R_{k-1}, r_k(\mathbf{x}^k))^T$ ,  $d_k = (d_{k-1} F_k(x_k), r_k(\mathbf{x}^k) f_k(x_k))$ , and using the definition of  $M_k^{-1}$  and

$l_k$  we can write:

$$q_k = F_k(x_k)q_{k-1} + f_k(x_k) (r_k(\mathbf{x}^k) - q_{k-1})^2 / (B_k(x_k) - q_{k-1}f_k(x_k)/F_k(x_k)), \quad (3.27)$$

Thus,  $B_{k+1}(v_{k+1}) - q_k \frac{f_{k+1}(v_{k+1})}{F_{k+1}(v_{k+1})} > 0$  is equivalent to show:

$$\left( B_k(v_{k+1}) \frac{F_k(v_{k+1})F_{k+1}(v_{k+1})}{f_k(x_k)f_{k+1}(v_{k+1})} - q_{k-1} \frac{F_k(x_k)}{f_k(x_k)} \right) \left( B_k(x_k) - q_{k-1} \frac{f_k(x_k)}{F_k(x_k)} \right) > (r_k - q_{k-1})^2$$

where  $B_{k+1}(v_{k+1}) = B_k(v_{k+1})F_k(v_{k+1})$  was used. By the existence proof we are only interested in  $v_{k+1} \geq x_i$ ; using this condition, that  $B_k(v)$  is decreasing in  $v$ , and condition (3.13) we find that  $(B_k(x_k)x_k - q_{k-1})^2$  is a lower bound for the LHS of the expression above. Lemma 8.1 shows  $B_i(x_k)x_k \geq r_k(\mathbf{x}^k)$ . Thus we just need to show that  $B_k(x_k)x_k - q_{k-1} \geq 0$ , which is done by proving  $r_k(\mathbf{x}^k) - q_{k-1} \geq 0$ . We do this by induction. Since  $q_0$  is not defined, we begin with  $i = 2$ . Using integration by parts  $r_2(\mathbf{x}^2) - q_1$  is equal to

$$x_1 F_1(x_1) + \int_{x_1}^{x_2} F_1(s) ds - (x_1)^2 f_1(x_1) > \int_{x_1}^{x_2} F_1(s) ds \geq 0$$

where condition (3.13) was used in the last step. Suppose we have shown  $r_j(\mathbf{x}^j) \geq q_{j-1}$  for  $j \leq i$ . We show  $r_{i+1}(\mathbf{x}^{i+1}) \geq q_i$ . Using equation (3.27), this is equivalent to:

$$r_{i+1}(\mathbf{x}^{i+1})/F_i(x_i) - q_{i-1} - (r_i(\mathbf{x}^i) - q_{i-1})^2 / \left( B_i(x_i) \frac{F_i(x_i)}{f_i(x_i)} - q_{i-1} \right) \geq 0.$$

Lemma 8.2 shows  $r_{i+1}(\mathbf{x}^{i+1})/F_i(x_i) \geq r_i(\mathbf{x}^i)$ . By induction hypothesis  $r_i(\mathbf{x}^i) \geq q_{i-1}$

and we can rewrite the condition as

$$1 \geq (r_i(\mathbf{x}^i) - q_{i-1}) \left/ \left( B_i(x_i) \frac{F_i(x_i)}{f_i(x_i)} - q_{i-1} \right) \right.$$

The result follows from condition (3.13) and Lemma 8.1. Thus  $r_{i+1}(\mathbf{x}^{i+1}) \geq q_i$ , which proves  $dh_{k+1}^{k+1}(v_{k+1})/dv_{k+1} > 0$  for all  $v_{k+1} \geq x_k$ . Notice that this result implies  $dh_{k+1}^n(v_{k+1}, \tilde{\mathbf{v}}^{k+2})/dv_{k+1} > 0$  for all  $\tilde{\mathbf{v}}^{k+2}$  and a unique herculean equilibrium exists.  $\square$

**Claim 6.** There is no non-herculean equilibria.

*Proof.* Let  $\mathbf{x} = (x_1, x_2, \dots, x_n)$  be an ordered vector of equilibrium cutoffs. beginning from the lower cutoff, let  $i$  be the first bidder to play a smaller cutoffs than a stronger bidder  $i+1$ ; i.e.,  $x_i < x_{i+1}$  but  $s_i > s_{i+1}$ . In other words, every bidder  $k \leq i$  have their cutoffs in the same order as their strength. Because of this, we can use our recursive construction in the existence proof and our induction argument in the uniqueness proof up to bidder  $i$ , so that best responses are uniquely defined for any vector  $\bar{x}^{i+1}$  that bidders may play.

