Spring 2021

Essays in Network Economics

Jian Xin Xin Heng
Yale University Graduate School of Arts and Sciences, jxheng87@gmail.com

Abstract

Essays in Network Economics

Jian Xin Heng

2021

Chapter 1 explores possibilities for non-parametrically identifying peer effects. In my model, outcomes associated with individuals are an unknown function of expected peer outcomes and other individual-specific attributes. Unobserved heterogeneity in outcomes across individuals is captured by an additively separable individual-specific error. The error is mean dependent on peer group, rendering expected peer outcome an endogenous covariate. Exogenous peer attributes are absent from the individuals' remaining covariates.

When the data is cross-sectional, I propose imposing a stringent but intuitive assumption on how individual outcomes depend on expected peer outcomes and remaining individual-specific attributes. Specifically, I assume one of the individual-specific characteristics indexes how strongly an individual interacts with his peers. If the index is zero, the individual is not directly affected by peer outcomes. Under this assumption, the model is identified, up to a normalization. When panel data is available, the assumption is unrequired, and the model is identified via a more traditional IV-based approach. The cross-sectional result leads to tests for whether peer effects estimates reflect forces associated with social interactions or not.

Chapter 2 studies how economies of scale and product differentiation affect manufacturer-supplier relationships. A model consisting of two manufacturers, each with pre-existing relationships to two separate suppliers, is analyzed. At the start, an unrelated manufacturer-supplier pair decides whether to invest in a new relationship. Based on the resulting network of manufacturer-supplier relationships, a manufacturer's input price is determined by Nash bargaining if it's related to one supplier, and a first-price auction if otherwise. When manufac-

turers are horizontally differentiated, hold-up of investment by neighbor manufacturers causes manufacturer-supplier network connectivity to be too low. On the other hand, overinvestment in outside option relationships causes network connectivity to be inefficiently high. Shocks to individual manufacturers or suppliers have disproportionately large (small) welfare consequences vis-a-vis ex-ante market shares when the network is under (overly) connected compared to the socially optimal network.

Chapter 2 also estimates a micro-founded model of firm-to-firm relationship formation, using prices, quantities and product-supplier-level network data for 2008-16 U.S. automobiles. The model incorporates the theoretical model's key elements - supplier-level economies of scale, downstream market product differentiation and relationship network contingent input pricing. To identify the model, I assume manufacturers Nash bargain with suppliers inherited from previous periods. I then exploit variation in these suppliers' quantities to identify how production costs vary with rival product output. I find, on average, main suppliers of chassis, exterior and combined inputs experience significant economies of scale. Ex-post and on average, manufacturers do not benefit from forming their chosen relationships, absent rents from outside option relationship overinvestment or compensation from other firms. In comparison, hold-up of relationship investment is less significant in affecting incentives to form relationships.

Essays in Network Economics

A Dissertation
Presented to the Faculty of the Graduate School
of
Yale University
in Candidacy for the Degree of
Doctor of Philosophy

by
Jian Xin Heng

June 2021

Contents

List of Figures

List of Tables

Chapter 1

Non-parametrically Identifying Peer Effects when Correlated Effects are Present but Exogenous Effects are Absent

1.1 Introduction

In this paper, social interactions within non-overlapping peer groups imply each individual's outcome is an unknown function of expected peer outcome, and other characteristics. Unobserved heterogeneity across individuals is captured by an individual-specific additively separable error. The challenge is to characterize assumptions on the data generating process that ensure the function is non-parametrically identified.

The primary motivation behind this paper is econometrical. Linear social effects models have been estimated in a wide variety of contexts[1]. Papers such as Bramoulle et al. (2009) understandably study identification of peer effects in similar settings. Quite often, sign restrictions on the regressors' coefficients are needed to identify linear social effects models, by implying

1. The surveys by Carney (2013) and dePaula (2015) review how the econometric and applied literatures in this field have evolved since Manski (1993), and significantly affected my understanding of the field.

a bijection between reduced form coefficients and the original model's structural parameters[2]. Whether peer effects are identified absent these restrictions or functional form assumptions is of interest.

The secondary motivation behind this paper is theoretical. Armengol et al. (2009) and Blume et al. (2015) explain how empirical social effects models can stem from best responses of simultaneous-moves games. However, payoff functions in these games are typically linear quadratic, in contrast to a range of studies documenting non-linear peer effects[3]. Non-parametric relationships between outcomes and characteristics in the empirical model allow for more general payoff functional forms in the implicit game underlying the model. This bridges the gap between theory and evidence.

Aside from its non-parametric formulation, several features of this paper's model should be noted. First, peer groups partition individual observations in the model, and thus don't overlap. As explained by Bramoulle et al. (2009), the shift towards overlapping peer group effects was partially motivated by the need to overcome Manski's (1993) identification problem. Yet historically, social effects have been estimated for non-overlapping peer groups such as schools or neighborhoods[4], in addition to more general social networks. I see this paper's results as stepping stones towards both non-parametric estimation of peer effects in appropriate contexts, and non-parametric identification of peer effects from more general network data.

2. For example, Bramoulle et al. (2009) requires the size of the peer effects coefficient to be strictly less than one.

3. See (Bayer & Ross (2009), Sund (2009), Alcalde (2013), Burke & Sass (2013), Kiss (2013), Tincani (2015) and Agostinelli (2018)). These papers stem from a formal literature survey on three well-known databases, using "non-linear" or "non-parametric", (with and without the hyphen) and "peer effects" as keywords. The first 100-125 hits in each database were extracted and pruned. The majority of papers studying peer effects non-linearly find evidence of peer effects operating non-linearly. I acknowledge Leila Bengali's efforts in this task.

4. See Gaviria & Raphael (2001) or Carrell et al. (2008) for schools. See Sacerdote (2001) for college dormitory roommates. See Bertrand et al. (2003) or Bobonis & Finan (2009) for neighborhoods.

Second, this paper's asymptotic results implicitly require each peer group to contain infinitely many individuals. The model is hence more applicable to situations where peer groups contain hundreds of people, as opposed to less. Typically, empirically studied peer groups feature fewer than 50 observations[5]. A justification behind this focus is that individuals in smaller social units interact more frequently[6]. Yet, it is also conceivable for individuals to be influenced by broader subsets of society, since their payoffs can depend on actions of people they do not personally know. In support of this view, Carrell et al. (2008) document peer effects in cheating, within peer groups containing more than 100 military academy graduates.

Finally, individual outcomes do not depend on peer characteristics. Hence, exogenous effects are formally absent. This is disappointing since one often conceives of social effects as an individual's choices being influenced by the chosen outcomes of his peers (the so called peer effect) and their characteristics (more commonly known as exogenous effects). Manski (1993) proves empirically separating peer and exogenous effects in the linear social effects model is impossible. However, identifying the combined effect of peer outcomes and attributes is still achievable, under appropriate restrictions. Such possibilities are explored in this paper, by extending the base model to encompass exogenous effects.

I analyze the model when available data is cross-sectional, and an appropriately modified model when the researcher observes panel data. Identification of peer effects is non-trivial because peer outcomes are endogenous regressors. A traditional remedy has been to use peer

5. For example, see Sacerdote (2001) for peer groups constructed from dormitory roommates, Pinto (2010) and Tincani (2015) for peer groups constructed from classmates, and Duflo & Saez (2003) and Dahl et al. (2014) for peer groups constructed from office colleagues.

6. Dahl et al. (2014) state this as their rationale for studying peer effects within small and medium sized firms. Duflo & Saez (2003) state individuals are more likely to know their peers under tighter peer group definitions, while suggesting attenuation bias may arise if social units are larger.

characteristics to instrument for peer outcomes[7]. This approach does not extend immediately to the studied context, since no reduced-form relationship between expected peer characteristics and expected peer outcome emerges naturally from the model. When the data is cross-sectional, I propose imposing a stringent but intuitive assumption on the relationship between an individual's outcome, characteristics, and expected peer outcome. The assumption posits the existence of a characteristic among the model's covariates, that indexes how strongly an individual interacts with his peers. Transmission of peer effects shuts down when the index equals zero. Under this assumption, peer effects are identified.

When the data tracks observations across time periods, peer effects are identified under more orthodox assumptions. Cross-period data variation means lagged peer outcomes can be used to instrument for peer outcomes. The interaction index restriction can thus be dispensed with.

These results contrast with less optimistic messages on peer effects identification from two other papers. Consider the non-parametric set-up in Manski (1993) that was shown to be unidentified. I find Manski's non-parametric model is unidentified because it essentially assumes random assignment of individuals to peer groups. When his model is generalized to allow for endogenous peer group formation, then identification is possible, and actually attained under the interaction index assumption. In addition, the paper's model suggests how to empirically discriminate between peer effects caused by individuals responding to peer influence, and effects reflecting generic correlation between individual and group average outcomes. As explained in Angrist (2014), the latter possibility threatens causal interpretations of peer effects estimates. This possibility is ruled out precisely by the interaction index assumption, which is testable from the data's distribution.

7. See the methodology collectively developed by Case & Katz (1991), Gaviria & Raphael (2001) and Carell et al. (2008).

Finally, to complement the identification results, I provide a consistent and asymptotically normal estimator for the cross-sectional model, and demonstrate how the model's structural relationships are interpretable as best responses of an infinite-player bayesian game. In particular, it is hoped that formally micro-founding the model aids interpretation of peer effects, while rationalizing some of the model's more novel or troublesome features.

This paper is mainly related to the peer effects identification literature. In response to the aforementioned identification problem elaborated upon in Manski (1993), various attempts have been made to identify peer effects under alternative assumptions[8]. Many of these studies suggest the researcher observes a large number of finite, potentially overlapping peer groups. Morover, these studies are typically conducted in parametric frameworks. One exception is Brock & Durlauf (2007), who study social effects in a semi-parametric setting. However, they assume outcomes are binary, and focus on exogenous rather than peer effects. More recently, Graham et al. (2010), Pinto (2010), Manski (2013) and Tincani (2015) analyze social effects non or semi-parametrically, in more specific or applied contexts. However none of these papers' empirical specifications incorporate endogenous peer effects too. Finally, a more general non-parametric identification literature exists. Papers written in this area include Newey et al. (1999), Chesher (2003), Imbens & Newey (2009), Matzkin (1992, 2008) and others. Typically, non-parametric models are identified up to constants or scaling factors. This literature is also not focused on peer effects.

In sum, this paper contributes to the literature in two ways. First, it suggests ways of identifying and interpreting peer effects without relying on parametric assumptions. Second, its results apply when peer groups are large and non-overlapping, such as schools. In what

8. See Moffitt (2001), Lee (2007), Brock & Durlauf (2007), Bramoulle et al. (2009), Blume et al. (2015) and Agostinelli (2018) for more details.

follows, the models for cross sectional and panel data are described and analyzed in sections 1.2 and 1.3. Microfoundations for Section 1.2's model are established in Section 1.4. Uninstructive proofs are located in the appendix.

1.2 Identification from Cross-Sectional Data

There are I individuals and S possible peer groups. For each individual $i \leq I$, a researcher observes $(y_i, \mathbf{x}_i, s_i) \in \mathbb{R}^{K+1} \times \mathbb{N}$. As in the peer effects literature, y_i denotes the individual's endogenous outcome of interest that is susceptible to peer effects, $\mathbf{x}_i$ is a vector of observable characteristics for the individual while s_i denotes his peer group. The data facing the researcher is hence $\{y_i, \mathbf{x}_i, s_i\}_{i \leq I}$.

Example 1.2.1. [Educational Achievement] Let y_i be a measure of individual i's academic performance, such as his test score on a nationwide exam. Let $\mathbf{x}_i$ and s_i be i's characteristics and school of choice respectively. The characteristics might include his family income or neighbourhood.

1.2.1 Model

The data generating process is modeled as follows. For each individual i, $(y_i, \mathbf{x}_i, s_i)$, together with an individual specific error ϵ_i is drawn from a joint distribution $F : \mathbb{R}^{K+2} \times \mathbb{N} \to [0, 1]$. The draws are i.i.d. over individuals. The errors reflect unobserved heterogeneity across individuals that contribute to variation in outcomes.

F is a primitive of the model. Model identification corresponds to identifying F. Identification is non-trivial because only the marginal distribution of the data variables $F_{y,\mathbf{x},s}$, is known to the researcher. Hence, additional restrictions must be imposed. Within the context of pure peer effects, F implies each draw $(y_i, \mathbf{x}_i, s_i, \epsilon_i)$ satisfies the following relationship:

$$y_i = m\left(\mathbb{E}\left[y|s_i\right], \mathbf{x}_i\right) + \epsilon_i \quad \forall i \leq I. \tag{1.1}$$

Here, $m : \mathbb{R}^{K+1} \to \mathbb{R}$ is measurable and unknown to the researcher[9]. Expected peer outcome $\mathbb{E}[y|s_i]$ for individual i is pinned down uniquely by F, and also known to the researcher from $F_{y,\mathbf{x},s}$. The challenge facing the researcher is thus to identify m under the weakest possible set of assumptions. This enables one to extract ϵ_i from each observation $(y_i, \mathbf{x}_i, s_i)$, and thus identify $F_{\epsilon|y,\mathbf{x},s}$ too.

Notice y_i is unaffected by expected peer characteristics $\mathbb{E}[\mathbf{x}|s_i]$. The focus will thus be on identifying peer effects as opposed to exogenous effects. Also, peer effects operate through expected peer outcomes $\mathbb{E}[y|s]$ rather than its sample analogue $\overline{y}_s$. Each individual is thus influenced by each of his peers in the population, rather than only his sampled peers.

The following assumptions aid identification of m. First,

Assumption 1.2.1. [Mean Independence] Residual outcomes ϵ are mean independent of individual characteristics $\mathbf{x}$, given peer group s: $\mathbb{E}[\epsilon|\mathbf{x}, s] = \mathbb{E}[\epsilon|s] \equiv \mu(s)$, where $\mu : \mathbb{N} \to \mathbb{R}$ is unknown.

The first equality in Assumption 1.2.1 is the peer effects counterpart to the usual exclusion restriction needed to identify non-parametric models with additive errors. In its absence, prospects for non-parametric identification of peer effects are severe. In practice, the exclusion restriction is unlikely hold. For example, if one of the characteristics in $\mathbf{x}$ is school attendance in Example 1.2.1, than this variable is likely to correlate with unobserved academic ability incorporated in ϵ. However, from a technical standpoint, harnessing instrumental variables

9. The sigma-field associated with measurability of m is $\sigma(y, \mathbf{x}, s, \epsilon)$ where $(y, \mathbf{x}, s, \epsilon)$ is a random vector with distribution F.

in a non-parametric setting is no longer an issue. Moreover, from an application standpoint, the issue of finding valid instruments to control for endogenous characteristics is a first order concern that practitioners have already expended much energy resolving. Hence, this paper focuses on endogeneity stemming from an alternative source.

The second equality introduces a new function $\mu : \mathbb{N} \to \mathbb{R}$. The dependence of $\mu(s)$ on s renders expected peer outcome $\mathbb{E}[y|s]$ endogenous. This dependence is called correlated effects in the literature. Notice that when $\mu(s) = 0$ for all $s \in \mathbb{N}$, the model collapses to the non-parametric framework discussed in Manski (1993). Manski argues changes in outcomes caused by peer effects are difficult to identify in such a model. Hence, I study a more general framework where mean error $\mathbb{E}[\epsilon|s]$ depends on peer group s instead. Also, observe peer group s does not directly enter as an argument of $m(e, \mathbf{x})$. Hence, given the model and Assumption 1.2.1, the effects of peer group selection based on unobservable individual attributes on outcomes, are captured by $\mathbb{E}[\epsilon|s]$. Absent the next assumption, the effects of unobserved peer group attributes are also captured by $\mathbb{E}[\epsilon|s]$. It is thus even more pertinent to allow $\mu(s)$ to depend on s.

Now, one approach in dealing with peer outcome endogeneity in linear social effects models uses peer characteristics as instruments. In particular, if m in equation (1.1) were linear:

$$y_i = \alpha\mathbb{E}[y|s] + \mathbf{x}_i\boldsymbol{\beta} + \epsilon_i,$$

then a one-to-one relationship between expected peer outcome $\mathbb{E}[y|s]$ and expected peer characteristics $\mathbb{E}[\mathbf{x}|s]$ arises:

$$\mathbb{E}[y|s] = \frac{1}{1-\alpha}\mathbb{E}[\mathbf{x}|s]\boldsymbol{\beta} + \frac{1}{1-\alpha}\mu(s).$$

This motivates the use of average peer characteristics $\overline{\mathbf{x}}_s$ as instruments for average peer outcome $\overline{y}_s$ when estimating the linear model. Two points are apparent with this approach. First, it requires that $\alpha \neq 1$. Second, no straightforward relationship between $\mathbb{E}[y|s]$ and $\mathbb{E}[\mathbf{x}|s]$ emerges when (1.1) is non-linear. Instead,

$$\mathbb{E}[y|s] = \int m\left(\mathbb{E}[y|s], \mathbf{x}\right) dF_{\mathbf{x}|s} + \mu(s).$$

Expected peer outcomes hence depends in a complicated way, on the entire distribution of peer characteristics $F_{\mathbf{x}|s}$, rather than any particular moment of $\mathbf{x}|s$. Additional assumptions are required for expected peer characteristics to serve as instruments.

This paper proposes an alternative approach to deal with peer outcome endogeneity.

Assumption 1.2.2. [Interaction Index] There exists $j \leq K$ and $\underline{x}^j \in \cap_{s' \in supp(s)} supp(x^j|s = s')$ known to the researcher such that

$$m(e, \mathbf{x}^{-j}, \underline{x}^j) = m(e', \mathbf{x}^{-j}, \underline{x}^j) \quad \forall\ (e, \mathbf{x}^{-j}), (e', \mathbf{x}^{-j}) \in supp(\mathbb{E}[y|s], \mathbf{x}^{-j}|x^j = \underline{x}^j).$$

Assumption 1.2.2 posits the existence of a regressor x^j, which shuts down transmission of peer effects when its value equals a known number $\underline{x}^j$. The variable x_i^j should be interpreted as indexing how strongly individual i interacts with his peers. In the context of Example 1.2.1, x_i^j could be the number of days i attends school while $\underline{x}^j = 0$. Since a student who never attends school is unlikely to be susceptible to peer effects, Assumption 1.2.2 is plausible. More generally, x_i^j might be an aggregated measure of how frequently i contacts his peers. To use another example, the widely studied Add Health survey queries how frequently respondents contact their five best friends for homework help, recreation or other topics. Appropriately aggregating each respondent's answers to these questions produces another candidate for x^j.

Now when Assumption 1.2.2 is imposed, $\mathbb{E}[\epsilon|s]$ fails to capture the effects of unobserved school attributes on academic outcomes in the education example, as it did under just Assumption 1.2.1. This is because students who don't attend school are unlikely to benefit from the school's attributes. More generally, the combination of both assumptions allows for endogenous peer group selection, but require the characteristics $\mathbf{x}_i$ to be sufficiently broad so as to explain the total effect of individual i's attributes, including his peer group, on his outcome. To reuse Example 1.2.1, if $\mathbf{x}_i$ contains i's past test score improvement, the issue of unobserved school attributes is potentially mitigated.

Assumption 1.2.3. [Rank] There exists a sequence of characteristics and peer groups $\{\mathbf{x}_n, s_n\}_{n\in\mathbb{N}}$ satisfying

$$(\mathbf{x}_n, s_n) \in supp(\mathbf{x}, s), \qquad (\mathbf{x}_n, s_{n+1}) \in supp(\mathbf{x}, s), \qquad x_n^j = \underline{x}^j$$

for each $n \in \mathbb{N}$, and

$$\cup_{\mathbf{x}' \in supp(\mathbf{x})}\{s' : (\mathbf{x}', s') \in supp(\mathbf{x}, s)\} \subseteq \cup_{n\in\mathbb{N}}\{s' : (\mathbf{x}_n^{-j}, \underline{x}^j, s') \in supp(\mathbf{x}, s)\}.$$

Assumption 1.2.3 is a restriction on the set of possible peer groups. Each peer group s under this restriction lies in one of a countable number of overlapping subsets. Each subset consists of peer groups for zero interaction individuals, whose other covariates belong to another sequence of attributes $\{\mathbf{x}_n^{-j}\}_{n\in\mathbb{N}}$. A special case of Assumption 1.2.3 occurs when

$$supp(s|\mathbf{x}) = supp(s),$$

a.s. on the support of $\mathbf{x}$. Hence, individuals of all types belong to each peer group in the population. Assumption 1.2.3 also holds when the population of peer groups is a finite, ordered set, $s_1 < s_2 < ... < s_N$, with adjacent peer groups containing zero interaction individuals of similar type:

$$supp(\mathbf{x}^{-j}|s = s_n, x^j = \underline{x}^j) \cap supp(\mathbf{x}^{-j}|s = s_{n+1}, x^j = \underline{x}^j) \neq \emptyset \quad \forall n \leq N.$$

If in addition, the support of attributes $\mathbf{x}^{-j}$ for zero interaction individuals $x^j = \underline{x}^j$ from each peer group s is convex, then the support of attributes and peer groups for zero interaction individuals graphed in $\mathbb{R}^{K-1} \times \mathbb{N}$, consists of a connected stack of overlapping blocks of $\mathbf{x}^{-j}$ over s.

Assumption 1.2.3 fails to hold if at least one peer group contains no zero interaction individuals, whose x^j equals $\underline{x}^j$. Assumption 1.2.3 also fails if one peer group s' is sufficiently distinctive from the rest, so that none of its zero interaction individuals share the same characteristics as zero interaction individuals from any other group. The latter situation could occur in the context of Example 1.2.1, if say, one of the covariates in $\mathbf{x}$ equals entry test scores, and one school in population features an exceedingly high entry score cut-off. A solution to this situation would be to drop the unique school from the sample, and identify peer effects for the remaining population of students.

Regardless, it turns out that assumptions 1.2.1 to 1.2.3 are sufficient to identify $m(e, \mathbf{x})$ and $\mu(s)$, up to constants summing to zero. Assumption 1.2.2 enables one to identify the correlated effect $\mu(s)$ of an individual's peer group s on his outcome y, independently from endogenous peer outcomes $\mathbb{E}[y|s]$. This is achieved by comparing outcomes for individuals who don't interact with their peers ($x^j = \underline{x}^j$) but with otherwise similar characteristics, across peer groups. Having identified this correlated effect, the portion of the overall effect of peers on outcomes attributed to $\mathbb{E}[y|s]$ can be identified by partialling out the correlated effect from the overall effect on outcomes. The latter step is done by comparing outcomes for individuals with similar characteristics but different peer groups, and subtracting the difference in outcomes caused by correlated effects.

Consistent estimation of the model is facilitated by additional assumptions. First,

Assumption 1.2.4. [Normalization] There exists a known vector $\underline{\mathbf{x}}^{-j}$ in $\cap_{s' \in supp(s)} supp(\mathbf{x}^{-j} | x_j = \underline{x}_j, s = s')$ such that $m(e, \underline{\mathbf{x}}^{-j}, \underline{x}^j) = 0$ when $(e, \underline{\mathbf{x}}^{-j}, \underline{x}^j) \in supp(\mathbb{E}[y|s], \mathbf{x})$.

Hence, the supports of zero interaction attributes $\mathbf{x}^{-j}$ across different peer groups, must have a common point $\underline{\mathbf{x}}^{-j}$. Moreover, expected outcomes for zero interaction individuals whose remaining covariates are $\underline{\mathbf{x}}^{-j}$, equal zero. Because $m(e, \mathbf{x})$ and $\mu(s)$ are only identified up to constants, the latter requirement should be interpreted as a normalization pinning down their values to specific numbers for estimation.

To cater to datasets typically used to empirically analyze peer effects,

Assumption 1.2.5. [Absolute Continuity] The regressors in $(\mathbf{x}, s)$ can be partitioned into C continuously distributed random variables $\mathbf{c}$, and D discretely distributed variables $\mathbf{d}$. For all $\mathbf{d} \in supp(\mathbf{d})$, $\mathbf{c}|\mathbf{d}$ is absolutely continuous on $\mathbb{R}^C$.

Assumption 1.2.5 enables multiple choice survey responses to be analyzed non-parametrically, alongside continuously distributed variables. It also implies $s \in \mathbf{d} \subseteq (\mathbf{x}, s)$.

Because the proposed estimation method is kernel based,

Assumption 1.2.6. [Kernel] Let $K : \mathbb{R}^C \to \mathbb{R}$ be an integrable function that is spherically symmetric about its origin, satisfying

$$\int_{\mathbb{R}^C} K(\mathbf{u})d\mathbf{u} = 1, \qquad \int_{\mathbb{R}^C} K(\mathbf{u})^2 d\mathbf{u} < \infty.$$

Assumption 1.2.7. [Continuity and Dominance] The conditional density $f_{\mathbf{c}|\mathbf{d}}(\mathbf{c}|\mathbf{d})$ is continuous in $\mathbf{c}$, and satisfies

$$\sup_{\mathbf{c}} \left| \int y f_{y,\mathbf{c}|\mathbf{d}}(y, \mathbf{c}|\mathbf{d}) dy \right| < \infty, \qquad \sup_{\mathbf{c}} \mathbb{V}[y|\mathbf{x}, s] < \infty, \qquad \sup_{\mathbf{c}} \left| f_{y,\mathbf{c}|\mathbf{d}}(y, \mathbf{c}|\mathbf{d}) \right| < \infty$$

for all $\mathbf{d}$ in its support[10].

These assumptions generalize from standard assumptions that ensure consistency of kernel-based estimation methods. Assumption 1.2.6 posits the existence of a well-behaved kernel that can be used to approximate marginal densities of various data variables, while the inequalities

10. Note that first and final inequalities imply $\sup_{\mathbf{c}} \left|\mathbb{E}[y^2|\mathbf{x}, s]\right| \leq \sup_{\mathbf{c}} \mathbb{V}[y|\mathbf{x}, s] + \sup_{\mathbf{c}} \left|\mathbb{E}[y|\mathbf{x}, s]\right| < \infty$.

in Assumption 1.2.7 enable one to interchange limits with integrals in the proof of consistency.

Finally, asymptotic normality of the estimator requires making a final set of additional assumptions. First and second,

Assumption 1.2.8. [Smoothness] The density $f_{\mathbf{x},s}(\mathbf{x}, s) \equiv f_{\mathbf{c}|\mathbf{d}}(\mathbf{c}|\mathbf{d})f_{\mathbf{d}}(\mathbf{d})$, and the conditional expectation $\mathbb{E}[y|\mathbf{x}, s] \equiv \mathbb{E}[y|\mathbf{c}, \mathbf{d}]$ are thrice continuously differentiable in $\mathbf{c}$, on $\mathbb{R}^C$ and $supp(\mathbf{x}, s)$ respectively.

Assumption 1.2.9. [Bandwidth] The bandwidth used for estimating the model b converges to zero sufficiently quickly so that $b^2\sqrt{Ib^C} \to 0$ as $Ib^C \to \infty$ and $b \to 0$.

Assumptions 1.2.8 and 1.2.9 ensure that the estimator converges in distribution at the usual rate $\sqrt{Ib^c}$ for kernel estimators. More specifically, Assumption 1.2.8 enables one to take Taylor expansions of $f_{\mathbf{x},s}(\mathbf{x}, s)$ and $\mathbb{E}[y|\mathbf{x}, s]$, a crucial step found in many proofs of asymptotic normality for econometric objects. Moreover, Assumption 1.2.9 ensures the estimator's finite sample bias vanishes at a rate faster than $\sqrt{Ib^c}$, the rate at which confidence intervals for the estimator collapses. Of course, the assumption is implied if b is less than proportional to $I^{-\frac{1}{C+4}}$, the usual condition ensuring asymptotic normality of kernel-based estimators.

Because establishing asymptotic normality requires use of Lyapunov's central limit theorem,

Assumption 1.2.10. [Dominance II] The kernel $K(\mathbf{c})$ and the the random variable η defined by $\eta \equiv y - m(\mathbb{E}[y|s], \mathbf{x}) - \mu(s)$ satisfies

$$\sup_{\mathbf{c}\in\mathbb{R}^C} |K(\mathbf{c})|^\delta < \infty \qquad \sup_{\mathbf{c}\in\mathbb{R}^C} \mathbb{E}[|\eta|^\delta|\mathbf{c}, \mathbf{d}] < \infty$$

for all $\mathbf{d}$ in its support, and $\delta > 0$.

Assumption 1.2.10 introduces new notation by defining the random variable η, and proposes inequalities that are stronger versions of inequalities already imposed by the previous assumptions. In particular, the first inequality in Assumption 1.2.10 generalizes a similar condition on the kernel $K(\mathbf{u})$ found in Assumption 1.2.6 for values of δ not equal to 2. Similar algebra reveals that the second inequality in Assumption 1.2.10 strengthens the analogous restriction on $\mathbb{E}[y^2|\mathbf{c}, \mathbf{d}]$ implied by Assumption 1.2.7[11]. The assumption also generalizes from conditions sufficient for many kernel-based estimators to be asymptotically normal.

Finally, to conserve on notation,

Assumption 1.2.11. [Normalization II] The vector of individual characteristics values $(\underline{\mathbf{x}}^{-j}, \underline{x}^j)$ equals $\mathbf{0}$.

Because the units by which each variable in $\mathbf{x}$ is measured is determined by the researcher, Assumption 1.2.11 can always be met by appropriately scaling and translating the data.

1.2.2 Cross-Sectional Identification

Proposition 1.2.1. *Under assumptions 1.2.1, 1.2.2 and 1.2.3, the functions $m : \mathbb{R}^{K+1} \to \mathbb{R}$ and $\mu : \mathbb{N} \to \mathbb{R}$ are identified on supp$(\mathbb{E}[y|s], \mathbf{x}, s)$, up to constants a and b satisfying $a + b = 0$.*

Proof. By Assumption 1.2.2, one can let $n(\mathbf{x}^{-j}) = m(e, \mathbf{x}^{-j}, \underline{x}^j)$ for all $(e, \mathbf{x}^{-j}) \in$ *supp*$(\mathbb{E}[y|s], \mathbf{x}^{-j}|x^j = \underline{x}^j)$. So $n(\mathbf{x}^{-j}) = m(e, \mathbf{x}^{-j}, \underline{x}^j)$ whenever $(e, \mathbf{x}^{-j}, \underline{x}^j) \in$ *supp*$(\mathbb{E}[y|s], \mathbf{x}^{-j}, \underline{x}^j)$ too.

Assume the claim is false. Then there exists $m' : \mathbb{R}^{K+1} \to \mathbb{R}$ and $\mu' : \mathbb{N} \to \mathbb{R}$ satisfying:

$$m\left(\mathbb{E}[y|s], \mathbf{x}\right) + \mu(s) = \mathbb{E}[y|\mathbf{x}, s] = m'\left(\mathbb{E}[y|s], \mathbf{x}\right) + \mu'(s) \tag{1.2}$$

a.s. on *supp*$(\mathbb{E}[y|s], \mathbf{x}, s)$, by Assumption 1.2.1. Also, by Assumption 1.2.2,

$$n\left(\mathbf{x}^{-j}\right) = m(\mathbb{E}[y|s], \mathbf{x}^{-j}, \underline{x}^j), \quad n'\left(\mathbf{x}^{-j}\right) = m'(\mathbb{E}[y|s], \mathbf{x}^{-j}, \underline{x}^j)$$

11. See the previous footnote for this analogous restriction.

a.s. on $supp(\mathbb{E}[y|s], \mathbf{x}^j | x^j = \underline{x}^j)$. Since s is discrete, the above is true on a.s. on $supp(\mathbb{E}[y|s], \mathbf{x}^j, s | x^j = \underline{x}^j)$ too. So (1.2) collapses to

$$n\left(\mathbf{x}^{-j}\right) + \mu(s) = \mathbb{E}[y|\mathbf{x}^{-j}, \underline{x}^j, s] = n'\left(\mathbf{x}^{-j}\right) + \mu'(s) \;\Rightarrow\; (\mu' - \mu)(s) = (n - n')\left(\mathbf{x}^{-j}\right) \tag{1.3}$$

a.s. on $supp(\mathbb{E}[y|s], \mathbf{x}^{-j}, s | x^j = \underline{x}^j)$. The implication of (1.3) hence holds a.s. on $supp(\mathbf{x}^{-j}, s | x^j = \underline{x}^j)$ too. It follows $(\mu' - \mu)(\tilde{s}) = (n - n')\left(\tilde{\mathbf{x}}^{-j}\right)$ whenever $(\tilde{\mathbf{x}}^{-j}, \underline{x}^j, \tilde{s}) \in supp(\mathbf{x}, s)$. In particular, since $(\mathbf{x}_1^{-j}, \underline{x}_j, s_1) \in supp(\mathbf{x}, s)$,

$$(\mu' - \mu)(s'') = (n - n')\left(\mathbf{x}_1^{-j}\right) = (\mu' - \mu)(s_1),$$

for all s'' satisfying $(\mathbf{x}_1^{-j}, \underline{x}^j, s'') \in supp(\mathbf{x}, s)$, where the second equality holds by Assumption 1.2.3. So $(\mu' - \mu)(s'')$ is constant in s'' on $\cup_{n \leq 1}\{s' : (\mathbf{x}_n^{-j}, \underline{x}^j, s') \in supp(\mathbf{x}, s)\}$.

Now suppose for some $N \in \mathbb{N}$,

$$(\mu' - \mu)(s'') = (\mu' - \mu)(s_N) = (\mu' - \mu)(s_{N-1}) = ... = (\mu' - \mu)(s_1)$$

for all $s'' \in \cup_{n \leq N}\{s' : (\mathbf{x}_n^{-j}, \underline{x}^j, s') \in supp(\mathbf{x}, s)\}$. Then Assumption 1.2.3 combined with (1.3) implies

$$(\mu' - \mu)(s_{N+1}) = (n - n')(\mathbf{x}_N^{-j}) = (\mu' - \mu)(s_N) = ... = (\mu' - \mu)(s_1),$$

Moreover, for each s'' satisfying $(s'', \mathbf{x}_{N+1}^{-j}, \underline{x}^j) \in supp(s, \mathbf{x})$, then

$$(\mu' - \mu)(s'') = (n - n')(\mathbf{x}_{N+1}^{-j}) = (\mu' - \mu)(s_{N+1}),$$

where the second equality holds by Assumption 1.2.3. It follows $(\mu' - \mu)(s'')$ is constant in s'' over $\cup_{n \leq N+1}\{s' : (\mathbf{x}_n^{-j}, \underline{x}^j, s') \in supp(\mathbf{x}, s)\}$ too.

By induction, one obtains $(\mu - \mu')(s)$ as being constant in s over $\cup_{n\in\mathbb{N}}\{s' : (\mathbf{x}_n^{-j}, \underline{x}^j, s') \in supp(\mathbf{x}, s)\}$. Let its common value equal b. From (1.2),

$$(m - m')\left(\mathbb{E}[y|s], \mathbf{x}\right) = (\mu' - \mu)(s) = -b \tag{1.4}$$

a.s. on $supp(\mathbb{E}[y|s], \mathbf{x}, s)$. It follows we can set $a = -b$, and the proof is completed. □

The proof exploits the non-linearity of the relationship between individual outcomes, characteristics and expected peer outcome. Intuitively, because peer effects are captured by expected peer outcome $\mathbb{E}[y|s]$ rather than its finite sample analogue $\overline{y}_s$, the issue of simultaneity bias traditionally impeding peer effects identification is no longer present. Instead, endogeneity stems from correlated effects $\mathbb{E}[\epsilon|s]$ confounding estimates of peer effects. Fortunately, Assumption 1.2.2 implies that when individuals are observed not to interact with their peers $x^j = \underline{x}^j$, transmission of peer effects shuts down. The correlated effect term is thus identified from cross peer group outcome variation for zero interaction individuals. But once correlated effects have been identified, they can then be partialled out from variation in cross peer group outcomes, for the entire population. The remaining variation is the peer effect.

As previously alluded to, contrasts such as $m(e, \mathbf{x}) - m(e', \mathbf{x})$ are unidentified, absent assumptions such as 1.2.2 and 1.2.3, and restrictions on the data's support. Consider the following counter-example. Suppose individuals were randomly assigned to peer groups so that $\mathbf{x}$ and ϵ are statistically independent to s. The model implies

$$\mathbb{E}\left[y|s\right] = \int m\left(\mathbb{E}\left[y|s\right], \mathbf{u}\right) dF_{\mathbf{X}}(\mathbf{u}) \qquad \forall s \in supp(s). \tag{1.5}$$

When the above pins down $\mathbb{E}\left[y|s\right]$ to a constant value for all s, then $m(e, \mathbf{x})$ is only identified on a lower dimensional set in $\mathbb{R}^{K+1}$, whose width for the dimension associated with e is zero. It follows that meaningful variation in outcomes caused by changes in peer outcomes, $m(e', \mathbf{x}) - m(e, \mathbf{x})$, cannot be inferred from the distribution of $(y, \mathbf{x}, s)$ unless $e' = e$. This argument proves

Manski's (1993) result, restated below for completeness.

Proposition 1.2.2. *Assume $\mu(s) = 0$, individual characteristics $\mathbf{x}$ are statistically independent to peer group s, and equation (1.5) has a unique solution $e^*(s) \in \mathbb{R}$ for $\mathbb{E}[y|s]$. Then $e^*(s)$ does not depend on s, and $m : \mathbb{R}^{K+1} \to \mathbb{R}$ is unidentified on points $(e, \mathbf{x}) \in \mathbb{R}^{K+1}$ where $e \neq e^*$. Contrasts of the form $m(e', \mathbf{x}) - m(e, \mathbf{x})$ where $e' \neq e$ are thus unidentified.*

The result implies random assignment of individuals to peer groups may be unhelpful for identifying peer effects. If peer effects are thought to operate within large peer groups, then the law of large numbers implies average peer outcome converges to expected peer outcome, which is constant by Proposition 1.2.2. Variation in average peer outcome needed to identify any effect it has on individual outcomes thus vanishes when peer groups are infinitely large.

Also, observe the relationship

$$y_i = \mathbb{E}[y|s_i] + \epsilon_i, \qquad \mathbb{E}[\epsilon_i|s_i] = 0, \tag{1.6}$$

or more generally,

$$y_i = \mathbb{E}[y|s_i] + n(\mathbf{x}_i) + \epsilon_i, \qquad \mathbb{E}[n(\mathbf{x}_i)|s_i] = \mathbb{E}[\epsilon_i|\mathbf{x}_i, s_i] = 0 \tag{1.7}$$

perilously satisfy the model's main equation $y_i = m(\mathbb{E}[y|s_i], \mathbf{x}_i) + \epsilon_i$. As explained in Angrist (2014), these relationships are trivial since they imply within peer group variation in outcomes are completely explained by chance. The second equation (1.7) corresponds to the situation where within peer group variation in individual outcomes are partially explained by individual characteristics. Again, this is a relationship that would be expected to hold in a variety of contexts where peer effects are not at play.

Fortunately, the prospect of (1.6) or (1.7) generating the data is ruled out by Assumption

1.2.2. When there is no interaction $x_i^j = \underline{x}^j$, outcome y_i no longer depends on expected peer outcome $\mathbb{E}[y|s_i]$, as it does in (1.6) or (1.7). Moreover, Assumption 1.2.2 is testable. If 1.2.2 is true, then average improvement in outcomes for zero interaction individuals due to changes in $\mathbf{x}^{-j}$ do not depend on s.

Proposition 1.2.3. *Assume* (1.1) *holds for all* $(y_i, \mathbf{x}_i, s_i) \in supp(y, \mathbf{x}, s)$*, and Assumption 1.2.1 holds.*

1. *Suppose Assumption 1.2.2 hold under the null hypothesis, but does not hold under the alternate. Under the null, cross characteristics differences in outcomes for zero interaction individuals do not depend on peer group:*

$$(\mathbb{E}[y|\underline{x}^j, \mathbf{x}_1^{-j}, s_1] - \mathbb{E}[y|\underline{x}^j, \mathbf{x}_2^{-j}, s_1]) - (\mathbb{E}[y|\underline{x}^j, \mathbf{x}_1^{-j}, s_2] - \mathbb{E}[y|\underline{x}^j, \mathbf{x}_2^{-j}, s_2]) = 0$$

   when $(\mathbf{x}_1^{-j}, s_1), (\mathbf{x}_2^{-j}, s_1), (\mathbf{x}_1^{-j}, s_2), (\mathbf{x}_2^{-j}, s_2) \in supp(\mathbf{x}^{-j}, s|x^j = \underline{x}^j)$.

2. *Let* (1.7) *denote the null hypothesis, and suppose* (1.7) *does not hold in the alternate. Under the null, cross-peer group differences in outcomes do not depend on individual characteristics:*

$$(\mathbb{E}[y|\mathbf{x}_1, s_1] - \mathbb{E}[y|\mathbf{x}_1, s_2]) - (\mathbb{E}[y|\mathbf{x}_2, s_1] - \mathbb{E}[y|\mathbf{x}_2, s_2]) = 0$$

   for all $(\mathbf{x}_1, s_1), (\mathbf{x}_1, s_2), (\mathbf{x}_2, s_1), (\mathbf{x}_2, s_2) \in supp(\mathbf{x}, s)$.

Two points should be clarified in relation to Proposition 1.2.3. First, that (1.6) and (1.7) are tautological is *not* the cause for concern. Observe $y = m(\mathbf{x}) + \epsilon$ where $\mathbb{E}[\epsilon|\mathbf{x}] = 0$ is tautological and nested by (1.1) too. Rather, the issue is that were (1.6) or (1.7) to generate the data, one would hesitate attributing cross-peer correlations in outcomes to social interactions, since (1.6) and (1.7) are too trivial to capture theoretical hypotheses explaining why social effects exist[12]. Second, the possibility of ruling out trivial peer effects interpretations is not cause for

12. These hypotheses are that individuals respond to social norms, information externalities, or strategic complementaries in their payoffs when exhibiting peer effects through chosen actions (See the introductions of Duflo & Saez (2003), Bobonis & Finan (2009) and Dahl et al. (2014)).

estimating peer effects non-parametrically. Interacting attendance with average peer outcomes in the linear peer effects model, and estimating its coefficient, serves the same purpose[13]. The point being made is that absent apriori restrictions on the linear model's coefficients[14], correctly interpreting peer effects requires at least a non-linear formulation of social interactions.

Finally, observe the proof of identification continues to hold if $\mathbb{E}[y|s]$ is replaced by $\mathbb{V}[y|s]$, or any known function $g(.)$ of s:

$$y_i = m\left(g(s_i), \mathbf{x}_i\right) + \epsilon_i \quad \forall i \leq I. \tag{1.8}$$

Moreover, even if $g(s)$ were vector valued, (such as $g(s) = (\mathbb{E}[y|s], \mathbb{E}[\mathbf{x}|s])$), m is still identified on the support of $g(s)$ and $\mathbf{x}$, under an appropriately modified Assumption 1.2.2. This opens up the possibility of studying *social effects* non-parametrically.

The issue is that if the entries of $g(s)$ are functionally dependent as functions of s, then the support on which m rests upon might be lower dimensional. For example, say $g(s) = (\mathbb{E}[y|s], \mathbb{E}[\mathbf{x}|s])$ but the family of distributions $\{F_{y,\mathbf{x}|s} : s \leq S\}$ implies a one-to-one relationship between $\mathbb{E}[y|s]$ and $\mathbb{E}[\mathbf{x}|s]$. The model is thus

$$y_i = m\left(\mathbb{E}[y|s_i], \mathbb{E}[\mathbf{x}|s_i], \mathbf{x}_i\right) + \epsilon_i \quad \forall i \leq I.$$

This is a full social effects model incorporating exogenous effects. However contrasts of the

13. Bertrand et al. (2000) document peer effects by estimating a similar model, where one's peers consists of same language speakers (living in the same area). In their model, individual welfare usage is linear in peer welfare usage interacted with the density of same language speakers per location. Bobonis & Finan (2009) also document how households with similar incomes to households rewarded for enrolling their children in school, mimic the rewarded households more strongly than households with different incomes. They attribute greater interaction levels between similar income households as a possible reason for the differential peer effect.

14. See Proposition 2 in Manski (1993), requiring $\beta \neq 1$ or Section 2 of Bramoulle et al. (2009), assuming $|\beta| < 1$, for examples of such restrictions on peer effects. Blume et al. (2015) model peer effects as stemming from conformity effects rather than strategic complementarities, implying a strictly positive peer effect coefficient: $\frac{1}{1+\phi} > 0$.

form

$$m(\mathbb{E}[y|s], \mathbb{E}[\mathbf{x}|s], \mathbf{x}) - m(\mathbb{E}[y|s'], \mathbb{E}[\mathbf{x}|s], \mathbf{x}) \quad \text{or} \quad m(\mathbb{E}[y|s], \mathbb{E}[\mathbf{x}|s], \mathbf{x}) - m(\mathbb{E}[y|s], \mathbb{E}[\mathbf{x}|s'], \mathbf{x})$$

remain unidentified. Absent a full-rank condition on $(\mathbb{E}[y|s], \mathbb{E}[\mathbf{x}|s])$, Manski's non-identification result rears its head - peer effects and exogenous effects confound each other, and cannot be separated empirically.

However, the aggregate social effect, $m(\mathbb{E}[y|s], \mathbb{E}[\mathbf{x}|s], \mathbf{x}) - m(\mathbb{E}[y|s'], \mathbb{E}[\mathbf{x}|s'], \mathbf{x})$, is still identifiable on the support of s and $\mathbf{x}$.

Proposition 1.2.4. *Suppose there exists $g : \mathbb{N} \to \mathbb{R}^G$, and $\underline{x}_j \in \cap_{s' \in supp(s)} supp(x^j|s = s')$ known to the researcher, such that $m(\mathbf{g}, \mathbf{x}^{-j}, \underline{x}^j) = m(\mathbf{g}', \mathbf{x}^{-j}, \underline{x}^j)$ for all $(\mathbf{g}, \mathbf{x}^{-j}), (\mathbf{g}', \mathbf{x}^{-j}) \in supp(g(s), \mathbf{x}^{-j}|x^j = \underline{x}^j)$. Then if the main equation is given by (1.8) and Assumption 1.2.1 holds, the composition $mg : \mathbb{N}_{\leq S} \times \mathbb{R}^K \to \mathbb{R}$ defined by*

$$mg(s, \mathbf{x}) \equiv m(g(s), \mathbf{x})$$

and $\mu(s)$ are identified on the support of $(s, \mathbf{x})$. Contrasts of the form $m(\mathbb{E}[y|s], \mathbb{E}[\mathbf{x}|s], \mathbf{x}) - m(\mathbb{E}[y|s'], \mathbb{E}[\mathbf{x}|s'], \mathbf{x})$ are thus identified from $mg(s, \mathbf{x}) - mg(s', \mathbf{x})$ on the same support.

The proof simply extends from the identification proof of the original model. When interpreted in Example 1.2.1's context, Proposition 1.2.4 implies the total social effect experienced by a student when his school is counterfactually changed is identified, inspite of school choice being endogenous[15].

15. This discussion stems from a question by Soonwoo Kwon at the Yale Prospectus Lunch on 25-09-2017. Also, notice that one possible critique of the model are that in the context of the education example, students who don't attend school should not be transmitting peer effects. This point is addressed if one sets $g(s) = \mathbb{E}[y|s, x^j > 0]$ in place of $\mathbb{E}[y|s]$ and recognizes that the model continues to be identified.

1.2.3 Cross-Sectional Estimation

As aforementioned, the identification strategy naturally leads to a simple procedure for estimating $m(\mathbb{E}[y|s], \mathbf{x})$ and $\mu(s)$, as functions $(\mathbf{x}, s)$ and s respectively.

Step 1: Estimate $\mu(s)$ by taking the kernel weighted mean of outcomes evaluated at $\mathbf{x} = \mathbf{0}$:

$$\hat{\mu}(s) = \sum_{i \leq I} \frac{\{\mathbf{d}_i = (\mathbf{0}, s)\} K\left(\frac{\mathbf{c}_i}{b}\right) y_i}{\sum_{i \leq I} \{\mathbf{d}_i = (\mathbf{0}, s)\} K\left(\frac{\mathbf{c}_i}{b}\right)}.$$

Step 2: Non-parametrically regress $\{y_i\}_{i \leq I}$ on $\{\mathbf{x}_i, s_i\}_{i \leq I}$ to estimate $h(\mathbf{x}, s) \equiv m(\mathbb{E}[y|s], \mathbf{x})$ via

$$\hat{h}(\mathbf{x}, s) = \sum_{i \leq I} \frac{\{\mathbf{d}_i = \mathbf{d}\} K\left(\frac{\mathbf{c}_i - \mathbf{c}}{b}\right) y_i}{\sum_{i \leq I} \{\mathbf{d}_i = \mathbf{d}\} K\left(\frac{\mathbf{c}_i - \mathbf{c}}{b}\right)} - \hat{\mu}(s).$$

Step 3: Estimate unobserved heterogeneity for all $i \leq I$ via

$$\hat{\epsilon}_i = y_i - \hat{h}(\mathbf{x}_i, s_i)$$

and estimate $F_{\epsilon|y,\mathbf{x},s}$ using standard kernel-based methods.

It turns out that Assumptions 1.2.1 to 1.2.7 plus Assumption 1.2.11, implies the above procedure produces consistent estimates for the non-parametric relationship between outcomes, peer groups and observable characteristics.

Proposition 1.2.5. *If assumptions 1.2.1, 1.2.2, 1.2.4, 1.2.5, 1.2.6, 1.2.7, and 1.2.11 hold, then $\hat{h}(\mathbf{x}, s)$ and $\hat{\mu}(s)$ are pointwise consistent for $m(\mathbb{E}[y|s], \mathbf{x})$ and $\mu(s)$:*

$$\hat{h}(\mathbf{x}, s) \xrightarrow{p} m(\mathbb{E}[y|s], \mathbf{x})$$
$$\hat{\mu}(s) \xrightarrow{p} \mu(s)$$

as $I \to \infty$, $b \to 0$ for each $(\mathbf{x}, s) \in supp(\mathbf{x}, s)$.

Moreover, standard errors for the estimators can be obtained under Assumptions 1.2.1 to 1.2.11.

Proposition 1.2.6. *If assumptions 1.2.1, 1.2.2, 1.2.4, 1.2.5, 1.2.6, 1.2.7, 1.2.8, 1.2.9, 1.2.10 and 1.2.11 hold, then $\hat{h}(\mathbf{x}, s)$ and $\hat{\mu}(s)$ are asymptotically normal:*

$$\sqrt{Ib^C}\left(\hat{h} - h + O(b^2)c\right)(\mathbf{x}, s) \xrightarrow{d} \mathcal{N}\left(0, \frac{R^K\sigma^2(\mathbf{x}, s)}{f_{\mathbf{x},s}(\mathbf{x}, s)} + \frac{R^K\sigma^2(\mathbf{0}, s)}{f_{\mathbf{x},s}(\mathbf{0}, s)} - 2\{\mathbf{x} = \mathbf{0}\}\frac{R^K\sigma^2(\mathbf{0}, s)}{f_{\mathbf{x},s}(\mathbf{0}, s)}\right)$$
$$\sqrt{Ib^C}\left(\hat{\mu} - \mu + O(b^2)c\right)(s) \xrightarrow{d} \mathcal{N}\left(0, \frac{R^K\sigma^2(\mathbf{0}, s)}{f_{\mathbf{x},s}(\mathbf{0}, s)}\right)$$

as $Ib^C \to \infty$, $b \to 0$ for each for each $(\mathbf{x}, s) \in supp(\mathbf{x}, s)$, where $c(\mathbf{x}, s)$ is a C^3 function on the support of $(\mathbf{x}, s)$, and

$$R^K \equiv \int K(\mathbf{c})^2 d\mathbf{c} \qquad \sigma^2(\mathbf{x}, s) \equiv \mathbb{E}[\eta^2|\mathbf{x}, s].$$

I estimate $h(s, \mathbf{x}) = m(\mathbb{E}[y|s], \mathbf{x})$ and $\mu(s)$ using a kernel-based approach instead of more sophisticated methods. Asymptotic properties for kernel based estimators are easier to establish. The proof of propositions 1.2.5 and 1.2.6 are adapted from the standard asymptotic arguments for Nadaraya-Watson estimation. The conceptual difficulties in the proofs lie in allowing for discretely-distributed regressors, and controlling for correlated effects that confound the effect of peer outcomes on individual outcomes. The first problem is basically dealt with by assuming the discretely-distributed regressors have finite support (as in Assumption 1.2.5), and conditioning out discretely-distributed regressors when establishing mean squared convergence of estimators. The second problem is overcome by using the interaction index to estimate $\mu(s)$ first, before estimating the non-parametric function $h(\mathbf{x}, s)$ in a second stage. The two-stage estimation approach works due to errors being additive in the model, thereby allowing μ to be estimated separately from h[16].

16. Also, observe that the model's separate treatment of discrete and continuously distributed random variables imply standard errors converging at a rate $\sqrt{Ib^C}$ faster than $\sqrt{Ib^K}$, since C and K are the dimensions of $\mathbf{c}$ and $\mathbf{x}$ respectively. There is an (unreviewed) literature on discrete random variable kernels that might help a researcher interested in further improving estimation along these lines.

The estimation method appears to circumvent Angrist's critique of peer effects estimates reflecting meaningless differences between IV and OLS estimands. In particular, the method is non-parametric while eschewing use of instruments for expected peer outcomes. Mechanical relationships between linear model estimands and weak instruments for average peer outcomes are thus non-issues. Also, observe Assumption 1.2.5 implies the expected number of individuals per peer group $s \in supp(s)$ tends to infinity as $I \to \infty$. The asymptotic situation thus features infinitely large peer groups. This should be contrasted with Proposition 1 in Bramoulle et al. (2009), where the asymptotic variable seems to be the number of networks or peer groups instead.

An attractive avenue for further research would be to understand the asymptotic behavior of $\hat{h}(s, \mathbf{x})$ and $\hat{\mu}(s)$ when the model is mis-specified and the true model incorporates both peer and exogenous effects:

$$y_i = m(\mathbb{E}[y|s_i], \mathbb{E}[\mathbf{x}|s_i], \mathbf{x}_i) + \epsilon_i.$$

In light of Proposition 1.2.4, it is natural to believe $\hat{h}(\mathbf{x}, s)$ is still consistent for the true model $m(\mathbb{E}[y|s], \mathbb{E}[\mathbf{x}|s], \mathbf{x})$. However, this statement still needs to be proved.

My main concern with the overall identification and estimation strategy is the lack of observations $i \leq I$ featuring $(x_i^j, s_i) = (\underline{x}^j, s)$ in step 1. A small number of such observations leads to large standard errors for $\hat{\mu}(s)$ in the first stage, implying imprecise second stage estimators. If the interaction index x^j is continuously distributed (i.e. $x^j \in \mathbf{c}$), then the estimation procedure partially circumvents the problem. Intuitively, because $\mathbb{E}[y|\underline{\boldsymbol{x}}, s]$ is continuous in $\mathbf{c}$ under Assumption 1.2.8, so observations of $(\mathbf{c}_i^{-j}, x_i^j)$ close to $(\underline{\mathbf{c}}_i^{-j}, \underline{x}^j)$ can be used to approximate $\mu(s)$ or equivalently, $\mathbb{E}[y|\underline{\boldsymbol{x}}, s]$ in stage one. Within the context of Example 1.2.1, the random variable x^j can be modeled as being continuously distributed if say x_i^j was individual i's attendance rate in school. However, if the interaction index is discrete, then the possibility of having no observations of $(x_i^j, s_i) = (\underline{x}^j, s)$ to estimate $\mu(s)$ in stage one rears its head.

1.3 Identification from Panel Data

Now, consider the situation where the unit of observation varies over individuals and time. There are still I individuals and S possible peer groups. However, outcomes and characteristics associated with each individual are observed over T periods. More specifically, for each individual $i \leq I$ and period $t \leq T$, a researcher observes $(y_{i,t}, \mathbf{x}_{i,t}, s_i) \in \mathbb{R}^{K+1} \times \mathbb{N}_{\leq S}$, where the notation in this section corresponds to the notation introduced in the previous section. The dataset is hence $\{y_{i,t}, \mathbf{x}_{i,t}, s_i\}_{(i,t)\leq(I,T)}$. Notice the individual's peer group s_i is assumed to be carefully defined by the researcher so as to be time invariant. Hence, in context of the education example introduced in Section 1.2, this would mean students never drop out from or enter into schools.

In what follows, I will write $\mathbf{y}_i^t$ to denote individual i's truncated private history of outcomes $\{y_{i,\tau}\}_{t-l\leq\tau\leq t}$, $\mathbf{y}_{s,t}$ to denote peer group s's contemporaneous outcomes $\{y_{i,t}\}_{t,i\in s}$, $\mathbf{y}_s^t$ to denote peer group s's (truncated) historical outcomes $\{\mathbf{y}_i^t\}_{t,i\in s}$, and $\mathbf{y}^t$ to denote the entire population's historical outcomes. Analogous notation is used for individual characteristics $\mathbf{x}_{i,t}$ and peer group s_i.

Example 1.3.1. [Network Externalities] I counties are partioned into S states in a global market for a word processing software, or any other good featuring network externalities. A software purchased in period t is compatible with softwares purchased from periods $t - l$ to $t + l$. Let $y_{i,t}$ be the amount of software purchased in county i in period t. This depends on the expected number of future software users $\sum_{l+t\geq\tau\geq t} \mathbb{E}[y_{i,\tau}|s_i, \mathbf{y}_s^{t-1}, \mathbf{x}_s^t]$ and past number of buyers $\sum_{t-l\leq\tau<t} \overline{y}_{s,\tau}$. $\mathbf{x}_{i,t}$ contains demand shifters such as county i's demographic information, and cost shifters such as the distance from county i to the nearest software distribution center.

1.3.1 Model

In the initial period, a vector of individual outcomes $\mathbf{y}_{-1}$, characteristics $\mathbf{x}_0$ and peer groups $\mathbf{s}$ is predetermined. In any arbitrary period $t \geq 1$ onwards, individual characteristics for peer group s, $\mathbf{x}_{s,t}$ are jointly drawn from $F_{\mathbf{x}_{s,t}|\mathbf{y}_{s,t-1},\mathbf{x}_{s,t-1},\ldots,\mathbf{y}_{s,-1},\mathbf{x}_{s,0},s}$, while outcomes and private heterogeneity are drawn i.i.d. from $F_{y_{i,t},\epsilon_{i,t}|\mathbf{y}_{s,t-1},\mathbf{x}_{s,t},\ldots,\mathbf{y}_{s,-1},\mathbf{x}_{s,0},s}$ across individuals in a given peer group. For each $(i,s) \leq (I,S)$, assume

$$F_{\mathbf{x}_{s,t}|\mathbf{y}_{s,t-1},\mathbf{x}_{s,t-1},\ldots,\mathbf{y}_{s,-1},\mathbf{x}_{s,0},s}(\mathbf{x}|\mathbf{y}_{s,t-1},\mathbf{x}_{s,t-1},\ldots,\mathbf{y}_{s,-1},\mathbf{x}_{s,0},s) = D(\mathbf{x}|\mathbf{y}_s^{t-1},\mathbf{x}_s^{t-1},s)$$
$$F_{y_{i,t},\epsilon_{i,t}|\mathbf{y}_{s,t-1},\mathbf{x}_{s,t},\ldots,\mathbf{y}_{s,-1},\mathbf{x}_{s,0},s}(y,\epsilon|\mathbf{y}_{s,t-1},\mathbf{x}_{s,t},\ldots,\mathbf{y}_{s,-1},\mathbf{x}_{s,0},s) = G(y,\epsilon|\mathbf{y}_s^{t-1},\mathbf{x}_s^{t},s).$$

So the distributions of $\mathbf{x}_{s,t}, y_{i,t}$ and $\epsilon_{i,t}$ conditional on the entire history of past information, only depend on information from the past l periods. The stochastic process governing each individual's outcome, error and peer characteristics can thus be said to be "lth-order markov". Moreover, the conditional distributions depends on past peer outcomes and characteristics via the same functions D and G, for all individuals and periods. Also, assume these distributions have full supports over sets that are independent of the conditioning variables. This enables the researcher to learn $D(\mathbf{x}|\mathbf{y}_s^{t-1},\mathbf{x}_s^{t-1},s)$ and $\int G(y,\epsilon|\mathbf{y}_s^{t-1},\mathbf{x}_s^{t},s)d\epsilon$ from an infinitely long panel of observations.

Finally, assume G implies the following relationship between the latent variables and the data:

$$y_{i,t} = m\left(\frac{1}{l+1}\sum_{t+l\geq\tau\geq t}\mathbb{E}\left[y_\tau|\mathbf{y}_s^{\tau-1},\mathbf{x}_s^{\tau},s\right] + \frac{1}{l}\sum_{t-l\leq\tau<t}\overline{y}_{s,\tau},\ \mathbf{x}_{i,t}\right) + \epsilon_{i,t}, \tag{1.9}$$

when $s = s_i$, for all $i \leq I$ and $t = 1,\ldots,T$.

So an individual's outcome is a non-parametric function of the average peer outcome over $2l+1$ periods, rather than merely the contemporaneous average peer outcome. Moreover, observe that the conditional expectations inside the function $m(e, \mathbf{x})$ depends on the truncated history of past peer outcomes and characteristics $(\mathbf{y}_s^{t-1}, \mathbf{x}_s^t)$. Since the conditioning set consists of peer-group specific variables, so as in the previous section's set-up, peer effects stem from all peers in the individual's peer group, rather than only those with similar characteristics to his. In the context of Example 1.3.1, (1.9) captures the equilibrium relationship between software sales in county i and county i's characteristics, in the software market. The error $\epsilon_{i,t}$ is an idiosyncratic taste shock specific to county i.

Consider the identification problem associated when panel data corresponding to the asymptotic situation whereby the number of periods T tends to infinity. The researcher knows l. In the context of Example 1.2.1, l can be set to the number of years a student spends in school. In the context of Example 1.3.1, l can be calibrated from the specifications of word-processing softwares released each year. The researcher also knows the marginal distributions of the data $F_{\mathbf{x}_t|\mathbf{x}_s^{t-1},s}$ and $F_{y_t|\mathbf{y}_s^{t-1},\mathbf{x}_s^t,s}$. These distributions are directly identifiable from cross-period variation in the data. The researcher does not know the joint distribution of the outcome and the latent error $F_{y_t,\epsilon_t|\mathbf{y}_s^{t-1},\mathbf{x}_s^{t-1},s} \equiv G$ or the function $m : \mathbb{R}^{K+1} \to \mathbb{R}$. The goal is to characterize conditions on the data generating process under which m and G are identified.

Let $\mathcal{X}$ denote the union over t of *supp* $\left(\frac{1}{l+1}\sum_{t+l\geq\tau\geq t}\mathbb{E}\left[y_\tau|\mathbf{y}^{\tau-1}, \mathbf{x}^\tau, s\right] + \frac{1}{l}\sum_{t-l\leq\tau<t}\overline{y}_{s,\tau}, \mathbf{x}_t\right)$ in what follows. Consider the following assumptions.

Assumption 1.3.1. [Mean Independence] Error reflecting unobserved heterogeneity ϵ is mean independent of historical characteristics $\mathbf{x}_s^t$ and outcomes $\mathbf{y}_s^{t-1}$, given peer group s: $\mathbb{E}[\epsilon_t|\mathbf{y}_s^{t-1}, \mathbf{x}_s^t, s] = \mathbb{E}[\epsilon_t|s] \equiv \mu(s)$, where $\mu : \mathbb{N} \to \mathbb{R}$ is unknown to the researcher.

The rationale behind Assumption 1.3.1 has already been elaborated upon in the previous section.

Assumption 1.3.2. [Function Differentiability] For each period $t \in \mathbb{N}$, the support of m :

$\mathbb{R}^{K+1} \to \mathbb{R}$'s arguments, $\mathcal{X}$, is open and convex. Also, $m(e, \mathbf{x})$ is continuously differentiable in $(e, \mathbf{x})$ on $\mathcal{X}$.

Assumption 1.3.2 ensures $m(e, \mathbf{x})$ is identified on the support of its arguments, instead of just the support of its instruments and exogenous regressors.

Assumption 1.3.3. [Rank Condition] There exists $z_{i,t} \in \{\mathbf{y}_s^{t-1}, \mathbf{x}_{-i,s}^t\}$ such that the derivative

$$\frac{\partial \mathbb{E}[y_t|\mathbf{y}_s^{t-1}, \mathbf{x}_s^t, s]}{\partial z_{i,t}}$$

exists and is non-zero for each individual $i \leq I$ in peer group $s \leq S$ and period $t \leq T$.

Assumption 1.3.3 corresponds to the usual rank condition for IV, if one sees lagged outcomes $y_{i,t-j}$ as instruments for contemporaneous and future expected peer outcomes $\mathbb{E}[y_{t+k}|\mathbf{y}_s^{t+k-1}, \mathbf{x}_s^{t+k}, s]$.

It turns out that these conditions are sufficient to identify the model, up to a constant.

1.3.2 Panel Identification

Proposition 1.3.1. *Under assumptions 1.3.1, 1.3.2 and 1.3.3, $m : \mathbb{R}^{K+1} \to \mathbb{R}$ and $\mu : \mathbb{N}_{\leq S} \to \mathbb{R}$ are identified up to constants that sum to zero on $\mathcal{X}$.*

Proof. Taking expectations on both sides obtains

$$\mathbb{E}[y_t|\mathbf{y}_s^{t-1}, \mathbf{x}_s^t, s] = m\left(\frac{1}{l+1}\sum_{t+l\geq\tau\geq t}\mathbb{E}\left[y_\tau|\mathbf{y}_s^{\tau-1}, \mathbf{x}_s^\tau, s\right] + \frac{1}{l}\sum_{t-l\leq\tau<t}\overline{y}_{s,\tau}, \mathbf{x}_t\right) + \mu(s). \tag{1.10}$$

The researcher can compute L.H.S. of the above from G, which he knows. Assume there exists $m' : \mathbb{R}^{K+1} \to \mathbb{R}$ and $\mu' : \mathbb{N}_{\leq S} \to \mathbb{R}$ that satisfy (1.10) too:

$$\mathbb{E}[y_t|\mathbf{y}_s^{t-1}, \mathbf{x}_s^t, s] = m'\left(\frac{1}{l+1}\sum_{t+l\geq\tau\geq t}\mathbb{E}\left[y_\tau|\mathbf{y}_s^{\tau-1}, \mathbf{x}_s^\tau, s\right] + \frac{1}{l}\sum_{t-l\leq\tau<t}\overline{y}_{s,\tau}, \mathbf{x}_t\right) + \mu'(s).$$

Then equating both sides and subtracting obtains

$$(m - m')\left(\frac{1}{l+1}\sum_{t+l\geq\tau\geq t}\mathbb{E}\left[y_\tau|\mathbf{y}_s^{\tau-1},\mathbf{x}_s^\tau,s\right]+\frac{1}{l}\sum_{t-l\leq\tau<t}\overline{y}_{s,\tau},\mathbf{x}_t\right)=(\mu-\mu')(s).$$

Moreover, differentiating obtains

$$\begin{aligned}0 &= \frac{\partial(m-m')}{\partial e}\left(\frac{1}{l+1}\sum_{t+l\geq\tau\geq t}\mathbb{E}\left[y_\tau|\mathbf{y}_s^{\tau-1},\mathbf{x}_s^\tau,s\right]+\frac{1}{l}\sum_{t-l\leq\tau<t}\overline{y}_{s,\tau},\mathbf{x}_t\right)\\ &\times\frac{1}{l+1}\sum_{t+l\geq\tau\geq t}\frac{\partial\mathbb{E}\left[y_\tau|\mathbf{y}_s^{\tau-1},\mathbf{x}_s^\tau,s\right]}{\partial z_t}.\end{aligned}$$

So $(m-m')(e,\mathbf{x})$ is constant in e. Also,

$$0=\frac{\partial(m-m')}{\partial\mathbf{x}}\left(\frac{1}{l+1}\sum_{t+l\geq\tau\geq t}\mathbb{E}\left[y_\tau|\mathbf{y}_s^{\tau-1},\mathbf{x}_s^\tau,s\right]+\frac{1}{l}\sum_{t-l\leq\tau<t}\overline{y}_{s,\tau},\mathbf{x}_t\right).$$

So $(m-m')(e,\mathbf{x})$ is constant in e. It follows $m-m'$ is a constant. Hence $\mu-\mu'$, by virtue of equaling $m'-m$ must also be a constant which when summed with $m-m'$ equals zero. □

The above result is interesting because i) no interaction index restriction analogous to Assumption 1.2.2 is imposed and ii) the result continues to hold even when $\mu(s) = 0$ for all $s \leq S$. Being able to observe individual outcomes over multiple time periods greatly weakens the assumptions needed to identify peer effects. Intuitively, the change in lagged peer group information provides the variation in expected peer outcome, orthogonal to the errors, needed to identify contrasts in $m(e,\mathbf{x})$ caused by changing e. Manski's non-identification result (restated in Proposition 1.2.2) is thus circumvented.

The main drawback in the set-up just presented lies in the functional form implicitly chosen and disguised in the non-parametric formula displayed in (1.9). This assumes outcomes depend on mean peer outcomes over $2l+1$ periods, via a composition of a non-parametric function

$m(e, \mathbf{x})$, and a linear transformation $L(e_{t-l}, \dots, e_{t+l})$:

$$y_{i,t} = m\left(L\left(\overline{y}_{t-l,s}, \dots, \overline{y}_{t-1,s}, \mathbb{E}[y_t|\mathbf{y}_s^{t-1}, \mathbf{x}_s^t, s], \dots, \mathbb{E}[y_{t+l}|\mathbf{y}_s^{t+l-1}, \mathbf{x}_s^{t+l}, s]\right), \mathbf{x}_{i,t}\right) + \epsilon_{i,t}.$$

In the particular model described in Subsection 1.3.1, L corresponds to pre-multiplying a vector of conditional expectations and sample averages by a vector of $\frac{1}{2l+1}$'s. Of course, there is no reason for the average outcomes for each period to receive equal weights in a non-parametric model.

However, given that L is linear or satisfies an even milder property, Proposition 1.3.1 is the best we can hope for in the following sense:

Proposition 1.3.2. *Let $\{L_{\boldsymbol{\theta}} : \mathbb{R}^{2l+1} \to \mathbb{R} : \boldsymbol{\theta}\}$ be a family of functions parameterized by $\boldsymbol{\theta}$. If assumptions 1.3.1 and 1.3.2, along with the following hold:*

i) *for each pair of parameters $\boldsymbol{\theta}, \boldsymbol{\theta}' \in \mathbb{R}^{2l+1}$ and point $r \in \mathbb{R}$, there exists $r' \in \mathbb{R}$ such that*

$$L_{\boldsymbol{\theta}}^{-1}(r) = L_{\boldsymbol{\theta}'}^{-1}(r'),$$

ii) *for each individual $i \leq I$ and period $t \leq T$, the data is generated by:*

$$y_{i,t} = m\left(L\left(\overline{y}_{t-l,s}, \dots, \overline{y}_{t-1,s}, \mathbb{E}[y_t|\mathbf{y}_s^{t-1}, \mathbf{x}_s^t, s], \dots, \mathbb{E}[y_{t+l}|\mathbf{y}_s^{t+l-1}, \mathbf{x}_s^{t+l}, s]; \boldsymbol{\theta}\right), \mathbf{x}_{i,t}\right) + \epsilon_{i,t},$$

iii) *the functions $L(\mathbf{e}; \boldsymbol{\theta})$ are known to the researcher (for each value of $\boldsymbol{\theta}$), but $\boldsymbol{\theta}$ is unknown,*

then $m : \mathbb{R}^{K+1} \to \mathbb{R}$ and $\boldsymbol{\theta}$ are not identified.

For intuition, suppose the family of parameterized functions in Proposition 1.3.2 were the set of linear transformations given by $L(\mathbf{e}; \boldsymbol{\theta}) = \boldsymbol{\theta}\mathbf{e}$. Then, scaling $\boldsymbol{\theta}$ by a constant c while dividing L in the argument of $m(L, \mathbf{x})$ by the same amount, produces functions $L(\mathbf{e}; c\boldsymbol{\theta})$ and $m\left(\frac{L}{c}, \mathbf{x}\right)$. These functions define a new model that effectively mimics the data generating process from the old model. Because both models cannot be distinguished by the marginal distribution of the data, m and $\boldsymbol{\theta}$ are hence unidentified.

Proposition 1.3.2 implies that the researcher is required to take a stance on the value of $\boldsymbol{\theta}$ if $L(\mathbf{e};\boldsymbol{\theta})$ is assumed to hail from a known family of mappings parameterized by the vector $\boldsymbol{\theta}$. If $\boldsymbol{\theta}$ corresponds to weights on $2l+1$ conditional expectations of outcomes given $\mathbf{x}$ and s while $L(\mathbf{e};\boldsymbol{\theta})$ is the associated linear mapping, then a choice of weights must be made by the researcher to identify the model up to a constant. One can interpret the model analyzed in this section as corresponding to the neutral stance whereby the expectations are weighted symmetrically.

1.4 Microfoundations for the Cross-Sectional Model

As explained in Section 2.1, one factor motivating this paper's non-parametric approach is the recognition that the optimality conditions associated with the theoretical model implicitly underlying peer effects, need not take on a specific functional form.

Providing explicit microfoundations is important for four reasons. First, a micro-founded model improves interpretation of peer effects. They are now conceived as the outcomes arising from an incomplete information simultaneous moves game. Second, the arguments of $m(e,\mathbf{x})$ in (1.1) contain the expected peer outcome $\mathbb{E}[y|s]$, rather than it's sample analogue, $\overline{y}_s$. One might regard this as a puzzle. In this section, (1.1) is derived in the limit from the best responses of a game, as the number of peers tends to infinity.

Third, from a non-parametric perspective, $F_{y,\mathbf{x},s,\epsilon}$ is a primitive of the econometric model, thus pinning down $\mathbb{E}[y|s]$ uniquely. But (1.1) is also a primitive, implying $\mathbb{E}[y|s]$ must solve

$$e = \int m(e,\mathbf{x})dF(\mathbf{x}|s) + \mu(s) \qquad \forall s \leq S \tag{1.11}$$

for the model to be coherent. In Section 1.2, this issue was basically assumed away. Microfoundations relates this issue to the concept of pure-strategy equilibria not existing. The assumption of $\mathbb{E}[y|s]$ satisfying (1.11) can thus be replaced with hopefully more interpretable restrictions on the primitives of a game theoretic model. Finally, observe that (1.11) may admit multiple solutions. If the econometric model is taken to be the main equation (1.1) coupled with the conditional distribution of errors $F_{\epsilon|\mathbf{x},s}$ rather than $F_{y,\mathbf{x},s,\epsilon}$, then the model is incomplete. It turns out that such incompleteness stems from multiple equilibria existing in the underlying game, but the econometric model being estimated doesn't depend on the equilibria being coordinated upon. Hence, microfoundations help again by showing such incompleteness should not affect how one interprets peer effects estimates.

1.4.1 Bayesian Game

There are N players or individuals. The following sequence of play is observed.

1. For each individual $i \leq N$, nature draws a private type $(\mathbf{x}_i, \epsilon_i)$ and a peer group s_i i.i.d. from $F_{\mathbf{x},s,\epsilon} : \mathbb{N}_{\leq S} \times \mathbb{R}^{K+1} \to \mathbb{R}$. The peer group s_i is disclosed to all players. Without loss of generality, assume s has full support on $\mathbb{N}_{\leq S}$. The private type $\mathbf{x}_i$ is disclosed only to player i. Observe $\mathbf{x}_i$ and ϵ_i are allowed to correlate with each other, and with the peer group s_i. So $F_{\mathbf{x},s,\epsilon}$ possibly masks an initial peer group formation stage where individuals choose their peer group. Moreover, the peer group s_i serves as a signal to i's rivals about i's type.

2. Each individual i then simultaneously chooses action a_i from a set $\mathcal{A} \subseteq \mathbb{R}$ common to players. Actions in conjunction with the ϵ component of private types determine outcomes:

$$y_i = a_i + \epsilon_i$$

   for all individuals $i \leq I$. In the education example, a_i might hence be the number of hours student i sets as a target for study and practice, while y_i might be the actual number of hours worked. Alternatively, a_i and y_i might be the student's targeted and actual grades on the final year exam. ϵ_i should thus be interpreted as reflecting student i's hidden abilities or measurement error.

3. Nature discloses the profile of outcomes $\mathbf{y}$ to all players, and $\mathbf{y}, \mathbf{x}, \mathbf{s}$ to the researcher.

Each player's payoff in stage 3 depends on his action a, type $(\mathbf{x}, \epsilon)$, peer group s and average peer outcome $\overline{y}_s$ in the following way:

$$u(a_i, \overline{y}_s, \mathbf{x}_i) + c(\overline{y}_s, \mathbf{x}_i, s_i, \epsilon_i).$$

Here, $s = s_i$. The function is symmetric across players so that $\{\mathbf{x}_i, s_i, \epsilon_i\}_{i \leq N}$ explain preference heterogeneity across individuals. Observe rival actions $\mathbf{a}_{-i,s}$ matter up to average outcomes $\overline{y}_s$. Hence, peer types $\mathbf{x}_s$ influence payoffs through the outcomes $\mathbf{y}_s$ they induce in $\overline{y}_s$, rather than by directly affecting the payoff function. When interpreted in the context of Example 1.2.1, the payoff function implies student utility is influenced by school average test scores, rather than the particular results of their schoolmates. Also, if a_i measures student i's effort, then peer effects operate through an observable measure of effort $\mathbf{y}_s$, rather than through $\mathbf{a}_s$. This makes sense, since if the researcher does not observe the true effort level for each student, than the students are unlikely to observe and be influenced by such hidden effort too. Finally, while a student's peer group s and hidden abilities in ϵ might affect his utility via the function c, these variables do not appear as arguments of u. Hence, the student's preferences over effort levels are uninfluenced by s and ϵ.

Because peer outcomes are disclosed to students only after they exert effort, peer effects strictly speaking, operate through the beliefs held by each student regarding their peers' outcomes, rather than directly via average peer outcome. Staying within the context of Example 1.2.1, the extensive form is plausible in that students are unlikely to know the personal commitments of each of their peers when studying, and only keep track of their peers' successes either on year-end tests or the frequency at which their peers are seen to be studying. But since these outcomes occur only after effort has been exerted, students are thus influenced by their peers through their beliefs over how successful, competitive or knowledgeable their peers will be on average, rather than directly by peer outcomes.

Player i chooses a_i to maximize

$$\mathbb{E}_{\overline{y}_s,\epsilon|s}\left[u(a_i, \overline{y}_s, \mathbf{x}_i)|\mathbf{x}_i, s_i\right] + \mathbb{E}_{\overline{y}_s,\epsilon|s}\left[c(\overline{y}_s, \mathbf{x}_i, s_i, \epsilon_i)|\mathbf{x}_i, s_i\right].$$

Since player i only knows $(\mathbf{x}_i, s_i)$ when choosing a_i, a strategy for him is thus a (measurable) map σ_i from the type space $\mathbb{R}^K \times \mathbb{N}_{\leq S}$ onto the action set $\mathcal{A}$. Because the draws of $(\mathbf{x}_i, s_i, \epsilon_i)$ are i.i.d., so $\mathbf{x}_i$ is uninformative about rival actions $a_j = \sigma_j(\mathbf{x}_j, s_j)$. Only s_i serves as a signal for peer types and thus average peer outcome $\overline{y}_{s,-i}$. In the following sections, I shall also use N_s to notate the number of individuals in peer group s.

1.4.2 Equilibrium Analysis

The solution concept I use is symmetric (Pure-Strategy) Bayes Nash Equilibrium. These are strategy profiles $\boldsymbol{\sigma}$ satisfying $\sigma_i = \sigma_j$, and solve

$$\sigma(\mathbf{x}_i, s_i) = \text{argmax}_a \mathbb{E}_{\overline{y}_s,\epsilon|s}\left[u(a, \overline{y}_s, \mathbf{x}_i)|\mathbf{x}_i, s_i\right] + \mathbb{E}_{\overline{y}_s,\epsilon|s}\left[c(\overline{y}_s, \mathbf{x}_i, s_i, \epsilon_i)|\mathbf{x}_i, s_i\right]$$

a.s. on $F_{y,\mathbf{x},s}$'s support, where $\overline{y}_s = \frac{a+\epsilon_i}{N_s} + \frac{N_s-1}{N_s}\overline{y}_{s,-i}$. Observe the expectations are taken with respect to the players' common posterior over rival outcomes $\mathbf{y}_{s,-i}|\mathbf{x}_i, s_i$. This is constructed from the players' common prior over types, $F_{\mathbf{x},s,\epsilon}$, strategy σ, and the number of players in the peer group N_s. Hence, the expectations are conditional on $\mathbf{x}_i, s_i$ and implicitly depend on σ and N_s. Also, a in the expression above is effectively a parameter set by player i. If a is treated as a (non-degenerate) random variable, than the expectations above implicitly condition on a too.

To derive the estimating equation of the econometric model as a best response equation, I assume the following:

Assumption 1.4.1. The action set $\mathcal{A}$ is a convex subset of $\mathbb{R}$.

Assumption 1.4.2. The payoff function $u(a, y, \mathbf{x}) + c(y, \mathbf{x}, s, \epsilon)$ is continuously differentiable in $(a, y, \mathbf{x})$ on $\mathcal{A}^0 \times \mathbb{R}^{K+1}$, while $u(a, y, \mathbf{x})$ is quasiconcave in a with positive slope at $\inf \mathcal{A}$ and negative slope at $\sup \mathcal{A}$.

Assumption 1.4.3. The quantities $\sup_{a,y} \mathbb{E}[|u(a, y, \mathbf{x})||\mathbf{x}, s]$ and $\sup_y \mathbb{E}[|c(y, \mathbf{x}, s, \epsilon)||\mathbf{x}, s]$ are finite for all $s_i \leq S$ and $\mathbf{x} \in \mathbb{R}^K$.

