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Essays in Behavioral Finance and Asset Pricing

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Abstract

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This dissertation studies a range of topics in behavioral finance and asset pricing. The three essays presented in this dissertation have the common theme of using economic theory to explain puzzling phenomena in financial markets. Chapters 1 and 2 focus on one of the most well-known theories in behavioral finance, the prospect theory, and its implications on asset returns in the U.S. stock market and options market. Chapter 3 switches the focus to the Chinese warrants market, and explores the pricing efficiency of option pricing models from an econometric perspective.

Chapter 1 asks the question "why do investors sometimes require higher expected returns from the stock market in compensation for bearing volatility, but sometimes do not?" We answer this question by referring to two important components of the prospect theory, namely decreasing sensitivity and loss aversion. On one hand, decreasing sensitivity suggests that after investors have experienced a prior loss, they will behave in a locally risk-seeking way, such that the higher the market volatility, the lower the expected return they require from the market. On the other hand, even after a prior loss, investors do not like too much volatility because the pain inflicted by extra losses exceed the joy coming from extra gains. Consistent with the theory, we find the mean-variance relation depends on the relative strength of decreasing sensitivity and loss aversion.

In Chapter 2, Jianfei Cao and I ask the question "do investors' preferences over risk change over time in terms of their degrees of loss aversion and probability weighting, and

if so, how do these preferences change with other economic variables?" To answer that question, we build a representative agent model based on the prospect theory, and in a dynamic setting, we estimate the structural parameters in the model using data on the U.S. stock market and the options market. Our results show that after the 2007-2008 financial crisis, investors became more loss averse, and had a weaker tendency to overweight right tail events of the market. We also find close relationships between the prospect theory parameters and investor sentiment.

Chapter 3 studies the performance of various option pricing models in the Chinese warrants market. To capture the negative skewness and heavy tails in the distribution of Chinese stock returns, we modify the canonical Black-Scholes model from two perspectives. First, we introduce stochastic volatility into stock price dynamics using GARCH (generalized autoregressive conditional heteroscedasticity) models. Second, we add Poisson jumps to reflect big shocks to stock prices. We then conduct Monte Carlo simulations to calculate theoretical warrant prices implied by different models, and compare them with the observed prices. Our results show that the more sophisticated models successfully explain a large part of the discrepancy between theoretical and real prices, but the differences remain non-negligible in some cases, suggesting the existence of bubbles in the Chinese warrants market.

Essays in Behavioral Finance and Asset Pricing

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Chapter 1

Prospect Theory and the Mean-Variance Relation

Abstract: This paper shows the impact of prospect theory preference, specifically the loss aversion and the decreasing sensitivity parameters, on the mean-variance relation of the stock market. We find that the market excess return is positively related to the market's conditional variance after a prior gain, but the mean-variance relation is nonlinear after a prior loss. When the conditional variance is large enough relative to the magnitude of the loss and the effect of loss aversion dominates, the mean-variance relation is positive. However, the relation reverses to be negative when the conditional variance is small, where decreasing sensitivity comes into play and investors are risk seeking. A simple market timing strategy based on past gains and losses achieves superior performance compared to the market.

1.1 Introduction

The mean-variance relation of the aggregate stock market is one of the most fundamental questions in finance, as it is crucial in understanding investors' preference over risk and return. The literature of rational asset pricing theory usually suggests a positive mean-variance relation, as the representative agents in these models are risk averse, and volatility increases the risk of their portfolio. For example, Merton (1973) establishes the Intertemporal Capital Asset Pricing Model (ICAPM), which suggests the following mean-variance relation:

$$E_t[R_{t+1}] = \mu + \gamma Var_t[R_{t+1}] \tag{1.1}$$

where $E_t[R_{t+1}]$ is the expected return of the overall market in the future, $Var_t[R_{t+1}]$ is the conditional variance of the future market return, $\mu = 0$, and γ is the coefficient of relative risk aversion of the representative agent. Intuitively, when investors expect high variability in the market return in the future, which is harmful to their wealth, they are less willing to hold the market portfolio, and require a higher expected return for them to do so in equilibrium.

On the other hand, despite efforts over the past decades to find supportive evidence for the positive mean-variance relation, the results are rather inconclusive. Previous studies have found both positive and negative mean-variance relation using different periods of data and different methods. To name a few, Ghysels et al. (2005) use past daily squared returns to forecast monthly return variance, and find a significantly positive mean-variance relation. Yu and Yuan (2011) find that the mean-variance relation is positive during periods of low investor sentiment, but there is no relation whatsoever during periods of high investor sentiment. Brandt and Kang (2004) use a latent VAR model to study the joint behavior of conditional mean and conditional volatility, and find a strong and robust negative relationship between the two. The inconsistency between theory and empirical evidence seems to imply that investors' preference over risk and return changes over time. A slight modification of the constant relative risk aversion (CRRA) utility for the investors in the model to allow for time-varying risk aversion is able to accommodate the fact that the positive mean-variance relation is sometimes strong and sometimes weak, but the documented negative relationship is still a puzzle.

Prospect theory provides a very natural explanation for this puzzling phenomenon because of its decreasing sensitivity component. Specifically, the theory suggests that people evaluate gains and losses relative to their reference point, and they become less sensitive to the gains and losses as the magnitude of the gain or loss increases. This means that after they suffer a loss, they become locally risk seeking instead of risk averse. Therefore, it is possible that they require a lower expected return from their portfolio when they expect their portfolio to be more volatile. In this paper, we explore the mean-variance tradeoff in periods when investors have suffered a loss and when they have achieved a gain, and we find that investors' risk appetite is closely related to their prior gain and loss as well as the magnitude of the loss. The results show that after a prior gain, investors require a higher expected return for a higher conditional variance of the market return, hence the mean-variance relation is positive; but after a significant prior loss when decreasing sensitivity comes into play, they indeed behave in a risk seeking fashion: they require a lower expected market return for a higher conditional variance and the mean-variance relation reverses.

Our paper contributes to three streams of literature. First, we provide an explanation for the time-varying mean-variance relation of the aggregate stock market from the perspective of behavioral finance theory. Previous studies have focused on finding alternative econometric methods to model conditional variance in order to better estimate the mean-variance relation. Yu and Yuan (2011) and Wang (2018) are the only papers to our best knowledge that relate the behavioral finance literature to the mean-variance relation. They divide the whole sample into sub-periods based on Baker and Wurgler (2006, 2007) investor sentiment, and institutional sentiment, respectively, to measure the mean-variance relation in different periods. Although sentiment captures investors' irrational behavior in the financial markets to some extent, it still does not help us fully understand what exactly in their preferences drives the time-varying mean-variance relation. This paper builds directly upon prospect theory, and provides strong evidence that prospect theory is an accurate modeling device for people's decision under uncertainty.

Second, we further demonstrate the importance of the decreasing sensitivity component of prospect theory. There has been growing interests in the literature in the relationship between decreasing sensitivity and the the well-known disposition effect, the tendency of investors to hold on to their losing stocks too long and to sell their winning stocks too soon. For instance, Grinblatt and Han (2005)'s model of equilibrium asset prices suggests that the disposition effect causes the momentum in the cross section of stock returns, and they find that once investors' capital gain overhang is controlled for, the momentum disappears. Frazzini (2006) argues that the disposition effect induces underreaction to news, and finds that the post-announcement drift is most severe when capital gains and the news event have the same sign. Li and Yang (2013) build a general equilibrium model to examine the implications of decreasing sensitivity for the disposition effect, asset prices, and trading volume. More recently, Wang et al. (2017) study the risk-return tradeoff in the cross section, and find a pronounced negative risk-return relation among firms in which investors face prior losses. Barberis et al. (2020) propose a model in which the average returns of stocks with a larger capital overhang (larger prior gains) are generally higher than those with a smaller capital overhang, all else being equal, because prospect theory investors have concave utility over gains but convex utility over losses. Their theory explains the negative risk-return relation found by Wang et al. (2017). Our results provide additional evidence in favor of the convex-concave prospect theory utility function structure from the aggregate stock market.

Third, this paper also adds to the literature of equity premium predictability. Over the past decades, a lot of macroeconomic variables have been suggested as being able to predict the risk premium of the stock market. These variables include valuation ratios, interest rates, interest rate spreads, etc.; see Rapach and Zhou (2013) for an extensive survey. Our study shows that even though neither past returns nor conditional variances alone predict future market returns, jointly they have strong predictive power for the equity premium both in sample and out of sample.

The rest of the paper is organized as follows. Section 1.2 develops our hypothesis from theory and specifies our model. Section 1.3 shows the main empirical results and examines the robustness of our results. Section 1.4 concludes.

1.2 Hypothesis Development

1.2.1 Prospect Theory Overview

Our model is motivated by prospect theory, proposed by Kahneman and Tversky (1979). Prospect theory is to date the most widely used non-expected utility theory that models people's decision under uncertainty. To illustrate the differences between the prospect theory and the expected utility theory, consider the following gamble:

$$(x, p; y, q)$$

in which the agent gains x with probability p and y with probability q, where $x \leq 0 \leq y$ and $p + q = 1$. The expected utility theory states that the agent evaluates the gamble by combining its prize with her current wealth level and taking expectations under the objective probability. Specifically, the agent has a utility function $u(\cdot)$ defined on her terminal wealth, and the expected utility associated with this gamble is:

$$pu(W + x) + qu(W + y)$$

where W is her current wealth level. The utility function $u(\cdot)$ is usually assumed to be concave, so that the agent behaves in a risk averse way. One popular functional form for the utility is the CRRA utility function, which leads to the constant positive mean-variance relation in (1.1).

The prospect theory, on the other hand, suggests that the agent considers the gamble in isolation, and evaluates its possible gain and loss with respect to a reference point, which in this situation is her current wealth level. She also behaves in three other important ways different than an expected utility maximizing agent. First, she treats gains and losses differently: she always suffers more from a loss than she enjoys a gain of the same magnitude, no matter how small the magnitude is. Notice that this is not true in expected utility theory. Even though a concave utility function allows the disutility of a loss to be greater than the utility of a gain, the utility function is locally linear everywhere, meaning that the marginal utility of a gain and the marginal disutility of a loss are approximately equal when their magnitude is small enough compared to the current wealth level. Second, the agent assigns weights to different states of the world not proportional to the objective probabilities associated with those states, but rather based on a probability weighting function which tends to overweight low probability states with extreme outcomes. Probability weighting explain many phenomena in financial markets, such as people's preference for lottery-like stocks as in Barberis and Huang (2008). However, for our purposes, probability weighting is less important than the other two features of prospect theory. Third, prospect theory states that people's sensitivity to both gains and losses decreases as the magnitude of gains and losses increases. Decreasing sensitivity in the gain region is consistent with expected utility theory when the utility function is concave. Nonetheless, it is the decreasing sensitivity on the loss region that distinguishes prospect theory and allows for the possibility of a negative mean-variance relation.

Formally, a prospect theory agent will evaluate the above gamble in the following way:

$$\pi(p)v(x) + \pi(q)v(y)$$

where $\pi(\cdot)$ is the probability function that associates decision weights with probabilities, and $v(\cdot)$ is the prospect theory value function. Notice how the gamble is evaluated in isolation, i.e., the value function takes as input the gains and losses x and y themselves, but not the wealth level W except for using it as the reference point. Figure 1.1 shows the value function proposed by Kahneman and Tversky (1979). The origin stands for the agent's reference point, and all gains with respect to the reference point are to its right while losses to its left. Loss aversion manifests itself as the kink at the origin, where a small loss incurs a larger pain to the agent than the joy brought about by a small gain of the same magnitude. The concavity of the value function in the gain region and its convexity in the loss region represent the decreasing sensitivity component. When the agent is in her gain region, she is risk averse and will behave similarly to a concave utility function maximizing agent. However, when she suffers a prior loss which puts her in the loss region, she becomes less sensitive to further losses and behaves in a risk seeking way.

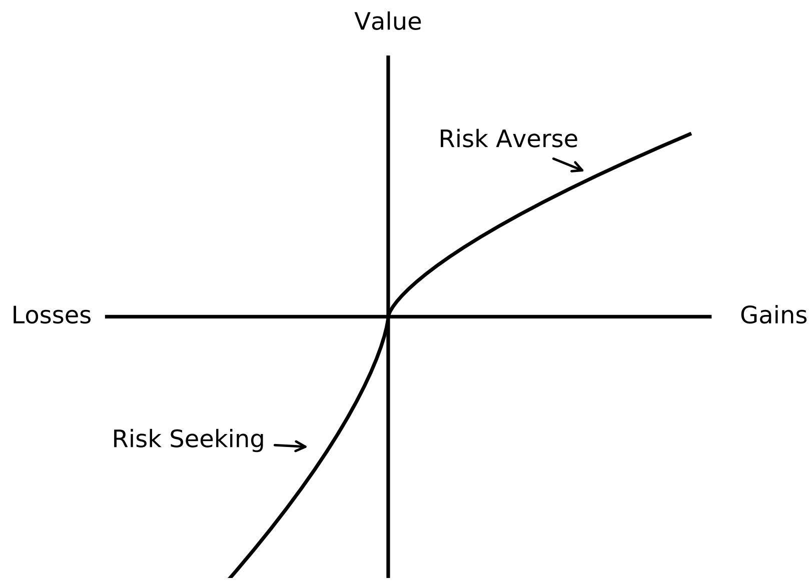


Figure 1.1: Kahneman and Tversky (1979) Value Function

Tversky and Kahneman (1992) suggest the following functional form for the value function:

$$v(x) = \begin{cases} x^{\alpha} & \text{if } x \geq 0 \\ -\lambda(-x)^{\beta} & \text{if } x < 0 \end{cases}$$

where $\lambda > 1$ is the coefficient of loss aversion, and $\alpha, \beta \in (0, 1)$ captures the decreasing sensitivity in the gain and loss region respectively. They use lab experiment data to estimate

$\lambda = 2.25$ and $\alpha = \beta = 0.88$. The value function specified by these estimates is close to being linear in both the gain and loss region, and therefore decreasing sensitivity was thought to be a weak force in driving people's preference compared to the loss aversion and the probability weighting components. However, recent evidence from both experiments and the field have suggested that α and β may be more like 0.7 or even lower still, which provides theoretical support for our claim that decreasing sensitivity results in a variable and possibly negative mean-variance relation.