Let's analyze  $h_{i+1}^n(\mathbf{v})$ . Because  $h_i^n(x_i, \tilde{\mathbf{v}}^{i+1}) = 0$  we know  $r_i(x_i, \tilde{\mathbf{v}}^{i+1}) = c_i/A_i^n$ . Substituting the vector of solutions  $\mathbf{x}^i$  we can write  $h_{i+1}^n(\mathbf{v})$  as  $h_{i+1}^n(\tilde{\mathbf{v}}^{i+1}) = A_{i+1}^n r_{i+1}(\tilde{\mathbf{v}}^{i+1}) - c_{i+1}$ . Take  $v_{i+1}$  to be value of that satisfies  $v_{i+1} = x_i(v_{i+1}, \tilde{\mathbf{v}}^{i+2})$  and notice that Lemma 8.2 implies  $r_{i+1}(x_i, \tilde{\mathbf{v}}^{i+2}) = F_i(x_i)r_i(x_i, x_i, \tilde{\mathbf{v}}^{i+2})$ . Then, using  $r_i(x_i, \tilde{\mathbf{v}}^{i+1}) = c_i/A_i^n$ , we can write  $h_{i+1}^n(x_i, \tilde{\mathbf{v}}^{i+2}) = c_i F_i(x_i)/F_{i+1}(x_i) - c_{i+1}$  which is *positive* under (3.12) and the condition that bidder  $i+1$  is stronger than bidder  $i$ . We need to show that there is no  $v_{i+1}^* > x_i$  such that  $h_{i+1}^n(v_{i+1}^*, \tilde{\mathbf{v}}^{i+2}) = 0$ . This follows from the proof

of uniqueness as condition (3.13) implies  $dh_{i+1}^n(v_{i+1}, \tilde{\mathbf{v}}^{i+2})/dv_{i+1} > 0$  for  $v_{i+1}^* > x_i$ , which implies the result. □ ■

**Proof of Proposition 6.** Define the *ex-ante* expected payoff of bidder  $i$  under the vector of cutoffs  $\mathbf{x} = (x_1, x_2)$  as:

$$U_i(\mathbf{x}) = \int_{x_i}^{\infty} \left( vF_j(\max\{v, x_j\}) - \int_{x_j}^{\max\{v, x_j\}} s dF_j(s) - c_i \right) dF_i(v)$$

Let  $x_1 < x_2$ , computing  $\Delta \equiv U_1(\mathbf{x}) - U_2(\mathbf{x})$  for the three different scenarios we obtain:

$$\Delta = \begin{cases} \int_{x_1}^{x_2} (v - c) dF(v) & \text{sym.} \\ \int_{x_1}^{x_2} (v - c_1) dF(v) + (c_2 - c_1)(1 - F(x_2)) & \text{cost} \\ \int_{x_1}^{x_2} (vF_2(x_2) - c) dF_1(v) + \int_{x_2}^{\infty} \Gamma_1(v, x_2) f_1(v) - \Gamma_2(v, x_1) f_2(v) dv & \text{FOSD} \end{cases}$$

where  $\Gamma_i(v, x) = xF_{3-i}(x) + \int_x^v F_{3-i}(s) ds - c$  which is increasing in  $v$  and  $x$  and positive if the value of  $x$  corresponds to an equilibrium cutoff for bidder  $i$ .<sup>22</sup> The first case corresponds to symmetric bidders ( $F_i(v) = F(v)$  and  $c_i = c$  for all  $i$ ). Since in equilibrium  $x_1 F(x_2) = c_1$ ,  $U_1(\mathbf{x}) > U_2(\mathbf{x})$  whenever  $x_1 < x_2$ . Thus, in a symmetric game, cutoff order implies expected payoff order. Similarly, for the second case, in a herculean equilibrium in which bidders are ordered by costs ( $c_2 > c_1$ ) and using the same argument above, bidders expected payoff are ordered. Lastly, in the third case, when bidders play a herculean equilibrium and bidders are ordered by first order stochastic dominance ( $F_1(v) \leq F_2(v)$  for all  $v$ ) we have that  $\Gamma_1(v, x) \geq \Gamma_2(v, x)$  for