Then taking the first order condition and interchanging expectations with derivatives obtains

$$0 = \mathbb{E}\left[u_a(a_i, \overline{y}_s, \mathbf{x}_i)|\mathbf{x}_i, s_i\right] + \frac{1}{N_s}\mathbb{E}\left[u_y(a_i, \overline{y}_s, \mathbf{x}_i)|\mathbf{x}_i, s_i\right] + \frac{1}{N_s}\mathbb{E}\left[c(\overline{y}_s, \mathbf{x}_i, s_i, \epsilon_i)|\mathbf{x}_i, s_i\right] \tag{1.12}$$

a.s. with respect to $F_{\mathbf{x},s}$. In the limit as $N \to \infty$, if F_s is full support, then $N_s \to \infty$ for each $s \leq S$ too. So by SLLN, $\overline{y}_s \to \mathbb{E}[y|s]$ a.s. Then because $u(.)$ and $c(.)$ are continuously differentiable, so by CMT,

$$\begin{aligned}
u_a(a_i, \overline{y}_s, \mathbf{x}_i) &\to u_a(a_i, \mathbb{E}[y|s], \mathbf{x}_i) \\
u_y(a_i, \overline{y}_s, \mathbf{x}_i) &\to u_y(a_i, \mathbb{E}[y|s], \mathbf{x}_i) \\
c_y(\overline{y}_s, \mathbf{x}_i, s_i, \epsilon_i) &\to c_y(\mathbb{E}[y|s], \mathbf{x}_i, s_i, \epsilon_i)
\end{aligned}$$

as $N \to \infty$ a.s.

So, via Assumption 1.4.3 and DCT, the FOC transforms into

$$0 = \mathbb{E}[u_a(a_i, \mathbb{E}[y|s], \mathbf{x}_i)|\mathbf{x}_i, s_i] = u_a(a_i, \mathbb{E}[y|s], \mathbf{x}_i)$$

as $N \to \infty$ a.s. But because $u_{aa} < 0$ at the optimal action, the above expression is invertible via IFT. This enables me to rewrite it as

$$a_i = m\left(\mathbb{E}[y|s_i], \mathbf{x}_i\right) \tag{1.13}$$

or equivalently,

$$y_i = m\left(\mathbb{E}[y|s_i], \mathbf{x}_i\right) + \epsilon_i$$

as observed by the researcher.

Proposition 1.4.1. *Assume assumptions 1.4.1, 1.4.2, and 1.4.3 hold. Consider the bayesian game in the limit as $N = \infty$. Suppose $\{y_i, \mathbf{x}_i, s_i, \epsilon_i\}_{i \leq I}$ is a random sample from individuals playing a symmetric BNE in the limiting game. Then $\{y_i, \mathbf{x}_i, s_i, \epsilon_i\}_{i \leq I}$ satisfies* (1.1) *a.s., and $m(e, \mathbf{x})$ is differentiable in a sufficiently small neighborhood of $(\mathbb{E}[y|s_i], \mathbf{x}_i)$ for each $i \leq I$.*

The function $m(e, \mathbf{x})$ analyzed in Section 1.2, can hence be interpreted as the best response of an infinite player bayesian game, in response to symmetric rival strategies.

Finally, to ensure existence of a symmetric (PS)BNE, recognize (1.13) defines a mapping from $\mathcal{A}$ into itself:

$$\mathbb{E}[a|s] = \int m\left(\mathbb{E}[a|s] + \mu(s), \mathbf{x}\right) dF(\mathbf{x}|s). \tag{1.14}$$

Observe $m(e, \mathbf{x})$ is continuous in e by IFT and Assumption 1.4.2. So if the following holds:

Assumption 1.4.4. The action set $\mathcal{A}$ is compact.

then the quantity $\sup_a \mathbb{E}[|m(a + \mu(s), \mathbf{x})||s]$ is finite for all $s \leq S$ since m maps into a compact set. Hence, $\int m\left(a + \mu(s), \mathbf{x}\right) dF(\mathbf{x}|s)$ is continuous in a too, by DCT.

It follows (1.14) is a continuous mapping from $\mathcal{A}$ onto itself, for all $s \leq S$ in their support. It thus possesses a fixed point, which can be set to $\mathbb{E}[a|s]$. Expected peer outcomes can thus be computed from $\mathbb{E}[y|s] = \mathbb{E}[a|s] + \mu(s)$, and hence, optimal equilibrium strategies can be computed from (1.13). This proves the following result.

Proposition 1.4.2. *Assume assumptions 1.4.1, 1.4.2, 1.4.3 and 1.4.4 hold. Consider the bayesian game in the limit as $N = \infty$. Then there exists a symmetric BNE in this limiting game.*

Finally, perhaps some words on the $N = \infty$ requirement in propositions 1.4.1 and 1.4.2 is merited. How onerous this requirement is probably depends on one's perspective. From the inference perspective of Section 1.2, one makes inferences about an asymptotic model as the sample size $I \to \infty$. Since the number of individuals playing the model's underlying game N, is necessarily larger than the number of observed players I, so N must equal infinity anyways.

1.4.3 Compatibility with Identification Requirements

To ensure the game described in 1.4.1 produces identifiable best responses (to symmetric rival strategies) $m(.,.)$, the following restrictions must be imposed.

Assumption 1.4.5. $F_{\mathbf{x},s,\epsilon}$ implies ϵ is mean independent of $\mathbf{x}$: $\mathbb{E}[\epsilon|\mathbf{x}, s] = \mathbb{E}[\epsilon|s]$.

Hence, observable individual attributes measuring diligence and persistence are uncorrelated with the contribution of hidden natural ability or measurement error to the outcome variable. Assumption 1.4.5 is analogous to the exclusion restriction 1.2.1. The only subtlety is that the assumption is imposed on the distribution of types $F_{\mathbf{x},s,\epsilon}$ across the population of individuals, rather than the full distribution $F_{y,\mathbf{x},s,\epsilon}$ associated with the econometric model.

Assumption 1.4.6. There exists a $j \leq K$ such that x_i^j measures how much player i's preferences depend on rival outcomes: $u(a, y, \underline{x}^j, \mathbf{x}^{-j}) = u(a, \mathbf{x}^{-j})$ for all $\mathbf{x}^{-j}$ in its support.

Under the game theoretic interpretation of peer effects, x_i^j now measures how much player i's preferences depend on his peers' choices, instead of simply his social interactions. In the context of peer effects, Assumption 1.4.6 implies that the preferences of individuals whose types mean they don't interact with their peers, are uninfluenced by the decisions of their peers, a rather reasonable property of any social interactions model. Obviously, Assumption 1.4.6 implies Assumption 1.2.2.

Proposition 1.2.1 implies that under assumptions 1.4.1 to 1.4.6, the best response function $m(.,.)$ is identified from a cross-section of individual-level observations. However, the previous subsection's derivation shows $m(\mathbb{E}[y|s], \mathbf{x})$ yields information only on the "strategic" component of player payoffs $u(a, y, \mathbf{x})$, and is entirely uninformative about $c(y, \mathbf{x}, s, \epsilon)$. Additional, possibly parametric assumptions might be useful for drawing normative implications from peer effects estimates.

1.4.4 Coherence and Incompleteness

As alluded to, whether the bayesian game, given by $\langle N, F_{\mathbf{x},s,\epsilon}, u(.,.,.), c(.,.,.,.)\rangle$ is coherent or complete depends on whether symmetric BNEs exist, and the number of such equilibria. In particular, when the game admits a set of such BNEs, then (1.14) is satisfied by at least one value for $\mathbb{E}[a|s]$. This enables calculation of $\mathbb{E}[y|s]$ that also coheres with condition (1.11). Hence, when the model is taken to be $\langle N, F_{\mathbf{x},s,\epsilon}, u(.,.,.), c(.,.,.,.)\rangle$ rather than $F_{y,\mathbf{x},s,\epsilon}$ and equation (1.1), than ad-hoc conditions on $m(e, \mathbf{x})$ are no longer needed to make $F_{y,\mathbf{x},s,\epsilon}$ cohere with (1.1). Instead, intuitive assumptions ensuring the existence of symmetric BNEs, as displayed in Proposition 1.4.2 ensure coherence.

However, (1.11) might still admit multiple solutions for $\mathbb{E}[y|s]$. At the risk of further abusing terminology from Tamer (2003), the econometric model in such a situation is said to be incomplete. The source of incompleteness is multiple equilibria in the microfounding bayesian game. This is because (1.11) is derived from (1.14) in the game. Hence the number of solutions to both equations are identical. However, each solution to (1.14) produces a distinct action profile $\mathbf{a}$ that is optimal for all players, given their type and peer group $(\mathbf{x}, s)$. So each solution to (1.14) corresponds to a separate symmetric BNE strategy profile. Of course, regardless of which BNE is being coordinated upon, estimation of (1.1) will continue to produce *meaningful* estimates, since $m(e, \mathbf{x})$ is directly derived from the utility function $u(a, y, \mathbf{x})$ common to all players. In other words, when the researcher estimates peer effects, he effectively estimates the game's best responses that depend only on a game's primitives, rather than the BNE being played.

Chapter 2

Forming Firm-to-Firm Relationships under Upstream Economies of Scale and Downstream Product Differentiation

2.1 Introduction

A concern during the Great Recession was the collapse of American automobile producers hurting other firms in their industry. Central to this fear was the industry's network of manufacturer-supplier relationships[1]. This network supposedly externalizes firm-specific disasters when highly connected. Yet as Oberfield (2018) and Taschereau-Dumochel (2020) observe, economic forces shape how manufacturers choose suppliers. In the auto industry, two such forces would likely be product differentiation and economies of scale. When manufacturers differentiate their products, they typically sell sophisticated goods with narrow market segments. Hence, they may not realize economies of scale. However, if their inputs remain simple and homogeneous, their suppliers can achieve scale economies by supplying multiple firms. From this viewpoint, highly connected production networks are desirable, since they reduce production costs.

1. See https://www.nytimes.com/2008/12/12/business/12rescue.html, http://mediamatters.org/video/2012/09/18/ford-ceo-tells-foxs-cavuto-without-auto-bailout/189981 and Hoffman (2012) for expressions of such fears in the media. See the introductions of Acemoglu et al. (2012) or Baqaee (2018) to gauge the extent to which this issue motivates macroeconomics and production networks research.

How do product differentiation and economies of scale affect manufacturer-supplier relationships? How do they distort supply chains from their socially optimal shapes? Three observations motivate these questions. First, as products become increasingly sophisticated and their supply chains more elaborate, vertical firm-to-firm relationships will arguably grow in number[2]. Analyses of such relationships are hence likely to be relevant to future industries. Second, studying vertically-related markets can reveal inefficiencies missed by studying markets in isolation[3]. Such inefficiencies are often more prevalent when product differentiation or economies of scale occur, compared to situations where welfare theorems apply. Finally, inefficiencies in vertical relationships possibly amplify economy-wide consequences of firm-specific predicaments. If so, logic suggests such inefficiencies should bear on optimal macroeconomic policy.

This paper introduces a simple model of manufacturer-supplier relationship formation. Although product differentiation and economies of scale are familiar notions, the model still helps in clarifying how they interact to create inefficiencies. Specifically, these forces produce two forms of inefficiencies, associated with overinvestment in vertical relationships and hold-up in relationship investment. This paper also compares the model to product-level production network data for U.S. auto parts. Amongst other patterns, the data shows auto producers sharing more suppliers after the Great Recession. This aligns with theoretical predictions. Finally, the data is combined with downstream automobile data, to estimate a more sophisticated model of relationship formation. The estimates imply significant overinvestment in auto manufacturer-supplier relationships.

2. For example, Shih (MIT Sloan Management Review, 2019) cites “increased product sophistication” and “manufacturing processes requiring specialists” as reasons for “increased use of subcontracting in, and deepening of global supply chains (paraphrased)”, in https://sloanreview.mit.edu/article/is-it-time-to-rethink-globalized-supply-chains/.

3. Literature on vertically-related markets including Ho (2008), document such inefficiencies in different contexts.

The simple model features two manufacturers, each separately affiliated to one of two suppliers. A single unaffiliated manufacturer-supplier pair then decides whether to form a new relationship. Based on the resulting network of manufacturer-supplier relationships, input prices are determined by Nash bargaining if their payer is linked to one supplier, or a first-price auction if otherwise. The model is analyzed under two cases, representing opposing ends of a spectrum measuring how differentiated the manufacturers' products are. This analysis indicates product differentiation causes manufacturers to share suppliers.

Product differentiation also creates two kinds of inefficiencies, unrelated to deadweight loss in the downstream market. First, adding another supplier to its network allows a manufacturer to pit suppliers against each other, reducing its input price. Thus, to protect their oligopoly rents, manufacturers waste resources into forming additional relationships. The result is inefficient "overinvestment" in relationships, á la Helper & Levine (1992) or Elliot (2015). Second, manufacturers benefit from cheaper inputs, when their suppliers realize scale economies by supplying rival firms. This externality holds up suppliers from investing in relationships, even when such relationships are valued by society at large. As far as I know, the resulting inefficiency has not been analyzed in the literature, except in considerably abstracter settings[4].

Depending on whether overinvestment or hold-up dominates incentives, the manufacturer-supplier network is either over or under connected with positive probability, vis-a-vis the social planner's network respectively[5]. Knowing which form of inefficiency dominates is important for various reasons. One motive is its surprising connection to whether firm-specific shocks imply welfare consequences in proportion to market shares. Hulten (1978) states that in a competitive economy, the effect of a firm's productivity on total surplus is basically its sales share. This

4. See Bloch & Jackson (2007) for example and details.

5. The manufacturer-supplier network however, is never simultaneously both over and under connected w.p.p. for a given configuration of model parameters.

well-studied theorem suggests the manufacturer-supplier network is - in a way- irrelevant for explaining total output fluctuations. One concern regarding this result is it assumes the network is exogenous[6]. But when product differentiation is absent or the allocation is socially optimal, my model equates marginal effects of firm-specific shocks to each firm's *expected* market share prior to network formation. So from this ex-ante partial equilibrium perspective, Hulten's result holds under network endogeneity.

Another concern regarding Hulten's theorem is its applicability to imperfectly competitive economies. When the downstream market features product differentiation, the model contradicts Hulten in counter-intuitive ways. In particular, marginal effects of firm-specific shocks on total surplus are above (below) what ex-ante market shares predict when supplier sharing is sub-optimal (excessive), or when the network is ex-ante insufficiently (excessively) connected. Welfare consequences of firm-specific shocks thus depend on the social planner's network, in addition to its equilibrium counterpart. Moreover, amplification of firm-specific shocks need not correlate with highly connected or asymmetric networks, as intuition or casually reading Acemoglu et al. (2012) and Carvalho (2014) might suggest. Instead, finding *negative* correlation between performance volatility and network connectivity from data spanning multiple industries, is possible.

Ex-ante, whether production networks are inefficiently over or under connected is theoretically ambiguous. The relevance of the model to actual supply chains is also of interest. The rest of this paper thus investigates how the model's mechanisms extrapolate to automobile production. To do so, prices and sales for US automobiles are combined with data regarding each vehicle's 10 most important suppliers, ranked by contract numerosity. Figure 2.1's left panel plots average price and sales, across domestic and imported auto models over 2008-16. The

6. In particular, each firm's ability to transact with another depends on the firms' respective industries, and their representative production functions.

right panel displays the average number of suppliers each model shares with another, over the same categories and years. Manufacturer-supplier network connectivity - measured by supplier sharing - visibly dips during 2008-10. These years mark the Great Recession, when fears of excessive network connectivity peaked. Whilst descriptive, Figure 2.1 underscores the importance of accounting for network endogeneity, when studying shock propagation through production networks.

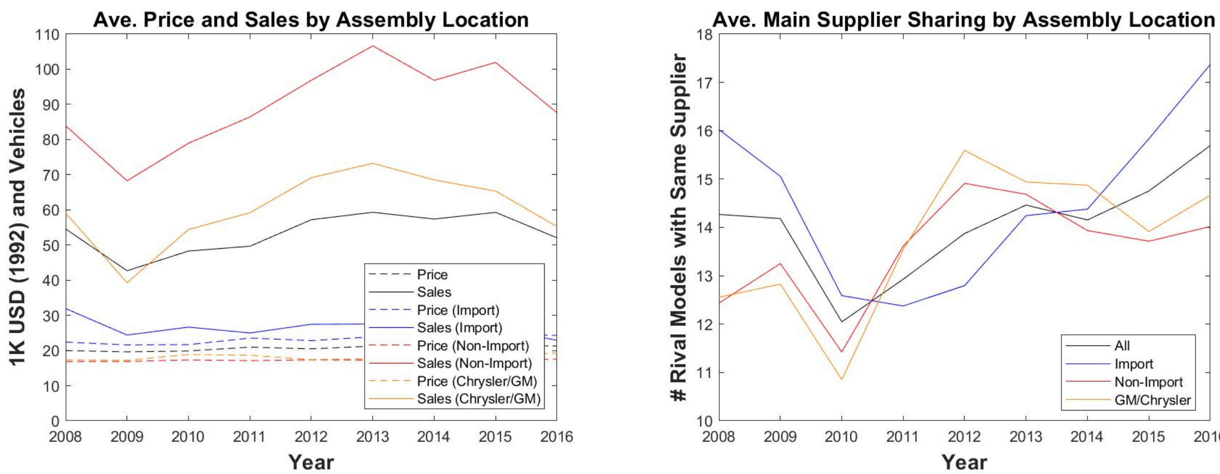


Figure 2.1: Prices, Sales & Supplier Sharing over 2008-16 US Automobile Models

Additional patterns in the data also align with the model's assumptions and predictions. However, an appropriate empirical framework is needed to obtain conclusions of more causal nature. This paper thus develops an empirical model of manufacturer-supplier relationship formation. This model does not strictly generalize the simpler one. However, it incorporates the simpler model's key elements - upstream economies of scale, downstream product differentiation, and relationship network contingent input pricing. Moreover, its key features are identified. The identified features form steps towards: i) evaluating how product differentiation and economies of scale affect auto production, and ii) quantifying inefficiencies in relationship formation highlighted by the simpler model - outside option overinvestment and neighbor manufacturer hold-up.

Quantifying the aforementioned distortions requires understanding how each firm's costs varies with its output. Unfortunately, two factors impede identification of the suppliers' cost curves. First, input prices are absent from the data. Second, suppliers are chosen by manufacturers, creating correlation between the manufacturers' costs and widely used "BLP instruments", introduced by Berry et al. (1995) to identify cost curvatures. These difficulties are circumvented by assuming manufacturers Nash bargain with suppliers they inherit from previous periods. One can then exploit variation in these suppliers' quantities, to identify how automobile production costs vary with rival product quantities.

The resulting estimates indicate suppliers of chassis and exterior inputs, along with suppliers for all input categories combined, experience significant economies of scale at their average output. Moreover, the manufacturers' average benefit from forming their chosen relationships is smaller than its distortion due to outside option overinvestment inefficiency. Hence, unless compensated by co-investing suppliers, manufacturers would not have benefited from their chosen relationships without overinvestment rents. In contrast, the average distortion to the suppliers' benefits from forming equilibrium relationships, due to rival manufacturer hold-up, is at most only 11% of the suppliers' overall relationship-forming benefits. Finally, the hold-up distortion actually acts in reverse to reinforce overinvestment in relationships with chassis and exterior suppliers. This is due to convexities in input marginal costs, causing average costs to rise over large quantity increments.

Whether auto industry relationships are dominated by overinvestment or hold-up is relevant to policy. The simple model predicts more relationship investment when manufacturers face greater demand. The empirical results indicate overinvestment dominates relationships. Hence, subsidizing US automakers during 2008-16 may only increase relationship formation to more inefficient levels. This observation opposes arguments for subsidizing the automakers based on their supplier network.

The aforementioned policy implication rests on several caveats. The empirical results' primary shortcoming is that their associated relationship-forming benefits fix output levels to equilibrium values as model-supplier relationships are altered. Thus, the benefits only approximate their ex-ante counterparts anticipated by firms when choosing whether to form new relationships. Also, the paper's scope neglects inefficiencies absent from the simple model plaguing firm-to-firm relationships empirically. On the other hand, the results are advantaged by their low reliance on stylized or unverifiable assumptions. In particular, identification does not rely on cost functions or utilities taking on particular functional forms. The results also do not depend on the specific information structure or sequence of moves governing relationship formation. Last, the results are uncontingent on the precise auction formats determining the models' suppliers and input prices.

In what follows, the theoretical model is analyzed in Section 2.2. Section 2.3 reviews relevant literature. Section 2.4 details data collection and descriptive analysis, while Section 2.5 describes the empirical model. Section 2.6 presents empirical results. Proofs are located in the appendix.

2.2 Simple Model

Consider an industry populated by a unit continuum of consumers, 2 manufacturers and 2 suppliers. Manufacturers start with pre-existing relationships to distinct suppliers. Without loss of generality, let $s = m$ be manufacturer m's affiliated supplier.

One can interpret these pre-existing relationships as due to the affiliated supplier being the manufacturer's past supplier. Alternatively, a pre-existing relationship might reflect the supplier being the manufacturer's in-house supplier, with the caveat that firms maximize their own profits.

Sequence of Play: The game between these participants unfolds as follows.

1. A fixed cost F of building a relationship between manufacturer 2 and supplier 1 is drawn. F is privately disclosed to both firms. The cost reflects liaison infrastructure connecting the firms, expenditures from ensuring compatibility of new automobile parts[7], or relocation costs from making one's plant closer to a supplier's or customer's location[8].

   Costly link formation and pre-existing relationships creates persistence in manufacturer-supplier relationships. Such persistence is consistent with US auto supplier survey results[9]. Although the model abstracts from repeated play, this consistency is still reassuring.

2. Manufacturer 2 and supplier 1 publicly and cooperatively decide whether to build a relationship. If a relationship is built, they bargain over the division of the costs incurred.

   i) The firms' decisions over whether to invest in a relationship are denoted by

   $$(a_1, a_2) \in \{I, N\} \times \{I, N\}.$$

   If monetary transfers allow both parties to benefit from investment, a link between $(m, s) = (2, 1)$ is added to the relationship network. If otherwise, no link is created, and the network is simply two disconnected edges connecting manufacturers to their affiliated suppliers. The model is thus consistent with pairwise stable network formation.

   ii) If both firms invest, the share of F each party bears is determined by Nash bargaining. Hence, usual hold-up associated with relationship-specific investments is not present[10]. Instead, inefficiencies will stem from link formation between $(m, s) = (2, 1)$ imposing

7. See Womack et al. (1990) and Ben-Shahar & White (2006)

8. Proximity to supplier or customer plants predict auto plant locations. (See Klier, 1999 and Rosenbaum, 2013).

9. See for example, Cusumano & Takeishi (1991, Table 6) and Choi & Hartley (1996, p. 341).

10. For an example of how this might arise, Ben-Shahar & White observe contractual clauses typically ameliorate hold-up caused by supplier switching costs in the auto industry.

externalities on *neighboring* firms in the network.

Note only one manufacturer-supplier pair is permitted to link up. This allows focused study of how product differentiation and economies of scale influence link formation, without interference from strategic linking behavior.

3. Each manufacturer $m \leq 2$ publicly announces a price

$$p_m \in \mathbb{R}_{\geq 0}$$

in the downstream market. Manufacturers commit to output prices before negotiating input prices with their suppliers. Downstream market competition is thus "harnessable" by manufacturers to reduce input prices. While in contrast to recent I.O. vertical markets research[11], the sequence of play is consistent with surveys of the auto industry[12].

4. Input prices are then determined either by Nash bargaining or a first-price auction:

   i) Suppose a manufacturer is connected to only one supplier in the relationship network. As the description of Stage 6 will clarify, the downstream prices $\mathbf{p}$ set in Stage 3 pin down demand for the manufacturers' output $\mathbf{q}(\mathbf{p})$. Each firm thus knows the surplus created from producing inputs and assembling them into outputs sold to consumers. When this surplus is positive, its division between both firms is determined by Nash bargaining.

   ii) Suppose both suppliers are connected to manufacturer 2 in the relationship network. Then each supplier $s \leq 2$ submits a bid

$$b_s \in \mathbb{R}_{\geq 0},$$

11. See Ghilli (2016), Ho & Lee (2016) for examples of simultaneous input and output pricing, Ho (2006) and Yang (2018) for models where input prices are determined before output prices, and Iozzi & Valletti (2016) for a more similar sequence of play to my framework, and a counter-example to the statement.

12. Womack et al. document American auto manufacturers threatening to switch suppliers (p. 148) and reducing input prices (p. 160) in response in downstream competition. Cusumano & Takeishi (1991) report how Japanese auto manufacturers set target prices based on a new model's sales price (p. 565), choose suppliers based on their ability to meet this target (Table 9), and implement the target post supplier selection (Table 11).

representing its desired price for the supply of inputs per output unit to the manufacturer.

5. i) If a manufacturer m is connected to a single supplier and there is non-negative surplus to divide, it operates. If otherwise, the manufacturer shuts down. Denote this status by

$$s_m \in \{\text{Operate}, \text{Shut Down}\}.$$

Let t_m denote the price manufacturer m pays its supplier for inputs needed to make each output unit, implied by Nash bargaining in the previous stage.

ii) Suppose both suppliers are connected to manufacturer 2 in the relationship network. Then based on output prices $\mathbf{p}$ and bids $\mathbf{b}$, manufacturer 2 decides whether to shut down, or to operate with a chosen supplier:

$$s_2 \in \{1, 2, \text{Shut Down}\}.$$

Its input price is the chosen supplier's bid: $t_2 = \sum_s \{s_2 = s\} b_s$.

Heterogeneity in input price determination is thus a consequence of the relationship network. This in turn depends on whether manufacturer 2 and supplier 1 build a relationship. The heterogeneity and its cause, reflect actual differences between American and Japanese automakers. As documented by several sources[13], American auto manufacturers often employ numerous suppliers, pitting them against each other when awarding contracts. In contrast, Japanese manufacturers work with smaller supplier bases, to improve inter-firm communication and cooperation.

6. Each consumer $i \in [0, 1]$ observes $\mathbf{s}$ and $\mathbf{p}$ before deciding whether to purchase a good, and

13. See Womack et al.'s description of the supplier selection processes for US and Japanese firms. Also, see tables 1, 4, 6 and 7 in Cusumano & Takeishi's survey of industry participants. Finally, see Mudambi & Helper (1998) for a more recent survey supporting the adverserial model of US auto supplier relations.

if so, which manufacturer to buy it from:

$$c_i \in \{1, 2, \text{Outside Option}\}, \quad s_m = \text{Shut Down} \Rightarrow c_i \neq m.$$

The stage 5 and 6 decisions create a new network of manufacturer-supplier relationships, based on production linkages. This will be referred to as the "production network", to distinguish it from the relationship network formed by $\mathbf{a}$.

Beliefs: All players share a common prior G over the cost F of building a manufacturer-supplier relationship. The prior quantifies the firms' relationship-forming incentives by the equilibrium frequency of investment. Also, no technology exists to authenticate F to manufacturer 1 and supplier 2. Hence, contracts stipulating F-contingent side payments by these firms to the parties linking up, cannot be written. Finally, F is absolutely continuous over support $[\underline{F}, \overline{F}] \subseteq \mathbb{R}_{>0}$. These are imposed for tractability.

Payoffs: Each consumer i has utility

$$u_i = v_{im} - p_m$$

from consuming a unit of output made by manufacturer m. Hence, v_{im} is effectively consumer i's willingness-to-pay for manufacturer m's product.

The model is analyzed under two special cases.

Assumption 2.2.1. [No Product Differentiation] Each consumer has identical willingness-to-pay for each product sold by either manufacturer:

$$v_{im} = v > 0 \quad \forall (i, m) \in [0, 1] \times \mathbb{N}_{\leq 2}.$$

Assumption 2.2.1 captures the situation where both manufacturers' products are perfect substitutes. Each manufacturer's demand is thus

$$q_m(s_{-m}, \mathbf{p}) \in \begin{cases} 1 & \text{if } p_m < (p_{-m} \div \{s_{-m} \neq \text{Shut Down}\}) \wedge v \\ [0,1] & \text{if } p_m = (p_{-m} \div \{s_{-m} \neq \text{Shut Down}\}) \wedge v \\ 0 & \text{if } p_m > (p_{-m} \div \{s_{-m} \neq \text{Shut Down}\}) \wedge v \end{cases},$$

depending on its downstream rival's status s_{-m} and price p_{-m}. When the rival shuts down, it should be clear $(p_{-m} \div \{s_{-m} \neq \text{Shut Down}\}) \wedge v = v$ in the expression above.

Assumption 2.2.2. [Full Product Differentiation] Half of the consumers are willing to pay up to v_1 for manufacturer 1's output and nothing for manufacturer 2's:

$$(v_{i1}, v_{i2}) = (v_1, 0) > \mathbf{0} \quad \forall i \in [0, 0.5].$$

The remaining half will pay up to v_2 for manufacturer 2's output and nothing for the rival product:

$$(v_{i1}, v_{i2}) = (0, v_2) > \mathbf{0} \quad \forall i \in [0.5, 1].$$

Under Assumption 2.2.2, both manufacturers' products are completely horizontally differentiated - the polar opposite of the previous assumption. Each manufacturer's demand is thus perfectly inelastic at any price below the consumers' willingness-to-pay:

$$q_m(\mathbf{p}) \in \begin{cases} \frac{1}{2} & \text{if } p_m < v_m \\ \left[0, \frac{1}{2}\right] & \text{if } p_m = v_m \\ 0 & \text{if otherwise} \end{cases}.$$

When $v_1 \neq v_2$, Assumption 2.2.2 also implies manufacturers are vertically differentiated.

While restrictive, assumptions 2.2.1 and 2.2.2 decontaminate the network-based effects of downstream product differentiation on social welfare, from (well understood) downstream market deadweight loss. Of course, stark but interpretable assumptions like these also promise sharper insights in environments where understanding is otherwise difficult to acquire.

Each manufacturer's payoff is its variable profit, less costs incurred from establishing relationships with unaffiliated suppliers in Stage 2:

$$\pi_m^M = q_m(p_m - t_m) - T_{21}\{\mathbf{a} = \mathbf{I}, m = 2\} \quad \forall m \leq 2.$$

Notice both manufacturers incur the same marginal cost from assembling inputs into any quantity of output. This cost in turn has been normalized to zero. v_m and p_m thus acquire alternate interpretations as the willingness-to-pay and output price *net of of marginal assembly costs* for manufacturer m respectively. Hence, the model's equilibrium response to changes in v_m also capture equilibrium effects of shocks to m's assembly costs. Also, manufacturer 2 incurs a sunk cost T_{21} from forming a relationship with supplier 1 in Stage 2. As the sequence of play clarifies, T_{21} is determined by Nash bargaining with supplier 1.

Each supplier's payoff is also its variable profit, less costs incurred from establishing relationships with unaffiliated manufacturers:

$$\pi_1^S = q_1t_1 - C(q_1) + \{s_2 = 1\}(q_2t_2 - C(q_1 + q_2) + C(q_1)) - \{\mathbf{a} = \mathbf{I}\}(F - T_{21})$$
$$\pi_2^S = \{s_2 = 2\}(q_2t_2 - C(q_2)).$$

Suppliers possess identical cost functions $C(Q)$, which are assumed to be strictly concave. Hence, upstream economies of scale are captured by how much average input production costs $AC(Q)$ declines in quantity Q.

Strategies: The sequence of play implies each player's strategy is contingent on past actions, many of which are irrelevant to the player's payoffs. To simplify analysis, I assume players adopt payoff relevant strategies. In particular, consumers choose what to consume, based only on output prices $\mathbf{p}$ and whether each manufacturer operates or shuts down, encoded in $\mathbf{s}$. A strategy for consumer i is hence of the form:

$$\sigma_i : \{\text{Operate}, \text{Shut Down}\}^2 \times \mathbb{R}^2_{\geq 0} \to \{1, 2, \text{Outside Option}\}. \tag{2.1}$$

The consumers' strategies imply market shares satisfy

$$q_m(s_{-m}, \mathbf{p}) = \int_{i=0}^{i=1} \{\sigma_i(\mathbf{s}, \mathbf{p}) = m\} di. \tag{2.2}$$

Assume each consumer consumes if indifferent between consuming its utility maximizing inside option and not consuming. Similarly, since manufacturer 2's optimal price does not depend on F, while its optimal supplier does not depend on F, $\mathbf{a}$ and p_1, its strategies are of the form

$$\sigma_2^M(F, \mathbf{a}, \mathbf{p}, \mathbf{b}) = (a_2(F), p_2(F, \mathbf{a}), s_2(F, \mathbf{a}, \mathbf{p}, \mathbf{b})),$$

satisfying

$$p_2(F, \mathbf{a}) = p_2(\mathbf{a}) \qquad s_2(F, \mathbf{a}, \mathbf{p}, \mathbf{b}) = s_2(p_2, \mathbf{b}). \tag{2.3}$$

Finally, since supplier 1's profit depends on F only through $\mathbf{a}$, its chosen strategy

$$\sigma_1^S(F, \mathbf{a}) = (a_1(F), b_1(F, \mathbf{a}, \mathbf{p}))$$

satisfies

$$b_1(F, \mathbf{a}, \mathbf{p}) = b_1(\mathbf{p}). \tag{2.4}$$

Both suppliers thus bid with reference only to output prices[14].

14. Without loss of generality, assume supplier 2's bid $b_2(\mathbf{a}, \mathbf{p}) = b_2(\mathbf{p})$ too, since supplier 2 bids only when $\mathbf{a} = \mathbf{I}$.

Terminology: Under the strategy sets considered, the players' actions from Stage 2 onwards are contingent only on public information. Moreover, the players' ex-post payoffs as functions of their actions, depend on private information F only through investment decisions $\mathbf{a}$ publicly chosen in Stage 2. Hence, by constructing a new game mirroring the extensive form originating from the investment decisions, one obtains a proper perfect information subgame.

I hence refer to the extensive form initiated by any profile of investment decisions $\mathbf{a}$ as a "subgame" from here on without apology. Notice any sensible solution concept thus requires its strategies $\boldsymbol{\sigma}$ when truncated in any of these subgames $\boldsymbol{\sigma}|_{\mathbf{a}}$, to be subgame perfect Nash equilibria.

Nash Bargaining: Completing the model requires specifying threat points and bargaining powers when manufacturers and suppliers Nash bargain. When a manufacturer and its affiliated supplier bargain in Stage 4 over how much the supplier receives for inputs, the manufacturer shuts down while the supplier loses a customer if they fail to agree. This defines their threat points, so that the negotiated input price equals

$$
\begin{aligned}
t_m(\mathbf{a},\mathbf{p}) = \operatorname*{argmax}_t &\left(\pi_m^M(\mathbf{a},\mathbf{p},\boldsymbol{\sigma}|_{\mathbf{a},\mathbf{p}} - s_m, \text{Operate}; t) - \pi_m^M(\mathbf{a},\mathbf{p},\boldsymbol{\sigma}|_{\mathbf{a},\mathbf{p}} - s_m, \text{Shut Down})\right)^{\beta_m} \\
&\times \left(\pi_m^S(\mathbf{a},\mathbf{p},\boldsymbol{\sigma}|_{\mathbf{a},\mathbf{p}} - s_m, \text{Operate}; t) - \pi_m^S(\mathbf{a},\mathbf{p},\boldsymbol{\sigma}|_{\mathbf{a},\mathbf{p}} - s_m, \text{Shut Down})\right)^{1-\beta_m}.
\end{aligned}
\tag{2.5}
$$

Here, $\pi_m^M(\boldsymbol{\sigma} - s_m, \text{Shut Down}) = -\{m = 2, \mathbf{a} = \mathbf{I}\}F$, while $\pi_s^S(\boldsymbol{\sigma} - s_m, \text{Shut Down})$ is supplier s's profit from either supplying the remaining manufacturer or not supplying any firm at all, depending on $\boldsymbol{\sigma}$. Observe any solution concept strategy profile $\boldsymbol{\sigma}$ thus ought to satisfy (2.5).

When manufacturer 2 and supplier 1 bargain over how much each contributes towards funding a new relationship between them, each firm's threat point is the profit incurred from unilaterally withdrawing investment. Hence

$$T_{21} = \underset{T}{\text{argmax}} \left(\pi_2^M(\mathbf{I}, \boldsymbol{\sigma}|_{\mathbf{I}}; T) - \pi_2^M(N, I, \boldsymbol{\sigma}|_{(N,I)})\right)^{\beta_{21}} \times \left(\pi_1^S(\mathbf{I}, \boldsymbol{\sigma}|_{\mathbf{I}}; T) - \pi_1^S(I, N, \boldsymbol{\sigma}|_{(I,N)})\right)^{1-\beta_{21}}. \tag{2.6}$$

Thus, any equilibrium strategy profile $\boldsymbol{\sigma}$ ought to satisfy (2.6) too.

Observe the manufacturer-supplier specific bargaining parameters $\boldsymbol{\beta}$ need not equal each other[15]. The only restrictions imposed are that $\beta_{21} \in (0, 1)$ and $\beta_m \in (0, 1)$ for each $m \leq 2$.

Solution Concept: The model is a dynamic game with private information disclosed to some players in Stage 1. Since beliefs are governed by a common prior, (pairwise stable) Bayes Nash equilibrium (BNE) is a reasonable starting point for a solution concept. The BNE strategies $\boldsymbol{\sigma}$ must also satisfy the informational requirements (2.1) to (2.4), and be consistent with the bargaining formulae in (2.5) and (2.6). To simplify exposition, one may also assume manufacturer 2 and supplier 1 invest in Stage 2 if indifferent between investing and not investing. Because F is absolutely continuous, this assumption binds only with zero probability.

Yet even with these restrictions, the model admits multiple equilibria. One reason why this occurs lies with the first-price auction embedded within its sequence of play. The obvious solution where both bidders bid the inefficient bidder's valuation is the auction's intuitive outcome. It is also the unique limit of weakly undominated Nash equilibria in discrete bidding first-price auction analogues, as shown by Alcalde and Dahm (2011). Unfortunately, alternate equilibria where both bidders bid common values located between their valuations also exist. These equilibria are unintuitive, but their existence impedes basic counterfactual analysis.

15. For example, Cusumano & Takeishi (p. 582)'s interviewees report lower profit margins for Japanese auto suppliers vis-a-vis their American counterparts while Dyer (1996, p. 281)'s descriptive statistics suggests Japanese manufacturers capture higher shares of joint profits with their suppliers compared to US manufacturers.

To deal with this issue, I assume:

Assumption 2.2.3. [Limit Bidding] Supplier 2 never submits a bid b_2 weakly dominated by bidding its lowest possible unit cost $AC(q_2(s_1, \mathbf{p}))$, across all feasible operate-or-quit decisions s_1 by manufacturer 1.

It is easily shown when both suppliers are restricted to bids weakly undominated by bidding their unit cost, the unique NE in the first-price auction is the intuitive solution. Assumption 2.2.3 is a weaker requirement that happens to yield the same outcome in this model.

Another reason why multiple equilibria arise is due to payoff complementarities associated with supplier-level economies of scale. Intuitively, when a manufacturer shuts down by setting an unreasonably high output price, its supplier's average cost of supplying the manufacturer's rival increases. This can cause the rival to shut down too, by depleting it of an affordable input sourcing option. The model thus admits equilibria featuring both manufacturers operating or both shutting down, when the relationship network is connected.

One solution to this problem assumes:

Assumption 2.2.4. [Pareto Efficient Bargaining] Manufacturer 2 and supplier 1 attain pareto efficient payoffs if the subgame originating from them investing $\mathbf{a} = \mathbf{I}$ is initiated, amongst the set of SPNE payoffs satisfying Assumption 2.2.3.

Assumption 2.2.4 corresponds to the pareto efficiency axiom associated with Nash bargaining in the following sense. Given manufacturer 2 and supplier 1 obviously communicate when bargaining over the splitting of their investment return, their bargained outcome ought to lie on the pareto efficient frontier for their payoffs. Requiring all four firms to coordinate on equilibria maximizing the bargaining firms' post-investment joint profit, is thus not unreasonable from this perspective.

An alternate approach refines equilibria by imposing the following assumptions:

Assumption 2.2.5. [Forward Induction] Let

$$\underline{\pi}_2^M = \min_{\boldsymbol{\sigma}|(N,I)} \pi_2^M((N,I), \boldsymbol{\sigma}|_{N,I}) \quad \underline{\pi}_1^S = \min_{\boldsymbol{\sigma}|(I,N)} \pi_1^S((I,N), \boldsymbol{\sigma}|_{I,N})$$

be the minimal possible payoffs manufacturer 2 and supplier 1 can attain from not investing in Stage 2, under all SPNEs satisfying Assumption 2.2.3. If there exists another SPNE satisfying Assumption 2.2.3 in the subgame initiated when $\mathbf{a} = \mathbf{I}$ and a $T_{21} \in \mathbb{R}$ where both firms attain payoffs pareto superior to $(\underline{\pi}_2^M, \underline{\pi}_1^S)$, their equilibrium payoffs in the subgame are not pareto inferior to $(\underline{\pi}_2^M, \underline{\pi}_1^S)$.

In my view, Assumption 2.2.5 resembles the forward induction refinement considered in games where a player effectively guarantees cross-player coordination on a desired outcome, by not preemptively enforcing an inferior outcome for himself. In this context, manufacturer 2 and supplier 1 sacrifice payoffs equal to $(\underline{\pi}_2^M, \underline{\pi}_1^S)$ when establishing a relationship. Rival firms thus anticipate manufacturer 2 and supplier 1 coordinating on strategies where they attain payoffs pareto undominated by $(\underline{\pi}_2^M, \underline{\pi}_1^S)$, if they invest in a relationship.

Assumption 2.2.6. [Time Consistency] Neither manufacturer can strictly improve its payoff by altering its downstream price p_m at the start of Stage 6, after its input price t_m has been determined.

As already discussed, manufacturers commit to output prices before negotiating input prices with connected suppliers. The issue with this sequence of play is that manufacturers may lack the required commitment ability. Moreover, manufacturers can often profit from raising output prices post input price determination, since product differentiation renders their demand schedules more inelastic. Assumption 2.2.6 should thus be interpreted as requiring output prices to remain incentive compatible, if manufacturers cannot commit to them during input price negotiations.

It turns out that making assumptions 2.2.3 and 2.2.4, or imposing 2.2.3, 2.2.5 and 2.2.6 yields a unique equilibrium relationship network for each realization of investment cost. Moreover, the equilibrium networks are identical irrespective of which assumption set is imposed. In what follows, I show why this is so, and compare the equilibrium allocations to their surplus-maximizing counterparts.

2.2.1 Planner's Allocation when Product Differentiation is Absent

Consider the problem facing a social planner attempting to maximize total surplus:

$$\max_{\mathbf{q},s_2} q_1 v + q_2 v - C(q_1) - C(q_2) - \{s_2 = 1\}\left(C(q_1 + q_2) - C(q_1) - C(q_2) + F\right)$$

subject to

$$q_1 + q_2 \leq 1, \quad \mathbf{q} \geq \mathbf{0}.$$

So the planner effectively allocates consumers to manufacturers through its choice of $\mathbf{q}$, and manufacturers to suppliers through its choice of s_2.

When $\mathbf{q} >> \mathbf{0}$, cost minimization implies $s_2 = 1$ iff

$$F \leq C(q_1) + C(q_2) - C(q_1 + q_2)$$

up to allocations indifferent to the planner. But by allocating manufacturer 2's customers to manufacturer 1, the planner avoids needing to link manufacturer 2 to another supplier. Under this allocation, the planner conserves costs from not forming relationships. It benefits from economies of scale during input production too. So the planner only produces when

$$v > AC(1)$$

and sets

$$\mathbf{q} \in \{(1,0),(0,1)\}, \quad s_2 = 2$$

if so. Total surplus of the planners' allocation is thus

$$(v - AC(1)) \vee 0$$

ex-ante and ex-post.

The socially optimal allocation features no supplier sharing when product differentiation is absent. Intuitively, when manufacturers sell homogeneous products, one can allocate consumers to a single manufacturer without reducing consumer utility. Thus, no costly investment in new relationships is needed to realize economies of scale via supplier sharing.

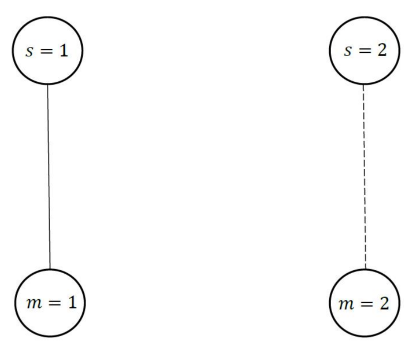


Figure 2.2: Example Social Planner's Network

2.2.2 Equilibrium Analysis when Product Differentiation is Absent

The equilibrium outcome is found using backward induction.

Stages 6 & 5: When $q_2(s_1(\mathbf{p}, \mathbf{b}), \mathbf{p}) > 0$, manufacturer 2 chooses the lowest bidder

$$s_2 = \text{argmin}_{s \leq 2} b_s$$

when $b_1 \wedge b_2 < v_2$. The manufacturer is indifferent between using that supplier or shutting down when $b_1 \wedge b_2 = v_2$. It thus behaves like a first-price auctioneer.

Stages 4 & 3: Suppose $\mathbf{a} \neq \mathbf{I}$ so that the relationship network is disconnected, as in Figure 2.3. Nash bargaining between each affiliated manufacturer-supplier pair implies their negotiated input price is a convex combination of average cost $AC(q_m(s_{-m}(\mathbf{p}), \mathbf{p}))$ and output price p_m:

$$t_m = \beta_m AC(q_m(s_{-m}(\mathbf{p}), \mathbf{p})) + (1 - \beta_m) p_m.$$

Such bargaining occurs when manufacturer 1 operates or equivalently, $p_m \geq AC(q_m(s_{-m}(\mathbf{p}), \mathbf{p}))$. As in Bertrand competition, each manufacturer's payoff is thus stepwise in price:

$$\pi_m^M = \begin{cases} \beta_m(p_m - AC(1)) \vee 0 & \text{if } p_m < (p_{-m} \div \{s_{-m} \neq \text{Shut Down}\}) \\ \beta_m q_m(s_{-m}, \mathbf{p})(p_m - AC(q_m(s_{-m}, \mathbf{p}))) \vee 0 & \text{if } p_m = (p_{-m} \div \{s_{-m} \neq \text{Shut Down}\}) \\ 0 & \text{if otherwise} \end{cases}$$

where $s_{-m} = s_{-m}(\mathbf{p})$.

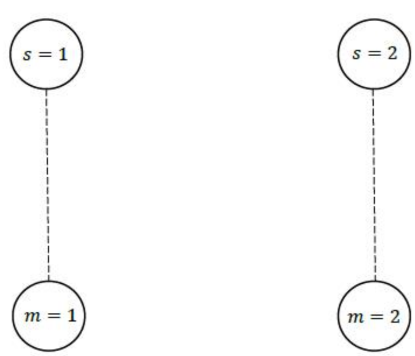


Figure 2.3: Disconnected Relationship Network

It follows neither manufacturer anticipates any profit when setting its output price in Stage 3.

Lemma 2.2.1. *Suppose Assumption 2.2.1 holds. Then neither manufacturer makes a strict profit*

$$\pi_1^M = 0 \qquad \pi_2^M = 0,$$

in any of the Nash equilibria of the subgame initiated by $\mathbf{a}$ *when* $\mathbf{a} = \mathbf{I}$ *in Stage 2.*

The proof simply requires routine checking of incentive compatibility for each feasible configuration of output prices. Intuitively, the logic of Bertrand competition equates downstream profits to zero.

Now suppose $\mathbf{a} = \mathbf{I}$ so that the relationship network is connected, as in Figure 2.4. Deriving this subgame's equilibria is less straightforward. Nash bargaining between manufacturer 1 and supplier 1 implies

$$t_1 = \beta_1 \left(AC(q_1(s_2(\mathbf{p}, \mathbf{b}), \mathbf{p}))\{s_2(\mathbf{p}, \mathbf{b}) \neq 1\} + \Delta C(\mathbf{q}(\mathbf{s}(\mathbf{p}, \mathbf{b}), \mathbf{p}))\{s_2(\mathbf{p}, \mathbf{b}) = 1\} \right) + (1 - \beta_1)p_1$$

whenever manufacturer 1 operates. Moreover, manufacturer 2's input price equals

$$t_2 = b_1 \wedge b_2$$

whenever it operates. Clearly, manufacturer 1's input price t_1, and its choice of whether to operate $s_1(\mathbf{p}, \mathbf{b})$, depends on manufacturer 2's chosen supplier $s_2(\mathbf{p}, \mathbf{b})$. Moreover, both manufacturers' input prices depend on bids $\mathbf{b}$ strategically chosen by suppliers in a first-price auction. Deriving the set of Nash equilibria outcomes thus requires evaluating the supplier's payoffs for every possible configuration of bids, incentive compatible supplier-or-quit choice maps $\mathbf{s}(\mathbf{p}, \mathbf{b})$, and incentive compatible market share functions $\mathbf{q}(\mathbf{s}, \mathbf{p})$. This allows derivation of all possible equilibrium bids and thus, all possible equilibrium input prices.

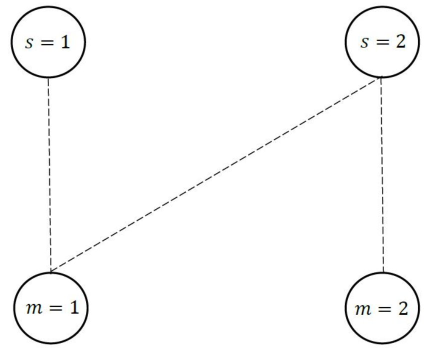


Figure 2.4: Connected Relationship Network

However, a simpler argument shows neither manufacturer makes any profit in equilibrium.

Lemma 2.2.2. *Suppose Assumption 2.2.1 holds. Then neither manufacturer strictly profits*

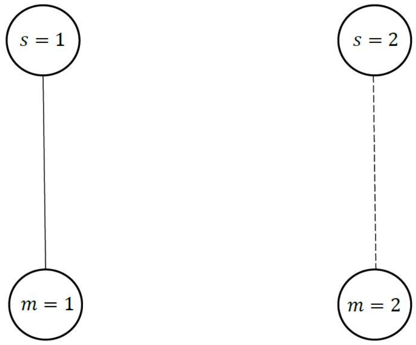
$$\pi_1^M \leq 0 \qquad \pi_2^M \leq 0,$$

in any of the Nash equilibria of the subgame initiated by $\mathbf{a}$ *when* $\mathbf{a} = \mathbf{I}$ *in Stage 2.*

As when $\mathbf{a} \neq \mathbf{I}$, Bertrand competition between homogeneous manufacturers equates their profits to zero.

Stage 2: The lemmas establish manufacturer 2 never benefits from forming a new relationship. The relationship network is thus disconnected, while supplier sharing never occurs in equilibrium.

Intuitively, price competition between homogeneous manufacturers concentrates production along a single manufacturer-supplier pair to fully realize economies of scale in input production. So supplier sharing never occurs, regardless of whether the relationship network is connected or not. Consequentially, additional manufacturer-supplier relationships, typically created to facilitate supplier sharing, do not increase any firm's payoff.



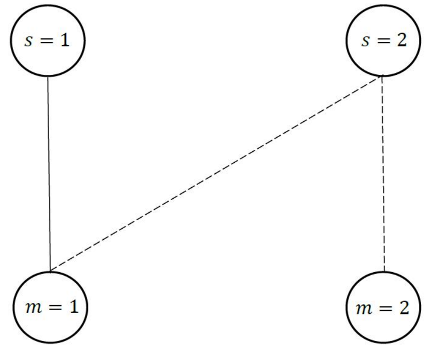


Figure 2.5: Production Network Examples when $\mathbf{a} \neq I$ (LHS) and $\mathbf{a} = I$ (RHS)

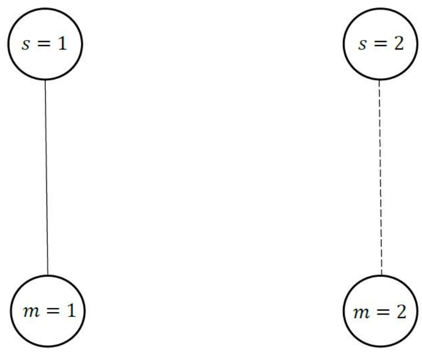


Figure 2.6: Equilibrium Path Production Network Example

Observe the allocation of consumers to manufacturers to suppliers on the equilibrium path, coincides with the social planner's allocation. Ex-ante, the total surplus it generates is hence

$$TS = (v - C(1)) \vee 0 = (q_1 v + q_2 v - C(1)) \vee 0$$

almost surely. The equilibrium allocation thus maximizes ex-ante total surplus.

Now, consider an infinitesimal increase in each manufacturer's ability to charge a markup above its assembly costs. Observe

$$\frac{\partial TS}{\partial v} = \{v > AC(1)\} = q_1 + q_2.$$

So the marginal effect of higher willingness-to-pay for the manufacturers' products (or equivalently lower assembly costs for both manufacturers) on total surplus is the manufacturers' combined market share. Also, consider a constant increase in the suppliers' marginal costs of making some quantity of inputs. Formally, suppose $C(Q)$ is increased to $C(Q) + cQ$ for all $Q \geq 0$. Then

$$\frac{\partial TS}{\partial c} = -\{v > AC(1)\} = -q_1^S - q_2^S.$$

Thus, the increase in marginal cost causes a loss of surplus equal to the suppliers' upstream market shares combined.

Now, Hulten (1978) basically equates marginal effects of firm-specific shocks on total output to the firms' respective shares of aggregated output. The marginal effects derived above are thus consistent with Hulten's theorem in a partial equilibrium sense. Specifically, they equate each firm's importance to total surplus to its share of its own market. Intuitively, this result follows from the envelope theorem. Due to downstream competition between manufacturers, conditional on any relationship network, the equilibrium allocation of consumers to manufacturers to suppliers maximizes total surplus. Moreover, any shock to a firm's markup, either via the willingness-to-pay for its product or its marginal costs, affects total surplus in only two ways. First, firm-specific markup shocks directly affect total surplus through the firm's portion of the surplus. This direct effect equals the firm's market share. Second, firm-specific shocks also influences total surplus by altering each firm's market share. However, since the equilibrium allocation maximizes total surplus, the second indirect effect is zero, implying the result.

2.2.3 Planner's Allocation when Product Differentiation is Complete

The social planner under Assumption 2.2.2 solves

$$\max_{\mathbf{q},s_2} q_1v_1 + q_2v_2 - C(q_1) - C(q_2) - \{s_2 = 1\}\left(C(q_1+q_2) - C(q_1) - C(q_2) + F\right)$$

subject to

$$0 \leq q_m \leq \frac{1}{2} \quad \forall m \leq 2.$$

The planner's allocation of manufacturers to suppliers is determined by the variable s_2 as before. However, in contrast to Subsection 2.2.1, the planner's allocation of consumers to manufacturers respects the market segments catered for by the manufacturer products.

Analyzing the planner's problem under various configurations of the parameters $\mathbf{v}$ and $AC\left(\frac{1}{2}\right)$ implies the following result.

Proposition 2.2.1. *Suppose Assumption 2.2.2 holds. Supplier sharing occurs only if*

$$F \leq \hat{F} \equiv \begin{cases} 2C\left(\frac{1}{2}\right) - C(1) & \text{if } AC\left(\frac{1}{2}\right) < v_1 \wedge v_2 \\ C\left(\frac{1}{2}\right) - C(1) + \frac{v_1 \wedge v_2}{2} & \text{if } v_1 \wedge v_2 \leq AC\left(\frac{1}{2}\right) \leq v_1 \vee v_2 \\ \frac{v_1+v_2}{2} - C(1) & \text{if } v_1 \vee v_2 < AC\left(\frac{1}{2}\right) \end{cases} \tag{2.7}$$

and occurs with frequency $G(\hat{F})$ under the social planner's allocation.

Proposition 2.2.1 says the socially optimal network depends on model primitives in intuitive ways. Figures 2.7 and 2.8 illustrate this. Briefly, the network is usually connected when the maximal willingness-to-pay for each product is high relative to the cost of building new vertical relationships. Moreover, production is often centralized along one manufacturer-supplier pair when only one market segment's willingness-to-pay is high.

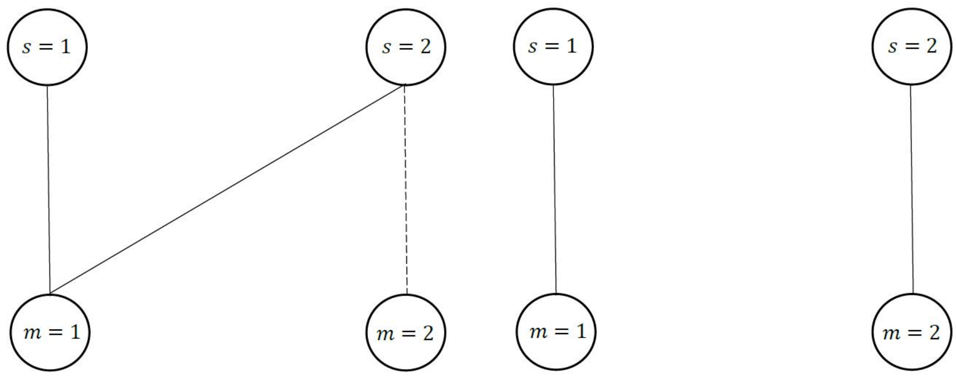


Figure 2.7: Planner's Production Network when F is Low (LHS) or High (RHS) versus $\mathbf{v}$

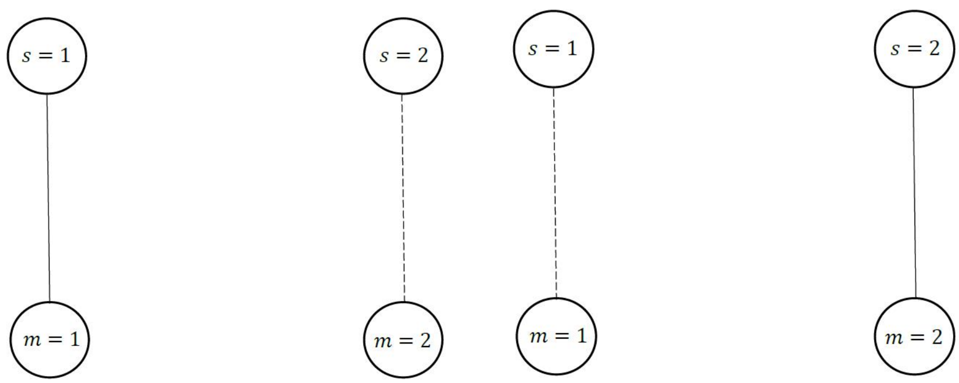


Figure 2.8: Planner's Production Network when v_1, F (LHS) or when v_2, F (RHS) are High

In contrast to the previous subsections, the production network endogenously depends on F. Hence, ex-ante and ex-post total surpluses are distinct. The effects of model primitives on ex-ante total surplus, thus depend on how they influence network formation, in addition to their effects on each firm's ex-post market share and markup.

Yet, the social planner's allocation still respects Hulten's theorem in an ex-ante sense. Differentiating ex-ante total surplus with respect to the model parameters obtains

Proposition 2.2.2. *Suppose Assumption 2.2.2 holds. Marginal effects of the willingness-to-pay for each manufacturer's product on ex-ante total surplus created by the social planner's* *allocation equal*

$$\frac{\partial TS}{\partial v_m} = \mathbb{E}q_m \quad \forall m \leq 2,$$

while the marginal effect of raising the suppliers' cost function $C(Q)$ by cQ is

$$\frac{\partial TS}{\partial c} = -\sum_{s\leq 2} \mathbb{E}q_s^S,$$

whenever these effects exist. These derivatives exist when $\mathbf{v} \neq AC\left(\frac{1}{2}\right)(1,1)$.

The intuition underlying this result is similar to the explanation given in Subsection 2.2.2. The planner's allocation maximizes ex-post total surplus and hence ex-ante surplus too. In particular, supplier sharing occurs only when it is socially optimal for manufacturers to share suppliers. The envelope theorem implies the marginal effect of a firm-specific shock on total surplus through relationship formation is nil. This marginal effect thus occurs directly from firm-specific parameters to total surplus. This effect in turn equals the firm's ex-ante market share.

2.2.4 Equilibrium Analysis when Product Differentiation is Complete

The game's equilibria are analyzed using backward induction as in Subsection 2.2.2. This involves deriving the suppliers' payoffs as functions of their bids, and rolling back to obtain the manufacturers' profits as functions of output prices. Solving for all equilbria obtains the following result.

Proposition 2.2.3. *Suppose assumptions 2.2.2 and 2.2.3 hold. The set of equilibrium relationship and production networks satisfying Assumption 2.2.4 or assumptions 2.2.5 and 2.2.6* *coinicide. Moreover, supplier sharing occurs in the relationship network with probability*

$$G\Bigg([1-\beta_1]\left[AC\left(\frac{1}{2}\right) - AC\left(1\right) + \frac{v_1 - AC(\frac{1}{2})}{2} \wedge 0\right] \vee 0 \\ + \frac{1}{2}\left[v_2 - AC\left(\frac{1}{2}\right)\right] - \frac{\beta_2}{2}\left[v_2 - AC\left(\frac{1}{2}\right)\right] \vee 0\Bigg) \tag{2.8}$$

in any of these equilibria.

In contrast to the equilibria of Subsection 2.2.2, supplier sharing can occur with positive probability. The intuition is twofold. First, Assumption 2.2.2 effectively locks consumers to separate manufacturers. Supplier 1 hence enjoys economies of scale only by supplying both manufacturers. Second, product differentiation implies manufacturers enjoy market power. Hence as in Helper & Levine (1992), manufacturer 2 is more willing to consider new suppliers. Doing so prevents supplier 1 from bargaining away a share of its oligopoly rent.

Proposition 2.2.3 also relates ex-ante relationship network connectivity to the model parameters. Ex-ante, supplier sharing is obviously independent of manufacturer 2's bargaining power β_{21} when it bargains with supplier 1 over the split of their joint investment's returns. Supplier sharing is also obviously decreasing in the distribution of investment costs G, when the latter is ordered by FOSD. Finally, supplier sharing relates to the remaining parameters as follows.

$\boldsymbol{\beta_m}$**:** Ex-ante, supplier sharing is non-increasing in manufacturer 1's bargaining power when negotiating input prices. For intuition, observe such negotiations occur after relationships are formed. Hence, manufacturer 1 extracts a share of any surplus created by manufacturer 2 and supplier 1 investing, via lower input prices. As β_1 increases, this share rises, leaving less surplus to split between the investing firms. Ex-ante, supplier sharing is also non-increasing in manufacturer 2's bargaining power. As β_2 rises, the input price manufacturer 2 pays when the relationship network is disconnected falls. In contrast, its input price when the network is connected is determined by an auction, and is thus independent of β_2.

$\mathbf{v_m}$: Ex-ante, supplier sharing is non-decreasing in the maximal willingness-to-pay for manufacturer 1's product. For intuition, observe the feasible payoffs during Nash bargaining over input prices enlarges as v_1 rises. Hence the input price manufacturer 1 negotiates with supplier 1 also increases. However, supplier-level economies of scale enables manufacturer 1 to operate when the relationship network is connected, even when it can't under a disconnected network. Supplier 1 earns this input price only when the network is connected under such circumstances. Hence, its incentive to form a new relationship increases in v_1.

Ex-ante, supplier sharing is also non-decreasing in manufacturer 2's ability to charge higher markups v_2 over its assembly costs. The underlying intuition depends on whether manufacturer 2's markup is sufficiently high to allow either supplier to fulfill its input needs. Suppose this is so and v_2 exceeds each supplier's average cost of supplying manufacturer 2 alone $AC\left(\frac{1}{2}\right)$. Then a procurement auction prevents suppliers from extracting manufacturer 2's oligopoly rent through Nash bargaining. Manufacturer 2 is thus more willing to invest in new supplier relationships when v_2 is high. Suppose v_2 is smaller than $AC\left(\frac{1}{2}\right)$ instead. Each supplier is incapable of supplying manufacturer 2 without also supplying manufacturer 1. Hence any increase in the markup enlarges the surplus created by manufacturers sharing suppliers. This encourages supplier sharing.

AC(Q): Supposed input production costs $C(Q)$ are increased to $C(Q) + cQ$ for all $Q \geq 0$. Proposition 2.2.3 implies this reduces the frequency of supplier sharing. Such a change leaves supplier-level economies of scale, measured by $AC(1) - AC\left(\frac{1}{2}\right)$, unchanged. It also implies higher equilibrium bids received by manufacturer 2 when the relationship network is connected. Manufacturer 2 is hence more reluctant to invest in relationships with new suppliers.

These comparative statics apply to typically unobserved relationships, as opposed to production linkages. Fortunately, the production network features similar comparative statics.

Proposition 2.2.4. *Suppose assumptions 2.2.2 to 2.2.4 or 2.2.2, 2.2.3, 2.2.5 and 2.2.6 hold. The probability of both manufacturers producing strictly positive quantities and sharing a supplier, is*

1. *weakly increasing in the willingness-to-pay for their products* $\mathbf{v}$*, and the distribution of relationship building costs* G *(ordered by FOSD),*
2. *weakly decreasing in their bargaining powers* $\boldsymbol{\beta}$*, and input production marginal cost* c *where* $C(Q)$ *is increased to* $C(Q) + cQ$*,*

in any equilibrium.

Hence, the frequency of supplier sharing occurring under the production network responds to model primitives in the same way as supplier sharing does under the relationship network. This yields testable predictions from production network data. Obviously, whether manufacturers operate correlates with whether they share suppliers. Hence, the aforementioned frequency differs from the probability of supplier sharing conditional on particular manufacturers operating. For example, suppose manufacturers sell products with equal willingness-to-pay. Corollary 2.9.3, located in the appendix, shows the latter conditional probabilities initially weakly increases as the willingness-to-pay parameters rise. Later, the probabilities discontinuously fall due to manufacturers entering the market without sharing suppliers. The non-decreasing relationship between willingness-to-pay and supplier sharing resumes after the jump.

In what follows, let $\Delta C\left(\frac{1}{2}, \frac{1}{2}\right) = 2\left(C(1) - C\left(\frac{1}{2}\right)\right)$ denote the average cost of producing $\frac{1}{2}$ output units of input, conditional on already producing the same amount.

To evaluate Assumption 2.2.2's welfare consequences, one can start with the difference in ex-ante surpluses for the social planner's and equilibrium allocations. One then characterizes its relationship to the model's parameters. However, investment costs are not distributed according to a convenient functional form. Hence, this exercise is unlikely to yield informative comparative statics. I thus assess the model's welfare implications via a less direct approach. Propositions 2.2.1 and 2.2.3 yield necessary and sufficient conditions for the equilibrium and socially optimal relationship networks to be connected. These conditions constitute upper thresholds for the cost of relationship creation F. Comparing the thresholds yields the following result.

Proposition 2.2.5. *Suppose assumptions 2.2.2 to 2.2.4 or 2.2.2, 2.2.3, 2.2.5 and 2.2.6 hold. The relationship and production networks are over or equally connected in comparison to their socially optimal counterparts if*

$$\beta_1 \left[\left(AC\left(\frac{1}{2}\right) - \Delta c\left(\frac{1}{2}, \frac{1}{2}\right) \right) + \left(v_1 - AC\left(\frac{1}{2}\right) \right) \wedge 0 \right] < (1-\beta_2)\left(v_2 - AC\left(\frac{1}{2}\right) \right) \vee 0.$$

The same networks are under or equally connected versus their social planner's counterparts if

$$(1-\beta_2)\left(v_2 - AC\left(\frac{1}{2}\right) \right) \vee 0 < \beta_1 \left[\left(AC\left(\frac{1}{2}\right) - \Delta c\left(\frac{1}{2}, \frac{1}{2}\right) \right) + \left(v_1 - AC\left(\frac{1}{2}\right) \right) \wedge 0 \right].$$

The equilibrium and social planner's allocations coincide almost surely if neither inequality holds.

For intuition regarding Proposition 2.2.5, observe inefficient network formation occurs for two reasons. First, Nash bargaining between manufacturer 1 and supplier 1 enables the former to extort a share

$$\beta_1 \left[\left(AC\left(\frac{1}{2}\right) - \Delta c\left(\frac{1}{2}, \frac{1}{2}\right) \right) + \left(v_1 - AC\left(\frac{1}{2}\right) \right) \wedge 0 \right] \tag{2.9}$$

of the surplus created when manufacturers share suppliers operating economies of scale technology. This holds up investment by supplier 1 and manufacturer 2 in a new relationship. Supplier sharing is thus inefficiently discouraged.

Second, by developing a new relationship with supplier 1, manufacturer 2 introduces competition to supplier 2. This erodes supplier 2's market power in the upstream market. Such competition typically benefits consumers by reducing costs and increasing output. However, the extreme nature of product differentiation considered under Assumption 2.2.2 means manufacturer 2 behaves like a downstream market monopolist. Thus, this competition merely transfers supplier 2's monopoly rent

$$(1-\beta_2)\left(v_2 - AC\left(\frac{1}{2}\right)\right) \vee 0 \tag{2.10}$$

to manufacturer 2. As Elliot (2015) explains, investment by manufacturer 2 can thus be seen rent-seeking. From this perspective, such investment is excessive.

Hence, whether the relationship network is over or under connected relative to its social optimum, depends on which of these two inefficiencies dominate. When hold-up of investment by manufacturer 1 is severe, the quantity in (2.9) exceeds that in (2.10). Thus, Proposition 2.2.5 implies the network is under connected when inefficient. Conversely, suppose rent-seeking overinvestment by manufacturer 2 is severe. The proposition implies the network is too connected when inefficient.

Finally, Proposition 2.2.5 also sheds light on when shocks to individual firms have disproportionately large welfare consequences. Computing the total surplus generated in equilibrium and taking derivatives yields the following result.

Proposition 2.2.6. *Suppose either assumptions 2.2.2 to 2.2.4 or 2.2.2, 2.2.3, 2.2.5 and 2.2.6 hold. Let* $\theta \equiv \frac{dG}{dF}\left(F(\boldsymbol{\sigma}|_{\mathbf{I}})\right)\left(\hat{F} - F(\boldsymbol{\sigma}|_{\mathbf{I}})\right)$*, where* $\hat{F}$ *is defined in* (2.7) *and* $F(\boldsymbol{\sigma}|_{\mathbf{I}})$ *is G's argument in* (2.8)*. Then*

1. *marginal effects of willingness-to-pay for the manufacturers' output on ex-ante total surplus are*

$$\frac{\partial TS}{\partial v_1} = \mathbb{E}q_1 + \frac{(1-\beta_1)\left\{\Delta c\left(\frac{1}{2},\frac{1}{2}\right) < v_1 < AC\left(\frac{1}{2}\right)\right\}}{2}\theta \qquad \frac{\partial TS}{\partial v_2} = \mathbb{E}q_2 + \frac{1-\left\{v_2 > AC\left(\frac{1}{2}\right)\right\}\beta_2}{2}\theta,$$

and strictly exceed (is less than) their manufacturers' ex-ante output only if the relationship and production networks are not over (under) connected w.p.p. vis-a-vis the social planner's networks,

2. *the marginal effect of raising the suppliers' cost function $C(Q)$ by cQ on total surplus is*

$$\frac{\partial TS}{\partial c} = -\mathbb{E}\left[q_1^S + q_2^S\right] - \frac{(1-\beta_1)\left\{\Delta c\left(\frac{1}{2},\frac{1}{2}\right) < v_1 < AC\left(\frac{1}{2}\right)\right\} + 1 - \beta_2\left\{v_2 < AC\left(\frac{1}{2}\right)\right\}}{2}\theta,$$

whose magnitude strictly exceeds (is less than) their combined ex-ante output only if the relationship and production networks aren't over (under) connected w.p.p. versus the planner's networks,

whenever the effects exist. The derivatives exist when $v_1 \neq \Delta C\left(\frac{1}{2},\frac{1}{2}\right)$ and $\mathbf{v} \neq AC\left(\frac{1}{2}\right)(1,1)$[16].

The marginal effects described by Proposition 2.2.6 do not respect Hulten's theorem. Moreover, these effects are (weakly) amplified rather than dampened vis-a-vis ex-ante market shares when the production network is ex-ante overly disconnected. The latter is surprising in light of recent macroeconomics research on production networks. The intuition one acquires from this literature is that shocks to small firms can have large welfare consequences. This is due to shock propagation across a highly connected network[17]. For example, Acemoglu et al. (2012) demonstrate aggregate output volatility caused by microeconomic shocks fails to vanish in large economies, when production network interconnectivity is highly asymmetric across nodes. Figure 2.7's LHS panel illustrates such a network. Acemoglu et al. establish their results via a multi-sector model of competitive markets. In contrast, this paper focuses on the effects of product differentiation and economies of scale on network formation. When such forces are accounted for, Proposition 2.2.6 shows the usual intuition correlating shock amplification to

16. The three derivatives' magnitudes also strictly exceed (is less then) ex-ante output only if the relationship network is under (over) connected w.p.p. See the proposition's proof for details.

17. This assertion is backed by Carvalho (2014)'s survey of the literature, and in particular, his comparison of "network multipliers" for horizontal, vertical and star production networks.

network connectivity may not hold, and is possibly reversed.

The intuition for why Hulten's theorem fails is simple. Marginal effects of shocks to the firms' markups on ex-ante total surplus equal their ex-ante market shares under the socially optimal allocation. This is due to the envelope theorem. However, downstream product differentiation distorts supplier 1 and manufacturer 2's relationship-forming incentives. Inefficient network formation hence causes Hulten's result to break. Understanding why firm-specific shocks have disproportionately large welfare consequences when the network is infrequently connected is less straightforward. Consider a shock to the maximal willingness-to-pay for manufacturer m's product, v_m. By the envelope theorem, the direct effect of such a shock on ex-ante total surplus is m's ex-ante market share $\mathbb{E}q_m$. However, the frequency of supplier sharing is also increasing in v_m by Proposition 2.2.3. So the indirect effect of v_m on total surplus due to network formation, reinforces the direct effect when supplier sharing is infrequent compared to the social optimum. Conversely, the indirect effect of v_m due to network formation dampens the original effect when supplier sharing is too frequent. The intuition correlating shock amplification with network connectivity is thus reversed. This intuition carries over to how total surplus responds to changes in supplier marginal costs c too.

We conclude this section by evaluating the welfare consequences of manufacturer-specific bailouts. The 2008 financial crisis saw Ford lobbying Congress to bailout its rivals, citing interdependence in auto manufacturers' fortunes through common suppliers. Baqaee (2018) suggests an explanation for Ford's unusual behavior. He uses a general equilibrium model similar to Acemoglu et al.'s, but featuring costly entry and exit. Ford's behavior is also rational in this model.

Corrollary 2.2.1. *Suppose either assumptions 2.2.2 to 2.2.4 or 2.2.2, 2.2.3, 2.2.5 and 2.2.6 hold. Let* $m \in \mathbb{N}_{\leq 2}$. *Manufacturer m's ex-ante profit* $\mathbb{E}\pi_m^M$ *is non-decreasing in its rival's willingness-to-pay* v_{-m}. $\mathbb{E}\pi_m^M$ *is strictly increasing in* v_{-m} *when* $\mathbf{v} >> AC\left(\frac{1}{2}\right)$, *and supplier* *sharing both occurs and doesn't with positive probabilities in the relationship network.*

However, the intuitions underlying Baqaee's result and Corollary 2.2.1 differ. In Baqaee's model, adverse shocks to a subset of downstream firms cause many of them to exit. This triggers a cascade of exits amongst their upstream suppliers too. Because manufacturers have Dixit-Stiglitz preferences over inputs, the surviving downstream firms incur costs when substituting from exiting suppliers to new ones. In my model, suppliers produce homogeneous inputs. Hence, technological economies of scale experienced by suppliers, rather than external economies of scale due to input variety, are responsible for shock propagation. The proof of Corollary 2.2.1 is simply a consequence of Proposition 2.2.3. Any increase in either manufacturer's markup through $\mathbf{v}$, causes more frequent supplier sharing. But when manufacturers share suppliers, manufacturer 1's ability to extract part of the surplus created by supplier sharing implies its (ex-post) profit increases by (2.9). Moreover, manufacturer 2's rent-seeking motive when investing in new relationships means its profit increases by (2.10). Each manufacturer thus benefits when a subsidy increases their rival's markup.

The difference in intuitions manifest in the way firm-specific shocks predict supplier entry and exit. Baqaee's intuition implies manufacturer-specific shocks are positively correlated with upstream entry[18]. The counterpart correlation in my model depends on the downstream market's state. Suppose manufacturers face low willingness-to-pay so that $v_1 \vee v_2$ is below the average cost of separately supplying the manufacturers $AC\left(\frac{1}{2}\right)$. Any increase in either manufacturer's markup increases the ex-ante number of operating suppliers. This is because neither supplier operates unless supplier sharing occurs, while higher downstream markups imply more frequent supplier sharing. Suppose $v_1 \wedge v_2 > AC\left(\frac{1}{2}\right)$ instead. Now manufacturers and suppliers can still operate profitably absent supplier sharing. Supplier sharing hence causes a supplier to shut down. Upstream entry thus is inversely correlated with manufacturer-specific shocks

18. when the elasticity of substitution across upstream market inputs is sufficiently small. See the explanation given for Example 1 in Baqaee (2018) for details.

instead.

2.3 Literature Review

Methodologically, this paper closely relates to the theoretical buyer-seller networks literature. A key question in this area is whether network formation is efficient when specialized to a buyer-seller context. Kranton & Minehart (2001) is a pioneering paper in this research strand. These authors show buyer-seller networks produce efficient outcomes when the more informed side of the market (buyers) incurs the cost of link formation, and prices are determined by Walrasian-style ascending bid auctions[19]. In my view, the paper in this literature most closely related to mine is Elliot (2015). Elliot points out when prices are determined by bilateral bargaining rather than auctions, hold-up in relationship-specific investments inefficiently discourages relationship formation. Moreover, when both buyers and sellers bear the cost of link formation, buyers are encouraged to over-invest in new relationships, as in my model.

Whilst being simpler, my paper departs from Elliot's in three ways. First, I highlight another source of inefficiency - investment hold-up by neighboring firms - in buyer-seller network formation. Because this inefficiency hinges on suppliers experiencing economies of scale when supplying multiple buyers, they do not arise in Elliot's model. Analyses of similar inefficiencies appear mainly located in more abstract networks research, such as by Bloch & Jackson (2007). Second, by basing my model on documented features of the auto industry, I account for heterogeneities in pricing protocols that Elliot's particular parameterizations miss. More specifically, a buyer's price in my model is determined by bilateral bargaining or a first-price auction, depending on its number of connected suppliers. This heterogeneity is significant be-

19. For another example, Condorelli and Galeotti (2012) highlight coordination failure associated with multiple equilibria and positive externalities when path connected agents serve as trading intermediaries, as causes for inefficient network formation. For a final example, Wang & Watts (2006) highlight how supplier vertical differentiation or sellers' associations cause ex-post mismatch when multiple buyers (sellers) ex-ante prefer to link with a common seller (buyer), but regret their choice after being rationed ex-post.

cause the buyer pays the less efficient supplier's cost under the auction's intuitive outcome. The payment is larger then what the buyer pays under Elliot's bargaining protocol. Incentives to invest in relationships are hence smaller. This suggests Elliot's computed overinvestment efficiency losses are upwards biased if applied to the auto industry. As far as I know, pricing protocol heterogeneity in buyer-seller networks is studied in only one other paper, Watts (2016). This paper is not focused on inefficient network formation. Finally, I introduce model-specific refinements to address the issue of multiple equilibria, instead of employing upper bounds for welfare losses. Multiple equilibria arise for different reasons in Elliot's model and mine. Due to payoff complementarities created by supplier-level economies of scale, multiple equilibria occur in my model. Because more than one buyer-seller pair can form links, strategic considerations imply multiple equilibrium *networks* in Elliot's. The hope is for my approach to be as useful for more general analyses of manufacturer-supplier networks.

This paper also relates to the literature on production networks. A portion of this literature attempts to explain distinct features of production networks seen in data. For example, Oberfield (2017) shows how slight dispersion in match-specific productivities creates highly efficient suppliers with unusually many customers. This is accomplished via a micro-founded model of efficient network formation[20]. Another portion of this literature concerns how network formation interacts with imperfect competition to create inefficiencies. These papers typically focus on a particular industry. For example, Ho (2009) studies insurer-hospital network formation when hospital heterogeneity confers market power to high quality hospitals[21]. A final portion of the

20. For another example, Atalay et al. (2016) show how realistically accounting for link formation when firms enter or exit, in a less micro-founded model, allows one to explain abnormally thick (thin) right (left) tails of the in-degree distribution for the U.S. economy's production network. For a final example, Mizuno et al. (2014) highlight how past manufacturer-supplier relationships determines future production networks, through network persistence in Japanese data.

21. Also, Ho & Lee (2015) highlight how greater downstream competition between insurance firms lead to higher hospital prices due to hospitals playing insurers off each other when bargaining with more than one insurer. And, Lee (2013) studies network formation between videogame developers and platform manufacturers, to evaluate the welfare consequences of vertical integration.

literature emphasize how inter-firm relationships propagate local shocks across an economy[22]. An important paper here is Acemoglu et al. (2012). These authors show how sectoral shocks cause aggregate output fluctuations through asymmetric production networks. In my view, the papers in this area closest in scope to mine are Baqaee (2018) and Taschereau-Dumochel (2020). These analyze how firm or industry-specific shocks trigger cascades of firm shutdowns through manufacturer-supplier relationships when imperfect competition is present. However, Baqaee focuses on the equilibrium network while Taschereau-Dumochel focuses on the socially optimal network.

My paper departs from Baqaee's and Tascehreau-Dumochel's in three ways. First, I consider a partial equilibrium model focused on a single industry. This approach allows me to better account for forces thought to be recently affecting the auto industry. Such forces include economies of scale and input pricing heterogeneity. Given the extent to which automaker bankruptcies during the Great Recession motivates macroeconomics research on production networks, I see my focused approach as being sensible and complementary. Second, I analyze formation of both equilibrium and socially optimal networks under a single, albeit simpler framework. This allows definition of when a network is "under" or "over" connected. Also, the effects of firm-specific shocks on overall welfare can be compared to the same effects under a socially optimal allocation. One can thus characterize how production network over-connectivity relates to amplification of firm-specific shocks beyond its socially optimal benchmark. Last, my empirical analysis uses product-level data from one industry, rather than more aggregated data. Hence, methods introduced by Berry et al. (1995) can be adapted to analyze how product differentiation affects network formation. Obviously, my empirical results are especially relevant to automobile production.

22. Acemoglu et al. (2012)'s main result has already been discussed. See Baqaee & Farhi (2018) for a restatement of Hulten's theorem, in addition to analysis of potentially large, second-order welfare consequences of firm-specific disasters. See Grassi (2018) for understanding shock propagation through production networks, when product differentiation exists in input and output markets. Finally, see Carvalho et al. (2014) for a purely empirical analysis of shock propagation using economy-wide firm-level data.