1.2.2 Model Specification

Two-Regime Model

Prospect theory clearly suggests that people will behave differently after a prior gain than a prior loss. In the financial market, prior gains and losses are captured by the previous performance of investors' portfolios. Consider two different cases for a prospect theory investor. In the first case, her portfolio did well in the past and achieved a gain with respect to her reference point, then she will reside in the gain region of her value function, where she is risk averse, and hence demands a higher expected return from her portfolio for a higher risk. In the second case, her portfolio did poorly and incurred a loss with respect to the reference point, she will be in the loss region of the value function where she is risk seeking, and hence demands a lower expected return from her portfolio given a higher risk. If all investors share a similar prospect theory preference, then their time-varying appetite for risk based on prior gains and losses will aggregate to the overall market. Specifically, when the overall market has had a loss, more investors, both stock pickers and index investors, are more likely to be in the loss region of their value function. They tend to pursue riskier assets because such assets, due to their volatile nature, provide investors a good chance of breaking even, i.e., bringing them back to their reference point. This joint behavior pushes up the overall price level of the stock market and lowers its expected return. Therefore, a possible negative mean-variance relation establishes.

Some argument against prospect theory claims that institutional investors are less subject to the type of psychological biases in the model, so the behavior described above of individual investors is not enough to drive the market outcome. However, Chevalier and Ellison (1997) demonstrated that active mutual fund managers tend to pursue riskier portfolios when they fall behind in order to attract fund flow. Therefore, the changes in risk appetite based on prior gains and losses are not only present among naïve retail investors but also among institutional investors.

Based on the theory, the first hypothesis we test in this paper is a two-regime mean-variance relation, where the regime is determined by investors' overall gains and losses in the past. In the one-regime model, the following equation is tested:

$$R_{t+1} = a + bVar_t(R_{t+1}) + \varepsilon_{t+1}$$

where R_{t+1} is the monthly excess return of the market, and $Var_t(R_{t+1})$ is its conditional variance. Conditional variance is not directly observable, but can be approximated by weighted average of daily squared returns in the past, Chicago Board Options Exchange's volatility index (VIX), or other more sophisticated volatility models such as GARCH models. Rational asset pricing theory such as Merton (1973)'s ICAPM implies a positive slope coefficient $b > 0$. The two-regime model suggests instead the following:

$$R_{t+1} = a_1 + b_1 Var_t(R_{t+1}) + a_2 D_t + b_2 D_t Var_t(R_{t+1}) + \varepsilon_{t+1} \tag{1.2}$$

where D_t is an indicator of whether a typical investor is in her gain region of the prospect theory value function. For example, one possible specification is:

$$D_t = \begin{cases} 1 & \text{if } R_t \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

where gain is defined by last month's market excess return being nonnegative. This is equivalent to saying the investor's reference point is her wealth level if she has invested all her money in the risk-free asset.

We test three hypotheses for this model. First, we expect $b_1 < 0$. That is, after a prior loss, investors require a lower return going forward for a higher conditional variance of the market portfolio. Second, $b_2 > 0$, meaning that a prior gain makes investors' tendency to pursue riskier portfolio weaker. Third, $b_1 + b_2 > 0$, so that the mean-variance relation is reversed to be positive when investors are in their gain region.

Nonlinear Mean-Variance Relation

In the two-regime model, only the sign of the past market excess return affects the future mean-variance relation. However, prospect theory predicts a more subtle mean-variance relation that also depends on the magnitude of prior gains and losses. In fact, two components, namely loss aversion and decreasing sensitivity, jointly determine investors' attitude toward risk. Specifically, suppose we fix the past loss level at 10%, then when the volatility increases from 5% to 10%, we may observe a negative mean-variance relation as the decreasing sensitivity is the main driving force. As the volatility continues to increase from 10% to a higher level, e.g., 20%, the agent is more likely to go beyond her reference point to the gain region, and the kink of the value function will make such gain not so attractive to the agent anymore. In other words, loss aversion becomes more relevant than decreasing sensitivity when the magnitude of the loss is relatively small compared to the volatility, and we may observe a positive mean-variance relation in this situation. Notice that when the agent is in her gain region and hence risk averse, decreasing sensitivity and loss aversion drive the mean-variance relation in the same way, so we expect the mean-variance relation to be always positive, except that when the volatility is high relative to the magnitude of the gain, loss aversion comes into play and the agent hates volatility even more than otherwise, and the mean-variance relation is expected to be steeper.

To test the nonlinear relationship between conditional variance and expected return, we augment our previous specification by adding square terms of the conditional variance to the model, and substitute R_t for the indicator D_t:

$$
\begin{aligned}
R_{t+1} = a_1 + b_1 Var_t(R_{t+1}) + c_1 Var_t^2(R_{t+1}) \\
+ a_2 R_t + b_2 R_t Var_t(R_{t+1}) + c_2 R_t Var_t^2(R_{t+1}) + \varepsilon_{t+1}
\end{aligned}
$$

1.2.3 Conditional Variance Models

Previous studies find that the empirical mean-variance relation relies heavily on the proxy for the conditional variance. In this paper, we use four conditional variance models following the literature: the rolling window model, the CBOE VIX, the GARCH (Generalized Autoregressive Conditional Heteroskedasticity) model, and the asymmetric GARCH model, and our results are fairly robust across different models.

Rolling window model

French et al. (1987) use a rolling window model to estimate the monthly volatility of stock market returns, and find a positive relation between the market risk premium and the predictable volatility. The idea of the rolling window model is very straightforward: it uses the current month's average daily squared returns to estimate the conditional variance of returns next month. Specifically,

$$Var_t(R_{t+1}) = \frac{22}{N_t} \sum_{d=1}^{N_t} r_{t+1-d}^2$$

where r_{t-d} is the daily excess return of the market d trading days before the current date t, N_t is the total number of trading days in the current month, and 22 is the average number of trading days in a typical month that scales the daily variance to monthly variance, by assuming returns are uncorrelated across days.

VIX

The CBOE volatility index, or VIX, is a measure of the market's expected annualized volatility of the S&P 500 index in the following 30 days. It is computed by the Chicago Board Options Exchange using a weighted portfolio of out-of-the-money European options on the S&P 500 index. Therefore, it is an option-implied volatility of the stock market in the next month, which we use as a proxy for the conditional volatility. There is empirical evidence (see Carr and Wu (2009) for example) showing that the option-implied volatility is systematically higher than the actual realized volatility over the same period, known as the volatility risk premium. Thus VIX can be a biased estimate of the conditional volatility of the stock market. For our purposes, we use it as a robustness check to show our result is consistent across different conditional volatility estimates.

GARCH and asymmetric GARCH

The ARCH (Autoregressive Conditional Heteroskedasticity) model by Engle (1982) and the GARCH (Generalized ARCH) model by Bollerslev (1986) have been commonly used in modeling the volatility of asset returns. Like the rolling window model, they assume that the conditional variance of returns are determined by past squared returns (for ARCH) as well as past conditional variance of returns (for GARCH). Moreover, the weights assigned to past observations are not equal across time, but estimated using data directly. We use the most popular model in the literature, GARCH(1, 1), to estimate conditional volatility. Specifically, the model assumes the daily return series $\{r_t\}$ follow the process below:

$$
\begin{aligned}
&r_t = \mu + \varepsilon_t \\
&\sigma_t^2 = \omega + \alpha\varepsilon_{t-1}^2 + \beta\sigma_{t-1}^2 \\
&\varepsilon_t = \sigma_t e_t, \quad e_t \sim N(0,1)
\end{aligned}
$$

where μ is the constant mean of daily returns, ε_t is the return residual, which has a conditional variance of σ_t^2. The set of parameters to be estimated is $\{\mu, \omega, \alpha, \beta\}$. We require $\alpha + \beta < 1$ for the process to be stationary.

Since the GARCH model captures the dynamics of daily returns while we require an estimate for the monthly conditional variance, we use the model together with its parameter estimates to forecast, at each end of month, the expected variance of sum of daily returns for the next month, and use that as our proxy for the conditional variance:

$$Var_t(R_{t+1}) = E_t\left(\sum_{d=1}^{22} \sigma_{t+d}^2\right) \tag{1.3}$$

where 22 is again the approximate number of trading days in a month. The conditional variance of daily returns over one day ahead depends on future realizations of returns which are not directly observable at time t. We take two approaches to estimate the expectation in equation (1.3). First, we use Monte Carlo simulation to generate a large number of paths of daily returns going forward based on the GARCH process, and use the sample mean of the conditional variance to approximate for its population mean in (1.3). Second, we solve for the analytical form of the expectation in (1.3) as a function of the GARCH model parameters $\{\mu, \omega, \alpha, \beta\}$, and then plug in their corresponding estimates. The analytical solution is provided in the appendix. The two approaches lead to very similar results, and we show only one set of the results in the following section. Full results are available upon request.

The canonical GARCH model assumes that innovations to the returns, either positive or negative, have the same impact on the conditional variance through the parameter α. However, empirical evidence establishes that negative innovations tend to have larger effect on the volatility than positive innovations, known as the "leverage effect" developed by Black (1976) and Christie (1982). Glosten et al. (1993) suggest an asymmetric GARCH model which takes into account the leverage effect, and find that it leads to different conclusions on the mean-variance relation than the symmetric GARCH. The asymmetric GARCH(1, 1) model is specified similarly to the GARCH(1, 1) model above, except for the process of conditional variance σ_t^2:

$$\sigma_t^2 = \omega + \alpha\varepsilon_{t-1}^2 + \gamma\varepsilon_{t-1}^2 I_{[\varepsilon_{t-1}<0]} + \beta\sigma_{t-1}^2$$

where $I_{[\varepsilon_{t-1}<0]}$ is an indicator taking value 1 if the innovation ε_{t-1} is negative. One extra parameter, γ is estimated for the asymmetric GARCH, and $\gamma > 0$ if the leverage effect holds true. The process is stationary if $\alpha + \gamma/2 + \beta < 1$. We derive the analytical solution for the monthly conditional variance in the appendix.

1.3 Main Empirical Results

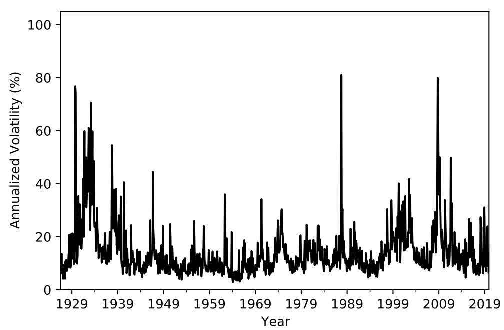
1.3.1 Data and Summary Statistics

The data we use in this paper come from several sources. We use as proxy for the aggregate stock market the value-weighted returns of all NYSE, AMEX, and NASDAQ stocks, and the one-month Treasury bill rate as the risk-free rate. We obtain these data from the Center for Research in Security Prices (CRSP) for the period from July 1926 to December 2019. The CBOE volatility index (VIX) data is published by the Chicago Board Options Exchange, and date back to January 1990. We also use other control variables in our regression analysis, which we will describe later.

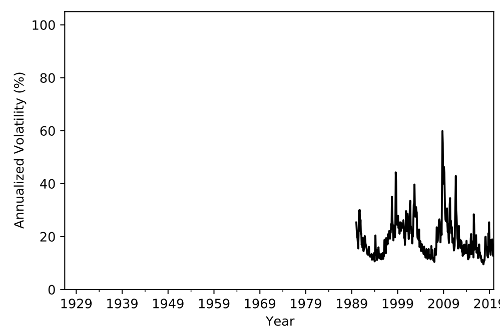
Table 1.1 presents the summary statistics for the monthly market excess return and its volatility. Both return and volatility are annualized and are in percentage terms. We show the results for the excess return in the current month (r_t^e) as well as in the next month (r_{t+1}^e), because we are interested in how the excess return in the future is determined by the conditional variance now. Four proxies for the conditional volatility are used in our analysis, as described in the previous section, namely the rolling window realized volatility, the VIX, the GARCH forecast, and the asymmetric GARCH forecast. As a comparison, we also show the summary statistics for the realized volatility in the next month. For each variable, we calculated its mean, standard deviation, skewness, kurtosis, its 5%, 25%, 50%, 75%, and 95% percentile, as well as its minimum and maximum in the sample. Panel A presents the results for the whole sample, Panel B shows the results for gain months only, in which the market excess returns are nonnegative, and Panel C shows the results for loss months.

We can see from the table that after a gain of the stock market with respect to the risk-free asset, the average annualized excess return in the next month is 10.67%, a lot higher than the average return of 3.79% after a loss in the current month. We will see in further analysis that the difference in average future excess returns in gain months and loss months depends also on the conditional variance of returns next month. The realized volatility is generally higher in a loss month than a gain month, with 17.15% versus 12.15%, which is probably due to the "leverage effect" on a daily frequency: in a loss month, negative daily return shocks are more likely to occur, causing higher volatility for the next trading day. Moreover, the realized volatility in the next month also tends to be higher if the current month experiences a loss (17.01%) than a gain (12.25%). The four proxies for conditional volatility, except for the VIX, closely match the realized volatility next month on average, both in gain months and loss months. The VIX is generally higher than other proxies as well as the realized volatility next month. For example, in a loss month, the VIX expects the next month's annualized volatility to be 22.57%, while other estimates are at most 18.31% (given by the asymmetric GARCH) and the realized volatility next month is 17.01% on average. This fact is not surprising due to the well known volatility risk premium puzzle. GARCH and asymmetric GARCH model achieve more stable estimates of the conditional volatility than the rolling window model in terms of their standard deviation, as they employ a smoother weighting scheme.

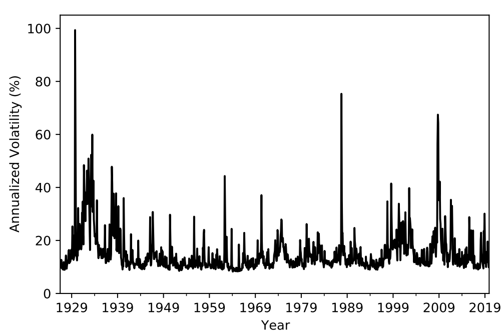
Figure 1.2 shows the plots of monthly estimates of annualized conditional volatility using four different models. The VIX series has a shorter time range due to data availability. These estimates are able to capture high volatility periods in history such as the 1929 and 1987 market crash as well as the 2008 financial crisis.