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<sup>22</sup> Integration by parts was used to obtain  $\Gamma_i(v, x)$

any  $v$  and  $x$ . Then,

$$\int_{x_2}^{\infty} \Gamma_1(v, x_2) f_1(v) dv > \int_{x_2}^{\infty} \Gamma_2(v, x_1) f_1(v) dv \geq \int_{x_2}^{\infty} \Gamma_2(v, x_1) f_2(v) dv$$

where the first inequality follows from the change in identity and herculean cutoffs ( $x_1 < x_2$ ), and the last inequality follows from integrating monotonic functions under stochastic dominance. Which proves that  $\Delta > 0$  in the three cases. For the order in the participation probability notice that when the distribution are symmetric, cutoff order and probability order are equivalent. In a herculean equilibrium  $x_1 < x_2$  if and only if  $F_1$  FOSD  $F_2$ . Thus,  $F_1(x_1) < F_1(x_2) \leq F_2(x_2)$  and the order follows. ■

**Proof of Lemma 6.** Pick two quasi-symmetric bidders, say 1 and 2, such that  $s_1 < s_2$ . Suppose we add a new potential bidder  $j$ . Let  $\alpha_i(v) = \prod_{k \neq i} F_k(v)$  which includes bidder  $j$ .  $\alpha_i(v)$  is strictly increasing in  $v$ . For cost order:  $s_1 < s_2$  implies  $c_1 < c_2$  and  $\alpha_1(v) = \alpha_2(v)$ . Then, using (3.5),  $\bar{s}_1 \alpha_1(\bar{s}_1) = c_1 < c_2 = \bar{s}_2 \alpha_2(\bar{s}_2)$ , which implies  $\bar{s}_1 < \bar{s}_2$  and the strength-order is preserved. For FOSD:  $s_1 < s_2$  implies  $c_1 = c_2 = c$  and  $\alpha_1(v) > \alpha_2(v)$ . Then  $\bar{s}_1 \alpha_1(\bar{s}_1) = c = \bar{s}_2 \alpha_2(\bar{s}_2)$ , which implies  $\bar{s}_1 < \bar{s}_2$  and strength order is preserved. For an example of strength reversal for non-quasi-symmetric bidders see the main text. ■

### 3.B Auxiliary Results

**Lemma 7.** *If a differentiable function  $f(x)$  satisfies  $xf'(x) \leq f(x)$  for  $x > c$ , then, for every  $x > c$ ,  $f(x)/x$  is weakly decreasing in  $x$ .*

*Proof.* Define  $\phi(x) = f(x)/x$ . Then,  $\phi'(x) = (f'(x)x - f(x))/x^2$ , which is non-positive by the condition.  $\square$

**Lemma 8.** *Let  $(x_1, x_2, \dots, x_n)$  be an ordered vector from smallest,  $x_1$ , to largest,  $x_n$ .*

*Then, the following properties hold.*

1.  $x_i B_i(x_i) \geq r_i(\mathbf{x}^i)$  and strict if exists  $j < i$  such that  $x_j < x_{j+1}$ .
2.  $r_i(\mathbf{x}^i) > F_{i-1}(x_{i-1})r_{i-1}(\mathbf{x}^{i-1})$  and with equality if  $x_i = x_{i-1}$ .

*Proof.* Recall the definition of  $r_i(\mathbf{x}^i)$  in equation (3.2). For the first claim simply observe,  $x_i B_i(x_i) - r_i(\mathbf{x}^i) = \sum_{k=1}^{i-1} \left( A_k^{i-1} \int_{x_k}^{x_{k+1}} s dB_{k+1}(s) \right)$  which is strictly positive if there exists a bidder  $j < i$  such that  $x_j < x_i$  or zero otherwise. For the second claim we show that  $r_i(\mathbf{x}^i) = F_{i-1}(x_{i-1})r_{i-1}(\mathbf{x}^{i-1}) + \int_{x_{i-1}}^{x_i} B_i(s) ds$ , which proofs the claim.