This paper also relates to the vast scholarship on the American auto industry. A literature on the industry's recent bailouts has recently developed[23]. However, none of these papers concern manufacturer-supplier network effects. Another literature focuses on empirically explaining when manufacturer-supplier relationships form. Benelli (2017), Fox (2017) and Badorf et al. (2019) typify this approach[24]. However, the papers do not formalize why relationship formation may be inefficient. Perhaps the approaches in this area most similar to mine, are theoretical studies of manufacturer-supplier contracting, based on documented aspects of the industry. For example, guided by international differences in auto manufacturer-supplier relationships, Helper & Levine (1992) theorize how variation in downstream market power causes heterogeneity in input market contracting. For another example, guided by dual-sourcing auto manufacturer behavior, Riordan (1996) shows how contracting under incomplete information causes manufacturers to build relationships with multiple suppliers. In these papers, inefficiencies arise due to relationship-specific investment hold-up and asymmetric information[25]. However, neither of these papers account for the supplier sharing behavior that partially rationalized the bailout[26]. Finally, a considerable amount of management research on auto suppliers also exists. Within this literature, a debate rages as to whether relationship-specific assets exist, and explain ownership structure or firm performance in the industry[27]. An obstacle towards answering these

23. In particular, Anginer & Warburton (2010) and Baumann & Thompson (2015) analyze the auto bailout's effect on credit and labor markets respectively, Hortacsu et al. (2011, 2013) analyze how financial distress prior to the bailout affected assembler sales, while Woollmann (2016) studies its effect on entry of new truck models.

24. Also, see Klier & Rubenstein (2008). Also, see Klier (1999), Buenstorf & Klepper (2009), Rosenbaum (2013), and Adams (2015) for studies of how manufacturer-supplier interactions explain plant locations in the auto industry.

25. In Helper & Levine, adverse selection is caused by suppliers not knowing their customers' oligopoly rents. Also, hold-up is present. In Riordan, moral hazard is created when manufacturers do not observe suppliers' efforts. Also, Riordan shows how endogenous supplier effort can create upstream economies of scale.

26. See Goolsbee & Krueger (2015) for details on policymaking during the crisis.

27. For a flavor of this debate, Monteverde & Teece (1982), Mastern et al. (1989), Dyer (1996) and Head, Reis & Spencer (2004) find evidence in favor of both claims, while Miwa and Ramseyer (2000) disagree with considerable conviction. Also, see Choi & Hartley (1996) for further details on how auto suppliers are selected, Ben-Shahar & White (2006) for how auto parts contracts are written, Mudambi & Helper (1998) for how relationships are perceived by U.S. auto suppliers, and Cusumano & Takeishi (1991) for comparisons of auto

questions is the difficulty of measuring specificity of any asset to a relationship, as Dyer (1996) notes. This paper implicitly presumes relationship-specific assets exist. However, its primary concern is the consequence of these assets on manufacturer-supplier network formation, rather than vertical integration or performance. In sum, my paper departs from this literature by focusing on the manufacturer-supplier network's shape, and its associated inefficiencies.

2.4 Data

The primary data sources are the collection of Ward's Auto Yearbooks for the years 1994-2016, and the Who Supplies Whom database over 2008-2016. As discussed in Berry et al. (1995), the Ward's yearbooks provide the characteristics of nearly every car model sold in the US. In particular, the dimensions, horsepower, weight, miles per gallon and import status of each model is tabulated at the back of every book, while the annual sales (the quantity sold) of the same models are provided in a different chapter. When the models' characteristics are matched to their sales, one obtains an unbalanced panel of 2971 model-year pairs, over the years 1994-2016.

As discussed in Rosenbaum (2013) and Benelli (2017), the Who Supplies Whom (WSW) database consists of auto parts contracts for selected model-supplier relationships, based on an online survey of auto manufacturers and suppliers. In particular, for each contract, the database provides the model's and supplier's name, the model's launch year and assembly plant, and brief descriptions of the auto part in varying levels of specificity. In addition, for a small subset of contracts, the relationship's start and end dates are also observed.

Matching the main WSW relationships to the Ward's models yields a subsample of 879 model-year observations over the years 2008-2016. Each model-year observation in this subsample is accompanied by its ten most important suppliers, judged by the number of contracts

supplier relationships by nationality.

the suppliers sign with the observed model. Each observation also yields its ten most important suppliers for auto parts in each of the five separate categories: powertrain, electrical, chassis, exterior and interior. Finally, the launch dates and assembly locations for the subsample's automobiles are also observed.

This model-level dataset is augmented with annual information from multiple external sources. This paper uses time series of average petrol prices and US auto manufacturing wages from the Energy Information Administration and the Bureau of Labor Statistics (BLS) respectively. Also used are the annual estimated number of US households, and producer price indices for various inputs from the Federal Reserve Economic Data (FRED) website. Finally, data on US inflation was obtained from the Organization for Economic Co-operation and Development while mean and median household incomes were obtained from the BLS Consumer Population Survey.

2.4.1 Data Preparation

The manner in which data were collected and transformed is now discussed in more detail. To obtain downstream market data, the sales and remaining specifications of each model-year enumerated in the Ward's yearbooks were scraped from library copies. The specifications collected was each model's name, year, brand, manufacturer, drive type, body style, length, breadth, height, horsepower, weight, estimated miles per gallon (separately for city and highway travel), whether the model was imported and the manufacturer's suggested retail price.

This process was complicated by the presence of multiple versions of a model-year, giving rise to multiple values for the characteristics to choose from. For example, the 2010 variant of the Honda Civic features sedan, coupe and luxury editions, each with unique dimensions and horsepower specifications. In line with Berry et al. (1995), I attempted using base version specifications for each model-year. When a model's base version was ambiguous, its cheapest version was used. Often, the chosen version of a model-year observation was simply the first version listed in the relevant yearbook. As will be discussed in further detail, I also googled the production history of many models to facilitate construction of my final dataset. When in this process it was discovered that a given model's base version was in error, its base version specifications were promptly rectified.

As aforementioned, the sales and specifications data were collected from different locations in the books and hence had to be matched. The matching was based on each model's name. For every model-year where sales datum was available (say the 2016 Mazda3), I located characteristics data for a model-year with the same name and year (say 2016 Mazda3 sedan specifications). On many occasions, no model-year with identical name and year could be found. For example, specifications for the 2005 Mercedes CLS are unavailable in the 2005 yearbook. In such situations, I googled the model's production history to locate its actual year of launch. The characteristics for the relevant model-year was then used, with its retail price appropriately deflated using inflation data (So the 2006 CLS coupe specifications were paired with 2005 CLS sales, with its 2006 price deflated to 1992 dollars). Often, the actual year in which a model-year was released was adjacent, if not equal to the year of the yearbooks where its sales datum was recorded. This process left only a handful of model-year observations with negligible market shares unmatched.

After assembling an unbalanced panel of model-year observations with their downstream market attributes, the data was transformed so that each model-year's size (length times breadth), volume, horsepower-to-weight and average miles per gallon (average of city and highway miles per gallon) were computed. OECD inflation data was also used to deflate each model-year's price, and the fuel price index. Each model-year's miles per dollar was thus computed from its average miles per gallon and the real fuel price for the model's year. Finally, models with inside market shares smaller than 0.05% were discarded. This process produces an unbalanced panel of 2971 model-year observations. The variables in this panel are brand, size, volume, horsepower-to-weight (hpw/wgt), miles per dollar (MPD), price, sales and import status.

To obtain the 2971 model-years' remaining information, each model-year was manually matched to model-supplier contracts in the WSW database. For a contract to be linked to a particular model-year, it must i) share the same model name or pseudonym, ii) feature a brand and assembly location compatible with the model-year's brand and import status, and iii) possess the latest launch year amongst WSW contracts satisfying i) and ii). For example, because there were no contracts for models launched during 2015-16 with model name and brand resembling Mazda3, the Mazda3 attributes in the 2016 Ward's yearbook were matched to the set of contracts whose model was listed as 3 (2014), under the brand Mazda, launched in 2014, and assembled in Salamanca, Mexico. For another example, if there was another contract with identical brand, model name and launch year, but whose assembly plant wasn't North American, it would remain unpaired with the Ward's attributes since the Mazda3 is classified as a non-import in the 2016 yearbook.

This matching process was complicated by models often possessing multiple aliases, unlisted in either the Ward's or WSW data. For example, the Subaru BRZ is also called the Toyota FT/GT 86 or the Scion FR-S, depending on the country in which the model was marketed. Wikipedia was thus employed to verify whether distinct model names were indeed aliases. As a result, model-years were sometimes matched to contracts with apparently different model names. To complete the example, the 2015 Ward's yearbook attributes of both the Subaru BRZ and the Scion FR-S were paired with supplier information for the Toyota (GT) 86 model launched in 2012.

After the matching process was completed, data summarizing each model's auto parts contracts was algorithmically extracted from the WSW database. As aforementioned, each contract in the data specifies names for a model and its supplier, along with the model's brand, assembly location and initial year of launch. Each contract also details the general function of the auto part supplied (contract area), the part's broad location in a typical automobile (contract sector and sub sector) and an ad verbatim description of the part. Finally, a small subset of contracts feature start and end dates.

Unfortunately, most models contain approximately 200-270 contracts in the WSW database, with little objective distinguishing information beyond broad contract area and sector. Also, most of the matched model-years are contracted to approximately 40-50 suppliers. Working with all contracts would thus produce a network of model-supplier relationships too dense and difficult to analyze. Hence, the empirical analysis that follows focuses on only the 10 most important suppliers, as judged by contract numerosity amongst all contracts and in 5 separately chosen contract areas. This yields up to of 60 distinct suppliers per model-year. The contract areas chosen were powertrain, electrical & electronics, chassis, exterior and interior (as per the WSW classification system). The areas omitted were infotainment, lighting, thermal management, user interface and miscellaneous.

Selection on number of contracts means the chosen suppliers are likely to be large firms benefiting from economies of scale. In addition, for each model-supplier relationship in the analyzed sample, the number, modal area and modal sector of the relevant contracts was used to characterize its nature. The analysis thus ignores input heterogeneity found *within* model-supplier relationships beyond the broad 5 contract areas analyzed. Finally, for the subset of relationships where this was possible, the modal start and end dates across contracts under each relationship was also noted. The start dates for these contracts align with the model's launch year. Moreover, the vast majority of contracts under these relationships have identical start and end dates. Hence, little information is lost by summarizing the relationships' durations via their modal dates.

Unfortunately, a mergers and acquisitions boom amongst auto suppliers during 2010-16, compromises the identities of multiple models' main suppliers[28] . To address this, each main supplier's company website and the first 10 pages from googling its name alongside the words "merger" and "acquisition", was manually combed. Each supplier's history of mergers, acquisitions and divestments was thus constructed. The compromised suppliers' IDs was then substituted with those of their acquirers or mergents, to improve data accuracy.

Merging the Ward's and WSW data produces 1286 model-year observations, with 879 of these observed during 2008-16[29]. Because the statistical analysis that follows assumes we observe close to the entire network of model-supplier relationships, its conclusions depend only on data spanning 2008-16. Finally, demographic and input price data were also used. These were downloaded from publicly available and more easily verifiable sources.

2.4.2 Descriptive Statistics

Tables 2.1 and 2.2 tabulate the means and standard deviations of the Ward's model-year attributes in each year. (Even numbered rows display standard deviations.) In particular, size (1000 inches^2), horsepower-to-weight (hpw/1000ibs) and price (1000 92US$) are measured in a way ensuring comparable magnitudes across all variables. The definitions of price, hpw/wgt, and MPD (miles per 92US$) are also compatible with the those featured in Table 1 of Berry et al. (1995).

The models exhibit substantial cross-sectional variation in prices and sales. This is consis-

28. See Foy's (2014) Financial Times article titled "The Age of Mega Suppliers Heralds Danger for Carmakers" at https://www.ft.com/content/50c272c4-dce9-11e3-ba13-00144feabdc0.

29. 1485 out of 2971 model-year observations have at least one supplier in the WSW database, of which 1286 of these have at least one unexpired supplier relationship. 879 of these model-years are found across 2008-2016.

Table 2.1: Descriptive Statistics, Model Attributes over 1994-2007

	# of Models	Size	Height	Hpw/Wgt	MPD	Price	Sales	Share	Import
1994	139.00	12.96	53.95	48.97	25.65	19.94	64.05	0.71	0.42
	0.00	1.62	2.17	10.00	4.89	12.01	69.32	0.77	0.50
1995	135.00	13.03	54.14	50.73	25.57	20.58	63.45	0.73	0.42
	0.00	1.63	2.11	9.83	4.53	11.87	68.95	0.80	0.50
1996	126.00	13.08	54.25	51.85	24.25	21.15	67.27	0.79	0.38
	0.00	1.65	1.97	10.54	4.39	12.12	76.36	0.90	0.49
1997	128.00	13.05	54.35	51.97	25.21	20.57	64.25	0.78	0.40
	0.00	1.54	2.01	11.45	4.64	11.31	74.50	0.90	0.49
1998	125.00	13.10	54.67	54.38	29.73	21.81	64.85	0.80	0.43
	0.00	1.49	2.00	11.39	5.57	11.67	75.98	0.93	0.50
1999	128.00	13.07	55.05	54.60	27.02	21.26	67.76	0.78	0.45
	0.00	1.50	2.31	11.32	4.42	11.91	73.90	0.85	0.50
2000	131.00	12.93	55.10	55.09	21.24	20.63	67.27	0.76	0.46
	0.00	1.57	2.40	12.04	3.65	11.22	72.15	0.82	0.50
2001	137.00	12.98	55.09	56.65	22.65	20.95	61.35	0.73	0.52
	0.00	1.42	2.14	12.38	4.79	11.28	70.45	0.84	0.50
2002	132.00	12.98	55.41	56.42	24.44	20.36	61.22	0.75	0.50
	0.00	1.39	2.39	12.41	4.14	10.68	68.09	0.84	0.50
2003	127.00	13.02	55.69	57.78	21.28	21.06	59.57	0.78	0.54
	0.00	1.36	2.35	13.24	3.66	11.04	70.27	0.92	0.50
2004	134.00	13.06	55.96	57.17	18.23	20.79	55.70	0.74	0.54
	0.00	1.35	2.49	13.49	3.25	11.23	72.78	0.97	0.50
2005	138.00	13.07	56.26	60.74	15.14	21.64	55.33	0.72	0.54
	0.00	1.45	2.50	15.72	2.68	11.96	71.20	0.93	0.50
2006	132.00	12.96	56.47	60.85	13.95	21.02	58.61	0.75	0.55
	0.00	1.48	2.69	15.67	2.48	11.62	69.64	0.89	0.50
2007	127.00	13.01	56.88	61.06	13.20	20.04	59.67	0.78	0.55
	0.00	1.43	3.08	16.35	2.41	11.73	74.98	0.98	0.50

tent with auto manufacturers vertically differentiating. Also, as shown in Figure 2.9's left panel, sales decline sharply over the GFC years (2007-09), and recover over the following years. This trend appears driven primarily by variation in domestic output[30]. Finally and in comparison, real prices (Figure 2.9, LHS, dashed) are temporally stabler, even during the GFC. Because sales *per model* changes over the same timeframe, attributing such stability to long-run average cost minimization is problematic. Rather, price stickiness during the GFC suggests either manufacturers face inelastic marginal cost curves, or that these curves shifted up during the

30. The latter includes models sold by General Motors (GM) or Chrysler, companies reportedly most affected by the GFC (Ingrassia, 2011)

Table 2.2: Descriptive Statistics, Model Attributes over 2008-2016

	# of Models	Size	Height	Hpw/Wgt	MPD	Price	Sales	Share	Import
2008	124.00	13.04	57.13	61.54	10.73	19.98	54.56	0.80	0.56
	0.00	1.55	2.66	15.70	1.82	11.10	70.89	1.04	0.50
2009	127.00	13.02	57.31	61.20	15.41	19.61	42.69	0.78	0.58
	0.00	1.58	2.59	15.63	2.64	10.92	59.30	1.09	0.50
2010	116.00	13.11	57.51	61.10	13.36	19.86	48.29	0.86	0.59
	0.00	1.61	2.30	16.65	2.50	11.11	64.02	1.14	0.49
2011	122.00	13.09	57.35	62.03	11.18	20.95	49.66	0.82	0.60
	0.00	1.45	2.50	15.72	2.68	11.96	71.20	0.93	0.50
2012	126.00	12.95	57.25	62.32	11.61	20.47	57.20	0.79	0.57
	0.00	1.48	2.69	15.67	2.48	11.62	69.64	0.89	0.50
2013	127.00	12.95	57.24	63.23	12.70	21.24	59.31	0.78	0.60
	0.00	1.43	3.08	16.35	2.41	11.73	74.98	0.98	0.50
2014	133.00	12.86	57.25	63.24	14.10	21.21	57.38	0.75	0.56
	0.00	1.55	2.66	15.70	1.82	11.10	70.89	1.04	0.50
2015	126.00	12.98	57.21	63.31	19.66	21.36	59.29	0.79	0.56
	0.00	1.58	2.59	15.63	2.64	10.92	59.30	1.09	0.50
2016	131.00	13.08	57.13	62.60	22.40	21.26	52.03	0.76	0.55
	0.00	1.61	2.30	16.65	2.50	11.11	64.02	1.14	0.49

financial crisis. The former explanation is consistent with external economies of scale reducing the slope of cost curves. The latter is consistent with financially distressed carmakers facing difficulties in financing their operations, as documented by Ingrassia (2011), Hoffman (2012), and Hortacsu et al. (2011, 2014).

The number of models appears to be falling, especially over 2006-10 (Figure 2.9, RHS). Deliberate adjustments to the manufacturers' product lineups in response to the GFC might be a cause. It is difficult however, to disentangle the alleged GFC decline in this variable from ordinary oscillations around its trend. Also, while the set of models shrinks only slightly over the sample's timeframe, its composition has evolved markedly to include more imports. Because the compositional trend predates the GFC, it likely reflects longer term changes in the economy.

Table 2.3 tabulates summary statistics for model-years over 2008-16 with at least one supplier in the WSW data. In particular, Column 1 displays the number of models with WSW information per year, while columns 2-6 summarize the models' age from launch year, assem-

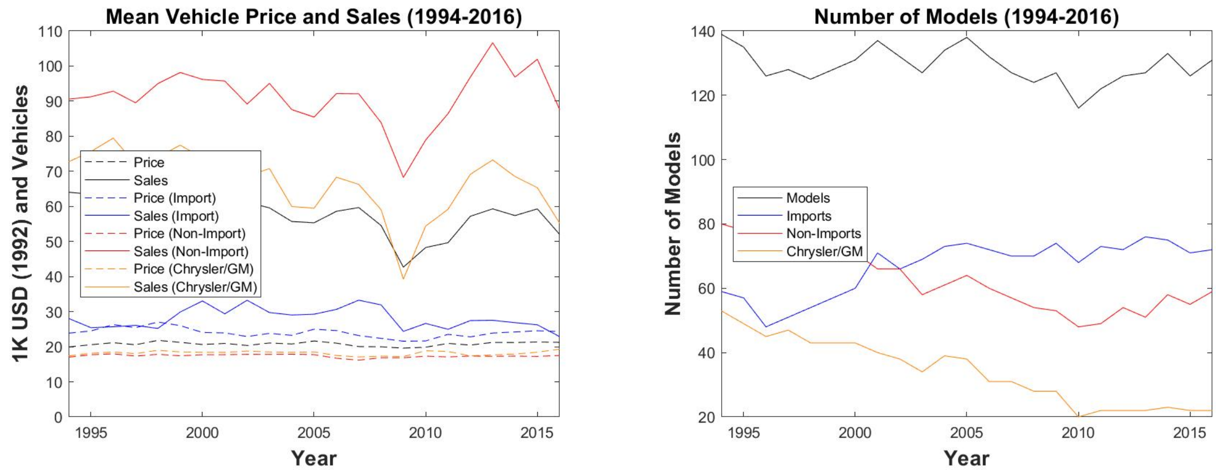


Figure 2.9: Price, Number of Models and Sales over 1994-2016

bly location, number of suppliers and number of auto part contracts. The last three columns describe data specific to the collection of the models' main 10 suppliers. Columns 7-8 count the number of distinct main suppliers and the parts sectors they inhabit. Column 9 counts the number of contracts (in thousands) with the main suppliers.

Table 2.3: Descriptive Statistics, Model-Supplier Attributes over 2008-16

	Models	Age	Latitude	Longitude	Suppliers	Contracts	Main Sup.	Main Sec.	Main Con.
2008	90.00	2.43	41.30	-28.33	54.01	264.14	209.00	41.00	12.22
	0.00	1.85	10.35	67.16	25.60	154.54	0.00	0.00	0.00
2009	95.00	2.73	40.08	-25.27	51.66	253.60	216.00	42.00	12.50
	0.00	2.07	10.96	72.83	27.28	164.68	0.00	0.00	0.00
2010	86.00	2.95	39.77	-20.69	52.71	259.34	202.00	43.00	11.50
	0.00	2.02	11.34	75.67	28.20	172.85	0.00	0.00	0.00
2011	94.00	2.36	40.02	-15.18	49.22	239.91	207.00	44.00	12.03
	0.00	2.10	11.00	78.15	26.74	165.44	0.00	0.00	0.00
2012	102.00	2.39	39.25	-14.27	47.29	221.67	198.00	45.00	12.22
	0.00	2.16	12.27	83.76	25.81	147.99	0.00	0.00	0.00
2013	103.00	2.68	38.87	-15.07	47.56	223.58	197.00	45.00	12.46
	0.00	2.16	12.69	84.25	25.43	151.22	0.00	0.00	0.00
2014	107.00	2.88	37.78	-20.47	47.21	222.09	194.00	45.00	12.91
	0.00	2.20	14.47	80.71	25.62	152.22	0.00	0.00	0.00
2015	100.00	2.86	37.35	-22.08	43.67	199.12	185.00	44.00	11.11
	0.00	2.33	14.36	81.55	23.98	137.19	0.00	0.00	0.00
2016	102.00	3.22	37.41	-20.12	41.71	191.92	179.00	44.00	11.19
	0.00	2.41	14.47	81.61	23.15	132.09	0.00	0.00	0.00

When plotted on the LHS of Figure 2.10, one finds the number of models with WSW data to be growing (and oscillating) in tandem with its Ward's yearbooks counterpart, over 2008-16. These numbers probably reflect recovery in auto industry conditions after the GFC. However, the number of distinct main suppliers falls over the same timeframe. Moreover, as Figure 2.10's right panel shows, the decline in suppliers relative to models holds regardless of auto part contract type (powertrain/electrical/etc.). When paired with the post-GFC recovery in automobile sales, these trends controvert theories positing higher input market profits and entry from higher output demand. As discussed in Section 2.2, I suspect Baqaee's (2018) model of cascading failures to be such a theory.

The trends however, are consistent with press accounts suggesting post-GFC consolidation in carmaker supplier networks[31]. These accounts are further supported by Column 5 of Table 2.3. The column reveals a significant drop in the average number of (main and minor) suppliers per model. This decline suggests Figure 2.10 is representative of the overall supplier population, rather than merely the subset of main suppliers. Column 6 provides a final piece of evidence that tips the scales further away from Baqaee's theory. This shows a decline in the average number of contracts per model. Hence, auto input variety as measured by contract multiplicity decreased while demand for inputs presumably rose, over 2008-16. This is inconsistent with theories relying on "love-of-variety" models of input demand to capture imperfect competition, such as Baqaee's.

Now, a declining survey response rate is potentially responsible for the aforementioned trends. In support of this view, Column 2 shows model-years ageing. It is unclear whether this is due to fewer responses to the WSW survey. Figure 2.11's LHS assesses this viewpoint.

31. See "Global Industry craves Mega Suppliers" in the June 17th, 2013 issue of Automotive News by David Sedgwick at https://www.autonews.com/assets/PDF/CA89220617.PDF. Also see Foy's (2014) Financial Times article titled "The Age of Mega Suppliers Heralds Danger for Carmakers" at https://www.ft.com/content/50c272c4-dce9-11e3-ba13-00144feabdc0.

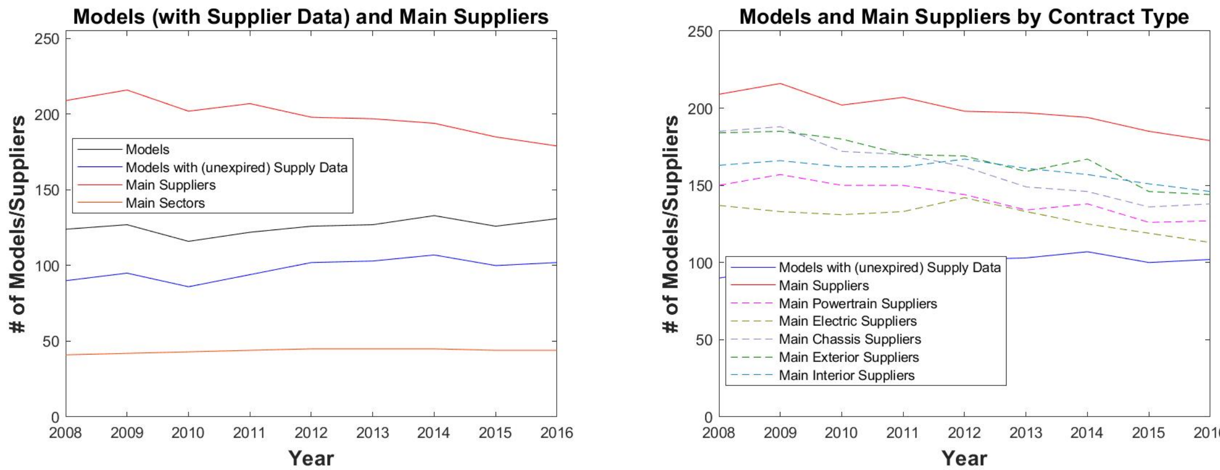


Figure 2.10: Number of 2008-16 Models (with Supply Network Data) & Suppliers

It does so by plotting the fraction of main Ward's model-supplier relationships with available WSW datum, assuming each Ward's model has at least 10 suppliers. The dashed line shows the fraction of main relationships covered by the WSW data when contract end dates are ignored. The red line shows the analogous fraction once expired links are deleted. Both lines trend up. Hence, equating the Ward's models with the population of US marketed models suggests the survey response rate is increasing rather than deteriorating. Since a WSW survey participant has little reason to hide part of its contracts with a counterparty while reporting agreement on another, survey response rates aren't likely causing contract numerosity to fall in Column 6 too.

The dashed lines in Figure 2.11's RHS replicate the fraction of main relationships covered by the (unexpired) data, but for differing contract types. Similar to before, each model is assumed to possess at least 10 suppliers for each contract category. Under this caveat, coverage appears to have declined across all but one category. It follows the falling number of non-powertrain suppliers shown on Figure 2.10's right, *are* potentially explained by declining WSW survey responses. Unfortunately, a model is more likely to have fewer than 10 suppliers of a given category, than 10 suppliers in total. Hence, it is unclear if declining survey responses or other, possibly technological factors are responsible for Figure 2.10's trends.

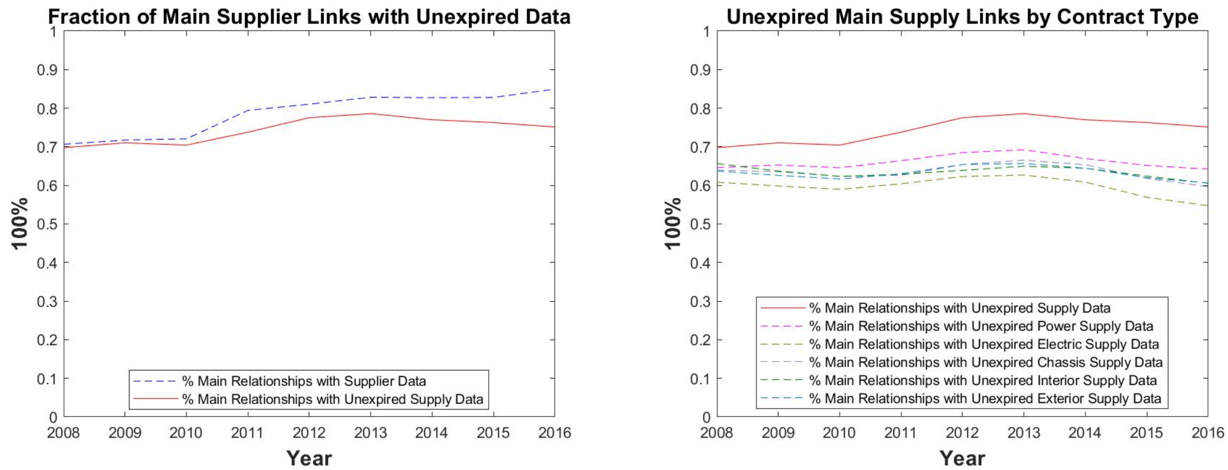


Figure 2.11: Fraction of Main Relationships Covered by Supply Network Data

In addition to survey response rates concerns, there is also the consideration of survey participants evolving over time. The aging of models in Column 2 has already been discussed. Columns 3-4 further reveal models being assembled in more southern and eastern locations over 2008-16. Obviously, this feature of the data is consistent with both the unstructured nature of the WSW survey, and more fundamental forces shifting assembly plant locations. The latter explanation is consistent with the rising number of imported models in the (purportedly) more reliable Ward's data. Moreover, when the plant coordinates are superimposed on a world map in Figure 2.12, we see plants persistently located in similar areas across time. From this perspective, the evolution of survey respondents according to plant locations is slight.

Table 2.4 displays summary statistics for supplier sharing by model-years through their main 10 suppliers amongst all auto part types. Columns 1-2 summarize the proportion of suppliers and contracts covered by the main suppliers. The remaining columns summarize the number of competing models contracting with another model's main supplier, in the same year.

Whilst making up only 27-39% of all suppliers, the main suppliers' contracts account for 60-67% of all sampled contracts. Moreover, although main suppliers and contracts fell in number over 2008-16 (Columns 7 and 9, Table 2.3), their shares of all sampled suppliers and contracts

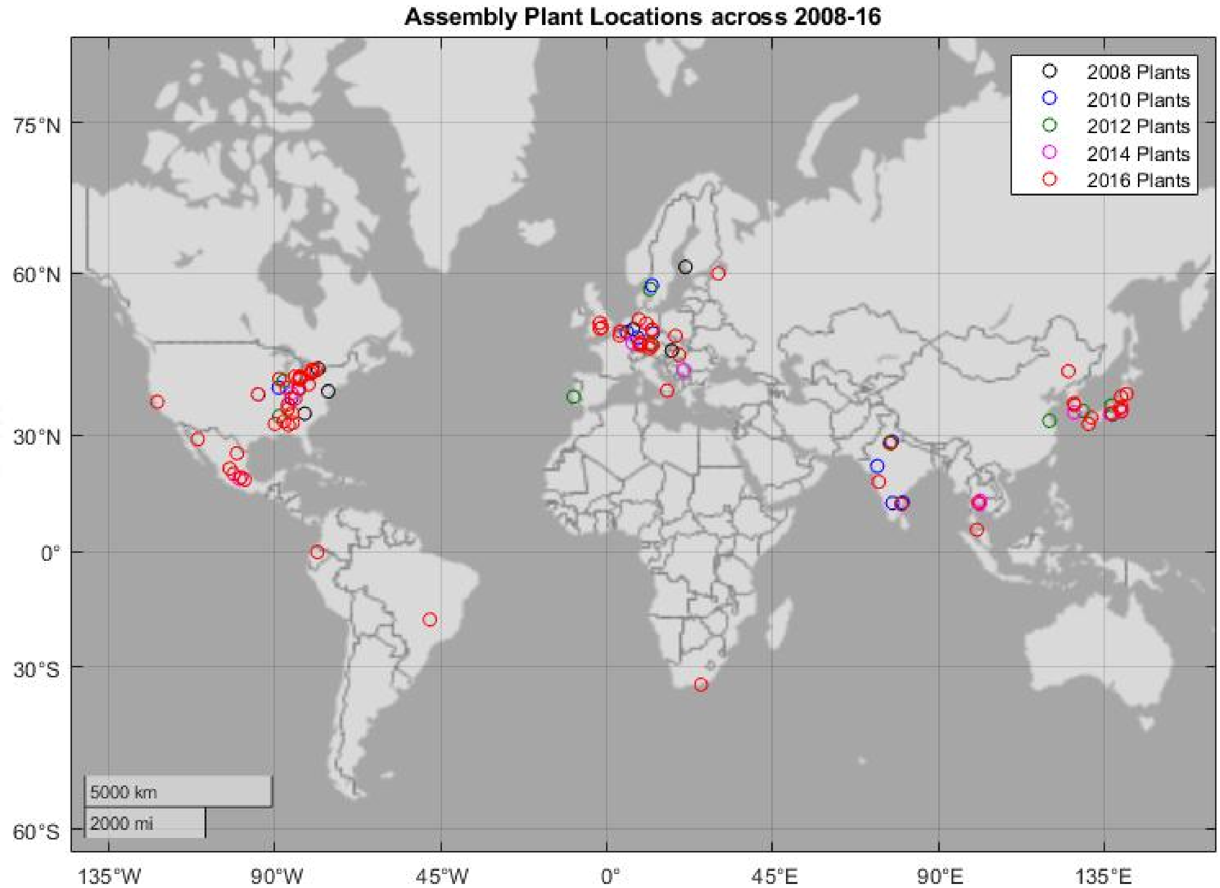


Figure 2.12: Assembly Plant Locations of 2008-16 Models (with Supply Network Data)

rose (Columns 1 and 2, Table 2.4). Hence, main suppliers appear to have grown in importance to the industry. Main suppliers and contracts also seem to comprise growing fractions of all Ward's models' suppliers and contracts. This is driven by the rising shares in columns 1 and 2 of Table 2.4, and the rising number of model-years with WSW data (Column 1, Table 2.3). Figure 2.13 illustrates this. The red line on its LHS plots the fraction of Ward's models matched to unexpired supplier data. The dashed lines plot the shares in columns 1 and 2, multiplied by the red line's height. The dashed lines' slopes thus suggest main suppliers and contracts increasingly represent all suppliers and contracts.

When focused on specific contract categories, one finds main contracts in each category comprise more of their population of sampled and unsampled contracts. More specifically, Figure

Table 2.4: Descriptive Statistics, Main Supplier Sharing over 2008-16

	Sup.%	Con.%	S1	S2	S3	S4	S5	S6	S7	S8	S9	S10
2008	0.27	0.59	21.93	24.74	16.48	19.17	11.92	10.86	9.14	9.83	12.25	9.80
	0.24	0.16	16.06	17.13	15.82	16.49	13.26	12.54	10.39	11.07	14.01	12.35
2009	0.30	0.61	24.11	24.55	15.30	18.41	11.62	10.60	9.82	10.26	12.05	9.51
	0.26	0.18	17.35	17.16	15.09	16.31	13.09	11.80	10.77	11.37	13.00	12.90
2010	0.30	0.61	20.83	21.99	12.79	14.76	9.59	10.61	8.41	8.23	9.26	8.00
	0.26	0.18	16.04	15.25	13.15	14.00	11.32	10.76	9.21	8.65	10.84	9.99
2011	0.32	0.63	22.89	19.57	15.89	12.95	12.17	9.67	9.33	11.12	10.06	9.21
	0.27	0.18	18.39	16.18	15.73	12.92	12.00	10.41	10.27	10.88	12.04	10.58
2012	0.34	0.64	23.29	18.80	16.43	13.34	14.12	11.31	10.48	12.27	10.48	10.34
	0.30	0.19	18.30	15.82	15.26	13.51	13.87	11.16	9.91	12.53	11.53	10.86
2013	0.34	0.64	23.00	18.47	16.54	13.50	15.17	12.61	11.55	13.20	10.46	11.09
	0.29	0.18	17.15	15.36	14.29	12.83	14.38	12.14	10.70	12.74	11.59	10.72
2014	0.34	0.64	20.64	17.78	16.57	13.27	15.64	12.88	11.45	13.27	10.31	11.16
	0.30	0.19	16.67	14.33	14.16	12.09	13.81	12.13	10.31	12.61	10.33	11.09
2015	0.38	0.66	21.09	18.43	15.66	15.78	16.59	12.47	11.22	14.54	10.47	12.51
	0.32	0.19	17.74	15.75	14.44	15.06	14.56	13.45	10.74	14.60	11.67	13.13
2016	0.39	0.67	22.63	19.27	16.23	15.82	16.89	14.57	11.93	15.44	11.29	12.77
	0.32	0.19	17.62	16.33	14.20	13.33	15.46	14.65	11.05	15.00	12.88	14.42

2.13's RHS displays the bold line from its LHS, and counterparts to the taller dashed line for specific contract areas. The RHS dashed lines are clearly higher than their LHS counterpart. Hence, main contracts represent more of all contracts in each category, compared to when all categories were combined. Figure 2.13's dashed lines are also uniformly rising. Hence, the share of main contracts amongst all contracts per category has increased. In sum, even as the number of distinct main suppliers declined, their importance vis-a-vis rival suppliers - even those missing from the data - appears to have grown. This partially justifies the paper's focus on main model-supplier relationships. It also eases concerns over declining survey responses raised earlier.

The main suppliers' distorted importance to the industry also manifests in the cross-sectional distribution of models and market shares to suppliers. Figure 2.14's LHS plots the frequency of individual suppliers appearing as main suppliers over 2008-16. Clearly, the returns in terms of customer popularity from superiority over rival suppliers are concentrated at the top. The figure's RHS shows the same is true when returns are measured in terms of output.

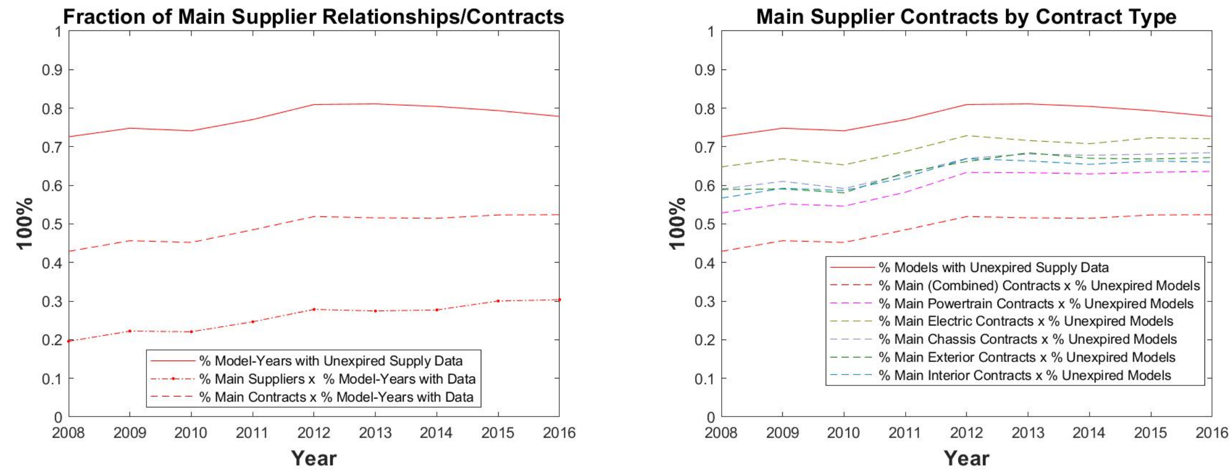


Figure 2.13: Fraction of Main Suppliers & Contracts over 2008-16 Models (with Supply Data)

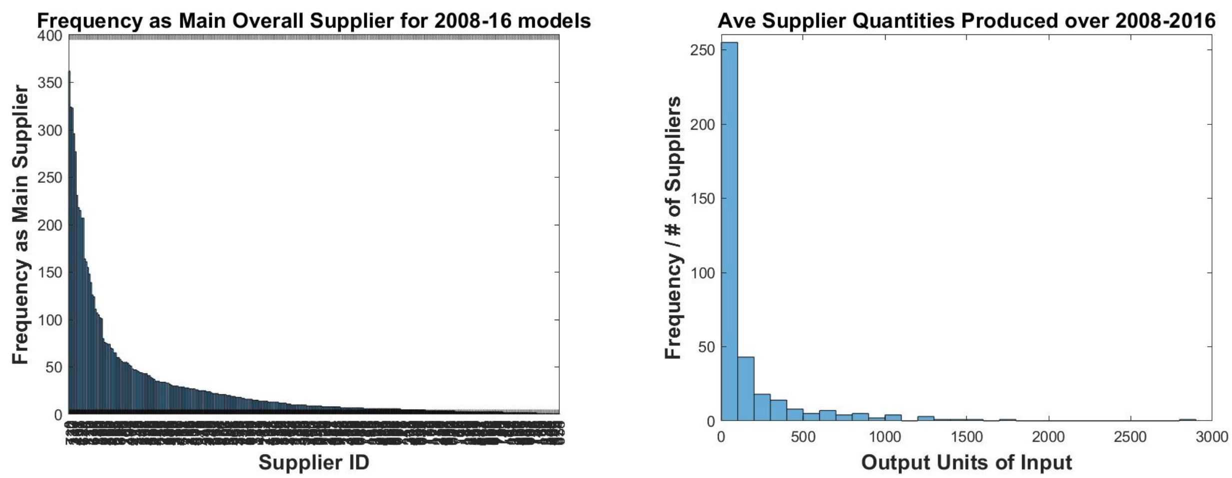


Figure 2.14: Distribution of 2008-16 Models & Output across Suppliers

Differences in input types sold by main suppliers are not solely responsible for these asymmetries. (See Figure 2.22 and accompanying explanation in the appendix for details.) Rather, the unusually high returns for top suppliers appears linked to the frequency of new models sharing suppliers with older rival models. Figure 2.15's LHS shows suppliers inheriting more customers from previous periods typically attract more entrant models. This suggests either economies of scale influence input production, or suppliers are vertically differentiated. Figure 2.15's right panel illustrates a weaker, but still positive relationship between the output suppliers sell to old and new customers. Since both variables are simultaneously chosen by firms, an additional consideration here is new model-supplier relationships expanding entrant model market share, at the older models' expense.

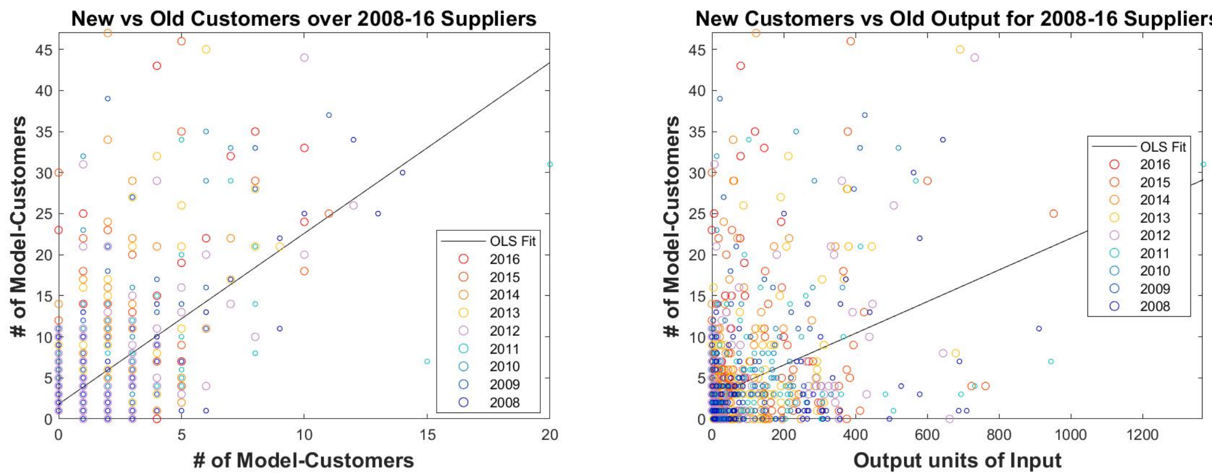


Figure 2.15: New Customers vs Old Customers & Output over 2008-16 Suppliers

Figures 2.16 and 2.17 investigate supplier sharing from more general viewpoints. In these plots, supplier sharing is measured by the average number of rivals using a model's main supplier. Figure 2.16's LHS depicts how supplier sharing is typically higher for models assembled closer to their rivals' locations. This correlation supports viewing these location choices as relationship-specific investments, as suggested in Section 2.2. As explained by Rosenbaum (2013), models are built closer to their suppliers' plants to conserve transportation costs. This in turn facilitates manufacturer-supplier transactions. To clarify whether the plotted relationship is confounded by extraneous effects of other location related variables, the panel also classifies the plotted observations by import status. Conditional on distance to rival location, a model's import status has a small, positive effect on supplier sharing. However, the relationship between supplier sharing and distance to rival plants remains negative regardless of whether models are imported.

Figure 2.16's RHS illustrates a positive relationship between the number of new suppliers chosen, and supplier sharing by entrant models. Under Section 2.2's simple model, a manufacturer switching its supplier automatically shares its rival's supplier. Hence, whilst loose, the supplier switching and sharing relationship on Figure 2.16's right supports Section 2.2's framework. To better understand what causes supplier switching, Figure 2.16's RHS also categorizes model observations by brand nationality. As discussed in Section 2.2, the simple model's asymmetric treatment of firms was partly motivated by documented cross-nationality differences in supplier selection practices[32]. These differences indicate Japanese automakers prefer working with fewer suppliers vis-a-vis US manufacturers. Such differences also appear in Figure 2.16's right panel. New Japanese models plotted in red typically retain more and share less suppliers, compared to their blue American counterparts.

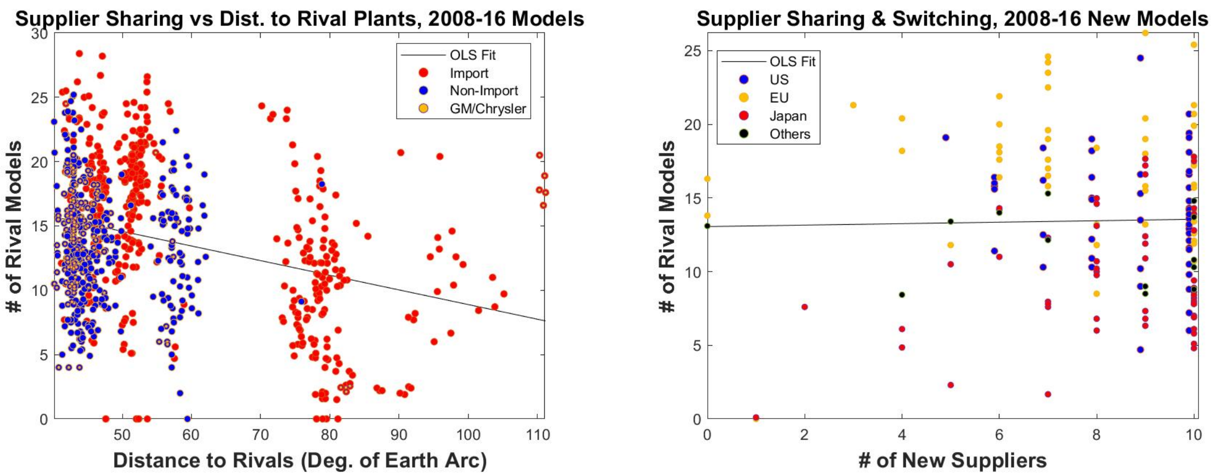


Figure 2.16: Supplier Sharing, Distance to Rival Models & Supplier Switching over 2008-16

In contrast to Section 2.2's assumption regarding economies of scale, Figure 2.17 illustrates supplier sharing correlating negatively with output (LHS), and positively with prices (RHS). Obviously, models with lower output but higher prices possibly appeal only to niche consumers. Hence, the simple model suggests such products may share more suppliers due to greater prod-

32. See p. 148 and p. 160 of Womack et al. (1990), or p. 565, and tables 9 and 11 of Cusumano & Takeishi (1991) as discussed in footnote 6. Also see tables 1, 4, 6 and 7 of Cusumano & Takeishi (as first mentioned in footnote 7).

uct differentiation. Moreover, once the models' nationalities are conditioned upon, the colored lines of Figure 2.17 feature slopes with opposite signs to their black counterparts. These signs are consistent with input production economies of scale. In sum, these panels emphasize a need to isolate exogenous variation in output and prices - such as the GFC induced drop in sales - to truly understand their relationship with supplier sharing.

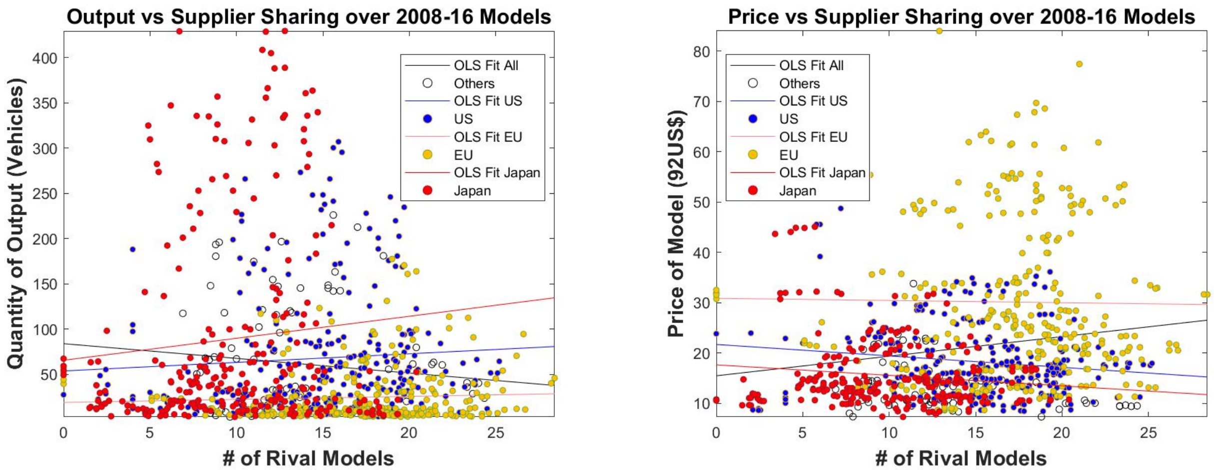


Figure 2.17: Supplier Sharing vs Output & Price over 2008-16

The evolution of supplier sharing over time is summarized by columns 3-9 of Table 2.4. As aforementioned, their odd-numbered rows cross-sectionally average the number of rivals a model shares suppliers with, through each of its main suppliers. (So each 2008 model on average shares its most important supplier with 21.93 own and rival firm models.) These statistics thus gauge the average level of supplier sharing across years. These statistics can also be multiplied by the total number of models with supply data (Column 1, Table 2.3), to obtain the total number of models sharing a rival's main supplier. (So there are approximately 855 models in 2008 connected by two model-supplier links to a rival via the latter's 10th main supplier). These totals thus measure the total amount of supplier sharing occurring in each period.

Both measures of supplier sharing are plotted in Figure 2.18, with the total measure on the LHS. Total supplier sharing oscillates but trends up, tracking the post-GFC recovery in sales. This is anticipated - as manufacturers become more profitable, Proposition 2.2.4 predicts more supplier sharing. In contrast, when averaged over the number of models (with supply data) operating per period, supplier sharing increases less markedly. This too, is unsurprising. Corollary 2.9.3 predict a non-monotone relationship between this measure of supplier sharing and the downstream market's state. Intuitively, better market conditions allows models to profitably operate without exploiting upstream market economies of scale.

Two additional features emerge from either panel of Figure 2.18. First, supplier sharing increases as model-supplier relationships become more important. This is possibly due to relationship importance being measured by the number of contracts embodied within the sampled relationships. Hence, higher relationship importance indicate wider supplier proficiency across differing tasks, attracting more customers to the supplier. Second, average supplier sharing trends less markedly for more important relationships. There are two possible explanations. First, highly important suppliers can fulfill too many contracts, and operate past their most efficient scales. These suppliers thus experience diseconomies of scale. Second, "high contract" firm-to-firm relationships may reflect more complicated notions of inputs between firms. Since such inputs are likely more differentiated, economies of scale are less applicable to their production.

Finally, one should note the aforementioned trends still hold when the data is broken down by assembly location, or when the data is disaggregated, and one focuses on powertrain, chassis and exterior supplier sharing. The details behind these statements are in the appendix.

2.5 Identification

Let $\{\mathbf{p}_t, \mathbf{q}_t, g_t, \mathcal{F}_t, \mathbf{x}_t, \mathbf{w}_t, \mathbf{e}_t, \mathbf{a}_t\}_{t\leq T}$ represent the data. For period $t \leq T$,

- g_t graphs the collection of model-supplier production links, and defines the set of operating

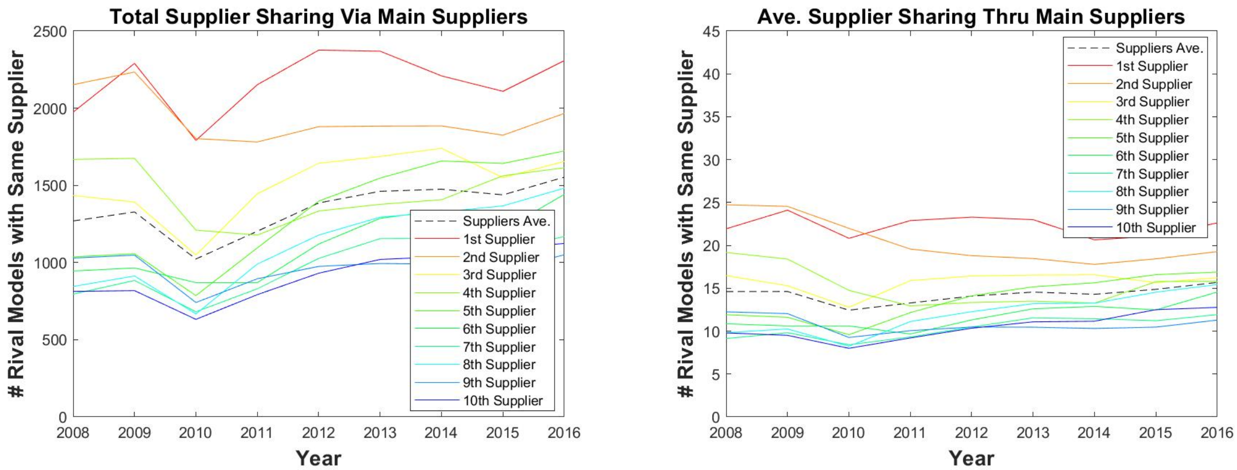


Figure 2.18: Supplier Sharing by Relationship Importance over 2008-16

models $\mathcal{M}_t$ and suppliers $\mathcal{S}_t$,

- $\mathcal{F}_t$ partitions models in $\mathcal{M}_t$ by manufacturer producership,
- $\mathbf{q}_t$ and $\mathbf{p}_t$ correspond to quantities and prices across models in $\mathcal{M}_t$,
- $\mathbf{w}_t$ and $\mathbf{x}_t$ contain cost and taste influencing attributes for models in $\mathcal{M}_t$,
- $\mathbf{e}_t$ contains supplier-specific covariates,
- $\mathbf{a}_t$ conveys period-specific (or annual) information such as mean and median household income.

Let $\mathcal{N}_t = \mathcal{M}_t - \mathcal{M}_{t-1}$ denote the entrant models in period t. These are referred to as new models, while their older counterparts are called old models. Also, for an arbitrary network g, let $s(m, g)$ denotes model m's suppliers, and $m(s, g)$ equal supplier s's model-customers under the network.

The rest of this paper aims to quantify inefficiencies analyzed in Section 2.2 in the auto industry. This is done by estimating distortions to the firms' relationship-forming incentives caused by the inefficiencies. Doing so requires organizing data according to a micro-founded model. When data on input prices are available, one can infer how input production marginal cost varies with output, through explicitly modeling input price determination. This helps quantify economies of scale in input production, in addition to relationship-forming incentives.

Unfortunately, no publicly available dataset of input prices paid by each automaker exists. Instead, auto part suppliers are chosen via idiosyncratic procurement auctions or customs. This impedes inferring assembly and input costs for three reasons. First, the relationship between each model manufacturer's costs to each supplier's quoted price is unknown, absent precise knowledge on the manufacturer's procurement customs. Hence, even if manufacturing marginal costs are identified from downstream market data, upstream prices remain unknown. Second, each supplier solves a complex optimization problem when quoting its price to a manufacturer. Hence, even if the quoted prices are observed and stem from a known auction format, recovering the suppliers' costs from such prices is challenging. Last, each supplier's optimal price likely depends on its competitors' attributes, in addition to its own. This invalidates using competing product attributes as "BLP instruments" for quantities when tracing out supplier marginal costs from input prices.

The aforementioned difficulties are circumvented by considering a feature of the network data in the context of Section 2.2's simpler model. In the data, automobile models are typically linked to identical suppliers over successive periods after their launch date. This presumably reflects long-term contracts signed by manufacturers and suppliers during the model's launch year. I assume input prices for models in periods after their launch, are determined by Nash-in-Nash bargaining between the models' manufacturers and suppliers. To the extent that manufacturers are contractually prevented from replacing their older incumbent models' suppliers, the assumption is valid.

Under the above assumption, the slopes of assembly and input marginal costs are identified. Estimating these cost functions allows measurement of economies of scale. The estimates also help quantify the firms' relationship-forming incentives. As noted in Section 2.1, the true benefit anticipated by a firm from forming new relationships is unidentified. However, proxies for these benefits - the average increase in profit when the production network is altered, after firms have already committed to prices and bids - are estimable. This allows even-handed empirical assessment of whether manufacturer-supplier relationship formation suffers from overinvestment or third party hold-up, inefficiencies highlighted by the simpler model.

Subsection 2.5.1 describes an empirical model of how firms form relationships with each other in the auto industry. The model does not generalize its simpler counterpart in three aspects:

i) Whilst input prices are determined after output prices are set, the former can depend on bids submitted before downstream market prices are determined.

ii) Manufacturers never pull products from the market before Nash bargaining occurs and after output prices are determined.

iii) Manufacturers pull their entire product line when failing to agree with any supplier during Nash bargaining.

Subsection 2.5.1 also provides reasons for these departures. Subsection 2.9.4 in the appendix states higher level assumptions needed to identify inefficiencies affecting relationship-formation in the empirical model. Finally, Subsection 2.5.2 outlines how such inefficiencies are empirically quantified.

2.5.1 Microfounded Emprical Model

Players: $\mathcal{S}$ denotes a set of suppliers. The suppliers produce inputs for assembling automobile models, $\mathcal{M}$. $\mathcal{M}$ is partitioned by $\mathcal{F}$, a producership structure for various model manufacturers. The models are also partitioned into new models $\mathcal{N}$, and older incumbent models launched in previous periods. Finally, a unit continuum of households are single unit consumers of automobiles.

Sequence of Play: Firms start each period with a network of model-supplier relationships inherited from the previous period, g^o. Let $\mathcal{M}^o$ denote g^o's models. Players know g^o, $\mathcal{M}^o$ and $\mathcal{M}^o$'s producership structure, in addition to $\mathcal{S}$, $\mathcal{M}$, $\mathcal{N}$ and $\mathcal{F}$.

The following sequence of play is then observed:

1. Nature draws a shock ω_{ms} to the cost of assembling inputs from each supplier $s \in \mathcal{S}$ into each model $m \in \mathcal{M}$. Nature also determines a taste shock ξ_m to each model.

   $\mathcal{I}_f$ denotes information disclosed to manufacturer f. Likewise, $\mathcal{I}_s$ denotes supplier s's information. These information sets across $\mathcal{F} \cup \mathcal{S}$ imply $\boldsymbol{\xi}$, and ω_{ms} when s supplies m's manufacturer under g^o, are public information.

2. New models in $\mathcal{N}$ are matched to suppliers from $\mathcal{S}$. This process reflects unmodeled choices by manufacturers and suppliers to invest in relationships with each other[33]. These relationships, coupled with the older models' past relationships in g^o, determines the current relationship network g^r.

3. Each supplier $s \in \mathcal{S}$ submits a bid b_{ms} to each new model m linked to it under the relationship network. The bid indicates its desired price for inputs needed to assemble an output unit. WLOG, let $b_{ms} = \infty$ when $m \notin \mathcal{N}$ or $(m, s) \notin g^r$.

   Because the bids are uncontingent on output levels chosen by manufacturers, input pricing is necessarily linear. This may appear unnatural. However, Novak & Kilbanoff (2003) provide some evidence of auto interior part prices being determined by piece rate bidding.

4. Each manufacturer $f \in \mathcal{F}$ announces a price $p_m > 0$ for each of its models $m \in f$.

33. See Section 2.2's description for examples of relationship-specific assets reflected by these relationships.

Notice the order in which bids and prices are determined is reversed in comparison to the simpler model. While unfortunate, this apparent contradiction is not necessarily important. The new model's sequence of play ensures prices remain optimal, even when all private information is made public at the game's end. Manufacturers thus never regret, as under the simpler model's solution concept.

Manufacturer f also chooses suppliers $s_m \subseteq \mathcal{S}$ for each model $m \in f$.

i) When model $m \notin \mathcal{N}$ is old, it inherits its previous production network suppliers: $s_m = s(m, g^o)$.

ii) When $m \in \mathcal{N}$, its suppliers are chosen from its relationship network suppliers: $s_m \subseteq s(m, g^r)$.

The collective supplier choices $\mathbf{s}$ determines the period's production network, g.

In contrast to the simple model, products are never removed from the market. This is because the new sequence of play ensures manufacturers correctly anticipate costs when setting prices. It is thus implicitly assumed that models in $\mathcal{M}$ are sufficiently price inelastic at high prices, allowing continual own price increments without demand vanishing.

5. Consumers choose models to consume while manufacturers pay suppliers for inputs.

   i) Each consumer $i \in [0, 1]$ purchases a model at its announced price, or consumes an outside option, $c_i \in \mathcal{M} \cup \{\text{Outside Option}\}$.

   ii) Each manufacturer $f \in \mathcal{F}$ pays t_{ms} to supplier s for inputs needed to make each model $m \in f$.

   – Suppose $m \notin \mathcal{N}$ is an old model. Then $s(m, g) = s(m, g^o) = s(m, g^r)$, and m inherits all of its past suppliers under the old production network g^o. Hence, when (m, s) lies in the current network g, t_{ms} is determined by Nash (-in-Nash) bargaining between f and s. If otherwise, assume $t_{ms} = 0$ WLOG.

– Suppose $m \in \mathcal{N}$ is newly launched. Then $s(m,g) \subseteq s(m,g^r)$, and t_{ms}'s determination depends on whether supplier s and model m's manufacturer f have previously collaborated:

a) s is paid according to an exogenous payment rule $t_{ms} = q_m \kappa^{ms}(\mathbf{b}_m, g, g^r)$, when s is completely new to f: $(n,s) \notin g^o$ for all $n \in f^o$. The rule serves as a primitive catering towards model-specific procurement customs.

Example 2.5.1. [First-Price Procurement Auction] Suppose model $m \in \mathcal{N}$ requires new suppliers for R different inputs. For each input $r \leq R$, let $a(r, g^r)$ denote the suppliers in $s(m, g^r)$ capable of producing it. Then when

$$\kappa^{ms}(\mathbf{b}_m, g, g^r) = \{s \in s(m,g)\} b_{ms}, \qquad s(m,g) = \cup_{r \in R} \text{argmin}_{k \in a(r,g^r)} b_{km},$$

the model's suppliers are chosen and paid via first-price auctions.

More generally, t_{ms} can be non-zero even when $(m,s) \notin g$. This could occur if m's suppliers were determined by all-pay auctions instead.

b) s Nash bargains with f if s previously supplied f: $\exists n \in f^o$ satisfying $(n,s) \in g^o$. This is consistent with the existence of an equilibrium in the simpler model featuring supplier sharing whenever a new manufacturer-supplier relationship is formed[34].

Payoffs: Each consumer $i \in [0,1]$ obtains u_{im} in utility from consuming model $m \in \mathcal{M}$, and 0 from not consuming. These utilities are drawn from a conditional cdf $F_{\mathbf{u}_i|\mathbf{p},\mathbf{x},\boldsymbol{\xi},y_i}$. Here, $\mathbf{x} \in \mathbb{R}^{\mathcal{M} \times X}$ denotes observable model attributes, $\boldsymbol{\xi} \in \mathbb{R}^{\mathcal{M}}$ are the Stage 1 taste shocks, and y_i is i's type.

Example 2.5.2. [Random Coefficient Logit Utility] For each consumer $i \in [0,1]$ and model

34. If there exists another model $n \in \mathcal{M}^o$ also produced by f and supplied by s, then one can also show the next subsection's identification results hold, even when t_{ms} is determined by the exogenous payment rule $\boldsymbol{\kappa}^m$ instead of bargaining.

$m \in \mathcal{M}$,

$$u_{im} = \beta_0 + \mathbf{x}_m \boldsymbol{\beta}_i - \frac{\alpha}{y_i} p_m + \xi_m + \epsilon_{im}$$

where ϵ_{im} is distributed type 1 Gumbel, $\boldsymbol{\beta}_i$ is normally distributed with mean $\boldsymbol{\beta}_{\mathbf{x}}$ and variance $diag(\boldsymbol{\sigma}_{\mathbf{x}}^2)$, and y_i is i's income.

Assume the consumers' types are independent of $(\mathbf{p}, \mathbf{x}, \boldsymbol{\xi})$, and distributed according to F_y. The implied distribution of utilities yields demand

$$q_m = q_m(\mathbf{p}, \mathbf{x}, \boldsymbol{\xi}) = \int_y \int_{\mathbf{u}_i} \{u_{im} \geq \mathbf{u}_{i,-m}\} dF_{\mathbf{u}_i|\mathbf{p},\mathbf{x},\boldsymbol{\xi},y} dF_y$$

for each model $m \in \mathcal{M}$ when consumers maximize utility.

Each manufacturer $f \in \mathcal{F}$ obtains a profit equal to

$$\pi_f = \sum_{m \in f} \left(p_m q_m - c^m(q_m) - \sum_s t_{ms} \right) - SC_f.$$

SC_f is its cost effectively sunk by Stage 3. This cost potentially reflect entry barriers, but also include relationship-forming expenses incurred in Stage 1. The cost of assembling inputs into q units of output is $c^m(q)$. Assume $c^m(q_m) = c(q_m, \mathbf{w}_m)$, where $\mathbf{w}_m \in \mathbb{R}^W$ denotes model-specific covariates. Obviously, this is WLOG since $\mathbf{w}_m$ can include model-specific dummies.

Example 2.5.3. [Quadratic Output Assembly] Suppose for each model $m \in \mathcal{M}$,

$$c^m(q) = c_0^m + c_1^m q + c_2^m \frac{q^2}{2}$$

where $c_0^m = c_0$, $c_1^m = \boldsymbol{\gamma}_{\mathbf{w}} \mathbf{w}_m$, and $c_2^m = \gamma_q$. Hence cross-model heterogeneity in fixed costs are sunk by Stage 3. Moreover, assembly costs exhibit either economies or diseconomies of scale.

Each supplier $s \in \mathcal{S}$ earns the following as profit:

$$\pi_s = \sum_{m \in m(s,g)} (t_{ms} - \omega_{ms} q_m) - C^s(Q_s) - SC_s.$$

SC_s equals the sunk cost for supplier s in Stage 2. As before, these costs also reflect investments in relationships allowing suppliers to acquire new customers. The cost of making inputs for Q amount of output is $C^s(Q)$. This equals $C^s(Q, \mathbf{e}_s)$ for supplier-specific covariates $\mathbf{e}_s \in \mathbb{R}^E$.

Example 2.5.4. [Cubic Input Production] Suppose for each supplier $s \in \mathcal{S}$,

$$C^s(Q) = c_0 + c_1 Q + c_2 \frac{Q^2}{2} + c_3 \frac{Q^3}{3}.$$

Suppliers thus operate homogeneous technology. They also face economies or diseconomies of scale, depending on their quantities.

Finally, suppliers also bear the match-specific cost shocks $\boldsymbol{\omega}_s$. Notice however, when supplier s knows $\boldsymbol{\omega}_s$ in Stage 2 and new suppliers are chosen via typical auctions, s can compensate for higher $\boldsymbol{\omega}_s$ by bargaining or bidding higher input prices.

Strategies: Fix the relationship network g^r, along with the information publicly disclosed in Stage 1. The sequence of play yields well defined strategies for each player in the (improper) subgame that follows. Each supplier s submits a vector of bids contingent on its private type:

$$\mathbf{b}_s(\mathcal{I}_s) = (b_{ms}(\mathcal{I}_s))_{m \in \mathcal{M}} \in \mathbb{R}^{\mathcal{M} \times supp(\mathcal{I}_s)},$$

satisfying $b_{ms}(\mathcal{I}_s) = \infty$ if $m \notin \mathcal{N}$ or $(m,s) \notin g^r$. Given bids $\mathbf{b}$ and information $\mathcal{I}_f$, manufacturer f chooses prices

$$\mathbf{p}_f(\mathbf{b}, \mathcal{I}_f) = (p_m(\mathbf{b}, \mathcal{I}_f))_{m \in f} \in \mathbb{R}_{\geq 0}^{|f| \times_{(m,s) \in \mathcal{M} \times \mathcal{S}} \mathbb{R} \times supp(\mathcal{I}_f)},$$

and suppliers

$$\mathbf{s}_f(\mathbf{b}, \mathcal{I}_f) = \left(s_m(\mathbf{b}, \mathcal{I}_f)\right)_{m\in f} \subseteq s(m, g^r)^{|f| \times_{(m,s)\in\mathcal{M}\times\mathcal{S}} \mathbb{R}\times supp(\mathcal{I}_f)}$$

for its models, satisfying $s_m(\mathbf{b}, \mathcal{I}_f) = s(m, g^r)$ when $m \notin \mathcal{N}$. Finally, each consumer i chooses not to consume or a model to purchase:

$$c_i(\mathbf{p}, \mathbf{b}, \mathbf{s}) \in \mathcal{M} \cup \{\text{Outside Option}\}.$$

Assume these decisions depend only on payoff relevant variables. Hence, write $\mathbf{c}(\mathbf{p}, \mathbf{b}, \mathbf{s}) = \mathbf{c}(\mathbf{p})$. In what follows, let $\boldsymbol{\sigma}|_{g^o, g^r, \mathcal{I}}$ denote the profile of these truncated strategies.

Nash Bargaining: The strategies $\boldsymbol{\sigma}|_{g^o, g^r, \mathcal{I}}$ enumerated above pins down the surplus created when a manufacturer and supplier combine to produce output for a model. Hence, the Nash bargaining payoff set is well-defined. Assume input prices satisfy

$$\begin{aligned} t_{ms} = \operatorname*{argmax}_{t} &\left(\sum_{n\in f}(p_n q_n - c^n(q_n) + c^n(0)) - t - \sum_{k\neq s} t_{mk} - \sum_{n\in f-m}\sum_{k} t_{nk}\right)^{\tau_f} \\ &\times \left(t - \omega_{ms} q_m + \sum_{n\in f-m}(t_{ns} - \omega_{ns} q_n) - C^s(Q_s) + C^s\left(Q_s - \sum_{n\in m(s,g)\cap f} q_n\right)\right)^{1-\tau_f} \end{aligned} \tag{2.11}$$

for each model-supplier pair $(m, s) \in f \times \mathcal{S}$ that "bargain" according to the sequence of play[35]. Observe manufacturer f enjoys equal bargaining power τ_f across each supplier. (2.11) also implies f shuts down and removes *all* its models from the market if negotiations with a supplier break down. Finally, f is excused from paying input prices for the deleted models' remaining suppliers.

35. Observe this problem admits multiple solutions, although it does pins down the total amount f pays s, $\sum_{n\in f} t_{ns}$, for manufacturer-supplier pairs (f, s) that Nash bargain under the sequence of play

The latter assumption arguably reflects renegotiation when products are removed. The former however, is slightly more controversial, since manufacturer f is unforced by technology to discard models unsupplied by s, when negotiations with s fail[36]. One potential justification recognizes the frequency of any two models m, n sharing a supplier, when assembled by the same firm. Another recognizes models m and n can be complements absent further restrictions on $F_{\mathbf{u}_i|\mathbf{p},\mathbf{x},\boldsymbol{\xi},y_i}$[37]. Finally, altering a manufacturer's product line at this stage may also inflict severe disruption costs[38]. These links between manufacturer f's models, suggests f might delay selling all its models until it can agree with each supplier in a future period.

Solution Concept: The strategies $\boldsymbol{\sigma}|_{g^o,g^r,\mathcal{I}}$, together with some system of beliefs, comprise a WPBE from Stage 2 onwards. Given its private type $\mathcal{I}_s$, each supplier s submits $\mathbf{b}_s(\mathcal{I}_s)$ to maximize $\mathbb{E}\pi_s|\mathcal{I}_s$ in response to $\mathbf{b}_{-s}(\boldsymbol{\mathcal{I}}_{-s})$, $\mathbf{p}(.,\mathbf{b}_{-s}(\boldsymbol{\mathcal{I}}_{-s}),\boldsymbol{\mathcal{I}}_{\mathcal{F}})$, $\mathbf{s}(.,\mathbf{b}_{-s}(\boldsymbol{\mathcal{I}}_{-s}),\boldsymbol{\mathcal{I}}_{\mathcal{F}})|\mathcal{I}_s$'s distribution. For each manufacturer f, its profit is known up to its own actions, given $\mathbf{b}, \mathbf{p}_{-f}, \mathbf{s}_{-f}$, and the publicly known cost shocks ω_{ms}. So f chooses $\mathbf{p}_f(\mathbf{b}(\boldsymbol{\mathcal{I}}_{\mathcal{S}}), \mathcal{I}_{public})$ and $\mathbf{s}_f(\mathbf{b}(\boldsymbol{\mathcal{I}}_{\mathcal{S}}), \mathcal{I}_{public})$ in response to maximize π_f.

2.5.2 Identification Strategy

Let $s(m)$ abbreviate model m's suppliers $s(m, g)$, when g is the equilibrium production network. Let $nb(f, g, g^o)$ denote the suppliers manufacturer f Nash bargains with under the sequence of play, when the current and past production networks equal g and g^o respectively. These suppliers are abbreviated by $nb(f)$, when it is understood (g, g^o) lie on the equilibrium path. Let $Q_{sf} = \sum_{m \in m(s,g) \cap f} q_m$ and $Q_{s-f} = Q_s - Q_{sf}$ denote the quantities supplier s sells to manufacturer f and f's rivals respectively. Finally, let $\kappa_{ms} = \kappa^{ms}(\mathbf{b}_m, g, g^r)$ for any model-supplier pair

36. Although a brief perusal of the Ford and General Motors websites shows these companies house firmwide procurement divisions, providing circumstantial evidence for firm-level bargaining over input prices.

37. My belief is Compiani (2017) points this out.

38. This would be captured in the model by incorporating alterations to f's product lineup as a variable in $\mathbf{w}_m$

(m, s) denote the "auction payment" made by m's manufacturer to s, under the payment rule κ^{ms}.

To assess how product differentiation affects automobile demand, one typically estimates demand elasticities implied by the model's demand schedules $\mathbf{q}(\mathbf{p}, \mathbf{x}, \boldsymbol{\xi})$. Berry & Haile (2015) discuss conditions under which these functions are non-parametrically identified. Assessing how economies of scale affects input production requires knowing how supplier marginal costs vary with production levels. Traditionally, the manufacturers' first-order conditions are manipulated to recover their marginal costs from downstream demand. However, retrieving information on the suppliers' costs from the manufacturers' marginal costs is complicated.

In particular, manufacturer f's first-order condition and the Nash bargaining condition (2.11), imply

$$\frac{\partial \mathbf{p}_f}{\partial \mathbf{q}_f}\mathbf{q}_f + \mathbf{p}_f = \left(c_q^m(q_m) + \sum_{s \in s(m) \cap nb(f)} \omega_{ms} + \sum_{s \in s(m) - nb(f)} \kappa_{ms} + \sum_{s \in s(m) \cap nb(f)} C_Q^s(Q_s) \right)_{m \in f} + \frac{\partial \mathbf{Q}_{nb(f),-f}}{\partial \mathbf{q}_f}\Big|_{\mathbf{p}_{-f}, \mathbf{x}, \boldsymbol{\xi}} \left(C_Q^s(Q_s) - C_Q^s(Q_{s,-f}) \right)_{s \in nb(f)}.$$

[39]The above equality's LHS equals f's marginal revenues. It is identified from downstream data. Ideally, one also identifies c_{qq}^m and C_{QQ}^S from variation in own model and supplier quantities, $q_m, \mathbf{Q}_{s(m)}$. However, these quantities are endogenous and correlate with the unobserved residuals $\boldsymbol{\omega}_f, \boldsymbol{\kappa}_f$. Typically, rival product attributes $(\mathbf{w}_{-f}, \mathbf{x}_{-f})$ are used as "BLP instruments". Unfortunately, these instruments are invalid for two reasons. First, the residuals $\boldsymbol{\kappa}_f$ represent payments made by manufacturer f to new suppliers. These payments thus depend on bids submitted by suppliers informed of rival model attributes. Second, the residuals $\boldsymbol{\omega}_f$ comprise of cost shocks for suppliers chosen in response to the same information. For these reasons, the BLP IVs are likely

39. More formally, note $\frac{\partial \mathbf{Q}_{nb(f),-f}}{\partial \mathbf{q}_f} = \frac{\partial \mathbf{q}_f}{\partial \mathbf{p}_f}^{-1} \frac{\partial \mathbf{Q}_{nb(f),-f}}{\partial \mathbf{p}_f}$.

correlated with the unobserved residual of marginal cost, $\sum_{s\in s(m)-nb(f)}\kappa_{ms}+\sum_{s\in s(m)\cap nb(f)}\omega_{ms}$.

However, when one focuses on the subset of incumbent model-years, a different picture emerges. For such models $m\in\mathcal{M}-\mathcal{N}$, marginal revenues equal

$$c_q^m(q_m)+\sum_{s\in s(m)}C_q^s(Q_s)+\sum_{s\in nb(f)}\frac{\partial Q_{s,-f}}{\partial q_m}\Big|_{\mathbf{p}_{-f},\mathbf{x},\boldsymbol{\xi}}\left(C_Q^s(Q_s)-C_Q^s(Q_s-Q_{sf})\right)+\sum_{s\in s(m)}\omega_{ms}. \quad (2.12)$$

Because these models exogenously inherit their previous period's suppliers, their input prices are determined entirely by Nash bargaining rather than bidding. Moreover, the residual marginal cost satisfies

$$\sum_{s\in s(m)}\omega_{ms}=\sum_{s\in s(m,g^o)}\omega_{ms}.$$

The residual thus comprises of supplier-specific cost shocks for predetermined suppliers. Hence, BLP IVs are valid instruments for the endogenous variables in (2.12): own-model quantities q_m, relevant supplier quantities $\mathbf{Q}_{(sm)\cup nb(f)}$, quantities sold by relevant suppliers to m's manufacturer $\mathbf{Q}_{nb(f),f}$, and the diversion ratios $\frac{\partial \mathbf{Q}_{nb(f),-f}}{\partial q_m}$.

When c^m and C^s satisfy examples 2.5.3 and 2.5.4, average variable costs

$$\begin{aligned} AVC_q^m(q_m) &= \frac{\partial}{\partial q_m}\frac{c^m(q_m)-c^m(0)}{q_m}=\frac{\gamma_q}{2} \\ AVC_Q^s(Q_s) &= \frac{\partial}{\partial Q_s}\frac{C^S(Q_s)-C^s(0)}{Q_s}=\frac{c_2}{2}+\frac{2c_3}{3}Q_s, \end{aligned} \quad (2.13)$$

can be estimated.

Example 2.5.5. [Quadratic and Cubic Costs Revisited] Suppose examples 2.5.3 and 2.5.4 hold.

Then for each incumbent model $m \in \mathcal{M} - \mathcal{N}$, its marginal revenue equals

$$mr_m = \boldsymbol{\gamma}_{\mathbf{w}} \mathbf{w}_m + \boldsymbol{\gamma}_q q_m + c_1 |s(m)| + c_2 \left(\sum_{s \in s(m)} Q_s + \sum_{s \in nb(f)} \frac{\partial Q_{s,-f}}{\partial q_m} |_{\mathbf{p}_{-f}, \mathbf{x}, \boldsymbol{\xi}} Q_{sf} \right) + c_3 \left(\sum_{s \in s(m)} Q_s^2 - \sum_{s \in nb(f)} \frac{\partial Q_{s,-f}}{\partial q_m} |_{\mathbf{p}_{-f}, \mathbf{x}, \boldsymbol{\xi}} \left[Q_{sf}^2 - 2 Q_s Q_{sf} \right] \right) + \sum_{s \in s(m)} \omega_{ms}.$$

Thus, $\boldsymbol{\gamma}$, c_1, c_2 and c_3 are identified when the RHS random variables are non-collinear.

However, even absent parametric restrictions, the curvatures of assembly costs for each model c_{qq}^m and those of their suppliers c_{QQ}^s, are identified under conditions (See appendix for details). These identification results serve the paper's aims in two ways. First, they capture how economies of scale affects input production and model assembly costs. Second, they help quantify inefficiencies in firm-to-firm relationship formation, spotlighted by the simpler model. The rest of this subsection explains how the latter is done.

Let $\mathcal{N}^o$ denote the set of new models in $\mathcal{N}$ with previous edition suppliers from g^o. Consider a manufacturer f of such a model $m \in \mathcal{N}^o$. Its expected benefit from forming m's chosen relationships in Stage 2, holding the remaining relationships fixed is

$$\mathbb{E}[\pi_f | \mathcal{I}_f, g^r] - \mathbb{E}[\pi_f | \mathcal{I}_f, g^r_{-m} + g^o_m]. \tag{2.14}$$

Observe the RHS term is f's expected profit when m's related suppliers are only its past suppliers in g^o, rather than also containing suppliers chosen by f. Likewise, for each supplier s of m, one can similarly define its expected benefit from forming a relationship to m:

$$\mathbb{E}[\pi_s | \mathcal{I}_s, g^r] - \mathbb{E}[\pi_s | \mathcal{I}_s, g^r_{-m} + g^o_m]. \tag{2.15}$$

Understanding how the model's parameters affect the returns in (2.14) and (2.15) are of interest. These comparative statics clarify how the returns are distorted by inefficiencies. Unfortunately, the incomplete description of the model in Subsection 2.5.1 leaves information sets $\mathcal{I}_f$ and $\mathcal{I}_s$ unspecified. Moreover, the relationship network g^r and its counterfactual $g^r_{-m} + g^o_m$ are unobserved in the data. Finally, the profits also depend on residuals $\boldsymbol{\kappa}$ and $\boldsymbol{\omega}$, whose distribution under the counterfactual network $g^r_{-m} + g^o_m$ does not equal its distribution given g^r. In particular, when manufacturers choose suppliers via procurement auctions, altering the relationship network causes suppliers to bid differently, changing the values of $\boldsymbol{\kappa}$. For these reasons, (2.14) and (2.15) are unidentified.