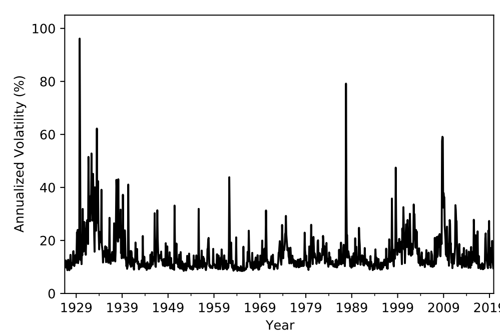
(a) Rolling Window



(b) VIX



(c) GARCH(1, 1)



(d) Asymmetric GARCH(1, 1)

Figure 1.2: Time Series Plots of Conditional Volatility

Table 1.1: Summary Statistics for Market Excess Return and Volatility

	Mean	SD	Skew	Kurt	Min	5%	25%	Median	75%	95%	Max
Panel A: Whole Sample											
r_t^e	8.00	63.87	0.18	7.88	−349.56	−94.31	−23.64	12.54	43.50	87.94	466.20
r_{t+1}^e	7.97	63.89	0.18	7.87	−349.56	−94.32	−23.64	12.36	43.56	87.96	466.20
Realized Volatility	14.11	9.55	2.88	11.91	2.83	5.86	8.55	11.20	15.99	31.99	81.10
VIX	19.18	7.36	1.77	4.90	9.51	11.46	13.66	17.14	23.02	31.93	59.89
GARCH	15.20	7.87	3.68	21.80	8.25	9.35	10.81	12.62	16.43	30.11	99.37
Asymmetric GARCH	14.88	7.61	3.77	22.87	8.47	9.26	10.69	12.39	16.01	28.55	96.15
Realized Volatility$_{t+1}$	14.12	9.55	2.88	11.90	2.83	5.86	8.56	11.21	16.00	32.00	81.10
Panel B: Gain Months											
r_t^e	44.29	43.39	4.24	31.98	0.00	3.60	16.92	36.48	58.56	105.60	466.20
r_{t+1}^e	10.67	57.05	0.53	14.45	−349.56	−71.86	−19.29	14.22	41.25	81.79	466.20
Realized Volatility	12.15	7.48	2.94	12.84	2.83	5.39	7.89	10.07	14.15	26.65	70.56
VIX	17.28	5.89	1.53	3.15	9.51	11.16	13.04	15.73	20.35	27.47	44.14
GARCH	13.39	5.23	3.02	12.38	8.25	9.12	10.41	11.83	14.47	23.41	52.31
Asymmetric GARCH	12.68	4.29	2.66	9.02	8.47	9.07	10.15	11.31	13.54	21.62	39.28
Realized Volatility$_{t+1}$	12.25	7.12	2.71	10.99	2.83	5.51	8.02	10.35	14.44	25.99	61.05
Panel C: Loss Months											
r_t^e	−48.46	47.54	−2.35	7.99	−349.56	−135.19	−65.46	−33.48	−17.64	−3.22	−0.12
r_{t+1}^e	3.79	73.14	−0.02	3.12	−278.88	−115.76	−34.08	8.16	47.82	106.97	406.08
Realized Volatility	17.15	11.45	2.54	8.70	3.76	6.89	9.98	14.12	20.36	39.27	81.10
VIX	22.57	8.46	1.70	4.12	11.73	13.03	16.44	20.97	25.42	39.57	59.89
GARCH	18.01	10.13	3.15	14.90	8.63	10.00	12.06	14.91	19.83	36.10	99.37
Asymmetric GARCH	18.31	10.01	3.06	13.92	8.98	10.27	12.30	15.24	20.41	37.27	96.15
Realized Volatility$_{t+1}$	17.01	11.86	2.46	8.01	3.51	6.22	9.46	13.77	19.99	38.93	81.10

1.3.2 Mean-Variance Relation

We estimate two sets of mean-variance relation models:

$$R_{t+1} = a + bVar_t(R_{t+1}) + \varepsilon_{t+1}$$

$$R_{t+1} = a_1 + b_1Var_t(R_{t+1}) + a_2D_t + b_2D_tVar_t(R_{t+1}) + \varepsilon_{t+1}$$

where R_{t+1} is the market excess return, $Var_t(R_{t+1})$ is its conditional variance at time t, D_t is an indicator for $R_t \geq 0$. The first model is a one-regime model while the second one is a two-regime model where the regime is determined by the current month's gain or loss. Table 1.2 shows the model estimates. Specifications (1)(3)(5)(7) correspond to the one-regime model, and specifications (2)(4)(6)(8) correspond to the two-regime model. Within the one-regime and two-regime specifications, models differ by the proxies used for conditional variance. Specifications (1)(2) use the rolling window model, (3)(4) use the VIX, (5)(6) use the GARCH(1, 1) model, and (7)(8) use the asymmetric GARCH(1, 1). t-statistics are given in parentheses.

Among the four one-regime models, no one has a significant coefficient on the conditional variance term, and the R^2's are extremely low (the highest being 0.2% given by the asymmetric GARCH model), meaning that there is no detectable mean-variance relation in our data. These results contradict the prediction by the rational asset pricing theory that the mean-variance relation is positive, but are not surprising given the inconclusive mean-variance relation in the empirical literature.

However, adding the gain/loss indicator D_t together with its interaction with the conditional variance completely changes the results. Specifically, in the two-regime models, the coefficients on the conditional variance term are all significant at 5% level with a negative

Table 1.2: Monthly Excess Returns against Conditional Variance

	Rolling Window		VIX		GARCH		Asymmetric GARCH	
	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
Intercept	0.7052	0.6094	0.6515	1.0254	0.7736	0.7047	0.8094	0.6418
	(3.955)	(2.114)	(1.988)	(1.837)	(4.211)	(2.429)	(4.408)	(2.174)
$Var_t(R_{t+1})$	−0.0017	−0.0083	0.0012	−0.0084	−0.0045	−0.0109	−0.0062	−0.0090
	(−0.504)	(−2.113)	(0.173)	(−0.969)	(−1.185)	(−2.675)	(−1.574)	(−2.130)
D_t		−0.0710		−0.9577		−0.6327		−0.4870
		(−0.192)		(−1.337)		(−1.596)		(−1.153)
$D_t \times Var_t(R_{t+1})$		**0.0290**		**0.0323**		**0.0584**		**0.0582**
		(3.802)		(2.104)		(5.197)		(3.730)
R^2	0.000	0.016	0.000	0.013	0.001	0.027	0.002	0.016

sign, except for the specification using VIX. These results are consistent with the prediction by the prospect theory that when investors are in their loss region of the value function after a prior loss, they become risk seeking, and hence demand a lower expected return for a higher conditional variance of the market portfolio. Moreover, the coefficients on the interaction term, $D_t \times Var_t(R_{t+1})$, are all highly significant across different specifications, with the lowest t-statistic being 2.104 for the specification using VIX. This means that for a given level of conditional variance, investors require a significantly higher expected return from the stock market when they have experienced a prior gain than a prior loss. The coefficient estimates do not directly tell us whether the mean-variance relation is positive after a past gain in the market. To answer that question, we do a F-test for the null hypothesis: $H_0 : b_1 + b_2 = 0$ against the alternative hypothesis: $H_1 : b_1 + b_2 \neq 0$, where b_1 and b_2 are the coefficients for the conditional variance term and the interaction term in the two-regime model, respectively. All the specifications except for the one using VIX reject the null hypothesis at a high significance level: all p-values are below 0.01. Even the VIX specification has a p-value of 0.06, which is marginally significant. Therefore, as expected, when investors are in their gain region and hence risk averse, they do demand a higher expected return for a higher conditional variance.

The impact of conditional variance on the future market excess return is not only statistically significant but also economically important. Take the rolling window model as an example, where the coefficient estimate is -0.0083 for the conditional variance term and 0.0290 for the interaction term. These estimates imply that in a loss month, a one standard deviation increase in the conditional variance is associated with a decrease of 0.40% (not annualized) in the next month's expected return, while in a gain month, the same increase in the conditional variance is associated with an increase of 0.99% in the next month's expected return. We will show in later part how the economically important alternating mean-variance relation is potentially turned into profitable market timing strategies.

Our results are strikingly robust across different conditional variance models, in contrast to the inconclusiveness in the empirical literature. The results also contradict sharply with traditional understanding of the mean-variance relation, and are highly consistent with the predictions of the prospect theory.

1.3.3 Other Explanatory Variables

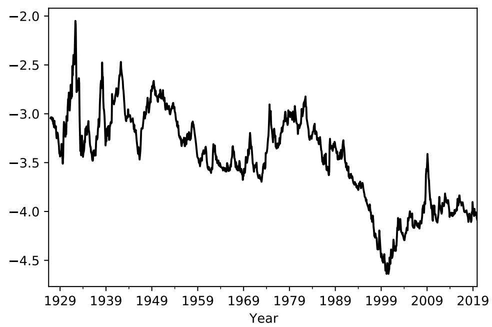
We have considered so far the univariate model relating expected return to conditional variance, and find that a mean-variance relation indeed exists for the aggregate stock market, but with alternating sign. Nonetheless, it could be the case that our conditional variance proxies actually capture the variation in other variables that explain the market expected returns. In this subsection, we examine the robustness of our model by considering control variables well studied in the market risk premium literature.

Ghysels et al. (2005) suggest that market variance is highly counter-cyclical and may proxy for business cycle variables, which are documented to predict the stock market. For example, Campbell (1991), Campbell and Shiller (1988), Chen et al. (1986), Fama (1990), Fama and French (1988, 1989), Ferson and Harvey (1991), and Keim and Stambaugh (1986), among many others, find evidence that the stock market can be predicted by variables related to the business cycle, such as the dividend-price ratio and the default spread. Even though these business cycle variables usually have a lower frequency than the monthly horizon we study, and they tend to predict the market over a longer window, we incorporate them into our model for completeness. The dividend-price ratio is computed as the ratio of dividends in the past 12 months to the CRSP value weighted index. We take natural log of the ratio to remove the positive skewness in the raw data. The default spread is defined as the difference in the yields of BAA and AAA rated corporate bonds. The bond yields data are obtained from FRED (Federal Reserve Economic Data).

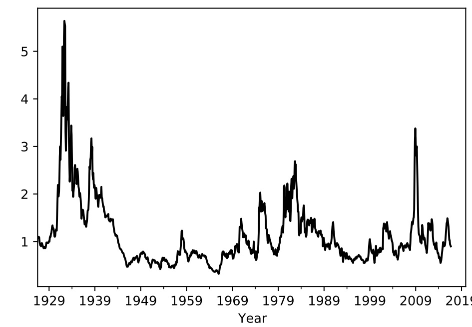
Yu and Yuan (2011) find a positive mean-variance relation in low investor sentiment periods, but no relation in high investor sentiment periods. Their argument is that during high sentiment periods, the market is dominated by irrational investors who are not able to properly evaluate how risk the market is, which obscures the risk-return relation of the overall market. If the low and high sentiment periods coincide with the gain and loss months in our study respectively, then the time-varying mean-variance relation we find may just be a phenomenon that has already been discovered, but under a different hood. Therefore, we take into account of investor sentiment in our analysis. We use the Baker and Wurgler (2006, 2007) sentiment index published on Jeffrey Wurgler's website, to be consistent with Yu and Yuan (2011). A month is defined as a low sentiment month if the sentiment index is below zero, and as a high sentiment month otherwise. The indicator for high sentiment as well as its interaction with conditional variance are used as control variables.

Lochstoer and Muir (2019) provide evidence that investors have slow-moving beliefs about stock market volatility, such that they underreact to volatility news initially and overreact with a delay. Therefore, the negative mean-variance relation may be a result of investors underreacting to volatility shocks together with the mean-reverting behavior of stock market volatility. To decompose the effects on expected returns of conditional variance and past variances, we add lagged terms of realized variances to our model. Specifically, lagged realized variance is calculated as the average of realized variances in the past 6 months, excluding the current month.

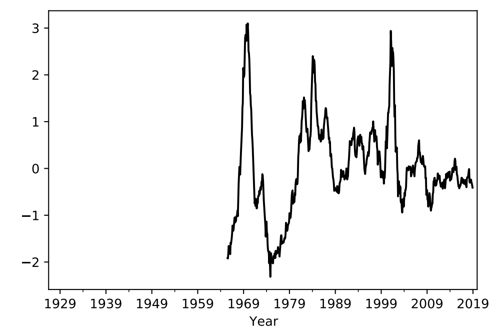
Figure 1.3 shows the time series plots of the control variables. The investor sentiment



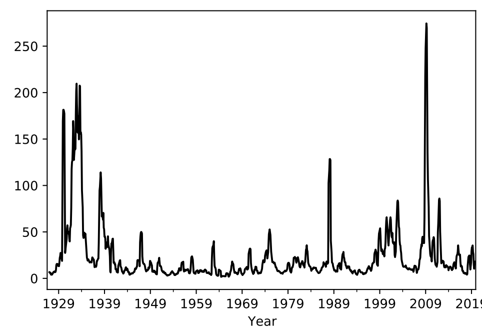
(a) (Log) Dividend-Price Ratio



(b) Default Spread (%)



(c) Investor Sentiment



(d) Lagged Realized Variance ($\times 10^{-4}$)

Figure 1.3: Time Series Plots of Control Variables

data is only available since 1965. All four series exhibit enough variations, which allows us to detect their predictability for stock market returns if it so exists. Table 1.3 presents the results for the following regression:

$$R_{t+1} = a_1 + b_1 Var_t(R_{t+1}) + a_2 D_t + b_2 D_t Var_t(R_{t+1}) + controls + \varepsilon_{t+1}$$

t-statistics are given in parentheses. Specifications (1)(2)(3)(4) control for dividend-price (D/P) ratio, default spread, investor sentiment, and lagged variance, respectively, and specification (5) include all the control variables together. We show here only the model estimates using the GARCH forecast as the proxy for conditional variance, but using other proxies generate very similar results. The reasons we choose the GARCH model as our benchmark are the following. First, the GARCH model gives less noisy estimates for the conditional variance than the rolling window model, which should in theory have smaller standard deviation than the realized variance. Second, it gives less biased estimates than the VIX, which tend to overestimate the realized variance due to the volatility risk premium. Third, it is nested in the asymmetric GARCH model, which has better fit in sample by construction, but a more complicated model could be capturing more noise than signal. In fact, the results without control variables in Table 1.2 show that the GARCH model leads to a better fit to the mean-variance relation as measured by R^2, with 0.027 compared to 0.016 by the asymmetric GARCH.

Table 1.3: Mean-Variance Relation with Control Variables

	(1)	(2)	(3)	(4)	(5)
Intercept	2.6941	0.5186	0.8262	0.7929	2.5478
	(2.262)	(1.269)	(2.647)	(2.578)	(1.597)
$Var_t(R_{t+1})$	−0.0112	−0.0116	−0.0111	−0.0097	−0.0105
	(−2.733)	(−2.707)	(−2.527)	(−2.261)	(−2.186)
D_t	−0.5733	−0.5717	−0.6559	−0.7102	−0.6525
	(−1.437)	(−1.380)	(−1.651)	(−1.740)	(−1.551)
$D_t \times Var_t(R_{t+1})$	**0.0576**	**0.0569**	**0.0584**	**0.0637**	**0.0637**
	(5.119)	(4.595)	(5.179)	(5.057)	(4.900)
D/P Ratio	Yes				Yes
Default Spread		Yes			Yes
Investor Sentiment			Yes		Yes
Lagged Variance				Yes	Yes
R^2	0.030	0.029	0.028	0.028	0.033

The results in Table 1.3 are strikingly similar to that of the GARCH specification without control variables. First, the coefficient estimates for the conditional variance term are all significantly negative at least at the 5% level. This means that the negative mean-variance relation following a loss month persists after accounting for business cycle variables, investor sentiment, and lagged variance. Second, the interaction term of the gain indicator and the conditional variance has a significantly positive estimate across different specifications. Moreover, the F-tests reject the null hypothesis $b_1 + b_2 = 0$ with a p-value less than 0.01. Therefore, the mean-variance relation is positive after a past gain, holding other variables constant. Our results show that none of the control variables is able to explain the alternating sign of the mean-variance relation, and prospect theory turns out to be the only explanation for this special phenomenon.