Rewriting (3.2):

$$r_{i-1}(\mathbf{x}^{i-1}) = x_i B_i(x_i) - F_{i-1}(x_{i-1}) \sum_{k=1}^{i-2} \left( A_k^{i-2} \int_{x_k}^{x_{k+1}} s dB_{k+1}(s) \right) - \int_{x_{i-1}}^{x_i} s dB_i(s).$$

Integrating by parts the last term we obtain:

$$r_{i-1}(\mathbf{x}^{i-1}) = x_{i-1} B_i(x_{i-1}) - F_{i-1}(x_{i-1}) \sum_{k=1}^{i-2} \left( A_k^{i-2} \int_{x_k}^{x_{k+1}} s dB_{k+1}(s) \right) + \int_{x_{i-1}}^{x_i} B_i(s) ds.$$

Since, by definition,  $B_i(x_{i-1}) = B_{i-1}(x_{i-1})F_{i-1}(x_{i-1})$ , the result follows.  $\square$

**Lemma 9.** *Symmetric bidders with concave CDF's must play symmetric equilibrium cutoffs.*



*Proof.* By contradiction. W.l.o.g. order bidders identities in terms of their cutoff order, with bidder 1 being the bidder with the lower cutoff. Suppose there exists an equilibrium such that bidders  $q < p$  are symmetric; i.e.,  $F_q = F_p = G$  and  $c_q = c_p = c$ , but play  $x_q < x_p$ . Integrating (3.3) by parts we obtain (see derivation in the Online Appendix):<sup>23</sup>

$$\sum_{j=1}^i \left\{ \prod_{k \geq j, k \neq i} F_k(x_k) \int_{x_{j-1}}^{x_j} \left( \prod_{\ell < j} F_\ell(y) \right) dy \right\} = c_i. \quad (3.28)$$

Subtracting (3.28) of  $q$  to that of  $p$  delivers

$$\begin{aligned} 0 = & \sum_{j=q+1}^p \left\{ \prod_{k \geq j, k \neq p} F_k(x_k) \int_{x_{j-1}}^{x_j} \left( \prod_{\ell < q} F_\ell(y) \right) G(y) \left( \prod_{\ell=q+1}^{j-1} F_\ell(y) \right) dy \right\} \\ & - (G(x_p) - G(x_q)) \sum_{j=1}^q \left\{ \prod_{k \geq j, k \neq q, p} F_k(x_k) \int_{x_{j-1}}^{x_j} \left( \prod_{\ell < j} F_\ell(y) \right) dy \right\} \end{aligned} \quad (3.29)$$

We show that a strict lower bound for the right-hand side of (3.29) is non-negative, a contradiction to (3.29). The first summation is strictly positive, we take a lower bound of this summation by taking a lower bound of its integrals in three steps: (i) for the terms in the first product ( $\ell < q$ ), replace  $F_\ell(y)$  by  $F_\ell(x_q)$ ; (ii) substitute  $G(y)$  by  $G(x_q)$  and; (iii) for the terms in the second product (ranging from  $q + 1$  to  $j - 1$ ), replace  $F_\ell(y)$  by  $F_\ell(x_\ell)$ . Hence, the following *strict* lower bound for the first summation is obtained<sup>24</sup>

$$(x_p - x_q)G(x_q) \prod_{\ell < q} F_\ell(x_q) \prod_{k > q, k \neq p} F_k(x_k)$$

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23. Recall the notation conventions:  $\sum_\emptyset = 0$ ,  $\prod_\emptyset = 1$  and  $x_0 = 0$ .

24. The strict inequality is guaranteed by taking  $G(x_q)$  as lower bound of  $G(y)$  over the range of integration  $x_q$  to  $x_p$  with  $x_q < x_p$ .