Hence, instead of the expected relationship-forming benefits in (2.14) and (2.15), the remainder of this section analyses the change in profits when a model's suppliers are altered, *after* equilibrium prices and quantities are determined. Let

$$\Delta\pi^a_{f,m} = \pi_f - \pi_f|_{g_{-m}+g^o_m} \tag{2.16}$$

equal manufacturer f's Stage 4 payoff from replacing model m's past production network suppliers with their equilibrium ones, holding the rest of the network g_{-m} fixed at equilibrium values. In contrast to (2.14), (2.16) ignores variation in equilibrium quantities $\mathbf{q}$ or auction payments to new suppliers $\boldsymbol{\kappa}$, caused by the relationship network changing when m's suppliers are replaced.

(2.16) admits a decomposition capturing how downstream oligopoly rents inefficiently enlarge manufacturer f's relationship-forming incentives. Let $\tau^f(\mathbf{g}) = \frac{\tau_f}{|nb(f,\mathbf{g})|-(|nb(f,\mathbf{g})|-1)\tau_f}$. Calculation shows

$$\Delta\pi^a_{f,m} = \underbrace{\left[\tau^f(\mathbf{g}) - \tau^f(g_{-m} + g^o_m, g^o)\right]\pi^{nb}_f}_{\longleftarrow \text{ overinvestment incentive } \longrightarrow} - \tau^f(g_{-m} + g^o_m, g^o)\Bigg[\sum_{s\in s(m)-nb(f)} \underbrace{\pi_{ms}}_{\leftarrow \text{rent to } s \rightarrow}$$
$$+ \underbrace{SC(q_m, \mathbf{w}_m, \mathbf{Q}_{s(m)}, \mathbf{e}_{s(m)}) - SC(q_m, \mathbf{w}_m, \mathbf{Q}_{s(m^o,g^o)} + \{s \notin s(m)\}q_m, \mathbf{e}_{s(m^o,g^o)})}_{\longleftarrow \text{ ex-ante change in production costs (for model } m\text{) to society } \longrightarrow} \tag{2.17}$$
$$+ \underbrace{q_m\left(\sum_{s\in s(m)} \omega_{ms} - \sum_{s\in s(m^o,g^o)} \omega_{ms}\right)}_{\leftarrow \text{ ex-post change in production costs } \rightarrow}\Bigg].$$

The first term on (2.17)'s RHS captures manufacturer f's incentive to inefficiently overinvest in relationships. As in the simpler model, f pits suppliers against each other when awarding procurement contracts, by forming relationships to new suppliers. This reduces the number of suppliers f bilaterally bargains with. Thus, fewer suppliers extract f's oligopoly rent during input price negotiations. Note

$$\pi^{nb}_f = \sum_{m\in f}\left[p_m q_m - c^m(q_m) + c^m(0) - \sum_{s\in s(m)-nb(f)} \kappa_{ms} q_m\right]$$
$$- \sum_{s\in nb(f)}\left[C^s(Q_s) - C^s\left(Q_{s,-f}\right) + \sum_{m:s\in s(m)} \omega_{ms} q_m\right]$$

in the first term, is simply the profit f shares with its bargaining counterparties. The following term contains

$$\pi_{ms} = \kappa_{ms} q_m - C^s(Q_s) + C^s(Q_s - q_m) - \omega_{ms} q_m,$$

the immediate increase in supplier s's profit from supplying model m. When m's suppliers are chosen via auctions, π_{ms} is also interpretable as s's information rent. Such rents discourage f from investing in relationships, unless f is compensated by its co-investors as in the simpler model. Finally, the remaining terms sum to the societal cost of altering m's suppliers, holding the rest of the production network g_{-m} and quantities $\mathbf{q}$ fixed. This arises from m relying on different suppliers with different cost structures and market shares. It should be noted

$$SC(q_m, \mathbf{w}_m, \mathbf{Q}_{s(m)}, \mathbf{e}_{s(m)}) = c(q_m, \mathbf{w}_m) - c(0, \mathbf{w}_m) + \sum_{s \in s(m)} [C(Q_s, \mathbf{e}_s) - C(Q_s - q_m, \mathbf{e}_s)]$$

here, is the cost to society from producing and assembling inputs into model m's output.

Under formal assumptions stated in the appendix, both the ex-post relationship-forming benefit, and its distortion due to oligopoly rents causing overinvestment in relationships, are identified up to the manufacturers' bargaining parameters and mean zero cost shocks.

Proposition 2.5.1. *Suppose assumptions 2.9.1 to 2.9.9 hold. Let $m \in \mathcal{N}^o \cap f$. The change in manufacturer f's profit from replacing model m's old suppliers $s(m^o, g^o)$ with its equilibrium ones $s(m)$, combined with m's residual production costs under $s(m^o, g^o)$*

$$\Delta\pi^a_{f,m} + \tau^f(g_{-m} + g^o_m, g^o) \sum_{s \in s(m^o, g^o)} \omega_{ms} q_m,$$

is degenerate given the data $\mathcal{D}$, and identified on $\mathcal{D}$'s support up to $\tau_f \in (0,1)$. The distortion of the above due to f bilaterally bargaining with fewer suppliers when $s(m^o, g^o)$ is replaced by $s(m)$

$$\left[\tau^f(\mathbf{g}) - \tau^f(g_{-m} + g^o_m, g^o)\right] \pi^{nb}_f,$$

is also degenerate given $\mathcal{D}$, and identified on $\mathcal{D}$'s support up to $\tau_f \in (0,1)$.

These results help quantify inefficient overinvestment in relationships. Let $F_{\pi,f,m|\mathcal{N}^o}$ denote the distribution of manufacturer f's shared profit per unit of model m's output $\frac{\pi^{nb}_f}{q_m}$, given the set of new models with previous editions $\mathcal{N}^o$. Also, let $F_{\Delta\pi,f,m|\mathcal{N}^o}$ equal the distribution of $\frac{\Delta\pi^a_{f,m}}{q_m}$ - manufacturer f's per output ex-post relationship-forming benefit - conditional on $\mathcal{N}^o$.

Then one can respectively define

$$
\begin{aligned}
F_{\pi} &= \sum_{\mathcal{N}^o \subseteq \mathcal{N}} \mathbb{P}_F(\mathcal{N}^o) \sum_{m \in \mathcal{N}^o} \frac{1}{|\mathcal{N}^o|} F_{\pi,f,m|\mathcal{N}^o} \\
F_{\Delta\pi} &= \sum_{\mathcal{N}^o \subseteq \mathcal{N}} \mathbb{P}_F(\mathcal{N}^o) \sum_{m \in \mathcal{N}^o} \frac{1}{|\mathcal{N}^o|} F_{\Delta\pi,f,m|\mathcal{N}^o}.
\end{aligned}
\tag{2.18}
$$

as the unconditional distributions of $\frac{\pi_f^{nb}}{q_m}$ and $\frac{\Delta\pi_{f,m}^a}{q_m}$ across all new models with predecessors.

Hence,

$$
\begin{aligned}
\frac{1}{T}\sum_{t\leq T}\frac{1}{|\mathcal{N}_t^o|}\sum_{m\in\mathcal{N}_t^o}\left(\frac{\Delta\pi_{f,m,t}^a}{q_{m,t}} + \sum_{s\in s(m,g_{t-1})}\omega_{m,s,t}\right) &\xrightarrow{p} \mathbb{E}_{F_{\Delta\pi}}\left[\frac{\Delta\pi_{f,m}^a}{q_m}\right], \\
\frac{1}{T}\sum_{t\leq T}\frac{1}{|\mathcal{N}_t^o|}\sum_{m\in\mathcal{N}_t^o}\frac{\tau^f(\mathbf{g}_t)-\tau^f(g_{-m,t}+g_{m,t-1},g_{t-1})}{q_{m,t}}\pi_{f,t}^{nb} &\xrightarrow{p} \mathbb{E}_{F_{\pi}}\left[\frac{\tau^f(\mathbf{g})-\tau^f(g_{-m}+g_m^o,g^o)}{q_m}\pi_f^{nb}\right]
\end{aligned}
$$

as $T\to\infty$. These probability limits imply

$$
\frac{\mathbb{E}_{F_{\pi}}\left[\frac{\tau^f(\mathbf{g})-\tau^f(g_{-m}+g_m^o,g^o)}{q_m}\pi_f^{nb}\right]}{\mathbb{E}_{F_{\Delta\pi}}\left[\frac{\Delta\pi_{f,m}^a}{q_m}\right]}
\tag{2.19}
$$

is estimable from the data. The extent to which the manufacturers' relationship-forming incentives are distorted by overinvestment associated inefficiencies is thus quantifiable, in this manner.

Next, consider supplier s's ex-post benefit when model m's production network suppliers are altered from $s(m^o,g^o)$ to $s(m,g)$

$$
\Delta\pi_{s,m}^a = \pi_s - \pi_s|_{g_{-m}+g_m^o},
\tag{2.20}
$$

holding prices and the rest of the network fixed at equilibrium values. When s is a new supplier to m's manufacturer, this benefit decomposes to

$$\begin{aligned}\Delta\pi^a_{s,m} = \underset{\leftarrow\text{rents to } s\rightarrow}{\pi_{ms}} &- \sum_{l\in nb(s)} \underset{\longleftarrow \quad \text{hold-up of investment by } l \quad \longrightarrow}{\left[1-\tau^{ls}(\mathbf{g})\right] h^s(Q_s - q_m, Q_{sl}, q_m)} \\ &+ \sum_{l\in nb(s)} \tau^{ls}(\mathbf{g}) \sum_{k\in nb(l)\cap[s(m)-s(m^o,g^o)]-s} \underset{\leftarrow\text{reduction in cost of supplying } l \text{ by } k\rightarrow}{h^k(Q_k - q_m, Q_{kl}, q_m)} \\ &- \sum_{l\in nb(s)} \tau^{ls}(\mathbf{g}) \sum_{k\in nb(l)\cap[s(m^o,g^o)-s(m)]} \underset{\leftarrow\text{increase in cost of supplying } l \text{ by } k\rightarrow}{h^k(Q_k, Q_{kl}, q_m)} .\end{aligned} \tag{2.21}$$

The first term π_{ms} is the rent paid to s by m's manufacturer. While this rent reduces the manufacturer's incentive to form a relationship, it enlarges s's incentive. The second term equals the reduction in input prices negotiated by s with other manufacturers $l \neq f$, due to economies of scale in input production. Here, $nb(s) = \{l \in \mathcal{F} : s \in nb(l)\}$ consists of customers such as l that bargain with s. Also, $\tau^{ls}(\mathbf{g}) = \frac{1-\tau^l(\mathbf{g})}{|nb(l,\mathbf{g})|}$ is s's share of the surplus split between l and l's suppliers under such bargaining. Finally,

$$h^s(Q_s, Q_{sl}, q_m) = C^s(Q_s) - C^s(Q_s - Q_{sl}) - C^s(Q_s + q_m) + C^s(Q_s + q_m - Q_{sl})$$

is the reduction in s's cost of producing Q_{sl} additional units, when it supplies model m in addition to l. As the following paragraph will explain, absent shape restrictions on $C^s(Q)$, the second term distorts s's relationship-forming incentive in either direction. The remaining terms consists of changes in s's revenues, resulting from its customers negotiating differing input prices with rival suppliers $k \neq s$ when the production network is altered. These terms also distort s's incentive in either direction.

Now, the second term in (2.21) captures the hold-up inefficiency illustrated by the simpler model, but in a more general way. Observe if $C^s(Q)$ is concave, $h^s(Q_s - q_m, Q_{sl}, q_m)$ is non-negative, and reduces s's incentive to form a new relationship with m. Third-party firms like l thus hold up relationship formation. However, when $C^s(Q)$ is convex, $h^s(Q_s - q_m, Q_{sl}, q_m)$ is non-positive and encourages s to form new relationships. The hold-up force then acts in reverse. It would thus be desirable to identify $\Delta\pi^a_{s,m}$ and the second term in its decomposition separately. Unfortunately, no result analogous to Proposition 2.5.1 for $\Delta\pi^a_{s,m}$ exists. Hence, no ratio analogous to (2.19) is usable for quantifying this hold-up inefficiency. However, under plausible restrictions, an upper bound for such a ratio is estimable. This statement's basis is the following result.

Proposition 2.5.2. *Suppose assumptions 2.9.1 to 2.9.9 hold. Let $m \in \mathcal{N}^o \cap f$. The change in combined profits for model m's new suppliers $s(m) - nb(f)$ from replacing m's old production network suppliers $s(m^o, g^o)$ with its equilibrium ones $s(m, g)$, combined with m's residual costs under g*

$$\sum_{s \in s(m) - nb(f)} \Delta\pi^a_{s,m} + \sum_{s \in s(m)} \omega_{ms} q_m,$$

is degenerate given the data $\mathcal{D}$, and identified on $\mathcal{D}$'s support up to $\boldsymbol{\tau} \in (0,1)^{\mathcal{F}}$. The distortion of the above due to supplier $s \in s(m) - nb(f)$ bilaterally negotiating a different input price with competing manufacturer $l \in nb(s) - f$ when supplying model m

$$\sum_{l \in nb(s)} \left(1 - \tau^{ls}(\mathbf{g})\right) h^s(Q_s - q_m, Q_{sl}, q_m),$$

is also degenerate given $\mathcal{D}$, and identified on $\mathcal{D}$'s support up to $\tau_l \in (0,1)$.

Now, one can define the cross-sectional distribution of new supplier relationship-forming benefits $\sum_{s \in s(m) - nb(f)} \Delta\frac{\pi^a_{s,m}}{q_m}$, across all new models m with predecessors, in similar fashion to (2.18). Let $F_{\Delta\pi,S}$ denote this distribution. Likewise, one can also construct the distribution of these benefits' distortions due to competing manufacturers holding up investment $\sum_{s \in s(m) - nb(f)} \sum_{l \in nb(s)} \frac{1-\tau^{ls}(\mathbf{g})}{q_m} h^s(Q_s - q_m, Q_{sl}, q_m)$, across the same set of models. Let F_h denote this distribution. Finally, one can also define the distribution of the residual marginal costs

$\sum_{s\in s(m)} \omega_{ms}$, across the same models. Let F_ω denote it. Hence,

$$\frac{1}{T}\sum_{t\leq T}\frac{1}{|\mathcal{N}_t^o|}\sum_{m\in\mathcal{N}_t^o}\left(\sum_{s\in s(m,g_t)-nb(f,\mathbf{g}_t)}\frac{\Delta\pi^a_{s,m,t}}{q_{m,t}}+\sum_{s\in s(m,g_t)}\omega_{m,s,t}\right)$$
$$\xrightarrow{p}\mathbb{E}_{F_{\Delta\pi,S}}\left[\sum_{s\in s(m)-nb(f)}\frac{\Delta\pi^a_{s,m}}{q_m}\right]+\mathbb{E}_{F_\omega}\left[\sum_{s\in s(m)}\omega_{ms}\right]$$

$$\frac{1}{T}\sum_{t\leq T}\frac{1}{|\mathcal{N}_t^o|}\sum_{m\in\mathcal{N}_t^o}\sum_{s\in s(m,g_t)-nb(f,\mathbf{g}_t)}\sum_{l\in nb(s,g_t)}\frac{1-\tau^{ls}(\mathbf{g}_t)}{q_{m,t}}h^s(Q_{s,t}-q_{m,t},Q_{s,l,t},q_{m,t})$$
$$\xrightarrow{p}\mathbb{E}_{F_h}\left[\sum_{s\in s(m)-nb(f)}\sum_{l\in nb(s)}\frac{1-\tau^{ls}(\mathbf{g})}{q_m}h^s(Q_s-q_m,Q_{sl},q_m)\right]$$

as $T\to\infty$. Observe each term on the LHS is effectively identified by Proposition 2.5.2.

Model m's suppliers are determined by investment and bidding decisions when m is an entrant. Hence $\mathbb{E}_{F_\omega}\left[\sum_{s\in s(m)}\omega_{ms}\right]$ is unlikely to equal zero. However, when manufacturers choose suppliers via procurement auctions, it is reasonable to assume their auctions are won by atypically efficient producers. Likewise, when manufacturers and suppliers form relationships, it is reasonable to assume they connect to abnormally efficient counterparties. For these reasons, one would expect

$$\sum_{s\in s(m)}\mathbb{E}[\omega_{ms}|s\in s(m),m\in\mathcal{N}^o]\leq 0\ \Rightarrow\ \mathbb{E}_{F_\omega}\left[\sum_{s\in s(m)}\omega_{ms}\right]\leq 0.$$

Hence, as long as suppliers benefit from the relationships they form in equilibrium

$$\mathbb{E}_{F_{\Delta\pi,S}}\left[\sum_{s\in s(m)-nb(f)}\frac{\Delta\pi^a_{s,m}}{q_m}\right]>0,$$

an upper bound exists for the proportion of the suppliers' relationship-forming incentives distorted by rival manufacturer hold-up, averaged across all models in $\mathcal{N}^o$, unweighted by model quantities:

$$\left|\frac{\mathbb{E}_{F_h}\left[\sum_{s\in s(m)-nb(f)}\sum_{l\in nb(s)}\frac{1-\tau^{ls}(\mathbf{g})}{q_m}h_{l,m,s}\right]}{\mathbb{E}_{F_{\Delta\pi,S}}\left[\sum_{s\in s(m)-nb(f)}\frac{\Delta\pi^a_{s,m}}{q_m}\right]}\right| \leq \left|\frac{\mathbb{E}_{F_h}\left[\sum_{s\in s(m)-nb(f)}\sum_{l\in nb(s)}\frac{1-\tau^{ls}(\mathbf{g})}{q_m}h_{l,m,s}\right]}{\mathbb{E}_{F_{\Delta\pi,S}}\left[\sum_{s\in s(m)-nb(f)}\frac{\Delta\pi^a_{s,m}}{q_m}\right]+\mathbb{E}_{F_\omega}\left[\sum_{s\in s(m)}\omega_{ms}\right]}\right|.$$

Moreover, the numerator and denominator of the bound

$$\frac{\mathbb{E}_{F_h}\left[\sum_{s\in s(m)-nb(f)}\sum_{l\in nb(s)}\frac{1-\tau^{ls}(\mathbf{g})}{q_m}h_{l,m,s}\right]}{\mathbb{E}_{F_{\Delta\pi,S}}\left[\sum_{s\in s(m)-nb(f)}\frac{\Delta\pi^a_{s,m}}{q_m}\right]+\mathbb{E}_{F_\omega}\left[\sum_{s\in s(m)}\omega_{ms}\right]}, \tag{2.22}$$

equal probability limits of terms identified by Proposition 2.5.2.

2.6 Estimation Results

Consumer utility in the estimated model is given by Example 2.5.2. Consumer types have a log-normal distribution, calibrated according to U.S. household income, $\mathbf{hh}_t$[40]. Assembly and input production costs satisfy examples 2.5.3 and 2.5.4 respectively. Finally, manufacturers enjoy identical bargaining power across all suppliers, so that $\tau_f = \tau$ for all $f \in \mathcal{F}$.

Estimation is summarized by the following steps.

1. Recover the taste coefficients $(\boldsymbol{\beta}, \boldsymbol{\sigma}_{\mathbf{x}}, \alpha)$ by implementing BLP on the entire sample of downstream data $(\mathbf{p}, \mathbf{q}, \mathbf{x}, \mathbf{hh})$. This requires instrumenting prices $\mathbf{p}$ by BLP IVs $\mathbf{z}^d_m$ satisfying $\mathbb{E}\epsilon_m|\mathbf{z}^d_m = 0$. Notice the random and price coefficients $(\boldsymbol{\sigma}_{\mathbf{x}}, \alpha)$ allow diversion ratios between each model and those produced by supplier sharing rival firms

$$\frac{\partial Q_{s,-f,t}}{\partial \mathbf{q}_{f,t}}\Big|_{\mathbf{p}_{-f,t},\mathbf{x}_t,\boldsymbol{\xi}_t,\mathbf{hh}_t} = \frac{\partial \mathbf{q}_{f,t}}{\partial \mathbf{p}_{f,t}}^{-1}\frac{\partial Q_{s,-f,t}}{\partial \mathbf{p}_{f,t}},$$

to be constructed for the subsample of model-years with available supplier data.

40. More specifically, the mean and median household income in each year was used to calibrate F_y.

2. Estimate the cost function parameters $\boldsymbol{\gamma}, \mathbf{c}$ by regressing

$$mr_{m,t} = \boldsymbol{\gamma}_{\mathbf{w}} \mathbf{w}_{m,t} + \boldsymbol{\gamma}_q q_{m,t} + c_1 |s(m,g_t)| + c_2 \left(\sum_{s \in s(m,g_t)} Q_{s,t} + \sum_{s \in nb(f,g_t)} \frac{\partial Q_{s,-f,t}}{\partial q_m}\Big|_{\mathbf{p}_{-f,t}, \mathbf{x}_t, \boldsymbol{\xi}_t, \mathbf{hh}_t} Q_{s,f,t} \right)$$
$$+ c_3 \left(\sum_{s \in s(m,g_t)} Q^2_{s,t} - \sum_{s \in nb(f,g_t)} \frac{\partial Q_{s,-f,t}}{\partial q_m}\Big|_{\mathbf{p}_{-f,t}, \mathbf{x}_t, \boldsymbol{\xi}_t, \mathbf{hh}_t} \left[Q^2_{s,f,t} - 2Q_{s,t} Q_{s,f,t} \right] \right) + \omega_{m,t}.$$

   across the subsample of incumbent models $m \in \cup_t (\mathcal{M}_t - \mathcal{N}_t)$[41]. Use $\mathbf{z}^c_m$ as instruments for $q_m, \mathbf{Q}_{nb(f)}, \frac{\partial \mathbf{Q}_{nb(f),-f}}{\partial q_m}$ in the regression.

3. Use previous estimates to construct empirical analogues of (2.19) and (2.22) across new models with predecessors $\cup_t \mathcal{N}^o_t$.

In practice, the estimation procedure introduced by Berry et al. (1995) allows steps 1 and 2 to be completed simultaneously. Optimally-weighted GMM can thus be applied on the moments

$$0 = \mathbb{E}[\xi_m \mathbf{z}^d_m] = \mathbb{E}[\omega_m \mathbf{z}^c_m | m \in \mathcal{M} - \mathcal{N}],$$

to estimate $\boldsymbol{\beta}$, $\boldsymbol{\sigma}_{\mathbf{x}}$, $\boldsymbol{\gamma}$, and $\mathbf{c}$[42].

Now, standard errors must be adjusted to account for i) the fewer number of incumbent model-year observations used to construct the cost residual moments above, and ii) the dependence of the regressors in Step 2 on the price and random coefficients $\alpha, \boldsymbol{\sigma}_{\mathbf{x}}$[43]. The estimates to

41. Observe $s(m) \subseteq nb(f)$ when m is an incumbent model.

42. The precise optimization algorithm used was based on Chris Conlon's MATLAB BLP code found at https://github.com/chrisconlon. In the absence of commercial KNITRO software, this in turn depends on Matlab's built in function minimizer 'fmincon'.

43. The former was accomplished by assuming the ratio λ of the number of incumbent model year observations $n^c = \sum_{m,t} \{m \in \mathcal{M}_t - \mathcal{N}_t\}$ to the total number of observations n is held constant as $n, n^c \to \infty$, and deriving the asymptotic variance of

$$\lambda n \mathbf{m}(\boldsymbol{\theta}) = \lambda n \left[\tfrac{1}{n} \textstyle\sum_{(m,t)\sim 1}^{n} \mathbf{z}^d_{m,t} \xi_{m,t}(\boldsymbol{\theta}, \mathbf{x}_t, \mathbf{q}_t, \mathbf{p}_t) \quad \tfrac{1}{\lambda n} \textstyle\sum_{(m,t)\sim 1}^{\lambda n} \{m \in \mathcal{M} - \mathcal{N}\} \mathbf{z}^c_{m,t} \omega_{m,t}(\boldsymbol{\theta}, \mathbf{w}_t, \mathbf{q}_t, \mathbf{p}_t) \right],$$

where $\boldsymbol{\theta} = (\boldsymbol{\beta}, \boldsymbol{\sigma}, \boldsymbol{\gamma}, \mathbf{c})$ (See Lynch & Wachter (2013) formulae (3) and (7)). This asymptotic variance can be presented reflect these adjustments. In principle, the standard errors should also be adjusted to account for simulation error involved in approximating each period's random coefficient and income distribution with a finite number of draws $\{\boldsymbol{\beta}_{i,t}, y_{i,t}\}_{i \leq ns}$. In particular, consumer heterogeneity is approximated with 23×1000 period-consumer types in the estimated model[44]. The estimates presented do not account for this error.

2.6.1 Parameter Estimates

Tables 2.5 and 2.6 display the taste and cost coefficient estimates respectively. Column 1 of both tables present estimates when the network data is formed from the models' main 10 suppliers over all input types. While utility is insignificantly falling in hpw/wgt, the remaining coefficients have anticipated signs. Demand for automobiles is significantly increasing in size, while significantly decreasing in price and during the GFC (2007-2009). Marginal costs are significantly higher for more voluminous or powerful vehicles, or for models produced during the GFC.

Models produced further (great-circle) distances from Detroit are significantly more costly to build. This is consistent with shipping costs affecting imported model production. Assembly costs are also (significantly) rising in proximity to Bavaria or Nagoya, the remaining auto manufacturing clusters in the data. This is inconsistent with location-specific externalities arising from knowledge spillovers. Finally, while the marginal cost of assembling automobiles significantly rises in own-model quantities, the suppliers' marginal costs are u-shaped. Moreover, each input production cost coefficient is significant. Economies of scale thus affect input production at low quantities, but not their assembly into automobiles.

The remaining columns tabulate the estimates when the network data stem from the mod-

then be plugged into the usual GMM formula to derive the desired standard errors. The latter adjustment is accomplished by adjusting $\frac{\partial}{\partial(\boldsymbol{\sigma},\alpha)}\mathbf{m}(\boldsymbol{\theta})$ to account for $\omega_{m,t}(\boldsymbol{\theta})$'s dependence on $(\boldsymbol{\sigma}, \alpha)$ through $\frac{\partial Q_{s,-f,t}}{\partial q_m}$.

44. Naive Monte Carlo integration was used to simulate market shares for given parameters $\boldsymbol{\theta}$.

els' main suppliers, per input type. In principle, since the downstream market associated with each input type is unchanged, the taste coefficients should equal those in the first column[45]. In practice, cross input type differences in the network data fed into the estimating model's supply-side imply differing taste coefficient estimates. The slightness of the differences in Table 2.5's columns is hence reassuring.

Aside from a handful of coefficients, the estimates for the exogenous cost coefficients have similar signs and significance to their Column 1 counterparts too. The differences in estimates lie mainly in the supplier cost function estimates. While chassis, exterior and interior suppliers exhibit u-shaped marginal cost curves, the shapes for the remaining suppliers' cost curves are inverted. Powertrain and electrical suppliers experience falling marginal costs - and thus economies of scale - only at high output levels. A cause for concern is also $\hat{c}_1$'s sign under the powertrain and electrical data. These estimates imply significantly negative marginal costs at low production levels. More sophisticated parameterizations of input production costs might resolve this issue.

2.6.2 Demand Elasticities & Average Variable Costs

To quantify product differentiation in the downstream market, average price elasticities were computed across separate timeframes. Table 2.7 displays these own and cross price elasticities in odd and even numbered rows respectively. Absent product differentiation, manufacturers' demand schedules are stepwise in price, implying zero variable manufacturing profit. The estimates in Table 2.7 imply manufacturers enjoy significant market power, that increases over time.

To understand how Table 2.6's estimates translate into economies of scale, Table 2.8 displays the mean derivatives of average variable input production costs across all operating suppliers[46].

45. Hence, one could estimate the estimating model described above jointly for all five separate input categories, and restrict their taste coefficients to equal each other.

46. See (2.13) for their formulae. Units adjusted so that the derivatives measure the effect of 1000 vehicles

Table 2.5: BLP Taste Coefficient Estimates

	Combined	Power	Electric	Chassis	Exterior	Interior
Constant	-11.232	-11.268	-11.008	-11.348	-11.165	-11.224
β_0	0.329	0.290	0.323	0.275	0.269	0.257
Size	0.355	0.359	0.307	0.353	0.347	0.346
β_{size}	0.016	0.017	0.066	0.039	0.044	0.029
Hpw/Wgt	-0.002	-0.001	0.001	0.000	-0.002	-0.000
β_{hpw}	0.010	0.007	0.002	0.002	0.007	0.002
MPD	0.042	0.042	0.041	0.044	0.042	0.041
β_{MPD}	0.004	0.004	0.011	0.006	0.005	0.004
GFC	-0.080	-0.112	-0.086	-0.046	0.009	-0.085
β_{GFC}	0.059	0.064	0.060	0.080	0.070	0.060
Price	-4.985	-5.364	-4.890	-5.048	-5.324	-4.899
$-\alpha$	0.678	0.685	0.605	0.601	0.609	0.586
Size	0.000	0.004	0.069	0.020	0.036	0.025
σ_{size}	0.064	0.052	0.050	0.084	0.058	0.042
Hpw/Wgt	0.007	0.005	0.000	0.001	0.007	0.000
σ_{hpw}	0.012	0.011	0.013	0.008	0.007	0.014
MPD	0.001	0.001	0.007	0.004	0.001	0.000
σ_{MPD}	0.040	0.040	0.033	0.043	0.039	0.039
n	2971.000	2971.000	2971.000	2971.000	2971.000	2971.000
n^c	704.000	678.000	685.000	685.000	675.000	688.000
ns	1000.000	1000.000	1000.000	1000.000	1000.000	1000.000
$\sigma^{(0)}_{size} \times \overline{Size}$	1.000	1.000	1.000	1.000	1.000	1.000
$\sigma^{(0)}_{hpw} \times \overline{Hpw/Wgt}$	1.000	1.000	1.000	1.000	1.000	1.000
$\sigma^{(0)}_{MPD} \times \overline{MPD}$	1.000	1.000	1.000	1.000	1.000	1.000
$\alpha^{(0)} \times \overline{p}$	100.000	100.000	100.000	100.000	100.000	100.000
Objective[a b]	0.763	0.669	0.714	0.679	0.689	0.731

a. n^c denotes the number of model-years used to construct the conditional moments $\mathbb{E}[\omega_m \mathbf{z}^c_m | m \in \mathcal{M} - \mathcal{N}]$

b. ns denotes number of simulations

Standard errors are provided in even numbered rows. On average, main suppliers for the combined category experience significant economies of scale, during and after the GFC. The same is true when the data is disaggregated to encompass only chassis or exterior suppliers. However, powertrain and electrical suppliers on average, experience significant diseconomies of scale, ignoring fixed costs. Finally, the units in which costs and quantities are measured in imply economies of scale are slight, even if statistically significant. For example, producing an additional 1000 vehicles imply a decline of 0.018US$ in average variable costs, across main

on 1US$.

Table 2.6: BLP Cost Coefficient Estimates

	Combined	Power	Electric	Chassis	Exterior	Interior
Constant	-1.697	-1.675	-1.864	-1.728	-1.475	-1.781
γ_0	0.142	0.126	0.162	0.138	0.120	0.130
Volume	1.193	1.440	1.429	1.358	1.262	1.305
γ_{vol}	0.121	0.111	0.146	0.129	0.125	0.119
Hpw/Wgt	0.013	0.013	0.014	0.014	0.013	0.014
γ_{hpw}	0.001	0.001	0.001	0.001	0.001	0.001
US	0.197	0.054	0.285	0.290	0.106	0.106
γ_{US}	0.040	0.033	0.052	0.051	0.039	0.039
EU	0.366	0.300	0.521	0.499	0.336	0.303
γ_{EU}	0.038	0.037	0.058	0.054	0.042	0.042
Japan	0.078	0.084	0.102	0.166	0.075	0.088
γ_{Jpn}	0.028	0.024	0.033	0.032	0.025	0.029
Detroit	0.003	0.003	0.004	0.002	0.002	0.003
γ_{Det}	0.001	0.000	0.001	0.001	0.000	0.000
Bavaria	-0.002	-0.002	-0.002	-0.002	-0.002	-0.002
γ_{Bav}	0.000	0.000	0.000	0.000	0.000	0.000
Nagoya	-0.000	0.000	-0.001	-0.000	-0.000	-0.000
γ_{Ngy}	0.001	0.000	0.000	0.000	0.000	0.000
GFC	0.103	0.056	0.147	0.192	0.125	0.125
γ_{GFC}	0.030	0.036	0.027	0.033	0.023	0.023
Qty	0.001	-0.000	0.002	0.001	0.000	0.000
γ_q	0.000	0.000	0.000	0.000	0.000	0.000
Supply Qty	0.026	-0.034	-0.038	0.006	-0.007	0.013
c_1	0.007	0.008	0.009	0.008	0.007	0.007
Supply Qty^2	-0.040	0.059	0.024	-0.084	-0.034	-0.013
c_2	0.015	0.013	0.020	0.023	0.018	0.020
Supply Qty^3	0.009	-0.015	-0.003	0.035	0.015	0.003
c_3	0.005	0.004	0.005	0.011	0.006	0.009
n	2971.000	2971.000	2971.000	2971.000	2971.000	2971.000
n^c	704.000	678.000	685.000	685.000	675.000	688.000
ns	1000.000	1000.000	1000.000	1000.000	1000.000	1000.000
$\sigma^{(0)}_{size} \times \overline{Size}$	1.000	1.000	1.000	1.000	1.000	1.000
$\sigma^{(0)}_{hpw} \times \overline{Hpw/Wgt}$	1.000	1.000	1.000	1.000	1.000	1.000
$\sigma^{(0)}_{MPD} \times \overline{MPD}$	1.000	1.000	1.000	1.000	1.000	1.000
$\alpha^{(0)} \times \overline{p}$	100.000	100.000	100.000	100.000	100.000	100.000
Objective$^{a\ b\ c}$	0.763	0.669	0.714	0.679	0.689	0.731

a. n^c denotes the number of model-years used to construct the conditional moments $\mathbb{E}[\omega_m \mathbf{z}^c_m | m \in \mathcal{M} - \mathcal{N}]$

b. ns denotes number of simulations

c. Supplier quantities and supplier quantities squared measured in $1K$ and $1mil$ units repsectively.

suppliers for the combined category.

Obviously, when fixed input production costs are present, Table 2.8's numbers underestimate the slopes of average costs. However, these numbers can be compared to the slope of average variable cost for assembling vehicles. The latter are given by dividing $\hat{\gamma}_q$ in Table 2.6 by 2. Observe $\hat{\gamma}_q$ is negative only for the powertrain data, and by an insignificant magnitude. The estimate is significantly positive when combined, electrical or chassis supplier data is used in estimation. Hence, there is little evidence of economies of scale in automobile assembly activity, ignoring fixed costs. In comparison, economies of scale affect upstream producers rather than downstream manufacturers.

Table 2.7: Own & Cross-Price Demand Elasticities

	Combined	Power	Electric	Chassis	Exterior	Interior
Pre-GFC	-1.510	-1.534	-1.513	-1.506	-1.555	-1.501
1994-2006	0.001	0.001	0.002	0.001	0.002	0.001
GFC	-1.392	-1.403	-1.435	-1.414	-1.492	-1.382
2007-09	0.001	0.001	0.001	0.001	0.001	0.001
Post-GFC	-1.378	-1.434	-1.409	-1.374	-1.455	-1.364
2010-16	0.001	0.001	0.001	0.001	0.001	0.001
All	-1.454	-1.487	-1.471	-1.454	-1.516	-1.444
1994-2016	0.001	0.001	0.002	0.001	0.001	0.001

Table 2.8: Average Variable Input Production Cost Slopes

	Combined	Power	Electric	Chassis	Exterior	Interior
GFC	-0.019	0.027	0.012	-0.037	-0.014	-0.006
2008-09	0.007	0.006	0.009	0.010	0.008	0.009
Post-GFC	-0.018	0.026	0.012	-0.034	-0.013	-0.006
2010-16	0.006	0.006	0.009	0.009	0.007	0.008
All	-0.018	0.026	0.012	-0.035	-0.014	-0.006
2008-16	0.006	0.006	0.009	0.009	0.008	0.008

2.6.3 Overinvestment & Hold-up in Relationships

Table 2.9 displays the portion of the manufacturers' benefits from forming relationships, labeled as overinvestment in (2.17). (Returns measured in millions of 92US$, standard errors located in even-numbered rows.) Recall when manufacturers form new relationships, old and new suppliers are pitted against each other, protecting the manufacturers' oligopoly rents. The manufacturers' relationship-forming incentives are thus enlarged. This distortion is identified up to the manufacturers' common bargaining power, τ. Hence, Table 2.9's rows display the distortion implied by various values of τ.

Observe this distortion differs significantly from zero, across input categories and bargaining powers. Moreover, higher manufacturer bargaining power implies greater overinvestment inducing incentive distortions. Finally, the overinvestment distortions are unambiguously large for high values of τ. In particular, when $\tau = 0.9$, the overinvestment inefficiency increases an average manufacturer's incentive to form relationships with suppliers of combined inputs by US$95 million.

Table 2.10 displays the portion of the suppliers' relationship-forming benefits, labeled as hold-up in (2.21). Recall when suppliers acquire new model customers, they experience economies or diseconomies of scale. This allows competing customers to negotiate lower or higher input prices with them, respectively. As with the overinvestment distortion, hold-up in supplier relationship investment is identified up to the manufacturers' bargaining parameter.

Observe hold-up is significantly positive only for powertrain suppliers. In contrast, hold-up is significantly negative for chassis and exterior suppliers. The latter is due to chassis and exterior suppliers possessing convex, asymptotically upwards sloping marginal cost curves. Hence whilst these suppliers experience economies of scale at the margin according to Table 2.8, their average costs increase significantly from supplying entire quantities of new models in equilibrium. Also, the hold-up distortions are amplified when manufacturers enjoy greater bargaining power. Intuitively, when their bargaining power is higher, manufacturers extract greater shares of the benefits (costs) from their suppliers experiencing (dis)economies of scale by supplying rival manufacturers. This discourages (encourages) suppliers from forming new relationships when experiencing (dis)economies of scale.

However, when directly compared to overinvestment, hold-up of relationship investment appears a smaller source of inefficiency. Table 2.11 divides the values in Table 2.9 by those in Table 2.10. For nearly all non-powertrain entries, this ratio exceeds one or is negative. Hence for non-powertrain suppliers, hold-up of relationship investment is either smaller than overinvestment in relationships, or acts in reverse to encourage relationship formation. Moreover, across all input categories, the importance of overinvestment in outside option relationships as an inefficiency grows as the manufacturers' bargaining power rises. Intuitively, this is connected to the slightness of the input production cost curvatures displayed in Table 2.8, compared to the departures of the demand elasticities in Table 2.7 from infinity.

Now, the ratios in Table 2.11 feature selection bias in that they apply only to model-supplier relationships actually formed in the data. These equilibrium relationships are likely excessively affected by overinvestment, while experiencing smaller or negative hold-up, compared to unformed relationships. Hence, that the ratios in Table 2.11 are small or negative is unsurprising.

To measure inefficient overinvestment in model-supplier relationships, Table 2.12 tabulates (2.19) instead. This ratio measures the proportion of average manufacturers' ex-post benefit from forming relationships for their new models, per unit of output, attributable as causing overinvestment. When positive, the ratio exceeds one across nearly all input types and values for τ. Hence, unless suppliers compensate their co-investing manufacturers when forming new relationships, the average model-supplier relationship in the sample would not be formed without overinvestment inducing distortions. Instead, on average, manufacturers would prefer retaining their models' past suppliers when new model editions are launched. From this viewpoint, overinvestment is an important driver of model-supplier relationship formation.

Likewise, to evaluate the importance of hold-up in affecting model-supplier relationships, Table 2.13 tabulates (2.22). Observe these share the same signs as their numerators in Table 2.10. It follows average supplier ex-post returns from relationship formation $\hat{\mathbb{E}}_{F_{\Delta\pi,S}}\left[\sum_{s\in s(m)-nb(f)}\frac{\Delta\pi^a_{s,m}}{q_m}\right]$ in (2.22) is positive. Thus, as explained in Subsection 2.5.2, Table 2.10's numbers' magnitudes are upper bounds for the proportion of the suppliers' relationship-forming benefits, per unit of output, distorted by third-party manufacturer hold-up. Across all values of τ, this upper bound is smaller than 1. The ratios also do not differ significantly from zero. From this viewpoint, hold-up is a smaller source of inefficiency compared to overinvestment.

In sum, estimates of the firms' relationship-forming incentives suggest too many model-supplier relationships are formed. Yet Section 2.2's comparative statics imply the frequency of relationship investment increases in willingness-to-pay for downstream output (Proposition 2.2.3). Hence, when viewed alongside theoretical results, the estimates suggest subsidies that increase automobile demand only serve to push model-supplier relationship formation towards more inefficient levels.

Table 2.9: Overinvestment Incentive Distortion

τ	Combined	Power	Electric	Chassis	Exterior	Interior
0.100	5.667	5.873	6.906	2.728	3.628	2.725
0.000	0.022	0.023	0.022	0.006	0.009	0.006
0.250	8.969	9.696	12.779	8.013	8.683	7.939
0.000	0.030	0.029	0.033	0.017	0.018	0.017
0.500	18.614	21.061	29.061	22.620	22.695	22.032
0.000	0.055	0.051	0.066	0.049	0.046	0.047
0.750	44.207	50.439	66.327	57.624	55.951	54.788
0.000	0.122	0.113	0.147	0.127	0.113	0.119
0.900	95.017	103.294	121.369	117.099	110.822	109.069
0.000	0.255	0.227	0.274	0.261	0.231	0.244

Table 2.10: Hold-Up Incentive Distortion

τ	Combined	Power	Electric	Chassis	Exterior	Interior
0.100	6.589	27.472	-10.253	-23.206	-21.419	2.068
0.000	4.486	8.556	6.518	10.388	7.101	6.552
0.250	6.590	27.488	-10.260	-23.226	-21.437	2.069
0.000	4.487	8.561	6.524	10.395	7.107	6.556
0.500	6.595	27.533	-10.278	-23.283	-21.487	2.072
0.000	4.491	8.574	6.541	10.417	7.124	6.569
0.750	6.609	27.652	-10.325	-23.417	-21.604	2.078
0.000	4.500	8.609	6.581	10.469	7.164	6.597
0.900	6.639	27.900	-10.415	-23.663	-21.816	2.090
0.000	4.522	8.681	6.656	10.567	7.236	6.650

Table 2.11: Overinvestment to Hold Up Ratio

τ	Combined	Power	Electric	Chassis	Exterior	Interior
0.100	0.860	0.214	-0.674	-0.118	-0.169	1.318
0.000	0.571	0.065	0.432	0.054	0.057	4.186
0.250	1.361	0.353	-1.246	-0.345	-0.405	3.840
0.000	0.905	0.107	0.799	0.159	0.137	12.205
0.500	2.825	0.767	-2.834	-0.975	-1.060	10.655
0.000	1.880	0.234	1.820	0.449	0.360	33.931
0.750	6.710	1.836	-6.469	-2.483	-2.612	26.497
0.000	4.477	0.564	4.179	1.149	0.892	84.734
0.900	14.421	3.760	-11.838	-5.046	-5.174	52.748
0.000	9.668	1.165	7.734	2.355	1.786	170.016

Table 2.12: Overinvestment to Manufacturer Benefit Ratio

τ	Combined	Power	Electric	Chassis	Exterior	Interior
0.100	4.688	1.896	4.146	-0.756	-1.270	-0.561
0.000	3.408	0.364	2.147	0.236	0.398	0.175
0.250	2.807	1.835	2.366	19.025	74.727	-7.337
0.000	0.995	0.314	0.548	61.296	772.354	12.402
0.500	1.960	1.727	1.722	1.978	2.372	2.465
0.000	0.371	0.241	0.221	0.349	0.475	0.744
0.750	1.666	1.670	1.542	1.559	1.768	1.761
0.000	0.221	0.203	0.152	0.159	0.205	0.278
0.900	1.654	1.786	1.613	1.552	1.774	1.748
0.000	0.218	0.248	0.179	0.156	0.210	0.269

2.7 Conclusion

This paper studies how downstream market product differentiation and upstream economies of scale affect manufacturer-supplier relationship formation. The question is motivated by the

Table 2.13: Upper Bounds for Hold-Up to Supplier Benefit Ratio

τ	Combined	Power	Electric	Chassis	Exterior	Interior
0.100	0.015	0.107	-0.031	-0.075	-0.070	0.006
0.000	0.056	0.041	0.040	0.042	0.030	0.048
0.250	0.015	0.107	-0.031	-0.075	-0.070	0.006
0.000	0.056	0.042	0.040	0.043	0.030	0.048
0.500	0.015	0.108	-0.031	-0.075	-0.070	0.006
0.000	0.056	0.042	0.040	0.043	0.030	0.048
0.750	0.015	0.109	-0.031	-0.075	-0.070	0.006
0.000	0.056	0.043	0.040	0.043	0.030	0.048
0.900	0.015	0.112	-0.031	-0.075	-0.071	0.006
0.000	0.056	0.045	0.040	0.043	0.030	0.048

2008 bailouts of General Motors and Chrysler. Based on surveys of the US automobile industry, I construct and analyze a simple model of manufacturer-supplier relationship formation. The model captures how product differentiation and economies of scale cause inefficient relationship formation in two different ways. These inefficiencies are overinvestment in relationships in service of protecting downstream manufacturers' rents, and hold-up of relationship investment by neigbor manufacturers due to upstream economies of scale. The model also implies shocks to individual firms have disproportionately smaller welfare consequences vis-a-vis ex-ante market shares when the network is ex-ante inefficiently over connected, as opposed to being too sparse. Finally, the model is qualitatively consistent with supplier sharing patterns exhibited by US marketed automobile producers over 2008-16.

To quantify the effects of the aforementioned inefficiencies on auto manufacturer-supplier relationships, I also estimate a micro-founded model of manufacturer-supplier network formation from product-supplier network data for 2008-16 US automobile varieties. The empirical model incorporates the theoretical model's key features. The estimates imply downstream oligopoly rents substantially enlarge relationship-forming benefits for auto manufacturers ex-post, likely contributing to overinvestment in relationships. In contrast, hold-up of relationship investment by neighbor manufacturers is less significant in affecting incentives. Moreover, for certain suppliers, the hold-up distortion even acts in reverse to reinforce overinvestment in relationships, due to input marginal cost nonlinearities. These estimates in conjunction with the simpler model's results, should give pause when considering arguments for bailing out future automakers, based solely on their industry's production network.

Appendix

2.8 Appendix I

Proof of Proposition 1.2.3. Assume the model under the null and alternative hypotheses is given by (1.1) and Assumption 1.2.1, but Assumption 1.2.2 only holds in the null. Under the null,

$$\mathbb{E}[y|\underline{x}^j, \mathbf{x}_1^j, s] - \mathbb{E}[y|\underline{x}^j, \mathbf{x}_2^j, s] = n(\mathbf{x}_1^{-j}) - n(\mathbf{x}_2^{-j}),$$

a quantity independent of s, a.s. Assume the model under the null is given by (1.7), while the model under the alternate is given by (1.1). Under the null, expected difference in outcomes caused by changes in $\mathbf{x}^j$ for zero interaction individuals for any distinct pair of peer groups $s, s,' \leq S$

$$\mathbb{E}[y|\mathbf{x}, s] - \mathbb{E}[y|\mathbf{x}, s'] = \mathbb{E}[y|s] - \mathbb{E}[y|s'].$$

$\square$

Proof of Proposition 1.2.4. Replace $m(\mathbb{E}[y|s], \mathbf{x})$ by $mg(s, \mathbf{x})$ in Proposition 1.2.1. Proceed as in the proof of that proposition. $\square$

Proof of Proposition 1.2.5. Let $\mathbf{w} \equiv (\mathbf{c}, \mathbf{d}) \equiv (\mathbf{x}, s) \in supp(\mathbf{x}, s)$, where $\mathbf{c}$ and $\mathbf{d}$ are the continuously and discretely distributed variables in $(\mathbf{x}, s)$ respectively, as defined in the main text.

Also define

$$\hat{l}(\mathbf{w}) \equiv \frac{\sum_{i\leq I}\{\mathbf{d}_i = \mathbf{d}\}K(\frac{\mathbf{c}_i-\mathbf{c}}{b})y_i}{\sum_{i\leq I}\{\mathbf{d}_i = \mathbf{d}\}K(\frac{\mathbf{c}_i-\mathbf{c}}{b})} \qquad l(w) \equiv \mathbb{E}[y|\mathbf{x}, s]$$

$$\hat{a}(\mathbf{w}) = \frac{1}{I}\sum_{i\leq I}\{\mathbf{d}_i = \mathbf{d}\}K\left(\frac{\mathbf{c}_i - \mathbf{c}}{b}\right)\frac{y_i}{b^c} \qquad a(\mathbf{w}) = \int yf(y, \mathbf{c}, \mathbf{d})dy$$

$$\hat{b}(\mathbf{w}) = \frac{1}{I}\sum_{i\leq I}\{\mathbf{d}_i = \mathbf{d}\}K\left(\frac{\mathbf{c}_i - \mathbf{c}}{b}\right)\frac{1}{b^c} \qquad b(\mathbf{w}) = f(\mathbf{c}, \mathbf{d}).$$

Let $f(\mathbf{d}_i)$ denote the probability mass of $\mathbf{d}_i$ for each $i \leq I$. (This doesn't depend on i since observations are i.i.d.) Since $(\mathbf{c}, \mathbf{d}) \in supp(\mathbf{c}_i, \mathbf{d}_i)$, so equivalently, $\mathbf{c} \in supp(\mathbf{c}_i|\mathbf{d}_i = \mathbf{d})$ and $\mathbf{d} \in supp(\mathbf{d}_i)$. The proposition is proved in parts.

Sub Claim: $\hat{a}(\mathbf{w}) \xrightarrow{L_2} a(\mathbf{w})$ as $Ib^c \to \infty$, $b^c \to 0$.

Proof. The sub claim is true if $\mathbb{E}a(\mathbf{w}) \to a(\mathbf{w})$ and $\mathbb{V}\hat{a}(\mathbf{w}) \to 0$. I need to show shrinking bias and variance. First (bias):

$$\begin{aligned}
\mathbb{E}\hat{a}(\mathbf{w}) &= \frac{1}{b^c}\mathbb{E}\left[K\left(\frac{\mathbf{c}_i - \mathbf{c}}{b}\right)y_i\Big|\mathbf{d}\right]f(\mathbf{d}) \\
&= \frac{1}{b^c}\mathbb{E}\left[K\left(\frac{\mathbf{c}_i - \mathbf{c}}{b}\right)\mathbb{E}[y_i|\mathbf{c}_i, \mathbf{d}]\Big|\mathbf{d}\right]f(\mathbf{d}) \\
&= \int_{\mathbf{c}_i}\frac{1}{b^c}K\left(\frac{\mathbf{c}_i - \mathbf{c}}{b}\right)\mathbb{E}[y_i|\mathbf{c}_i, \mathbf{d}]f(\mathbf{c}_i|\mathbf{d})d\mathbf{c}_i f(\mathbf{d}) \\
&= \int_{\mathbf{u}}K(\mathbf{u})\mathbb{E}[y_i|\mathbf{c} + b\mathbf{u}, \mathbf{d}]f(\mathbf{c} + b\mathbf{u}, \mathbf{d})d\mathbf{u} \to \int_{\mathbf{u}}K(\mathbf{u})d\mathbf{u}\mathbb{E}[y_i|\mathbf{c}, \mathbf{d}]f(\mathbf{c}, \mathbf{d}) = \int yf(y, \mathbf{c}, \mathbf{d})dy
\end{aligned}$$

as $b^c \to 0$, via DCT, Assumption 1.2.7 and 1.2.6. Next (variance):

$$
\begin{aligned}
\mathbb{V}\hat{a}(\mathbf{w}) &\leq \mathbb{E}\hat{a}(\mathbf{w})^2 \\
&= \frac{1}{Ib^{2c}}\mathbb{E}\left[\{\mathbf{d}_i = \mathbf{d}\}K\left(\frac{\mathbf{c}_i - \mathbf{c}}{b}\right)^2 y_i^2\right] \\
&= \frac{1}{Ib^{2c}}\int\{\mathbf{d}_i = \mathbf{d}\}K\left(\frac{\mathbf{c}_i - \mathbf{c}}{b}\right)^2 \mathbb{E}[y_i^2|\mathbf{c}_i, \mathbf{d}]dF(\mathbf{c}_i, \mathbf{d}_i) \\
&= \frac{1}{Ib^{2c}}\int K\left(\frac{\mathbf{c}_i - \mathbf{c}}{b}\right)^2 \mathbb{E}[y_i^2|\mathbf{c}_i, \mathbf{d}]f(\mathbf{c}_i|\mathbf{d})d\mathbf{c}_i f(\mathbf{d}) \\
&= \frac{1}{Ib^c}\int K\left(\mathbf{u}\right)^2 \mathbb{E}[y_i^2|\mathbf{c} + b\mathbf{u}, \mathbf{d}]f(\mathbf{c} + b\mathbf{u}, \mathbf{d})d\mathbf{u} \to \frac{1}{\infty}\int K(\mathbf{u})^2 d\mathbf{u}\mathbb{E}[y^2|\mathbf{c}, \mathbf{d}]f(\mathbf{c}, \mathbf{d}) = 0
\end{aligned}
$$

as $Ib^c \to \infty$ and $b^c \to 0$, via DCT, Assumption 1.2.7 and 1.2.6. □

Sub Claim: $\hat{b}(\mathbf{w}) \xrightarrow{L_2} b(\mathbf{w})$ as $Ib^c \to \infty$, $b^c \to 0$.

Proof. Using the same proof strategy as $\hat{a}(w)$ case, first (bias):

$$
\begin{aligned}
\mathbb{E}\hat{b}(\mathbf{w}) &= \int_{\mathbf{c}_i} K\left(\frac{\mathbf{c}_i - \mathbf{c}}{b}\right)\frac{1}{b^c}f(\mathbf{c}_i|\mathbf{d})d\mathbf{c}_i f(\mathbf{d}) \\
&= \int_{\mathbf{c}_i} K\left(\mathbf{u}\right)f(\mathbf{c} + b\mathbf{u}|\mathbf{d})d\mathbf{u}f(\mathbf{d}) \to \int_{\mathbf{u}} K(\mathbf{u})f(\mathbf{c}, \mathbf{d})d\mathbf{u} = f(\mathbf{c}, \mathbf{d})
\end{aligned}
$$

as $Ib^c \to \infty$, $b \to 0$ via DCT, Asumption 1.2.7 and 1.2.6. Second (variance):

$$
\begin{aligned}
\mathbb{V}\hat{b}(\mathbf{w}) &\leq \mathbb{E}\hat{b}(\mathbf{w})^2 \\
&= \frac{1}{Ib^{2c}}\int K\left(\frac{\mathbf{c}_i - \mathbf{c}}{b}\right)^2 \{\mathbf{d}_i = \mathbf{d}\}dF(\mathbf{c}_i, \mathbf{d}_i) \\
&= \frac{1}{Ib^{2c}}\int K\left(\frac{\mathbf{c}_i - \mathbf{c}}{b}\right)^2 f(\mathbf{c}_i, \mathbf{d}_i)d\mathbf{c}_i \\
&= \frac{1}{Ib^c}\int_{\mathbf{u}} K\left(\mathbf{u}\right)^2 f(\mathbf{c} + b\mathbf{u}, \mathbf{d})d\mathbf{u} \longrightarrow \frac{1}{\infty}\int_{\mathbf{u}} K(\mathbf{u})^2 d\mathbf{u}f(\mathbf{c}, \mathbf{d}) = 0
\end{aligned}
$$

as $Ib^c \to \infty$, $b \to 0$ via DCT, Assumption 1.2.7 and 1.2.6. □

Sub Claim: $\hat{l}(\mathbf{w})$ is consistent for $l(\mathbf{w})$.

Proof. Since $\mathbf{w} \in supp(\mathbf{x}, s)$, $b(\mathbf{w}) = f(\mathbf{c}, \mathbf{d}) \neq 0$. Mean square convergence implies convergence in probability. So by Slutsky:

$$\hat{l}(\mathbf{w}) = \frac{\hat{a}(\mathbf{w})}{\hat{b}(\mathbf{w})} \xrightarrow{p} \frac{a(\mathbf{w})}{b(\mathbf{w})} = \int y f(y|\mathbf{c}, \mathbf{d}) dy = \mathbb{E}[y|\mathbf{x}, s]$$

as $Ib^c \to \infty$ and $b^c \to 0$. $\square$

Sub Claim: $\hat{\mu}(s)$ is consistent for $\mu(s)$.

Proof. By assumptions 1.2.1, 1.2.2, 1.2.4 and 1.2.11,

$$y_i = m(\mathbb{E}[y|s = s_i], \underline{\mathbf{x}}_i^{-j}) + \epsilon_i = \epsilon_i \Rightarrow \mathbb{E}[y|\mathbf{0}, s] = \mathbb{E}[\epsilon|\mathbf{0}, s] = \mu(s).$$

So proving the claim boils down to showing

$$\hat{\mu}(s) = \frac{\sum_{i \leq I} K(\frac{\mathbf{c}_i}{b})\{\mathbf{d}_i = (\mathbf{0}, s)\} y_i}{\sum_{i \leq I} K(\frac{\mathbf{c}_i}{b})\{\mathbf{d}_i = (\mathbf{0}, s)\}} \xrightarrow{p} \mathbb{E}[y|\mathbf{0}, s]$$

as $Ib^c \to \infty$ and $b \to 0$. But this follows from the 1st subclaim, since $(\mathbf{0}, s) = (\underline{\mathbf{x}}, s) \in supp(\mathbf{x}_i, s_i)$ by assumptions 1.2.2, 1.2.4 and 1.2.11, and recognizing $f(\mathbf{c}, \mathbf{d}) > 0 \Rightarrow f(\mathbf{d}) > 0 \Rightarrow f(s) > 0$. $\square$

Applying Slutsky one more time:

$$\hat{h}(\mathbf{x}, s) = \hat{l}(\mathbf{x}, s) - \hat{\mu}(s) \xrightarrow{p} \mathbb{E}[y|\mathbf{x}, s] - \mu(s) = m(\mathbb{E}[y|s], \mathbf{x})$$

as $Ib^c \to \infty$ and $b \to 0$. $\square$

Proof of Proposition 1.2.6. The proof is done in several steps. First, define $l(\mathbf{x}, s) \equiv h(\mathbf{x}, s) +$

$\mu(s)$ and $\hat{l}(\mathbf{x},s) \equiv \hat{h}(\mathbf{x},s) + \hat{\mu}(s)$. Then recognize $\hat{l}(\mathbf{0},s) = \hat{\mu}(s)$, and

$$\begin{aligned}\frac{1}{Ib^c}\sum_{i\leq I} K\left(\frac{\mathbf{c}_i-\mathbf{c}}{b}\right)\{\mathbf{d}_i=\mathbf{d}\}y_i = &\frac{1}{Ib^c}\sum_{i\leq I} K\left(\frac{\mathbf{c}_i-\mathbf{c}}{b}\right)\{\mathbf{d}_i=\mathbf{d}\}l(\mathbf{x},s)\\ &-\frac{1}{Ib^c}\sum_{i\leq I} K\left(\frac{\mathbf{c}_i-\mathbf{c}}{b}\right)\{\mathbf{d}_i=\mathbf{d}\}\left(l(\mathbf{x},s)-l(\mathbf{x}_i,s_i)\right)\\ &+\frac{1}{Ib^c}\sum_{i\leq I} K\left(\frac{\mathbf{c}_i-\mathbf{c}}{b}\right)\{\mathbf{d}_i=\mathbf{d}\}\eta_i, \quad \mathbb{E}[\eta|\mathbf{c},\mathbf{d}]=0,\end{aligned}$$

where $\eta_i \equiv y_i - l(\mathbf{x}_i,s_i)$. From the definitions, it follows,

$$\begin{aligned}\hat{l}(\mathbf{x},s)-l(\mathbf{x},s) = &\frac{\frac{1}{Ib^c}\sum_{i\leq I} K\left(\frac{\mathbf{c}-\mathbf{c}_i}{b}\right)\{\mathbf{d}_i=\mathbf{d}\}\eta_i}{\frac{1}{Ib^c}\sum_{i\leq I} K\left(\frac{\mathbf{c}-\mathbf{c}_i}{b}\right)\{\mathbf{d}_i=\mathbf{d}\}}\\ &-\frac{\frac{1}{Ib^c}\sum_{i\leq I} K\left(\frac{\mathbf{c}-\mathbf{c}_i}{b}\right)\{\mathbf{d}_i=\mathbf{d}\}\left(l(\mathbf{x},s)-l(\mathbf{x}_i,s_i)\right)}{\frac{1}{Ib^c}\sum_{i\leq I} K\left(\frac{\mathbf{c}-\mathbf{c}_i}{b}\right)\{\mathbf{d}_i=\mathbf{d}\}}.\end{aligned}$$

Write: $\hat{f}(\mathbf{x},s) \equiv \hat{f}(\mathbf{c},\mathbf{d}) = \frac{1}{Ib^c}\sum_{i\leq I} K\left(\frac{\mathbf{c}-\mathbf{c}_i}{b}\right)\{\mathbf{d}_i=\mathbf{d}\}$. Fix $(\mathbf{c},\mathbf{d}) = (\mathbf{x},s) \in supp(\mathbf{x},s)$. The proof adapts Hansen's notes on assymptotic normality of Nadaraya-Watson estimation.

1. Use Lyapunov CLT to show

$$\sqrt{Ib^c}\frac{1}{Ib^c}\sum_{i\leq I}\begin{pmatrix} K\left(\frac{\mathbf{c}-\mathbf{c}_i}{b}\right)\{\mathbf{d}_i=\mathbf{d}\}\eta_i \\ K\left(\frac{\mathbf{c}_i}{b}\right)\{\mathbf{d}_i=(\mathbf{0},s)\}\eta_i\end{pmatrix} \xrightarrow{d} \mathcal{N}(0,\mathbf{\Sigma});$$

$$\mathbf{\Sigma} \equiv \begin{pmatrix} R^K\sigma^2(\mathbf{x},s)f(\mathbf{x},s) & \{\mathbf{x}=\mathbf{0}\}R^K\sigma^2(\mathbf{0},s)f(\mathbf{0},s) \\ \{\mathbf{x}=\mathbf{0}\}R^K\sigma^2(\mathbf{0},s)f(\mathbf{0},s) & R^K f(\mathbf{0},s)\end{pmatrix}.$$

The idea here will be to use Cramer-Wald by establishing that for all $\alpha,\beta > 0$,

$$\begin{aligned}&\frac{\alpha}{\sqrt{Ib^c}}\sum_{i\leq I} K\left(\frac{\mathbf{c}_i-\mathbf{c}}{b}\right)\{\mathbf{d}_i=\mathbf{d}\}\eta_i + \frac{\beta}{\sqrt{Ib^c}}\sum_{i\leq I} K\left(\frac{\mathbf{c}_i}{b}\right)\{\mathbf{d}_i=(\mathbf{0},s)\}\eta_i\\ &\qquad\xrightarrow{d} \mathcal{N}\left(0, \alpha^2(R^K\sigma^2 f)(\mathbf{x},s) + \beta^2(R^K\sigma^2 fs)(\mathbf{0},s) - 2\alpha\beta\{\mathbf{x}=\mathbf{0}\}(R^K\sigma^2 f)(\mathbf{0},s)\right)\end{aligned}$$

as $Ib^c \to \infty$, $b^c \to 0$, and recognizing the above is distribution of $\alpha z_1 + \beta z_2$, where $\mathbf{z} \sim$

$\mathcal{N}(0, \mathbf{\Sigma})$.

2. Show there exists some function $B(\mathbf{x}, s)$ such that as $Ib^c \to \infty$ and $b \to 0$,

$$\sqrt{Ib^c}\left(\frac{1}{Ib^c}\sum_{i\le I} K\left(\frac{\mathbf{c}_i - \mathbf{c}}{b}\right)\{\mathbf{d}_i = \mathbf{d}\}(l(\mathbf{x}_i, s_i) - l(\mathbf{x}, s)) - b^2 B(\mathbf{x}, s)\right) \xrightarrow{p} 0.$$

3. Recognize that we then have

$$\begin{aligned}\sqrt{Ib^c}\begin{pmatrix}(\hat{l} - l)(\mathbf{x}, s) - b^2 B(\mathbf{x}, s) \\ (\hat{\mu} - \mu)(\mathbf{x}, s) - b^2 B(\mathbf{0}, s)\end{pmatrix} &= \sqrt{Ib^c}\begin{pmatrix}(\hat{l} - l)(\mathbf{x}, s) - b^2 B(\mathbf{x}, s) \\ (\hat{l} - l)(\mathbf{0}, s) - b^2 B(\mathbf{0}, s)\end{pmatrix} \\ &\xrightarrow{d} \begin{pmatrix}\frac{1}{f(\mathbf{x},s)} & 0 \\ 0 & \frac{1}{f(\mathbf{0},s)}\end{pmatrix}\mathcal{N}\left(0, \mathbf{\Sigma}\right) + \mathbf{0}\end{aligned}$$

as $Ib^c \to \infty$, $b^c \to 0$.

4. Recognize $\sqrt{Ib^c}(\hat{h} - h)(\mathbf{x}, s) = \sqrt{Ib^c}(\hat{l} - l)(\mathbf{x}, s) - \sqrt{Ib^c}(\hat{\mu} - \mu)(s)$ to derive its limiting distribution.

Sub Claim: $\sqrt{Ib^c}\frac{1}{Ib^c}\sum_{i\le I}\begin{pmatrix}K\left(\frac{\mathbf{c}-\mathbf{c}_i}{b}\right)\{\mathbf{d}_i = \mathbf{d}\}\eta_i \\ K\left(\frac{\mathbf{c}_i}{b}\right)\{\mathbf{d}_i = (\mathbf{0}, s)\}\eta_i\end{pmatrix} \xrightarrow{d} \mathcal{N}(0, \mathbf{\Sigma})$ as $Ib^c \to \infty$ and $b^c \to 0$.

Proof. Idea is to use Cramer-Wald as stated above. Fix $\alpha > 0$ and $\beta > 0$. Let

$$\mathbf{g}_{bi} = b^{-\frac{c}{2}}\begin{pmatrix}K\left(\frac{\mathbf{c}-\mathbf{c}_i}{b}\right)\{\mathbf{d}_i = \mathbf{d}\}\eta_i \\ K\left(\frac{\mathbf{c}_i}{b}\right)\{\mathbf{d}_i = (\mathbf{0}, s)\}\eta_i\end{pmatrix}, \qquad \begin{pmatrix}z_1 \\ z_2\end{pmatrix} \sim \mathcal{N}(\mathbf{0}, \mathbf{\Sigma}).$$

The idea is to use Lyapunov CLT to show $\frac{1}{\sqrt{I}}\sum_{i\le I}(\alpha g_{bi}^1 + \beta g_{bi}^2) \overset{a}{\sim} \alpha z_1 + \beta z_2$. First, observe $\{\alpha g_{bi}^1 + \beta g_{bi}^2\}_{i\le I}$ produces an independently distributed triangular array as $(I, b) \to (\infty, 0)$. Next, since $\mathbb{E}[\eta_i|\mathbf{x}_i, s_i] = 0$, so $\mathbb{E}\mathbf{g}_{bi} = \mathbf{0}$. Third,

$$\frac{1}{I}\sum_{i\le I}\mathbb{E}[\alpha g_{bi}^1 + \beta g_{bi}^2]^2 = \alpha\mathbb{E}[g_{bi}^1]^2 + \beta\mathbb{E}[g_{bi}^2]^2 + 2\alpha\beta\mathbb{E}[g_{bi}^1 g_{bi}^2]. \tag{2.23}$$

Examining the first term of (2.23),

$$\begin{aligned}
\mathbb{E}[g_{bi}^1] &= \mathbb{E}\left[b^{-c}K\left(\frac{\mathbf{c}_i-\mathbf{c}}{b}\right)^2\{\mathbf{d}_i=\mathbf{d}\}\eta_i^2\right] \\
&= b^{-c}\int K\left(\frac{\mathbf{c}_i-\mathbf{c}}{b}\right)^2\mathbb{E}[\eta_i^2|\mathbf{c}_i,\mathbf{d}]f(\mathbf{c}_i|\mathbf{d})d\mathbf{c}_i f(\mathbf{d}) \\
&= \int K(\mathbf{u})^2\sigma^2(\mathbf{c}+b\mathbf{u},\mathbf{d})f(\mathbf{c}+b\mathbf{u},\mathbf{d})d\mathbf{u} \\
&\to \int K(\mathbf{u})^2 d\mathbf{u}\sigma^2(\mathbf{c},\mathbf{d})f(\mathbf{c},\mathbf{d}) = R^K\sigma^2(\mathbf{c},\mathbf{d})f(\mathbf{c},\mathbf{d})
\end{aligned}$$

as $b\to 0$. Examing the second term in (2.23),

$$\mathbb{E}[g_{bi}^2]^2 \to R^K\sigma^2(\mathbf{0},s)f(\mathbf{0},s).$$

Examining the final term of (2.23), we see that there are two cases to consider. In the first case, (where $\{\mathbf{d}_i=\mathbf{d}\}\{\mathbf{d}_i=(\mathbf{0},s)\}=1 \Leftrightarrow \mathbf{d}=(\mathbf{0},s)$):

$$\begin{aligned}
\mathbb{E}[g_{bi}^1g_{bi}^2] &= b^c\int_{\mathbf{c}_i}K\left(\frac{\mathbf{c}_i-\mathbf{c}}{b}\right)K\left(\frac{\mathbf{c}_i}{b}\right)\mathbb{E}[\eta_i^2|\mathbf{c}_i,\mathbf{0},s]f(\mathbf{c}_i|\mathbf{0},s)d\mathbf{c}_i f(\mathbf{0},s) \\
&= b^c\int_{\mathbf{c}_i}K(\mathbf{u})K\left(\frac{\mathbf{c}}{b}+\mathbf{u}\right)\sigma^2(\mathbf{c}+b\mathbf{u},\mathbf{0},s)f(\mathbf{c}+b\mathbf{u},\mathbf{0},s)d\mathbf{c}_i \\
&= \int_{\mathbf{u}}K(\mathbf{u})K\left(\frac{\mathbf{c}}{b}+\mathbf{u}\right)\sigma^2(\mathbf{c}+b\mathbf{u},\mathbf{0},s)f(\mathbf{c}+b\mathbf{u},\mathbf{0},s)d\mathbf{u} \\
&\to \int_{\mathbf{u}}K(\mathbf{u})K(\infty)\{\mathbf{c}\neq 0\}\sigma^2(\mathbf{c},\mathbf{0},s)f(\mathbf{c},\mathbf{0},s)d\mathbf{u} \\
&\qquad + R^K\sigma^2(\mathbf{c},\mathbf{0},s)f(\mathbf{c},\mathbf{0},s)\{\mathbf{c}=\mathbf{0}\} = R^K\sigma^2(\mathbf{0},s)f(\mathbf{0},s)\{\mathbf{c}=\mathbf{0}\}.
\end{aligned}$$

via DCT, 1.2.7 and 1.2.6. In the second case, (where $\{\mathbf{d}_i=\mathbf{d}\}\{\mathbf{d}_i=(\mathbf{0},s)\}=0 \Leftrightarrow \mathbf{d}\neq(\mathbf{0},s)$), $\mathbb{E}[g_{bi}^1g_{bi}^2]=0$. So

$$\frac{1}{I}\sum_{i\leq I}\mathbb{E}[\alpha g_{bi}^1+\beta g_{bi}^2]^2 \to \alpha^2R^K\sigma^2(\mathbf{x},s)f(\mathbf{x},s)+\beta^2R^K\sigma^2(\mathbf{0},s)f(\mathbf{0},s)+2\alpha\beta R^K\sigma^2(\mathbf{0},s)f(\mathbf{0},s)\{\mathbf{x}=\mathbf{0}\}.$$

Finally, for Lyapunov to work, one has to show $\sum_{i\leq I}\mathbb{E}\left|\frac{\alpha g_{bi}^1+\beta g_{bi}^2}{\sqrt{I}}\right|^{2+\delta} \to \infty$ as $Ib^c\to\infty$ and

$b \to 0$. Now, starting from the L.H.S. of the inequality,

$$\begin{aligned}
\sum_{i\le I}\mathbb{E}\left|\frac{\alpha g_{bi}^1+\beta g_{bi}^2}{\sqrt{I}}\right|^{2+\delta} &= I\mathbb{E}\left[\left(\frac{1}{\sqrt{I}}\right)\left|\frac{\alpha}{b^{c/2}}\{\mathbf{d}_i=\mathbf{d}\}K\left(\frac{\mathbf{c}_i-\mathbf{c}}{b}\right)\eta_i+\frac{\beta}{b^{c/2}}\{\mathbf{d}_i=(\mathbf{0},s)\}K\left(\frac{\mathbf{c}_i}{b}\right)\eta_i\right|\right]^{2+\delta}\\
(\text{ via } C_r \text{ inequality}) &\le \left(\frac{1}{\sqrt{I}}\right)^{2+\delta}2^{1+\delta}\mathbb{E}\left|\frac{\alpha}{b^{c/2}}\{\mathbf{d}_i=\mathbf{d}\}K\left(\frac{\mathbf{c}_i-\mathbf{c}}{b}\right)\eta_i\right|^{2+\delta}\\
&\quad+\left(\frac{1}{\sqrt{I}}\right)^{2+\delta}2^{1+\delta}\mathbb{E}\left|\frac{\beta}{b^{c/2}}\{\mathbf{d}_i=(\mathbf{0},s)\}K\left(\frac{\mathbf{c}_i}{b}\right)\eta_i\right|^{2+\delta}\\
&\equiv \left(\frac{1}{\sqrt{I}}\right)^{2+\delta}2^{1+\delta}A+\left(\frac{1}{\sqrt{I}}\right)^{2+\delta}2^{1+\delta}B
\end{aligned}$$

where A and B have been appropriately defined. Observe,

$$\begin{aligned}
A &= \int\int\left|\frac{\alpha}{b^{c/2}}\right|^{2+\delta}\left|K\left(\frac{\mathbf{c}_i-\mathbf{c}}{b}\right)\right|^{2+\delta}\mathbb{E}[\eta_i^{2+\delta}|\mathbf{c}_i,\mathbf{d}_i]\{\mathbf{d}_i=\mathbf{d}\}f(\mathbf{c}_i,\mathbf{d}_i)d\mathbf{d}_id\mathbf{c}_i\\
&= \frac{\alpha^{2+\delta}}{b^{c+c\delta/2}}\int K(\mathbf{u})^{2+\delta}\mathbb{E}[\eta_i^{2+\delta}|\mathbf{c}+b\mathbf{u},\mathbf{d}]f(\mathbf{c}+b\mathbf{u},\mathbf{d})d\mathbf{c}_i\\
&= \frac{\alpha^{2+\delta}}{b^{c\delta/2}}\int K(\mathbf{u})^{2+\delta}\mathbb{E}[\eta_i^{2+\delta}|\mathbf{c}+b\mathbf{u},\mathbf{d}]f(\mathbf{c}+b\mathbf{u},\mathbf{d})d\mathbf{u}.
\end{aligned}$$

And so

$$\begin{aligned}
2^{1+\delta}\left(\frac{1}{\sqrt{I}}\right)^{2+\delta}A =&\frac{1}{I}\left(\frac{1}{\sqrt{Ib^c}}\right)^{\delta}\alpha^{2+\delta}2^{1+\delta}\int K(\mathbf{u})^{2+\delta}\mathbb{E}[\eta_i^{2+\delta}|\mathbf{c}+b\mathbf{u},\mathbf{d}]f(\mathbf{c}+b\mathbf{u},\mathbf{d})d\mathbf{u}\\
\to&\frac{1}{\infty}\left(\frac{1}{\infty}\right)^{\delta}\alpha^{2+\delta}2^{1+\delta}\int K(\mathbf{u})^{2+\delta}\mathbb{E}[\eta_i^{2+\delta}|\mathbf{c},\mathbf{d}]f(\mathbf{c},\mathbf{d})d\mathbf{u}=0
\end{aligned}$$

as $Ib^c \to \infty$, $b \to 0$ using DCT and assumptions 1.2.8 to 1.2.10. Also observe,

$$\begin{aligned}
B &= \int\int\left|\frac{\beta}{b^{c/2}}\right|^{2+\delta}\left|K\left(\frac{\mathbf{c}_i}{b}\right)\right|^{2+\delta}\mathbb{E}[\eta_i^{2+\delta}|\mathbf{c}_i,\mathbf{d}_i]\{\mathbf{d}_i=(\mathbf{0},s)\}f(\mathbf{c}_i,\mathbf{d}_i)d\mathbf{d}_id\mathbf{c}_i\\
&= \frac{\beta^{2+\delta}}{b^{c\delta/2}}\int|K(\mathbf{u})|^{2+\delta}\mathbb{E}[\eta_i^{2+\delta}|b\mathbf{u},\mathbf{0},s]f(b\mathbf{u},\mathbf{0},s)d\mathbf{u}.
\end{aligned}$$

And so

$$
\begin{aligned}
2^{1+\delta}\left(\frac{1}{\sqrt{I}}\right)^{2+\delta} B &= \frac{1}{I}\left(\frac{1}{\sqrt{Ib^c}}\right)^{\delta}\beta^{2+\delta}\int K(\mathbf{u})^{2+\delta}\mathbb{E}[\eta_i^{2+\delta}|b\mathbf{u},\mathbf{0},s]f(b\mathbf{u},\mathbf{0},s)d\mathbf{u} \\
&\to \frac{1}{\infty}\left(\frac{1}{\infty}\right)^{\delta}\beta^{2+\delta}\int K(\mathbf{u})^{2+\delta}\mathbb{E}[\eta_i^{2+\delta}|\mathbf{0},s]f(\mathbf{0},s)d\mathbf{u} = 0
\end{aligned}
$$

as $Ib^c \to \infty$ $b \to 0$ using DCT and assumptions 1.2.8 to 1.2.10. It follows $\sum_{i\leq I}\mathbb{E}\left|\frac{\alpha g_{bi}^1+\beta g_{bi}^2}{\sqrt{I}}\right|^{2+\delta} \to 0$ as $Ib^c \to \infty$ and $b \to 0$. Since all conditions for Lyapunov holds, so by Cramer-Wald, claim also holds. $\square$

Sub Claim: There exists some function $B(\mathbf{x}, s)$ such that

$$
\sqrt{Ib^c}\left(\frac{1}{Ib^c}\sum_{i\leq I}K\left(\frac{\mathbf{c}_i-\mathbf{c}}{b}\right)\{\mathbf{d}_i=\mathbf{d}\}\left(l(\mathbf{x}_i,s_i)-l(\mathbf{x},s)\right)-b^2B(\mathbf{x},s)\right) \xrightarrow{p} 0
$$

as $Ib^c \to \infty$ and $b \to 0$.

Proof. The idea is to show the term inside the paranthesis has mean squared error smaller than $\sqrt{Ib^c}$. First, (bias):

$$
\begin{aligned}
&\frac{1}{b^c}\mathbb{E}\left[K\left(\frac{\mathbf{c}_i-\mathbf{c}}{b}\right)\{\mathbf{d}_i=\mathbf{d}\}\left(l(\mathbf{x}_i,s_i)-l(\mathbf{x},s)\right)\right] \\
&\quad = \frac{1}{b^c}\int K\left(\frac{\mathbf{c}_i-\mathbf{c}}{b}\right)\left(l(\mathbf{c}_i,\mathbf{d}_i)-l(\mathbf{c},\mathbf{d}\right)f(\mathbf{c},\mathbf{d})d\mathbf{c}_i \\
&\quad = \int K\left(\mathbf{u}\right)\left(l(\mathbf{c}_i,\mathbf{d}_i)-l(\mathbf{c},\mathbf{d})\right)f(\mathbf{c}+b\mathbf{u},\mathbf{d})d\mathbf{u} \\
&\quad = \int K\left(\mathbf{u}\right)\left(b\frac{\partial l(\mathbf{c},\mathbf{d})}{\partial \mathbf{c}'}\mathbf{u}+b^2\mathbf{u}^T\frac{\partial^2 l}{\partial\mathbf{c}'\partial\mathbf{c}}(\mathbf{c},\mathbf{d})\mathbf{u}+O(b^3)\right) \\
&\qquad\qquad \times\left(f(\mathbf{c},\mathbf{d})+b\frac{\partial f(\mathbf{c},\mathbf{d})}{\partial\mathbf{c}'}\mathbf{u}+b^2\mathbf{u}^T\frac{\partial^2 f(\mathbf{c},\mathbf{d})}{\partial\mathbf{c}'\partial\mathbf{c}}\mathbf{u}+O(b^3)\right)d\mathbf{u} \\
&\quad = bf(\mathbf{c},\mathbf{d})\frac{\partial l(\mathbf{c},\mathbf{d})}{\partial\mathbf{c}'}\int K(\mathbf{u})\mathbf{u}d\mathbf{u}+b^2 f(\mathbf{c},\mathbf{d})\int\mathbf{u}^T\frac{\partial^2 l(\mathbf{c},\mathbf{d})}{\partial\mathbf{c}'\partial\mathbf{c}}(\mathbf{c},\mathbf{d})\mathbf{u}d\mathbf{u} \\
&\qquad\qquad + b^2\int\mathbf{u}^T\frac{\partial l(\mathbf{c},\mathbf{d})}{\partial\mathbf{c}}\frac{\partial f(\mathbf{c},\mathbf{d})}{\partial\mathbf{c}'}\mathbf{u}d\mathbf{u}+O(b^3),
\end{aligned}
$$

where the $O(b^3)$ terms are linear in b^3, b^4 and so forth. Let $B(\mathbf{x},s)\equiv f(\mathbf{c},\mathbf{d})\int\mathbf{u}^T\frac{\partial^2 l(\mathbf{c},\mathbf{d})}{\partial\mathbf{c}'\partial\mathbf{c}}(\mathbf{c},\mathbf{d})\mathbf{u}d\mathbf{u}+$

$\int \mathbf{u}^T \frac{\partial l(\mathbf{c},\mathbf{d})}{\partial \mathbf{c}} \frac{\partial f(\mathbf{c},\mathbf{d})}{\partial \mathbf{c}'} \mathbf{u} d\mathbf{u}$. The above collapses to

$$\frac{1}{b^c}\mathbb{E}\left[K\left(\frac{\mathbf{c}_i - \mathbf{c}}{b}\right)\{\mathbf{d}_i = \mathbf{d}\}\left(l(\mathbf{x}_i, s_i) - l(\mathbf{x}, s)\right)\right] = 0 + b^2 B(\mathbf{x}, s) + O(b^3).$$

But since $Bias^2(O(b^3)) = O(b^9)$, so $MSE(\sqrt{Ib^c}O(b^3)) = Ib^{9+c} \to 0$ as $Ib^c \to \infty$ and $b \to 0$ sufficiently quickly.