To evaluate the economic importance of our findings, consider the model with all control variables included. The coefficient estimate is -0.0097 for the conditional variance term, and 0.0637 for the interaction term. These estimates imply that if the market has experienced a loss in the current month, then a one standard deviation increase in the conditional variance is associated with a decrease of -0.44% (not annualized) in the expected return next month, all else being equal, while a same increase in the conditional variance is associated with an increase of 2.25% in the expected return next month, if the current month achieves a gain. Again, our results are both statistically significant and economically important after controlling for other explanatory variables, and demonstrate that the direction of the mean-variance relation depends on the market's past gain or loss, consistent with the predictions of prospect theory.

1.3.4 Gain Proxy

We have so far used as proxy for a typical investor's gain or loss with respect to her reference point the current month's market excess return. Therefore, we make the assumption that the typical investor evaluates her portfolio performance at the end of each month, with respect to her wealth level at the end of last month (adjusted for the risk-free return), and her willingness to pay for the market portfolio depends on both her gain/loss and the volatility of the portfolio going forward. Then her wealth level at the end of current month (adjusted for the risk-free return) becomes her reference point for the next month, and so on so forth. Although the one month evaluation frequency is a natural choice, prospect theory itself does not provide us any clear guidance where investors' reference point should be, and therefore our choice of the gain proxy is somewhat arbitrary in nature. In this subsection we explore the robustness of our findings to the choice of gain proxy. Specifically, we redefine our indicator for gain as:

$$D_{t-l,t} = \begin{cases} 1 & \text{if } R_{t-l,t} \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

where $R_{t-l,t}$ is the market excess return from month $t-l$ to month t, and l is the look-back horizon. For example, when $l = 1$, this definition coincides with our previous one, i.e., $D_{t-1,t} = D_t$, as both indicate whether the current month is a gain month or not. We take l to be 1, 2, 3, 6, and 12, which correspond to a monthly, bimonthly, quarterly, half-yearly, and yearly evaluation frequency, respectively. Table 1.4 presents the results for the following regression equation:

$$R_{t+1} = a_1 + b_1 Var_t(R_{t+1}) + a_2 D_{t-l,t} + b_2 D_{t-l,t} Var_t(R_{t+1}) + controls + \varepsilon_{t+1}$$

t-statistics are shown in the parentheses. We use the GARCH forecast as the proxy for conditional variance, and include the full set of control variables for all the specifications.

The only difference here than specification (5) in Table 1.3 is the proxy used for the gain indicator. The second column uses a lag of one month for the indicator, which is identical to specification (5) in Table 1.3, and we show it here for comparison purposes.

Table 1.4: Mean-Variance Relation Using Different Gain Proxies

Lag (months)	1	2	3	6	12
Intercept	2.5478	1.9735	1.9081	1.6130	1.6674
	(1.597)	(1.235)	(1.198)	(1.009)	(1.052)
$Var_t(R_{t+1})$	−0.0105	−0.0126	−0.0129	−0.0106	−0.0192
	(−2.186)	(−2.558)	(−2.619)	(−2.132)	(−2.724)
$D_{t-l,t}$	−0.6525	−0.7733	−0.6602	−0.0760	0.7331
	(−1.551)	(−1.840)	(−1.554)	(−0.168)	(1.610)
$D_{t-l,t} \times Var_t(R_{t+1})$	**0.0637**	**0.0319**	**0.0362**	**0.0173**	**0.0171**
	(4.900)	(2.818)	(3.231)	(1.587)	(2.069)
Controls	Yes	Yes	Yes	Yes	Yes
R^2	0.033	0.016	0.018	0.012	0.021

We observe in Table 1.4 very similar mean-variance patterns seen in Table 1.3. Again, following a period of loss, investors demand a lower expected return for a higher conditional variance, as the coefficient estimates on the conditional variance term are all highly significantly negative. Additionally, all the interaction terms except the one with a lag of 6 months are significantly positive, meaning that investors always require a higher expected return from the market when they face a prior gain than a loss no matter how far backward (up to a year) the gain/loss is calculated, given a fixed level of conditional variance going forward. Notice that even the specification with a 6-month lag barely misses the significance test. Given these robust results, we conclude that even though investors may use different reference points while evaluating their portfolio performance, they always behave in a risk seeking way after a prior loss, and less so after a prior gain.

For completeness, we include in Table 1.5 the regression results of the two-regime model using gain proxies with even longer look-back period up to five years. Interestingly, neither the coefficients on the conditional variance nor those on the interaction term are significantly different from zero in any of the specifications. Our interpretation is that investors tend to change their reference points quite frequently, and they do not seem to look back too far in the past to define gains and losses. However, we do recognize that the literature on the reference points investors use to evaluate past performance is rather scarce, and we leave the question for future research.

Table 1.5: Mean-Variance Relation Using Different Gain Proxies (Continued)

Lag (months)	24	36	48	60
Intercept	1.1212	1.7569	0.6908	0.8778
	(0.689)	(1.047)	(0.401)	(0.525)
$Var_t(R_{t+1})$	-0.0052	-0.0021	-0.0056	-0.0056
	(-0.660)	(-0.279)	(-1.101)	(-1.104)
D_t	0.4479	-0.1059	0.7069	0.5337
	(0.892)	(-0.205)	(1.317)	(1.025)
$D_t \times Var_t(R_{t+1})$	-0.0051	-0.0106	-0.0202	-0.0207
	(-0.566)	(-1.196)	(-1.797)	(-1.830)
Controls	Yes	Yes	Yes	Yes
R^2	0.010	0.011	0.012	0.012

1.3.5 Nonlinear Mean-Variance Relation

The analysis we have done so far has focused on the decreasing sensitivity component of the prospect theory utility function. We demonstrated that when investors are in their loss region and hence risk seeking, they are more willing to be exposed to a more volatile stock market, and hence the mean-variance relation is generally negative after investors have experienced past losses. However, there is another subtle implication of the prospect theory when we take into account the magnitude of the past loss. Specifically, the loss aversion component of the prospect theory suggests that there is a kink in the utility function at the reference point, and that the marginal utility an investor gets from an extra gain beyond the kink is smaller than the marginal disutility she suffers from an extra loss. Thus, even though the prospect theory utility function is convex in the loss region, when a future gain is so large that it brings the investor back to her previous reference point and even beyond, the extra boost in her utility will become smaller and smaller. In this situation, she will not value the volatility of the market as much as when the volatility is smaller.

The loss aversion component therefore drives the mean-variance relation towards the positive direction, the opposite to the decreasing sensitivity component. The overall direction of the mean-variance relation depends on which of the two driving forces dominates, which in turn depends on the magnitude of the conditional variance compared to the distance the investor is away from her reference point. When the market's conditional variance is small relative to the magnitude of the investor's prior loss, it is rather unlikely that a future gain brings the investor back and past the previous reference point. When this happens, the decreasing sensitivity dominates the loss aversion, and the mean-variance relation is negative. On the other hand, when the conditional variance is relatively large, there is higher chance that the kink at the reference point will come into play, so the loss aversion dominates and the mean-variance relation will become positive.

In this subsection, we test whether the relationship between conditional mean and conditional variance of market returns are indeed nonlinear as predicted by the prospect theory. To be specific, we run the following regressions, adding quadratic terms of conditional variance to our previous specification, and replace the gain indicator D_t with R_t:

$$
\begin{aligned}
R_{t+1} = {} & a_1 + b_1 Var_t(R_{t+1}) + c_1 Var_t^2(R_{t+1}) \\
& + a_2 R_t + b_2 R_t Var_t(R_{t+1}) + c_2 R_t Var_t^2(R_{t+1}) + controls + \varepsilon_{t+1}
\end{aligned}
$$

The reason we use R_t instead of D_t is that we want to explore the impact of the magnitude of past losses on the mean-variance relation. Before we move on to the results, it is worth noting a numerical issue in the regression. The issue is caused by the heavy-tail distribution of both the market excess returns and the conditional variance estimates, especially during market downturns, as shown by their high kurtosis in Table 1.1. Therefore, it is possible that our results will be driven mainly by a few outliers in the sample. For example, our sample includes the 1929 and 1987 market crashes, as well as the recent financial crisis. This issue is more serious here than in our previous specifications because now we have the square terms of conditional variances, the interaction of excess returns and conditional variances, and the interaction of excess returns and the squared conditional variances. The high correlation between the magnitude of current month's market return and the conditional variance makes things even worse. To make our results more robust to outliers, we make the following adjustments. First, we exclude the interaction term between R_t and $Var_t^2(R_{t+1})$, which is the most vulnerable to extreme observations. In fact, this interaction term turns out to be not significant in most of our specifications, so there is no loss of generality here. Second, we use the rolling window model as our proxy for conditional variances, as it generates less heavy-tailed estimates. For instance, in a loss month, the conditional variance estimates given by the two GARCH models have a kurtosis of 14.90 and 13.92 respectively, while the rolling window model estimates only have a kurtosis of 8.70. Third, we re-estimate our model after removing the most extreme data points and check the robustness of our results.

Table 1.6 presents the regression results after the adjustments mentioned above, where t-statistics are shown in parentheses. Specification (1) uses the full sample, while specifications (2) through (6) leave out the lower 1%, 2%, 3%, 4%, 5% months in terms of market excess returns. The corresponding cutoff points for the monthly excess returns are −20.36%, −16.13%, −13.23%, −12.74%, and −11.27%, respectively.

Table 1.6: Nonlinear Mean-Variance Relation

	(1)	(2)	(3)	(4)	(5)	(6)
Intercept	1.9001	1.9875	1.5249	1.6162	1.6931	1.4263
	(1.191)	(1.249)	(0.962)	(1.021)	(1.069)	(0.902)
$Var_t(R_{t+1})$	−0.0078	−0.0192	−0.0299	−0.0271	−0.0274	−0.0343
	(−0.717)	(−1.587)	(−2.350)	(−2.118)	(−2.120)	(−2.595)
R_t	−0.0082	0.0050	0.0141	0.0294	0.0281	0.0402
	(−0.179)	(0.111)	(0.309)	(0.641)	(0.604)	(0.858)
$R_t \times Var_t(R_{t+1})$	**0.1523**	**0.1824**	**0.1391**	**0.1368**	**0.1337**	**0.1264**
$(\times 10^{-2})$	(3.715)	(4.351)	(3.114)	(3.029)	(2.898)	(2.657)
$Var_t^2(R_{t+1})$	**0.4458**	**0.7986**	**1.1693**	**1.1242**	**1.1316**	**1.3806**
$(\times 10^{-4})$	(1.588)	(2.559)	(3.317)	(3.162)	(3.168)	(3.522)
Controls	Yes	Yes	Yes	Yes	Yes	Yes
R^2	0.031	0.039	0.034	0.035	0.034	0.038

The results in Table 1.6 show several interesting findings that are consistent with our conjectures. First, the coefficient estimates for the squared conditional variance is significantly positive for all but one specification. This documents that there is indeed nonlinearity in the mean-variance relation. The only specification where the quadratic term does not have a significant coefficient is the one with the full sample, which again corroborates our belief that outliers may alter the results. However, by eliminating different portions of the left tail observations, our results are fairly robust across specifications (2) through (6). Second, the R^2 for the regressions on different subsamples increase first and then decrease, with the highest being for specification (2). This means that our model achieves the best fit when the lower 1% observations are left out. Third, for specification (2), the coefficient for $Var_t(R_{t+1})$ is not significantly different from zero. Notice that this number together with the quadratic coefficient determines the axis of symmetry of the quadratic function when the current month's excess return is zero. An estimate close to zero indicates that the axis of symmetry stays at $Var_t(R_{t+1}) = 0$ when there is no gain or loss. Therefore, as conditional variance increases, the required return also increases, establishing a positive mean-variance relation. This is not surprising because when investors are at the reference point, loss aversion plays the major role in determining her appetite for risk.

Last but not least, the coefficients for the interaction term of current excess return and conditional variance are all highly significantly positive across all specifications. This number determines how the axis of symmetry of the quadratic function moves around when the current month's return changes. A positive estimate implies that when the current month experiences a gain, the axis of symmetry is to the left of zero, and hence the mean-variance relation is positive. This is again consistent with prospect theory's prediction, because while in the gain region, both loss aversion and decreasing sensitivity make the investor unwilling to hold volatile assets. More importantly, when the current month is a loss month, the axis of symmetry is to the right of zero. With a positive quadratic estimate, this means that the expected return investors demand from the market decrease first with the conditional variance, while it starts to increase when the conditional variance goes beyond the axis of symmetry. Moreover, the turning point where the mean-variance relation switches sign depends on the magnitude of the current month's loss. As the magnitude of the loss increases, the axis of symmetry shifts further to the right, requiring a higher conditional variance to attain the positive mean-variance relation.

Figure 1.4 helps better visualize our results by plotting the mean-variance relation when the magnitude of prior gains and losses differs. If investors face a prior gain of 5%, our estimates suggest a mean-variance relation given by the solid line, a quadratic function with the axis of symmetry to the left of the vertical axis. Since conditional variance can only be positive, our results imply a positive mean-variance relation. The dashed line corresponds to the case where investors have experienced a moderate past loss of 5%. Now the axis of symmetry resides to the right of the vertical axis, such that the market return required by investors first decrease and then increase with conditional variance, as the importance of loss aversion becomes stronger and that of decreasing sensitivity becomes weaker. If the past loss is at a higher level of 10%, the axis of symmetry moves further away to the right. The expected return still goes down first and up afterwards as conditional variance increases. However, since now the investor is farther away from her reference point, decreasing sensitivity dominates loss aversion for the most part, until conditional variance is high enough to reach the turning point. All these results are exactly the same as the prospect theory predicts: when the conditional variance is relatively small compared to the past loss, decreasing sensitivity dominates loss aversion, and investors are willing to take the market risk in order to break even; when the conditional variance is too large and loss aversion becomes more relevant, any extra possible gain is not as attractive any more, and investors become averse to the market risk.