Now we construct an upper bound to the subtracting term in (3.29) by substituting in the integral  $F_\ell(x_j)$  for  $F_\ell(y)$ . Then, the second summation in equation (3.29) becomes

$$\prod_{k>q, k \neq p} F_k(x_k) \left( \sum_{j=1}^{q-1} \left\{ x_j \prod_{k=j+1}^{q-1} F_k(x_k) \left( \prod_{\ell \leq j} F_\ell(x_j) - \prod_{\ell \leq j} F_\ell(x_{j+1}) \right) \right\} + x_q \prod_{\ell < q} F_\ell(x_q) \right)$$

Since  $x_j \leq x_{j+1}$ , the summation in the previous expression is over non-positive terms. We can obtain an upper bound by replacing the summation with zero. Then, our strict lower bound for the right-hand side of (3.29) is

$$(x_p G(x_q) - x_q G(x_p)) \prod_{\ell < q} F_\ell(x_q) \prod_{k>q, k \neq p} F_k(x_k).$$

Because the products are positive, the previous expression is non-negative if and only if  $G(x_q)/x_q > G(x_p)/x_p$ . The result follows from condition (3.10), Lemma 7 and  $x_q < x_p$ . □

### 3.C Additional Appendix

**Lemma 10.** *For any given bidder  $i$ , equilibrium condition  $A_i^n r_i(\mathbf{x}^i) = c_i$  is equivalent to*

$$\sum_{j=1}^i \left\{ \prod_{k \geq j, k \neq i} F_k(x_k) \int_{x_{j-1}}^{x_j} \left( \prod_{\ell < j} F_\ell(y) \right) dy \right\} = c_i. \quad (\text{OA } 1)$$

**Proof.** Recall

$$r_i(\mathbf{x}^i) = x_i B_i(x_i) - \sum_{j=1}^{i-1} \left( A_j^{i-1} \int_{x_j}^{x_{j+1}} s dB_{j+1}(s) \right),$$

where  $A_i^n = \prod_{k=i+1}^n F_k(x_k)$  and  $B_i(v) = \prod_{k<i} F_k(v)$ . Integrating  $r_i(\mathbf{x}^i)$  by parts

$$\begin{aligned} r_i(\mathbf{x}^i) &= x_i B_i(x_i) - \sum_{j=1}^{i-1} \left\{ A_j^{i-1} \left( x_{j+1} B_{j+1}(x_{j+1}) - x_j B_{j+1}(x_j) - \int_{x_j}^{x_{j+1}} B_{j+1}(s) ds \right) \right\} \\ &= x_i A_1^{i-1} F_1(x_1) + \sum_{j=1}^{i-1} A_j^{i-1} \int_{x_j}^{x_{j+1}} B_{j+1}(s) ds, \end{aligned}$$

where  $A_j^{i-1} B_{j+1}(x_{j+1}) = A_{j+1}^{i-1} B_{j+2}(x_{j+1})$  was used. Observing that  $x_1 = \int_{x_0}^{x_1} B_1(s) ds$ , and multiplying  $r_i(\mathbf{x}^i)$  by  $A_i^n$ , we obtain (OA 1). ■

**Proof of Proposition 7.** We begin by proving existence of a herculean equilibrium.

Suppose that both bidders are equally strong ( $s_1 = s_2 = s$ ). Set  $x_1 = x_2 = s$  and observe that equations in (3.16) hold and an equilibrium exists.

Suppose w.l.o.g. that  $s_1 < s_2$ ; i.e., bidder 1 is the strong bidder of the game. We construct an equilibrium where  $x_1 < x_2$ . Define  $g(v) = r + c_1/F_2(v)$  to be the equilibrium cutoff played by bidder 1 when bidder 2 plays  $x_2 = v$ . Observe that  $g(v) > r$  and  $g'(v) = -(g(v) - r) f_2(v)/F_2(v) < 0$ . Define the function  $h : [s_1, \infty) \rightarrow \mathbb{R}$  by

$$h(v) = vF_1(v) - rF_1(g(v)) - \int_{g(v)}^v x f_1(x) dx - c_2$$

which is a continuous function of  $v$ . The function  $h(v)$  represents bidder 2's revenue of drawing valuation  $v$  when she plays a cutoff  $x_2 = v$  and 1 plays the cutoff  $x_1 =$

$g(v)$ . To have a herculean equilibrium we need a value  $x_2$  satisfying  $h(x_2) = 0$  and  $x_2 > g(x_2)$ . The next two claims prove the result.