Second (variance):

$$\begin{aligned}
&\frac{1}{Ib^c}\mathbb{V}\left[K\left(\frac{\mathbf{c}_i - \mathbf{c}}{b}\right)\{\mathbf{d}_i = \mathbf{d}\}\left(l(\mathbf{x}_i, s_i) - l(\mathbf{x}, s)\right)\right] \\
&\quad \leq \frac{1}{Ib^{2c}}\int K\left(\frac{\mathbf{c}_i - \mathbf{c}}{b}\right)^2 \left(l(\mathbf{x}_i, s_i) - l(\mathbf{x}, s)\right)^2 \{\mathbf{d}_i = \mathbf{d}\} f(\mathbf{c}_i, \mathbf{d}) d\mathbf{c}_i \\
&\quad = \frac{1}{Ib^c}\int K\left(\mathbf{u}\right)^2 \left(l(\mathbf{c} + b\mathbf{u}, \mathbf{d}) - l(\mathbf{c}, \mathbf{d})\right)^2 f(\mathbf{c} + b\mathbf{u}, \mathbf{d}) d\mathbf{u} \\
&\quad = \frac{1}{Ib^c}\int K\left(\mathbf{u}\right)^2 \left(b\frac{\partial l(\mathbf{c}, \mathbf{d})}{\partial \mathbf{c}'}\mathbf{u} + b^2\mathbf{u}^T\frac{\partial l(\mathbf{c}, \mathbf{d})}{\partial \mathbf{c}'}\mathbf{u} + O(b^3)\right)^2 \\
&\qquad\qquad \times \left(f(\mathbf{c}, \mathbf{d}) + b\frac{\partial l(\mathbf{c}, \mathbf{d})}{\partial \mathbf{c}'}\mathbf{u} + O(b^2)\right) d\mathbf{u} \\
&\quad = \frac{1}{Ib^c}\int K\left(\mathbf{u}\right)^2 \left(b^2\left(\frac{\partial l(\mathbf{c}, \mathbf{d})}{\partial \mathbf{c}'}\mathbf{u}\right)^2 + 2b^3\frac{\partial l(\mathbf{c}, \mathbf{d})}{\partial \mathbf{c}'}\mathbf{u}\mathbf{u}^T\frac{\partial^2 l(\mathbf{c}, \mathbf{d})}{\partial \mathbf{c}'\partial \mathbf{c}}\mathbf{u} + b^4\left(\mathbf{u}^T\frac{\partial^2 l(\mathbf{c}, \mathbf{d})}{\partial \mathbf{c}'\partial \mathbf{c}}\mathbf{u}\right)^2 + O(b^4)\right) \\
&\qquad\qquad \times \left(f(\mathbf{c}, \mathbf{d}) + b\frac{\partial f(\mathbf{c}, \mathbf{d})}{\partial \mathbf{c}'}\mathbf{u} + O(b^2)\right) d\mathbf{u} \\
&\quad = \frac{1}{Ib^c}\int K(\mathbf{u})^2 d\mathbf{u} = O\left(\frac{b^2}{Ib^c}\right)
\end{aligned}$$

where $O(b^2)$ and $O(b^3)$ are linear in b^2, b^3 ecetera. So

$$\mathbb{V}\left[\sqrt{Ib^c}\frac{1}{Ib^c}\sum_{i \leq I} K\left(\frac{\mathbf{c}_i - \mathbf{c}}{b}\right)\{\mathbf{d}_i = \mathbf{d}\}\left(l(\mathbf{x}_i, s_i) - l(\mathbf{x}, s)\right)\right] = O(b^2) \to 0$$

as $Ib^c \to \infty$ and $b \to 0$. So

$$MSE\left(\sqrt{Ib^c}\left(\frac{1}{Ib^c}\sum_{i \leq I} K\left(\frac{\mathbf{c}_i - \mathbf{c}}{b}\right)\{\mathbf{d}_i = \mathbf{d}\}\left(l(\mathbf{x}_i, s_i) - l(\mathbf{x}, s)\right) - b^2 B(\mathbf{x}, s)\right)\right) = O(b^2) \to 0$$

as $Ib^c \to \infty$ and $b \to 0$. □

Sub Claim: $MSE(\hat{f} - f) = O(b^4) + O\left(\frac{1}{nb^c}\right)$.

Proof. By Assumption 1.2.10, 1.2.8 and the standard argument provided in my Econometrics III lecture notes, the claim holds. □

Sub Claim: There exists a function $k(\mathbf{x}, s)$ such that

$$\sqrt{Ib^c}\begin{pmatrix}(\hat{l} - l)(\mathbf{x}, s) - b^2 k(\mathbf{x}, s) \\ (\hat{\mu} - \mu)(\mathbf{x}, s) - b^2 k(\mathbf{0}, s)\end{pmatrix} \xrightarrow{d} \begin{pmatrix}\frac{1}{f(\mathbf{x},s)} & 0 \\ 0 & \frac{1}{f(\mathbf{0},s)}\end{pmatrix}\mathcal{N}(0, \mathbf{\Sigma})$$

as $Ib^c \to \infty$, $b^c \to 0$.

Proof. Since $(\mathbf{x}, s), (\mathbf{0}, s) \in supp(\mathbf{x}, s)$, $f(\mathbf{0}, s), f(\mathbf{x}, s) > 0$. Let $k(\mathbf{x}, s) = \frac{B(\mathbf{x},s)}{f(\mathbf{x},s)}$ and likewise for the special case of $\mathbf{x} = \mathbf{0}$. Let $\hat{a}(\mathbf{x}, s) = \hat{a}(\mathbf{c}, \mathbf{d}) = \frac{1}{Ib^c}\sum_{i \leq I} K\left(\frac{\mathbf{c}_i - \mathbf{c}}{b}\right)\{\mathbf{d}_i = \mathbf{d}\}\left(l(\mathbf{x}_i, s_i) - l(\mathbf{x}, s)\right)$. Then

$$\sqrt{Ib^c}\begin{pmatrix}(\hat{l} - l)(\mathbf{x}, s) - b^2 k(\mathbf{x}, s) \\ (\hat{\mu} - \mu)(\mathbf{x}, s) - b^2 k(\mathbf{0}, s)\end{pmatrix}$$
$$= \sqrt{Ib^c}\begin{pmatrix}\frac{\hat{a}(\mathbf{x},s)}{\hat{f}(\mathbf{x},s)} - b^2 k(\mathbf{x}, s) \\ \frac{\hat{a}(\mathbf{0},s)}{\hat{f}(\mathbf{0},s)} - b^2 k(\mathbf{0}, s)\end{pmatrix} + \frac{1}{\sqrt{Ib^c}}\sum_{i \leq I}\begin{pmatrix}K\left(\frac{\mathbf{c}_i - \mathbf{c}}{b}\right)\{\mathbf{d}_i = \mathbf{d}\}\eta_i / \hat{f}(\mathbf{x}, s) \\ K\left(\frac{\mathbf{c}_i}{b}\right)\{\mathbf{d}_i = (\mathbf{0}, s)\}\eta_i / \hat{f}(\mathbf{0}, s)\end{pmatrix}.$$

But the above equals

$$\sqrt{Ib^c}\begin{pmatrix}\frac{\hat{a}(\mathbf{x},s) - b^2 B(\mathbf{x},s)}{\hat{f}(\mathbf{x},s)} \\ \frac{\hat{a}(\mathbf{0},s) - b^2 B(\mathbf{0},s)}{\hat{f}(\mathbf{0},s)}\end{pmatrix} + b^2 k(\mathbf{x}, s)\begin{pmatrix}\frac{f(\mathbf{x},s) - \hat{f}(\mathbf{x},s)}{\hat{f}(\mathbf{x},s)} \\ \frac{f(\mathbf{0},s) - \hat{f}(\mathbf{0},s)}{\hat{f}(\mathbf{0},s)}\end{pmatrix} + \begin{pmatrix}\frac{1}{\hat{f}(\mathbf{x},s)} & 0 \\ 0 & \frac{1}{\hat{f}(\mathbf{0},s)}\end{pmatrix}\frac{1}{\sqrt{Ib^c}}\sum_{i \leq I}\begin{pmatrix}K\left(\frac{\mathbf{c}_i - \mathbf{c}}{b}\right)\{\mathbf{d}_i = \mathbf{d}\}\eta_i \\ K\left(\frac{\mathbf{c}_i}{b}\right)\{\mathbf{d}_i = (\mathbf{0}, s)\}\eta_i\end{pmatrix}$$
$$\xrightarrow{d}\begin{pmatrix}\frac{0}{f(\mathbf{x},s)} \\ \frac{0}{f(\mathbf{0},s)}\end{pmatrix} + \begin{pmatrix}\frac{1}{\hat{f}(\mathbf{x},s)} & 0 \\ 0 & \frac{1}{\hat{f}(\mathbf{0},s)}\end{pmatrix}\mathcal{N}(\mathbf{0}, \mathbf{\Sigma}),$$

which proves the sub claim. □

Then observe

$$\begin{pmatrix} \frac{1}{f(\mathbf{x},s)} & 0 \\ 0 & \frac{1}{f(\mathbf{0},s)} \end{pmatrix} \begin{pmatrix} R^k\sigma^2(\mathbf{x},s)f(\mathbf{x},s) & \{\mathbf{x}=\mathbf{0}\}R^K\sigma^2(\mathbf{x},s)f(\mathbf{0},s) \\ \{\mathbf{x}=\mathbf{0}\}R^K\sigma^2(\mathbf{x},s)f(\mathbf{0},s) & R^K\sigma^2(\mathbf{0},s)f(\mathbf{0},s) \end{pmatrix} \begin{pmatrix} \frac{1}{f(\mathbf{x},s)} & 0 \\ 0 & \frac{1}{f(\mathbf{0},s)} \end{pmatrix}$$
$$= \begin{pmatrix} \frac{R^k\sigma^2(\mathbf{x},s)}{f(\mathbf{x},s)} & \{\mathbf{x}=\mathbf{0}\}\frac{R^K\sigma^2(\mathbf{x},s)}{f(\mathbf{0},s)} \\ \{\mathbf{x}=\mathbf{0}\}\frac{R^K\sigma^2(\mathbf{x},s)}{f(\mathbf{0},s)} & \frac{R^K\sigma^2(\mathbf{0},s)}{f(\mathbf{0},s)} \end{pmatrix},$$

is the (assymptotic) variance-covariance matrix of

$$\sqrt{Ib^c}\begin{pmatrix} \hat{l}(\mathbf{x},s) - l(\mathbf{x},s) + b^2k(\mathbf{x},s) \\ \hat{\mu}(s) - \mu(s) + b^2k(\mathbf{0},s) \end{pmatrix}.$$

So

$$\sqrt{Ib^c}\left((\hat{\mu}-\mu)(s) - b^2k(\mathbf{0},s)\right) \xrightarrow{d} \mathcal{N}\left(0, \frac{R^K\sigma^2(\mathbf{0},s)}{f(\mathbf{0},s)}\right)$$

and

$$\begin{aligned} &\sqrt{Ib^c}\left((\hat{h}-h)(\mathbf{x},s) - b^2(k(\mathbf{x},s)-k(\mathbf{0},s))\right) \\ &= \sqrt{Ib^c}\left((\hat{l}-l)(\mathbf{x},s) - b^2(k(\mathbf{x},s)-k(\mathbf{0},s))\right) - \sqrt{Ib^c}((\hat{\mu}-\mu)(s) - b^2k(\mathbf{0},s)) \\ &\xrightarrow{d} \mathcal{N}\left(0, \frac{R^K\sigma^2(\mathbf{x},s)}{f(\mathbf{x},s)} + \frac{R^K\sigma^2(\mathbf{0},s)}{f(\mathbf{0},s)} - 2\{\mathbf{x}=\mathbf{0}\}\frac{R^K\sigma^2(\mathbf{0},s)}{f(\mathbf{0},s)}\right) \end{aligned}$$

as $Ib^c \to \infty$ and $b \to 0$.

Because $B(\mathbf{x},s)$ and $B(\mathbf{0},s)$ are integrals of the remainder of a Taylor series expansion of C^2 functions, so by FTC, these are C^3. Define $c(\mathbf{x},s)$ appropriately so that the proposition holds. □

Proof of Proposition 1.3.2. Fix $\boldsymbol{\theta}$ and $\boldsymbol{\theta}'$ as the two distinct parameters. Assume m, μ and $\boldsymbol{\theta}$ represent the true model parameters. Condition i) ensures the pre-images of $L_{\boldsymbol{\theta}}$ partition the support of it's arguments in a way that exactly aligns with the pre-images of each map $L_{\boldsymbol{\theta}'}$.

Define $m' : \mathbb{R}^{K+1} \to \mathbb{R}$ by

$$m'(r, \mathbf{x}) \equiv m\left(L\left(L^{-1}(r; \boldsymbol{\theta}'); \boldsymbol{\theta}\right), \mathbf{x}\right).$$

By construction,

$$\begin{aligned} m'(L(\mathbf{e}; \boldsymbol{\theta}'), \mathbf{x}) &\equiv m\left(L\left(L^{-1}(r; \boldsymbol{\theta}'); \boldsymbol{\theta}\right), \mathbf{x}\right); \quad r = L(\mathbf{e}; \boldsymbol{\theta}') \\ &= m(L(\mathbf{e}; \boldsymbol{\theta}), \mathbf{x}), \end{aligned}$$

for all $(\mathbf{e}, \mathbf{x}) \in \mathbb{R}^{2l+1+K}$. It follows the model

$$y_{i,t} = m'\left(L\left(\bar{\mathbf{y}}_{s,t-l\leq r<t}, \mathbb{E}\left[\mathbf{y}_{t+l\geq r\geq t} | \mathbf{y}_s^r, \mathbf{x}_s^{r-1}, s\right], \boldsymbol{\theta}'\right), \mathbf{x}_{i,t}\right) + \epsilon_{i,t}$$

where $s = s_i$, is observationally equivalent to the original model. $\square$

2.9 Appendix II

Proofs of results employ extra notation. In keeping with convention, $\Gamma(\mathbf{a})$, $\Gamma(\mathbf{a}, \mathbf{p})$ and $\Gamma(\mathbf{a}, \mathbf{p}, \mathbf{b})$ denotes the subgames initiated by investments equaling $\mathbf{a}$, output prices $\mathbf{p}$, and bids $\mathbf{b}$. For clarity, I also use $c_i(\mathbf{s}, \mathbf{p})$ in place of $\sigma_i(\mathbf{s}, \mathbf{p})$ (as in the main text) to denote consumer i's decision, when the manufacturers' supplier-or-quit choices and prices are given by $(\mathbf{s}, \mathbf{p})$. Hence, $\mathbf{c}(\mathbf{s}, \mathbf{p})$ denotes the family of consumption choices $\{c_i(\mathbf{s}, \mathbf{p}) : 0 \leq i \leq 1\}$. Last, for brevity, the manufacturers' Stage 5 choices, "Operate" and "Shut Down" are abbreviated by Op and SD.

Also, certain actions are termed incentive compatible in the proofs. Incentive compatibility acquires subtly different meanings in differing contexts. In this paper, a player's action is incentive compatible iff no profitable alternative action exist. So for example, $c_i(\mathbf{s}, \mathbf{p}) = 1$ is IC (in $\Gamma(\mathbf{a}, \mathbf{p}, \mathbf{b}, \mathbf{s})$) only if consumer i cannot increase his payoff by shopping from a different manufacturer *or* consuming the outside option. More colloquially, demand functions $(q_m(s_{-m}, \mathbf{p}))_{m\leq 2}$ are termed as "incentive compatible" only if consumption rules $\mathbf{c}(\mathbf{s}, \mathbf{p})$ used to define them via (2.2) are likewise IC.

Finally let $\Delta C(q, Q) \equiv \frac{C(q+Q)-C(Q)}{q}$ be the marginal cost of going from Q to $Q + q$ output units of input. Notice $\Delta C(q, Q) < AC(q)$ by strict concavity of $C(Q)$.

2.9.1 Proofs for Results under Assumption 2.2.1

In what follows, $*$ is used to distinguish equilibrium variable values from ordinary values. So equilibrium prices notated by $\mathbf{p}$ in the main text are denoted by $\mathbf{p}^*$ in the proofs. Similarly, equilibrium path quantities $\mathbf{q}(\mathbf{p}^*) = \mathbf{q}(\mathbf{s}(\mathbf{p}^*, \mathbf{b}(\mathbf{p}^*)), \mathbf{p}^*)$ when prices equal $\mathbf{p}^*$ are denoted by $\mathbf{q}^*$.

The following lemmas apply when $\mathbf{a} \neq (1, 1)$ (when the relationship network is disconnected).

Lemma 2.9.1. *Suppose Assumption 2.2.1 holds. When $v > AC(1)$, both manufacturers' output* *prices are*

$$p_1 = p_2 = AC(1)$$

while their market share satisfy

$$\mathbf{q} \in \{(1,0),(0,1)\}$$

in the Nash equilibria of the subgame initiated by $\mathbf{a}$ *when* $\mathbf{a} \neq \mathbf{I}$ *in Stage 2.*

Proof of Lemma 2.9.1. This is proved via a sequence of sub claims.

Claim: $v \geq p^*_m$ for at least one $m \leq 2$.

Proof. Assume otherwise. Neither manufacturer attracts customers in equilibrium (So $\mathbf{q}^* = 0$.). So neither would make a profit. Either manufacturer $m \leq 2$ can strictly increase its payoff by

$$v - AC(1) > 0$$

by defecting to $p^*_m = v$. $\square$

Claim: $\mathbf{p}^* >> AC(1)(1,1)$ implies $p^*_1 = p^*_2$.

Proof. Without loss of generality (WLOG), assume $p^*_1 > p^*_2 > AC(1)$. Then the previous claim implies $v \geq p^*_2$. Then manufacturer 1 can strictly increase its payoff by

$$\beta_1(p^*_2 - AC(1) - \epsilon) > 0$$

by defecting to a new price $p_1 = p^*_2 - \epsilon$ for $\epsilon > 0$ sufficiently small. $\square$

Claim: $\mathbf{p}^* \geq AC(1)(1,1)$ implies $p^*_m = AC(1)$ for some $m \leq 2$.

Proof. Observe if $\mathbf{p}^* >> AC(1)(1,1)$, then $v \geq p^*_1 = p^*_2$ by prior sub claims. But then each manufacturer m can strictly increase its payoff by

$$\beta_m(p^* - AC(1) - \epsilon) > 0$$

by defecting to $p_m = p^* - \epsilon$ (where p^* obviously notates the common value of $\mathbf{p}^*$), for $\epsilon > 0$ sufficiently small. So we must have that $p^*_m = AC(1)$ for some $m \leq 2$. $\square$

Claim: $\mathbf{p}^* \geq AC(1)(1,1)$ implies $p^*_m = AC(1)$ for both $m \leq 2$.

Proof. By prior claims, one can assume $p^*_1 \wedge v \geq p^*_2 = AC(1)$ WLOG. Observe $p_1 > p_2$ implies manufacturer 2 can strictly improve profits by

$$\beta_2(p^*_1 \wedge v - p^*_2 - \epsilon) > 0$$

by increasing its price by that same amount, for ϵ sufficiently small. $\square$

Notice that $p_1 = p_2 = AC(1)$, $\mathbf{q}^* \in \{(1,0),(0,1)\}$ is clearly an NE outcome - neither manufacturer has a profitable deviation, and consumers are indifferent between either manufacturer when prices are equal. I proceed to show that it is the only NE in $\Gamma(\mathbf{a})$.

Claim: $p^*_1 = p^*_2 = AC(1)$ implies $\mathbf{q}^* \in \{(1,0),(0,1)\}$.

Proof. Assume otherwise. Then there exists some manufacturer $m \leq 2$ such that

$$q_m \in (0,1) \Rightarrow p^*_m < AC(q_m) \Rightarrow \pi^M_m = \beta_m(p^*_m - AC(q_m)) < 0.$$

Manufacturer m can strictly improve its payoff by increasing p_m so that it effectively shuts down. $\square$

Claim: $p^*_1 = p^*_2 = AC(1)$ is the unique NE in $\Gamma(\mathbf{a})$.

Proof. From a previous claim,

$$\mathbf{p}^* \geq AC(1)(1,1) \Rightarrow p^*_1 = p^*_2 = AC(1).$$

So if an alternate $\mathbf{p}^*$ exists, there exists some manufacturer $m \leq 2$ whose price satisfies $p_m^* < AC(1) \leq AC(q_m)$. But that would imply input price t_m is undefined. So

$$s_m(\mathbf{p}^*) = \text{Shut Down}, \quad p_{-m}^* = v$$

by profit maximization by its supplier and rival. But this is not an NE since manufacturer m can strictly increase its profit by $\beta_m(v - AC(1) - \epsilon)$ by defecting to $p_m = v - \epsilon$, for $\epsilon > 0$ sufficiently small. $\square$

$\square$

As an aside, observe the statement "$p_1 = p_2 = AC(1)$, $\mathbf{q}^* \in \{(1,0),(0,1)\}$ is an NE outcome" in the proof implies an NE exists. Hence, Lemma 2.9.1 is not vacuous. Also, Lemma 2.9.1 implies firms make no profit in $\Gamma(\mathbf{a})$'s NE when $\mathbf{a} \neq \mathbf{I}$ - $\pi_m^M = \pi_s^S = 0\ \forall (m,s) \in \mathbb{N}_{\leq 2} \times \mathbb{N}_{\leq 2}$.

Proof of Lemma 2.2.1. To see why, first suppose $v \leq AC(1)$. Then a maximum of only one manufacturer operates since

$$0 < q_m(s_{-m}(\mathbf{p}), \mathbf{p}) < 1 \Rightarrow p_m \leq v \leq AC(1) < AC(q_m(s_{-m}(\mathbf{p}), \mathbf{p})) \Rightarrow s_m = \text{Shut Down}.$$

But then $AC(1) < p_{-m} \leq v \leq AC(1)$ must occur for the operating manufacturer $-m$ to make any profit. Next, assume $AC(1) < v$. Then Lemma 2.9.1 implies the result. $\square$

The following lemmas apply when $\mathbf{a} \neq (1,1)$, or when the relationship network is disconnected.

Lemma 2.9.2. *Suppose $AC(1) < p_2$. Consider the game played by suppliers when they bid, defined by the subgame $\Gamma(\mathbf{I}, \mathbf{p})$, and incentive compatible truncated strategies $\boldsymbol{\sigma}|_{\mathbf{I},\mathbf{p},\mathbf{b}} = (\mathbf{s}(\mathbf{p},\mathbf{b}), \mathbf{c}(\mathbf{s},\mathbf{p}))$ satisfying $q_2(s_1, \mathbf{p}) = 1$ for all $s_1 \in \{Op, SD\}$. Both suppliers bid $b_s = AC(1)$ for $s \leq 2$ in its NE.*

Proof. (As an aside, because $\mathbf{c}(\mathbf{s},\mathbf{p})$ is IC, we know $p_2 \leq v \wedge p_1$.) Observe $q_2(s_1,\mathbf{p}) = 1$ implies manufacturer 1 has zero market share while manufacturer 2's output is 1, if the latter operates. Moreover, manufacturer 2's variable profit is proportional to $p_2 - \{s_2 = 1\}b_1 - \{s_2 = 2\}b_2$ when it operates. So incentive compatibility implies its choice of which supplier and whether to operate satisfies

$$s_2(\mathbf{p},\mathbf{b}) = \begin{cases} 1 & \text{if } b_1 < b_2 \wedge p_2 \\ 2 & \text{if } b_2 < b_1 \wedge p_2 \\ SD & \text{if } b_2 \wedge b_1 > p_2 \end{cases} . \tag{2.24}$$

Sub Claim: $b_1^* = b_2^* = AC(1)$ is an NE.

Proof. Observe $AC(1) < p_2$ implies $s_2(\mathbf{p},\mathbf{b}) \neq$ SD by incentive compatibility (2.24). So $\mathbf{q}(\mathbf{s}(\mathbf{p},\mathbf{b}^*),\mathbf{p}) = (0,1)$ by $q_2(s_1,\mathbf{p}) = 1$ being hypothesized. Supplier 1's variable profit is thus zero since manufacturer 1 has zero market share while manufacturer 2 operates:

$$\begin{aligned}
\pi_1^S &= \{s_2(\mathbf{p},\mathbf{b}^*) = 1\}q_2(\mathbf{s}(\mathbf{p},\mathbf{b}^*),\mathbf{p})(b_1^* - AC(q_2(\mathbf{s}(\mathbf{p},\mathbf{b}^*),\mathbf{p}))) + T_{21} - F \quad (\text{by } q_1(\mathbf{s}(\mathbf{p},\mathbf{b}^*),\mathbf{p}) = 0) \\
&= \{s_2(\mathbf{p},\mathbf{b}^*) = 1\}(b_1^* - AC(1)) + T_{21} - F \quad (\text{by } q_2(s_1,\mathbf{p}) = 1) \\
&= T_{21} - F \quad (\text{by } b_1^* = AC(1)).
\end{aligned}$$

If supplier 1 bids $b_1 > b_1^*$, it loses the auction by (2.24) and still makes zero variable profit since $b_2^* < p_2$ means manufacturer 2 still operates by (2.24), and $q_2(s_1,\mathbf{p}) = 1$ implies $q_1(\mathbf{s}(\mathbf{p},b_1,b_2^*),\mathbf{p}) = 0$. If supplier 1 defects to $b_1 < b_1^*$, it wins the auction by (2.24), but its payoff is now strictly decreased:

$$\begin{aligned}
\pi_1^S &= \{s_2(\mathbf{p},b_1,b_2^*) = 1\}q_2(\mathbf{s}(\mathbf{p},b_1,b_2^*),\mathbf{p})(b_1 - AC(q_2(\mathbf{s}(\mathbf{p},b_1,b_2^*),\mathbf{p}))) + T_{21} - F \quad (\text{by } b_2^* \leq p_2) \\
&= \{s_2(\mathbf{p},b_1,b_2^*) = 1\}(b_1 - AC(1)) + T_{21} - F \quad (\text{by } q_2(s_1,\mathbf{p}) = 1) \\
&= b_1 - AC(1) + T_{21} - F \quad (\text{by (2.24)}) \\
&< T_{21} - F.
\end{aligned}$$

Supplier 2's profit is likewise zero:

$$\pi_2^S = \{s_2(\mathbf{p}, \mathbf{b}^*) = 2\} q_2(\mathbf{s}(\mathbf{p}, \mathbf{b}^*), \mathbf{p})(b_2^* - AC(q_2(\mathbf{s}(\mathbf{p}, \mathbf{b}^*), \mathbf{p})))$$
$$= 0 \quad \text{(by } q_2(s_1, \mathbf{p}) = 1 \text{ and } b_2^* = AC(1)\text{)}.$$

Bidding $b_2 > b_2^*$ implies it loses the auction via (2.24), and makes no profit. Defecting to $b_2 < b_2^*$ implies it wins the auction by (2.24), but its payoff is now strictly decreased:

$$\pi_2^S = \{s_2(\mathbf{p}, b_1^*, b_2) = 2\} q_2(\mathbf{s}(\mathbf{p}, b_1^*, b_2), \mathbf{p})(b_2 - AC(q_2(\mathbf{s}(\mathbf{p}, b_1^*, b_2), \mathbf{p})))$$
$$= b_2 - AC(1) < 0 \quad \text{(by } q_2(s_1, \mathbf{p}) = 1 \text{ and (2.24))}.$$

So neither supplier has a profitable deviation. □

Sub Claim: $b_1^* = b_2^* = AC(1)$ in any NE.

Proof. Suppose $\mathbf{b}^* >> \mathbf{p}$. From (2.24), supplier 2 makes no profit. Under $q_2(s_1, \mathbf{p}) = 1$ and (2.24), it can increase its profit by

$$p - AC(1) > 0$$

by bidding $b_2 = p$. So at least one bid, must lie weakly below manufacturer 2's price in an NE - $b_1^* \wedge b_2^* \leq p_2$.

Suppose $\exists s \leq 2$ such that $b_s^* < AC(1)$. WLOG, assume $b_s^* < b_{-s}^*$. There are two possibilities.

- Suppose $s = 1$. Then (2.24) implies $s_2(\mathbf{p}, \mathbf{b}^*) = 1$. Moreover, $q_2(s_1, \mathbf{p}) = 1$ by hypothesis, implying $\mathbf{q}(\mathbf{s}(\mathbf{p}, \mathbf{b}^*), \mathbf{p}) = (0, 1)$. Hence supplier 1's profit is

$$\pi_1^S = \{s_2(\mathbf{p}, \mathbf{b}^*) = 1\} q_2(\mathbf{s}(\mathbf{p}, \mathbf{b}^*), \mathbf{p})(b_1^* - AC(q_2(\mathbf{s}(\mathbf{p}, \mathbf{b}^*)) + T_{21} - F \quad \text{(by } q_1(\mathbf{s}(\mathbf{p}, \mathbf{b}^*), \mathbf{p}) = 0\text{)}$$
$$= b_1^* - AC(1) + T_{21} - F < T_{21} - F.$$

Supplier 1 can avoid its loss by bidding in excess of p_2:

$$
\begin{aligned}
b_1 > p_2 \Rightarrow & \; s_2(\mathbf{p}, b_1, b_2^*) \neq 1 \quad \text{(via (2.24))} \\
& \Rightarrow \pi_1^S = \{s_1(\mathbf{p}, b_1, b_2^*) = Op\}(1-\beta_1)q_1(\mathbf{s}(\mathbf{p}, b_1, b_2^*), \mathbf{p})(p_1 - \Delta C(\mathbf{q}(\mathbf{s}(\mathbf{p}, b_1, b_2^*), \mathbf{p}))) \\
& \qquad + \{s_2(\mathbf{p}, b_1, b_2^*) = 1\}q_2(\mathbf{s}(\mathbf{p}, b_1, b_2^*), \mathbf{p})(b_1 - AC(q_2(\mathbf{s}(\mathbf{p}, b_1, b_2^*), \mathbf{p}))) + T_{21} - F \\
& \quad = (1-\beta_1)q_1(\mathbf{s}(\mathbf{p}, b_1, b_2^*), \mathbf{p})(p_1 - AC(q_1(\mathbf{s}(\mathbf{p}, b_1, b_2^*), \mathbf{p}))) \vee 0 \\
& \qquad + \{s_2(\mathbf{p}, b_1, b_2^*) = 1\}(b_1 - AC(1)) + T_{21} - F \\
& \quad \geq T_{21} - F.
\end{aligned}
$$

Note the second equality holds because $s_1(\mathbf{p}, \mathbf{b})$ is IC and $s_2(\mathbf{p}, b_1, b_2^*) = 1$ implies $q_2(\mathbf{s}(\mathbf{p}, b_1, b_2^*), \mathbf{p}) = q_2(s_1(\mathbf{p}, b_1, b_2^*), \mathbf{p}) = 1$.

- Suppose $s = 2$. Then (2.24) implies $s_2(\mathbf{p}, \mathbf{b}^*) = 2$. Moreover, the hypothesis still implies $\mathbf{q}(\mathbf{s}(\mathbf{p}, \mathbf{b}^*), \mathbf{p}) = (0, 1)$. Hence supplier 2's profit is

$$
\pi_2^S = \{s_2(\mathbf{p}, \mathbf{b}^*) = 2\}q_2(\mathbf{s}(\mathbf{p}, \mathbf{b}^*), \mathbf{p})(b_2^* - AC(q_2(\mathbf{s}(\mathbf{p}, \mathbf{b}^*), \mathbf{p}))) = b_2^* - AC(1) < 0.
$$

Supplier 2 can avoid its loss by bidding in excess of p_2:

$$
\begin{aligned}
b_2 > p_2 \Rightarrow & \; s_2(\mathbf{p}, b_1^*, b_2) \neq 2 \quad \text{(via (2.24))} \\
& \Rightarrow \pi_2^S = \{s_2(\mathbf{p}, b_1^*, b_2) = 2\}q_2(\mathbf{s}(\mathbf{p}, b_1^*, b_2), \mathbf{p})(b_2 - AC(q_2(\mathbf{s}(\mathbf{p}, b_1^*, b_2), \mathbf{p}))) = 0.
\end{aligned}
$$

It follows $\mathbf{b}^* \geq AC(1)$.

Observe if $b_2^* > p_2$, (2.24) implies $s_2(\mathbf{p}, \mathbf{b}^*) \neq 2$ and hence supplier 2 makes no profit. By defecting to $b_2 = b_1^* \wedge p_2 - \epsilon$ where $\epsilon > 0$, supplier 2 can assure itself of victory in the auction by (2.24). Its profit is thus

$$\begin{aligned}\pi_2^S &= q_2(\mathbf{s}(\mathbf{p}, b_1^*, b_2), \mathbf{p})(b_2 - AC(q_2(\mathbf{s}(\mathbf{p}, b_1^*, b_2), \mathbf{p}))) \\ &= b_2 - AC(1) \qquad (\text{by } q_2(s_1, \mathbf{p}) = 1) \\ &= b_1^* \wedge p_2 - AC(1) - \epsilon > 0\end{aligned}$$

for ϵ sufficiently small, unless $b_1^* = AC(1)$. But if $b_1^* \leq p_2 < b_2^*$, then since supplier 1 wins the auction by (2.24), its profit is

$$\begin{aligned}\pi_1^S = &\{s_1(\mathbf{p}, \mathbf{b}^*) = Op\}(1-\beta_1)q_1(\mathbf{s}(\mathbf{p}, \mathbf{b}^*), \mathbf{p})(p_1 - \Delta C(\mathbf{q}(\mathbf{s}(\mathbf{p}, \mathbf{b}^*), \mathbf{p}))) \\ &+ \{s_2(\mathbf{p}, \mathbf{b}^*) = 1\}q_2(\mathbf{s}(\mathbf{p}, \mathbf{b}^*), \mathbf{p})(b_1^* - AC(q_2(\mathbf{s}(\mathbf{p}, \mathbf{b}^*), \mathbf{p}))) + T_{21} - F \\ = &\ b_1^* - AC(1) + T_{21} - F \qquad (\text{by (2.24) and } q_2(s_1, \mathbf{p}) = 1).\end{aligned}$$

Bidding $AC(1)$ is strictly dominated by bidding p_2, implying $\mathbf{b}^*$ are not best responses. The contradiction implies $b_2^* \leq p_2$. Hence, for any $b_1 \geq 0$,

$$s_2(\mathbf{p}, b_1, b_2^*) \neq SD \quad \mathbf{q}(\mathbf{s}(\mathbf{p}, b_1, b_2^*), \mathbf{p}) = (0, 1). \tag{2.25}$$

The final equality holds by $q_2(s_1, \mathbf{p}) = 1$ being assumed in the hypothesis.

Suppose $b_1^* > p_2$ instead. Then (2.24) implies $s_2(\mathbf{p}, \mathbf{b}^*) \neq 1$. Moreover, (2.25) implies supplier 1 doesn't supply manufacturer 1 either. Supplier 1's variable profit is thus zero

$$\pi_1^S = T_{21} - F.$$

Observe by defecting to $b_1 = p_2 \wedge b_2^* - \epsilon$ where $\epsilon > 0$, (2.24) implies it wins the auction. Moreover, (2.25) implies its profit is increased

$$\pi_1^S = p_2 \wedge b_2^* - AC(1) - \epsilon + T_{21} - F > T_{21} - F,$$

for ϵ sufficiently small, unless $b_2^* = AC(1)$. But if $b_2^* \leq p_2 < b_1^*$, then since supplier 2 wins the auction by (2.24), its profit is

$$\pi_2^S = b_2^* - AC(1) \qquad \text{(by (2.24) and } q_2(s_1, \mathbf{p}) = 1).$$

Bidding $AC(1)$ is strictly dominated by bidding p_2 for supplier 2, implying $\mathbf{b}^*$ are not best responses. The contradiction implies $b_1^* \leq p_2$.

It follows $\mathbf{b}^* \in [AC(1), p_2]^2$. Hence (2.24) implies $s_2(\mathbf{p}, b_1, b_2^*) \neq SD$ and $s_2(\mathbf{p}, b_1^*, b_2) \neq SD$. It follows

$$\mathbf{q}(\mathbf{s}(\mathbf{p}, b_1, b_2^*), \mathbf{p}) = \mathbf{q}(\mathbf{s}(\mathbf{p}, b_1^*, b_2), \mathbf{p}) = (0, 1). \tag{2.26}$$

Hence for either supplier $s \leq 2$, its profit equals

$$\pi_s^S(b_s, b_{-s}^*) = \{s_2(\mathbf{p}, b_s, b_{-s}^*) = s\}(b_s - AC(1)) + \{s = 1\}(T_{21} - F),$$

as a function of its own bid when its rival submits the equilibrium bid. We thus have the following.

- Suppose $b_1^* > AC(1)$. If $s_2(\mathbf{p}, \mathbf{b}^*) = 1$, then $\pi_2^S(b_2, b_1^*)$ and (2.24) implies supplier 2's best response to b_1^* is empty. If $s_2(\mathbf{p}, \mathbf{b}^*) = 2$, then supplier 2's best response is to set $b_2 = b_1^* > AC(1)$. Hence $b_2^* = b_1^* > AC(1)$. But then $\pi_1^S(b_1, b_2^*)$ and (2.24) implies supplier 1's best response to b_2^* is empty.
- Suppose $b_2^* > AC(1)$. A symmetrical argument shows this produces a contradiction.

It follows $b_1^* = b_2^* = AC(1)$. □

□

Notice in the bidding game induced by $\Gamma(\mathbf{I}, \mathbf{p})$ and $\boldsymbol{\sigma}|_{\mathbf{I},\mathbf{p},\mathbf{b}} = (\mathbf{s}(\mathbf{p}, \mathbf{b}), \mathbf{c}(\mathbf{s}, \mathbf{p}))$, prices $\mathbf{p}$ are fixed. Hence restricting $q_2(s_1, \mathbf{p}) = 1$ for all $s_1 \in \{Op, SD\}$ has the same effect on the suppliers' payoffs as assuming $q_2(s_1(\mathbf{p}, \mathbf{b}), \mathbf{p})$ is constantly equal to 1 for all $\mathbf{b} \geq \mathbf{0}$. This implies the following result.

Corrollary 2.9.1. *Suppose $AC(1) < p_2$. Consider the game played by suppliers when they bid, defined by the sub game $\Gamma(\mathbf{I}, \mathbf{p})$, and incentive compatible truncated strategies $\boldsymbol{\sigma}|_{\mathbf{I},\mathbf{p},\mathbf{b}} = (\mathbf{s}(\mathbf{p}, \mathbf{b}), \mathbf{c}(\mathbf{s}, \mathbf{p}))$ satisfying $q_2(s_1(\mathbf{p}, \mathbf{b}), \mathbf{p}) = 1$ for all $\mathbf{b} \geq \mathbf{0}$. Both suppliers bid $b_s = AC(1)$ for $s \leq 2$ in its NE.*

It also follows from Lemma 2.9.2 that neither manufacturer makes any variable profit when $\mathbf{a} = \mathbf{I}$.

Lemma 2.9.3. *Suppose Assumption 2.2.1 holds. If $v > AC(1)$, the firms' output prices satisfy*

$$\mathbf{p} \leq (AC(1), AC(1))$$

while no firm makes any variable profit

$$\pi_1^M = 0, \;\; \pi_2^M = -T_{21}, \;\; \pi_1^S = T_{21} - F, \;\; \pi_2^S = 0$$

in any Nash equilibrium of the subgame initiated when $\mathbf{a} = \mathbf{I}$ in Stage 2.

Proof of Lemma 2.9.3. The lemma is proved in a sequence of claims. The first two verify manufacturer m's stage 3 profit, as a function of prices $\mathbf{p}$ and the truncated players' strategies $\boldsymbol{\sigma}_{\mathbf{p},\mathbf{I}}$, satisfy

$$\{m = 2\}T_{21} + \pi_m^M(\mathbf{p}) = \begin{cases} (\{m = 1\}\beta_1 + \{m = 2\})\,(p_m - AC(1)) \vee 0 & \text{if } p_m \leq v \\ 0 & \text{if otherwise,} \end{cases} \tag{2.27}$$

for two configurations of $\mathbf{p}$. The reader should feel free to check how the 1st two claims prove the remaining claims before returning to the proofs of the original two claims.

Claim: For any $m \leq 2$, $p^*_{-m} > v$ implies manufacturer m's stage 3 profit as a function of $\mathbf{p}$ and $\boldsymbol{\sigma}|_{\mathbf{p},\mathbf{I}}$ satisfies (2.27), implying its best response is $p_m = v$.

Proof. **Sub Claim:** Claim true when $m = 1$.

Proof. Observe

$$
\begin{aligned}
p_2 > v \ &\Rightarrow q_2(\mathbf{s}, \mathbf{p}) = 0 \quad \forall \mathbf{s} \in \{\text{Op}, \text{SD}\} \times \{\text{SD}, 1, 2\} \\
&\Rightarrow q_1(s_2, \mathbf{p}) = \{p_1 \leq v\} \\
&\Rightarrow \pi_1^M(\mathbf{p}) = \begin{cases} \beta_1(p_1 - AC(1)) \vee 0 & \text{if } p_1 \leq v \\ 0 & \text{if otherwise.} \end{cases}
\end{aligned}
$$

Compare expressions in the final rows displayed above with those in the lemma to complete the proof. ∎

Sub Claim: Claim true when $m = 2$.

Proof. Observe

$$
\begin{aligned}
p_1 > v \Rightarrow \ & q_1(\mathbf{s}, \mathbf{p}) = 0 \quad \forall \mathbf{s} \in \{\text{Op}, \text{SD}\} \times \{\text{SD}, 1, 2\} \\
\Rightarrow \ & q_2(s_1, \mathbf{p}) = \{p_2 \leq v\}.
\end{aligned}
$$

So $\pi_2^M(\mathbf{p}) = -T_{21}$ when $p_2 > v$. When $AC(1) < p_2 \leq v$, Lemma 2.9.2 implies $\mathbf{b}(\mathbf{p}) = AC(1)(1, 1)$. Finally, when $p_2 \leq AC(1) < v$, observe each supplier's profit is

$$
b_s - AC(1) + \{s = 1\}(T_{21} - F) < \{s = 1\}(T_{21} - F)
$$

from bidding any bid $b_s < p_2 \wedge b_{-s}$ that wins it the auction. Hence,

$$
\pi_2^M(\mathbf{p}) = \begin{cases} (p_2 - AC(1)) \vee 0 - T_{21} & \text{if } p_2 \leq v \\ -T_{21} & \text{if otherwise.} \end{cases}
$$

Compare expressions in the final rows above with those in the lemma to complete the proof. $\square$

Verify that $p_m = v$ maximizes $\pi_m^M(\mathbf{p})$ to complete the proof. $\square$

Claim: For any $m \leq 2$, $p_m < p_{-m}$ implies manufacturer m's Stage 3 profit as a function of $\mathbf{p}$ and $\boldsymbol{\sigma}|_{\mathbf{p},\mathbf{I}}$ satisfies (2.27).

Proof. **Sub Claim:** Claim true when $m = 1$.

Proof. Utility maximization by consumers imply

$$\begin{aligned} p_1 < p_2 &\Rightarrow q_2(\mathrm{Op}, \mathbf{p}) = q_2(\mathrm{Op}, s_2, \mathbf{p}) = 0 \\ &\Rightarrow q_1(s_2, \mathbf{p}) = \{p_1 \leq v\} \quad \forall s_2 \in \{\mathrm{SD}, 1, 2\}. \end{aligned}$$

It follows $\pi_1^M(\mathbf{p}) = \beta_1\{p_1 \leq v\}(p_1 - AC(1)) \vee 0$. $\square$

Sub Claim: Claim true when $m = 2$.

Proof. Utility maximization by consumers imply

$$\begin{aligned} p_2 < p_1 &\Rightarrow q_1(\mathrm{Operate}, \mathbf{p}) = q_1(\mathrm{Op}, s_2, \mathbf{p}) = 0 \\ &\Rightarrow q_2(s_1, \mathbf{p}) = \{p_2 \leq v\} \end{aligned}$$

for all $s_1 \in \{\mathrm{Op}, \mathrm{SD}\}$. It follows $\pi_2^M(\mathbf{p}) = -T_{21}$ when $p_2 > v$, partly verifying the claim. Suppose $AC(1) < p_2 \leq v$. Then Lemma 2.9.2 implies $\mathbf{b}(\mathbf{p}) = AC(1)(1,1)$. Now suppose $p_2 \leq AC(1)$ instead. Either supplier incurs a loss in payoff equal to

$$b_s - AC(1) \leq p_2 - AC(1) < 0,$$

from bidding $b_s < p_2 \wedge b_{-s}$. So

$$p_2 \leq AC(1) \Rightarrow b_1(\mathbf{p}) \wedge b_2(\mathbf{p}) \geq p_2. \tag{2.28}$$

It follows $\pi_2^M(\mathbf{p})$ equals $(p_2 - AC(1)) \vee 0 - T_{21}$ if $p_2 \leq v$ from (2.28). $\square$

$\square$

Claim: $p_m^* \leq v$ for $m \leq 2$.

Proof. Assume otherwise. Then either i) $p_1^* > v$ or ii) $p_2^* > v$ must occur. Suppose i) holds. Then

$$p_1^* > v \Rightarrow p_2^* = v > AC(1)$$

since manufacturer 2 maximizes its profit at $p_2 = v$ when $p_1 > v$. Then manufacturer 1 can strictly increase its profit by

$$\beta_1(p_2^* - AC(1) - \epsilon) > 0$$

for $\epsilon > 0$ sufficiently small, by defecting to $p_1 = p_2^* - \epsilon$. Suppose ii) holds. Then

$$p_2^* > v \Rightarrow p_1^* = v > AC(1)$$

since manufacturer 1 maximizes its profit at $p_1 = v$ when $p_2 > v$. Manufacturer 2 can then strictly increase its profit by

$$p_1^* - AC(1) - \epsilon > 0$$

by defecting to $p_2 = p_1^* - \epsilon$ for $\epsilon > 0$ sufficiently small. $\square$

Claim: Suppose $p_m^* > AC(1)$ for some $m \leq 2$. Then $p_1^* = p_2^* = p^*$.

Proof. Assume otherwise. Then either i) $AC(1) < p_m^* < p_{-m}^*$ or ii) $AC(1) \wedge p_{-m}^* < p_m^*$. Suppose i) holds. Previous claim implies $p_{-m}^* \leq v$. So manufacturer m can strictly increase its profit by

$$(\{m=1\}\beta_1 + \{m=2\}) \left(p_{-m}^* - AC(1) - \epsilon\right) > 0$$

for $\epsilon > 0$ sufficiently small, by defecting to $p_m = p_{-m}^* - \epsilon$. Suppose ii) holds instead. Then $-m$

can strictly increase its profit by

$$(\{m=2\}\beta_1 + \{m=1\})\,(p_m^* - AC(1) - \epsilon) > 0$$

for $\epsilon > 0$ sufficiently small, by defecting to $p_{-m} = p_m^* - \epsilon$. □

Claim: Suppose $p_m^* \geq AC(1)$ for some $m \leq 2$. Then $p_m^* = AC(1)$.

Proof. Assume otherwise. Then from the previous claim,

$$p_m^* > AC(1) \Rightarrow p_m^* = p_{-m}^* = p^* > AC(1).$$

Observe p^* in the expression above, denotes the common value of $\mathbf{p}^*$ implied by the previous claim. From the 3rd claim in this lemma's proof, $AC(1) < p^* \leq v$. Observes manufacturer 2 can attain at least $p^* - AC(1) - \epsilon - T_{21}$ by defecting to $p_2 = p^* - \epsilon$. So on the equilibrium path,

$$\pi_2^M \geq p^* - AC(1) - \epsilon - T_{21} \ \forall \epsilon > 0.$$

Sending $\epsilon \to 0$ obtains a lower bound for manufacturer 2's payoff, $\pi_2^M \geq p^* - AC(1) - T_{21}$, in equilibrium. Observe also:

i) Manufacturer 1 can attain at least 0 from defecting to $p_1 > v$, so $\pi_1^M \geq 0$ in equilibrium.

ii) Supplier 1 can attain strictly more than $T_{21} - F$ from defecting to $b_1 > p_2$ - its payoff from such a defection satisfies

$$\pi_1^S \geq (1 - \beta_1)(p^* - AC(1)) + T_{21} - F > T_{21} - F.$$

So $\pi_1^S > T_{21} - F$ in equilibrium.

iii) Supplier 2 can attain at least 0 from defecting to $b_2 > p_2$, so $\pi_2^S \geq 0$ in equilibrium.

Finally, the social planner's attained surplus implies total surplus generated by equilibrium play is bounded from above by

$$TS \le v - C(1) - F = v - AC(1) - F.$$

Also, utility maximization implies $CS = v - p^*$. Hence,

$$\pi_1^M + \pi_2^M + \pi_1^S + \pi_2^S = TS - CS \le v - AC(1) - F - (v - p^*) = p^* - AC(1) - F.$$

But the lower bounds in i) to iii) and for π_2^M imply

$$p^* - AC(1) - T_{21} \le \pi_2^M < p^* - AC(1) - F - T_{21} + F = p^* - AC(1) - T_{21},$$

a contradiction. So we must have $p_m^* = AC(1)$. □

Claim: $q_m^* \in (0,1) \Rightarrow q_{-m}^* = 1 - q_m^* > 0$ for $m \le 2$.

Proof. As stated in Section 2.2, consumers consume when indifferent between m's product and its outside option in equilibrium. □

Claim: $\pi_1^M \ge 0,\ \pi_2^M \ge -T_{21},\ \pi_1^S \ge T_2 - F,\ \pi_2^S \ge 0$ on the equilibrium path.

Proof. Manufacturer $m \le 2$ can guarantee itself a payoff of $-\{m = 2\}T_{21}$ from setting p_m to any number strictly bigger than v. Thus

$$\pi_1^M \ge 0 \quad \pi_2^M \ge -T_{21}.$$

Because $s_1(\mathbf{p}, \mathbf{b})$ is IC, supplier 1 can guarantee itself a payoff of

$$(1-\beta_1)q_1(\mathbf{s}(\mathbf{p}^*, b_1, b_2^*), \mathbf{p}^*)(p_1^* - AC(q_1(\mathbf{s}(\mathbf{p}^*, b_1, b_2^*), \mathbf{p}^*))) \vee 0 + T_{21} - F \ge T_{21} - F$$

by submitting any bid b_1 strictly bigger than p_2^*. Thus $\pi_1^S \geq T_{21} - F$. Supplier 2 can guarantee itself a payoff equal to 0 by submitting $b_2 > p_2^*$. Thus $\pi_2^S \geq 0$. $\square$

Claim: $\pi_1^M = 0$, $\pi_2^M = -T_{21}$, $\pi_1^S = T_{21} - F$, $\pi_2^S = 0$ on the equilibrium path when $\mathbf{p}^* \leq AC(1)(1,1)$.

Proof. Observe $\mathbf{p}^* \leq AC(1)(1,1)$ implies consumer surplus satisfies

$$CS = v - p_1^* \wedge p_2^* \geq v - AC(1) \geq 0.$$

The planner's allocation when $\mathbf{a} = \mathbf{I}$ implies $TS \leq v - AC(1) - F$. Hence

$$\pi_1^M + \pi_2^M + \pi_1^S + \pi_2^S = TS - CS \leq -F.$$

But from the previous claim,

$$\pi_1^M \geq 0 \quad \pi_2^S \geq 0$$
$$\Rightarrow \pi_2^M + \pi_1^S \leq -F - \pi_1^M - \pi_2^S \leq -F.$$

But from the previous claim,

$$\pi_2^M \geq -T_{21},\ \pi_1^S \geq T_{21} - F \ \Rightarrow \pi_2^M + \pi_1^S \geq -F.$$

Combining both inequalities thus yields $\pi_2^M + \pi_1^S = -F$. But from the previous claim again, and the equality above,

$$\pi_2^M \geq -T_{21},\ T_{21} - F \leq \pi_1^S = -F - \pi_2^M \leq T_{21} - F,$$
$$\Rightarrow \ \pi_1^S = T_{21} - F,\ \pi_2^M = -T_{21}.$$

Likewise,

$$\pi_1^M + \pi_2^S \leq TS - CS - \pi_2^M - \pi_1^S \leq F - (-F) = 0.$$

But the previous claim implies both π_1^M and π_2^S are non-negative. $\square$

The lemma holds from combining the fifth and final claims of its proof. $\square$

Proof of Lemma 2.2.2. Suppose $v \leq AC(1)$. Then because total surplus under the social planner's allocation is $-F$ when $\mathbf{a} = \mathbf{I}$,

$$\pi_1^M + \pi_2^M + \pi_1^S + \pi_2^S \leq -F \Rightarrow \pi_1^M + \left(\pi_2^M + T_{21}\right) + \left(\pi_1^S + F - T_{21}\right) + \pi_2^S \leq 0.$$

But each manufacturer can ensure itself zero variable profit by setting an unreasonably high output price. Because $\mathbf{s}(\mathbf{p}, \mathbf{b})$ is IC, each supplier can ensure itself at least zero variable profit by submitting an unacceptably high bid. So each of the four terms in the sum above are non-negative. Hence

$$\pi_1^M = \pi_2^S = 0, \quad \pi_2^M = -T_{21}, \quad \pi_1^S = T_{21} - F.$$

Then suppose $v > AC(1)$. Lemma 2.9.3 completes the argument. $\square$

It remains to show the existence of a (subgame perfect) BNE. I claim such a BNE exists when $v > AC\,(1)$, or equivalently, there is surplus to be made from producing inputs and assembling them into output. As aforementioned, Lemma 2.9.1's proof establishes $p_1 = p_2 = AC(1)$, $\mathbf{q}^* \in \{(1,0),(0,1)\}$ as NE outcomes for $\Gamma(\mathbf{a})$ when $\mathbf{a} \neq \mathbf{I}$. Example 2.9.1 is exhibited as an NE outcome, when $\mathbf{a} = \mathbf{I}$.

Example 2.9.1. [Monopoly Equilibrium] If $v > AC(1)$, then there exists subgame perfect equilibria in the subgame initiated when manufacturer 2 and supplier 1 invest, such that output prices and downstream market shares satisfy

$$\mathbf{p} = (AC(1), AC(1)), \quad \mathbf{q} \in \{(1,0),(0,1)\}$$

on the equilibrium path.

The following lemma proves this.

Lemma 2.9.4. *There exists an NE for the subgame* $\Gamma(\mathbf{a})$ *when* $\mathbf{a} = \mathbf{I}$ *and* $AC(1) < v$. *Prices and quantities on its equilibrium path are specified by Example 2.9.1.*

Proof. The lemma is proved in 3 claims, the first of which formalizes the trivial.

Claim: $\Gamma(\mathbf{I}, \mathbf{p}, \mathbf{b})$ admits an SPNE in supplier-or-quit and consumption choices, $(\mathbf{s}(\mathbf{p}, \mathbf{b}),$ $\mathbf{c}(\mathbf{s}, \mathbf{p}))$.

Proof. Each consumer chooses from discrete sets. His payoff is independent of rival consumers' choices. Existence of incentive compatible $\mathbf{c}(\mathbf{s}, \mathbf{p})$ follows trivially $\forall \mathbf{s} \in \{\text{Operate}, \text{Shut Down}\} \times \{\text{Operate}, 1, 2\}$.

Set manufacturer 2's supplier-or-quit choice to

$$s_2(\mathbf{p}, \mathbf{b}) \in \begin{cases} \text{argmin}_s b_s & \text{if } b_1 \wedge b_2 \leq p_2 \\ \text{Shut Down} & \text{if otherwise.} \end{cases}$$

Observe $s_2(\mathbf{p}, \mathbf{b})$ is incentive compatible. Observe manufacturer 1's BR to $s_2(\mathbf{p}, \mathbf{b})$ is chosen from a discrete set and is hence non-empty. The BR is trivially incentive compatible too. $\square$

The next two claims are technical and have to do with the following identities that define supplier-or-quit choices

$$s_1(\mathbf{p}, \mathbf{b}) = \begin{cases} \text{Operate} & \text{if } p \geq AC(1) \\ \text{Shut Down} & \text{if otherwise} \end{cases} \quad \forall (\mathbf{p}, \mathbf{b}) \in \mathbb{R}^2_{\geq 0} \tag{2.29}$$

$$s_2(\mathbf{p},\mathbf{b}) = \begin{cases} \text{argmin}_s b_s & \text{if } b_1 \wedge b_2 \leq p_2, b_1 \neq b_2 \\ 1 & \text{if } b_1 \wedge b_2 \leq p_2, b_1 = b_2 \\ \text{Shut Down} & \text{if otherwise} \end{cases} \quad \forall (\mathbf{p},\mathbf{b}) \in \mathbb{R}^2_{\geq 0}, \tag{2.30}$$

and the following equalities that define demands when prices are equal

$$\begin{aligned} q_1(s_2,p,p) =& \begin{cases} 0 & \text{if } s_2 = \text{Operate} \\ \{p \leq v\} & \text{if otherwise} \end{cases} \\ q_2(s_2,p,p) =& \{p \leq v\} \quad \forall s_1 \in \{\text{Operate}, \text{Shut Down}\}. \end{aligned} \tag{2.31}$$

$$\begin{aligned} q_1(s_2,p,p) =& \{p \leq v\} \quad \forall s_2 \in \{1, 2, \text{Shut Down}\} \\ q_2(s_1,p,p) =& \begin{cases} 0 & \text{if } s_1 = \text{Operate} \\ \{p \leq v\} & \text{if otherwise.} \end{cases} \end{aligned} \tag{2.32}$$

Finally, the following claims also denote the demand functions $(q_m(s_{-m},\mathbf{p}))_{m\leq 2}$ by $\overline{\mathbf{q}}(\mathbf{s},\mathbf{p})$. Their statements are as follows.

Claims: Fix $\delta > 0$. Fix $\overline{\mathbf{q}} \in \{0,1\} \cap \Delta$ for the second bullet.

- Suppose $p_1 \neq p_2$. $\forall(\exists)$ supplier-or-quit and consumption choices $(\mathbf{s}(\mathbf{p},\mathbf{b}),\mathbf{c}(\mathbf{s},\mathbf{p}))$ forming a SPNE in $\Gamma(\mathbf{I},\mathbf{p},\mathbf{b})$ for all $\mathbf{b} \in \mathbb{R}^2_{\geq 0}$, $\exists$ bids $\mathbf{b}(\mathbf{p})$ satisfying

$$b_1 = b_2 = AC(q_2(s_1(\mathbf{p},b_1,b_2),\mathbf{p})) \wedge (v+\delta) \tag{2.33}$$

  that together with $(\mathbf{s}(\mathbf{p},\mathbf{b}),\mathbf{c}(\mathbf{s},\mathbf{p}))$, form a SPNE in $\Gamma(\mathbf{p},\mathbf{b})$.

- Suppose $p_1 = p_2$. Then $\exists(\forall)$ supplier-or-quit and consumption choices $(\mathbf{s}(\mathbf{p},\mathbf{b}),\mathbf{c}(\mathbf{s},\mathbf{p}))$ (that form a SPNE of $\Gamma(\mathbf{I},\mathbf{p},\mathbf{b})$ while satisfying (2.30) and (2.31) if $\overline{\mathbf{q}} = (0,1)$, and (2.29), (2.30) and (2.32) if $\overline{\mathbf{q}} = (1,0)$ for all $\mathbf{b} \in \mathbb{R}^2_{\geq 0}$), and $\exists$ bids $\mathbf{b}(\mathbf{p})$ such that

$$\begin{aligned} v < p_1 = p_2 &\Rightarrow b_1(\mathbf{p}) = b_2(\mathbf{p}) = AC(0) \wedge (v+\delta),\ \overline{q}(\mathbf{s},\mathbf{p}) = \mathbf{q}(\mathbf{s}(\mathbf{p},\mathbf{b}(\mathbf{p})),\mathbf{p}) \\ p_1 = p_2 \leq v &\Rightarrow b_1(\mathbf{p}) = b_2(\mathbf{p}) = AC(q_2(s_1(\mathbf{p},\mathbf{b}(\mathbf{p})),\mathbf{p})) \wedge (v+\delta) \\ AC(1) \leq p_1 = p_2 \leq v &\Rightarrow \overline{\mathbf{q}}(\mathbf{s}(\mathbf{p},\mathbf{b}(\mathbf{p})),\mathbf{p}) = \mathbf{q}(\mathbf{s}(\mathbf{p},\mathbf{b}(\mathbf{p})),\mathbf{p}) = \overline{\mathbf{q}} \end{aligned} \tag{2.34}$$

hold, that (together with $(\mathbf{s}(\mathbf{p},\mathbf{b}),\mathbf{c}(\mathbf{s},\mathbf{p}))$) form an SPNE of $\Gamma(\mathbf{I},\mathbf{p})$.

These claims are proved together in what follows.

Proof of the two above claims. Denote $s_2 \in \{1,2\}$ by $s_2 =$ Operate.

Sub Claim: If $p_1 < p_2$, $\forall$ SPNEs $(\mathbf{s}(\mathbf{p},\mathbf{b}),\mathbf{c}(\mathbf{p},\mathbf{b}))$ in $\Gamma(\mathbf{I},\mathbf{p},\mathbf{b})$, $\exists$ $b(\mathbf{p}) \geq 0$ satisfying (2.33).

Proof. Fix SPNE $(\mathbf{s}(\mathbf{p},\mathbf{b}),\mathbf{c}(\mathbf{p},\mathbf{b}))$. Suppose $p_1 \leq v$. Then since $p_1 < p_2$,

$$q_1(s_2,\mathbf{p}) = 1,\ q_2(Op,\mathbf{p}) = 0 \quad \forall s_2 \in \{Op, SD\}$$

via consumer incentive compatibility. Manufacturer incentive compatibility implies

$$s_1(\mathbf{p},\mathbf{b}) \in \begin{cases} Op & \text{if } p_1 > AC(1) \\ \{Op, SD\} & \text{if } p_1 = AC(1) \\ SD & \text{if } p_1 < AC(1). \end{cases}$$

$s_1(\mathbf{p},\mathbf{b})$ is thus independent of $\mathbf{b} \in \mathbb{R}^2_{\geq 0}$. Write $s_1(\mathbf{p},\mathbf{b}) = s_1(p_1)$. Thus there trivially exists $b \geq 0$ satisfying $b = AC(q_2(s_1(\mathbf{p},b,b),\mathbf{p})) \wedge (v+\delta)$. Suppose $p_1 > v$ instead. Then since $p_1 < p_2$,

$$v < p_1 < p_2 \Rightarrow q_2(s_1,\mathbf{p}) = 0 \quad \forall s_1 \in \{Op, SD\}.$$

Set $b = AC(0) \wedge (v+\delta)$ to see the existence of $b \geq 0$ satisfying $b = AC(q_2(s_1(\mathbf{p}, b, b), \mathbf{p})) \wedge (v+\delta)$. Set $b(\mathbf{p}) = b(1,1)$ to complete the proof. $\square$

Sub Claim: If $p_1 < p_2$, then $b_1(\mathbf{p}) = b_2(\mathbf{p}) = b(\mathbf{p})$ as defined by the previous subgame, together with IC $(\mathbf{s}(\mathbf{p}, \mathbf{b}), \mathbf{c}(\mathbf{s}, \mathbf{p}))$ in $\Gamma(\mathbf{I}, \mathbf{p}, \mathbf{b})$ constitute an SPNE.

Proof. The sub claim's proof reduces to showing $\mathbf{b}(\mathbf{p})$ are incentive compatible for the suppliers. Suppose $p_1 \leq v$. The previous sub claim's proof shows $s_1(\mathbf{p}, \mathbf{b}) = s_1(p_1)$ and $q_1(s_2, \mathbf{p}) = 1$. Therefore

$$s_1(p_1) = Op \Rightarrow q_2(s_1(\mathbf{p}, \mathbf{b}), \mathbf{p}) = 0 \ \forall \mathbf{b} \in \mathbb{R}^2_{\geq 0}.$$

Hence $s_1(p_1) = Op$ implies both suppliers are indifferent between winning and losing the auction, and thus indifferent between any bid:

- For $s = 2$, this is because $q_2(s_1(\mathbf{p}, \mathbf{b}), \mathbf{p}) = 0$ for all $\mathbf{b} \in \mathbb{R}^2_{\geq 0}$.

- For $s = 1$, this is because $q_2(s_1(\mathbf{p}, \mathbf{b}), \mathbf{p}) = 0$, but also because

$$s_1(\mathbf{p}, \mathbf{b}) = Op, \ q_1(s_2, \mathbf{p}) = 1 \ \forall s_2 \in \{1, 2, SD\} \ \Rightarrow q_1(\mathbf{s}(\mathbf{p}, \mathbf{b}), \mathbf{p}) = 1 \ \forall \mathbf{b} \geq \mathbf{0}.$$

Hence, the sub claim holds for all $p_1 \geq 0$ satisfying $s_1(p_1) = Op$. Also,

$$s_1(p_1) = SD \Rightarrow q_2(s_1(\mathbf{p}, \mathbf{b}), \mathbf{p}) = q_2(SD, \mathbf{p}) = \{p_2 \leq v\}.$$

So when $s_1(p_1) = SD$, if:

- $p_2 \leq v$, $q_2(s_1(\mathbf{p}, \mathbf{b}), \mathbf{p}) = 1$ for all $\mathbf{b} \in \mathbb{R}^2_{\geq 0}$. So if

  - $AC(1) < p_2 < v$, then the usual FPA argument from Corollary 2.9.1 applies and $b_1(\mathbf{p}) = b_2(\mathbf{p}) = AC(1) = b(\mathbf{p})$ in the unique bidding NE.

  - $p_2 \leq AC(1) < v$, neither supplier can submit a bid acceptable to manufacturer 2 without incurring non-positive variable profit from winning. Bidding $b(\mathbf{p})$ as defined in the previous sub claim ensures the bid covers both suppliers' costs, or is rejected for exceeding v and thus p_2. So $\mathbf{b} = b(\mathbf{p})(1,1)$ is an NE.

- $p_2 > v$, $q_2(s_1(\mathbf{p}, \mathbf{b}), \mathbf{p}) = 0$ for all $\mathbf{b} \in \mathbb{R}^2_{\geq 0}$. Since $s_1(p_1) = SD$, so both suppliers are indifferent between any bid in $\mathbb{R}_{\geq 0}$. Hence $\mathbf{b}(\mathbf{p}) = b(\mathbf{p})(1,1)$ is also an NE.

Hence, the sub claim holds for all $p_1 \geq 0$ satisfying $s_1(p_1) = SD$.

Suppose $p_1 > v$. Then $q_1(s_2, \mathbf{p}) = 0$ for all $s_2 \in \{1, 2, SD\}$. Moreover,

$$p_2 > p_1 > v \Rightarrow q_2(s_1, \mathbf{p}) = 0 \ \forall s_1 \in \{SD, Op\}.$$

It follows that both suppliers are indifferent between submitting any bid since neither manufacturer has any market share regardless of the bids - $q_m(s_{-m}(\mathbf{p}, \mathbf{b}), \mathbf{p}) = 0$ for all $\mathbf{b} \in \mathbb{R}^2_{\geq 0}$ and $m \leq 2$. So setting both bids to equal

$$b(\mathbf{p}) = AC(q_2(s_1(\mathbf{p}, b(\mathbf{p}), b(\mathbf{p})), \mathbf{p})) \wedge (v + \delta) = AC(0) \wedge (v + \delta)$$

as implicitly defined by the previous sub claim trivially yields NE bids. □

Main claim thus holds when $p_1 < p_2$. In what follows, the claim is verified when $p_2 < p_1$.

Sub Claim: If $p_2 < p_1$, for any IC $(q_m(s_{-m}, \mathbf{p}))_{m \leq 2}$ and $\mathbf{s}(\mathbf{p}, \mathbf{b})$ in $\Gamma(\mathbf{I}, \mathbf{p}, \mathbf{b})$, $\exists b(\mathbf{p})$ satisfying (2.33).

Proof. Observe $p_2 < p_1$ implies $q_2(s_1, \mathbf{p}) = \{p_2 \leq v\}$ for all $s_1 \in \{Op, SD\}$. Set $b(\mathbf{p})$ equal to

$$b(\mathbf{p}) = AC(\{p_2 \leq v\}) \wedge (v + \delta) = AC(q_2(s_1(\mathbf{p}, b(\mathbf{p}), b(\mathbf{p})), \mathbf{p})) \wedge (v + \delta) > 0$$

to complete the proof. □

Sub Claim: If $p_2 < p_1$, then $b_1(\mathbf{p}) = b_2(\mathbf{p}) = b(\mathbf{p})$ defined in previous sub claim together with IC $((q_m(s_{-m}, \mathbf{p}))_{m \leq 2}, \mathbf{s}(\mathbf{p}, \mathbf{b}))$ in $\Gamma(\mathbf{I}, \mathbf{p}, \mathbf{b})$ constitute an SPNE.

Proof. Suppose $p_2 \le v$. The previous sub claim and its proof implies $q_2(s_1, \mathbf{p}) = 1$ for all $s_1 \in \{Op, SD\}$. Also, $b(\mathbf{p}) = AC(1) \wedge (v + \delta)$. Finally, $p_2 < p_1$ also implies $q_1(s_2, \mathbf{p}) = 0$ if $s_2 = \mathit{Operate}$. Now, if

- $AC(1) < p_2$, then $AC(1) < p_2 \le v$ implies $b(\mathbf{p}) = AC(1) < v$. The usual FPA argument from Corollary 2.9.1 shows $b_1(\mathbf{p}) = b_2(\mathbf{p}) = b(\mathbf{p})$ is the unique NE outcome in the bidding game between suppliers. So sub claim holds when $AC(1) < p_2 \le v$.

- $p_2 \le AC(1)$, then $b_1(\mathbf{p}) = b_2(\mathbf{p}) = b(\mathbf{p}) = AC(1) \wedge (v + \delta) \ge p_2 \wedge (v + \delta) > p_2$. Hence,

$$s_2(\mathbf{p}, \mathbf{b}(\mathbf{p})) = SD$$

  because $s_2(\mathbf{p}, \mathbf{b})$ is IC and $q_2(s_1, \mathbf{p}) = 1$ for all s_1. It follows that $q_1(s_2(\mathbf{p}, \mathbf{b}(\mathbf{p})), \mathbf{p}) = \{p_1 \le v\}$, and that the suppliers' profits are

$$\pi_1^S = (1 - \beta_1)(p_1 - AC(1)) \vee 0 + T_{21} - F \quad \pi_2^S = 0$$

  when bidding $b(\mathbf{p})$. Consider a deivation by supplier 2. Bidding $b_2 > p_2$ implies it loses the auction while bidding $b_2 \le p_2$ implies

$$b_1 \wedge b_2 \le p_2 \le AC(1) \Rightarrow \pi_2^S = \{s_2(\mathbf{p}, \mathbf{b}(\mathbf{p})) = 2\}(b_2 - AC(1)) \le 0$$

  where the final equality recognizes $q_2(s_1, \mathbf{p}) = 1$ for all s_1 when $p_2 < p_1$. Consider a deviation by supplier 1. Bidding $b_1 > p_2$ implies it loses the auction, and its profit is thus bounded from above by

$$(1 - \beta_1)(p_1 - AC(1)) \vee 0 + T_{21} - F,$$

  its profit from only supplying manufacturer 1 when $q_1(s_2(\mathbf{p}, \mathbf{b}(\mathbf{p})), \mathbf{p}) = 1$. Biding $b_1 \le p_2$

implies $b_1 \leq AC(1)$. Hence supplier 1's profit equals

$$\begin{aligned}\pi_1^S &= \{s_2(\mathbf{p}, \mathbf{b}(\mathbf{p})) = 1\}(b_1 - AC(1)) \\ &\qquad + \{s_2(\mathbf{p}, \mathbf{b}(\mathbf{p})) = SD\}(1-\beta_1)(p_1 - AC(1)) \vee 0 + T_{21} - F \\ &\leq (1-\beta_1)(p_1 - AC(1)) \vee 0 + T_{21} - F.\end{aligned}$$

So supplier 1 has no incentive to defect either. So the sub claim holds when $p_2 \leq AC(1) \wedge v$.

Suppose $p_2 > v$. Observe then $v < p_2 < p_1$ implying

$$q_1(s_2, \mathbf{p}) = q_2(s_1, \mathbf{p}) = 0 \ \forall \mathbf{s} \in \{Op, SD\} \times \{1, 2, SD\}.$$

So neither supplier makes any variable profit when submitting any bid

$$\mathbf{b} \in \mathbb{R}^2_{\geq 0} \Rightarrow (\pi_1^S, \pi_2^S) = (T_{21} - F, 0),$$

and $b_1(\mathbf{p}) = b_2(\mathbf{p}) = b(\mathbf{p})$ does not violate IC for either supplier. So the sub claim holds when $p_2 > v$. $\square$

The remaining sub claims address the case when $p_1 = p_2$. Recall $\overline{\mathbf{q}}(\mathbf{s}, \mathbf{p})$ denotes $(q_m(s_{-m}, \mathbf{p}))_{m \leq 2}$.

Sub Claim: If $p_1 = p_2$, then $\forall \delta > 0, \overline{\mathbf{q}} \in \Delta$, $\exists$ IC $(\overline{\mathbf{q}}(\mathbf{s}, \mathbf{p}), \mathbf{s}(\mathbf{p}, \mathbf{b}))$ such that a solution $\mathbf{b}(\mathbf{p})$ to (2.33) exists, constitutes an NE of the suppliers' bidding game, and satisfies

$$AC(1) \leq p_1 = p_2 \leq v \Rightarrow \overline{\mathbf{q}}(\mathbf{s}(\mathbf{p}, \mathbf{b}(\mathbf{p})), \mathbf{p}) = \overline{\mathbf{q}} \tag{2.35}$$

$$p_1 = p_2 > v \Rightarrow \overline{\mathbf{q}}(\mathbf{s}(\mathbf{p}, \mathbf{b}(\mathbf{p})), \mathbf{p}) = \mathbf{0} \tag{2.36}$$

with $(\overline{\mathbf{q}}(\mathbf{s}, \mathbf{p}), \mathbf{s}(\mathbf{p}, \mathbf{b}))$.

Proof. In keeping with previously used notation, let $b(\mathbf{p})$ denote the solution to (2.33). Let p denote the common value of $\mathbf{p}$.

Observe we can set $s_2(\mathbf{p}, \mathbf{b})$ according to (2.30) for all $\mathbf{b} \in \mathbb{R}^2_{\geq 0}$ when $p_1 = p_2$ without violating manufacturer 2's IC constraint. Under (2.30), notice when $b_1 = b_2$, neither supplier alters $s_2(\mathbf{p}, b_1, b_2) \in \{Op, SD\}$ and thus, $q_1(s_2(\mathbf{p}, \mathbf{b}(\mathbf{p})), \mathbf{p})$ by raising its bid. Observe we can also set $q_2(s_1, \mathbf{p})$ according to (2.31) without violating any consumers' IC constraint either, since $p_1 = p_2$. Finally, we can set $q_1(s_2, \mathbf{p})$ according to (2.32) without violating any consumers' IC constraint when $p_1 = p_2$.

Suppose $v < p$. Then $q_1(s_2, \mathbf{p}) = q_2(s_1, \mathbf{p}) = 0$ for all $\mathbf{s} \in \{Op, SD\} \times \{1, 2, SD\}$ regardless of whether (2.31) or (2.32) holds. So both suppliers are indifferent between any bid $b_s \in \mathbb{R}_{\geq 0}$. Therefore

$$b_1(\mathbf{p}) = b_2(\mathbf{p}) = b(\mathbf{p}) \equiv AC(0) \wedge (v + \delta)$$

is an NE of the suppliers' bidding game and (2.33) and (2.36) both hold; regardless of whether (2.30) and (2.31), or (2.30) and (2.32) hold.

Suppose $p \leq v$.

a) Consider the case where $\overline{\mathbf{q}} = (0, 1)$. Assume (2.30) and (2.31) hold. Observe $b(\mathbf{p}) = AC(1) \wedge (v + \delta)$ satisfies (2.33) by (2.31). Moreover, $b_1(\mathbf{p}) = b_2(\mathbf{p}) = b(\mathbf{p})$ is an NE too.

   Suppose $AC(1) \leq v + \delta$. I proceed to show bidding $b(\mathbf{p})$ is indeed an NE for the suppliers' bidding game. There are two possibilities.

   First, assume $AC(1) \leq p$. Then

$$\begin{aligned} b_1 = b_2 = b(\mathbf{p}) = AC(1) \leq p &\Rightarrow s_2(\mathbf{p}, \mathbf{b}) = 1 \quad \text{(via (2.30))} \\ &\Rightarrow \mathbf{q}(\mathbf{s}(\mathbf{p}, \mathbf{b}), \mathbf{p}) = (0, 1) \quad \text{(via (2.31) and } p \leq v) \\ &\Rightarrow \pi_1^S = T_{21} - F \quad \pi_2^S = 0 \end{aligned}$$

   when both suppliers bid $b(\mathbf{p})$. Suppliers make zero variable profit along the proposed equilibrium's path. Neither supplier will deviate from $b_1(\mathbf{p}) = b_2(\mathbf{p}) = b(\mathbf{p})$ when (2.30) and (2.31) hold:

- If supplier 1 increases its bid, manufacturer 2 still operates. So $q_1(s_2(\mathbf{p}, \mathbf{b}(\mathbf{p})), \mathbf{p}) = 0$. Supplier 1 loses the auction without any gain in sales to manufacturer 1.
- If supplier 1 reduces its bid, manufacturer 2 still chooses supplier 1, and supplier 1's profit is reduced by $AC(1) - b_1$.
- If supplier 2 increases its bid, it loses the auction and makes no profit as before.
- If supplier 2 shades its bid, manufacturer 2 chooses supplier 2, implying a loss for supplier 2 equal to $b_2 - AC(1) < 0$.

Second, assume $p < AC(1)$. Observe

$$\begin{aligned} p < AC(1), p \leq v \Rightarrow\ & p < AC(1) \wedge (v + \delta) \\ & \Rightarrow p_2 < b_1(\mathbf{p}) = b_2(\mathbf{p}) \Rightarrow s_2(\mathbf{p}, \mathbf{b}(\mathbf{p})) = SD. \quad \text{(via (2.30))} \\ p < AC(1) \Rightarrow\ & p_1 < AC(1) = AC(q_1(SD, \mathbf{p})) \quad \text{(via (2.31))} \\ & \Rightarrow s_1(\mathbf{p}, \mathbf{b}) = SD \quad \forall \mathbf{b} \in \mathbb{R}^2_{\geq 0}. \end{aligned}$$

Hence the suppliers' variable profits are zero

$$\pi_1^S = T_{21} - F \quad \pi_2^S = 0$$

as when $AC(1) \leq p$. Hence, neither supplier will deviate under (2.30) and (2.31):

- If supplier 1 increases its bid, manufacturer 2 still shuts down since $b_1 > AC(1) = b(\mathbf{p})$ implies $s_2(\mathbf{p}, b_1, b(\mathbf{p})) = SD = s_2(\mathbf{p}, b(\mathbf{p}), b(\mathbf{p}))$. Moreover, $p < AC(1) \Rightarrow s_1(\mathbf{p}, b_1, b(\mathbf{p})) = SD$ as explained above. So supplier 1 cannot increase its sales to manufacturer 1, and doesn't win the auction to supply manufacturer 2.
- If supplier 1 reduces its bid, defecting to $b_1 > p$ implies no change in its profit since

$s_2(\mathbf{p}, b_1, b(\mathbf{p})) = SD = s_2(\mathbf{p}, b(\mathbf{p}), b(\mathbf{p}))$. Defecting to $b_1 \leq p$ weakly decreases its profit:

$$\begin{aligned} b_1 \leq p &\Rightarrow s_2(\mathbf{p}, b_1, b(\mathbf{p})) = 1 \quad \text{(via (2.30))} \\ &\Rightarrow \mathbf{q}(\mathbf{s}(\mathbf{p}, b_1, b(\mathbf{p})), \mathbf{p}) = (0, 1) \quad \text{(via (2.31))} \\ &\Rightarrow \pi_1^S = b_1 - AC(1) + T_{21} - F \leq T_{21} - F. \end{aligned}$$

- If supplier 2 increases its bid, it loses the auction and earns no profit.
- If supplier 2 decreases its bid, reducing its bid to $b_2 > p$, it still loses the auction. Cutting its bid to $b_2 \leq p$ implies it wins the auction but at a loss:

$$\begin{aligned} b_2 \leq p &\Rightarrow s_2(\mathbf{p}, b(\mathbf{p}), b_2) = 2 \quad \text{(via (2.30))} \\ &\Rightarrow \mathbf{q}(\mathbf{s}(\mathbf{p}, b(\mathbf{p}), b_2), \mathbf{p}) = (0, 1) \quad \text{(via (2.31))} \\ &\Rightarrow \pi_2^S = b_2 - AC(1) \leq 0. \end{aligned}$$

So $\mathbf{b}(\mathbf{p}) = b(\mathbf{p})(1, 1) = AC(1)(1, 1)$ is an NE when $AC(1) \leq v + \delta$.