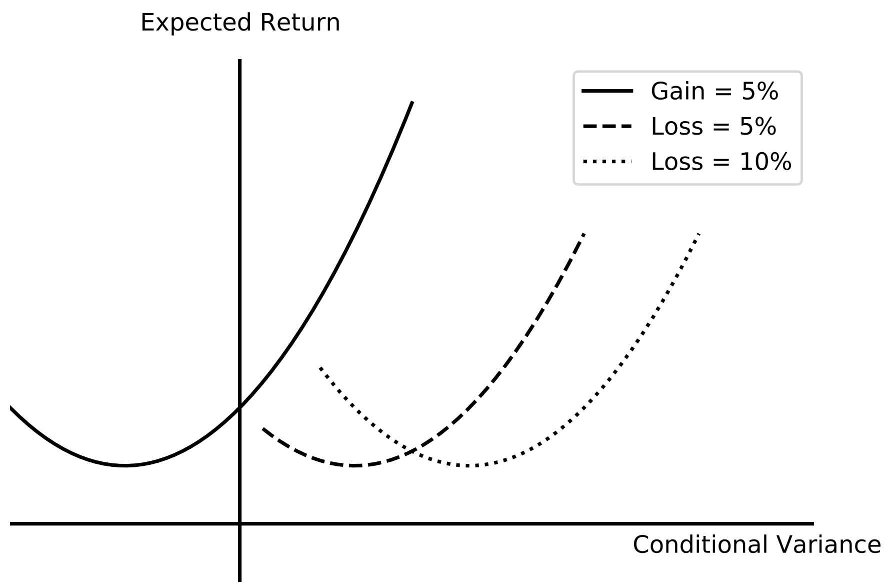


Figure 1.4: Nonlinear Mean-Variance Relation

1.3.6 Out-of-sample Prediction

Our results have shown that although neither past returns nor conditional variances alone can predict future market returns, jointly they have strong predictive power for the equity premium, at least in sample. Table 1.2 shows that the in-sample R^2 of predictive regressions when using different conditional variance proxies range from 1.3% to 2.7%, while the corresponding univariate regressions have essentially no predictive power whatsoever. It is unfair to compare the in-sample R^2 of nested regressions with different numbers of predictors, as the high R^2 associated with more predictors could well be an artifact of overfitting noises. In fact, Welch and Goyal (2008) test the performance of various equity premium predictors suggested by the literature, and find that they have done very poorly out of sample. In this subsection, we address the concern of overfitting, and examine our predictive regressions in a similar way to Welch and Goyal (2008). Specifically, we estimate the parameters of our predictive regression every period, using information only up to that period, make prediction of the equity premium for the next period, and calculate the out-of-sample R^2 using the following formula suggested by Campbell and Thompson (2008):

$$R^2_{OS} = 1 - \frac{\sum_{t=1}^{T}(r_t - \hat{r}_t)^2}{\sum_{t=1}^{T}(r_t - \bar{r}_t)^2}$$

where $\hat{r}_t$ is the fitted value from a predictive regression estimated through period $t-1$, and $\bar{r}_t$ is the historical average return estimated through period $t-1$. The out-of-sample R^2 thus directly measures the proportional decrease in mean-squared-error of a predictive model compared to simple expanding window averages. A positive R^2_{OS} implies that the predictive model does better than simple average, and a negative value means the opposite.

We require at least 30 years of data to obtain the regression coefficient estimates. Therefore, VIX is excluded from our analysis and we only use the other three conditional variance proxies. Table 1.7 shows the out-of-sample R^2 in the one-regime and the two-regime models with different conditional variance proxies. Columns 2 through 4 correspond to the R^2 for the full sample, the subsample following gain months, and the subsample following loss months, respectively. It is not surprising that the magnitude of these out-of-sample R^2's are fairly small, because monthly returns contain a substantial unpredictable component. However, a monthly R^2 around 0.5% represents an economically important degree of equity premium predictability, as illustrated in Campbell and Thompson (2008). We can see from Table 1.7 that over the full sample, the two-regime model achieves a much higher out-of-sample R^2 (at a level around 0.5%) than the one-regime model across different specifications. Moreover, the R^2_{OS}'s of the two-regime model are all positive, meaning that the gain indicator together with the conditional variance predicts market excess returns better than simple historical averages. Therefore, the two-regime model suggested by prospect theory not only explains the time-varying mean-variance relation in sample, but also attains superior predictive performance out of sample compared to the one-regime model and the constant equity premium model.

Table 1.7: Out-of-sample R^2 of Predictive Regressions

	Full Sample	Gain Months	Loss Months
Panel A: Rolling Window			
One-Regime	−0.43%	−0.07%	−0.67%
Two-Regime	0.49%	−0.87%	1.39%
Panel B: GARCH			
One-Regime	0.07%	−0.20%	0.25%
Two-Regime	0.56%	−0.80%	1.46%
Panel C: Asymmetric GARCH			
One-Regime	0.20%	−0.23%	0.48%
Two-Regime	0.56%	−0.31%	1.14%

Another interesting finding is that the higher predictive power of the two-regime model concentrates on the subsample following loss months, with a R^2 greater than 1% across different specifications. In contrast, following gain months, the two-regime model does no better than the one-regime model nor the simple average. This fact again corroborates our belief that decreasing sensitivity is important in understanding investors' attitude toward risk, because it is exactly after prior losses that decreasing sensitivity drives the negative mean-variance relation, which in turn leads to the better out-of-sample predictive performance of our model.

1.3.7 Gain/Loss Timing

We have demonstrated the importance of the decreasing sensitivity component of prospect theory in determining the mean-variance relation of the overall stock market. To sum up, investors are only compensated with a higher expected return for a higher conditional variance after the market has had a prior gain, but there is no commensurate compensation after prior losses. Another way to test whether this is true in the data is to look at the profitability of a market timing strategy based on prior gains and losses. Specifically, if one is not well compensated for bearing risk after prior losses, she should avoid the market in such situations. A very simple but natural strategy is to hold the market portfolio after a gain month, but to switch to the risk-free asset after a loss month. Figure 1.5 gives the reader an intuitive idea why this strategy might work. We divide the whole sample into three parts based on monthly market excess returns, with the cutoff points at the lower and upper 30% quantiles. Then we calculate the Sharpe ratios of the market portfolio in the next month in each of these three subsamples. We can see from Figure 1.5 that the Sharpe ratio increases monotonically from 0.25 (annualized) following months with lowest returns, to as high as 0.70 following months with highest returns. Therefore, our simple strategy gets exposed to the market when exactly its Sharpe ratio is high.

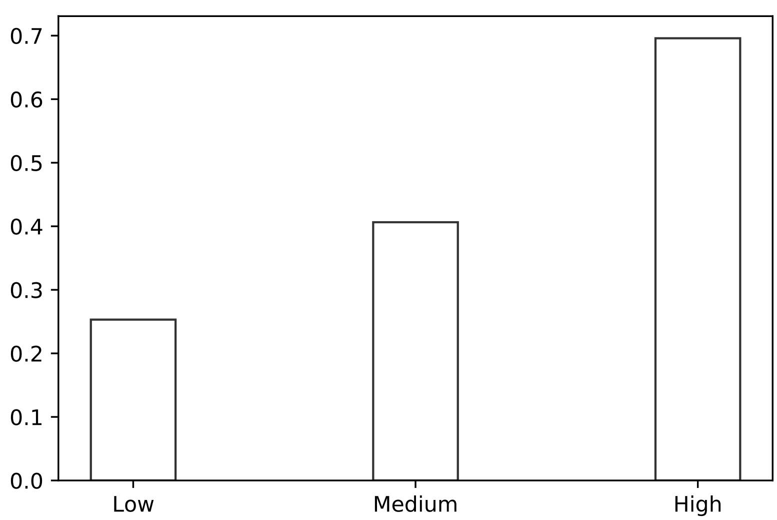


Figure 1.5: Sharpe Ratio by Previous Month's Return

We further evaluate the performance of our gain/loss timing strategy within benchmark factor models. Table 1.8 shows the decomposition of our gain/loss timing strategy in CAPM and the Fama-French three factor model. Even though the strategy does not involve stock picking, and hence we do not expect our returns to depend on the size and value factors, we include the Fama-French results here for completeness. We can see from Table 1.8 that our timing strategy generates statistically significant alphas in both of the models. In the market model, the gain/loss timing strategy has a positive alpha of 0.22% per month, which amounts to 2.59% per year. The strategy does have a significant loading on the market factor, which is not surprising given it holds the market portfolio over half of the time. In the Fama-French model, the loading on the market factor does not change much. Even though our strategy loads positively on the value factor, it still generates a significantly positive alpha of 0.19% per month, or 2.27% per year. Besides the positive alphas in factor models, our strategy also has an annualized Sharpe ratio of 0.50, higher than the market Sharpe ratio of 0.43 over the same period. Therefore, the strategy successfully utilizes the positive mean-variance relation after gain months and avoids the flat or negative mean-variance relation after loss months, and achieve a better risk-return tradeoff.

Table 1.8: Gain/Loss Timing against Factor Models

	α	Market	Size	Value	R^2
CAPM	0.2160	0.4891			0.487
	(2.505)	(10.469)			
Fama-French	0.1892	0.4631	0.0672	0.0852	0.496
	(2.230)	(10.949)	(1.210)	(1.937)	

We emphasize here that our strategy is different from several other market timing strategies in the literature. Moskowitz et al. (2012) find that the past 12-month excess return of equity indexes positively predicts its future return, and thus a timing strategy based on the market excess return over the past year generates alpha. Our strategy differs from theirs in two dimensions. First, the role of past gains and losses in our model is in predicting the tradeoff between risk and return going forward, rather than future returns themselves. In fact, neither the gain indicator D_t nor the raw excess return R_t has significant predicting power for future returns R_{t+1} in any of our specifications. Instead, past gains and losses do predict the direction of future mean-variance relation. Second, we use a one-month look-back period rather than a one-year horizon. It is worth noting that the time-varying mean-variance relation still holds when we use a 12-month gain proxy, as shown in Table 1.4. Moreover, a gain/loss timing strategy based on the 12-month gain proxy also generates positive alpha.[1] However, we stick to the one-month look-back period to distinguish our strategy from the time series momentum strategy, because their strategy does not work over such horizon.

Another strategy we want to compare ours with is the volatility timing strategy proposed by Moreira and Muir (2017). They find that the risk-return tradeoff is weak when market volatility is high, so they suggest leveraging the market portfolio in inverse proportion to the current month's realized volatility. Their strategy is similar to ours in the sense that it is based on the prediction of future risk-return tradeoff rather than return itself. Additionally, it is well known that when market goes down, its volatility tends to be high, the so-called "leverage effect". Thus, by holding the market when it goes up while avoiding it when it goes down, it could be that we are simply timing the market based on realized volatility indirectly. To investigate the relationship between our gain/loss timing strategy and the volatility timing strategy, we divide the whole sample into four parts based on the gain dummy and the low volatility dummy, where low volatility is defined as the volatility being below its unconditional median. Table 1.9 shows the summary statistics for these four

1. The results are not shown here, but are available upon request.

subsamples. We can see from the table that gain tends to coincide with low volatility while loss tends to coincide with high volatility, which is consistent with the "leverage effect". However, notice that the correlation between the gain and the low volatility indicator is not very high: we still have fair amount of observations of high volatility with a gain (275), and of low volatility with a loss (153). Row three shows the average realized volatility in the four subsamples, with higher values in high volatility periods by construction. Row four shows the average leverage that the volatility timing strategy assigns to the market portfolio in the four different scenarios, and row five shows the Sharpe ratios of the market in the months following these scenarios. Clearly, the volatility timing strategy applies a high leverage after a low-volatility month and a low leverage after a high-volatility month, which enables it to harvest the high Sharpe ratio of 0.75 following a gain and low-volatility month, while escaping the low Sharpe ratio of 0.07 following a loss and high volatility month. However, it misses the chance after a gain and high volatility month, by placing only a leverage of 0.49 on average, when the Sharpe ratio of the market is reasonably high, at 0.62 in such situation. In contrast, our results suggest that after a prior gain, even if the current realized volatility is high, the mean-variance relation is still positive going forward, so we should take the market risk as we will be well compensated by a commensurate expected return. Therefore, although both our strategy and the volatility timing strategy try to time the market based on the prediction of the future mean-variance relation, their exposure to the market differs substantially over various subperiods of our sample.

Table 1.9: Gain/Loss v.s. Realized Volatility

	Gain & Low Vol	Gain & High Vol	Loss & Low Vol	Loss & High Vol
# of Obs.	408	275	153	286
Realized Vol	8.13%	19.62%	8.77%	24.11%
Avg. Leverage	2.18	**0.49**	**1.69**	0.40
Sharpe Ratio	0.75	**0.62**	**0.56**	0.07

The fact that both our gain/loss timing strategy and the volatility timing strategy invest heavily in the market when the risk-return tradeoff is superior and that they do not overlap with each other too much motivates us to combine the two strategies to achieve better performance. Here, we examine one simple strategy that incorporates both signals. Specifically, we apply a leverage to the market portfolio that is inversely proportional to current month's realized variance as in Moreira and Muir (2017), and we double this leverage when current month has had a gain rather than a loss. Table 1.10 evaluates the performance of both the volatility timing alone and the gain/loss augmented volatility timing strategy within the framework of factor models. Both strategies are scaled such that they have the same unconditional volatility as the market. We can see that these two strategies both generate significantly positive alphas against CAPM and the Fama-French three factor model. Moreover, in both benchmarks, the volatility and gain/loss combined timing strategy has alphas that are about 0.1% per month higher than the volatility timing strategy alone, equivalent to a 1.2% boost in annualized alpha. In terms of Sharpe ratio, the combined timing strategy is superior to either single strategy alone, with a Sharpe ratio of 0.55 compared to 0.52 of the volatility timing strategy and 0.50 of the gain/loss timing strategy.

The differences in alpha as well as in Sharpe ratio turn out to matter a lot for the portfolio performance. Figure 1.6 shows the compounded excess return (over risk-free return) of different strategies over our sample period, from 1926 to 2019. The solid line corresponds to the market portfolio, the dashed line to the volatility timing strategy, and the dotted line to the volatility and gain/loss combined timing strategy. All returns are in log scales on the base of 10. One dollar invested in the market since 1926 would have turned into \$345.80 in 2019. In comparison, the volatility timing strategy would have turned one dollar in 1926 to \$1582.66 in 2019. Moreover, combining our gain/loss timing strategy with the volatility

Table 1.10: Volatility and Gain/Loss Timing

Model	α	Market	Size	Value	R^2
Panel A: Vol Timing					
CAPM	0.3951	0.6103			0.372
	(2.887)	(7.444)			
Fama-French	0.4363	0.6429	−0.0533	−0.1455	0.382
	(3.134)	(9.187)	(−1.024)	(−1.715)	
Panel B: Vol & G/L Timing					
CAPM	0.5069	0.5124			0.263
	(3.388)	(7.051)			
Fama-French	0.5354	0.5389	(−0.0637)	(−0.0933)	0.268
	(3.577)	(8.307)	(−1.239)	(−1.179)	

timing strategy boosts the portfolio even further, turning one dollar to $2595.46 over the same period. Therefore, the small differences in annualized alpha and Sharpe ratio make a huge difference over the horizon of nearly 100 years. Notice also that the combined timing strategy actually started to outperform the volatility timing strategy in late 1960s. The two timing strategies also did well during several market downturns in the history, such as the Great Depression, and the recent 2007-08 financial crisis.