**Claim 7.**  $x_2 \in (s_1, \infty)$  is necessary and sufficient to have herculean cutoffs.

*Proof.* Observe that  $g(v)$  is weakly decreasing in  $v$  and, using equation (3.15),  $g(s_1) = s_1$ . Therefore,  $x_2 > g(x_2)$  if and only if  $x_2 \in (s_1, \infty)$ .  $\square$

**Claim 8.**  $h(s_1) < 0$  and  $h(v)$  is unbounded above.

*Proof.* Bidder 2 being weak ( $s_1 < s_2$ ) implies

$$h(s_1) = (s_1 - r) F_1(s_1) - c_2 < (s_2 - r) F_1(s_2) - c_2 = 0.$$

On the other hand,  $h(v)$  is unbounded above as  $vF_1(v)$  is unbounded and the finite expectation assumption.  $\square$

Claim 8 plus continuity imply that there exists  $x^* > s_1$  such that  $h(x^*) = 0$ . On the other hand,  $h(x^*) = 0$  holds if and only if equations (3.16) are satisfied. Therefore,  $x_1 = g(x^*)$  and  $x_2 = x^*$ , constitute a herculean equilibrium.

Now we prove uniqueness. We begin by showing that among the herculean class the equilibrium is unique. Then we extend the uniqueness result among all equilibria. In order to have a unique equilibrium in the herculean class it is sufficient to show that  $h'(v) > 0$  for all  $v > s_1$ , so that  $h(v)$  single crosses zero at  $x^*$  from below. Differentiating and then using  $g'(v)$

$$h'(v) = F_1(v) + g'(v) (g(v) - r) f_1(g(v)) = F_1(v) - (g(v) - r)^2 \frac{f_2(v)}{F_2(v)} f_1(g(v)).$$

We show that a lower bound for  $h'(v)$  is positive. Using  $0 < g(v) - r < g(v)$ , that  $vf_2(v) \leq F_2(v)$  for  $v > s_1$ , and  $vf_1(v) \leq F_1(v)$  for  $v > c_2$  we can write  $h'(v)$  as

$$h'(v) > F_1(v) - \frac{g(v)}{v} F_1(g(v)).$$

Since we are only interested in  $v \geq s_1$ , Claim 7 implies  $v > g(v)$ . Thus  $h'(v) > 0$  proving uniqueness withing the herculean class.<sup>25</sup>

To prove that the only equilibrium is the herculean, suppose we have a non-herculean equilibrium; i.e.,  $x_1 \geq x_2$ . Define  $\bar{g}(v) = r + c_2/F_1(v)$  to be the equilibrium cutoff played by bidder 2 when bidder 1 plays  $x_1 = v$ , and let

$$\bar{h}(v) = vF_2(v) - rF_2(\bar{g}(v)) - \int_{\bar{g}(v)}^v xf_2(x) dx - c_1$$

represent bidder 1's revenue of drawing valuation  $v$  when she plays a cutoff  $x_1 = v$  and 2 plays the cutoff  $x_2 = \bar{g}(x_1)$ . As before, because  $\bar{g}(s_2) = s_2$  and  $\bar{g}(v)$  being decreasing, in order to have an non-herculean equilibrium  $\bar{h}$  has to be defined on  $[s_2, \infty)$ . Now observe that  $\bar{h}(s_2) = (s_2 - r)F_2(s_2) - c_1 > 0$ . By repeating the argument above  $\bar{h}'(v) > 0$  and  $\bar{h}(v) > 0$  for all  $v \in (s_2, \infty)$ , so there is no  $x^*$  such that  $\bar{h}(x^*) = 0$  and no non-herculean equilibrium exists. ■

**Lemma 11.** *For any  $x, y \in \mathbb{R}^{++}$ , the function  $\pi_i(x, y)$  defined by equation (3.18)*

*satisfies: (i)  $\pi_i(x, y) < x$ ; (ii)  $\pi_i(0, y) = 0$ ; and (iii)  $\pi'_i(x, y) > 0$ .*<sup>26</sup>

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<sup>25.</sup> Observe that uniqueness within the herculean class needs a weaker condition than (3.17). In particular, it needs that the weak player satisfies  $vf_2(v) \leq F_2(v)$  for  $v > s_1$  but  $s_1 > r + c_1$ .