Suppose $v + \delta < AC(1)$ instead. Then $p \leq v$ implies

$$\begin{aligned} p_1 = p_2 < v + \delta < AC(1) &\Rightarrow p_2 < v + \delta = b(\mathbf{p}) \\ &\Rightarrow s_2(\mathbf{p}, b(\mathbf{p}), b(\mathbf{p})) = SD \quad \text{(via (2.30))} \\ &\Rightarrow q_1(s_2(\mathbf{p}, b(\mathbf{p}), b(\mathbf{p})), \mathbf{p}) = 1 \quad \text{(via (2.31))} \\ &\Rightarrow p_1 < A(1) = AC(q_1(s_2(\mathbf{p}, b(\mathbf{p}), b(\mathbf{p})), \mathbf{p})) \\ &\Rightarrow s_1(\mathbf{p}, b(\mathbf{p}), b(\mathbf{p})) = SD \quad \text{(via IC)}. \end{aligned}$$

It follows that neither manufacturer operates and neither supplier makes any variable profit

$$\pi_1^S = T_{21} - F \quad \pi_2^S = 0$$

under $\mathbf{b} = \mathbf{b}(\mathbf{p})$. Neither supplier will deviate:

- If supplier 1 defects to $b_1 > p_2$, manufacturer 2 continues to shut down

$$s_2(\mathbf{p}, b_1, b(\mathbf{p})) = SD = s_2(\mathbf{p}, b(\mathbf{p}), b(\mathbf{p})) \quad \text{(via (2.30))}$$

  and supplier 1's profit is unchanged.

- If supplier 1 defects to $b_1 \leq p_2$, manufacturer 2 chooses supplier 1:

$$s_2(\mathbf{p}, b_1, b_2(\mathbf{p})) = 1 \neq SD = s_2(\mathbf{p}, \mathbf{b}(\mathbf{p})).$$

  Hence, supplier 1's profit is decreased:

$$\begin{aligned}\mathbf{q}(\mathbf{s}(\mathbf{p}, b_1, b_2(\mathbf{p})), \mathbf{p}) = (0,1) \Rightarrow \pi_1^S &= b_1 - AC(1) + T_{21} - F \\ &\leq p_2 - AC(1) + T_{21} - F \leq T_{21} - F.\end{aligned}$$

- If supplier 2 defects to $b_2 > p_2$, its bid remains rejected and its profit is still 0.

- If supplier 2 defects to $b_2 \leq p_2$, its bid is accepted and

$$q_2(s_1(\mathbf{p}, b_1, b_2(\mathbf{p})), \mathbf{p}) = 1 \Rightarrow \pi_2^S = b_2 - AC(1) < 0.$$

  So $\mathbf{b} = \mathbf{b}(\mathbf{p})$ is an NE in the bidding game.

So $\mathbf{b}(\mathbf{p}) = (v + \delta)(1,1)$ are NE strategies when $(v + \delta) \leq AC(1)$. Now observe that if $AC(1) \leq p$ is assumed in addition to $p \leq v$:

$$\begin{aligned}AC(1) \leq p \Rightarrow b_1(\mathbf{p}) = b_2(\mathbf{p}) &= AC(1) \wedge (v + \delta) \\ &= AC(1) \quad (\because AC(1) \leq p \leq v) \\ &\leq p = p_2 \\ \Rightarrow s_2(\mathbf{p}, \mathbf{b}(\mathbf{p})) &= Op.\end{aligned}$$

Hence (2.31) implies (2.35) holds.

b) Suppose $\overline{\mathbf{q}} = (1,0)$ instead. Assume (2.29), (2.30) and (2.32) hold. Observe the cost of supplying manufacturer 1 equals

$$\begin{aligned}
&\{s_2(\mathbf{p},\mathbf{b}) = 1\}\Delta C(q_1(Op,\mathbf{p}), q_2(Op,\mathbf{p})) + \{s_2(\mathbf{p},\mathbf{b}) = 2\}AC(q_1(Op,\mathbf{p})) \\
&\quad + \{s_2(\mathbf{p},\mathbf{b}) = SD\}AC(q_1(SD,\mathbf{p})) \\
&= \{s_2(\mathbf{p},\mathbf{b}) = 1\}\Delta C(1,0) + \{s_2(\mathbf{p},\mathbf{b}) \neq 1\}AC(1) = AC(1).
\end{aligned}$$

by (2.32). So (2.29) doesn't violate manufacturer 1's IC constraints.

Suppose $AC(1) \leq p$. Then $s_2(\mathbf{p},\mathbf{b}) = Op$ by (2.29), and

$$\begin{aligned}
AC(0) \wedge (v+\delta) &= AC(q_2(Op,\mathbf{p})) \wedge (v+\delta) \quad \text{(by (2.32))} \\
&= AC(q_2(s_1(\mathbf{p},b,b),\mathbf{p})) \wedge (v+\delta) = b
\end{aligned}$$

is solved by $b = AC(0) \wedge (v+\delta)$. So (2.33) holds. Moreover,

$$\begin{aligned}
&s_1(\mathbf{p},\mathbf{b}) = Op \Rightarrow q_2(s_1(\mathbf{p},\mathbf{b}),\mathbf{p}) = 0 \\
&s_2(\mathbf{p},\mathbf{b}) = Op \Rightarrow q_1(s_2(\mathbf{p},\mathbf{b}),\mathbf{p}) = \{p \leq v\} = 1
\end{aligned}$$

by (2.32) and $p \leq v$ being hypothesized. So (2.35) holds. Also

$$\pi_1^S = (1-\beta_1)(p_1 - AC(1)) + T_{21} - F \quad \pi_2^S = 0$$

for all $\mathbf{b} \in \mathbb{R}^2_{\geq 0}$. So both suppliers are indifferent between any bid $b_s \geq 0$, implying $\mathbf{b} = b(\mathbf{p})(1,1)$ is an NE for the suppliers' bidding game.

Suppose $p < AC(1)$ instead. Then $s_1(\mathbf{p}, \mathbf{b}(\mathbf{p})) = SD$ by (2.29). Moreover,

$$\begin{aligned} AC(1) \wedge (v+\delta) &= AC(q_2(SD, \mathbf{p})) \wedge (v+\delta) \quad \text{(by (2.32))} \\ &= AC(q_2(s_1(\mathbf{p}, b, b), \mathbf{p})) \wedge (v+\delta) \quad \text{(by (2.29) and } p < AC(1)\text{)}. \end{aligned}$$

Setting the above quantity to $\mathbf{b}(\mathbf{p})$ shows (2.33) holds. Moreover,

$$s_1(\mathbf{p}, \mathbf{b}(\mathbf{p})) = SD \Rightarrow q_1(\mathbf{s}(\mathbf{p}, \mathbf{b}(\mathbf{p})), \mathbf{p}) = 1$$
$$s_1(\mathbf{p}, \mathbf{b}(\mathbf{p})) = SD, \ (2.32) \ \Rightarrow q_2(s_1(\mathbf{p}, \mathbf{b}(\mathbf{p})), \mathbf{p}) = 1$$

since $p \le v$. The above implications and $p < AC(1)$ thus imply

$$s_1(\mathbf{p}, \mathbf{b}(\mathbf{p})) = SD, \quad \mathbf{q}(\mathbf{s}(\mathbf{p}, \mathbf{b}(\mathbf{p})), \mathbf{p}) = (0, \{s_2(\mathbf{p}, \mathbf{b}(\mathbf{p})) \neq SD\}).$$

But observe

$$\begin{aligned} AC(1) \le v + \delta \Rightarrow \ & p < AC(1) = b(\mathbf{p}) \\ \Rightarrow \ & s_2(\mathbf{p}, \mathbf{b}(\mathbf{p})) = SD \quad \text{(by (2.30))} \\ \Rightarrow \ & \mathbf{q}(\mathbf{s}(\mathbf{p}, \mathbf{b}(\mathbf{p})), \mathbf{p}) = \mathbf{0}. \end{aligned}$$

So neither supplier makes any variable profit from $\mathbf{b} = \mathbf{b}(\mathbf{p})$. Observe neither supplier has incentives to defect:

– If supplier 1 defects to $b_1 > p_1$, then manufacturer 2 still shuts down and supplier 1 loses the auction without gaining from selling more inputs to manufacturer 1:

$$\begin{aligned} & s_2(\mathbf{p}, b_1, b(\mathbf{p})) = SD = s_2(\mathbf{p}, \mathbf{b}(\mathbf{p})) \quad \text{(by (2.30))} \\ \Rightarrow \ & \mathbf{q}(\mathbf{s}(\mathbf{p}, \mathbf{b}(\mathbf{p})), \mathbf{p}) = \mathbf{0}. \end{aligned}$$

– If supplier 1 defects to $b_1 \leq p_1$, then manufacturer 2 accepts its bid,

$$\begin{aligned} & s_2(\mathbf{p}, b_1, b(\mathbf{p})) = 1 \neq SD = s_2(\mathbf{p}, \mathbf{b}(\mathbf{p})) \quad \text{(by (2.30))} \\ \Rightarrow\ & \mathbf{q}(\mathbf{s}(\mathbf{p}, \mathbf{b}(\mathbf{p})), \mathbf{p}) = (0, \{p \leq v\}) \\ \Rightarrow\ & \pi_2^S = \{p \leq v\}(b_1 - AC(1)) + T_{21} - F < T_{21} - F. \end{aligned}$$

– If supplier 2 defects to $b_2 \leq p$, then manufacturer 2 still shuts down and supplier 2 makes no profit.

– If supplier 2 defects to $b_2 < p$, then its bid is accepted. Moreover, since $p < AC(1)$,

$$\begin{aligned} s_1(\mathbf{p}, b(\mathbf{p}), b_2) = SD \Rightarrow\ & q_2(s_1(\mathbf{p}, b(\mathbf{p}), b_2), \mathbf{p}) = 1 \\ \Rightarrow\ & \pi_2^S = \{p \leq v\}(b_2 - AC(1)) + T_{21} - F < T_{21} - F. \end{aligned}$$

It follows $\mathbf{b} = b(\mathbf{p})(1, 1)$ is an NE for the suppliers' bidding game.

□

When $p_1 = p_2 > v$, the previous sub claim's proof makes clear $\overline{\mathbf{q}}(\mathbf{s}, \mathbf{p})$ and thus $\mathbf{q}(\mathbf{s}, \mathbf{p})$ equal zero. Also, observe how the expressions for $\mathbf{b}(\mathbf{p})$ in (2.34) collapse into (2.33). The previous sub claim thus implies first two lines of (2.33) hold too. It remains to show its third line holds.

Sub Claim: When $AC(1) \leq p_1 = p_2 \leq v$, then the IC $\overline{\mathbf{q}}(\mathbf{s}, \mathbf{p})$, $\mathbf{s}(\mathbf{b}, \mathbf{p})$ and $\mathbf{b}(\mathbf{p})$ constructed to satisfy the previous sub claim also satisfies $\mathbf{q}(\mathbf{s}(\mathbf{b}(\mathbf{p}), \mathbf{p}), \mathbf{p}) = \overline{\mathbf{q}}$.

Proof. Suppose that $\overline{\mathbf{q}} = (0, 1)$. Then $(\overline{\mathbf{q}}(\mathbf{s}, \mathbf{p}), \mathbf{s}(\mathbf{b}, \mathbf{p}))$ satisfy (2.30) and (2.31). Also, the previous sub claim implies

$$\overline{\mathbf{q}}(\mathbf{s}(\mathbf{p}, \mathbf{b}(\mathbf{p})), \mathbf{p}) = \overline{\mathbf{q}} = (0, 1) \Rightarrow q_2(s_1(\mathbf{p}, \mathbf{b}(\mathbf{p})), \mathbf{p}) = 1$$

(when $\mathbf{p} = p(1,1)$). The previous sub claim also implies

$$b_1(\mathbf{p}) = b_2(\mathbf{p}) = AC(q_2(s_1(\mathbf{p}, \mathbf{b}(\mathbf{p})), \mathbf{p})) \wedge (v+\delta) = AC(1) \wedge (v+\delta) = AC(1) \leq p$$

since $AC(1) \leq p \leq v < v+\delta$. It follows $s_2(\mathbf{p}, \mathbf{b}(\mathbf{p})) = 1$ by (2.30). Thus,

$$q_2(\mathbf{s}(\mathbf{p}, \mathbf{b}(\mathbf{p})), \mathbf{p}) = q_2(s_1(\mathbf{p}, \mathbf{b}(\mathbf{p})), \mathbf{p}) = 1 \;\Rightarrow\; q_1(\mathbf{s}(\mathbf{p}, \mathbf{b}(\mathbf{p})), \mathbf{p}) = 0.$$

The implication above recognizes $q_1 + q_2 \in [0,1]$ on the equilibrium path.

Suppose that $\overline{\mathbf{q}} = (1,0)$ instead. Then $\overline{\mathbf{q}}(\mathbf{s}, \mathbf{p})$ and $\mathbf{s}(\mathbf{p}, \mathbf{b})$ satisfy (2.29), (2.30) and (2.32) instead. Also, the previous sub claim implies

$$\overline{\mathbf{q}}(\mathbf{s}(\mathbf{p}, \mathbf{b}(\mathbf{p})), \mathbf{p}) = \overline{\mathbf{q}} = (1,0) \Rightarrow q_2(s_1(\mathbf{p}, \mathbf{b}(\mathbf{p})), \mathbf{p}) = 0$$

and

$$b_1(\mathbf{p}) = b_2(\mathbf{p}) = AC(q_2(s_1(\mathbf{p}, \mathbf{b}(\mathbf{p})), \mathbf{p})) \wedge (v+\delta) = AC(0) \wedge (v+\delta).$$

From (2.29),

$$s_1(p(1,1), b(\mathbf{p})(1,1)) = \begin{cases} Op & \text{if } p \geq AC(1) \\ SD & \text{if otherwise} \end{cases} = Op$$

since $AC(1) \leq p \leq v$ by hypothesis. Hence

$$q_1(\mathbf{s}(\mathbf{p}, \mathbf{b}(\mathbf{p})), \mathbf{p}) = q_1(s_2(\mathbf{p}, \mathbf{b}(\mathbf{p})), \mathbf{p}) = 1$$

$$q_2(\mathbf{s}(\mathbf{p}, \mathbf{b}(\mathbf{p})), \mathbf{p}) = 0 \Rightarrow q_2(s_1(\mathbf{p}, \mathbf{b}(\mathbf{p})), \mathbf{p}) = 0,$$

completing the proof when $\overline{\mathbf{q}} = (1,0)$. □

□

Claim: Suppose $AC(1) < v$. For any $\mathbf{q}^* \in \{(0,1),(1,0)\}$ there exists a SPNE $\boldsymbol{\sigma}_{\mathbf{I}}$ in $\Gamma(\mathbf{I})$

such that $p_1^* = p_2^* = AC(1)$ and $\mathbf{q}(\mathbf{s}(\mathbf{p}^*, \mathbf{b}(\mathbf{p}^*)), \mathbf{p}^*) = \mathbf{q}^*$.

Proof. **Sub Claim:** $p_1^* = p_2^* = AC(1) < v$ implies $\exists$ an SPNE $(\mathbf{b}(\mathbf{p}^*), \mathbf{s}(\mathbf{p}^*, \mathbf{b}), \mathbf{c}(\mathbf{s}, \mathbf{p}^*))$ in $\Gamma(\mathbf{I}, \mathbf{p}^*)$ satisfying $\mathbf{q}(\mathbf{s}(\mathbf{p}^*, \mathbf{b}(\mathbf{p}^*)), \mathbf{p}^*) = \mathbf{q}^*$ and $\mathbf{b}(\mathbf{p}^*) = AC(q_2^*) \wedge (v + \delta)$.

Proof. See previous main claim. $\square$

Sub Claim: Suppose $p_1^* = p_2^* = AC(1)$ and $\mathbf{q}^* \in \{(0,1), (1,0)\}$. The previous sub claim's SPNE $(\mathbf{b}(\mathbf{p}^*), \mathbf{s}(\mathbf{p}^*, \mathbf{b}), \mathbf{c}(\mathbf{s}, \mathbf{p}^*))$ implies $\pi_1^M = 0$ and $\pi_2^M = -T_{21}$ on the equilibrium path.

Proof. Suppose $\mathbf{q}^* = (0,1)$. Then previous sub claim implies $b_1^* \wedge b_2^* = AC(1) < v + \delta$ for some $\delta > 0$. Hence

$$\pi_1^M = q_1^* \beta_1 (p_1^* - \{s_2^* = 1\} \Delta C(\mathbf{q}^*) - \{s_2^* \neq 1\} AC(q_1^*)) = 0 \quad \pi_2^M + T_{21} = q_2^* (p_2^* - b_1^* \wedge b_2^*) = 0.$$

Suppose $\mathbf{q}^* = (1,0)$. Then rather trivially, $\pi_1^M = 0$ and $\pi_2^M = -T_{21}$. $\square$

Sub Claim: Suppose $p_1 \neq p_2$. Then $\exists$ SPNE $(\mathbf{b}(\mathbf{p}), \mathbf{s}(\mathbf{p}, \mathbf{b}), \mathbf{c}(\mathbf{s}, \mathbf{p}))$ such that $b_1(\mathbf{p}) = b_2(\mathbf{p})$, (2.33) holds,

$$\begin{aligned} s_1(\mathbf{p}, \mathbf{b}) = Op \quad \text{if } p_1 = \{s_2(\mathbf{p}, \mathbf{b}) = 1\} \Delta C(q_1(Op, s_2(\mathbf{p}, \mathbf{b}), \mathbf{p}), q_2(Op, Op, \mathbf{p})) \\ + \{s_2(\mathbf{p}, \mathbf{b}) \neq 1\} AC(q_1(Op, s_2(\mathbf{p}, \mathbf{b}), \mathbf{p})) \end{aligned} \tag{2.37}$$

and

$$s_2(\mathbf{p}, \mathbf{b}) \in \text{argmin}_{s \leq 2} b_s \quad \text{if } \text{argmin}_{s \leq 2} b_s = p_2. \tag{2.38}$$

Proof. Previous main claims shows $\forall$ SPNE $(\mathbf{s}(\mathbf{p}, \mathbf{b}), \mathbf{c}(\mathbf{s}, \mathbf{p}))$ in $\Gamma(\mathbf{I}, \mathbf{p}, \mathbf{b})$, can find $\mathbf{b}(\mathbf{p})$ satisfying $b_1(\mathbf{p}) = b_2(\mathbf{p})$, and (2.33). It thus remains to show the requirements (2.37), (2.38) on $\mathbf{s}(\mathbf{p}, \mathbf{b})$ are IC.

(2.37) is IC for manufacturer 1, who is indifferent between operating or shutting down when

$$p_1 = \{s_2(\mathbf{p},\mathbf{b}) = 1\}\Delta C(q_1(Op, s_2(\mathbf{p},\mathbf{b}),\mathbf{p}), q_2(Op, Op, \mathbf{p})) + \{s_2(\mathbf{p},\mathbf{b}) \neq 1\}AC(q_1(Op, s_2(\mathbf{p},\mathbf{b}),\mathbf{p}))$$

Price Average Cost

given $s_2(\mathbf{p},\mathbf{b})$, regardless of $\mathbf{q}(\mathbf{b},\mathbf{p})$.

(2.38) is completely consistent with incentive compatibility since manufacturer 2 is indifferent between operating through its cheapest supplier or shutting down when

$$p_2 = b_1 \wedge b_2$$

Price Average Cost

regardless of $s_1(\mathbf{p},\mathbf{b})$ and $\mathbf{q}(\mathbf{s},\mathbf{b})$. □

Let $(\mathbf{b}(\mathbf{p}), \mathbf{s}(\mathbf{p},\mathbf{b}), \mathbf{c}(\mathbf{s},\mathbf{p}))$ satisfy the conditions stated in first and third sub claims above. These sub claims imply the former constitutes a SPNE in $\Gamma(\mathbf{I},\mathbf{p})$ for all $\mathbf{p} \in \mathbb{R}^2_{\geq 0}$. Sub claim 2's proof also shows manufacturers make no variable profit when $p_1 = p_2 < v$. It remains to show neither manufacturer can improve its profit by defecting to $p_m \neq p_m^*$.

Sub Claim: Neither manufacturer $m \leq 2$ can strictly increase payoffs from setting $p_m > p_m^*$.

Proof. Let $m = 1$. Then since consumers consume when indifferent and $AC(1) \leq v$:

$$p_1 > p_2^* \Rightarrow q_2(s_1, p_1, p_2^*) = 1 \quad \forall s_1 \in \{Op, SD\}.$$

The 3rd sub claim in this main claim's proof implies

$$b_1(p_1, p_2^*) = b_2(p_1, p_2^*) = AC(q_2(s_1(p_1, p_2^*, \mathbf{b}(p_1, p_2^*)), p_1, p_2^*)) = AC(1) \wedge (v + \delta) = AC(1) = p_2^*.$$

The same sub claim also implies

$$\begin{aligned} s_2(p_1, p_2^*, \mathbf{b}(p_1, p_2^*)) \in \{1, 2\} &\Rightarrow q_1(s_2(p_1, p_2^*, \mathbf{b}(p_1, p_2^*)), p_1, p_2^*) = 0 \\ &\Rightarrow q_1(\mathbf{s}(p_1, p_2^*, \mathbf{b}(p_1, p_2^*)), p_1, p_2^*) = 0 \end{aligned}$$

and manufacturer 1's profit is still zero.

Let $m = 2$. Then

$$p_2 > p_1^* \Rightarrow q_1(s_2, p_1^*, p_2) = 1 \quad \forall s_2 \in \{1, 2, SD\}.$$

Hence,

$$p_1^* = AC(1) = AC(q_1(s_2(p_1^*, p_2, \mathbf{b}(p_1^*, p_2)), p_1^*, p_2))$$
$$p_1^* = AC(1) = \Delta C(1, 0) = \Delta C(\mathbf{q}(Op, Op, p_1^*, p_2)) \quad (\because q_1(s_2, p_1^*, p_2) = 1 \quad \forall s_2 \in \{1, 2, SD\}).$$

So the 3rd sub claim implies

$$s_1(p_1^*, p_2, \mathbf{b}(p_1^*, p_2)) = Op \Rightarrow q_2(\mathbf{s}(p_1^*, p_2, \mathbf{b}(p_1^*, p_2)), p_1^*, p_2) = 0.$$

Manufacturer 2's profit post defection is $-T_{21}$, the same as before. □

Sub Claim: Neither manufacturer $m \leq 2$ can strictly increase its profit by setting $p_m < p_m^*$ when $p_m^* = AC(1)$ under $(\mathbf{b}(\mathbf{p}), \mathbf{s}(\mathbf{p}, \mathbf{b}), \mathbf{c}(\mathbf{s}, \mathbf{p}))$ for all $m \leq 2$.

Proof. If $m = 1$, observe

$$\begin{aligned} p_1 < p_1^* = AC(1) = p_2^* &\Rightarrow q_1(s_2, p_1, p_2^*) = 1 \quad \forall s_2 \in \{1, 2, SD\} \\ &\Rightarrow p_1 < AC(1) = AC(q_1(s_2(p_1, p_2^*, \mathbf{b}(p_1, p_2^*)), p_1, p_2^*)) \\ &\Rightarrow s_1(p_1, p_2^*, \mathbf{b}(p_1, p_2^*)) = SD. \end{aligned}$$

So manufacturer 1's profit is still zero.

Let $m = 2$ instead. Then

$$
\begin{aligned}
p_2 < p_2^* = p_1^* \Rightarrow q_2(s_1, p_1^*, p_2) = 1 \quad \forall s_1 \in \{Op, SD\} \\
\Rightarrow p_2 < AC(1) = AC(q_2(s_1(p_1^*, p_2, \mathbf{b}(p_1^*, p_2)), p_1^*, p_2)) \wedge (v + \delta).
\end{aligned}
$$

So the 3rd sub claim implies

$$
\begin{aligned}
b_1(p_1^*, p_2) = b_2(p_1^*, p_2) &= AC(q_2(s_1(p_1^*, p_2, \mathbf{b}(p_1^*, p_2)), p_1^*, p_2)) \\
&= AC(1) \wedge (v + \delta) = AC(1) = p_2^* > p_2.
\end{aligned}
$$

Hence $s_2(p_1^*, p_2, \mathbf{b}(p_1^*, p_2)) = SD$. Manufacturer 2's profit is still equal to $-T_{21}$ when it defects.

□

□

□

2.9.2 Proofs for Results under Assumption 2.2.2

The following results apply to the social planner's allocation described in Subsection 2.2.3.

Proof of Proposition 2.2.1. The planner's allocation can be expressed as instructions to each player for each decision node in the game, since each feasible allocation is derived from some strategy profile $\boldsymbol{\sigma}$.

Suppose $AC\left(\frac{1}{2}\right) < v_1 \wedge v_2$. Then the gain in total surplus from sharing suppliers must equal the reduction in the cost of making inputs $2C(\frac{1}{2}) - C(1)$, since both manufacturers operate regardless of whether they share suppliers in the socially optimal allocation. The planner thus instructs supplier sharing when its gain is smaller than its cost, the cost of forming a new relationship between manufacturer 2 and supplier 1:

$$
F < 2C\left(\frac{1}{2}\right) - C(1) = \hat{F}.
$$

The planner avoids supplier sharing when the inequality holds in reverse. The planner is indifferent when both sides of the inequality equal.

Suppose $v_1 \wedge v_2 \leq AC\left(\frac{1}{2}\right) \leq v_1 \vee v_2$. There are two possibilities.

1. Suppose $q_{argmax(v)} > 0$ in the planner's instructed strategies, when $\mathbf{a} \neq \mathbf{I}$ or equivalently when supplier sharing doesn't occur. There are two sub cases to consider.

   - Suppose $q_{argmin(v)} > 0$ in the planner's chosen strategies when $\mathbf{a} \neq \mathbf{I}$. Then $v_1 \wedge v_2 = AC\left(\frac{1}{2}\right)$. Also the gain in total surplus from sharing suppliers is $2C(\frac{1}{2}) - C(1)$. So supplier sharing occurs only if

     $$F < 2C\left(\frac{1}{2}\right) - C(1).$$

     The planner avoids supplier sharing when the inequality holds in reverse, and is indifferent when both sides of the inequality equal.

   - Suppose $q_{argmin(v)} = 0$ when supplier sharing doesn't occur. The gain in total surplus from sharing suppliers is $\frac{v_1 \wedge v_2}{2} - C(1) + C(\frac{1}{2})$. So supplier sharing occurs only if

     $$F < \frac{v_1 \wedge v_2}{2} - C(1) + C\left(\frac{1}{2}\right).$$

     The planner avoids supplier sharing when the inequality holds in reverse, and is indifferent when both sides of the inequality equal.

   The inequalities above are identical when $v_1 \wedge v_2 = AC\left(\frac{1}{2}\right)$. So the planner instructs supplier sharing only if

   $$F < \frac{v_1 \wedge v_2}{2} - C(1) + C\left(\frac{1}{2}\right), \tag{2.39}$$

   avoids supplier sharing when the inequality holds in reverse, and is indifferent when both sides of the inequality equal.

2. Suppose $q_{argmax(v)} = 0$ in the planner's instructed strategies, when $\mathbf{a} \neq \mathbf{I}$. Then $AC\left(\frac{1}{2}\right) = v_1 \vee v_2$. Moreover, supplier sharing occurs via the following rules.

- Suppose $q_{argmin(v)} > 0$ in the planner's chosen strategies when $\mathbf{a} \neq \mathbf{I}$. Then $v_1 = v_2 = AC\left(\frac{1}{2}\right)$, and the gain in total surplus from sharing suppliers is $C(1) - C(\frac{1}{2})$ and supplier sharing occurs only if
$$F < \frac{v_1 \vee v_2}{2} + C\left(\frac{1}{2}\right) - C(1).$$
The planner avoids supplier sharing when the inequality holds in reverse, and is indifferent when both sides of the inequality equal.

- Suppose $q_{argmin(v)} = 0$ when supplier sharing doesn't occur. The gain in total surplus from sharing suppliers is $\frac{v_1+v_2}{2} - C(1)$ and supplier sharing occurs only if
$$F < \frac{v_1 + v_2}{2} - C(1).$$
The planner avoids supplier sharing when the inequality holds in reverse, and is indifferent when both sides of the inequality equal.

Notice when $v_1 = v_2 = AC\left(\frac{1}{2}\right)$, both inequalities displayed above equal each other. So the planner instructs supplier sharing only if
$$F < \frac{v_1 + v_2}{2} - C(1), \tag{2.40}$$
avoids supplier sharing when the inequality holds in reverse, and is indifferent when both sides of the inequality equal.

The expressions in (2.39) and (2.40) equal each other when $v_1 \vee v_2 = AC\left(\frac{1}{2}\right)$. The planner thus instructs supplier sharing if
$$F < \frac{v_1 \wedge v_2}{2} - C(1) + C\left(\frac{1}{2}\right) = \hat{F},$$
avoids supplier sharing if the above inequality is reversed, and is indifferent when both sides of the inequality equal.

Suppose $v_1 \vee v_2 < AC\left(\frac{1}{2}\right)$. Then both manufacturers operate only when supplier sharing occurs by the planner's instructions. The planner instructs supplier sharing if the gain from doing so $\frac{v_1+v_2}{2} - C(1)$ outweighs the cost

$$F < \frac{v_1 + v_2}{2} - C(1) = \hat{F},$$

avoids supplier sharing if the inequality above is reversed, and is indifferent if both sides equal.

□

Proof of Proposition 2.2.2. Let $\underline{\boldsymbol{\pi}}$ denote the firms' profits when $\mathbf{a} \neq \mathbf{I}$ and $\mathbf{p} = \mathbf{v}$. These profits are formally defined by (2.46) and (2.47), displayed later. Write total surplus as

$$TS = \sum_m \underline{\pi}_m^M + \sum_s \underline{\pi}_s^S + \int_{F=\underline{F}}^{F=\hat{F}} \hat{F} - F dG = \sum_m \frac{1}{2}\left(v_m - AC\left(\frac{1}{2}\right)\right) \vee 0 + \int_{F=\underline{F}}^{F=\hat{F}} \hat{F} - F dG.$$

Proposition 2.2.1 implies $\hat{F}$ is differentiable in $(\mathbf{v}, c)$ whenever $v_m \neq AC\left(\frac{1}{2}\right)$ for $m \leq 2$. It follows TS as written above is differentiable in $(\mathbf{v}, c) >> 0$ unless $v_m = AC\left(\frac{1}{2}\right)$ for some $m \leq 2$. Moreover, at all such points,

$$\begin{aligned}
\frac{\partial TS}{\partial v_m} &= \frac{1}{2}\left\{v_m > AC\left(\frac{1}{2}\right)\right\} + \int_{F=\underline{F}}^{F=\hat{F}} \frac{\partial \hat{F}}{\partial v_n} dG \\
\frac{\partial TS}{\partial c} &= -\frac{1}{2}\sum_m \left\{v_m > AC\left(\frac{1}{2}\right)\right\} + \int_{F=\underline{F}}^{F=\hat{F}} \frac{\partial \hat{F}}{\partial c} dG.
\end{aligned}$$

But Proposition 2.2.1 implies

$$\frac{\partial \hat{F}}{\partial v_m} = \frac{1}{2}\left\{v_m < AC\left(\frac{1}{2}\right)\right\} \quad \frac{\partial \hat{F}}{\partial c} = -\frac{1}{2}\left\{v_1 \wedge v_2 < AC\left(\frac{1}{2}\right) < v_1 \vee v_2\right\} - \left\{v_1 \vee v_2 < AC\left(\frac{1}{2}\right)\right\}$$

at all differentiable points of $\hat{F}$ or equivalently, TS. Hence

$$\begin{aligned}
\frac{\partial TS}{\partial v_m} &= \frac{\left\{v_m > AC\left(\frac{1}{2}\right)\right\}}{2} + \frac{\left\{v_m < AC\left(\frac{1}{2}\right)\right\}}{2} G(\hat{F}) = \mathbb{E} q_m \\
\frac{\partial TS}{\partial c} &= -\sum_m \frac{\left\{v_m > AC\left(\frac{1}{2}\right)\right\}}{2} - \frac{\left\{v_1 \wedge v_2 < AC\left(\frac{1}{2}\right) < v_1 \vee v_2\right\}}{2} G(\hat{F}) - \left\{v_1 \vee v_2 < AC\left(\frac{1}{2}\right)\right\} G(\hat{F}) \\
&= -\sum_m \mathbb{E} q_m = -\sum_s \mathbb{E} q_s^S,
\end{aligned}$$

for both $m \leq 2$. □

The following results concern the game's (Bayes Nash) equilibria. The equilibria of the game are analyzed using backward induction as in Subsection 2.2.2.

Stage 6: The restrictions imposed on the consumers' strategies imply each manufacturer's demand when operating is

$$q_m = \frac{1}{2}\{p_m \leq v_m\}.$$

The manufacturers' demand schedules thus depend only on their own output prices. In particular, demand for their products conditional on entry simplifies to

$$q_m(s_{-m}, \mathbf{p}) = q_m(p_m) = \frac{1}{2}\{p_m \leq v_m\} \quad \forall m \leq 2.$$

Stage 5: Manufacturer 1 operates when (non-negative) surplus is created by converting its affiliated supplier's inputs into output sold at price p_1 to its consumers:

$$s_1 = \begin{cases} \text{Operate} & \text{if } p_1 \geq \{s_2 = 1\}\Delta C(q_1(p_1), q_2(p_2)) + \{s_2 \neq 1\} AC(q_1(p_1)) \\ \text{Shut Down} & \text{if } p_1 < \{s_2 = 1\}\Delta C(q_1(p_1), q_2(p_2)) + \{s_2 \neq 1\} AC(q_1(p_1)) \end{cases}. \tag{2.41}$$

Manufacturer 2's chosen supplier or shut down decision satisfies

$$s_2 \in \begin{cases} \text{argmin}_s b_s & \text{if } b_1 \wedge b_2 < p_2,\ q_1(p_2) > 0 \\ \text{Shut Down} & \text{if } b_1 \wedge b_2 > p_2 \\ \{\text{argmin}_s b_s, \text{Shut Down}\} & \text{if otherwise} \end{cases} . \tag{2.42}$$

Note when $q_m(p_m) = 0$, manufacturer m effectively shuts down, regardless of s_m's value.

Stage 4: Suppose $\mathbf{a} \neq \mathbf{I}$ so that the relationship network is disconnected. Then Nash bargaining between manufacturer m and its affiliated supplier implies their input price is

$$t_m = \beta_m AC\left(\frac{1}{2}\right) + (1-\beta_m)p_m \quad \forall m \leq 2,$$

whenever the manufacturer operates, or $AC\left(\frac{1}{2}\right) \leq p_m \leq v_m$. Each manufacturer's profit when setting its output price in Stage 3 thus equals

$$\pi_m^M = \begin{cases} \frac{1}{2}\beta_m\left(p_m - AC\left(\frac{1}{2}\right)\right) \vee 0 & \text{if } p_m \leq v \\ 0 & \text{if otherwise} \end{cases} .$$

Notice each manufacturer's profit does not depend on it's downstream rival's output price.

Suppose $\mathbf{a} = \mathbf{I}$ so that the relationship network is connected. Then Nash bargaining between manufacturer 1 and supplier 1 implies

$$t_1 = \beta_1\left(\{s_2(\mathbf{p},\mathbf{b}) = 1\}\Delta C\left(\frac{1}{2}, \frac{\{p_2 \leq v_2\}}{2}\right) + \{s_2(\mathbf{p},\mathbf{b}) \neq 1\}AC\left(\frac{1}{2}\right)\right) + (1-\beta_1)p_1 \tag{2.43}$$

whenever there is non-negative surplus to split between both firms or equivalently, if the expression for t_1 above is non-negative and $p_1 \leq v_1$.

The input price paid by manufacturer 2 - when it operates - depends on bids $\mathbf{b}$ submitted by the suppliers. In equilibrium, these bids in turn, depend on output prices $\mathbf{p}$ and the suppliers' common cost structure. However, by analyzing the suppliers' payoffs as functions of $\mathbf{b}$ and $\mathbf{p}$, one can enumerate the Nash equilibria bids $\mathbf{b}(\mathbf{p})$ for each possible configuration of $\mathbf{p}$. One thus learns how manufacturer 2's input price and consequentially, profit, depends on $\mathbf{p}$. Doing so obtains the following lemma, applicable in the subgame $\Gamma(\mathbf{I}, \mathbf{p})$.

Lemma 2.9.5. *Let $H(p) \equiv (1-\beta_1)\left(p \wedge AC\left(\frac{1}{2}\right) - \Delta C\left(\frac{1}{2}, \frac{1}{2}\right)\right) \vee 0$. Suppose assumptions 2.2.2 and 2.2.3 hold. When $\mathbf{a} = \mathbf{I}$, the manufacturers' Stage 3 payoffs equal*

$$\pi_1^M(\mathbf{p}) = \begin{cases} \frac{\beta_1}{2}\left(p_1 - \{s_2 = 1\}\Delta C\left(\frac{1}{2}, \frac{\{p_2 \le v_2\}}{2}\right) + \{s_2 \neq 1\} AC\left(\frac{1}{2}\right)\right) \vee 0 & \text{if } p_1 \le v_1 \\ 0 & \text{if otherwise} \end{cases}$$

$$\pi_2^M(\mathbf{p}) = \begin{cases} \frac{\beta_2}{2}\left(p_2 - AC\left(\frac{1}{2}\right)\right) \vee 0 - T_{21} & \text{if } p_2 \le v_2 \\ -T_{21} & \text{if otherwise} \end{cases}$$

where $s_2 = s_2(\mathbf{p}, \mathbf{b}(\mathbf{p}))$. Manufacturer 2's supplier-or-shut down decision satisfies

$$s_2(\mathbf{p}, \mathbf{b}(\mathbf{p})) = \begin{cases} \text{Shut Down} & \Rightarrow p_2 \le AC\left(\frac{1}{2}\right) - H(p_1) \text{ or } p_2 > v_2 \\ 1 & \Rightarrow AC\left(\frac{1}{2}\right) - H(p_1) \le p_2 \le v_2 \\ 2 & \Rightarrow AC\left(\frac{1}{2}\right) - H(p_1) = AC\left(\frac{1}{2}\right) = p_2 \end{cases}$$

when $p_1 \le v_1$.

Proof of lemma 2.9.5. Nash bargaining between manufacturer 1 and supplier 1 implies (2.43) when there is surplus to split between both firms:

$$p_1 > \{s_2 = 1\}\Delta C\left(\frac{1}{2}, q_1(p_1)\right) + \{s_2 \neq 1\} AC\left(\frac{1}{2}\right); \quad s_2 = s_2(\mathbf{p}, \mathbf{b}).$$

So manufacturer 1's profit is

$$\pi_1^M(\mathbf{p}) = \begin{cases} \frac{\beta_1}{2}\left(p_1 - \{s_2 = 1\}\Delta C\left(\frac{1}{2}, \frac{1}{2}\{p_1 \leq v_1\}\right) + \{s_2 \neq 1\}AC\left(\frac{1}{2}\right)\right) \vee 0 & \text{if } p_1 \leq v_1 \\ 0 & \text{if otherwise.} \end{cases}$$

where $s_2 = s_2(\mathbf{p}, \mathbf{b}(\mathbf{p}))$, verifying the first identity of Lemma 2.9.5.

Each supplier's profit (and hence its optimal bid) depends on $\mathbf{p}$. There are two cases two consider. First, suppose $p_1 > v$. Then $q_1(p_1) = 0$, implying the suppliers' profits are

$$\pi_s^S = \begin{cases} \frac{1}{2}(b_2 - AC(\frac{1}{2})) + \{s = 1\}(T_{21} - F) & \text{if } s_2 = s, p_2 \leq v_2 \\ \{s = 1\}(T_{21} - F) & \text{if otherwise.} \end{cases} \tag{2.44}$$

The outcomes of NEs in $\mathbf{b}$ and $\mathbf{s}(\mathbf{p}, \mathbf{b})$ are easy to enumerate.

1. Suppose $v_2 < p_2$. Then each supplier $s \leq 2$ is indifferent between any bid $b_s \geq 0$ since $q_2(p_2) = 0$. Manufacturer 2 trivially makes zero variable profit.

2. Suppose $AC\left(\frac{1}{2}\right) < p_2 \leq v_2$. The situation facing bidders is analogous to a FPA. By (2.41), both suppliers bid their unit cost

$$b_1 = b_2 = AC\left(\frac{1}{2}\right)$$

   while manufacturer 2 chooses either supplier on the equilibrium path - $s_2(\mathbf{p}, \mathbf{b}) \in \{1, 2\}$ in the subgame's unique NE. This is the auction's intuitive outcome. Thus, it satisfies Assumption 2.2.3.

3. Suppose $AC\left(\frac{1}{2}\right) = p_2 \leq v_2$. There are two possible classes of NEs.

   - At least one supplier submitting the only bid that is simultaneously acceptable while covering its own cost

$$b_1 \wedge b_2 = AC\left(\frac{1}{2}\right)$$

is an NE for any choice by manufacturer 2 over its suppliers and shutting down $s_2(\mathbf{p}, \mathbf{b}) \in \{1, 2, SD\}$ on the equilibrium path. Manufacturer 2 makes zero variable profit in these equilibria.

- Neither supplier submitting an acceptable bid

$$b_1 \wedge b_2 > AC\left(\frac{1}{2}\right)$$

with manufacturer 2 shutting down on the equilibrium path, is also an NE. This yields the same payoffs to manufacturers as in the first NE class.

4. Suppose $p_2 < AC\left(\frac{1}{2}\right), p_2 \leq v_2$. Clearly, neither supplier wishes to win the auction since doing so incurs a loss. It follows $b_1 \wedge b_2 \geq p_2$ in any NE.

- Observe any bids that cover each supplier's average cost

$$b_1 \wedge b_2 \geq AC\left(\frac{1}{2}\right)$$

is an NE. Manufacturer 2 shuts down on the equilibrium path and makes no variable profit. These bids also satisfy Assumption 2.2.3 since they are weakly undominated by bidding $AC\left(\frac{1}{2}\right)$.

- When either supplier submits a bid smaller than its average cost but exceeding the effective reserve bid p_2, so that

$$p_2 \leq b_1 \wedge b_2 < AC\left(\frac{1}{2}\right)$$

(and manufacturer 2 shuts down on the equiibirum path when $p_2 < b_1 \wedge b_2$), then the bids and consequential manufacturers' decisions under (2.41) and (2.42), are an NE too. Since $p_2 \leq b_s < AC\left(\frac{1}{2}\right)$ is weakly dominated by $b_s = AC\left(\frac{1}{2}\right)$, these NEs do not satisfy Assumption 2.2.3 when $p_2 \leq b_2 < AC\left(\frac{1}{2}\right)$. However, irrespetive of whether Assumption 2.2.3 is satisfied, manufacturer 2's payoff remain the same as in the first NE class.

So manufacturer 2's profit is simply

$$\pi_2^M(\mathbf{p}) = \begin{cases} \frac{\beta_2}{2}\left(p_2 - AC\left(\frac{1}{2}\right)\right) \vee 0 - T_{21} & \text{if } p_2 \leq v_2 \\ -T_{21} & \text{if otherwise.} \end{cases}$$

So the second identity of Lemma 2.9.5 holds when $p_1 > v_1$.

Second, suppose $p_1 \leq v$. Then $q_1(p_1) = \frac{1}{2}$, implying suppliers' profits are

$$\begin{aligned} \pi_1^S &= \begin{cases} \frac{1-\beta_1}{2}\left(p_1 - \Delta C\left(\frac{1}{2}, \frac{1}{2}\right)\right) + \frac{1}{2}\left(b_2 - AC\left(\frac{1}{2}\right)\right) + T_{21} - F & \text{if } s_2 = 2, p_2 \leq v_2 \\ \frac{1-\beta_1}{2}\left(p_1 - AC\left(\frac{1}{2}\right)\right) \vee 0 + T_{21} - F & \text{if otherwise.} \end{cases} \\ \pi_2^S &= \begin{cases} \frac{1}{2}\left(p_2 - AC\left(\frac{1}{2}\right)\right) \vee 0 & \text{if } s_2 = 1, p_2 \leq v_2 \\ 0 & \text{if otherwise.} \end{cases} \end{aligned} \tag{2.45}$$

One can hence enumerate the NE bids $\mathbf{b}$ in the subgame $\Gamma(\mathbf{I}, \mathbf{p})$ for varying $\mathbf{p}$ satisfying $p_1 \leq v_1$. (2.41) and (2.42) already pin down $\mathbf{s}(\mathbf{p}, \mathbf{b})$ in any such NE when $b_1 \neq b_2$. Moreover, observe when (2.41) doesn't define $s_1(\mathbf{p}, \mathbf{b})$, manufacturer 1 is indifferent between operating and shutting down, and makes no profit either way. Likewise, when $b_1 \wedge b_2 = p_2$, manufacturer 2 makes no profit regardless of how $s_2(\mathbf{p}, \mathbf{b})$ is defined. Thus, all that remains is to enumerate NE bids $\mathbf{b}$ and manufacturer 2's supplier-or-quit decision $s_2(\mathbf{p}, \mathbf{b})$ when $b_1 = b_2$.

1. Suppose $v_2 < p_2$. Then each supplier is indifferent between any bid $b_s \geq 0$ while $q_2 = q_2(p_2) = 0$ by (2.42).

2. Suppose $AC(\frac{1}{2}) < p_2 \leq v_2$.

    - Observe the intuitive solution where both suppliers bid the less efficient unit cost $b_1 = b_2 = AC(\frac{1}{2})$ and the more efficient supplier wins the auction when manufacturer 1 operates

$$p_1 > \Delta C\left(\frac{1}{2}, \frac{1}{2}\right) \Rightarrow s_2 = 1,$$

is an NE outcome of $\Gamma(\mathbf{I}, \mathbf{p})$. It satisfies Assumption 2.2.3.

- Observe for each b satisfying $AC\left(\frac{1}{2}\right) - H(p_1) \leq b \leq AC(\frac{1}{2})$, both suppliers bidding $b_1 = b_2 = b$, and manufacturer 2 choosing supplier 1 $s_2 = 1$ when that occurs, is also (part of) an NE. To see why, consider supplier 1's profit from supplying both manufacturers at input price b_2:

$$\frac{b_2}{2} - AC\left(\frac{1}{2}\right) + \frac{1-\beta_1}{2}\left(p_1 - \Delta C\left(\frac{1}{2}, \frac{1}{2}\right)\right) = \frac{1}{2}\left(b_2 - AC\left(\frac{1}{2}\right) + H(p_1)\right).$$

It is increasing in b_2. So supplier 1 cannot increase its profit by defecting to $b_2 \leq b$. It is non-negative at $b_2 = b$. So supplier 1 finds it weakly profitable to supply both suppliers vis-a-vis defecting to $b_2 > b$. Likewise, by (2.42), supplier 2 makes zero variable profit and cannot increase it by defecting. Finally, note these NEs do not satisfy Assumption 2.2.3.

3. Suppose $p_2 \leq AC\left(\frac{1}{2}\right) \wedge v_2$.

- Consider NEs whereby supplier 1 supplies an operating manufacturer 2. Then

$$AC\left(\frac{1}{2}\right) - H(p_1) \leq b_1 = p_2 \wedge b_2$$

for supplier 1's bid to be IC, and for manufacturer 2 to choose supplier 1 on the equilibrium path. But since $b_2 < p_2$ implies $b_2 < AC(\frac{1}{2})$, the NE does not satisfy Assumption 2.2.3 when $b_2 < p_2$. So the only NEs in this class feature

$$AC\left(\frac{1}{2}\right) - H(p_1) \leq b_1 = p_2 \leq b_2.$$

Manufacturer 2 thus makes no variable profit from operating in these equilibria.

- Consider NEs whereby supplier 2 supplies an operating manufacturer 2. Then

$$AC\left(\frac{1}{2}\right) \leq b_2 = p_2 \wedge b_1 \Rightarrow b_2 = p_2 = AC\left(\frac{1}{2}\right)$$

for supplier 2's bid to be IC. Moreover, $H(p_1) = 0$ or equivalently, $p_1 \leq \Delta C\left(\frac{1}{2}, \frac{1}{2}\right)$ for supplier 1 to have no incentive to undercut its rival. It follows

$$p_1 \leq \Delta C\left(\frac{1}{2}, \frac{1}{2}\right), \quad p_2 = AC\left(\frac{1}{2}\right) = AC\left(\frac{1}{2}\right) - H(p_1) \leq v_2$$

for this to occur - neither supplier makes a positive profit from supplying manufacturer 2, which makes zero variable profit.

- Consider NEs where neither supplier supplies manufacturer 2. Then $b_1 \wedge b_2 \geq p_2$ for manufacturer 2 to have no incentive to operate. Also $p_2 \leq AC\left(\frac{1}{2}\right) - H(p_1)$ for both suppliers to have no incentive to shade their bids. Moreover, for these NEs to satisfy Assumption 2.2.3, $AC\left(\frac{1}{2}\right) \leq b_2$ so that supplier 2's bid is not weakly dominated by bidding its own unit cost. Again, manufacturer 2 makes zero variable profit.

It follows from the enumeration of NEs satisfying Assumption 2.2.3 that

$$\pi_2^M(\mathbf{p}) = \begin{cases} \frac{\beta_2}{2}\left(p_2 - AC\left(\frac{1}{2}\right)\right) \vee 0 - T_{21} & \text{if } p_2 \leq v_2 \\ -T_{21} & \text{if otherwise.} \end{cases}$$

So the second identity of Lemma 2.9.5 holds when $p_1 \leq v_1$. Moreover, the exhaustive analysis of all possible NE production networks above implies

$$s_2(\mathbf{p}, \mathbf{b}(\mathbf{p})) = \begin{cases} \text{Shut Down} & \Rightarrow p_2 \leq AC\left(\frac{1}{2}\right) - H(p_1) \text{ or } p_2 > v_2 \\ 1 & \Rightarrow AC\left(\frac{1}{2}\right) - H(p_1) \leq p_2 \leq v_2 \\ 2 & \Rightarrow AC\left(\frac{1}{2}\right) - H(p_1) = AC\left(\frac{1}{2}\right) = p_2 \end{cases}$$

when $p_1 \leq v_1$. □

Stage 3: Suppose $\mathbf{a} \neq \mathbf{I}$ so that the relationship network is disconnected. Each manufacturer's best response is

$$BR(p_{-m}) \in \begin{cases} v_m & \text{if } AC\left(\frac{1}{2}\right) < v_m \\ \mathbb{R}_{\geq 0} & \text{if } AC\left(\frac{1}{2}\right) = v_m \\ \left[0, AC\left(\frac{1}{2}\right)\right) \cup (v_m, \infty) & \text{if otherwise} \end{cases} = \begin{cases} v_m & \text{if } AC\left(\frac{1}{2}\right) < v_m \\ \mathbb{R}_{\geq 0} & \text{if otherwise} \end{cases}.$$

So each manufacturer $m \leq 2$ sets the monopolist's price

$$p_m = v_m$$

when its market segment's willingness-to-pay v_m strictly exceeds the average cost of supplying the segment $AC\left(\frac{1}{2}\right)$, and is indifferent between any price if otherwise.

Each manufacturer's profit in the $\mathbf{a} \neq \mathbf{I}$ subgame is thus uniquely pinned down by

$$\underline{\pi}_m^M = \frac{\beta_m}{2}\left(v_m - AC\left(\frac{1}{2}\right)\right) \vee 0 \quad \forall m \leq 2. \tag{2.46}$$

In contrast to the no product differentiation environment, the manufacturers' profit margins are that of a monopolist's. Each supplier's profit is also uniquely determined by

$$\underline{\pi}_s^S = \frac{1-\beta_s}{2}\left(v_s - AC\left(\frac{1}{2}\right)\right) \vee 0 \quad \forall s \leq 2. \tag{2.47}$$

Product differentiation thus enlarges the suppliers' profit too.

Suppose $\mathbf{a} = \mathbf{I}$ so that the relationship network is connected instead. Observe each configuration of output prices $\mathbf{p}$ implies a distinct set of suppliers' bids $\mathbf{b}(\mathbf{p})$ prescribed by the equilibrium strategies $\boldsymbol{\sigma}|_{\mathbf{p},\mathbf{I}}$. However, by analyzing the manufacturers' Stage 3 payoff functions derived above, one can enumerate the set of equilibrium $\mathbf{p}$. The next set of lemmas apply in $\Gamma(\mathbf{I})$.

Lemma 2.9.6. *Suppose assumptions 2.2.2 and 2.2.3 hold, and define $H(v)$ as in Lemma 2.9.5.*

If $v_1 \wedge v_2 > AC\left(\frac{1}{2}\right)$*, then manufacturer* 2 *chooses supplier* 1*,* $s_2 = 1$*, and each firm's payoff equals*

$$\pi_1^M = \frac{\beta_1}{2}\left(v_1 - \Delta C\left(\frac{1}{2},\frac{1}{2}\right)\right) \qquad \pi_2^M = \frac{1}{2}\left(v_2 - AC\left(\frac{1}{2}\right)\right) - T_{21}$$
$$\pi_1^S = \frac{1-\beta_1}{2}\left(v_1 - \Delta C\left(\frac{1}{2},\frac{1}{2}\right)\right) + T_{21} - F \qquad \pi_2^S = 0$$

in any equilibrium of the subgame initiated when Stage 2 investment decisions $\mathbf{a}$ *equal* $\mathbf{I}$*.*

Proof of Lemma 2.9.6. Each manufacturer's best response is well defined. Because $v_1 > AC\left(\frac{1}{2}\right) \geq \Delta C\left(\frac{1}{2},\frac{1}{2}\right)$, $\pi_1^M(\mathbf{p})$ implies manufacturer 1's BR is to set $p_1 = v_1$. Similarly, $v_2 > AC\left(\frac{1}{2}\right)$ implies manufacturer 2's BR is to set $p_1 = v_1$. It follows $\boldsymbol{\pi}$ satisfy their stated expressions in the lemma. Notice neither manufacturer has an incentive to alter its price after Stage 5. Assumption 2.2.6 thus holds. Since the NE is unique, Assumptions 2.2.4 and 2.2.5 also hold. Lemma 2.9.5 also implies $s_2 = 1$ since $H(v_1) > 0$ and $p_m = v_m$ for $m \leq 2$. □

Lemma 2.9.7. *Suppose* $v_2 \leq AC\left(\frac{1}{2}\right) < v_1$ *and assumptions 2.2.2 to 2.2.3 hold. Then the firms' Nash equilibria payoffs for the subgame initiated by* $\mathbf{a} = \mathbf{I}$ *under Assumption 2.2.4, coincide with those obtained under assumptions 2.2.5 to 2.2.6. Moreover,*

i) $AC(1/2) - H(v_1) \leq v_2$ *implies manufacturer* 2 *chooses supplier* 1*,* $s_2 = 1$*, and profits equal*

$$\pi_1^M = \frac{\beta_1}{2}\left(v_1 - \Delta C\left(\frac{1}{2},\frac{1}{2}\right)\right) \qquad \pi_2^M = -T_{21}$$
$$\pi_1^S = \frac{1-\beta_1}{2}\left(v_1 - \Delta C\left(\frac{1}{2},\frac{1}{2}\right)\right) + \frac{1}{2}\left(v_2 - AC\left(\frac{1}{2}\right)\right) + T_{21} - F \qquad \pi_2^S = 0$$

ii) $v_2 < AC(1/2) - H(v_1)$ *implies manufacturer* 2 *has no market share,* $q_2 = 0$*, and profits equal*

$$\pi_1^M = \frac{\beta_1}{2}\left(v_1 - AC\left(\frac{1}{2}\right)\right) \qquad \pi_2^M = -T_{21}$$
$$\pi_1^S = \frac{1-\beta_1}{2}\left(v_1 - AC\left(\frac{1}{2}\right)\right) + T_{21} - F \qquad \pi_2^S = 0$$

from any of these equilibria's paths, where $H(v)$ is defined as in Lemma 2.9.5.

Proof of Lemma 2.9.7. Since $\Delta C\left(\frac{1}{2}, \frac{1}{2}\right) \leq AC\left(\frac{1}{2}\right) < v_1$, manufacturer 1 sets the monopolist's price in any NE

$$p_1 = v_1 \qquad s_1(\mathbf{p}, \mathbf{b}(\mathbf{p})) = \mathrm{Op} \qquad (\Rightarrow H(p_1) \neq 0).$$

Since only one supplier is capable of supplying manufacturer 2 when $v_2 < AC\left(\frac{1}{2}\right)$, manufacturer 2 makes zero variable profit, regardless of how its price is set. This yields multiple NEs, which are enumerated below.

1. Suppose $AC\left(\frac{1}{2}\right) - H(v_1) \leq v_2$.

   - In one NE, manufacturer 2 operates and sets the monopolist price:

   $$p_m = v_m \quad \forall m \leq 2.$$

   Since $AC\left(\frac{1}{2}\right) > \Delta C\left(\frac{1}{2}, \frac{1}{2}\right)$, manufacturer 2 is supplied by supplier 1 (on the equilibrium path):

   $$H(p_1) = H(v_1) \neq 0 \Rightarrow s_2(\mathbf{p}, \mathbf{b}(\mathbf{p})) = 1, \qquad \left(b_1 \geq AC\left(\frac{1}{2}\right), b_2 = p_2 = v_2\right).$$

   Hence, each firm's equilibrium profit is

   $$\begin{aligned} \pi_1^M &= \frac{\beta_1}{2}\left(v_1 - \Delta C\left(\frac{1}{2}, \frac{1}{2}\right)\right) & \pi_2^M &= -T_{21} \\ \pi_1^S &= \frac{1-\beta_1}{2}\left(v_1 - \Delta C\left(\frac{1}{2}, \frac{1}{2}\right)\right) + \frac{1}{2}\left(v_2 - AC\left(\frac{1}{2}\right)\right) + T_{21} - F & \pi_2^S &= 0. \end{aligned}$$

   Since neither manufacturer has an incentive to alter its price after Stage 5. Because

   $$\pi_1^S + \pi_2^M = \frac{1-\beta_1}{2}\left(v_1 - \Delta C\left(\frac{1}{2}, \frac{1}{2}\right)\right) + \frac{1}{2}\left(v_2 - AC\left(\frac{1}{2}\right)\right) - F$$

   is the maximal possible joint surplus manufacturer 2 and supplier 1 can possibly realize across all $(\mathbf{p}, \mathbf{b}, \mathbf{s})$ (by (2.44), (2.45) and π_2^M as described in Section 2.2), it is pareto efficient amongst SPNEs for $\Gamma(\mathbf{I})$. So Assumption 2.2.4 holds. Assumption 2.2.5 also holds. Finally, since prices equal $\mathbf{v}$, Assumption 2.2.6 also holds.

- In an alternate class of NEs, manufacturer 2 operates but doesn't set the monopolist's price. It's price is sufficiently high to allow a supplier (supplier 1) to submit an acceptable bid while covering its own cost:

$$p_1 = v_1, \qquad AC\left(\frac{1}{2}\right) - H(v_1) \leq p_2 \leq v_2.$$

Manufacturer 2 is always supplied by supplier 1, since $p_2 < AC\left(\frac{1}{2}\right)$:

$$H(p_1) = H(v_1) \neq 0 \Rightarrow s_2(\mathbf{p}, \mathbf{b}(\mathbf{p})) = 1 \quad \left(b_1 \geq AC\left(\frac{1}{2}\right), b_2 = p_2\right)$$

Each firm's profit is

$$\begin{aligned} \pi_1^M &= \frac{\beta_1}{2}\left(v_1 - \Delta C\left(\frac{1}{2}, \frac{1}{2}\right)\right) & \pi_2^M &= -T_{21} \\ \pi_1^S &= \frac{1-\beta_1}{2}\left(v_1 - \Delta C\left(\frac{1}{2}, \frac{1}{2}\right)\right) + \frac{1}{2}\left(p_2 - AC\left(\frac{1}{2}\right)\right) + T_{21} - F & \pi_2^S &= 0 \end{aligned}$$

in equilibrium. Unfortunately, $p_2 < v_2$ implies manufacturer 2 strictly improves its payoff by setting $p_2 = v_2$ in Stage 5. These NEs thus do not satisfy Assumption 2.2.6. Moreover, manufacturer 2 and supplier 1's joint payoff is pareto dominated by the previous NE's payoff. Assumption 2.2.4 is thus violated too.

- In a final class of NEs, manufacturer 2 effectively shuts down by setting a price too low for supplier 1 to supply it or too high for consumers to frequent it, without incurring losses. More specifically,

$$p_1 = v_1 \qquad p_2 \in \left(0,\ AC\left(\frac{1}{2}\right) - H(v_1)\right) \cup (v_2,\ \infty).$$

Manufacturer 2 either shuts down formally in Stage 5, or shuts down implicitly by setting an unreasonably high price:

$$p_2 < AC\left(\frac{1}{2}\right) - H(v_1) \Rightarrow s_2(\mathbf{p}, \mathbf{b}(\mathbf{p})) = SD, \quad p_2 > v_2 \Rightarrow q_2(p_2) = 0.$$

Each firm's equilibrium profit is thus

$$\begin{aligned} \pi_1^M &= \frac{\beta_1}{2}\left(v_1 - AC\left(\frac{1}{2}\right)\right) & \pi_2^M &= -T_{21} \\ \pi_1^S &= \frac{1-\beta_1}{2}\left(v_1 - AC\left(\frac{1}{2}\right)\right) + T_{21} - F & \pi_2^S &= 0. \end{aligned}$$

Since $F > 0$, either supplier 1 or manufacturer 2 suffers a loss as compared to when it doesn't invest. The NE described is thus inconsistent with Assumption 2.2.5. The NE also yields a pareto inefficient outcome to manufacturer 2 and supplier 1, compared to the 1st NE. Assumption 2.2.4 is thus violated.

The above cases exhaust the possibilities for p_2's value.

2. Suppose $v_2 < AC\left(\frac{1}{2}\right) - H(v_1)$. So long as $p_2 \leq v_2$, supplier 1 cannot submit an acceptable bid without incurring a loss from winning the right to supply manufacturer 2. Manufacturer 2 thus shuts down in any of this subgame's NEs. Its price is either too low for supplier 1 to profitably supply it, or too high for its consumers to benefit from purchasing its product:

$$p_1 = v_1, \quad p_2 \geq 0 \qquad \left(b_2 \geq AC\left(\frac{1}{2}\right)\right).$$

Manufacturer 2 either shuts down formally in Stage 5, or shuts down implicitly on the equilibrium path:

$$p_2 \leq v_2 \Rightarrow s_2(\mathbf{p}, \mathbf{b}(\mathbf{p})) = SD, \qquad p_2 > v_2 \Rightarrow q_2(p_2) = 0.$$

Each firm's profit is thus

$$\pi_1^M = \frac{\beta_1}{2}\left(v_1 - AC\left(\frac{1}{2}\right)\right) \qquad\qquad \pi_2^M = -T_{21}$$
$$\pi_1^S = \frac{1-\beta_1}{2}\left(v_1 - AC\left(\frac{1}{2}\right)\right) + T_{21} - F \qquad\qquad \pi_2^S = 0.$$

Since no other NE can possibly improve supplier 1 and manufacturer 2's joint payoff when $v_2 < AC\left(\frac{1}{2}\right) - H(v_1)$, Assumption 2.2.4 is satisfied. Since no other NE (yielding pareto superior outcomes for manufacturer 2 and supplier 1 compared to when they don't invest) exists, Assumption 2.2.5 holds. To verify which of these NEs satisfy Assumption 2.2.6, recognize either $p_2 > v_2$ or $s_2 = SD$ on the equilibrium paths.

i) If $p_2 > v_2$ and $b_1 \wedge b_2 < v_2$, than a profitable deviation for manufacturer 2 does exist in Stage 5 - by defecting to $p_2 = v_2$. Assumption 2.2.6 is violated.

ii) When $p_2 > v_2$ and $b_1 \wedge b_2 \geq v_2$, manufacturer 2 cannot make a strict profit by operating even if granted the ability to do so after Stage 5. The NEs thus satisfy Assumption 2.2.6.

iii) If $s_2(\mathbf{p}, \mathbf{b}(\mathbf{p})) = SD$, its input price is not defined. The NEs thus satisfy Assumption 2.2.6.

So NEs satisfying all three assumptions exist in $\Gamma(\mathbf{I})$ under this configuration of parameters.

Whilst the set of NEs under Assumption 2.2.4 don't coincide with those under assumption 2.2.5 and 2.2.6, their equilibrium path firm payoffs do, and equal the expressions stated in the lemma. Moreover, $s_2 = 1$ if $s_2 \neq SD$. □

Lemma 2.9.8. *Suppose $v_1 \leq AC(\frac{1}{2}) < v_2$ and assumptions 2.2.2 to 2.2.3 hold. Then the firms' Nash equilibria of the subgame initiated when $\mathbf{a} = \mathbf{I}$ obtained under Assumption 2.2.4, coincide with those obtained under assumptions 2.2.5 to 2.2.6. Moreover, in any of these equilibria paths,* *profits equal*

$$\pi_1^M = \frac{\beta_1}{2}\left(v_1 - \Delta C\left(\frac{1}{2},\frac{1}{2}\right)\right) \vee 0 \qquad \pi_2^M = \frac{1}{2}\left(v_2 - AC\left(\frac{1}{2}\right)\right) - T_{21}$$
$$\pi_1^S = \frac{1-\beta_1}{2}\left(v_1 - \Delta C\left(\frac{1}{2},\frac{1}{2}\right)\right) \vee 0 + T_{21} - F \qquad \pi_2^S = 0.$$

Either manufacturer 2 *chooses supplier* 1 *(*$s_2 = 1$*) or* $v_1 < \Delta c\left(\frac{1}{2},\frac{1}{2}\right)$ *in these paths. For any such equilibrium featuring* $q_2 > 0, s_2 = 2$ *on its path, there exists another implying* $q_2 > 0, s_2 = 1$.

Proof of Lemma 2.9.8. Since $v_2 > AC\left(\frac{1}{2}\right)$, manufacturer 2 always sets the monopolist's price

$$p_2 = v_2, \qquad s_2(\mathbf{p}, \mathbf{b}(\mathbf{p})) \neq SD$$

in any NE. By Lemma 2.9.5, whether manufacturer 1 operates (produces positive market share) in turn depends on whether v_1 exceeds $\{s_2 = 1\}\Delta C\left(\frac{1}{2},\frac{1}{2}\right) + \{s_2 \neq 1\}AC\left(\frac{1}{2}\right)$, which in turn, depends on whether manufacturer 2 is supplied by supplier 1 or 2.

1. Suppose $\Delta C\left(\frac{1}{2},\frac{1}{2}\right) \leq v_1$. Then the NEs are as follows:

   - In one NE, both manufacturers operate by setting the monopolist's price

     $$p_m = v_m \quad \forall m \leq 2.$$

     Manufacturer 1 can only be feasibly supplied by a supplier of both manufacturers when $v_1 < AC\left(\frac{1}{2}\right)$. Moreover, $v_1 = AC\left(\frac{1}{2}\right)$ implies $v_1 > \Delta C\left(\frac{1}{2},\frac{1}{2}\right)$, implying supplier 1 can undercut its rival's bid and strictly profit. Hence

     $$s_2(\mathbf{p}, \mathbf{b}(\mathbf{p})) = 1 \qquad \left(b_1 = b_2 = AC\left(\frac{1}{2}\right)\right),$$

and

$$\pi_1^M = \frac{\beta_1}{2}\left(v_1 - \Delta C\left(\frac{1}{2}, \frac{1}{2}\right)\right) \qquad \pi_2^M = \frac{1}{2}\left(v_2 - AC\left(\frac{1}{2}\right)\right) - T_{21}$$
$$\pi_1^S = \frac{1-\beta_1}{2}\left(v_1 - \Delta C\left(\frac{1}{2}, \frac{1}{2}\right)\right) + T_{21} - F \qquad \pi_2^S = 0$$

in equilibrium. Since either manufacturer sets the monopolist's price, neither has an incentive to defect after Stage 5. Assumption 2.2.6 holds. $\pi_1^S + \pi_2^M$ cannot be possibly larger for alternate values of $(\mathbf{p}, \mathbf{b})$ (by (2.44), (2.45) and π_2^M descibed in Section 2.2). So assumptions 2.2.4 and 2.2.5 hold.

- When $\Delta C\left(\frac{1}{2}, \frac{1}{2}\right) = v_1$, alternate NEs where only manufacturer 2 operates exists. Manufacturer 2 sets the monopolist's price while manufacturer 1 is indifferent between any price:

$$p_1 \geq 0 \qquad p_2 = v_2.$$

Since $AC(\frac{1}{2}) < p_2$, both suppliers must bid the average cost, rendering manufacturer 2 indifferent between either supplier:

$$s_2(\mathbf{p}, \mathbf{b}(\mathbf{p})) = s_2\left(\mathbf{p}, AC\left(\frac{1}{2}\right)(1,1)\right) \neq SD.$$

Manufacturer 1 implicitly or explicitly shuts down whenever $p_1 < \Delta C\left(\frac{1}{2}, \frac{1}{2}\right)$ or $s_2(\mathbf{p}, \mathbf{b}(\mathbf{p})) \neq 1$, and implicitly shuts down when $\Delta C\left(\frac{1}{2}, \frac{1}{2}\right) < p_1$. Suppose it chooses to shut down when $p_1 = \Delta C\left(\frac{1}{2}, \frac{1}{2}\right)$, when it is indifferent between operating and shutting down:

$$s_1(\mathbf{p}, \mathbf{b}(\mathbf{p})) = SD.$$

Clearly an NE exists where $s_2 = 1$. The same is true for $s_2 = 2$. Moreover, each firm's equilibrium profit is always

$$\pi_1^M = 0 \qquad \pi_2^M = \frac{1}{2}\left(v_2 - AC\left(\frac{1}{2}\right)\right) - T_{21}$$
$$\pi_1^S = T_{21} - F \qquad \pi_2^S = 0,$$

and manufacturer 2 operates ($q_2 > 0$) in the NE path. Because $v_1 = \Delta C\left(\frac{1}{2}, \frac{1}{2}\right)$, manufacturer 2 and supplier 1's profits are pareto efficient across all possible $\mathbf{p}, \mathbf{b}$. Their payoffs are actually a special case of those of the previous NE class. Hence, assumptions 2.2.4 and 2.2.5 hold. Neither manufacturer has an incentive to adjust prices after Stage 5. So Assumption 2.2.6 also holds.

2. Suppose $v_1 < \Delta C\left(\frac{1}{2}, \frac{1}{2}\right)$. Then $\pi_1^M(\mathbf{p})$ from Lemma 2.9.5 implies manufacturer 1 never operates (produces positive market share) in any NE. Manufacturer 2 sets the monopolist's price, while manufacturer 1 is indifferent between any price:

$$p_1 \geq 0 \qquad p_2 = v_2 \qquad \left(b_1 = b_2 = AC\left(\frac{1}{2}\right)\right).$$

Manufacturer 1 explicitly shuts down or implicitly does so by setting $p_1 > v_1$. Since both suppliers have symmetric cost structures, manufacturer 2 operates, but is indifferent between either supplier:

$$p_1 \leq v_1 \Rightarrow s_1(\mathbf{p}, \mathbf{b}(\mathbf{p})) = SD, \qquad p_1 > v_1 \Rightarrow q_1(p_1) = 0$$
$$s_2(\mathbf{p}, \mathbf{b}(\mathbf{p})) = s_2\left(\mathbf{p}, AC\left(\left(\frac{1}{2}\right)(1,1)\right)\right) \neq SD.$$

So there exists an NE where $s_1 = 1$ occurs. Likewise, there exists an NE where $s_2 = 2$. Also, the firms' profits are

$$\pi_1^M = 0 \qquad\qquad \pi_2^M = \frac{1}{2}\left(v_2 - AC\left(\frac{1}{2}\right)\right) - T_{21}$$
$$\pi_1^S = T_{21} - F \qquad\qquad \pi_2^S = 0.$$

(and $q_2 > 0$) in any NE. Since these are the unique NE payoffs, Assumption 2.2.5 is trivially satisfied. Moreover, the pareto efficient joint payoff for manufacturer 2 and supplier 1 is actually attained over all $\mathbf{b}, \mathbf{p}$. Assumption 2.2.4 holds. Finally, neither manufacturer has an incentive to defect in Stage 5 - manufacturer 1 and supplier 1 cannot feasibly produce inputs and assemble output while manufacturer 2 is already setting the monopolist's price. So Assumption 2.2.6 holds.

Thus, the NEs satisfying Assumption 2.2.4 are precisely those satisfying assumptions 2.2.5 and 2.2.6. When $\Delta C\left(\frac{1}{2}, \frac{1}{2}\right) < v_1$, $s_2 = 1$. When $v_1 \leq \Delta C\left(\frac{1}{2}, \frac{1}{2}\right)$, there exists NEs satisfying all three assumptions where $q_2 > 0, s_2 = 1$ or $q_2 > 0, s_2 = 2$ occurs. □

Lemma 2.9.9. *Suppose $v_1 \vee v_2 \leq AC\left(\frac{1}{2}\right)$ and assumptions 2.2.2 to 2.2.3 hold. Then the firms' Nash equilibria payoffs for the subgame initiated when $\mathbf{a} = \mathbf{I}$ under Assumption 2.2.4, coincide with those obtained under assumptions 2.2.5 to 2.2.6. Moreover,*

i) $AC\left(\frac{1}{2}\right) - H(v_1) < v_2$ implies manufacturer 2 chooses supplier 1 $s_2 = 1$, and profits equal

$$\pi_1^M = \frac{\beta_1}{2}\left(v_1 - \Delta C\left(\frac{1}{2}, \frac{1}{2}\right)\right) \qquad\qquad \pi_2^M = -T_{21}$$
$$\pi_1^S = \frac{1-\beta_1}{2}\left(v_1 - \Delta C\left(\frac{1}{2}, \frac{1}{2}\right)\right) + \frac{1}{2}\left(v_2 - AC\left(\frac{1}{2}\right)\right) + T_{21} - F \qquad \pi_2^S = 0$$

ii) $v_2 \leq AC\left(\frac{1}{2}\right) - H(v_1)$ implies either $s_2 = 1$ or $v_1 \leq \Delta C\left(\frac{1}{2}, \frac{1}{2}\right)$ and manufacturer 2 produces nothing $q_2 = 0$, while profits equal

$$\pi_1^M = 0 \quad \pi_1^S = T_{21} - F \quad \pi_2^M = -T_{21} \quad \pi_2^S = 0$$

on the equilibria paths, where $H(v)$ is defined as in Lemma 2.9.5. For any such equilibrium featuring $q_2 > 0, s_2 = 2$ on its path, there exists another where $q_2 > 0, s_2 = 1$.

Proof of Lemma 2.9.9. Whether both manufacturers operate, share a single supplier in an NE depends on whether supplier 1 can supply both firms at a profit. When

$$p_1 \leq v_1, \qquad AC\left(\frac{1}{2}\right) - H(p_1) \leq p_2 \leq v_2,$$

supplier 1 can supply both firms positive quantities of inputs without incurring a loss by (2.45).

1. Suppose $AC\left(\frac{1}{2}\right) - H(v_1) < v_2$. Then the NEs can be categorized as follows:

   - Consider any NE where both manufacturers operate (produce positive market share) and use supplier 1. Observe

   $$AC\left(\frac{1}{2}\right) - H(v_1) < p_2 \leq v_2 \leq AC\left(\frac{1}{2}\right) \Rightarrow H(v_1) \neq 0 \Rightarrow \Delta\left(\frac{1}{2}, \frac{1}{2}\right) < v_1.$$

   So manufacturer 1 always sets the monopolist's price in any such NE.

   i) Suppose both manufacturers set monopolists' prices

   $$p_1 = v_1, \qquad p_2 = v_2.$$

   Equilibrium bids (satisfying Assumption (2.2.3)) are thus

   $$b_1 = v_2, \qquad b_2 \geq AC\left(\frac{1}{2}\right)$$

   to ensure supplier 1 is chosen while neither supplier defects. On the equilibrium path,

$s_2 = 1$ and each firm's profit is thus

$$\pi_1^M = \frac{\beta_1}{2}\left(v_1 - \Delta C\left(\frac{1}{2}, \frac{1}{2}\right)\right) \qquad \pi_2^M = -T_{21}$$
$$\pi_1^S = \frac{1-\beta_1}{2}\left(v_1 - \Delta C\left(\frac{1}{2}, \frac{1}{2}\right)\right) + \frac{1}{2}\left(v_2 - AC\left(\frac{1}{2}\right)\right) + T_{21} - F \qquad \pi_2^S = 0.$$

Observe $\pi_1^S + \pi_2^M$ is maximized over all $\mathbf{p}, \mathbf{b}$ (by (2.45) and π_2^M described in Section 2.2). So assumptions 2.2.4 and 2.2.5 hold. Moreover, neither manufacturer has any incentive to alter its price. So Assumption 2.2.6 holds too.

ii) Suppose only manufacturer 1 sets the monopolist's price:

$$p_1 = v_1, \qquad p_2 < v_2.$$

Bids on the equilibrium path thus satisfy

$$b_1 = p_2, \qquad b_2 \geq AC\left(\frac{1}{2}\right)$$

by Assumption 2.2.3. Hence,

$$\pi_1^M = \frac{\beta_1}{2}\left(v_1 - \Delta C\left(\frac{1}{2}, \frac{1}{2}\right)\right) \qquad \pi_2^M = -T_{21}$$
$$\pi_1^S = \frac{1-\beta_1}{2}\left(v_1 - \Delta C\left(\frac{1}{2}, \frac{1}{2}\right)\right) + \frac{1}{2}\left(p_2 - AC\left(\frac{1}{2}\right)\right) + T_{21} - F \qquad \pi_2^S = 0.$$

Observe Assumption 2.2.4 is violated, since π_1^S is strictly smaller than in Case i), while π_1^M is the same. Also, because $b_1 = p_2 < v_2$, Assumption 2.2.6 is violated.

- Consider any NE in which both manufacturers operate, but use different suppliers. Manufacturers' prices are thus

$$p_1 = v_1 = AC\left(\frac{1}{2}\right), \qquad p_2 = v_2 = AC\left(\frac{1}{2}\right)$$

to ensure each supplier s can supply manufacturer $m = s$ without incurring a loss. Moreover, observe supplier 1 can undercut supplier 2's minimum acceptable bid ($b_2 = AC\left(\frac{1}{2}\right)$) and win the auction unless

$$p_1 = \Delta C\left(\frac{1}{2}, \frac{1}{2}\right) \Rightarrow H(p_1) = 0 \Rightarrow AC\left(\frac{1}{2}\right) - H(p_1) = AC\left(\frac{1}{2}\right) \geq v_2,$$

a contradiction to the hypothesis assumed. It follows no such an NE exists.

- Consider any NE in which only one manufacturer - manufacturer 1 - operates. Profit maximization by manufacturer 1 and supplier 1 implies

$$p_1 = v_1 = AC\left(\frac{1}{2}\right), \qquad p_2 < AC\left(\frac{1}{2}\right) - H(v_1) \text{ or } p_2 > v_2.$$

The equilibrium path bids (satisfying Asumption 2.2.3) thus satisfy

$$b_1 \geq p_2 \text{ if } p_2 < AC\left(\frac{1}{2}\right) - H(v_1), \qquad b_2 \geq AC\left(\frac{1}{2}\right).$$

Each firm thus makes zero variable profit

$$\begin{aligned} \pi_1^M &= \frac{\beta_1}{2}\left(v_1 - AC\left(\frac{1}{2}\right)\right) & \pi_2^M &= -T_{21} \\ \pi_1^S &= T_{21} - F & \pi_2^S &= 0. \end{aligned}$$

Notice (π_1^S, π_2^M) is pareto dominated by those of the first NE class, so Assumption 2.2.4 does not hold. The NE is also inconsistent with Assumption 2.2.5, since either manufacturer 2 or supplier 1 can improve its payoff by not investing, unlike in the 1st equilibria class.

- Consider any NE in which only manufacturer 2 operates. Then $p_2 = v_2 = AC\left(\frac{1}{2}\right)$ for manufacturer 2 and its supplier to avoid making a loss. Since $v_1 \geq p_1 > \Delta C\left(\frac{1}{2}, \frac{1}{2}\right)$ would induce supplier 1 to outbid its rival, we obtain from profit maximization,

$$v_1 \leq \Delta C\left(\frac{1}{2}, \frac{1}{2}\right), \qquad p_2 = v_2 = AC\left(\frac{1}{2}\right).$$

But this yields a contradiction since

$$v_1 \leq \Delta C\left(\frac{1}{2}, \frac{1}{2}\right) \Rightarrow H(v_1) = 0 \Rightarrow AC\left(\frac{1}{2}\right) - H(v_1) \geq v_2.$$

Hence no such NE exists.

- Consider any NE in which neither manufacturer operates. For supplier 1 to have no profitable deviations, downstream prices satisfy

$$p_1 \geq 0, \qquad p_2 \leq AC\left(\frac{1}{2}\right) - H(p_1) \text{ or } p_2 > v_2.$$

Equilibrium bids (satisfying Assumption 2.2.3) satisfy

$$b_2 \geq AC\left(\frac{1}{2}\right), \qquad b_1 \geq 0, \quad b_1 \geq p_2 \text{ if } p_2 \leq v_2$$

on the equilibrium path. Each thus makes no variable profit

$$\begin{aligned} \pi_1^M &= 0 & \pi_2^M &= -T_{21} \\ \pi_1^S &= T_{21} - F & \pi_2^S &= 0 \end{aligned}$$

in equilibrium. Observe (π_1^S, π_2^M) are pareto dominated by those of the first NE class. So Assumption 2.2.4 is violated. Moreover, Assumption 2.2.5 is also violated, since at least one of manufacturer 2 or supplier 1 can strictly increase its payoff from not investing, whereas the same argument doesn't apply to first NE class.

We see that manufacturer 2 always operates and chooses $s_2 = 1$.