The superior returns of our gain/loss timing strategy as well as the volatility augmented gain/loss timing strategy demonstrate indirectly the importance of the time-varying mean-variance relation over our sample period. By investing heavily in the market after past gains while decreasing market exposure after past losses, investors are able to exploit better risk-return tradeoff and accumulate substantially more wealth than if they have held the market passively.

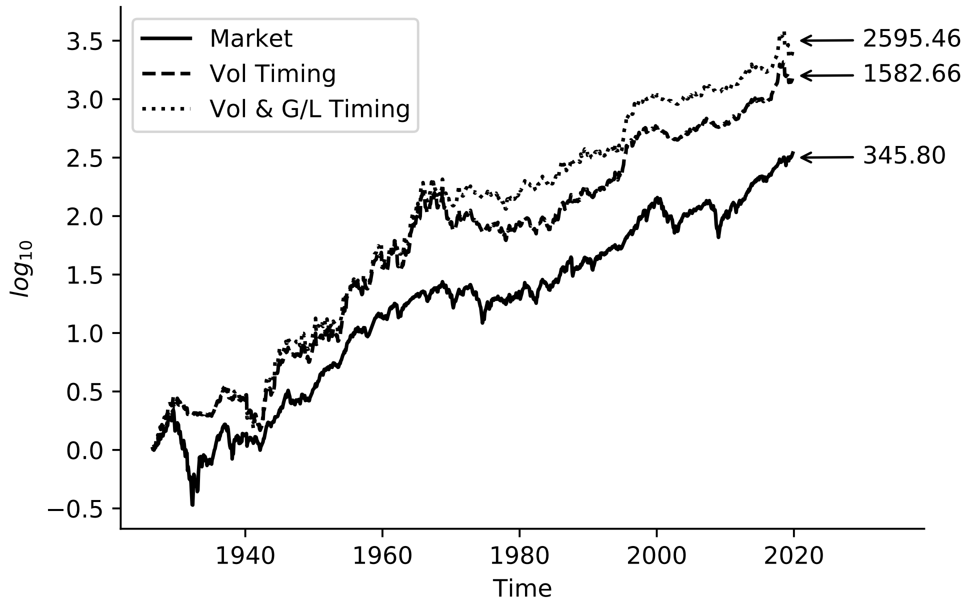


Figure 1.6: Compounded Excess Return (Log Scale)

1.3.8 Sector-Level Analysis

We have so far focused our analysis on the mean-variance relation for the overall stock market. However, it could well be the case that investors evaluate their gains and losses in different segments of the stock market separately. Consequently, decreasing sensitivity will cause a time-varying mean-variance relation in each segment similar to that we observe for the aggregate market. This kind of narrow framing/mental accounting argument is widely used in the literature to study the implications of prospect theory in the cross section; see Frazzini (2006), Li and Yang (2013), and Wang et al. (2017), to name a few. In this subsection, we examine the mean-variance relation at the sector level, and propose a sector timing strategy that utilizes this time-varying relation.

We obtain the data on the daily and monthly returns of 12 industry portfolios from Kenneth French's website. Industries are defined by companies' four-digit SIC codes. We choose the number of 12 because it is the closest to the division of S&P 500 stocks into 11 sectors, which is the basis of sector timing strategies used by a lot investors. The reason that we do not use the S&P 500 sector returns directly is because the division began in 1999, while our data dates back to 1927, and we want to study the sector level mean-variance relation over a longer period that is comparable to our findings about the aggregate market. Table 1.11 shows a detailed description of the 12 industry portfolios.

Table 1.11: Definitions of Industry Portfolios

Industries	Abbreviation
Consumer Nondurables – Food, Tobacco, Textiles, Apparel, Leather, Toys	NoDur
Consumer Durables – Cars, TVs, Furniture, Household Appliances	Durbl
Manufacturing – Machinery, Trucks, Planes, Off Furn, Paper, Com Printing	Manuf
Oil, Gas, and Coal Extraction and Products	Enrgy
Chemicals and Allied Products	Chems
Business Equipment – Computers, Software, and Electronic Equipment	BusEq
Telephone and Television Transmission	Telcm
Utilities	Utils
Wholesale, Retail, and Some Services (Laundries, Repair Shops)	Shops
Healthcare, Medical Equipment, and Drugs	Hlth
Finance	Money
Other – Mines, Constr, BldMt, Trans, Hotels, Bus Serv, Entertainment	Other

To test whether the mean-variance relation within each sector also depends on prior gains and losses in that sector, we run the regression in (1.2), with the excess returns and the corresponding gain indicator defined for each sector separately. Table 1.12 presents the regression results. We can see from the table that in all but two industries, the coefficient estimates on the interaction term of the gain indicator and the conditional variances are significantly positive. On the other hand, the coefficients on the conditional variances alone are all negative, some of which significantly so. This means that at the sector level, the mean-variance relation is either negative or flat after prior sector losses, and that investors in these sectors are better compensated for bearing volatility risk after prior gains than prior losses. Therefore, the results are very similar to our previous findings about the overall stock market, and are also consistent with the prediction made by prospect theory together with narrow framing.

Our results then motivate a natural strategy that refrains from investing in industries that have experienced recent losses. Specifically, we put equal weights on industries that have gains in the past month, and zero weight on industries with losses. When all industries have losses, we switch to the risk-free asset. Figure 1.7 shows the cumulative excess returns for this sector gain/loss timing strategy over the past one hundred years. As a comparison, we also include the performance of the market, and a equally weighted sector portfolio. Our sector gain/loss timing strategy achieves a higher Sharpe ratio of 0.53 during this period, compared to 0.43 of the market, and 0.47 of the equally weighted portfolio. One dollar invested in 1927 in the sector timing portfolio would have turned into $2115.72 at the end of 2019, substantially higher than $345.80 provided by the market and $654.00 by the equally weighted portfolio.

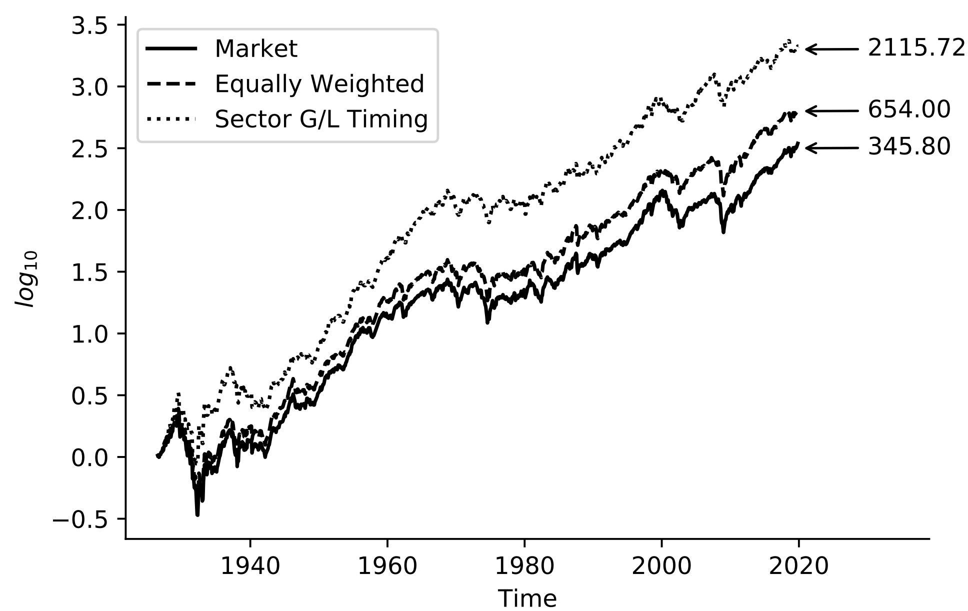


Figure 1.7: Compounded Excess Return (Log Scale)

Table 1.12: Mean-Variance Relation at Sector Level

	NoDur	Durbl	Manuf	Enrgy	Chems	BusEq	Telcm	Utils	Shops	Hlth	Money	Other
Intercept	0.6560	0.2253	0.7895	0.8430	0.9992	0.9222	0.6288	0.1130	0.5929	0.8419	0.3257	0.0302
	(2.794)	(0.576)	(2.340)	(2.778)	(3.476)	(2.363)	(2.660)	(0.419)	(1.938)	(2.832)	(0.958)	(0.090)
$Var_t(R_{t+1})$	−0.0122	−0.0006	−0.0089	−0.0043	−0.0057	−0.0062	−0.0076	−0.0034	−0.0073	−0.0042	−0.0041	−0.0044
	(−2.520)	(−0.175)	(−2.554)	(−1.491)	(−1.717)	(−1.944)	(−2.280)	(−1.068)	(−1.600)	(−0.997)	(−1.543)	(−1.129)
D_t	−0.4281	0.1778	−0.7440	−0.2875	−0.8199	−0.3209	0.2118	0.6089	−0.0671	−0.1895	0.6910	0.5908
	(−1.381)	(0.342)	(−1.694)	(−0.660)	(−2.151)	(−0.622)	(0.668)	(1.713)	(−0.167)	(−0.482)	(1.558)	(1.326)
$D_t \times Var_t(R_{t+1})$	**0.0685**	**0.0211**	**0.0469**	**0.0120**	**0.0312**	**0.0195**	**0.0008**	**0.0205**	**0.0334**	**0.0165**	**0.0107**	**0.0277**
	(6.856)	(4.111)	(7.943)	(1.614)	(4.828)	(3.698)	(0.116)	(3.551)	(4.169)	(1.999)	(2.156)	(3.744)
R^2	0.044	0.030	0.061	0.003	0.021	0.014	0.007	0.021	0.019	0.004	0.010	0.023

1.4 Conclusion

This paper investigates one of the most fundamental questions in finance, the mean-variance relation of the overall stock market. Traditional asset pricing theories have hypothesized a positive relationship between the expected return of the market and its conditional variance, as investors require higher compensation for bearing more risk. However, the empirical evidence in the literature has been mixed at best. We study the mean-variance relation from the perspective of prospect theory, or its decreasing sensitivity and loss aversion components to be more specific. Prospect theory suggests that investors' attitude toward risk depends on their current state with respect to their reference point. They are risk averse when they have had a past gain and hence to the right of their reference point, but become risk seeking when they suffer a prior loss. Thus, whether investors demand a higher return for greater risk from the market depends on which side the investors are with respect to their reference point. Our empirical results are strikingly consistent with prospect theory's prediction. We find that the future mean-variance relation is negative if the current month has experienced a loss, and it reverses to be positive when the current month achieves a gain. Our findings are robust to controlling for business cycle variables, lagged realized variance, as well as investor sentiment. Moreover, using different look-back windows to define past gains and losses does not alter the results either.

We also test another more subtle prediction of the prospect theory in terms of the mean-variance relation, that not only decreasing sensitivity but also loss aversion drive investors' appetite toward risk. These two forces drive the mean-variance to opposite directions when the investors are in their loss region. We find a nonlinear mean-variance relation that is exactly the same as the prediction of prospect theory. When the magnitude of the market's conditional variance is relatively small compared to investors' prior losses, decreasing sensitivity dominates loss aversion, and the mean-variance relation is negative. However, when the magnitude of the conditional variance is large relative to past losses, loss aversion plays a major role, which reverses the mean-variance relation to be positive.

The time-varying pattern of the mean-variance relation has its practical importance for investors as well. We propose a simple market timing strategy based on the market's past gains and losses. The strategy suggests holding the market when the current month has experienced a gain and the future mean-variance relation is expected to be positive, and switching to the risk-free asset when the current month has suffered a loss and the mean-variance relation is weak. This simple strategy achieves a higher Sharpe ratio than the market, and generates significantly positive alphas in CAPM and Fama-French three factor models. Our gain/loss timing strategy is different in various dimensions from several other timing strategies in the literature such as the time series momentum strategy and the volatility timing strategy. Augmenting our timing strategy with the volatility timing strengthens its performance even further. Investors would have accumulated much more wealth if they had employed the combined timing strategy instead of holding the market passively.

1.A Proofs

We derive here the analytical solution for the monthly conditional variance, $Var_t(R_{t+1})$, assuming daily returns follow an asymmetric GARCH(1, 1) process below:

$$
\begin{aligned}
& r_t = \mu + \varepsilon_t \\
& \sigma_t^2 = \omega + \alpha\varepsilon_{t-1}^2 + \gamma\varepsilon_{t-1}^2 I_{[\varepsilon_{t-1}<0]} + \beta\sigma_{t-1}^2 \\
& \varepsilon_t = \sigma_t e_t, \quad e_t \sim N(0,1)
\end{aligned}
$$

The solution when the daily returns follow a symmetric GARCH(1, 1) process is a special case where $\gamma = 0$ and hence its derivation is omitted here.

The conditional variance of daily return d days ahead, i.e., r_{t+d} is:

$$
\sigma_{t+d}^2 = \omega + \alpha\varepsilon_{t+d-1}^2 + \gamma\varepsilon_{t+d-1}^2 I_{[\varepsilon_{t+d-1}<0]} + \beta\sigma_{t+d-1}^2
$$

Take expectation of both sides conditional on date t gives:

$$
\begin{aligned}
E_t(\sigma_{t+d}^2) &= \omega + \alpha E_t[E_{t+d-2}(\varepsilon_{t+d-1}^2)] + \gamma E_t[E_{t+d-2}(\varepsilon_{t+d-1}^2 I_{[\varepsilon_{t+d-1}<0]})] + \beta E_t[E_{t+d-2}(\sigma_{t+d-1}^2)] \\
&= \omega + (\alpha + \gamma/2 + \beta)E_t(\sigma_{t+d-1}^2) \qquad (1.4)
\end{aligned}
$$

where the first equation follows by the law of iterated law of expectation. The second equation holds because: first, $\varepsilon_{t+d-1} \sim N(0, \sigma_{t+d-1}^2)$; second, σ_{t+d-1}^2 is know at $t+d-2$; and third, by symmetry we have:

$$
E_{t+d-2}(\varepsilon_{t+d-1}^2 I_{[\varepsilon_{t+d-1}<0]}) = E_{t+d-2}(\varepsilon_{t+d-1}^2 I_{[\varepsilon_{t+d-1}\geq 0]}) = E_{t+d-2}(\varepsilon_{t+d-1}^2)/2
$$

Solve for equation (1.4) iteratively until time t, we get:

$$E_t(\sigma^2_{t+d}) = \frac{\omega[1-(\alpha+\gamma/2+\beta)^{d-1}]}{1-(\alpha+\gamma/2+\beta)} + (\alpha+\gamma/2+\beta)^{d-1}\sigma^2_{t+1}$$

The conditional variance at time t for the next month is given by the sum of the expected daily return variance for each of the D trading days next month:

$$\begin{aligned}
Var_t(R_{t+1}) &= E_t\left(\sum_{d=1}^{D}\sigma^2_{t+d}\right) \\
&= \sum_{d=1}^{22} E_t(\sigma^2_{t+d}) \\
&= \frac{\omega D}{1-(\alpha+\gamma/2+\beta)} - \frac{\omega[1-(\alpha+\gamma/2+\beta)^D]}{[1-(\alpha+\gamma/2+\beta)]^2} + \frac{1-(\alpha+\gamma/2+\beta)^D}{1-(\alpha+\gamma/2+\beta)}\sigma^2_{t+1}
\end{aligned}$$

Since σ^2_{t+1} is observable at time t, plugging in the estimates for $\{\mu,\omega,\alpha,\gamma,\beta\}$ gives the estimate for the monthly conditional variance.