<sup>26.</sup> Where prime denotes derivative with respect the first dimension.

**Proof.** For the first claim, notice that equation (3.18) satisfies

$$\pi_i(x, y) < x \int_0^\infty \varepsilon_i \Phi_j\left(\frac{x\varepsilon_i}{y}\right) d\Phi(\varepsilon_i) < x$$

where the first inequality follows from taking the subtracting term to zero and the second inequality by making  $\Phi_j(v_i\varepsilon_i/v_j)$  equal to 1 and integrating. The second claim follows from simply taking  $x = 0$  in equation (3.18). For the last claim, take Leibnitz differentiation of equation (3.18) and obtain

$$\pi'_i(x, y) = \int_0^\infty \left( \int_0^{x\varepsilon_i} \varepsilon_i d\Phi_j\left(\frac{s}{y}\right) \right) d\Phi_i(\varepsilon_i) > 0$$

which is positive as it is the integral of positive functions on a positive support. ■

**Proof of Proposition 8.** *Preliminaries:* Suppose first that  $s_1 = s_2 = s$ . Then, by the definition of strength, we know that  $x_1 = x_2 = s$  corresponds to a herculean equilibrium. Henceforth, assume without loss of generality that  $s_1 < s_2$ . Define

$$H_i(x, y) = xF_j(y) + \int_y^\infty \pi_i(x, v) dF_j(v) - c_i.$$

and let  $g(x)$  be given by  $H_1(g(x), x) = 0$ . That is,  $g(x)$  corresponds to bidder 1's best response to bidder 2, when bidder 2 plays the cutoff strategy  $x$ . Using Lemma 11 in the Auxiliary Results section of the appendix,  $H_i(0, y) = -c_i$  and  $H_i$  is strictly increasing. Because  $H_i$  is also unbounded above in  $x$ , the value  $g(x)$  exists and is uniquely defined. Notice that  $H_i(x, y)$  is differentiable and strictly increasing in  $y$ .<sup>27</sup>

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<sup>27</sup>.  $dH_i/dy = f_j(y)[x - \pi_i(x, y)]$  but  $\pi_i(x, y) < x$  for every value of  $y$  by Lemma 11.

**Claim 9.**  $g(s_1) = s_1$  and

$$0 > g'(x) > -\frac{f_2(x)g(x)}{F_2(x)}. \quad (3.30)$$

*Proof.* First part follows by definition of strength. For the inequalities in equation (3.30), we use implicit differentiation:

$$g'(x) = -\frac{f_2(x)(g(x) - \pi_1(g(x), x))}{F_2(x) + \int_x^\infty \pi_1'(g(x), y)dF_2(y)}.$$

By Lemma 11,  $g'(x) < 0$ , as for any  $x$  and  $y$ ,  $\pi_1(x, y) < x$  and  $\pi_1'(x, y) > 0$ . For the lower bound of  $g'(x)$  observe that the integral term in the denominator is positive. Then, taking the subtracting term in the numerator and the integral in the denominator to zero gives the lower bound.  $\square$

*Existence of herculean equilibrium:* Define the continuous function  $h : [s_1, \infty) \rightarrow \mathbb{R}$  by  $h(x) \equiv H_2(x, g(x))$ . This function corresponds to the expected payoff of bidder 2 when bidder 1 best-responds to  $x$ ; i.e., bidder 1 plays  $x_1 = g(x)$ . Define  $x_2$  to be the value satisfying  $h(x_2) = 0$ , we prove that  $x_2$  exists and that it is an herculean equilibrium. Observe that, because  $g(s_1) = s_1$  and  $g(x)$  is weakly decreasing in  $x$ ,  $x_2 \in (s_1, \infty)$  is necessary and sufficient for  $x_1 < x_2$  (herculean cutoffs). Also, because  $h(x)$  is unbounded above in  $x$ , we prove existence by showing  $h(s_1) < 0$ . Because  $s_1 < s_2$ , we have  $h(s_1) = H_2(s_1, s_1) < H_2(s_2, s_2) = 0$ , where the inequality follows  $H_i$  being increasing in both dimensions; and the equality follows from the definition of strength. Therefore, a herculean equilibrium exists.