2. Suppose $v_2 \leq AC\left(\frac{1}{2}\right) - H(v_1)$.

- Consider any NE whereby both manufacturers operate. If both manufacturers operate using different suppliers, then incentive compatibility for manufacturer 1, and loss avoidance by manufacturer 2 and supplier 2 requires

$$v_1 = v_2 = AC\left(\frac{1}{2}\right).$$

But since $v_2 \leq AC\left(\frac{1}{2}\right) - H(v_1)$,

$$v_2 = AC\left(\frac{1}{2}\right) \Rightarrow H(v_1) = 0 \Rightarrow v_1 \leq \Delta C\left(\frac{1}{2}, \frac{1}{2}\right) < AC\left(\frac{1}{2}\right),$$

a contradiction. So both manufacturers must use supplier 1 on the equilibrium path. Hence, profit maximization implies

$$p_1 = v_1, \qquad AC\left(\frac{1}{2}\right) - H(v_1) \leq p_2 \leq v_2 \Rightarrow p_2 = v_2 = AC\left(\frac{1}{2}\right) \text{ and } H(v_1) = 0.$$

By Assumption 2.2.3, suppliers submit bids satisfying

$$b_1 = p_2, \quad b_2 \geq AC\left(\frac{1}{2}\right).$$

Each firm's equilibrium profit is thus

$$\pi_1^M = \frac{\beta_1}{2}\left(v_1 - \Delta C\left(\frac{1}{2}, \frac{1}{2}\right)\right) = 0 \qquad \pi_2^M = -T_{21}$$
$$\pi_1^S = T_{21} - F \qquad \pi_2^S = 0.$$

(And as stated above, $s_2 = 1$ on the NE path.)[47] It follows from the profit expressions that

47. To see why $\pi_1^M = 0$, observe

$$0 = H(v_1) = (1-\beta_1)\left(v_1 \wedge AC(1) - \Delta C\left(\frac{1}{2}, \frac{1}{2}\right)\right) \vee 0 \Rightarrow 0 = \left(v_1 - \Delta C\left(\frac{1}{2}, \frac{1}{2}\right)\right) \vee 0 \Rightarrow v_1 \leq \Delta C\left(\frac{1}{2}, \frac{1}{2}\right).$$

manufacturer 2 and supplier 1 attain their maximal possible payoffs over $\mathbf{p}, \mathbf{b}$. Assumption 2.2.4 thus holds. Assumption 2.2.5 also holds, since no alternate NE with a superior payoff to one of manufacturer 2 and supplier 1 exists. Assumption 2.2.6 also holds, since neither manufacturer has an incentive to alter its price.

- Consider any NE in which only manufacturer 1 operates. Profit maximization by manufacturer 1 and incentive compatibility for supplier 1 implies

$$p_1 = v_1 = AC\left(\frac{1}{2}\right) \qquad p_2 \leq AC\left(\frac{1}{2}\right) - H(v_1) \text{ or } p_2 > v_2,$$

which are equivalent to

$$p_1 = v_1 = AC\left(\frac{1}{2}\right) \qquad p_2 \geq v_2,$$

under $v_2 \leq AC\left(\frac{1}{2}\right) - H(v_1)$. Each supplier submits an unacceptable bid, with supplier 2 bidding above unit cost to satisfy Assumption 2.2.3:

$$b_1 \geq p_2, \qquad b_2 \geq AC\left(\frac{1}{2}\right).$$

Each firm thus makes zero variable profit, since $v_1 = AC\left(\frac{1}{2}\right)$ and $q_2 = 0$:

$$\pi_1^M = 0 \qquad \pi_2^M = -T_{21}$$
$$\pi_1^S = T_{21} - F \qquad \pi_2^S = 0.$$

As with the 1st class of NEs, assumptions 2.2.4 and 2.2.5 must hold. Whether Assumption 2.2.6 holds depends on $\mathbf{b}$:

i) For $p_2 \leq b_1 \wedge b_2$, the NE satisfies Assumption 2.2.6.

ii) For $p_2 > v_2 > b_1 \wedge b_2$ and $s_2(\mathbf{p}, \mathbf{b}) \neq SD$, the NE does not satisfy Assumption 2.2.6 - manufacturer 2 can defect to $p_2 = v_2$, increase $q_2(p_2)$ by $\frac{1}{2}$ and strictly profit.

So $v_1 = \Delta C\left(\frac{1}{2}, \frac{1}{2}\right)$ for manufacturer 1 to not incur losses and violate IC.

Importantly, NEs satisfying all three assumptions exist. The payoffs of supplier 2 and manufacturer 1 are identical to those for the 1st NE class.

- Consider any NE in which only manufacturer 2 operates. Since only manufacturer 2 operates,
$$p_1 \leq AC\left(\frac{1}{2}\right) \quad \text{or } p_1 > v_1 \qquad p_2 = v_2 = AC\left(\frac{1}{2}\right)$$
to ensure manufacturer 1 shuts down (implicitly or explicitly) while manufacturer 2's supplier doesn't make a loss. But then
$$AC\left(\frac{1}{2}\right) - H(v_1) \geq v_2 = AC\left(\frac{1}{2}\right) \Rightarrow H(v_1) = 0 \Rightarrow v_1 \leq \Delta C\left(\frac{1}{2}, \frac{1}{2}\right)$$
(which doesn't yield any contradiction with the hypothesis $v_2 \leq AC\left(\frac{1}{2}\right) - H(v_1)$). Bids thus satisfy
$$b_1 \wedge b_2 = p_2 = v_2 = AC\left(\frac{1}{2}\right)$$
on the equilibrium path. Manufacturer 2 either chooses $s_2 = 1$ or $s_2 = 2$. Each firm's equilibrium variable profit is thus zero
$$\begin{aligned} \pi_1^M &= 0 & \qquad \pi_2^M &= -T_{21} \\ \pi_1^S &= T_{21} - F & \qquad \pi_2^S &= 0. \end{aligned}$$
Notice manufacturer 2 and supplier 1 are attaining pareto efficient payoffs over all $\mathbf{p}, \mathbf{b}$. Assumptions 2.2.4 and 2.2.5 hold. Moreover, neither manufacturer has an incentive to adjust its price after Stage 5 since $b_1 \wedge b_2 = v_2$. So Assumption 2.2.6 holds.

- Consider any NE in which neither manufacturer operates. So downstream prices satisfy
$$p_1 \leq AC\left(\frac{1}{2}\right) \text{ or } p_1 > v_1, \qquad p_2 \leq AC\left(\frac{1}{2}\right) - H(p_1) \text{ or } p_2 > v_2.$$

These conditions are equivalent to $\mathbf{p} \geq \mathbf{0}$. Equilibrium path bids must satisfy

$$v_1 \geq p_2, \qquad b_2 \geq AC\left(\frac{1}{2}\right) \qquad (\text{with } s_2(\mathbf{p}, \mathbf{b}) = SD \text{ if } b_1 = p_1).$$

by Assumption 2.2.3. So each firm makes zero variable profit:

$$\pi_1^M = 0 \qquad \pi_2^M = -T_{21}$$
$$\pi_1^S = T_{21} - F \qquad \pi_2^S = 0$$

in equilibrium. Manufacturer 2 and supplier 1 are attaining pareto efficient payoffs over all $\mathbf{p}, \mathbf{b}$. Assumptions 2.2.4 and 2.2.5 hold. Moreover, neither manufacturer has an incentive to adjust its price after Stage 5 since $b_1 \wedge b_2 = v_2$. So Assumption 2.2.6 holds.

We see that $\boldsymbol{\pi}$ are the same across all NEs. Also, when manufacturer 2 operates, either $s_2 = 1$ or $v_1 \leq \Delta C\left(\frac{1}{2}, \frac{1}{2}\right), q_1 = 0$.

Whilst the set of NEs under Assumption 2.2.4 do not coincide with those under assumption 2.2.5 and 2.2.6, their equilibrium path firm payoffs do, and equal the expressions stated in the lemma. When $\Delta C\left(\frac{1}{2}, \frac{1}{2}\right) < v_1$, $s_2 = 1$. When $\Delta C\left(\frac{1}{2}, \frac{1}{2}\right) \geq v_1$, whenever there exists NEs satisfying all three assumptions where $q_2 > 0, s_2 = 1$, the same is true for $q_2 > 0, s_2 = 2$. $\square$

Corrollary 2.9.2. *Consider any BNE satisfying assumptions 2.2.2 and 2.2.4, or 2.2.2, 2.2.5 and 2.2.6.*

i) Consider the subgame $\Gamma(\mathbf{I})$. Suppose manufacturer 2 produces a positive quantity ($q_2 > 0$) and chooses supplier 2 ($s_2 = 2$) in equilibrium. Then there exists another NE where $q_2 > 0$ but $s_2 = 1$.

ii) Suppose the willingness-to-pay for manufacturer 1's product strictly exceeds the cost of producing it under supplier sharing ($\Delta C\left(\frac{1}{2}, \frac{1}{2}\right) < v_1$) and $q_2 > 0$. Then $s_2 = 1$ iff supplier 1 and manufacturer 2 invests in a new relationship ($\mathbf{a} = \mathbf{I}$).

iii) Suppose $v_1 \leq \Delta C\left(\frac{1}{2},\frac{1}{2}\right)$ and $\mathbf{q} >> \mathbf{0}$. Then $s_2 = 1$ iff $\mathbf{a} = \mathbf{I}$.

Suppose $\mathbf{q} >> \mathbf{0}$. Then manufacturers share suppliers in the production network iff $\mathbf{a} = \mathbf{I}$.

Proof. Consider the subgame $\Gamma(\mathbf{I})$. Lemmas 2.9.6 to 2.9.8 show when $v_1 > C\left(\frac{1}{2},\frac{1}{2}\right)$, then $q_2 > 0$ implies $s_2 = 1$ under $\Gamma(\mathbf{I})$. When $\mathbf{a} \neq \mathbf{I}$, manufacturer 2 never chooses supplier 1. This establishes ii). When $v_1 \leq C\left(\frac{1}{2},\frac{1}{2}\right)$, Lemmas 2.9.8 and 2.9.9 imply equilibria featuring $q_2 > 0, s_2 = 2$ exist only when equilibria featuring $s_2 = 1$ occur, across all parameter configurations. This completes the argument for i). Finally, suppose $v_1 \leq \Delta C\left(\frac{1}{2},\frac{1}{2}\right)$ and $\mathbf{q} >> \mathbf{0}$. Then $\pi_1^M(\mathbf{p})$ from Lemma 2.9.5 implies manufacturer 1 can weakly profit (avoid a strict loss) iff it shares a supplier with manufacturer 2 and manufacturer 2 produces a positive quantity of output[48]. So manufacturer 1 and 2 share suppliers when $\mathbf{a} = \mathbf{I}$. Conversely, if neither manufacturer operates, then no supplier sharing can occur in the production network. This establishes iii).

To prove the final statement, observe it holds when $v_1 \leq \Delta C\left(\frac{1}{2},\frac{1}{2}\right)$ by iii). Suppose $\Delta C\left(\frac{1}{2},\frac{1}{2}\right) < v_1$. Then $\mathbf{q} >> \mathbf{0}$ implies $q_2 > 0$. So ii) implies statement holds. □

Stage 2 & 1: Equation (2.6) implies manufacturer 2 and supplier 1 invests iff

$$\underline{\pi}_2^M + \underline{\pi}_1^S + F \leq \pi_2^M(\mathbf{I}, \boldsymbol{\sigma}|_{\mathbf{I}}) + \pi_1^S(\mathbf{I}, \boldsymbol{\sigma}|_{\mathbf{I}}), \tag{2.48}$$

or their joint return to their investment is positive.

Proof of Proposition 2.2.3. The claim is proved in two parts. The first step establishes $\mathbf{a} = \mathbf{I}$

48. To see this more formally, observe when manufacturer 2 doesn't share a supplier with manufacturer 1 or manufacturer 1 doesn't produce positive output,

$$\pi_1^M = \frac{\beta_1}{2}\left(p_1 - AC\left(\frac{1}{2}\right)\right) \leq \frac{\beta_1}{2}\left(v_1 - AC\left(\frac{1}{2}\right)\right) < \frac{\beta_1}{2}\left(v_1 - \Delta C\left(\frac{1}{2},\frac{1}{2}\right)\right) \leq 0$$

(according the expressions for π_1^M when $\mathbf{a} \neq \mathbf{I}$ described in the main text).

iff

$$F \leq [1-\beta_1]\left[AC\left(\frac{1}{2}\right) - AC\left(1\right) + \frac{v_1 - AC(\frac{1}{2})}{2} \wedge 0\right] \vee 0 \\ + \frac{1}{2}\left[v_2 - AC\left(\frac{1}{2}\right)\right] - \frac{\beta_2}{2}\left[v_2 - AC\left(\frac{1}{2}\right)\right] \vee 0. \tag{2.49}$$

holds.

Claim: $\mathbf{a} = \mathbf{I}$ iff (2.49) holds.

Proof. Manufacturer 2 and supplier 1 invests iff (2.48) holds. (2.46) and (2.47) define $\underline{\pi}_1^S + \underline{\pi}_2^M$ for all possible parameter configurations. Lemmas 2.9.6 to 2.9.8 similarly define $\pi_1^S + \pi_2^M$ for all parameter values. Hence, whether the relationship network is connected does not depend on whether Assumption 2.2.4 or assumptions 2.2.5 and 2.2.6 are imposed.

Sub Claim: Proposition holds when $v_1 \wedge v_2 \geq AC\left(\frac{1}{2}\right)$.

Proof. Lemmas 2.9.6 and 2.9.9 imply both manufacturer 2 and supplier 1 invest iff

$$F \leq \frac{1-\beta_1}{2}\left(AC\left(\frac{1}{2}\right) - \Delta C\left(\frac{1}{2}, \frac{1}{2}\right)\right) + \frac{1-\beta_2}{2}\left(v_2 - AC\left(\frac{1}{2}\right)\right)$$

when $\mathbf{v} >> AC\left(\frac{1}{2}\right)(1,1)$ or when $\mathbf{v} = AC\left(\frac{1}{2}\right)(1,1)$ and $AC\left(\frac{1}{2}\right) - H(v_1) \leq v_2$. The RHS of the expression above collapses to (2.49)'s RHS. Lemma 2.9.9 implies $\pi_1^S + \pi_2^M = -F$ when investment occurs if $\mathbf{v} = AC\left(\frac{1}{2}\right)(1,1)$ and $AC\left(\frac{1}{2}\right) - H(v_1) > v_2$. Hence investment never occurs if $\mathbf{v} = AC\left(\frac{1}{2}\right)(1,1)$ and $AC\left(\frac{1}{2}\right) - H(v_1) > v_2$. But then

$$AC\left(\frac{1}{2}\right) - H(v_1) > v_2 \Rightarrow \frac{1-\beta_1}{2}\left(AC\left(\frac{1}{2}\right) - \Delta C\left(\frac{1}{2}, \frac{1}{2}\right)\right) + \frac{1-\beta_2}{2}\left(v_2 - AC\left(\frac{1}{2}\right)\right) < 0 \leq \underline{F}.$$

So investment never occurs (with positive probability) under the proposition too. □

Sub Claim: Proposition holds when $v_2 \leq AC\left(\frac{1}{2}\right) < v_1$.

Proof. Lemma 2.9.7 implies manufacturer 2 and supplier 1 invests iff

$$F \leq \frac{1-\beta_1}{2}\left(AC\left(\frac{1}{2}\right) - \Delta C\left(\frac{1}{2},\frac{1}{2}\right)\right) + \frac{1}{2}\left(v_2 - AC\left(\frac{1}{2}\right)\right).$$

Use $v_2 \leq AC\left(\frac{1}{2}\right) < v_1$ to show the above holds iff (2.49) holds. $\square$

Sub Claim: Proposition holds when $v_1 \leq AC\left(\frac{1}{2}\right) < v_2$.

Proof. Lemma 2.9.8 implies manufacturer 2 and supplier 1 invests iff

$$F \leq \frac{1-\beta_1}{2}\left(v_1 - \Delta C\left(\frac{1}{2},\frac{1}{2}\right)\right) \vee 0 + \frac{1-\beta_2}{2}\left(v_2 - AC\left(\frac{1}{2}\right)\right).$$

Under $v_1 \leq AC\left(\frac{1}{2}\right) < v_2$, the above holds iff (2.49) holds. $\square$

Sub Claim: Proposition holds when $v_1 \vee v_2 < AC\left(\frac{1}{2}\right)$.

Proof. Lemma 2.9.9 implies manufacturer 2 and supplier 1 invests iff

$$F \leq \frac{1-\beta_1}{2}\left(v_1 - \Delta C\left(\frac{1}{2},\frac{1}{2}\right)\right) \vee 0 + \frac{1}{2}\left(v_2 - AC\left(\frac{1}{2}\right)\right).$$

Under $v_1 \vee v_2 < AC\left(\frac{1}{2}\right)$, the above holds iff (2.49) holds. $\square$

(2.49) thus holds whenever $\mathbf{a} = \mathbf{I}$. $\square$

Observe the relationship network is connected and features supplier sharing occurs iff $\mathbf{a} = \mathbf{I}$. $\square$

Proof of Proposition 2.2.4. Let $\boldsymbol{\phi} = (\mathbf{v}, 1-\beta_1, 1-\beta_2, -c)$. Let SS be the event that two manufacturers operate, share a supplier, and produce strictly positive quantities. Formally,

$$SS = \{\mathbf{q} >> \mathbf{0}, s_2 = 1\} = \{\mathbf{q}(\mathbf{s}(\mathbf{p}, \mathbf{b}(\mathbf{p}(\mathbf{I}))), \mathbf{p}(\mathbf{I})) >> \mathbf{0}, s_2(\mathbf{p}(\mathbf{I}), \mathbf{b}(\mathbf{p}(\mathbf{I}))) = 1, \mathbf{a} = \mathbf{I}\}.$$

where $\mathbf{s}, \mathbf{b}, \mathbf{p}$ should be understood as being part of $\boldsymbol{\sigma}|_{\mathbf{I}}$ in the the final term. Observe when $\mathbf{a} \neq \mathbf{I}$, SS never occurs. Hence $SS \subseteq \{\mathbf{a} = \mathbf{I}\}$, as the expression above makes clear.

Let $F(\boldsymbol{\sigma}|_{\mathbf{I}})$ denote the threshold for investment occurring $\mathbf{a} = \mathbf{I}$ and the manufacturers sharing suppliers in the relationship network. That is, as defined in (2.49) or equivalently, Proposition 2.2.3. Observe this threshold is also a function of $\boldsymbol{\phi}$, whose form doesn't depend on the equilibria $\boldsymbol{\sigma}|_{\mathbf{I}}$ played. Write $F(\boldsymbol{\sigma}|_{\mathbf{I}}) = F(\boldsymbol{\phi})$. Notice $F(\boldsymbol{\phi})$ is weakly increasing in $\boldsymbol{\phi}$. The result is proved in parts.

Claim 1: Desired result holds for $AC\left(\frac{1}{2}\right) + c < v_1 \wedge v_2$, and $\mathbb{P}SS = G(F(\boldsymbol{\phi}))$.

Proof. Fix $\mathbf{a} = \mathbf{I}$. When $AC\left(\frac{1}{2}\right) < v_1 \wedge v_2$, Lemma 2.9.6 implies $s_2 = 1$ on the equilibrium path. Moreover, its stated expression for $\boldsymbol{\pi}^M$ (or the lemma's proof) implies $\mathbf{q} >> \mathbf{0}$ (in equilibrium). So supplier sharing between positive market share manufacturers occur whenever $\mathbf{a} = \mathbf{I}$. So $\mathbb{P}SS = G(F(\boldsymbol{\phi}))$, which is weakly increasing in $\boldsymbol{\phi}$, and G when the latter is ordered via FOSD. $\square$

Write $H(v_1) = H(\boldsymbol{\phi})$ in what follows. Observe

$$H(v_1) = (1 - \beta_1)\left(v_1 \wedge \left(AC\left(\frac{1}{2}\right) + c\right) - \Delta C\left(\frac{1}{2}, \frac{1}{2}\right) - c\right)$$

is weakly increasing in each entry of $\boldsymbol{\phi}$. In what follows, the following inequality is useful:

$$H(\boldsymbol{\phi}) + v_2 - AC\left(\frac{1}{2}\right) - c < 0. \tag{2.50}$$

To interpret this condition, note the above occurs when supplier 1 cannot avoid losses from supplying both manufacturers with positive quantities of inputs when $\mathbf{p} = \mathbf{v}$, for all bids $\mathbf{b}$.

Claim 2: Desired result holds when $v_2 \leq AC\left(\frac{1}{2}\right) + c < v_1$. $\mathbb{P}SS = 0$ when (2.50) holds, $\mathbb{P}SS = G(F(\boldsymbol{\phi}))$ if otherwise.

Proof. There are two cases to consider. When $AC\left(\frac{1}{2}\right) + c \leq H(\boldsymbol{\phi}) + v_2$, then Lemma 2.9.7

implies $s_2 = 1$ on the equilibrium path when $\mathbf{a} = \mathbf{I}$, while its stated expression for π_1^M implies $\mathbf{q} >> \mathbf{0}$. So $\mathbb{P}SS = G(F(\boldsymbol{\phi}))$. Suppose $AC\left(\frac{1}{2}\right) + c > H(\boldsymbol{\phi}) + v_2$. Then Lemma 2.9.7 implies manufacturer 1 never produces a positive quantity of output. Hence, $\mathbb{P}SS = 0$.

It follows $\mathbb{P}SS$ equals 0 when (2.50) holds. When (2.50) doesn't hold (weak inequality in reverse direction), $\mathbb{P}SS = G(F(\boldsymbol{\phi}))$. Now (2.50)'s LHS is weakly increasing in $\boldsymbol{\phi}$. Moreover, $G(F(\boldsymbol{\phi}))$ is obviously weakly increasing in $\boldsymbol{\phi}$. So for any entry in $\boldsymbol{\phi}$, $\mathbb{P}SS$ is constant as that entry increases until (2.50) binds, or the hypothesis $v_1 \leq AC\left(\frac{1}{2}\right) + c < v_2$ is violated. In the case where (2.50) binds, $\mathbb{P}SS = G(F(\boldsymbol{\phi}))$ is weakly increasing in that entry. Moreover, holding $\boldsymbol{\phi}$ constant, $\mathbb{P}SS$ is obviously weakly increasing in G via FOSD ordering. □

Claim 3: Desired result holds when $v_1 \leq AC\left(\frac{1}{2}\right) + c < v_2$. $\mathbb{P}SS = 0$ when $v_1 < \Delta C\left(\frac{1}{2}, \frac{1}{2}\right) + c$, $\mathbb{P}SS = G(F(\boldsymbol{\phi}))$ when $v_1 > \Delta C\left(\frac{1}{2}, \frac{1}{2}\right) + c$, and $\mathbb{P}SS \in \{0, G(F(\boldsymbol{\phi}))\}$ if otherwise.

Proof. There are three cases to consider. These are labeled i) $v_1 < \Delta C\left(\frac{1}{2}, \frac{1}{2}\right) + c$, ii) $v_1 = \Delta C\left(\frac{1}{2}, \frac{1}{2}\right) + c$, and $v_1 > \Delta C\left(\frac{1}{2}, \frac{1}{2}\right) + c$. Under Case i), manufacturer 1 clearly cannot profitably produce positive market share, even when sharing suppliers. (Lemma 2.9.8's proof makes this clear.). So $\mathbb{P}SS = 0$. Under Case ii), at least two equilibria classes can exist whenever $\mathbf{a} = \mathbf{I}$, from the lemma's statment or proof. Equilibria in which both manufacturers operate, produce positive quanitities and share suppliers exist alongside equilibria where manufacturer 1 shuts down. So $\mathbb{P}SS \in \{G(F(\boldsymbol{\phi})), 0\}$. In Case iii), the payoffs for manufacturer 1 in the lemma or its proof shows both manufacturers share suppliers and produce positive market share whenever $\mathbf{a} = \mathbf{I}$.

So as any entry in $\boldsymbol{\phi}$ increases, $\mathbb{P}SS$ initially equals 0, before jumping discontinuously to $G(F(\boldsymbol{\phi}))$, whereupon it is weakly increasing in that entry. Likewise, holding $\boldsymbol{\phi}$ constant, $\mathbb{P}SS$ is obviously weakly increasing in G via FOSD. □

Claim 4: Desired result holds when $v_1 \wedge v_2 \leq AC\left(\frac{1}{2}\right) + c$. $\mathbb{P}SS = 0$ when the inequality in (2.50) holds or binds, $\mathbb{P}(SS) = G(F(\boldsymbol{\phi}))$ if otherwise.

Proof. When (2.50) holds or binds, Lemma 2.9.9 shows each firm's variable profit is zero. Supplier 1 and manufacturer 2 thus never invest, and $\mathbb{P}SS = 0$. When the inequality in (2.50) is reversed (strict opposite inequality), observe

$$v_2 - \Delta C\left(\frac{1}{2}, \frac{1}{2}\right) - c \geq H(\boldsymbol{\phi}) + v_2 - AC\left(\frac{1}{2}\right) - c > 0. \tag{2.51}$$

So the expression for π_1^M in Lemma 2.9.9, or the lemma's proof shows manufacturer 1 must be producing a positive quantity, and sharing a supplier with manufacturer 2. So $\mathbb{P}SS = G(F(\boldsymbol{\phi}))$.

So $\mathbb{P}SS$ is initially constant in zero as each entry in $\boldsymbol{\phi}$ increases, until (2.50) binds, whereupon it equals $G(F(\boldsymbol{\phi}))$ thereafter. $\mathbb{P}SS$ is thus always weakly increasing in that entry. Likewise, holding $\boldsymbol{\phi}$ constant, $\mathbb{P}SS$ is obviously weakly increasing in G via FOSD. $\square$

Each of the prior claims addresses what happens to $\mathbb{P}SS$ when $\boldsymbol{\phi}$ is increased to $\boldsymbol{\phi}'$, in four, mutually exclusive parameter configuration sets. It remains to show that when we compare $\mathbb{P}SS$ across $\boldsymbol{\phi}$ that cross the boundaries of these four separate cases in an increasing direction, the desired result also holds. Any northeastern movement from a $\boldsymbol{\phi}$ located in the last three claim cases, to $\boldsymbol{\phi}'$ the first claim case, must result in either a discontinuous jump from 0 to $\mathbb{P}SS = G(F(\boldsymbol{\phi}'))$, or a change from $\mathbb{P}SS = G(F(\boldsymbol{\phi}))$ to $G(F(\boldsymbol{\phi}'))$ where $\boldsymbol{\phi} \leq \boldsymbol{\phi}'$. $\mathbb{P}SS$ is clearly weakly increasing in this change.

This leaves northeast movements from a $\boldsymbol{\phi}$ located in the last claim case, to a $\boldsymbol{\phi}'$ in claim 2 or 3 case. Suppose $\boldsymbol{\phi}'$ lies in Claim 2's case. Suppose for contradiction, $\mathbb{P}SS(\boldsymbol{\phi}') < \mathbb{P}SS(\boldsymbol{\phi})$. Then

$$\begin{aligned} &0 = \mathbb{P}SS(\boldsymbol{\phi}') < \mathbb{P}SS(\boldsymbol{\phi}) = G(F(\boldsymbol{\phi})) \\ \Rightarrow\ & H(\boldsymbol{\phi}') + v_2' - AC\left(\frac{1}{2}\right) - c' \leq 0 < H(\boldsymbol{\phi}) + v_2 - AC\left(\frac{1}{2}\right) - c. \end{aligned}$$

The implication hold by the proofs of the 4th and 2nd claims above. The final row is a contradiction to (2.50)'s LHS being non-increasing in its arguments, since $\boldsymbol{\phi} < \boldsymbol{\phi}'$.

Suppose $\boldsymbol{\phi}'$ lies in Claim 3's case. Suppose for contradiction, $\mathbb{P}SS(\boldsymbol{\phi}') < \mathbb{P}SS(\boldsymbol{\phi})$. Then

$$\begin{aligned} & 0 = \mathbb{P}SS(\boldsymbol{\phi}') < \mathbb{P}SS(\boldsymbol{\phi}) = G(F(\boldsymbol{\phi})) \\ \Rightarrow \ & v_1' - \Delta C\left(\frac{1}{2}, \frac{1}{2}\right) - c' \leq 0 < H(\boldsymbol{\phi}) + v_2 - AC\left(\frac{1}{2}\right) - c \\ \Rightarrow \ & H(\boldsymbol{\phi}') = 0 < H(\boldsymbol{\phi}) + v_2 - AC\left(\frac{1}{2}\right) - c \leq H(\boldsymbol{\phi}). \end{aligned}$$

The first implication holds by the proofs of the 4th and 3rd claims above. The final inequality holds since in the 4th claim's case, $v_2 \leq AC\left(\frac{1}{2}\right) + c$. The final row is a contradiction to (2.50)'s LHS being non-increasing in its arguments, since $\boldsymbol{\phi} < \boldsymbol{\phi}'$. $\square$

Corrollary 2.9.3. *Suppose assumptions 2.2.2 to 2.2.4 or assumptions 2.2.2, 2.2.3, 2.2.5 and 2.2.6 hold. The probabilities of both manufacturers producing strictly positive quantities and sharing suppliers conditional on particular manufacturers producing strictly positive output*

$$\mathbb{P}(\mathbf{q} >> \mathbf{0}, s_2 = 1 | \mathbf{q} >> \mathbf{0}) \quad \mathbb{P}(\mathbf{q} >> \mathbf{0}, s_2 = 1 | q_1 > 0) \quad \mathbb{P}(\mathbf{q} >> \mathbf{0}, s_2 = 1 | q_2 > 0),$$

equal (2.8) *when* $v_1 \wedge v_2 > AC\left(\frac{1}{2}\right)$, *and equal* 1 *when* $v_1 \vee v_2 < AC\left(\frac{1}{2}\right)$.

Proof of Corollary 2.9.3. Let SS denote the event of both manufacturers producing positive quantities ($\mathbf{q} >> \mathbf{0}$) and sharing suppliers ($s_2 = 1$). Suppose $v_1 \wedge v_2 > AC\left(\frac{1}{2}\right)$. Then $\mathbf{q} >> \mathbf{0}$ a.s. and

$$\mathbb{P}SS = \mathbb{P}(SS | \mathbf{q} >> \mathbf{0}) = \mathbb{P}(SS | q_1 > 0) = \mathbb{P}(SS | q_2 > 0).$$

Moreover, since $\mathbf{q} >> \mathbf{0}$, so SS occurs when $\mathbf{a} = \mathbf{I}$, or when the relationship network is connected. Suppose $v_1 \vee v_2 < AC\left(\frac{1}{2}\right)$. Neither manufacturer can operate without sharing a supplier with a rival manufacturer producing positive quantity. The conditional probabilities thus all equal 1. $\square$

Proof of Proposition 2.2.5. The proposition asserts similar properties for the equilibria and social planner's relationship and production networks. I prove the relevant properties hold for the relationship networks first.

Observe manufacturer 2 and supplier 1 invests in equilibrium when they don't under the social planner's model if

$$\hat{F} < F < \pi_2^M(\mathbf{I}, \boldsymbol{\sigma}|_{\mathbf{I}}) - \underline{\pi}_2^M + \pi_1^S(\mathbf{I}, \boldsymbol{\sigma}|_{\mathbf{I}}) - \underline{\pi}_1^S \tag{2.52}$$

where $\hat{F}$ is defined by Proposition 2.2.1. When one of the inequalities in (2.52) holds only weakly, the equilibrium relationship network is either over connected or socially optimal. Investment doesn't occur when it should if

$$\pi_2^M(\mathbf{I}, \boldsymbol{\sigma}|_{\mathbf{I}}) - \underline{\pi}_2^M + \pi_1^S(\mathbf{I}, \boldsymbol{\sigma}|_{\mathbf{I}}) - \underline{\pi}_1^S < F < \hat{F}. \tag{2.53}$$

When one of the inequalities in (2.53) holds weakly, the relationship network is either under connected or socially optimal. Hence, the relationship network is either excessively connected relative to the planner's network with positive probability, or under connected with positive probability, but never both. Hence, proving the proposition reduces to asserting

$$(1-\beta_2)\left(v_2 - AC\left(\frac{1}{2}\right)\right)\vee 0 > \beta_1\left[\left(AC\left(\frac{1}{2}\right) - \Delta c\left(\frac{1}{2},\frac{1}{2}\right)\right) + \left(v_1 - AC\left(\frac{1}{2}\right)\right)\wedge 0\right] \tag{2.54}$$

iff (2.52) occurs w.p.p., and

$$\beta_1\left[\left(AC\left(\frac{1}{2}\right) - \Delta c\left(\frac{1}{2},\frac{1}{2}\right)\right) + \left(v_1 - AC\left(\frac{1}{2}\right)\right)\wedge 0\right] > (1-\beta_2)\left(v_2 - AC\left(\frac{1}{2}\right)\right)\vee 0 \tag{2.55}$$

iff (2.53) occurs w.p.p.

Claim: Proposition holds for the relationship network.

Proof. Suppose $v_1 \wedge v_2 \geq AC\left(\frac{1}{2}\right)$. Proposition 2.2.3 implies (2.52) occurs w.p.p. iff

$$\begin{aligned}
&\frac{1}{2}\left(AC\left(\frac{1}{2}\right) - \Delta C\left(\frac{1}{2}, \frac{1}{2}\right)\right) < \frac{1-\beta_1}{2}\left(AC\left(\frac{1}{2}\right) - \Delta C\left(\frac{1}{2}, \frac{1}{2}\right)\right) + \frac{1-\beta_2}{2}\left(v_2 - AC\left(\frac{1}{2}\right)\right) \\
\Leftrightarrow\ &\frac{\beta_1}{2}\left(AC\left(\frac{1}{2}\right) - \Delta C\left(\frac{1}{2}, \frac{1}{2}\right)\right) < \frac{1-\beta_2}{2}\left(v_2 - AC\left(\frac{1}{2}\right)\right),
\end{aligned}$$

while (2.53) occurs w.p.p. iff the inequality displayed above is reversed. But the above is equivalent to (2.54) under $v_1 \wedge v_2 \geq AC\left(\frac{1}{2}\right)$. Likewise, the reverse inequality is equivalent to (2.55).

Suppose $v_2 \leq AC\left(\frac{1}{2}\right) < v_1$. Proposition 2.2.3 implies (2.52) occurs with positive probability iff

$$\begin{aligned}
&\frac{1}{2}\left(v_2 - \Delta C\left(\frac{1}{2}, \frac{1}{2}\right)\right) < \frac{1-\beta_1}{2}\left(AC\left(\frac{1}{2}\right) - \Delta C\left(\frac{1}{2}, \frac{1}{2}\right)\right) + \frac{1}{2}\left(v_2 - AC\left(\frac{1}{2}\right)\right) \\
\Leftrightarrow\ &\frac{\beta_1}{2}\left(AC\left(\frac{1}{2}\right) - \Delta C\left(\frac{1}{2}, \frac{1}{2}\right)\right) < 0
\end{aligned}$$

while (2.53) occurs w.p.p. iff the inequality displayed above holds is reversed. But the above is equivalent to (2.54) under $v_2 \leq AC\left(\frac{1}{2}\right) < v_1$. Likewise, the reverse inequality is equivalent to (2.55). Notice in this situation, (2.52) never occurs and investment is always suboptimal, due to strict concavity of $C(Q)$.

Suppose $v_1 \leq AC\left(\frac{1}{2}\right) < v_2$. Proposition 2.2.3 implies (2.52) occurs w.p.p. iff

$$\begin{aligned}
&\frac{1}{2}\left(v_1 - \Delta C\left(\frac{1}{2}, \frac{1}{2}\right)\right) < \frac{1-\beta_1}{2}\left(v_1 - \Delta C\left(\frac{1}{2}, \frac{1}{2}\right)\right) + \frac{1-\beta_2}{2}\left(v_2 - AC\left(\frac{1}{2}\right)\right) \\
\Leftrightarrow\ &\frac{\beta_1}{2}\left(v_1 - \Delta C\left(\frac{1}{2}, \frac{1}{2}\right)\right) < \frac{1-\beta_2}{2}\left(v_2 - AC\left(\frac{1}{2}\right)\right),
\end{aligned}$$

while (2.53) occurs w.p.p. iff the inequality displayed above holds is reversed. But the above is equivalent to (2.54) under $v_1 \leq AC\left(\frac{1}{2}\right) < v_2$. Likewise, the reverse inequality is equivalent to (2.55).

Finally, suppose $v_1 \vee v_2 < AC\left(\frac{1}{2}\right)$. Proposition 2.2.3 implies (2.52) occurs (with one inequality holding strictly) w.p.p. iff

$$\frac{1}{2}\left(v_1+v_2-AC\left(\frac{1}{2}\right)-\Delta C\left(\frac{1}{2},\frac{1}{2}\right)\right)<\frac{1-\beta_1}{2}\left(v_1-\Delta C\left(\frac{1}{2},\frac{1}{2}\right)\right)+\frac{1}{2}\left(v_2-AC\left(\frac{1}{2}\right)\right)$$
$$\Leftrightarrow\frac{\beta_1}{2}\left(v_1-\Delta C\left(\frac{1}{2},\frac{1}{2}\right)\right)<0,$$

while (2.53) occurs w.p.p. iff the inequality displayed above is reversed. But the above is equivalent to (2.54) under $v_1 \vee v_2 < AC\left(\frac{1}{2}\right)$. Likewise, the reverse inequality is equivalent to (2.55). Again, (2.52) clearly never occurs with positive probability and investment is always suboptimal. □

Claim: Fix $F \in [\underline{F}, \overline{F}]$. When the equilibrium and social planner's relationship networks coincide, the equilibrium and planner's production networks coincide too.

Proof. Suppose both relationship networks are disconnected. The production network likewise coincides with its socially optimal counterpart, since Nash bargaining implies manufacturer m produce efficient quantities of output - $q_m = \frac{1}{2}$ when $AC\left(\frac{1}{2}\right) < v_m$, $q_2 = 0$ otherwise. Suppose both are connected. The planner's optimality conditions and the support restrictions for G imply

$$\frac{v_1 \wedge v_2}{2} - C(1) + C\left(\frac{1}{2}\right) > F \Rightarrow v_1 \geq v_1 \wedge v_2 > \Delta C\left(\frac{1}{2},\frac{1}{2}\right).$$

Corollary 2.9.2 implies $s_2 = 1$ whenever $q_2 > 0$. By Lemma 2.9.5, π_1^M is strictly increasing in $p_1 \in [0, v_1]$ when $s_2 = 1, q_2 > 0$. So by Assumption 2.2.6, $p_1 = v_1$ and manufacturer 1 produces positive market share whenever manufacturer 2 also does so. So if $q_2 > 0$, the production network coincides with its planner's counterpart. Suppose $q_2 = 0$. Then (2.44) and (2.45) imply supplier 1's profit is

$$\pi_1^S = \frac{1-\beta_1}{2}\left(p_1 - AC\left(\frac{1}{2}\right)\right) \vee 0\{p_1 \leq v_1\} + T_{21} - F \leq \frac{1-\beta_1}{2}\left(v_1 - AC\left(\frac{1}{2}\right)\right) + T_{21} - F,$$

smaller than when $\mathbf{a} \neq \mathbf{I}$. Obviously, $q_2 = 0$ implies $\pi_2^M = -T_{21}$. It follows that $\mathbf{a} \neq \mathbf{I}$ on the equilibrium path, and the relationship network was never connected to begin with. $\square$

Notice this claim proves the proposition's final statement. When both sides of the proposition's inequalities equal, $\mathbf{a} = \mathbf{I}$ occurs whenever it is socially optimal. The relationship network is thus socially optimal a.s. Hence the production network is socially optimal a.s. too.

Claim: Proposition holds for the equilibrium production network.

Proof. Suppose

$$\hat{F} < \pi_2^M(\mathbf{I}, \boldsymbol{\sigma}|_{\mathbf{I}}) - \underline{\pi}_2^M + \pi_1^S(\mathbf{I}, \boldsymbol{\sigma}|_{\mathbf{I}}) - \underline{\pi}_1^S \tag{2.56}$$

holds. I claim the production network (graph of positive market share manufacturers with their suppliers and relationships) is weakly over-connected (over or equally connected) or socially optimal. Fix a value of F. Assume (2.52) holds. Then the the (equilibrium) relationship network is connected and the social planner's network is disconnected. It follows that the planner's production network is either i) disconnected, consisting of two (strictly) positive market share manufacturers connected to two separate suppliers ii) disconnected and empty or iii) connected but consisting of only one positive market share manufacturer connected to a single supplier. Now the relationship network is connected. Suppose for contradiction that the production network was disconnected. As with the planner's production network, the possibilities are it is i) disconnected, consisting of two positive market share manufacturers connected to two separate suppliers ii) disconnected and empty. If the production network were empty, each firm makes no variable profit. Thus $\mathbf{a} \neq \mathbf{I}$ and the relationship network was disconnected in equilibrium to begin with. The production network is thus properly disconnected - it has two positive market share manufacturers, connected to two separate suppliers. But Corollary 2.9.2 implies that when $\mathbf{q} >> \mathbf{0}$, supplier sharing occurs in $\Gamma(\mathbf{I})$, a contradiction. So the production network is connected, whereas its planner's counterpart is either connected or disconnected. Assume (2.52) doesn't hold. Then the equilibrium relationship network coincides with its socially optimal counterpart. Thus the equilibrium production network is socially optimal too, by the previous claim.

Suppose (2.56) holds in reverse instead (strict reverse inequality). I claim the production network is weakly under connected or socially optimal. Fix a value of F. Assume (2.53) holds. Then the relationship network is disconnected when its planner's counterpart is connected. The planner never instructs investment in new relationships unless instructing supplier sharing between positive market share manufacturers too. So the planner's production network must feature supplier sharing between two positive market share manufacturers. The planner's network can only be connected, while in comparison the equilibrium production is either connected or disconnected. Moreover (as an aside), the possibilities are that the equilibrium production network is i) disconnected and consisting of two unlinked operating manufacturers, ii) disconnected and empty, or iii) is connected, with only one manufacturer producing positive market share. Assume (2.53) doesn't hold. The equilibrium relationship network thus coincides with its socially optimal counterpart. Thus the equilibrium production network is socially optimal too, by the previous claim. □

□

Proof of Proposition 2.2.6. The total surplus generated in equilibrium equals

$$TS = \sum_m \underline{\pi}_m^M + \sum_s \underline{\pi}_s^S + \int_{F=\underline{F}}^{F=\pi_1^S(\mathbf{I},\boldsymbol{\sigma}|_{\mathbf{I}})-\underline{\pi}_1^S+\pi_2^M(\mathbf{I},\boldsymbol{\sigma}|_{\mathbf{I}})-\underline{\pi}_2^M} \sum_m \left(\pi_m^M(\mathbf{I},\boldsymbol{\sigma}|_{\mathbf{I}}) - \underline{\pi}_m^M\right) + \sum_s \left(\pi_s^S(\mathbf{I},\boldsymbol{\sigma}|_{\mathbf{I}}) - \underline{\pi}_s^S\right) - F dG.$$

One can plug in the expressions for $\underline{\boldsymbol{\pi}}$ from (2.46), (2.47), and $\boldsymbol{\pi}(\mathbf{I},\boldsymbol{\sigma}|_{\mathbf{I}})$ from lemmas 2.9.6 to 2.9.8 into the expression above and differentiate, to verify the proposition. Alternatively, by substituting $\hat{F}$ defined in Proposition 2.2.1 for the relevant terms in TS above, one obtains

$$TS = \sum_m \underline{\pi}_m^M + \sum_s \underline{\pi}_s^S + \int_{F=\underline{F}}^{F=\pi_1^S(\mathbf{I},\boldsymbol{\sigma}|_{\mathbf{I}})-\underline{\pi}_1^S+\pi_2^M(\mathbf{I},\boldsymbol{\sigma}|_{\mathbf{I}})-\underline{\pi}_2^M} \hat{F} - F dG = \widehat{TS} + \int_{F=\hat{F}}^{F=F(\boldsymbol{\sigma}|_{\mathbf{I}})} \hat{F} - F dG, \tag{2.57}$$

where

$$\widehat{TS} \equiv \sum_m \underline{\pi}_m^M + \sum_s \underline{\pi}_s^S + \int_{F=\underline{F}}^{F=\hat{F}} \hat{F} - F dG$$

is the surplus achieved under the planner's allocation, and

$$F \leq F(\boldsymbol{\sigma}|_{\mathbf{I}}) \equiv \pi_1^S(\mathbf{I}, \boldsymbol{\sigma}|_{\mathbf{I}}) - \underline{\pi}_1^S + \pi_2^M(\mathbf{I}, \boldsymbol{\sigma}|_{\mathbf{I}}) - \underline{\pi}_2^M$$

is manufacturer 2 and supplier 1's threshold for the network being connected ((2.49) defines this in terms of the model's primitives). By propositions 2.2.1 and 2.2.2, $\hat{F}$ and $\widehat{TS}$ are differentiable in $(\mathbf{v}, c)$ whenever $\mathbf{v} \neq AC\left(\frac{1}{2}\right)(1,1)$. Also, Proposition 2.2.3 or (2.49) establishes $F(\boldsymbol{\sigma}|_{\mathbf{I}})$'s differentiability in $(\mathbf{v}, c)$, whenever $\mathbf{v} \neq AC\left(\frac{1}{2}\right)(1,1)$ and $v_1 \neq \Delta C\left(\frac{1}{2}, \frac{1}{2}\right)$. Hence, by (2.57), TS is differentiable in $(\mathbf{v}, c)$ whenever $\mathbf{v} \neq AC\left(\frac{1}{2}\right)(1,1)$ and $v_1 \neq \Delta C\left(\frac{1}{2}, \frac{1}{2}\right)$. This verifies the proposition's final statement.

Then taking derivatives on (2.57) and applying Proposition 2.2.2 obtains

$$\begin{aligned}
\frac{\partial TS}{\partial v_n} &= \mathbb{E}\hat{q}_n + \theta \frac{\partial F(\boldsymbol{\sigma}|_{\mathbf{I}})}{\partial v_n} + \int_{F=\hat{F}}^{F=F(\boldsymbol{\sigma}|_{\mathbf{I}})} \frac{\partial \hat{F}}{\partial v_n} dG \\
&= \frac{\left\{v_m > AC\left(\frac{1}{2}\right)\right\} + G(\hat{F})\left\{v_n < AC\left(\frac{1}{2}\right)\right\}}{2} + \theta \frac{\partial F(\boldsymbol{\sigma}|_{\mathbf{I}})}{\partial v_n} + \int_{F=\hat{F}}^{F=F(\boldsymbol{\sigma}|_{\mathbf{I}})} \frac{\partial \hat{F}}{\partial v_n} dG,
\end{aligned}$$

where $\hat{q}_n$ is manufacturer n's market share in the planner's allocation, for $n \leq 2$. Hence,

$$\begin{aligned}
\frac{\partial TS}{\partial v_1} = &\frac{\left\{v_1 > AC\left(\frac{1}{2}\right)\right\}}{2} + \theta \frac{(1-\beta_1)\left\{\Delta C\left(\frac{1}{2}, \frac{1}{2}\right) < v_1 < AC\left(\frac{1}{2}\right)\right\}}{2} \\
&+ \frac{\left\{v_1 < AC\left(\frac{1}{2}\right)\right\}}{2} G(\hat{F}) + \int_{F=\hat{F}}^{F=F(\boldsymbol{\sigma}|_{\mathbf{I}})} \frac{\left\{v_1 < AC\left(\frac{1}{2}\right)\right\}}{2} dG \\
= &\frac{\left\{v_1 > AC\left(\frac{1}{2}\right)\right\}}{2} + G(F(\boldsymbol{\sigma}|_{\mathbf{I}})) \frac{\left\{v_1 < AC\left(\frac{1}{2}\right)\right\}}{2} + \theta \frac{(1-\beta_1)\left\{\Delta C\left(\frac{1}{2}, \frac{1}{2}\right) < v_1 < AC\left(\frac{1}{2}\right)\right\}}{2} \\
= &\mathbb{E}q_1 + \theta \frac{(1-\beta_1)\left\{\Delta C\left(\frac{1}{2}, \frac{1}{2}\right) < v_1 < AC\left(\frac{1}{2}\right)\right\}}{2}.
\end{aligned}$$

Notice the final equality recognizes that when $v_1 < AC\left(\frac{1}{2}\right)$, manufacturer 1 operates iff it shares a supplier. This establishes the proposition's first identity. Applying the same sequence of steps when $n = 2$ obtains

$$\begin{aligned}\frac{\partial TS}{\partial v_2} &= \frac{\left\{v_2 > AC\left(\frac{1}{2}\right)\right\}}{2} + G(F(\boldsymbol{\sigma}|_{\mathbf{I}}))\frac{\left\{v_2 < AC\left(\frac{1}{2}\right)\right\}}{2} + \theta\frac{1-\beta_2\left\{v_2 < AC\left(\frac{1}{2}\right)\right\}}{2} \\ &= \mathbb{E}q_2 + \theta\frac{1-\beta_2\left\{v_2 < AC\left(\frac{1}{2}\right)\right\}}{2}.\end{aligned}$$

Notice the final equality recognizes that when $v_2 < AC\left(\frac{1}{2}\right)$, manufacturer 2 operates iff it shares a supplier. Finally,

$$\frac{\partial TS}{\partial c} = -\sum_s \mathbb{E}\hat{q}_s^S + \theta\frac{\partial F(\boldsymbol{\sigma}|_{\mathbf{I}})}{\partial c} + \int_{F=\hat{F}}^{F=F(\boldsymbol{\sigma}|_{\mathbf{I}})} \frac{\partial \hat{F}}{\partial c} dG$$

by Proposition 2.2.2. Observe $\hat{\mathbf{q}}^S$ has been used in place of $\mathbf{q}^S$ in the proposition to denote the suppliers' market shares under the planner's allocation. Then since a manufacturer whose willingness-to-pay lies strictly below $AC\left(\frac{1}{2}\right)$, operates only if it shares a supplier,

$$\begin{aligned}\frac{\partial F(\boldsymbol{\sigma}|_{\mathbf{I}})}{\partial c} &= -\frac{(1-\beta_1)\left\{\Delta C\left(\frac{1}{2},\frac{1}{2}\right) < v_1 < AC\left(\frac{1}{2}\right)\right\}}{2} - \frac{1-\beta_2\left\{v_2 > AC\left(\frac{1}{2}\right)\right\}}{2} \\ \frac{\partial \hat{F}}{\partial c} &= -\left\{v_1 \vee v_2 < AC\left(\frac{1}{2}\right)\right\} - \left\{v_1 \wedge v_2 < AC\left(\frac{1}{2}\right) < v_1 \vee v_2\right\}\frac{1}{2} \\ \sum_s \mathbb{E}\hat{q}_s^S &= -\left\{v_1 \vee v_2 < AC\left(\frac{1}{2}\right)\right\}G(\hat{F}) - \left\{v_1 \wedge v_2 < AC\left(\frac{1}{2}\right) < v_1 \vee v_2\right\}\frac{1+G(\hat{F})}{2} \\ &\quad - \left\{v_1 \wedge v_2 > AC\left(\frac{1}{2}\right)\right\}.\end{aligned}$$

Plugging these partial derivatives back into the previously worked out expression for $\frac{\partial TS}{\partial c}$ obtains

$$\begin{aligned}\frac{\partial TS}{\partial c} &= -\left\{v_1 \vee v_2 < AC\left(\frac{1}{2}\right)\right\}G(F(\boldsymbol{\sigma}|_{\mathbf{I}})) - \left\{v_1 \wedge v_2 < AC\left(\frac{1}{2}\right) < v_1 \vee v_2\right\}\frac{1+G(F(\boldsymbol{\sigma}|_{\mathbf{I}}))}{2} \\ &\quad - \left\{v_1 \wedge v_2 > AC\left(\frac{1}{2}\right)\right\} - \frac{(1-\beta_1)\left\{\Delta c\left(\frac{1}{2},\frac{1}{2}\right) < v_1 < AC\left(\frac{1}{2}\right)\right\} + 1 - \beta_2\left\{v_2 < AC\left(\frac{1}{2}\right)\right\}}{2}\theta \\ &= -\sum_s \mathbb{E}q_s^S - \frac{(1-\beta_1)\left\{\Delta c\left(\frac{1}{2},\frac{1}{2}\right) < v_1 < AC\left(\frac{1}{2}\right)\right\} + 1 - \beta_2\left\{v_2 < AC\left(\frac{1}{2}\right)\right\}}{2}\theta,\end{aligned}$$

This verifies the proposition's third identity. Note the final equality above recognizes if $v_m <$

$AC\left(\frac{1}{2}\right)$, manufacturer m operates only when sharing suppliers.

Finally, by absolute continuity of G on support $[\underline{F}, \overline{F}]$,

$$\theta = g(F(\boldsymbol{\sigma}|_{\mathbf{I}}))(\hat{F} - F(\boldsymbol{\sigma}|_{\mathbf{I}})) < 0 \;\Rightarrow\; \hat{F} \vee \underline{F} < F(\boldsymbol{\sigma}|_{\mathbf{I}}) - \epsilon < F(\boldsymbol{\sigma}|_{\mathbf{I}}) \leq \hat{F}$$

for $\epsilon > 0$ sufficiently small. So (2.52) holds w.p.p. Given the definitions of $\hat{F}$ and $F(\boldsymbol{\sigma}|_{\mathbf{I}})$ in (2.7) and (2.8) as thresholds for supplier sharing under the planner's and equilibrium relationship networks, $\theta < 0$ implies the relationship network is inefficiently over connected with positive probability. So by Proposition 2.2.5, the relationship and production networks are not under connected compared to their planner's counterparts w.p.p. Likewise,

$$\theta = g(F(\boldsymbol{\sigma}|_{\mathbf{I}}))(\hat{F} - F(\boldsymbol{\sigma}|_{\mathbf{I}})) > 0 \;\Rightarrow\; \underline{F} \leq F(\boldsymbol{\sigma}|_{\mathbf{I}}) < F(\boldsymbol{\sigma}|_{\mathbf{I}}) + \epsilon < \hat{F} \wedge \overline{F}$$

for $\epsilon > 0$ sufficiently small. So (2.53) holds w.p.p. So $\theta < 0$ implies the relationship network is under connected with positive probability. So by Proposition 2.2.5, the relationship and production networks are not over connected compared to their planner's counterparts w.p.p. $\square$

Proof of Corollary 2.2.1. Let $\mathbb{E}\pi_m^M$ be manufacturer m's ex-ante profit. Proposition 2.2.3 means one can write

$$\mathbb{E}\pi_2^M = \beta_{21}\left(\underline{\pi}_1^S + \underline{\pi}_2^M + \int_{F=\underline{F}}^{F=F(\boldsymbol{\sigma}|_{\mathbf{I}})} F(\boldsymbol{\sigma}|_{\mathbf{I}}) - F dG\right). \tag{2.58}$$

where $F(\boldsymbol{\sigma}|_{\mathbf{I}})$ is by the RHS of (2.49). Observe

$$\underline{\pi}_1^S = \frac{1-\beta_1}{2}\left(v_1 - AC\left(\frac{1}{2}\right)\right) \vee 0 \quad \underline{\pi}_2^M = \frac{\beta_2}{2}\left(v_2 - AC\left(\frac{1}{2}\right)\right) \vee 0$$

are weakly increasing in v_1. They are also differentiable in v_1 when $v_1 \neq AC\left(\frac{1}{2}\right)$. Also, $F(\boldsymbol{\sigma}|_{\mathbf{I}})$ is weakly increasing in v_1, and differentiable in the parameter when $v_1 \neq AC\left(\frac{1}{2}\right), \Delta C\left(\frac{1}{2}, \frac{1}{2}\right)$.

So by (2.58), $\mathbb{E}\pi_2^M$ is weakly increasing in v_1 and

$$\begin{aligned}\frac{\mathbb{E}\pi_2^M}{\partial v_1} &= \beta_{21}\left(\frac{\partial \underline{\pi}_1^S}{\partial v_1} + \int_{F=0}^{F=F(\boldsymbol{\sigma}|_{\mathbf{I}})} \frac{\partial F(\boldsymbol{\sigma}|_{\mathbf{I}})}{\partial v_1} dG\right) \\ &= \frac{\left\{AC\left(\frac{1}{2}\right) < v_1\right\} + \left\{\Delta C\left(\frac{1}{2},\frac{1}{2}\right) < v_1 < AC\left(\frac{1}{2}\right)\right\}(1-\beta_1)\,G\left(F(\boldsymbol{\sigma}|_{\mathbf{I}})\right)}{2} \geq 0\end{aligned}$$

at all differentiable points - $v_1 \neq AC\left(\frac{1}{2}\right), \Delta C\left(\frac{1}{2},\frac{1}{2}\right)$. (2.58) also inherits continuity of $\underline{\boldsymbol{\pi}}$ and $F(\boldsymbol{\sigma}|_{\mathbf{I}})$ in v_1. This completes the argument for $\mathbb{E}\pi_2^M$ being weakly increasing in v_1. Moreover, the derivative above is also strictly positive iff $AC\left(\frac{1}{2}\right) < v_1$ or $\Delta C\left(\frac{1}{2},\frac{1}{2}\right) < v_1 < AC\left(\frac{1}{2}\right)$ and $G\left(F(\boldsymbol{\sigma}|_{\mathbf{I}})\right) > 0$. $G(F(\boldsymbol{\sigma}|_{\mathbf{I}}))$ is the probability of supplier sharing in the relationship network by Proposition 2.2.3. Hence, $\mathbb{E}\pi_2^M$ strictly increases in v_1 on $[\Delta C\left(\frac{1}{2},\frac{1}{2}\right))$ when supplier sharing occurs w.p.p. under the relationship network.

Proposition 2.2.3 also means one can write

$$\mathbb{E}\pi_1^M = \underline{\pi}_1^M + \int_{F=\underline{F}}^{F=F(\boldsymbol{\sigma}|_{\mathbf{I}})} h(\boldsymbol{\sigma}|_{\mathbf{I}})dG, \tag{2.59}$$

where $h(\boldsymbol{\sigma}|_{\mathbf{I}})$ is the additional profit manufacturer 1 makes when $\mathbf{a} = \mathbf{I}$. $\underline{\pi}_1^M$ is clearly constant in v_2. (2.59) also implies $\mathbb{E}\pi_1^M$ is weakly increasing in $F(\boldsymbol{\sigma}|_{\mathbf{I}})$ and $h(\boldsymbol{\sigma}|_{\mathbf{I}})$. $F(\boldsymbol{\sigma}|_{\mathbf{I}})$ is weakly increasing in v_2. Moreover, Lemmas 2.9.6 to 2.9.9 show

$$
\begin{aligned}
h(\boldsymbol{\sigma}|_{\mathbf{I}}) &= \begin{cases}
\frac{\beta_1}{2}\left(AC\left(\frac{1}{2}\right) - \Delta C\left(\frac{1}{2},\frac{1}{2}\right)\right) & \text{if } AC\left(\frac{1}{2}\right) - H(v_1) < v_2 \leq AC\left(\frac{1}{2}\right) < v_1 \\
\frac{\beta_1}{2}\left(AC\left(\frac{1}{2}\right) - \Delta C\left(\frac{1}{2},\frac{1}{2}\right)\right) & \text{if } v_2 \leq AC\left(\frac{1}{2}\right) < v_1 \wedge (H(v_1) + v_2) \\
\frac{\beta_1}{2}\left(v_1 - \Delta C\left(\frac{1}{2},\frac{1}{2}\right)\right) \vee 0 & \text{if } v_1 \leq AC\left(\frac{1}{2}\right) < v_2 \\
\frac{\beta_1}{2}\left(v_1 - \Delta C\left(\frac{1}{2},\frac{1}{2}\right)\right) & \text{if } v_1 \vee v_2 \leq AC\left(\frac{1}{2}\right) < H(v_1) + v_2 \\
0 & \text{if otherwise.}
\end{cases} \\
&= \begin{cases}
\frac{\beta_1}{2}\left(AC\left(\frac{1}{2}\right) - \Delta C\left(\frac{1}{2},\frac{1}{2}\right)\right) & \text{if } AC\left(\frac{1}{2}\right) < v_1 \wedge (H(v_1) + v_2) \\
\frac{\beta_1}{2}\left(v_1 - \Delta C\left(\frac{1}{2},\frac{1}{2}\right)\right) \vee 0 & \text{if } v_1 \leq AC\left(\frac{1}{2}\right) < H(v_1) + v_2 \\
0 & \text{if otherwise.}
\end{cases},
\end{aligned}
\tag{2.60}
$$

where $H(v)$ is defined in Lemma 2.9.5. So $h(\boldsymbol{\sigma}|_{\mathbf{I}})$ is constant in v_2 except when $v_2 + H(v_1) = AC\left(\frac{1}{2}\right)$, in which case it weakly but discontinuously jumps from 0 to a higher value - $\frac{\beta_1}{2}\left(AC\left(\frac{1}{2}\right) - \Delta C\left(\frac{1}{2},\frac{1}{2}\right)\right)$ or $\frac{\beta_1}{2}\left(v_1 - \Delta C\left(\frac{1}{2},\frac{1}{2}\right)\right) \vee 0$, depending on on v_1's value. So $\mathbb{E}\pi_1^M$ is weakly increasing in v_2. $\mathbb{E}\pi_1^M$ is also differentiable in v_2:

$$
\frac{\partial \mathbb{E}\pi_1^M}{\partial v_2} = h(\boldsymbol{\sigma}|_{\mathbf{I}})g(F(\boldsymbol{\sigma}|_{\mathbf{I}}))\frac{\partial F(\boldsymbol{\sigma}|_{\mathbf{I}})}{\partial v_2} + \int_{F=\underline{F}}^{F=F(\boldsymbol{\sigma}|_{\mathbf{I}})} \frac{\partial h(\boldsymbol{\sigma}|_{\mathbf{I}})}{\partial v_1} dG
$$

wherever $h(\boldsymbol{\sigma}|_{\mathbf{I}})$ and $F(\boldsymbol{\sigma}|_{\mathbf{I}})$ are likewise differentiable. Using (2.49) and (2.60), the first term in the derivative simplifies to

$$
\begin{aligned}
h(\boldsymbol{\sigma}|_{\mathbf{I}})\frac{\partial F(\boldsymbol{\sigma}|_{\mathbf{I}})}{\partial v_2}g(F(\boldsymbol{\sigma}|_{\mathbf{I}})) &= \frac{\beta_1\left(v_1 \wedge AC\left(\frac{1}{2}\right) - \Delta C\left(\frac{1}{2},\frac{1}{2}\right)\right) \vee 0 \left\{H(v_1) + v_2 > AC\left(\frac{1}{2}\right)\right\}}{2} \\
&\quad \times \frac{1 - \beta_2\left\{v_2 > AC\left(\frac{1}{2}\right)\right\}}{2} g(F(\boldsymbol{\sigma}|_{\mathbf{I}})) \\
&= \frac{\beta_1 H(v_1)\left\{H(v_1) + v_2 > AC\left(\frac{1}{2}\right)\right\}}{2(1-\beta_1)} \frac{1 - \beta_2\left\{v_2 > AC\left(\frac{1}{2}\right)\right\}}{2} g(F(\boldsymbol{\sigma}|_{\mathbf{I}})),
\end{aligned}
$$

whenever $v_2 \neq AC\left(\frac{1}{2}\right)$. From (2.60), the second term is

$$\int_{F=\underline{F}}^{F=F(\boldsymbol{\sigma}|_{\mathbf{I}})} \frac{\partial h(\boldsymbol{\sigma}|_{\mathbf{I}})}{\partial v_1} dG = 0$$

whenever $v_2 \neq AC\left(\frac{1}{2}\right) - H(v_1)$, or $v_1 \leq \Delta C\left(\frac{1}{2}, \frac{1}{2}\right)$, $v_2 = AC\left(\frac{1}{2}\right) - H(v_1)$, or $G(F(\boldsymbol{\sigma}|_{\mathbf{I}})) = 0$. The second term is undefined (or equals ∞) if otherwise. So $\mathbb{E}\pi_1^M$ is strictly increasing in v_2 iff $g(F(\boldsymbol{\sigma}|_{\mathbf{I}})) > 0$, $H(v_1) > 0$, and $v_2 + H(v_1) > AC\left(\frac{1}{2}\right)$ (via effects from the first term), or when $v_2 = AC\left(\frac{1}{2}\right) - H(v_1)$, $v_1 > \Delta C\left(\frac{1}{2}, \frac{1}{2}\right)$, and $G(F(\boldsymbol{\sigma}|_{\mathbf{I}})) > 0$ (via effects from the second term).

Thus, both derivatives are strictly positive when $\mathbf{v} >> AC\left(\frac{1}{2}\right)(1,1)$ and $\underline{F} < F(\boldsymbol{\sigma}|_{\mathbf{I}}) < \overline{F}$. But $\underline{F} < F(\boldsymbol{\sigma}|_{\mathbf{I}}) < \overline{F}$ holds when supplier sharing both occurs and doesn't under the relationship network w.p.p. This establishes the corollary's final statement. □

2.9.3 Auxilliary Statistics

Model-specific Attributes: Figure 2.19 shows automobiles have not been changing in size, but are getting taller. The LHS of Figure 2.20 also reveals models becoming increasingly powerful. These properties hold regardless of the models' import statuses or brands. Hence, they likely reflect technological improvements made by auto manufacturers in increasing their products' desirability. In contrast, the middle years (2000-2008) of the sample show a reduction in fuel efficiency as measured by MPD. The right panel however, suggests this might be due to soaring fuel prices over the same timeframe, rather than regression in the vehicles' characteristics. In sum, cross-period variation in automobile size, volume, power, fuel efficiency and import status appears driven by separate factors, weakening their temporal correlation with each other. This suggests these attributes are also unlikely to strongly correlate with other, unobserved automobile demand determinants.

Annual Demographics: The right panel of Figure 2.20 plots the cost of gasoline. Observe this correlates with MPD on the left of Figure 2.20, and thus affects demand for automobiles.

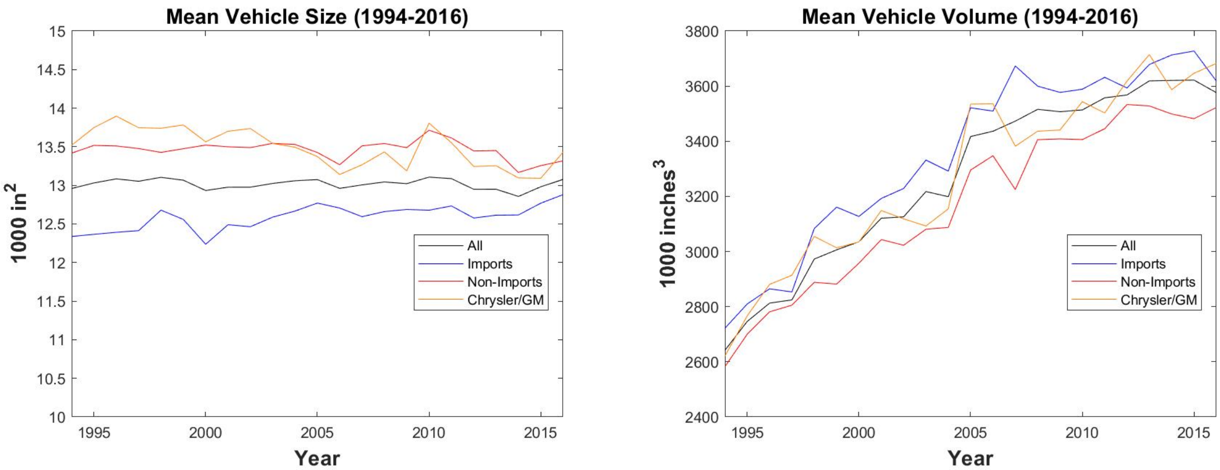


Figure 2.19: Size and Volume over 1994-2016

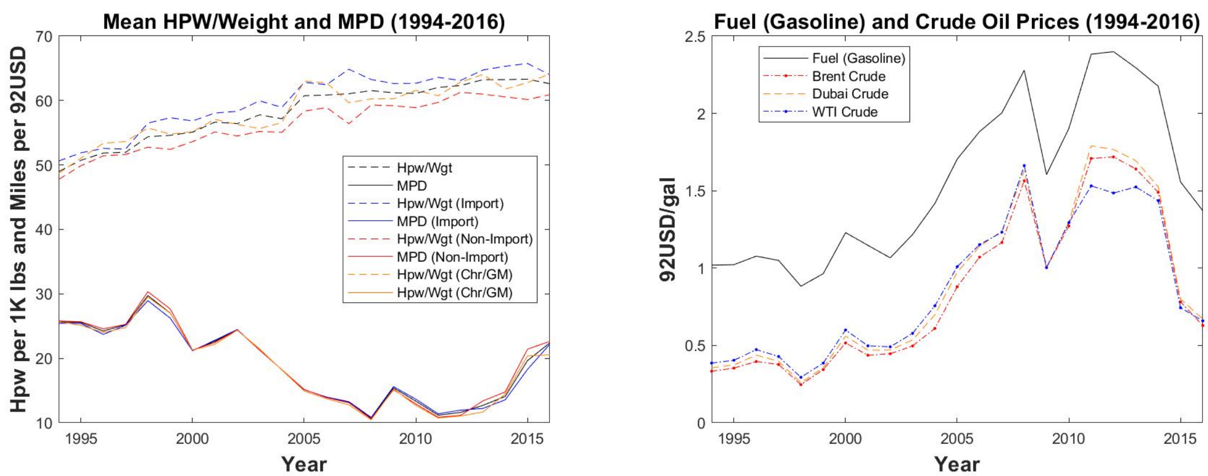


Figure 2.20: Horsepower, Mile per Dollar, Fuel and Crude Oil Prices over 1994-2016

The remaining lines on Figure 2.20's RHS plot the price indices for crude oil sold at various locations. These lines also correlate with the price of gasoline, and affect automobile manufacturing costs.

Figure 2.21 plots manufacturing wages for relevant sectors in the left panel[49], and relevant input prices on the right. As illustrated by the LHS plot, average wages paid by auto manufac-

49. with annual household income measured in 1992 US$ per hour by calibrating working hours to 2087, the US Office of Personnel Management's recommended number, available at https://www.opm.gov/policy-data-oversight/pay-leave/pay-administration/fact-sheets/how-to-compute-rates-of-pay/

turers declined over 1994-2016, even as their national counterparts soared. This is consistent with the oft portrayed narrative of US auto worker unions accepting pay cuts, so as to keep their employers competitive vis-a-vis foreign manufacturers and financially sound. The wages paid by auto suppliers track their auto manufacturing counterparts well. However, auto supplier employees are consistently paid less than their downstream peers. Differences in the nature of tasks faced by workers making inputs and employees assembling output, might explain this discrepancy.

Figure 2.21's RHS plots price indices of various inputs into auto production, alongside CPI. Each index has been normalized to equal 1 in 1992. Relative to consumables, crude oil (average petroleum spot price), shipping (deep sea freight), steel (cold rolled strips/sheets) and plastics (plastic material and resins) became more expensive over 1999-2008, and cheaper during 2015-16. The increase in pre-GFC shipping prices is particularly surprising, since it coincides with the proliferation of imported models over the same timeframe (Figure 2.9, RHS, blue). The first order effect of lower transportation costs should be to increase international trade. A possible explanation for this contradiction involves foreign manufacturers needing to ship inputs from foreign suppliers, even when assembling models in the US. Such needs arise when manufacturers and suppliers develop significant relationship-specific capital from working together. Another explanation considers lower tariffs offsetting rising shipping prices. China's entry into the World Trade Organization in 2001 exemplifies this hypothesis. A plot of recent assembly plant locations however, reveals that they mainly still lie in Germany and Japan (Figure 2.12, pink and red).

Supplier Popularity: Figure 2.22 summarizes the number of customers across main suppliers for each contract category. The left panel shows suppliers of all categories exist in their sample's upper tertile when ranked by customer popularity. Although suppliers in the lower tertile often sell only one type of input, the right panel also illustrates the numerosity of lower

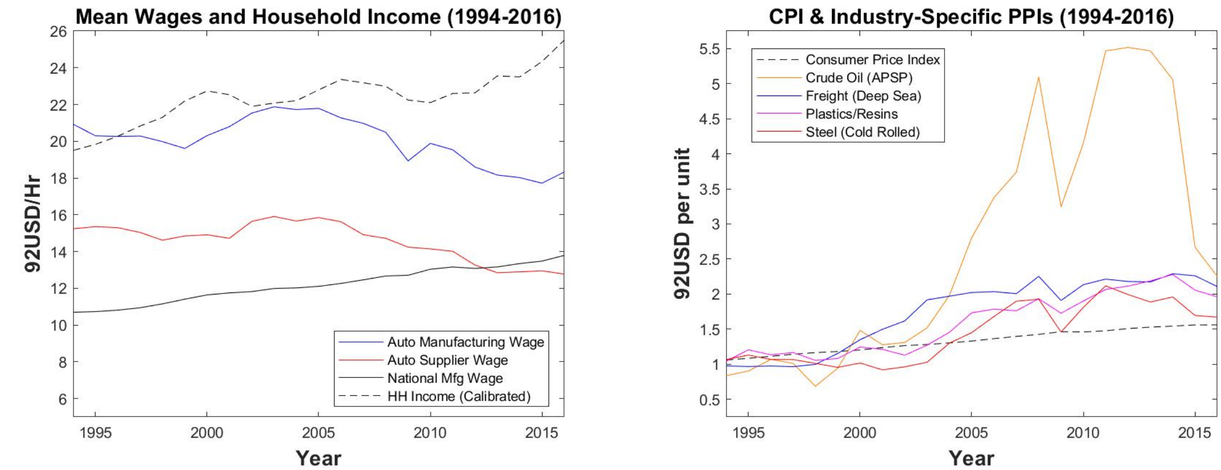


Figure 2.21: Income, Wages and Input Price Indices over 1994-2016

tertile suppliers per type. Finally, within each panel or tertile, suppliers attract more customers per contract type as their popularity across all types increases.

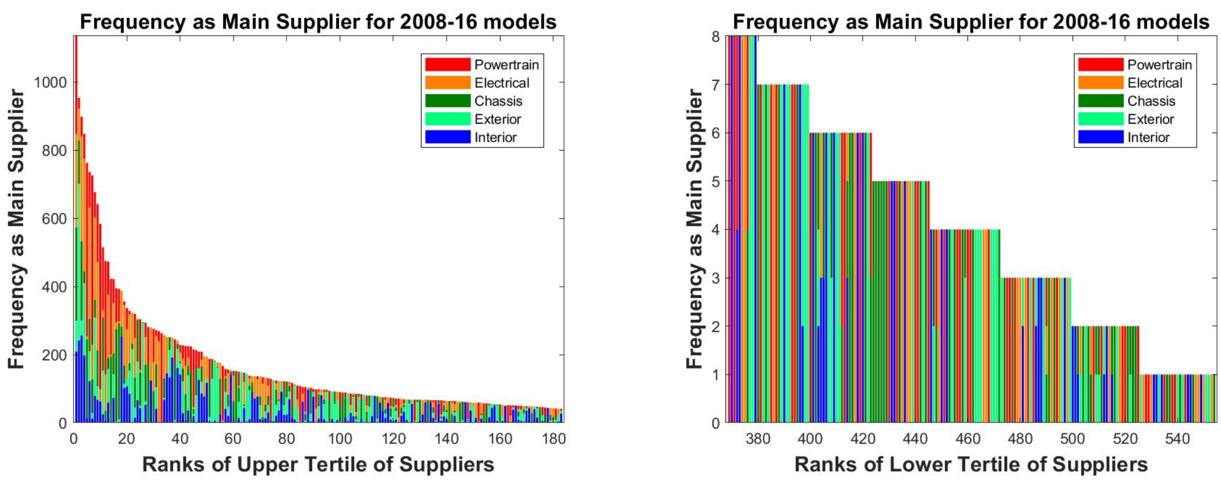


Figure 2.22: Distribution of 2008-16 Models across Suppliers by Contract Type

Supplier Sharing: Figure 2.23 breaks down supplier sharing by assembly location. As with the previous figures, total supplier sharing by models trends upwards regardless of whether the models were imported. When attention is focused on GM and Chrysler models, one can check from the data that total supplier sharing also oscillates up, albeit at a slower rate. These trajectories possibly reflect the increasing number of imports relative to GM and Chrysler models depicted on the RHS of Figure 2.9. When supplier sharing is averaged across operating models, one finds a sharper trend for GM and Chrysler models relative to their rivals, mirroring the post-GFC growth in their sales vis-a-vis imported output, depicted on Figure 2.9's LHS.

Figure 2.24 breaks down supplier sharing by contract type. Its LHS panel illustrates how total supplier sharing is typically increasing across all but two categories. While each plotted line exhibits the same sideways-S shape, the lines corresponding to electrical and chassis contracts trend down. This contradiction with Proposition 2.2.4 probably reflects factors specific to electrical or chassis producers unaccounted for in Section 2, emphasizing the simplicity of Section 2's analysis. The RHS panel illustrates how average supplier sharing across each contract type track their counterparts on the left, albeit with flatter trajectories. In particular, average supplier sharing is still increasing for the powertrain, exterior and interior categories, but at slower rates.

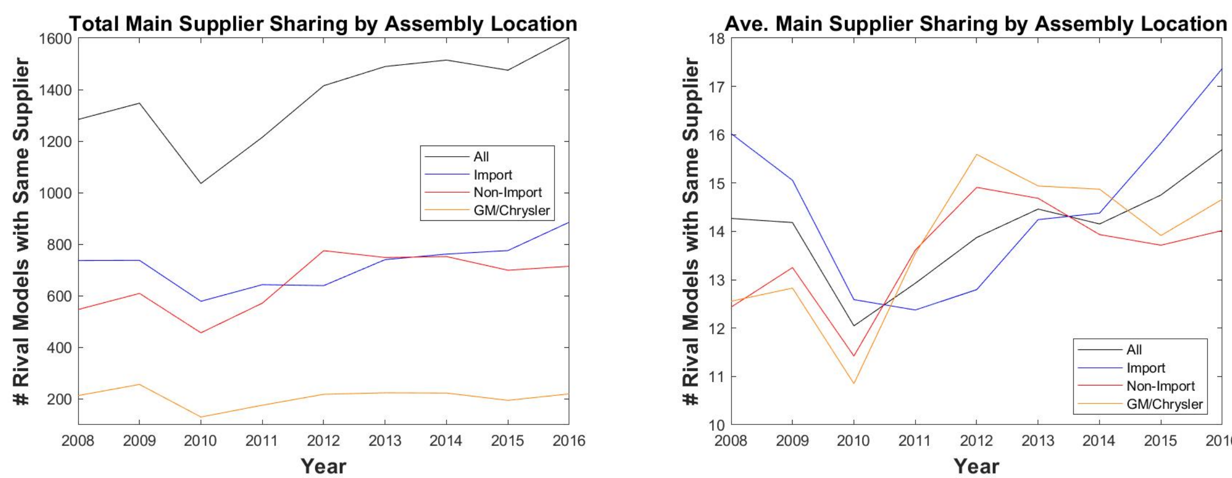


Figure 2.23: Supplier Sharing by Assembly Location over 2008-16

2.9.4 Identifying Assumptions

Let g graph the collection of model-supplier production relationships, across all models in some population $\mathcal{M}$ and suppliers in $\mathcal{S}$. Let g^o denote the production network for an earlier period

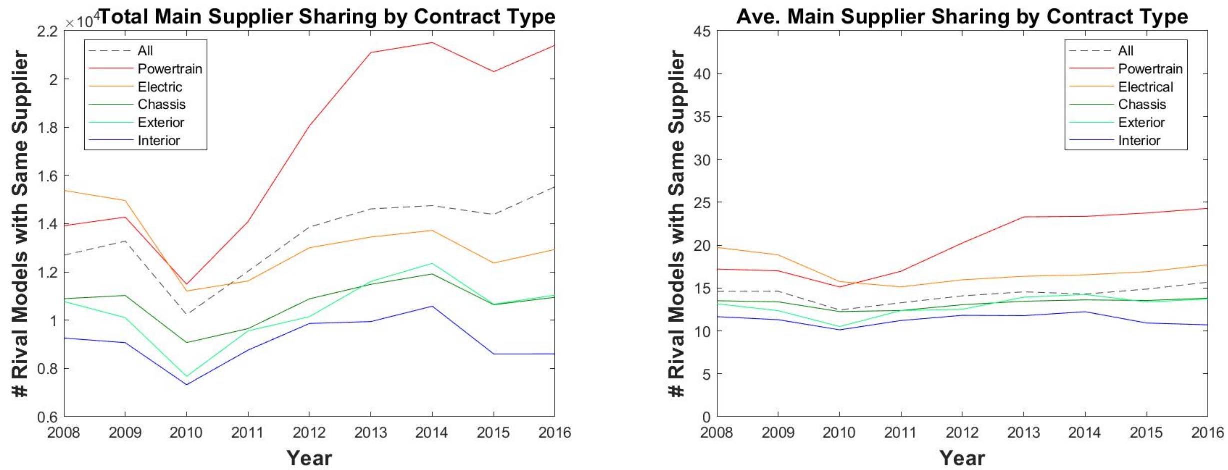


Figure 2.24: Total Supplier Sharing by Autopart/Contract Type over 2008-16

whose vertices equal $\mathcal{M}^o \cup \mathcal{S}$. Finally, let $\mathcal{N} = \mathcal{M} - \mathcal{M}^o$ denote the set of model entrants, and $\mathcal{F}^o$ the previous period's producership structure. In what follows, $s(m)$ denotes the suppliers of model $m \in \mathcal{M}$ under the production network g, while $m(s)$ denotes supplier s's model-customers.

When T, $|\mathcal{M}_t|$ and $|\mathcal{S}_t|$ are large across periods $t \in \mathbb{N}$, the data allows the distribution of $\mathcal{D} = (\mathbf{p}, \mathbf{q}, g, \mathcal{F}, \mathbf{x}, \mathbf{w}, \mathbf{a}, \mathbf{e}, g^o, \mathcal{F}^o)$, (conditional on $\mathcal{M}, \mathcal{S}$) to be approximated. $F_{\mathcal{D}}$ is thus assumed to be known. Additional variables $(\boldsymbol{\xi}, \boldsymbol{\omega}, \mathbf{t}, \boldsymbol{\kappa})$ are unobserved. Let $\mathcal{I}$ denote $(\mathcal{D}, \boldsymbol{\xi}, \boldsymbol{\omega}, \boldsymbol{\kappa})$. $F_{\mathcal{I}}$ is thus unknown.

As in Subsection 2.5.1 or Berry & Haile (2014), this framework eschews explicit modeling of entry and exit. Relatedly, the underlying population of models is time invariant. Assuming the researcher samples time-varying attributes of the same models with sufficient frequency is problematic. However, as Section 2.6's estimation procedure shows, imposing symmetric functional form restrictions across models partially address this issue. Moreover, associating each model with a previous period predecessor, so that a 2010 Camry is equated with its 2008 edition, also addresses this concern.

As explained in Subsection 2.5.2, the aim is to empirically approximate incentives facing firms when they form new relationships with other firms, and their distortions due to inefficiencies highlighted by the simpler model. These estimands are identified under the following assumptions. First, demand is well-behaved and identified.