Chapter 2

Time-Varying Loss Aversion and Probability Weighting

Abstract: The prospect theory suggests that people are subject to loss aversion and probability weighting when they make decisions involving risk, the degrees of which have been well documented in the experimental setting to vary in different situations. This paper estimates investors' time-varying loss aversion and probability weighting using financial data. We find that after the 2007-2008 financial crisis, investors are more averse to losses, and their tendency to overweight right tail events becomes weaker. Additionally, both loss aversion and probability weighting have a close relationship with investor sentiment.

2.1 Introduction

The prospect theory developed by Kahneman and Tversky (1979) and Tversky and Kahneman (1992) has been the major alternative to the expected utility theory as a model to describe economic agents' attitude towards risk. In an experimental setting, the authors find that people differ in two[1] main perspectives from the assumptions of the expected utility theory when making decisions involving risk. First, they derive value not from their absolute wealth level but from gains and losses defined relative to some reference point, and they suffer more from losses than they enjoy gains of the same magnitude. Second, people tend to overweight tail events even if they know the true probabilities of these events are small.

These two building blocks of the prospect theory, loss aversion and probability weighting respectively, have helped solve many puzzles in asset pricing that are not easily explained by the traditional "rational" asset pricing models, which are built upon the expected utility theory. To name a few, Benartzi and Thaler (1995) attribute the equity premium to investors' loss aversion, because the highly disperse distribution of stock market returns is unappealing to loss averse investors and they have to be compensated to hold the stocks by a much higher expected return than the risk-free return. In Barberis and Huang (2008)'s model, even idiosyncratic skewness that is not correlated with the market return is priced when the economy is populated with prospect theory investors. They claim that assets with positive skewness such as out-of-the-money options are attractive to investors because they overweight the states of the world that are not very likely to realize, and consequently these assets have high price and low expected returns.

Nonetheless, literature on gauging the parameters of the prospect theory is scarce. Most research has been in the experimental setting, which may not well describe how investors behave in the financial markets when real money is involved. To our best understanding, Baele et al. (2019) is the only paper that estimates the parameters of loss aversion and

1. The third feature of the prospect theory, namely diminishing sensitivity, is less important in our paper, so we omit the discussion here.

probability weighting using financial data, but the main focus of that paper is not on the dynamics of the parameters across time. Although the authors did calibrate their model allowing for time-varying loss aversion and probability weighting, they found that the improvement in the fit of the model was quantitatively small, and that static estimates were sufficient for their purposes, namely explaining the variance risk premium. However, as will be clearer in later part of this paper, a dynamic specification is crucial to understanding interesting phenomena other than variance premium.

At least two strands of literature motivate us to believe how averse people are to losses and how heavily they overweight tail events do vary in different situations. For instance, Thaler and Johnson (1990) find that people are more likely to take gambles they otherwise would not have taken if they have experienced prior gains, and the authors suggest the reason be that losses are less painful after prior gains, but more so after prior losses. A recent paper by Cohen et al. (2020) argues that people make choices relying on small samples of past experiences with similar decision tasks, and thus the probability weighting tendency also depends on their experiences.

On the other hand, the empirical asset pricing literature provides indirect evidence for time-varying loss aversion and probability weighting. For example, Yu and Yuan (2011) find that the positive relationship between the stock market's expected excess return and the market's conditional variance is stronger when the investor sentiment is low than otherwise. One explanation would be that investors are more loss averse when sentiment is low, and a higher variance of the market is more likely to put them in the loss region with respect to their reference point, so they require being compensated by a higher expected return; when the sentiment is high, investors are less loss averse, so they do not care about the variance so much and the correlation is weaker in consequence. Another example is Han (2008), who finds that the slope of the implied volatility of index options is steeper when market sentiment is low. In fact, the negative slope of the volatility smile itself shows evidence for probability weighting: out-of-the-money put options are valued more than their in-the-money counterparts by prospect theory investors because they overweight the tail event of a sharp market downturn, which the out-of-the-money options provide protection against at a relatively low cost. If the degree of probability weighting of investors changes over time and is heavier when the sentiment is lower, then the relationship between the slope of the volatility smile and the sentiment could be well explained. Green and Hwang (2012) also find a close relationship between investor sentiment and their preference over tails. They show that IPOs with high expected skewness experience significantly higher first-day returns during periods of high investor sentiment than otherwise.

Therefore, we believe loss aversion and probability weighting are indeed changing over time and are closely related to investor sentiment, and that we will be able to demystify a lot interesting phenomena like those mentioned above if we can learn more about the prospect theory parameters.

Since the analysis framework of this paper bears some similarity with Baele et al. (2019), we make clear here the major differences between the two papers. First, the main bulk of their paper focuses on using prospect theory to explain the variance premium, which can be expressed as a weighted average of expected returns from out-of-the-money put and call options. The variance premium puzzle is of course an interesting topic by itself, but by aggregating the returns of puts and calls across different moneyness into a weighted average, one misses the chance to see other patterns in the return data, e.g., the difference in returns between puts and calls, and between out-of-the-money options and in-the-money options[2]. In fact, these differences in returns are very important in helping us identify investors' degrees of loss aversion and probability weighting, as we will show later.

And that brings us to the second difference between our paper and Baele et al. (2019), that we put more emphasis on the time variation in investors' prospect theory preference. Though the authors did some analysis on the time-varying parameters, they found that the dynamic setting only improved the fit to the variance premium marginally compared to the static setting. Admittedly, if the variance premium puzzle is the main interest, constant prospect theory parameters suffice to match the data. However, since we aim to explain other patterns like those mentioned in the first point, the dynamics of these parameters are as important as, if not more so than their general levels.

Third, because of the importance of the time variation in loss aversion and probability weighting parameters, we take a closer look at how the parameters change with other economic variables. Specifically, besides raw returns of the market in the past as in their paper, we also studied other key features such as the skewness and kurtosis of past returns, though the results turned out to be not statistically significant. Nonetheless, we did find close relationships between prospect theory parameters and investor sentiment as well as its several components.

Last but not least, we propose a new estimation procedure that is very different from Baele et al. (2019)'s. In their paper, the authors did calibration instead of estimation in the dynamic setting, because their method falls out of the GMM (Generalized Method of Moments) framework, and consequently, standard errors of estimators are not available. By contrast, the new procedure of our paper still fits into the GMM framework, which allows us to draw more meaningful statistical inference on the structural parameters of the model.

2. Baele et al. (2019) discussed the latter point in the introduction of their paper, but did not continue to study its time-varying pattern in later analysis.

The rest of this paper is organized as follows. Section 2.2 introduces our representative agent model, section 2.3 describes our main empirical findings, and section 2.4 concludes.

2.2 The Model

We use a two-period representative agent model to describe the economy. The representative agent makes investment decisions in period 0 facing J risky assets and one risk-free asset. Following Barberis et al. (2001), and Baele et al. (2019), her total utility comes from two sources: the expected utility defined on her terminal wealth in period 1, and the cumulative prospect theory (CPT) utility defined on her gain (loss) in period 1 with respect to some reference level. That is, she solves the following maximization problem:

$$\max_{\alpha} EU(W_T) + b_0 V(X_T) \tag{2.1}$$

s.t. $W_T = W_0[R_f + \alpha'(R - R_f)]$, and $X_T = W_T - W_{Ref}$, where $EU(\cdot)$ is her expected utility defined on her terminal wealth W_T, $V(\cdot)$ is her prospect theory utility defined on her gain/loss X_T, W_0 is her initial wealth, and W_{Ref} is the reference level of wealth on which gain and loss are defined. We use $W_{Ref} = W_0 R_f$, or the wealth the agent will have if she invests in the risk-free asset only, with R_f being the risk-free rate. The representative agent chooses her allocation α to the J risky assets with gross return vector R, to maximize the weighted sum of the expected utility and the CPT utility, and b_0 is the relative weight she puts on the CPT utility. This utility specification avoids a common issue in modelling investors' preference with prospect theory alone, that the demand for risky assets is either infinity or zero. Also, it makes more intuitive sense that the investor cares not only about her gain or loss in her investment, but also about the absolute level of her wealth which is available for consumption.

Notice that our model is very similar to that in Baele et al. (2019), but one key difference is that the agent in our model faces J risky assets instead of one. Introducing multiple risky assets allows us to use assets other than the market index and its derivatives as test assets for the GMM estimation. Although in theory any asset could be used in the estimation, the availability of index options data is rather limited compared to stocks and portfolios. The data we have for index options (and in Baele et al. (2019) as well) only dates back to 1996, which greatly limits the possibility to use the data to study investors' time-varying preferences over a longer period of time. As we will see in section 2.3, the extra test assets in the stock market successfully help us capture the dynamics of prospect theory parameters and their relationship with investor sentiment.

The expected utility part satisfies constant relative risk aversion (CRRA) of degree γ, i.e., the utility function is given by:

$$U(W_T) = \begin{cases} \frac{W_T^{1-\gamma}}{1-\gamma} & \text{if } \gamma \neq 1 \\ \ln W_T & \text{if } \gamma = 1 \end{cases}$$

For the CPT part, the agent derives utility from gain/loss X_T in the following way:

$$v(X_T) = \begin{cases} X_T & \text{if } X_T \geq 0 \\ \lambda X_T & \text{if } X_T < 0 \end{cases}$$

where λ is the parameter of loss aversion and it is usually greater than 1. That is, the agent is more sensitive to losses than to gains: she suffers more from a loss than she enjoys a gain of the same magnitude, even when the magnitude is small. We do not model decreasing sensitivity here fore tractability purposes, but we study its implications in another chapter of the thesis (see "Prospect Theory and the Mean-Variance Relation"). The utilities in different states of the world are then weighted by decision weights instead of objective probabilities. To determine the decision weights, the agent first sorts the states of the world based on her wealth level: $W_1 \leq \cdots \leq W_{Ref} \leq \cdots \leq W_N$, with corresponding objective probabilities p_i, then she applies the following transformation to the cumulative probabilities to obtain decision weights π_i:

$$\pi_i = \begin{cases} w^-(p_1 + \cdots + p_i) - w^-(p_1 + \cdots + p_{i-1}) & \text{if } W_i < W_{Ref} \\ w^+(p_i + \cdots + p_N) - w^+(p_{i+1} + \cdots + p_N) & \text{if } W_i \geq W_{Ref} \end{cases}$$

w^- and w^+ are the probability weighting functions in the loss and gain region, respectively:

$$w^-(p) = \frac{p^{c_1}}{[p^{c_1} + (1-p)^{c_1}]^{1/c_1}}$$

$$w^+(p) = \frac{p^{c_2}}{[p^{c_2} + (1-p)^{c_2}]^{1/c_2}}$$

where $c_1, c_2 \in [0.28, 1]$ to ensure w^- and w^+ are monotonically increasing and that decision weights are positive. Figure 2.1 plots the probability weighting function for $c = 1$ (the solid line, no probability weighting), for $c = 0.65$ (the dashed line), and for $c = 0.4$ (the dot-dash line). The weighting functions are steeper in the low and high probability regions, and flatter in the moderate probability region. Intuitively, the more extreme an outcome is in terms of ranking of states, the more distortion the agent applies to that state relative to its objective probability.

We allow the states of the world to be continuous, while in comparison, Baele et al. (2019)'s model has finite states. Because of the discrete nature of their model, the authors used a discrete approximation to the distribution of market returns. However, we caution

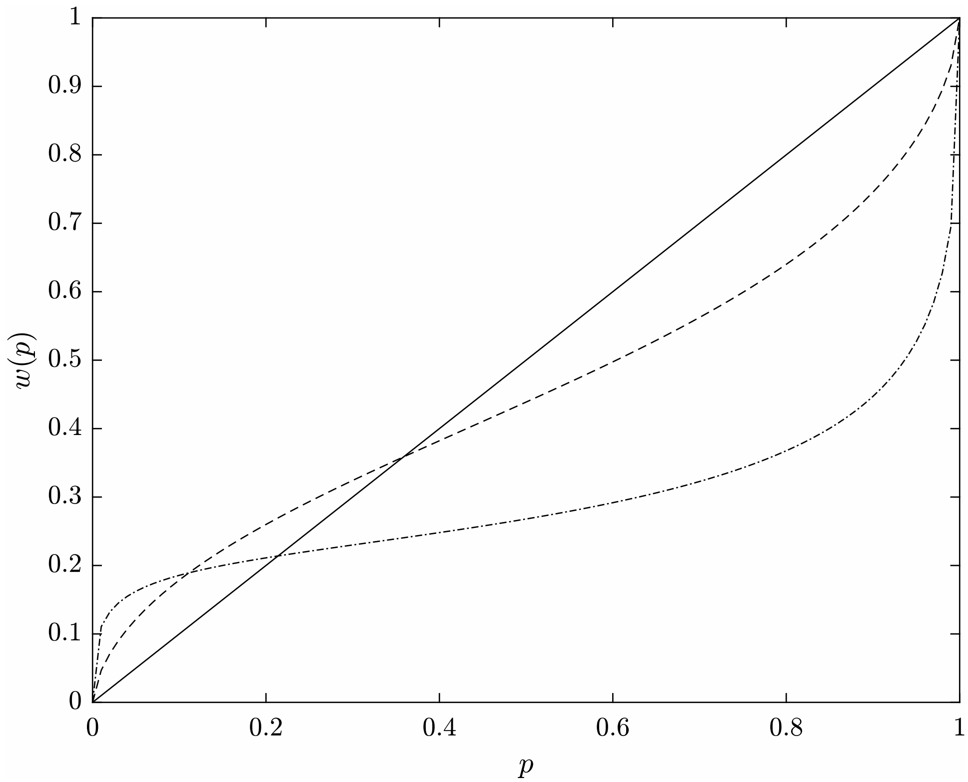


Figure 2.1: Probability Weighting Function

Note: Probability weighting functions by Tversky and Kahneman (1992) are defined on cumulative probabilities, $w(p) = p^c/(p^c + (1-p)^c)^{1/c}$. The solid line represents $c = 1$, where there is no overweighting of tails. The dashed line is for $c = 0.65$, which is obtained in some experimental settings. The dot-dash line is for $c = 0.4$. The closer c is to 0, the more heavily the agent weights tails.

against doing so especially when modeling probability weighting, as tail events are extremely important. For example, suppose in practice there are ten observations of the market returns, each corresponding to a possible state. Now consider the lowest value of the returns, which has a cumulative probability of 0.1. Based on Figure 2.1, a moderate probability of 0.1 does not lead to much distortion in decision weight, no matter how extreme the lowest value is. Thus, discretization may hide the true impact of probability weighting. Even though this is a hypothetical example, and it will be less of an issue when we have more observations, we want to avoid any unnecessary approximation errors as much as possible. We also re-estimated the model after discretizing the states, and the results did not change much, but the benefits of having continuous states will be more obvious if one wants to do a similar analysis on a shorter period.