*Uniqueness:* The uniqueness proof is divided into two claims. Condition (3.20)

used in each of them.

**Claim 10.** There exists a unique herculean equilibrium.

*Proof.* To prove uniqueness within the herculean class, it is shown that  $h'(x) > 0$  so that  $h(x)$  single-crosses zero from below. The derivative of  $h(x)$  is:

$$h'(x) = F_1(g(x)) + \int_{g(x)}^{\infty} \pi'_2(x, y) dF_j(y) + g'(x)f_1(g(x)) [x - \pi_2(x, g(x))].$$

The first two terms of  $h'(x)$  are positive. The term containing  $g'(x)$  is negative. We show that a lower bound of  $h'(x)$  is positive. For the lower bound, we take the subtracting term in the square brackets to zero and replace the lower bound (3.30), and find

$$h'(x) > F_1(g(x)) \left( 1 - \frac{x f_2(x) g(x) f_1(g(x))}{F_2(x) F_1(g(x))} \right) + \int_{g(x)}^{\infty} \pi'_2(x, y) dF_1(y) > 0$$

Condition (3.20) implies that the term in parenthesis is positive and the results follows. □

**Claim 11.** There is no equilibrium in which the strong bidder plays a higher cutoff than the weak bidder.

*Proof.* To prove that the only equilibrium is the herculean equilibrium, suppose we have a non-herculean equilibrium—i.e.,  $x_1 > x_2$  but  $s_1 < s_2$ . Define  $\bar{g}(x)$  to be the function that satisfies  $H_2(\bar{g}(x), x) = 0$ .  $\bar{g}(x)$  corresponds to bidder 2's best-response to the cutoff of bidder 1 when  $x_1 = x$ . As before,  $\bar{g}(x)$  is well defined. Similarly,



following the steps of Claim 9, it can be shown:  $\bar{g}(s_2) = s_2$ ,  $\bar{g}'(x) < 0$ , and that  $\bar{g}'(x)$  is bounded below by

$$-\frac{f_1(x)\bar{g}(x)}{F_1(x)}. \quad (3.31)$$

Define the continuous function  $\bar{h}(x) = H_1(x, \bar{g}(x))$  which corresponds to bidder 1's expected profits of participating with signal  $x$  when bidder 2 best-responds to  $x$ . We show that there is no  $x$  such that  $x_1 = x > \bar{g}(x) = x_2$  and  $\bar{h}(x) = 0$ ; i.e., no non-herculean equilibrium exists. Begin by observing that  $x > \bar{g}(x)$  if and only if  $x \in (s_2, \infty)$ . Because  $H_1(x, y)$  is increasing in both dimensions,  $H_1(s, s)$  is strictly increasing in  $s$ . Then, by the definition of strength and by bidder 2 being the weak bidder,

$$0 = H_1(s_1, s_1) < H_1(s_2, s_2) = H_1(s_2, \bar{g}(s_2)) = \bar{h}(s_2).$$

Following analogous steps to those in Claim 10, which requires to use the lower bound (3.31), it is possible to show that  $\bar{h}'(x) > 0$ . Then, because  $\bar{h}(s_2) > 0$  and  $\bar{h}'(x) > 0$ ,  $\bar{h}(x)$  never crosses zero when  $x > s_2$  and the result holds. □ ■

**Proposition 9.** *In two-bidder symmetric games in which the distributions of valuations are convex for valuations  $v \in [0, \bar{v}]$  (see eq. (3.11)), the symmetric equilibrium is never efficient.*

**Proof.** Campbell (1998) shows that convexity implies multiple equilibria. Under symmetric bidder playing a symmetric equilibrium, the Hessian (and its determinant)

of the welfare function becomes

$$H(\mathbf{x}) = - \begin{pmatrix} f(x)F(x) & xf(x)^2 \\ xf(x)^2 & f(x)F(x) \end{pmatrix} \quad \det(H) = f(x)^2(F(x)-xf(x))(F(x)+xf(x)).$$

Convexity of the distribution function implies  $xf(x) > F(x)$ . Thus, the determinant is negative, which implies that the symmetric equilibrium is a saddle point. ■

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