Assumption 2.9.1. [Identified & Smooth Demand] Output equals demand $\mathbf{q}_t = \mathbf{q}(\mathbf{p}_t, \mathbf{x}_t, \boldsymbol{\xi}_t, \mathbf{hh}_t)$ for all periods $t \leq T$, where $\mathbf{hh}_t$ are the variables in $\mathbf{a}_t$ affecting demand. The demand function $\mathbf{q}(\mathbf{p}, \mathbf{x}, \boldsymbol{\xi}, \mathbf{hh})$ is C^2 in $\mathbf{p} \in \mathbb{R}^{\mathcal{M}}_{>0}$, and has an invertible Hessian $\frac{\partial^2 \mathbf{q}_f}{\partial \mathbf{p}_f \partial \mathbf{p}_f'}$ on $(\mathbf{p}, \mathbf{x}, \boldsymbol{\xi}, \mathbf{hh})$'s support. The distribution of $\boldsymbol{\xi}|\mathcal{D}$ is degenerate on $\mathcal{D}$'s support and is identified. $\mathbf{q}(\mathbf{p}, \mathbf{x}, \boldsymbol{\xi}, \mathbf{hh})$ is also identified on the same support.

In the context of the microfounded model, Example 2.5.2 satisfies Assumption 2.9.1. Berry & Haile (2014) produce more general conditions under which $\mathbf{q}(\mathbf{p}, \mathbf{x}, \boldsymbol{\xi}, \mathbf{hh})$ is identified.

Next, manufacturers and suppliers operate smooth, product specific production technology.

Assumption 2.9.2. [Smooth Costs] For each model-supplier pair $(m, s) \in \mathcal{M} \times \mathcal{S}$ and quantity $q > 0$, let $C^s(q) = C(q, \mathbf{e}_s)$ denote supplier s's expected cost of making inputs for q units of output, conditional on its attributes $\mathbf{e}_s$, and $c^m(q) = c(q, \mathbf{w}_m)$ equal m's expected cost of assembling inputs into q output units, given $\mathbf{w}_m$. These functions are C^∞ in $q \in \mathbb{R}_{>0}$ for all $(m, s) \in \mathcal{M} \times \mathcal{S}$.

Examples 2.5.3 and 2.5.4 satisfy Assumption 2.9.2. Since any continuous function on $[0, 1]$ is uniformly approximable by (infinitely differentiable) polynomials, the assumption is less onerous than it appears.

Finally, downstream market prices are profit maximizing. The remaining assumptions specify how each manufacturer's profit relates to the primitives of a model economy. First,

Assumption 2.9.3. [Proportionate Profits] For each manufacturer $f \in \mathcal{F}$, $\mathbf{p}_f >> \mathbf{0}$ and maximizes

$$\sum_{m\in f}\left[p_m q_m - c^m(q_m) - \sum_{s\in s(m)\cap nb(f)} \omega_{ms} q_m - \sum_{s\in s(m)\cap au(f)} t_{ms}\right] - \sum_{s\in nb(f)} \left[C^s(Q_s) - C^s\left(Q_s - Q_{sf}\right)\right],$$

where $Q_{sf} = \sum_{m\in m(s)\cap f} q_m$ is the quantity supplied by supplier s to f, $nb(f) \subseteq \cup_{m\in f} s(m)$ is an identified set of suppliers satisfying $nb(f)\cap s(m) = s(m)$ if $m \in \mathcal{M}-\mathcal{N}$, and $au(f) = \mathcal{S}-nb(f)$.

When the data is generated by the microfounded model and

$$\begin{aligned} nb(f) &= nb(f,\mathbf{g}) = \cup_{m\in f}\{s \in s(m,g) : \exists n \in f^o\ (s,n) \in g^o\}, \\ au(f) &= au(f,\mathbf{g}) = \mathcal{S} - nb(f,\mathbf{g}), \end{aligned} \tag{2.61}$$

one can show each manufacturer's profit is affine in Assumption 2.9.3's maximand[50]. Intuitively, Nash bargaining between manufacturer f and its suppliers in $nb(f,\mathbf{g})$ under the micro-founded model, allows the firms to redivide their combined profit according to fixed bargaining parameters, for any vector of prices $\mathbf{p}_f$ chosen by f. Prices thus maximize revenues less expected production costs incurred by m and $nb(f)$, residual costs $\boldsymbol{\omega}_{f,nb(f)}\mathbf{q}_f$ incurred when f's models are matched to suppliers in $nb(f)$, and payments for inputs $\mathbf{t}_{f,au(f)}$ made to suppliers outside $nb(f)$.

Second,

Assumption 2.9.4. [Payment Rule] Manufacturer $f \in \mathcal{F}$ pays t_{ms} to supplier $s \in au(f)$ in exchange for inputs used to assemble q units of model $m \in f$, where $au(f)$ is defined as in Assumption 2.9.3. The payments satisfy $t_{ms} = \kappa_{ms} q$ for all $(m,s) \in f \times au(f)$ and $q \in \mathbb{R}_{\geq 0}$.

Observe the objective in Assumption 2.9.3 clarifies how $(t_{ms})_{s\in au(f)}$ constitute payments by manufacturer f and its "Nash bargaining set" of suppliers $nb(f)$, to suppliers in $au(f)$. Under

50. The slope coefficient is $\tau^f(\mathbf{g}) = \frac{\tau_f}{|nb(f,\mathbf{g})|-(|nb(f,\mathbf{g})|-1)\tau_f}$, and the constant coefficient is $\tau^f(\mathbf{g})\sum_{n\in f} c^n(0) + SC_f$. Assumption 2.9.3 thus holds when $supp(\mathbf{p}) \subseteq \mathbb{R}^M_{>0}$, even when $\boldsymbol{\tau}$ and $\mathbf{SC}$ change over periods.

(2.61), $au(f)$ is simply the complement of $nb(f)$. Hence, it would be sensible to interpret these payments as determined by competitive procurement auctions, rather than bilateral bargaining.

However, since the payments depend on endogenously chosen bids, they potentially depend on the quantities supplied by all suppliers to all models, in addition to information specific to each payment's model and supplier. This is problematic because when manufacturers set prices taking the dependencies into account, their optimization problems become forbiddingly complicated. Such complications hinder use of their first-order conditions to recover the suppliers' marginal costs from downstream market data.

Assumption 2.9.4 addresses this issue by assuming the payments are linear in own-model output. Because their slopes $\boldsymbol{\kappa}$ can still correlate with rival model output, the payments are potentially statistically related to rival output, even if they functionally depend only on own-model quantities when manufacturers set prices. These requirements are satisfied when suppliers are chosen via first-price auctions where bidders quote piece rate prices, prior to downstream market pricing, as in the microfounded model. Setting $\kappa_{ms} = \{s \in s(m)\} b_{sm}(\mathcal{I})$ in the model obtains $t_{ms} = \kappa_{ms} q_m$. Model m's production cost due to $\mathbf{t}_m$ is thus linear in q_m. However, because bidding precedes downstream pricing, κ_{ms} can still depend on supplier s's beliefs regarding its overall output Q_s, in addition to q_m.

Third,

Assumption 2.9.5. [Exclusion Restriction] ω_{ms} in Assumption 2.9.3 is a shock to the cost of producing output for model $m \in \mathcal{M}$ using inputs from supplier $s \in \mathcal{S}$ where $m \in f \in \mathcal{F}$. There exists instrumental variables $\mathbf{z}_m^c \subseteq (\mathbf{x}, \mathbf{w}_{-m}, \mathbf{a})$ satisfying

$$\mathbb{E}[\omega_{ms} | \mathbf{z}_m^c, g^o, m \in \mathcal{M} - \mathcal{N}] = 0$$

a.s. for all $(m,s) \in \mathcal{M} \times \mathcal{S}$.

The objective in Assumption 2.9.3 highlights how $(\omega_{ms})_{s \in s(m)}$ constitute the residual portion of model m's marginal cost, after its assembly and input production costs $(c^m(q_m), \{C^s(q_m)\}_{s \in nb(f)})$ have been taken into account. The residuals' indices clarify their idiosyncrasy across all possible model-supplier matches induced by the production network.

When these residuals are independent of prices, the subset of incumbent model marginal costs can be used to estimate cost parameters governing $c(q, \mathbf{w}_m)$ and $C(Q, \mathbf{e}_s)$ respectively. This is because the cost of producing such models $m \in \mathcal{M} - \mathcal{N}$, depend only on the cost of assembling them $c^m(q_m)$, producing inputs for their manufacturer $\{C^s(Q_s) - C^s(Q_s - Q_{sf})\}_{s \in nb(f)}$, and the residuals $(\omega_{ms})_{s \in nb(f)}$ themselves. In particular, since the models inherit past suppliers, their marginal costs do not contain payments $\boldsymbol{\kappa}$ determined via procurement auctions, that likely correlate with quantities determining $c^m(q_m)$ and $\{C^s(Q_s) - C^s(Q_s - Q_{sf})\}_{s \in nb(f)}$, and the attributes $(\mathbf{x}, \mathbf{w})$ of all models.

Obviously, since prices are chosen by manufacturers with $\boldsymbol{\omega}$ in mind, $\boldsymbol{\omega}$ is likely correlated with prices and thus quantities. However, when model attributes $(\mathbf{x}, \mathbf{w})$ and *past* production relationships g^o are mean independent of residual costs, these variables can serve as instruments for estimating $c(q, \mathbf{w}_m)$ and $C(Q, \mathbf{e}_s)$. Assumption 2.9.5 stipulates the existence of such an instrument set. In principle, the variety of models in each period means there are at least $(|\mathcal{M}| - 1) \times dim(\mathbf{x}_m \cup \mathbf{w}_m)$ candidates for $\mathbf{z}^c_m$. Recognition that the reduced form of $\mathbf{q}$ is symmetric across competing models explains why $(\overline{\mathbf{x}}_{f(m)-m}, \overline{\mathbf{x}}_{-f(m)}, \overline{\mathbf{w}}_{f(m)-m}, \overline{\mathbf{w}}_{-f(m)})$ is typically used as instruments in the BLP literature. When supplier-level data is available however, the reduced-form symmetries may fail to hold[51]. This suggests basing $\mathbf{z}^c_m$ on $((\overline{\mathbf{x}}_s)_{s \in s(m,g^o)}, (\overline{\mathbf{x}}_s)_{s \notin s(m,g^o)}, (\overline{\mathbf{w}}_{s,-m})_{s \in s(m,g^o)}, (\overline{\mathbf{w}}_s)_{s \notin s(m,g^o)})$ instead[52], yielding a larger number of

51. In particular, it is no longer trivially true that each product's cost depends only on own-cost covariates. It's dependence on rival product attributes depends on its suppliers.

52. The averages are taken over models in g, linked to s under g^o, since past attributes such as "$\mathbf{x}^o$" are

instruments[53].

Fourth,

Assumption 2.9.6. [Completeness] For any model $m \in \mathcal{M}$ and its manufacturer $f \in \mathcal{F}$, let

$$\mathcal{D}_m = \left\{ q_m, \mathbf{w}_m, \left\{ Q_s, \mathbf{e}_s, Q_{sf}, \left.\frac{\partial Q_{s,-f}}{\partial q_m}\right|_{\mathbf{p}_{-f},\mathbf{x},\boldsymbol{\xi},\mathbf{hh}} \right\}_{s \in nb(f)}, s(m) \right\},$$

denote the model's cost-shifting variables, where $nb(f)$ is as defined in Assumption 2.9.3, Q_{sf} is the quantity supplier s sells to f, and $Q_{s,-f} = Q_s - Q_{sf}$ is the quantity s sells to f's rivals. Let $s(\mathcal{D}_m) = \{\mathcal{D}_m^* : \exists \overline{\mathcal{D}} \in supp(\mathcal{D}),\ \overline{\mathcal{D}}_m = \mathcal{D}_m^*\}$ denote the set of possible $\mathcal{D}_m$ values. For any map $C : s(\mathcal{D}_m) \to \mathbb{R}$, $\mathbb{E}[C(\mathcal{D}_m)|\mathbf{z}_m^c] = 0$ a.s. implies $C = 0$, where $\mathbf{z}_m^c$ is defined as in Assumption 2.9.5.

The previous assumption and the comments that follow, clarify why instruments $\mathbf{z}_m^c$ are needed to identify any feature of assembly or input production costs. Clearly, for such identification to occur, the endogenous determinants of marginal costs must vary with the instruments as per a rank condition. Assumption 2.9.6 is the equivalent condition in the context of this paper. In particular, it mirrors the completeness criterion introduced by Newey & Powell (2003), for identifying triangular systems of equations in endogenous variables. The endogenous variables in question are encoded by $\mathcal{D}_m$. These include the market shares $\mathbf{q}_f$ associated with a given manufacturer, the market shares of the manufacturer's retained suppliers $\mathbf{Q}_{nb(f)}$, and the responsiveness of the latter to the former, captured by the cross-market diversion ratios $\frac{\partial \mathbf{Q}_{nb(f),-f}}{\partial \mathbf{q}_f} = \frac{\partial \mathbf{q}_f}{\partial \mathbf{p}_f}^{-1} \frac{\partial \mathbf{Q}_{nb(f),-f}}{\partial \mathbf{p}_f}$. The twist here is that $\mathcal{D}_m$ does not have constant dimension unless $nb(f)$ is conditioned upon. Assumption 2.9.6 has thus been written in a way to reflect this.

unspecified under $F_{\mathbf{p},\mathbf{q},g,\mathcal{F},\mathbf{x},\mathbf{w},\mathbf{a},\mathbf{e},g^o,\mathcal{F}^o}$.

53. One can obviously incude variables in $\mathbf{x}_m$ excluded from $\mathbf{w}_m$, and household demographic information $\mathbf{hh}$ in the instrument set too.

Unfortunately, because the dimension of $\mathcal{D}_m$ grows with $|nb(f)|$, Assumption 2.9.6 is unlikely to hold when $|nb(f)|$ is large but $dim(\mathbf{z}_m^c)$ is small. Fortunately, the previously suggested network-based instrument set produces more instrumental variables. Alternatively, functional form restrictions along the lines of examples 2.5.3 and 2.5.4 can also be imposed.

Finally, three assumptions of more technical nature are needed. First,

Assumption 2.9.7. [Dense-in-itself Support] For any model $m \in \mathcal{M}$ produced by manufacturer $f \in \mathcal{F}$, the set of its possible prices $\{p_m^* : \exists \mathcal{D}^*,\ (p_m^*, \mathcal{D}^* - p_m^*) \in supp(\mathcal{D})\}$ contains no isolated points. Define $\mathcal{D}_m$ as in Assumption 2.9.6. Suppose m's cost-shifters equal $\mathcal{D}_m^* \in s(\mathcal{D}_m)$. Choose one of the continuous cost-shifting variables:

$$d_m \in \left\{ \mathbf{q}_f, \mathbf{Q}_{nb(f)}, \mathbf{Q}_{nb(f),f}, \left.\frac{\partial \mathbf{Q}_{nb(f),-f}}{\partial q_m}\right|_{\mathbf{p}_{-f},\mathbf{x},\boldsymbol{\xi},\mathbf{hh}} \right\}.$$

The set of possible values for the chosen cost-shifter $\{d_m : (d_m, \mathcal{D}_m^* - d_m^*) \in s(\mathcal{D}_m)\}$ contains no isolated points too.

Assumption 2.9.7 basically requires each point in the support of $\mathcal{D}_m$ to be approximated by a sequence of alternate possible values of $\mathcal{D}_m$, in each direction corresponding to $\mathcal{D}_m$'s continuous variables. This allows derivatives of marginal revenues with respect to the continuous entries of $\mathcal{D}_m$ to be defined and identified on $\mathcal{D}_m$'s support.

Next,

Assumption 2.9.8. [Product Lineup Persistence] Suppose there exists a realization for the data $\mathcal{D}$ in its support such that m is a new model

$$m \in \mathcal{N} \cap f, \quad f \in \mathcal{F}.$$

Let $s(m)^*$ denote m's suppliers, $\mathbf{w}_m^*$ denote m's attributes, and $\mathbf{e}^*$ denote the suppliers' attributes under this realization. Then there exist another realization $\mathcal{D}'$ in the data's support, where m is an older model

$$m \in \mathcal{M} \cap f - \mathcal{N}, \quad f \in \mathcal{F},$$

whose suppliers and attributes are still $s(m)^*$ and $\mathbf{w}_m^*$ respectively. Moreover, when f's Nash bargaining set of suppliers under $\mathcal{D}'$ are $nb(f)'$, their attributes are $\mathbf{e}^*_{nb(f)'}$.

Assumption 2.9.8 basically requires each new model introduced in a given period, to appear as an older model in the following periods. Once models with insignificant market shares are removed from the sample, virtually zero models in the data are sold only in a single year. The assumption allows marginal costs for incumbent models to be extrapolated to newer models, thus identifying the newer models' assembly and input production cost structures.

Last,

Assumption 2.9.9. [Product Predecessors] For any possible realization of old and new models $(\mathcal{M}^o, \mathcal{N})$, let $\mathcal{N}^o$ denote a subset of new models satisfying the following. For any model $m \in \mathcal{N}^o$, there exists a preceding model $m^o \in \mathcal{M}^o$ such that

$$\mathbb{P}\{m \in \mathcal{N}^o\} > 0 \Rightarrow \mathbb{P}\{m^o \in \mathcal{M} - \mathcal{N}\} > 0, \tag{2.62}$$

and

$$(c(q, \mathbf{w}_m) - c(0, \mathbf{w}_m)) - (c(q, \mathbf{w}_{m^o}) - c(0, \mathbf{w}_{m^o})) \tag{2.63}$$

is known to the researcher for all $q > 0$.

The set $\mathcal{N}^o$ introduced by Assumption 2.9.9, should be interpreted as the subset of new models $m \in \mathcal{N}$ with a previous edition or predecessor m^o, in the data. Hence if m represents the 2008 edition of a Toyota Camry, m^o equals the Camry's 2007 version. (2.62) thus implies that each model's predecessor is sampled with positive probability. This condition is trivially satisfied when $\mathcal{N}^o$ is appropriately defined with respect to the data[54].

Assumption 2.9.9's second condition dictates that the researcher knows the difference-in-difference in assembly costs (2.63), across a model and its predecessor for a given quantity. This is an easily satisfiable high-level restriction. For example, when $\mathbf{w}_{m,t}$ in the data consists only of model-specific (instead of model-year specific) dummies, then $\mathbf{w}_{m,t} = \mathbf{w}_{m^o,t-1}$. Thus, (2.63) collapses to zero. Alternatively, suppose $\mathbf{w}_m$ is continuously distributed and $c(q, \mathbf{w}_m)$ is infinitely differentiable in $\mathbf{w}_m$ instead. Then Taylor's theorem relates (2.63) to the derivatives

$$\frac{\partial^k}{\partial \mathbf{w}_m^k} \left(c(q, \mathbf{w}_m) - c(0, \mathbf{w}_m)\right) = \int_0^{q_m} \frac{\partial^k}{\partial \mathbf{w}_m^k} c_q(u, \mathbf{w}_m) du,$$

for each $k \in \mathbb{N}$. Basic manipulation of the manufacturers' first-order conditions reveal the integrand above is identified from the derivatives of marginal revenue with respect to $\mathbf{w}_m$.

2.9.5 Proofs of Results in Subsection 2.5.2

In what follows, let

$$\begin{aligned} mr(\mathcal{D}_m) = c_q(q_m, \mathbf{w}_m) &+ \sum_{s \in s(m)} C_q(Q_s, \mathbf{e}_s) \\ &+ \sum_{s \in nb(f)} \frac{\partial Q_{s,-f}}{\partial q_m}\Big|_{\mathbf{p}_{-f}, \mathbf{x}, \boldsymbol{\xi}, \mathbf{hh}} \left(C_Q(Q_s, \mathbf{e}_s) - C_Q^s(Q_s - Q_{sf}, \mathbf{e}_s)\right) \end{aligned} \tag{2.64}$$

be model m's ex-ante marginal revenue,

$$\kappa_m = \sum_{s \in s(m) \cap au(f)} \kappa_{ms} + \sum_{s \in s(m) \cap nb(f)} \omega_{ms} = \sum_{s \in s(m) \cap au(f)} (\kappa_{ms} - \omega_{ms}) + \omega_m \tag{2.65}$$

54. For example, when each new model in a given period t cannot be linked to a preceding model in past periods, while satisfying Assumption 2.9.9's conditions, one can simply set $m(\mathcal{N}_t, \mathcal{M}_t^o)) = \emptyset$

be m's residual marginal cost and

$$\omega_m = \sum_{s \in s(m,g)} \omega_{ms} = \sum_{s \in s(m)} \omega_{ms} \tag{2.66}$$

be m's ex-post marginal cost shock arising from match-specific factors. Then manufacturer f's first-order conditions imply

$$\begin{aligned}\frac{\partial \mathbf{p}_f}{\partial \mathbf{q}_f}\mathbf{q}_f + \mathbf{p}_f &= \left(c_q^m(q_m) + \sum_{s \in s(m) \cap nb(f)} \omega_{ms} + \sum_{s \in s(m) \cap au(f)} \kappa_{ms} + \sum_{s \in s(m) \cap nb(f)} C_q^s(Q_s) \right)_{m \in f} \\ &\qquad + \frac{\partial \mathbf{Q}_{nb(f),-f}}{\partial \mathbf{q}_f}\Big|_{\mathbf{p}_{-f}, \mathbf{x}, \boldsymbol{\xi}, \mathbf{hh}} \left(C_Q^s(Q_s) - C_Q^s(Q_s - Q_{sf}) \right)_{s \in nb(f)} \\ &= \left(mr(\mathcal{D}_m) + \kappa_m - \sum_{s \in s(m) \cap au(f)} C_q(Q_s, \mathbf{e}_s) \right)_{m \in f}\end{aligned} \tag{2.67}$$

a.s. on the support of the data variables $\mathcal{D}$.

As outlined in Subsection 2.5.2, the initial focus of this paper's identification strategy is on identifying features of production costs associated with old models in $\mathcal{M} - \mathcal{N}$. This allows one to learn about profits created from producing these models. The following results explain this.

Lemma 2.9.10. *Suppose assumptions 2.9.1 to 2.9.7 hold. Then* $mr(\mathcal{D}_m)$ *is identified on* $\{\mathcal{D}_m^* : \exists \overline{\mathcal{D}} \in supp(\mathcal{D} | m \in \mathcal{M} - \mathcal{N}),\ \overline{\mathcal{D}}_m = \mathcal{D}_m^*\}$*. Moreover,* ω_m *is degenerate and identified on the same set.*

Proof. Suppose $\exists$ an alternate mapping $mr^{alt}(.)$ not equal to $mr(.)$, and $\boldsymbol{\omega}_m^{alt} \in \mathbb{R}^{\mathcal{S}}$ satisfying the model. Observe

$$\omega_m + mr(\mathcal{D}_m) = mr_m = \omega_m^{alt} + mr^{alt}(\mathcal{D}_m).$$

Since $s(m) = s(m, g^o)$, hence

$$\mathbb{E}[\omega_m|\mathbf{z}_m^c, g^o] = \sum_{s \in s(m,g^o)} \mathbb{E}[\omega_{ms}|\mathbf{z}_m^c, g^o] = 0 = \sum_{s \in s(m,g^o)} \mathbb{E}[\omega_{ms}^{alt}|\mathbf{z}_m^c, g^o] = \mathbb{E}[\omega_m^{alt}|\mathbf{z}_m^c, g^o]$$

a.s. under Assumption 2.9.5. Hence,

$$\mathbb{E}[(mr - mr^{alt})(\mathcal{D}_m)|\mathbf{z}_m^c] = \mathbb{E}[\mathbb{E}[(mr - mr^{alt})(\mathcal{D}_m)|\mathbf{z}_m^c, g^o]|\mathbf{z}_m^c] = 0.$$

But then Assumption 2.9.6 implies $mr = mr^{alt}$ a.s. It follows $\omega_m = \omega_m^{alt}$. □

Let

$$SC(q_m, \mathbf{w}_m, \mathbf{Q}_{s(m)}, \mathbf{e}_{s(m)}) = c(q_m, \mathbf{w}_m) + \sum_{s \in s(m)} [C(Q_s, \mathbf{e}_s) - C(Q_s - q_m, \mathbf{e}_s)] \tag{2.68}$$

equal the ex-ante social cost of producing model m's output given $\mathcal{D}_m$. Let

$$smc(q, q_m, \mathbf{w}_m, \mathbf{Q}_{s(m)}, \mathbf{e}_{s(m)}) = c_q(q, \mathbf{w}_m) + \sum_{s \in s(m)} C_Q(Q_s - q_m + q, \mathbf{e}_s) \tag{2.69}$$

equal the (ex-ante) social marginal cost of producing q units of model m given $\mathcal{D}_m$.

Lemma 2.9.11. *Suppose assumptions 2.9.1 to 2.9.7 hold. Then*

$$c_{qq}(q, \mathbf{w}_m) \quad C_{QQ}(Q, \mathbf{e}_s) \quad SC(q_m, \mathbf{w}_m, \mathbf{Q}_{s(m)}, \mathbf{e}_{s(m)}) \quad smc(q, q_m, \mathbf{w}_m, \mathbf{Q}_{s(m)}, \mathbf{e}_{s(m)})$$

are identified for all $(q, Q) \in \mathbb{R}^2_{>0}$ on the support of $\mathcal{D}$, conditional on $m \in \mathcal{M} \cap f - \mathcal{N}$, $s \in nb(f)$ and $f \in \mathcal{F}$.

Proof. Let $\mathbf{D}_{nb(f)}$ denote the relevant diversion ratios. Taking derivatives of $mr(\mathcal{D}_m)$ implies

$$\begin{aligned}
\frac{\partial mr(\mathcal{D}_m)}{\partial q_m} &= c_{qq}(q_m, \mathbf{w}_m) \\
\frac{\partial mr(\mathcal{D}_m)}{\partial D_{nb(f),m,s}} &= C_Q(Q_s, \mathbf{e}_s) - C_Q(Q_s - Q_{sf}, \mathbf{e}_s) \\
\frac{\partial^2 mr(\mathcal{D}_m)}{\partial q_m \partial D_{nb(f),m,s}} &= C_{QQ}(Q_s, \mathbf{e}_s)
\end{aligned} \tag{2.70}$$

under Assumption 2.9.2. By Lemma 2.9.10 and Assumption 2.9.7, these derivatives are identified on $supp(\mathcal{D}|m \in \mathcal{M} \cap f - \mathcal{N}, s \in nb(f), f \in \mathcal{F})$.

The 1st and 3rd rows of (2.70) imply $c_{qq}(q_m, \mathbf{w}_m)$ and $C_{QQ}(Q_s, \mathbf{e}_s)$ are identified on $supp(\mathcal{D}|m \in \mathcal{M} \cap f - \mathcal{N}, s \in nb(f), f \in \mathcal{F})$. Assumptions 2.9.2 and 2.9.7 imply higher order derivatives of these functions with respect to q_m and Q_s are also identified on the same support.

Hence, by Taylor's theorem,

$$\begin{aligned}
c_{qq}(q, \mathbf{w}_m) &= c_{qq}(q_m, \mathbf{w}_m) + \sum_{k \geq 1} \frac{\partial^k}{\partial q^k} c_{qq}(q_m, \mathbf{w}_m) \frac{(q - q_m)^k}{k!} \\
C_{QQ}(Q, \mathbf{e}_s) &= C_{QQ}(Q_s, \mathbf{e}_s) + \sum_{k \geq 1} \frac{\partial^k}{\partial Q^k} C_{QQ}(Q_s, \mathbf{e}_s) \frac{(Q - Q_s)^k}{k!}
\end{aligned}$$

are identified on the same support for all $q, Q > 0$. Plugging Row 2 of (2.70) across all $s \in s(m)$ into (2.64) identifies

$$c_q(q_m, \mathbf{w}_m) + \sum_{s \in s(m)} C_Q^s(Q_s)$$

on the same support. Hence by Assumption 2.9.2 and Taylor's theorem,

$$\begin{aligned}
smc(q, q_m, \mathbf{w}_m, \mathbf{Q}_{s(m)}, \mathbf{e}_{s(m)}) &= c_q(q_m, \mathbf{w}_m) + \sum_{s \in s(m)} C_Q^s(Q_s) \\
&\quad + \sum_{k \geq 0} \frac{\partial^k}{\partial q^k} c_{qq}(q_m, \mathbf{w}_m) \frac{(q - q_m)^k}{k!} + \sum_{s \in s(m)} \sum_{k \geq 0} \frac{\partial^k}{\partial Q^k} C_{QQ}(Q_s, \mathbf{e}_s) \frac{(q - q_m)^k}{k!}.
\end{aligned}$$

is identified on the same support for arbitrary $q > 0$. It follows by Assumption 2.9.2 and FTC that

$$SC(q_m, \mathbf{w}_m, \mathbf{Q}_{s(m)}, \mathbf{e}_{s(m)}) = \int_0^{q_m} smc(q, q_m, \mathbf{w}_m, \mathbf{Q}_{s(m)}, \mathbf{e}_{s(m)})dq$$

is also identified on the same support. □

Proposition 2.9.1. *Suppose assumptions 2.9.1 to 2.9.7 hold. Consider any $\mathcal{D}$ in its support where $m \in \mathcal{M} - \mathcal{N}$ occurs. The cost function curvatures of model m and those of its suppliers $s \in s(m)$*

$$c_{qq}^m(q) \qquad C_{QQ}^s(Q)$$

are identified for all $q, Q > 0$. The unobserved cost shock ω_m and the joint variable profit shared by m with its suppliers

$$\pi_m^j = p_m q_m - c^m(q_m) + c^m(0) - \sum_{s \in s(m)} [C^s(Q_s) - C^s(Q_s - q_m)] - \sum_{s \in s(m)} \omega_{ms} q_m$$

are degenerate given $\mathcal{D}$ and identified too.

Proof. Lemma 2.9.11 implies $c_{qq}(q, \mathbf{w}_m)$ is identified. Observe $s(m) \subseteq nb(f)$ when $m \in \mathcal{M} \cap f - \mathcal{N}$. So the same lemma implies $C_{QQ}(Q, \mathbf{e}_s)$ is identified too. Identification of ω_m follows from Lemma 2.9.10. Identification of π_m^j then follows from

$$\pi_m^j = p_m q_m - SC(q_m, \mathbf{w}_m, \mathbf{Q}_{s(m)}, \mathbf{e}_{s(m)}) - \omega_m q_m$$

and Lemma 2.9.11. □

The previous results identify assembly and input cost curvatures for incumbent models in $\mathcal{M} - \mathcal{N}$. The new two results apply to new models.

Lemma 2.9.12. *Suppose assumptions 2.9.1 to 2.9.8 hold. Then*

$$c_{qq}(q, \mathbf{w}_m) \quad C_{QQ}(Q, \mathbf{e}_s) \quad SC(q_m, \mathbf{w}_m, \mathbf{Q}_{s(m)}, \mathbf{e}_{s(m)}) \quad smc(q, q_m, \mathbf{w}_m, \mathbf{Q}_{s(m)}, \mathbf{e}_{s(m)})$$

are identified for all $(q,Q) \in \mathbb{R}^2_{>0}$ *on supp*$(\mathcal{D}|m \in \mathcal{N} \cap f, s \in nb(f), f \in \mathcal{F})$*. Also,*

$$\sum_{s \in s(m) \cap au(f)} \pi_{ms} + \omega_m q_m$$

is degenerate and identified a.s. on supp$(\mathcal{D}|m \in \mathcal{N} \cap f, f \in \mathcal{F})$*.*

Proof. For the rest of the proof, fix a value for $\mathcal{D}$ in its support such that $m \in \mathcal{N} \cap f$ and $f \in \mathcal{F}$.

Suppose $s(m) = s(m)'$, $\mathbf{w}_m = \mathbf{w}'_m$ and $\mathbf{e} = \mathbf{e}'$. Assumption 2.9.8 implies $\exists \mathcal{D}'$ in its support where $m \in \mathcal{M} \cap f - \mathcal{N}$ occurs. Moreover, $s(m) = s(m)'$, $\mathbf{w}_m = \mathbf{w}'_m$ and $\mathbf{e}_{nb(f)'} = \mathbf{e}'_{nb(f)'}$ also occur, where $nb(f)'$ denotes f's Nash bargaining set of suppliers under $\mathcal{D}'$. So Lemma 2.9.11 implies $C_{QQ}(Q, \mathbf{e}_s) = C_{QQ}(Q, \mathbf{e}'_s)$ is identified on $[0,1]$ for all $s \in nb(f)'$. Since $m \in \mathcal{M} \cap f - \mathcal{N}$ implies $s(m)' \subseteq nb(f)'$, so $C_{QQ}(Q, \mathbf{e}_s) = C_{QQ}(Q, \mathbf{e}'_s)$ is identified on $[0,1]$ for all $s \in s(m)'$ too.

Consider any supplier $s \in nb(f)$ such that $s \in s(n)$ for some $n \in \mathcal{M} \cap f - \mathcal{N}$. Lemma 2.9.11 implies $C_{QQ}(Q, \mathbf{e}_s)$ is identified for all $Q > 0$. Consider any supplier $s \in nb(f)$ such that $s \in s(n)$ for some $n \in \mathcal{N} \cap f$. Suppose $s(n) = s(n)''$, $\mathbf{w}_n = \mathbf{w}''_n$ and $\mathbf{e} = \mathbf{e}''$. Assumption 2.9.8 implies $\exists \mathcal{D}''$ in its support where $m \in \mathcal{M} \cap f - \mathcal{N}$ occurs. Moreover, $s(n) = s(n)''$, $\mathbf{w}_n = \mathbf{w}''_n$ and $\mathbf{e}_{nb(f)''} = \mathbf{e}''_{nb(f)''}$ occur too, where $nb(f)''$ denotes f's Nash bargaining set of suppliers under $\mathcal{D}''$. Because $s \in s(n)'' \subseteq nb(f)''$, Lemma 2.9.11 implies $C_{QQ}(Q, \mathbf{e}_s) = C_{QQ}(Q, \mathbf{e}''_s)$ is identified on $[0,1]$.

Now observe (2.67) implies m's marginal revenue equals

$$mr_m = mr(\mathcal{D}_m) + \sum_{s \in s(m) \cap au(f)} \left[\kappa_{ms} - C^s_Q(Q_s) - \omega_{ms}\right] + \omega_m.$$

Assumption 2.9.8 implies there exists $\mathcal{D}'$ in the data's support, where m is an old incumbent model produced by f, whose attributes and suppliers equal those in $\mathcal{D}$, and whose Nash bargaining set of suppliers are $nb(f)'$. Assumption 2.9.8 implies the suppliers in $nb(f)'$ have identical attributes under $\mathcal{D}$ and $\mathcal{D}'$. Let

$$\mathcal{D}'_m = \left\{ q'_m, \mathbf{w}_m, \mathbf{e}_{s(m)}, \mathbf{e}'_{nb(f)'-s(m)}, \mathbf{Q}'_{nb(f)'}, \frac{\partial \mathbf{Q}'_{nb(f)'-f}}{\partial q_m}, s(m) \right\}.$$

denote the marginal cost shifting variables for model m in $\mathcal{D}'$. Then

$$mr(\mathcal{D}'_m) = mr(q'_m, \mathbf{w}_m, \mathbf{Q}'_{nb(f)'}, \frac{\partial \mathbf{Q}'_{nb(f)'-f}}{\partial q_m}, \mathbf{e}'_{nb(f)'-s(m)}, \mathbf{e}_{s(m)})$$

is identified. Observe Assumption 2.9.2 and Taylor's theorem implies

$$\begin{aligned}
mr(\mathcal{D}_m) = mr(\mathcal{D}'_m) &+ \sum_{k\geq 1} \frac{\partial^k}{\partial q^k} c^m_q(q)\big|_{q=q'_m} \frac{(q_m - q'_m)^k}{k!} + \sum_{k\geq 1} \frac{\partial^k}{\partial Q^k} \sum_{s\in s(m)} C^s_Q(Q)\big|_{Q=Q'_s} \frac{(Q_s - Q'_s)^k}{k!} \\
&+ \sum_{s\in nb(f)\cap nb(f)'} \frac{\partial}{\partial q_m} \left(Q_{s,-f} - Q'_{s,-f}\right) \left[C^s_Q(Q_s) - C^s_Q(Q_s - Q_{sf})\right] \\
&+ \sum_{s\in nb(f)\cap nb(f)'} \frac{\partial Q'_{s,-f}}{\partial q_m} \sum_{k\geq 1} \frac{\partial^k}{\partial Q^k} C^s_Q(Q)|_{Q=Q'_s} \frac{(Q_s - Q'_s)^k}{k!} \\
&- \sum_{s\in nb(f)\cap nb(f)'} \frac{\partial Q'_{s,-f}}{\partial q_m} \sum_{k\geq 1} \frac{\partial^k}{\partial Q^k} C^s_Q(Q)|_{Q=Q'_s - Q'_{sf}} \frac{(Q_s - Q_{sf} - Q'_s + Q'_{sf})^k}{k!} \\
&+ \sum_{s\in nb(f)-nb(f)'} \frac{\partial Q_{s,-f}}{\partial q_m} \left[C^s_Q(Q_s) - C^s_Q(Q_s - Q_{sf})\right] \\
&- \sum_{s\in nb(f)'-nb(f)} \frac{\partial Q'_{s,-f}}{\partial q_m} \left[C^s_Q(Q'_s) - C^s_Q(Q'_s - Q'_{sf})\right].
\end{aligned}$$

This simplifies to

$$
\begin{aligned}
&mr(\mathcal{D}'_m) + \sum_{k\geq 0} \frac{\partial^k}{\partial q^k} c^m_{qq}(q'_m) \frac{(q_m - q'_m)^k}{k!} + \sum_{k\geq 0} \frac{\partial^k}{\partial Q^k} \sum_{s\in s(m)} C^s_Q(Q'_s) \frac{(Q_s - Q'_s)^k}{k!} \\
&\quad + \sum_{s\in nb(f)\cap nb(f)'} \frac{\partial}{\partial q_m} \left(Q_{s,-f} - Q'_{s,-f}\right) \left[C^s_Q(Q_s) - C^s_Q(Q_s - Q_{sf})\right] \\
&\quad + \sum_{s\in nb(f)\cap nb(f)'} \frac{\partial Q'_{s,-f}}{\partial q_m} \sum_{k\geq 0} \left[\frac{\partial^k C^s_{QQ}}{\partial Q^k}(Q'_s) \frac{(Q_s - Q'_s)^k}{k!} - \frac{\partial^k C^s_{QQ}}{\partial Q^k}(Q'_s - Q'_{sf}) \frac{(Q_s - Q_{sf} - Q'_s - Qsf')^k}{k!} \right] \\
&\quad + \sum_{s\in nb(f)-nb(f)'} \frac{\partial Q_{s,-f}}{\partial q_m} \sum_{k\geq 0} \frac{\partial^k}{\partial Q^k} C^s_{QQ}(Q_s - Q_{sf}) \frac{Q^k_{sf}}{k!} - \sum_{s\in nb(f)'-nb(f)} \frac{\partial Q'_{s,-f}}{\partial q_m} \sum_{k\geq 0} \frac{\partial^k}{\partial Q^k} C^s_{QQ}(Q'_s - Q'_{sf}) \frac{Q'^k_{sf}}{k!}.
\end{aligned}
$$

The argument in the first two paragraphs of the proof identify $C^s_{QQ}(Q)$ for each $s \in nb(f)$ and $s \in nb(f)'$. So each term on the RHS is identified by Assumption 2.9.7. Hence, $mr(\mathcal{D}_m)$ is identified on $supp(\mathcal{D}|m \in \mathcal{N} \cap f, f \in \mathcal{F})$.

Now for all $(m, s) \in \mathcal{N} \cap f \times \mathcal{S}$, observe

$$
\begin{aligned}
\sum_{s\in s(m)\cap au(f)} \pi_{ms} &= \sum_{s\in s(m)\cap au(f)} \left[\kappa_{ms} q_m - C^s(Q_s) + C^s(Q_s - q_m) - \omega_{ms} q_m\right] \\
&= \sum_{s\in s(m)\cap au(f)} \left[\kappa_{ms} - C^s_Q(Q_s) - \omega_{ms}\right] q_m + \sum_{s\in s(m)\cap au(f)} \sum_{k\geq 0} \frac{\partial^k}{\partial Q^k} C^s_{QQ}(Q_s) \frac{(-q_m)^k}{k!} \\
&= (mr_m - mr(\mathcal{D}_m) - \omega_m) q_m + \sum_{s\in s(m)\cap au(f)} \sum_{k\geq 0} \frac{\partial^k}{\partial Q^k} C^s_{QQ}(Q_s) \frac{(-q_m)^k}{k!}
\end{aligned}
$$

But mr_m and $mr(\mathcal{D}_m)$ are identified a.s. on $supp(\mathcal{D}|m \in \mathcal{N} \cap f, f \in \mathcal{F})$. So by Assumption 2.9.7,

$$
\sum_{s\in s(m)\cap au(f)} \pi_{ms} + \omega_m q_m = (mr_m - mr(\mathcal{D}_m)) q_m + \sum_{s\in s(m)\cap au(f)} \sum_{k\geq 0} \frac{\partial^k}{\partial Q^k} C^s_{QQ}(Q_s) \frac{(-q_m)^k}{k!}
$$

is likewise identified on $supp(\mathcal{D}|m \in \mathcal{N} \cap f, f \in \mathcal{F})$.

Finally, because $mr(\mathcal{D}_m)$ is identified on $supp(\mathcal{D}|m \in \mathcal{N} \cap f, f \in \mathcal{F})$, so by (2.70) and the same arguments used to prove Lemma 2.9.11,

$$smc(q, q_m, \mathbf{w}_m, \mathbf{Q}_{s(m)}, \mathbf{e}_{s(m)}) = c_q(q_m, \mathbf{w}_m) + \sum_{s \in s(m)} C_Q^s(Q_s) + \sum_{k \geq 0} \frac{\partial^k}{\partial q^k} c_{qq}(q_m, \mathbf{w}_m) \frac{(q - q_m)^k}{k!} + \sum_{s \in s(m)} \sum_{k \geq 0} \frac{\partial^k}{\partial Q^k} C_{QQ}(Q_s, \mathbf{e}_s) \frac{(q - q_m)^k}{k!}$$

and

$$SC(q_m, \mathbf{w}_m, \mathbf{Q}_{s(m)}, \mathbf{e}_{s(m)}) = \int_0^{q_m} smc(q, q_m, \mathbf{w}_m, \mathbf{Q}_{s(m)}, \mathbf{e}_{s(m)}) dq$$

are likewise identified on $supp(\mathcal{D} | m \in \mathcal{N} \cap f, f \in \mathcal{F})$. □

Proposition 2.9.2. *Suppose assumptions 2.9.1 to 2.9.8 hold. Consider any $\mathcal{D}$ in its support where $m \in \mathcal{N}$ occurs. The cost function curvatures of model m and those of its suppliers $s \in s(m)$*

$$c_{qq}^m(q) \qquad C_{QQ}^s(Q)$$

are identified for all $q, Q > 0$. When m is assembled by $f \in \mathcal{F}$, m's joint profit with the suppliers f "Nash bargains" with

$$\pi_m^j = p_m q_m - c^m(q_m) + c^m(0) - \sum_{s \in s(m) \in nb(f)} [C^s(Q_s) - C^s(Q_s - q_m) + \omega_{ms} q_m] - \sum_{s \in s(m) \cap au(f)} \kappa_{ms} q_m,$$

and the portion of these profits lost to suppliers through auctions or to cost shocks

$$\pi_{ms} + \omega_m q_m = \sum_{s \in s(m) \cap au(f)} [\kappa_{ms} q_m - C^s(Q_s) + C^s(Q_s - q_m) - \omega_{ms} q_m] + \sum_{s \in s(m)} \omega_{ms} q_m$$

are degenerate given $\mathcal{D}$ and identified too.

Proof. Lemma 2.9.12 identifies $c^m_{qq}(q)$, $C^s_{QQ}(Q)$ and $\sum_{s\in s(m)\cap au(f)} \pi_{ms} + \omega_m q_m$. Also

$$\begin{aligned}
\pi^j_m &= p_m q_m - c^m(q_m) + c^m(0) - \sum_{s\in s(m)} \left[C^s(Q_s) - C^s(Q_s - q_m) + \omega_{ms} q_m\right] \\
&\quad - \sum_{s\in s(m)\cap au(f)} \left[\kappa_{ms} q_m - C^s(Q_s) + C^s(Q_s - q_m) - \omega_{ms} q_m\right] \\
&= p_m q_m - SC(q_m, \mathbf{w}_m, \mathbf{Q}_{s(m)}, \mathbf{e}_{s(m)}) - \omega_m q_m - \sum_{s\in s(m)\cap au(f)} \pi_{ms}.
\end{aligned}$$

But the same lemma identifies $SC(q_m, \mathbf{w}_m, \mathbf{Q}_{s(m)}, \mathbf{e}_{s(m)})$ too. So π^j_m is identified[55]. □

The following results apply to both incumbent and new models.

Lemma 2.9.13. *Suppose assumptions 2.9.1 to 2.9.8 hold. For any $I \in \mathbb{N}$, fix $\mathbf{s} \in \mathcal{S}^I$. Then for all $(q, \mathbf{U_s}) \in \mathbb{R}^{I+1}_{>0}$, $SC(q, \mathbf{w}_m, \mathbf{U_s}, \mathbf{e_s})$ is identified a.s. on supp$(\mathcal{D}|m \in \mathcal{M}\cap f, s(m) = \mathbf{s}, f \in \mathcal{F})$.*

Proof. Fix $m \in \mathcal{M} \cap f$, $f \in \mathcal{F}$, $s(m) = \mathbf{s}$, and $(q, \mathbf{U_s}) \in \mathbb{R}^{I+1}_{>0}$. Let $SC_m = SC(q_m, \mathbf{w}_m, \mathbf{Q_s}, \mathbf{e_s})$.

When $m \in \mathcal{M} - \mathcal{N}$, Lemma 2.9.11 implies SC_m is degenerate and identified. When $m \in \mathcal{N}$, Lemma 2.9.12 implies SC_m is degenerate and identified. Hence, SC_m is identified regardless of whether m is old or new under $\mathcal{D}$.

Now Assumption 2.9.2 and Taylor's theorem implies

$$\begin{aligned}
SC(q, \mathbf{w}_m, \mathbf{U_s}, \mathbf{e_s}) &= SC_m + \sum_{k\geq 1} \frac{\partial^k}{\partial q^k} c^m(q_m) \frac{(q - q_m)^k}{k!} \\
&\quad + \sum_{s\in\mathbf{s}} \sum_{k\geq 1} \frac{\partial^k}{\partial Q^k_s} \left[C^s(Q_s) \frac{(U_s - Q_s)^k}{k!} - C^s(Q_s - q_m) \frac{(U_s - q - Q_s + q_m)^k}{k!}\right].
\end{aligned}$$

55. Alternatively, permute the models' indices so that $m = \max\{n \in f\}$ in Lemma 2.9.14. This identifies π^j_m.

The above in turn equals

$$
\begin{aligned}
SC_m &+ \sum_{k\geq 0} \frac{\partial^k}{\partial q^k}\left[c^m(q) + \sum_{s\in\mathbf{s}} C_Q^s(Q_s - q)\right]\Big|_{q=q_m}\frac{(q-q_m)^k}{k!} \\
&+ \sum_{k\geq 0}\frac{\partial^k}{\partial Q^k}\sum_{s\in\mathbf{s}}\left[C^s(Q) - C^s(Q-q_m)\right]\Big|_{Q=U_s}\frac{(U_s-Q_s)^k}{k!} \\
&- \sum_{k\geq 0}\frac{\partial^k}{\partial Q_s^k}\sum_{s\in\mathbf{s}} C_{QQ}^s(Q_s - q_m)\binom{k}{j}\sum_{j=1}^{k-1}\frac{(U_s-Q_s)^k(q-q_m)^{k-j}}{k!},
\end{aligned}
$$

which simplfies to

$$
\begin{aligned}
SC(q_m, \mathbf{w}_m, \mathbf{Q_s}, \mathbf{e_s}) &+ \sum_{k\geq 0}\frac{\partial^k}{\partial v^k} smc(v, q_m, \mathbf{w}_m, \mathbf{Q_s}, \mathbf{e_s})\big|_{v=q_m}\frac{(q-q_m)^k}{k!} \\
&- \sum_{k\geq 0}\frac{\partial^k}{\partial v^k}\sum_{j\geq 0}\frac{\partial^j}{\partial u^j}\sum_{s\in\mathbf{s}} C_{QQ}^s(v - q_m + u)\big|_{u=q_m, v=Q_s}\frac{(U_s-Q_s)^j(q-q_m)^k}{j!k!} \\
&+ \sum_{k\geq 0}\frac{\partial^k}{\partial v^k}\sum_{j\geq 0}\frac{\partial^j}{\partial u^j}\sum_{s\in\mathbf{s}} C_{QQ}^s(Q_s - q_m + u + v)\big|_{u=0, v=Q_s}\frac{(q_m-2v)^j(q-q_m)^k}{j!k!} \\
&- \sum_{k\geq 0}\frac{\partial^k}{\partial v^k}\sum_{s\in\mathbf{s}} C_{QQ}^s(Q_s - q_m + v)\big|_{v=0}\sum_{j=1}^{k-1}\binom{k}{j}\frac{(q-q_m)^{k-j}(U_s-Q_s)^j}{k!}.
\end{aligned}
$$

Assumption 2.9.7, together with lemmas 2.9.11 and 2.9.12, imply the above is identified on $supp(\mathcal{D}|m \in \mathcal{M}\cap f, s(m) = \mathbf{s}, f \in \mathcal{F})$. Hence $SC(q, \mathbf{w}_m, \mathbf{U_s}, \mathbf{e_s})$ is likewise identified on the same support. □

Observe for each $(s, m, f) \in \mathcal{S}\times\mathcal{M}\times\mathcal{F}$, one can define

$$
Q_{s,f\leq m} = Q_{s,-f} + \sum_{n\in m(s), n\leq m} q_n \tag{2.71}
$$

as the amount supplied by supplier s to models owned manufacturer f, indexed by integers smaller or equal to m.

Lemma 2.9.14. *Suppose assumptions 2.9.1 to 2.9.8 hold. Then*

$$\pi^j_{m,f\le m} = p_m q_m - c^m(q_m) + c^m(0) - \sum_{s\in s(m)\cap nb(f)} [C^s(Q_{s,f\le m}) - C^s(Q_{s,f\le m} - q_m)] - \kappa_m q_m$$

is degenerate and identified a.s. on supp($\mathcal{D}|m \in \mathcal{M}\cap f, f\in\mathcal{F}$).

Proof. Fix $\mathcal{D}$ in its support. Suppose $m \in f \in \mathcal{F}$. Setting

$$\mathbf{s} = s(m), \quad q = q_m, \quad \mathbf{U_s} = \mathbf{Q}_{\mathbf{s},f\le m}$$

in Lemma 2.9.13 identifies $SC(q_m, \mathbf{w}_m, \mathbf{Q}_{s(m),f\le m}, \mathbf{e}_{s(m)})$.

When $m \in \mathcal{M} - \mathcal{N}$ under $\mathcal{D}$,

$$\begin{aligned}\pi^j_{m,f\le m} &= p_m q_m - c^m(q_m) + c^m(0) - \sum_{s\in s(m)} [C^s(Q_{s,f\le m}) - C^s(Q_{s,f\le m} - q_m) + \omega_{ms} q_m] \\ &= p_m q_m - SC(q_m, \mathbf{w}_m, \mathbf{Q}_{s(m),f\le m}, \mathbf{e}_{s(m)}) - \omega_m q_m.\end{aligned}$$

Lemma 2.9.10 identifies ω_m. So the argument in the first paragraph completes the proof of the RHS being identified.

When $m \in \mathcal{N}$, assumptions 2.9.2 and 2.9.7, together with Lemma 2.9.12 imply

$$\begin{aligned}&\sum_{s\in s(m)\cap au(f)} [C^s(Q_s) - C^s(Q_s - q_m) - C^s(Q_{s,f\le m}) + C^s(Q_{s,f\le m} - q_m)] \\ &\quad = \sum_{k\ge 0} \frac{\partial^k}{\partial Q^k} \sum_{s\in s(m)\cap au(f)} \left[C^s_Q(Q_{s,f\le m}) - C^s_Q(Q_{s,f\le m} - q_m)\right] \frac{(Q_s - Q_{s,f\le m})^k}{k!} \\ &\quad = -\sum_{k\ge 0} \frac{\partial^k}{\partial Q^k} \sum_{j\ge 0} \frac{\partial^j}{\partial q^j} \sum_{s\in s(m)\cap au(f)} C^s_{QQ}(Q_{s,f\le m}) \frac{(-q_m)^j (Q_s - Q_{s,f\le m})^k}{j!k!}\end{aligned}$$

and $\sum_{s\in s(m)\cap au(f)} \pi_{ms} + \omega_m q_m$ are identified. Hence, the first paragraph completes the argument establishing

$$\pi^j_{m,f\leq m} = p_m q_m - c^m(q_m) + c^m(0) - \sum_{s\in s(m)} [C^s(Q_{s,f\leq m}) - C^s(Q_{s,f\leq m} - q_m) + \omega_{ms} q_m]$$
$$- \sum_{s\in s(m)\cap au(f)} [\kappa_{ms} q_m - C^s(Q_{s,f\leq m}) + C^s(Q_{s,f\leq m} - q_m) - \omega_{ms} q_m]$$
$$= p_m q_m - SC(q_m, \mathbf{w}_m, \mathbf{Q}_{s(m),f\leq m}, \mathbf{e}_{s(m)}) - \omega_m q_m - \sum_{s\in s(m)\cap au(f)} \pi_{ms}$$
$$+ \sum_{s\in s(m)\cap au(f)} [C^s(Q_s) - C^s(Q_s - q_m) - C^s(Q_{s,f\leq m}) + C^s(Q_{s,f\leq m} - q_m)]$$

is identified. □

Proposition 2.9.3. *Suppose assumptions 2.9.1 to 2.9.8 hold. The variable profit manufacturer $f \in \mathcal{F}$ shares with suppliers it "Nash bargains" with*

$$\pi^{nb}_f = \sum_{m\in f} [p_m q_m - c^m(q_m) + c^m(0) - \kappa_m q_m] - \sum_{s\in nb(f)} [C^s(Q_s) - C^s(Q_s - Q_{sf})],$$

is degenerate given $\mathcal{D}$ and identified on $\mathcal{D}$'s support.

Proof. Observe $\pi^{nb}_f = \sum_{m\in f} \pi^j_{m,f\leq m}$. The result follows from Lemma 2.9.14. □

In what follows, let $\mathbf{g}_c = (g_{-m} + g^o_m, g^o)$ be the current and past production networks, under the counterfactual where model m's manufacturer retains m's past production network suppliers instead of switching to m's equilibrium ones.

Proof of Proposition 2.5.1. Fix τ_f to a known value in $(0,1)$. Then $\tau^f(\mathbf{g})$ and $\tau^f(\mathbf{g}_c)$ are known, since g and g^o are known. Proposition 2.9.3 implies π^{nb}_f is also identified given $\mathcal{D}$. Hence, the distortion of $\Delta\pi^a_{f,m}$ due to f bilaterally bargaining with fewer suppliers under $\mathbf{g}$ versus $\mathbf{g}_c$, $(\tau^f(\mathbf{g}) - \tau^f(\mathbf{g}_c))\pi^{nb}_f$, is identified on $\mathcal{D}$'s support.

Also, Proposition 2.9.2 implies $\pi_{ms} + \sum_{s\in s(m)} \omega_{ms} q_m$ is identified on $\mathcal{D}$'s support. Lemma 2.9.13 identfies $SC(q_m, \mathbf{w}_m, \mathbf{Q}_{s(m)}, \mathbf{e}_{s(m)})$ on the same support. The lemma and the first part of Assumption 2.9.9 imply $SC(q_m, \mathbf{w}_{m^o}, \mathbf{Q}_{s(m^o,g^o)} + \{s \notin s(m)\}q_m, \mathbf{e}_{s(m^o,g^o)})$ is identified too. Hence, the second part of Assumption 2.9.9 identifies $SC(q_m, \mathbf{w}_m, \mathbf{Q}_{s(m^o,g^o)} + \{s \notin s(m)\}q_m, \mathbf{e}_{s(m^o,g^o)})$ too. It follows each term in the decomposition (2.17), except for $\sum_{s\in s(m^o,g^o)} \omega_{ms}q_m$, is identified on $\mathcal{D}$'s support. $\Delta\pi^a_{f,m} - \tau^f(\mathbf{g}_c)\sum_{s\in s(m^o,g^o)} \omega_{ms}q_m$ is thus likewise identified on the same support. □

Proof of Proposition 2.5.2. Fix $\boldsymbol{\tau}$ to known values in $(0,1)^{\mathcal{F}}$. Then $\tau^f(\mathbf{g})$ and $\tau^{ls}(\mathbf{g})$ are known, since $\mathbf{g}$ is known.

Proposition 2.9.2 implies $\pi_{ms} + \sum_{s\in s(m)} \omega_{ms}q_m$ is identified on $\mathcal{D}$'s support. Assumption 2.9.2 implies

$$\begin{aligned} h^s(Q_s - q_m, Q_{sl}, q_m) &= C^s(Q_s - q_m) - C^s(Q_s - Q_{sl} - q_m) - C^s(Q_s) + C^s(Q_s - Q_{sl}) \\ &= -\int_0^{Q_{sl}}\int_0^{q_m} C^s_{QQ}(Q_s - Q_{sl} - q_m + q + Q)\,dq\,dQ \end{aligned}$$

for each $s \in \mathcal{S}$. Assumption 2.9.7 combined with propositions 2.9.1 and 2.9.2 imply the above is identified whenever $s \in nb(l)$ for some $l \in \mathcal{F}$. It follows $\big(1 - \tau^{ls}(\mathbf{g})\big)\sum_{l\in nb(s)} h^s(Q_s - q_m, Q_{sl}, q_m)$ is identified on $\mathcal{D}$'s support. Likewise, $h^k(Q_k - q_m, Q_{kl}, q_m)$ and $h^k(Q_k, Q_{kl}, q_m)$ in $\sum_{s\in s(m)\cap au(f)} \Delta\pi^a_{s,m}$'s decomposition (2.21) are also identified on the support of $\mathcal{D}$. Hence, $\sum_{s\in s(m)\cap au(f)} \Delta\pi^a_{s,m} + \sum_{s\in s(m)} \omega_{ms}q_m$ is identified on the same support. □

Bibliography

Acemoglu, D., U. Akcigit, and W. Kerr (2016): "Networks and the macroeconomy: An empirical exploration," *NBER Macroeconomics Annual*, 30(1), 273–335.

Acemoglu, D., and P. D. Azar (2020): "Endogenous production networks," *Econometrica*, 88(1), 33–82.

Acemoglu, D., V. M. Carvalho, A. Ozdaglar, and A. Tahbaz-Salehi (2012): "The network origins of aggregate fluctuations," *Econometrica*, 80(5), 1977–2016.

Ackerburg, D., X. Chen, and J. Hahn (2012): "A Practical Asymptotic Variance Estimator for Two-Step Semiparametric Estimators," *The Review of Economics and Statistics*, 94, 481–498.

Adams, B. (2015): "Migration and agglomeration among motor vehicle parts suppliers," *work*.

Agostinelli, F. (2018): "Investing in Children's Skills - An Equilibrium Analysis of Social Interactions and Parental Investments," *Arizona State University Working Paper*.

Alcalde, J., and M. Dahm (2011): "On the complete information first-price auction and its intuitive solution," *International Game Theory Review*, 13(03), 353–361.

Alcalde, M. (2013): "A Behavioral Model of Non-linear Peer Effects in Cognitive Achievement," *University of Pennsylvania Working Paper*.

Andrews, I., and D. Barron (2016): "The allocation of future business: Dynamic relational contracts with multiple agents," *American Economic Review*, 106(9), 2742–59.

Anginer, D., and A. J. Warburton (2014): "The Chrysler effect: The impact of government intervention on borrowing costs," *Journal of Banking & Finance*, 40, 62–79.

Angrist, J. (2014): "The Perils of Peer Effects," *Labour Economics*, 30, 98–108.

Arcidiacono, P., G. Foster, N. Goodpaster, and J. Kinsler (2012): "Estimating Spillovers using Panel Data, with an Application to the Classroom," *Quantitative Economics*, 3, 421–470.

Atalay, E., A. Hortacsu, J. Roberts, and C. Syverson (2011): "Network structure of production," *Proceedings of the National Academy of Sciences*, 108(13), 5199–5202.

Atalay, E., A. Hortacsu, and C. Syverson (2014): "Vertical integration and input flows," *American Economic Review*, 104(4), 1120–48.

Athey, S., and P. A. Haile (2002): "Identification of standard auction models," *Econometrica*, 70(6), 2107–2140.

——— (2007): "Nonparametric approaches to auctions," *Handbook of econometrics*, 6, 3847–3965.

Badorf, F., S. M. Wagner, K. Hoberg, and F. Papier (2019): "How Supplier Economies of Scale Drive Supplier Selection Decisions," *Journal of Supply Chain Management*, 55(3), 45–67.

Baqaee, D. R. (2018): "Cascading failures in production networks," *Econometrica*, 86(5), 1819–1838.

Baqaee, D. R., and E. Farhi (2019): "The macroeconomic impact of microeconomic shocks: beyond Hulten's Theorem," *Econometrica*, 87(4), 1155–1203.

Barrot, J.-N., and J. Sauvagnat (2016): "Input specificity and the propagation of idiosyncratic shocks in production networks," *The Quarterly Journal of Economics*, 131(3), 1543–1592.

Baumann, R., and A. Thompson (2015): "Unemployment and Public Budget Impacts of the Auto Bailout," *Holy Cross Working Paper Series*, (15-04).

Bayer, P., and S. Ross (2009): "Identifying Individual and Group Effects in the Presence of Sorting: A Neighborhood Effects Application," *NBER Working Paper*.

Beckert, W., H. Smith, and Y. Takahashi (2015): "Competitive price discrimination in a spatially differentiated intermediate goods market," Discussion paper, Technical report, Univ. London.

Ben-Shahar, O., and J. J. White (2005): "Boilerplate and economic power in auto manufacturing contracts," *Mich. L. Rev.*, 104, 953.

Benelli, N.-A. (2017): "Default Risk in the Automotive Supply Network," Ph.D. thesis, Duke University.

Bernard, A. B., and A. Moxnes (2018): "Networks and trade," *Annual Review of Economics*, 10, 65–85.

Bernard, A. B., A. Moxnes, and Y. U. Saito (2019): "Production networks, geography, and firm performance," *Journal of Political Economy*, 127(2), 639–688.

Berry, S., and P. Haile (2013): "Identification of Simultaneous Nonparametric Models with a Residual Index Structure," *Cenmap Working Paper*.

Berry, S., J. Levinsohn, and A. Pakes (1995): "Automobile prices in market equilibrium," *Econometrica: Journal of the Econometric Society*, pp. 841–890.

Berry, S. T., and P. A. Haile (2014): "Identification in differentiated products markets using market level data," *Econometrica*, 82(5), 1749–1797.

Bertrand, M., E. Luttmer, and S. Mullainathan (2000): "Network Effects and Welfare Cultures," *Quarterly Journal of Economics*, 115, 1019–1055.

BLAND, J., AND S. LOERTSCHER (2012): "Monotonicity, Non-Participation, and Directed Search Equilibria," *University of Melbourne Economics Department Working Paper Series*, (1147).

BLOCH, F., AND M. O. JACKSON (2007): "The formation of networks with transfers among players," *Journal of Economic Theory*, 133(1), 83–110.

BLUME, L., W. BROCK, S. DURLAUF, AND R. JAYARAMAN (2015): "Linear Social Interaction Models," *Journal of Political Economy*, 123, 444–496.

BOBONIS, G., AND F. FREDERICO (2009): "Neigborhood Peer Effects in Secondary School Enrollment Decisions," *The Review of Economics and Statistics*, 91, 695–1716.

BRAMOULLE, Y., H. DJEBBARI, AND B. FORTIN (2009): "Identification of Peer Effects through Social Networks," *Journal of Econometrics*, 150, 41–55.

BROCK, W., AND S. DURLAUF (2007): "Identification of Binary Choice Models with Social Interactions," *Journal of Econometrics*, 140, 52–75.

BUENSTORF, G., AND S. KLEPPER (2009): "Heritage and agglomeration: the Akron tyre cluster revisited," *The Economic Journal*, 119(537), 705–733.

BURKE, M., AND T. SASS (2013): "Classroom Peer Effects and Student Achievement," *Journal of Labor Economics*, 31, 51–82.

CABALLERO, R. J., AND R. K. LYONS (1990): "Internal versus external economies in European industry," *European Economic Review*, 34(4), 805–826.

CALVO-ARMENGOL, A., AND M. O. JACKSON (2004): "The effects of social networks on employment and inequality," *American Economic Review*, 94(3), 426–454.

CALVO-ARMENGOL, A., E. PATACCHINI, AND Y. ZENOU (2009): "Peer Effects and Social Networks in Education," *Review of Economic Studies*, 76, 1239–1267.

Card, D., and L. Giuliano (2013): "Peer Effects and Multiple Equilibria in the Risky Behavior of Friends," *The Review of Economics and Statistics*, 95, 1130–1149.

Carney, M. (2013): "Identifying Peer Effects: Thinking Outside the 'Linear-in-Means' Box," *Working Paper*.

Carrell, S., F. Malmstrom, and J. West (2007): "Peer Effects in Academic Cheating," *Journal of Human Resources*, 43, 173–207.

Carvalho, V., M. Nirei, Y. Saito, and A. Tabhaz-Salehi (2016): "Supply Chain Disruptions: Evidence from the Great East Japan Earthquake," *Becker Friedman Institute for Research in Economics Working Paper*.

Carvalho, V. M. (2014): "From micro to macro via production networks," *Journal of Economic Perspectives*, 28(4), 23–48.

Carvalho, V. M., M. Nirei, Y. Saito, and A. Tahbaz-Salehi (2016): "Supply chain disruptions: Evidence from the great east japan earthquake," *Columbia Business School Research Paper*, (17-5).

Case, A., and L. Katz (1991): "The Company You Keep: The Effects of Family and Neighborhood on Disadvantaged Youths," *NBER Working Paper*.

Chen, X., R. Kanbur, and X. Zhang (2012): "Peer Effects, Risk Pooling, and Status Seeking: What Explains Gift Spending Escalation in Rural China?," *Charles H. Dyson School of Applied Economics and Management Working Paper*.

Chen, X., and J. Qiu (2016): "Methods for Nonparametric and Semiparametric Regressions with Endogeneity: A Gentle Guide," *Cowles Foundation Discussion Paper*.

Chesher, A. (2003): "Identification in Nonseparable Models," *Econometrica*, 71, 1405–1441.

Choi, T. Y., and J. L. Hartley (1996): "An exploration of supplier selection practices across the supply chain," *Journal of operations management*, 14(4), 333–343.

Collard-Wexler, A., G. Gowrisankaran, and R. S. Lee (2019): "Nash-in-Nash bargaining: A microfoundation for applied work," *Journal of Political Economy*, 127(1), 163–195.

Compiani, G. (2017): "Nonparametric Demand Estimation in Differentiated Products Markets," *Yale University Dissertation Chapter*.

Condorelli, D., and A. Galeotti (2012): "Endogenous trading networks," *University of Essex Economics Discussion Papers*.

——— (2016): "Strategic models of intermediation networks," *The Oxford Handbook of the Economics of Networks*.

Conlon, C., and J. Gortmaker (2020): "Best practices for differentiated products demand estimation with PyBLP," *The RAND Journal of Economics*, 51(4), 1108–1161.

Covert, T. R., and R. L. Sweeney (2019): "Relinquishing Riches: Auctions vs Informal Negotiations in Texas Oil and Gas Leasing," Discussion paper, National Bureau of Economic Research.

Crawford, V. P. (1990): "Relationship-specific investment," *The Quarterly Journal of Economics*, 105(2), 561–574.

Cusumano, M. A., and A. Takeishi (1991): "Supplier relations and management: a survey of Japanese, Japanese-transplant, and US auto plants," *Strategic Management Journal*, 12(8), 563–588.

Dahl, G., K. Loken, and M. Mogstad (2014): "Peer Effects in Program Participation," *American Economics Review*, 104, 2049–2074.

Donald, S. G., and H. J. Paarsch (1996): "Identification, estimation, and testing in parametric empirical models of auctions within the independent private values paradigm," *Econometric Theory*, pp. 517–567.

Duflo, E., and E. Saez (2003): "The Role of Information and Social Interactions in Retirement Plan Decisions: Evidence from a Randomized Experiment," *Quarterly Journal of Economics*, 118, 815–842.

Dworczak, P. (2015): "The effects of post-auction bargaining between bidders," *Available at SSRN 2550653*.

Dyer, J. H. (1996): "Specialized supplier networks as a source of competitive advantage: Evidence from the auto industry," *Strategic management journal*, 17(4), 271–291.

Eaton, J., and S. Kortum (2002): "Technology, geography, and trade," *Econometrica*, 70(5), 1741–1779.

Elliott, M. (2015): "Inefficiencies in networked markets," *American Economic Journal: Microeconomics*, 7(4), 43–82.

Fox, J. T. (2018): "Estimating matching games with transfers," *Quantitative Economics*, 9(1), 1–38.

Foy, H. (2014): "Age of mega supplier heralds danger for carmakers," *Financial Times*, May 18th.

Fruehwirth, J. (2013): "Identifying Peer Achievement Spillovers: Implications for Desegregation and the Achievement Gap," *Quantitative Economics*, 4, 85–124.

Ganapati, S. (2017): "The Modern Wholesaler: Global Sourcing, Domestic Distribution, and Scale Economies," *Yale University Dissertation Chapter*.

Gaviria, A., and S. Raphael (2001): "School-Based Peer Effects and Juvenile Behavior," *Review of Economics and Statistics*, 83, 257–268.

Ghili, S. (2016): "Network formation and bargaining in vertical markets: The case of narrow networks in health insurance," *Available at SSRN 2857305*.

Glaeser, E., B. Sacerdote, and J. Scheinkman (2003): "The Social Mutiplier," *Journal of the European Economic Association*, 1, 345–353.

Golman, R., and S. Klepper (2016): "Spinoffs and clustering," *The RAND Journal of Economics*, 47(2), 341–365.

Goolsbee, A. D., and A. B. Krueger (2015): "A retrospective look at rescuing and restructuring General Motors and Chrysler," *Journal of Economic Perspectives*, 29(2), 3–24.

Graham, B., G. Imbens, and G. Ridder (2010): "Measuring the Effects of Segregation in the Presence of Spillovers - a Nonparametric Approach," *NBER Working Paper*.

Grassi, B. (2017): "Io in I-O: Size, industrial organization, and the input-output network make a firm structurally important," *Work. Pap., Bocconi Univ., Milan, Italy.*

Hansen, B. (2018): "Econometrics," *Online Book*.

Head, K., J. Ries, and B. J. Spencer (2004): "Vertical networks and US auto parts exports: is Japan different?," *Journal of Economics & Management Strategy*, 13(1), 37–67.

Helper, S., and D. I. Levine (1992): "Long-term supplier relations and product-market structure," *Journal of Law, Economics, & Organization*, 8(3), 561–581.

Ho, K. (2009): "Insurer-provider networks in the medical care market," *American Economic Review*, 99(1), 393–430.

Ho, K., and R. S. Lee (2017): "Insurer competition in health care markets," *Econometrica*, 85(2), 379–417.

——— (2019): "Equilibrium provider networks: Bargaining and exclusion in health care markets," *American Economic Review*, 109(2), 473–522.

Hoffman, B. G. (2012): *American Icon: Allan Mullaly and the fight to save Ford Motor Company*. Crown Business.

Holmes, T., and E. Singer (2017): "Indivisibilities in Distribution," *Society for Economic Dynamics Meeting Papers*, (1535).

Hortacsu, A., G. Matvos, C. Shin, C. Syverson, and S. Venkataraman (2011): "Is an automaker's road to bankruptcy paved with customers' beliefs?," *American Economic Review*, 101(3), 93–97.

Hortacsu, A., G. Matvos, C. Syverson, and S. Venkataraman (2013): "Indirect costs of financial distress in durable goods industries: The case of auto manufacturers," *The Review of Financial Studies*, 26(5), 1248–1290.

Hortacsu, A., and C. Syverson (2007): "Cementing relationships: Vertical integration, foreclosure, productivity, and prices," *Journal of Political Economy*, 115(2), 250–301.

Huh, W. T., and K. S. Park (2010): "A sequential auction-bargaining procurement model," *Naval Research Logistics (NRL)*, 57(1), 13–32.

Hulten, C. R. (1978): "Growth accounting with intermediate inputs," *The Review of Economic Studies*, 45(3), 511–518.

Huneeus, F. (2018): "Production network dynamics and the propagation of shocks," *Princeton University*.

Imbens, G., and W. Newey (2009): "Identification and Estimation of Triangular Simultaneous Equations Models Without Additivity," *Econometrica*, 77, 1481–1512.

Ingrassia, P. (2011): *Crash Course - The American Automobile Industry's road to Bankruptcy and Bailout and beyond*. Random House Trade Paperbacks.

Iozzi, A., and T. Valletti (2014): "Vertical bargaining and countervailing power," *American Economic Journal: Microeconomics*, 6(3), 106–35.

Jackson, M. (2005): "A survey of network formation models: stability and efficiency," *Group formation in Economics: Networks, Clubs, and Coalitions*, 664, 11–49.

Jacoby, D. (2009): *The Economist - Guide to Supply Chain Management*. Profile Books.

Kee, H. (2015): "Local intermediate inputs and the shared supplier spillovers of foreign direct investment," *Journal of Development Economics*, 112, 56–71.

Khawand, C. (2015): "Peering Past Linearity: Nonlinear Peer Effects in High School Classrooms," *Michigan State University Working Paper*.

Kiss, D. (2013): "The Impact of Peer Achievement and Peer Heterogeneity on Own Achievement Growth: Evidence from School Transitions," *BGPE Discussion Paper*.

Klibanoff, P., and S. Novak (2003): "Contracting, directed parts, and complexity in automotive outsourcing decisions," *Unpublished working paper, Northwestern University*.

Klier, T., and J. M. Rubenstein (2012): "Detroit back from the brink? Auto industry crisis and restructuring, 2008-11," *Economic Perspectives*, 36(2).

Klier, T. H., et al. (1999): "Agglomeration in the US auto supplier industry," *Economic Perspectives-Federal Reserve Bank of Chicago*, 23, 18–34.

Kranton, R., and D. Minehart (2001): "A theory of buyer-seller networks," *American Economic Review*, 91, 485–508.

Krauth, B. (2006): "Simulation-based Estimation of Peer Effects," *Journal of Econometrics*, 133, 243–271.

Kuhn, P., P. Kooreman, A. Soetevent, and A. Kapteyn (2011): "The Effects of Lottery Prizes on Winners and Their Neighbours: Evidence from the Dutch Postcode Lottery," *American Economic Review*, 133, 2226–2247.

Langford, R. P. (2015): *Estimating the Environmental Benefits of Gasoline Taxes and the Market Benefits of Hybrid Vehicles*. Yale University.

LARSEN, B. (2013): "The efficiency of dynamic, post-auction bargaining: Evidence from wholesale used-auto auctions," *Post-Auction Bargaining: Evidence from Wholesale Used-Auto Auctions (January 24, 2013).*

LEE, L. (2007): "Identification and Estimation of Econometric Models with Group Interactions, Contextual Factors and Fixed Effects," *Journal of Econometrics*, 140, 333–374.

LEE, R. S. (2013): "Vertical integration and exclusivity in platform and two-sided markets," *American Economic Review*, 103(7), 2960–3000.

LEE, R. S., AND K. FONG (2013): "Markov-Perfect Network Formation," *NYU Stern working.*

LIM, K. (2013): "Endogenous Production Networks and the Business Cycle," *Princeton working.*

LIU, E. (2019): "Industrial policies in production networks," *The Quarterly Journal of Economics*, 134(4), 1883–1948.

LYNCH, A. W., AND J. A. WACHTER (2013): "Using samples of unequal length in generalized method of moments estimation," *Journal of Financial and Quantitative Analysis*, 48(1), 277307.

M. GROSSMAN, G., AND E. ROSSI-HANSBERG (2010): "External economies and international trade redux," *The Quarterly journal of economics*, 125(2), 829–858.

MANEA, M. (2016): "Models of Bilateral Trade in Networks," *Oxford Handbook of the Economics of Networks.*

MANRESA, E. (2015): "Estimating the Structure of Social Interactions Using Panel Data," *Centro de Estudios Monetarios y Financieros Working Paper.*

MANSKI, C. (2013): "Identification of Treatment Response with Social Interactions," *The Econometrics Journal*, 16, 1–23.

Martin, V., S. Hurn, and D. Harris (2012): *Econometric Modelling with Time Series: Specification, Estimation and Testing*, Themes in Modern Econometrics. Cambridge University Press.

Masten, S. E., J. W. Meehan Jr, and E. A. Snyder (1989): "Vertical integration in the US auto industry: A note on the influence of transaction specific assets," *Journal of Economic Behavior & Organization*, 12(2), 265–273.

Matzkin, R. (1992): "Nonparametric and Distribution-Free Estimation of the Binary Threshold Crossing and The Binary Choice Models," *Econometrica*, 60, 239–270.

——— (2008): "Identification in Non-Parametric Simultaneous Equations Models," *Econometrica*, 76, 945–978.

Meilijson, I. (1981): "Estimation of the lifetime distribution of the parts from the autopsy statistics of the machine," *Journal of Applied Probability*, pp. 829–838.

Miwa, Y., and J. M. Ramseyer (2000): "Rethinking relationship-specific investments: subcontracting in the Japanese automobile industry," *Michigan Law Review*, pp. 2636–2667.

Mizuno, T., W. Souma, and T. Watanabe (2014): "The structure and evolution buyer-seller networks," *PLOS One*, 9(7).

Moffitt, R. (2001): "Chapter 3: Policy Interventions, Low-Level Equilibria, and Social Interactions," *Social Dynamics*, 4, 45–82.

Monteverde, K., and D. J. Teece (1982): "Supplier switching costs and vertical integration in the automobile industry," *The Bell Journal of Economics*, pp. 206–213.

Mudambi, R., and S. Helper (1998): "The close but adversarial model of supplier relations in the US auto industry," *Strategic Management Journal*, 19(8), 775–792.

Newey, W., and J. Powell (2003): "Instrumental Variable Estimation of Nonparametric Models," *Econometrica*, 71, 1565–1578.

Newey, W., J. Powell, and F. Vella (1999): "Nonparametric Estimation of Triangular Simultaneous Equations Models," *Econometrica*, 67, 565–603.

Oberfield, E. (2018): "A theory of input–output architecture," *Econometrica*, 86(2), 559–589.

O'Leary, L. (2020): "The modern supply chain is snapping," *The Atlantic*, March 19th.

Ostrovsky, M. (2008): "Stability in supply chain networks," *American Economic Review*, 98(3), 897–923.

Ozdagli, A., and M. Weber (2017): "Monetary policy through production networks: Evidence from the stock market," Discussion paper, National Bureau of Economic Research.

Paarsch, H. J. (1992): "Deciding between the common and private value paradigms in empirical models of auctions," *Journal of econometrics*, 51(1-2), 191–215.

Paula, A. (2015): "Econometrics of Network Models," *Econometrics Society World Congress Working Paper*.

Petrin, A., and K. Train (2010): "A control function approach to endogeneity in consumer choice models," *Journal of marketing research*, 47(1), 3–13.

Pinto, C. (2010): "Semiparametric Estimation of Peer Effects in Classrooms - Evidence for Brazilian Schools in 2003," *University of California Berkeley Working Paper*.

Riordan, M. H. (1996): "Contracting with qualified suppliers," *International Economic Review*, pp. 115–128.

Rosenbaum, T. (2012): "Where do automotive suppliers locate and why," *New Haven, CT, USA: Yale University (mimeo)*.

Sacerdote, B. (2001): "Peer Effects with Random Assignment: Results for Dartmouth Roomates," *Quarterly Journal of Economics*, 116, 681–704.

——— (2013): "Chapter 4: Peer Effects in Education: How Might They Work, How Big Are They and How Much Do We Know Thus Far," *Handbook of the Economics of Education*, 3, 249–277.

Sedgwick, D. (2013): "Global industry craves megasuppliers," *Automotive News*, June 17th, 3–4.

Shih, W. (2020): "Is it time to rethink globalized supply chains," *MIT Sloan Management Review*, March 19th.

Shultz, K. (2013): "Peer Effects, Sports Teams and Academic Performance," *Purdue University Working Paper*.

Sund, K. (2009): "Estimating Peer Effects in Swedish High School using School, Teacher, and Student Fixed Effects," *Economics of Education Review*.

Tamer, E. (2003): "Incomplete Simultaneous Discrete Response Model with Multiple Equilibria," *The Review of Economic Studies*, 70, 147–165.

Taschereau-Dumouchel, M. (2020): "Cascades and fluctuations in an economy with an endogenous production network," *Available at SSRN 3115854*.

Tincani, M. (2015): "Heterogeneous Peer Effects and Rank Concerns - Theory and Evidence," *University of Pennsylvania Working Paper*.

Verdier, V. (2015): "Identification and Estimation of Peer Effects with Panel Data," *University of North Carolina Working Paper*.

Vlasic, B., and L. Wayne (2008): "Auto suppliers share anxiety over a bailout," *The New York Times*, Dec 11.

Wang, P., and A. Watts (2006): "Formation of buyer-seller trade networks in a quality-differentiated product market," *Canadian Journal of Economics*, 39(3), 971–1004.

Watts, A. (2016): "Auctions versus private negotiations in buyer-seller networks," *Games*, 7(3), 22.

——— (2018): "Generalized second price auctions over a network," *Games*, 9(3), 67.

Webber, T. (2012): "Scale matters in aviation," *Sydney Morning Herald*, February 20th.

Wollmann, T. G. (2018): "Trucks without bailouts: Equilibrium product characteristics for commercial vehicles," *American Economic Review*, 108(6), 1364–1406.

Womack, J., D. T. Jones, D. Roos, and D. S. Carpenter (1990): *The Machine that Changed the World - the story of Lean Production, Toyota's Secret Weapon in the Global Car Wars that is Revolutionizing World Industry (2007 edition)*. Free Press.

Wooldridge, J. M. (2014): "Quasi-maximum likelihood estimation and testing for nonlinear models with endogenous explanatory variables," *Journal of Econometrics*, 182(1), 226–234.

Yang, C. (2017): "Could Vertical Integration Increase Innovation?," *University of Michigan Dissertation Chapter*.