In out setting, the CPT utility the agent derives from her gain/loss X_T is given by

$$V(X_T) = V^+(X_T) + V^-(X_T)$$

where

$$\begin{aligned} V^+(X_T) &= -\int_0^{\infty} v(X)\,dw^+(1 - P(X)) \\ V^-(X_T) &= \int_{-\infty}^{0} v(X)\,dw^-(P(X)) \end{aligned}$$

$v(\cdot)$, $w^+(\cdot)$, and $w^-(\cdot)$ are the value function and the probability weighting functions defined previously, and $P(\cdot)$ is the cumulative distribution function of the gain/loss. Put simply, the CPT utility is a weighted sum of the agent's values in different states of the world, with the weights assigned according to the probability weighting functions.

Given the representative agent's preference described above, Proposition 1 gives the moment conditions that have to be satisfied in equilibrium, which are the basis of our empirical analysis in the next section.

Proposition 2.1. *Assume $b_0 = \hat{b}W_0^{-\gamma}$, and that the representative agent ranks the states of the world based on the market return, i.e., she cannot endogenously alter the rankings by choosing portfolio weights*[3]. *In equilibrium, the optimality condition for* (2.1) *together with*

3. This assumption implies a "order-constrained" optimization problem as in Ingersoll et al. (2016). A full-blown model with prospect theory preference usually generates multiple global optima, which makes the analysis less tractable.

market clearing implies:

$$E[(R_j - R_f)(m^{CRRA} + m^{CPT})] = 0,\ j = 1, \ldots, J \tag{2.2}$$

where $m^{CRRA} = R_M^{-\gamma}$, *and*

$$m^{CPT} = \mathbb{1}_{\{R_M \geq R_f\}}(w^+)'(1 - P(R_M - R_f)) + \lambda \mathbb{1}_{\{R_M < R_f\}}(w^-)'(P(R_M - R_f)) \tag{2.3}$$

j *ranges over all risky assets in the economy, and* M *is the market portfolio.*

Proof. See Appendix.

The pricing kernel in our model consists of two parts, one part from the expected utility m^{CRRA}, and the other from the CPT utility m^{CPT}. The CPT part of the pricing kernel tells us two things. First, assets that pay well when the market experiences losses are valued more by investors than assets that pay well when the market experiences gains if $\lambda > 1$. Second, investors prefer assets that pay well when the market is in extreme conditions, be it extremely good or extremely bad, because of the probability weighting. As will be seen in the next section, the CPT pricing kernel is essential in explaining certain patterns in the data of asset prices.

2.3 Empirical Results

Our empirical analysis focuses mainly on the GMM estimation of the two key parameters in prospect theory, namely loss aversion and probability weighting, and how they vary over time with other factors such as investor sentiment.

2.3.1 Data

The GMM estimators are based on the moment conditions derived in (2.2). Theoretically speaking, any risky asset j must satisfy the orthogonality condition in equilibrium. One of the candidates is index options, as their payoffs directly depend on the market return, which in turn enters both parts of the pricing kernel. Barberis and Huang (2008) show that prospect theory investors tend to overvalue positively-skewed assets such as out-of-the-money options because they overweight the low probability events that those options pay off. Therefore, options data should be able to help identify the degree of probability weighting. In fact, as we will show in the next subsection, loss aversion could also be identified from option return data. Moreover, Baele et al. (2019) use index options in their analysis, which provides us a benchmark for comparison. We obtain from the OptionMetrics database the options surface constructed using S&P 500 index options from January 1996 to December 2017. These options are standardized in that they have a fixed maturity (30/60/90/etc. days), and evenly spaced moneyness defined by the option delta.

We calculate the buy-and-hold-until-maturity returns for these options, and present the summary statistics for options with maturity of 30 days in Table 2.1. The table replicates Table 1 of Baele et al. (2019), but provides more detailed information. Specifically, for each option, we compute the skewness, kurtosis, median, and the 25% and 75% percentiles of their returns, besides the mean and standard deviation already shown in Baele et al. (2019). We emphasize the importance of skewness here because as Barberis and Huang (2008) suggest, prospect theory investors have a preference for skewness and hence it earns a negative risk premium. Panel A shows the results for 13 put options, with delta ranging from -0.8 to -0.2. The option deltas are negative because they measure the sensitivity of option prices to the price of the underlying, and the put options become less valuable when the index is higher, everything else being equal. Also, the more out of the money an option is, the less sensitive its price is to the underlying index, and the smaller the magnitude of the option delta, as we move down the table.

Two interesting facts are worth noting from the table. First, the 25% percentile of the option returns is uniformly -100% across different moneyness, except for the most in-the-money option (delta $= -0.8$). This means that more than a quarter of the time in our sample periods, these options lose their entire value at maturity. Moreover, from the last two columns we can see that options with delta over -0.55 become valueless at maturity more than half of the time; and for the most out-of-the-money put options with delta over -0.35, that happens more than 75% of the time. This fact is mainly because the stock market generally had a upward trend over a 30-day horizon during our sample periods.

Second and more importantly, as the option delta increases, the skewness of the options is monotonically increasing, and at the same time, the average return of the options is monotonically decreasing. This is consistent with Barberis and Huang (2008)'s theory that assets with positive skewness are attractive to prospect theory investors because they overweight the tail events that the assets realize a big payoff. In Panel B, we obtain similar results for the 13 call options. Notice that call options are more valuable when the underlying index is higher, so the option deltas are positive; and the calls are more in the money towards the bottom of the table, as they are more sensitive to the underlying and hence have a higher delta. Again, the skewness of the option returns increases monotonically as the options are more out-of-the money, and correspondingly, the average return decreases monotonically. These results also add evidence to Boyer and Vorkink (2014), who find a similar negative relationship between total skewness and average return of individual equity options.

Table 2.1: Summary Statistics for SPX Option Returns (Maturity = 30 Days)

Panel A: Put Options

Delta	Mean	Std	Skew	Kurt	25%	Median	75%
-0.80	-17.29%	84.08%	1.2420	1.7560	-92.61%	-38.55%	30.85%
-0.75	-19.48%	91.93%	1.3982	2.1848	-100.00%	-48.89%	31.02%
-0.70	-21.66%	98.87%	1.5621	2.7508	-100.00%	-61.54%	29.17%
-0.65	-23.96%	105.43%	1.7374	3.4653	-100.00%	-76.80%	25.68%
-0.60	-26.36%	111.79%	1.9316	4.3802	-100.00%	-96.17%	19.36%
-0.55	-28.95%	118.17%	2.1494	5.5441	-100.00%	-100.00%	9.52%
-0.50	-31.71%	124.58%	2.4009	7.0694	-100.00%	-100.00%	-4.99%
-0.45	-34.68%	130.98%	2.7033	9.1502	-100.00%	-100.00%	-26.27%
-0.40	-38.27%	137.49%	3.0777	12.0817	-100.00%	-100.00%	-57.77%
-0.35	-42.66%	144.17%	3.5545	16.3947	-100.00%	-100.00%	-100.00%
-0.30	-47.69%	150.90%	4.2046	23.2891	-100.00%	-100.00%	-100.00%
-0.25	-54.02%	157.46%	5.1672	35.5073	-100.00%	-100.00%	-100.00%
-0.20	-62.07%	163.29%	6.7383	60.3333	-100.00%	-100.00%	-100.00%

Panel B: Call Options

Delta	Mean	Std	Skew	Kurt	25%	Median	75%
0.20	-15.66%	220.51%	3.8824	21.4966	-100.00%	-100.00%	-100.00%
0.25	-10.67%	193.99%	3.0607	12.9510	-100.00%	-100.00%	-28.79%
0.30	-7.21%	173.05%	2.4822	8.1686	-100.00%	-100.00%	30.55%
0.35	-4.62%	155.97%	2.0465	5.2446	-100.00%	-100.00%	58.34%
0.40	-2.51%	141.43%	1.7000	3.3438	-100.00%	-100.00%	72.41%
0.45	-0.94%	128.83%	1.4104	2.0416	-100.00%	-69.19%	77.75%
0.50	0.37%	117.54%	1.1633	1.1379	-100.00%	-42.45%	78.89%
0.55	1.47%	107.23%	0.9444	0.5012	-100.00%	-24.30%	76.68%
0.60	2.16%	97.82%	0.7388	0.0326	-100.00%	-11.75%	72.83%
0.65	2.63%	88.98%	0.5460	-0.2964	-96.61%	-2.91%	68.32%
0.70	3.04%	80.34%	0.3659	-0.4900	-75.48%	3.09%	62.84%
0.75	3.25%	71.84%	0.1899	-0.5562	-58.58%	6.88%	56.27%
0.80	3.44%	62.98%	0.0264	-0.4716	-44.41%	8.82%	49.33%

One of the restrictions of Baele et al. (2019) is that there is only one risky asset, which limits the choices of possible test assets to the index and its options. Our model eliminates the limitation by introducing multiple risky assets into the economy, so more test assets are available. A natural candidate is individual stocks, but they may greatly increase the standard errors of our estimates because of their high idiosyncratic risk. We therefore use industry portfolios in Kenneth French's data library instead. The portfolio data also has an advantage that it dates back much earlier than the options data, which enables us to extract information about investors' prospect theory preference over a longer period and to study how it evolves over time.

2.3.2 Identification

The option return data provides the most intuitive explanation for our identification strategy. We compute the buy-and-hold returns for all the 30-day maturity options every day, and obtain a time series of daily frequency. Figure 2.2 plots in circles the average returns of put and call options over the full sample against the moneyness (delta) of the options. Put options have negative deltas and call options have positive deltas, so the left part of the graph represents the returns for puts, and the right part for calls. Besides, options with delta closer to 0 are more out-of-the-money, and options with absolute value of delta closer to 1 are more in-the-money. As a comparison, we also show as plus signs the expected option returns required by investors who have a pure CRRA utility function. Figure 2.2 is a replication of Figure 1 of Baele et al. (2019), but we use delta as the horizontal axis in order to show put and call returns in the same graph, so that we can see more clearly the sharp difference in put and call returns. As we will discuss shortly, this return difference is crucial in the identification of the degree of loss aversion.

We observe at least two interesting patterns from the plot. First, both out-of-the-money puts and calls have substantially lower average returns than their in-the-money counterparts. Note that the rational asset pricing framework also predicts lower expected returns from out-of-the-money puts than in-the-money puts, because the pricing kernel is monotonically decreasing, and the out-of-the-money puts tend to pay off well when the market goes down a lot and investors need money the most. However, for a reasonable

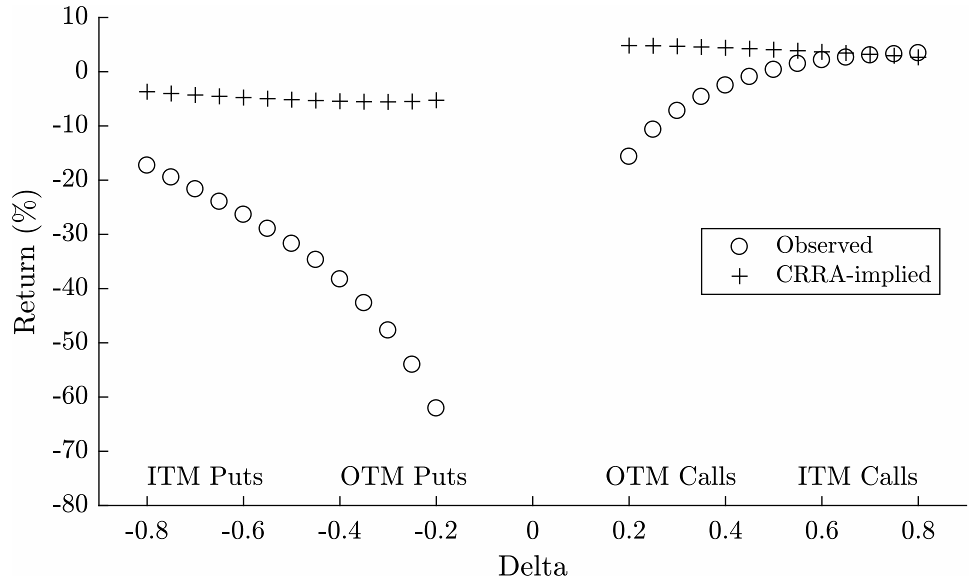


Figure 2.2: Average Returns for Puts and Calls

Note: The graph plots the full-sample average returns for put and call options of different moneyness (in circle), alongside with the corresponding expected returns required by a CRRA investor (in plus sign). Moneyness is measured by delta, the partial derivative of option prices with respect to the underlying prices. The left part of the graph shows returns for 13 puts, with delta ranging from -0.8 to -0.2 as the puts get more and more out of the money, and the right part shows returns for 13 calls, with delta ranging from 0.2 to 0.8 as the calls get more and more in the money.

level of risk aversion, the slope of the pricing kernel is not high enough to justify the huge magnitude of difference in returns between out-of-the-money and in-the-money puts as we see in Figure 2.2: a put option with a delta of -0.8 loses 17.29% on average, whereas a put with a delta of -0.2 loses 62.07% on average; in contrast, the returns required by a CRRA investor are almost flat.

On the other hand, the rational framework even gets the sign wrong when it comes to the average returns of calls. Since call option returns are positively correlated with the underlying index and hence negatively correlated with the pricing kernel, calls are not appealing to investors with a pure CRRA utility, so they earn positive expected returns regardless of their moneyness. Also, out-of-the-money calls are even less appealing because they pay off well when investors need money least. These predictions are completely in contradiction to what we observe on the right part of Figure 2.2. The average call returns in reality actually decrease monotonically as the calls move towards the out-of-the-money region. Additionally, the out-of-the-money calls earn an average return that is so much less than the in-the-money calls that it even becomes negative: the call option with a delta of 0.2 has an average return of -15.66%!

Therefore, rational asset pricing models fail to explain the sharp discrepancy between out-of-the-money and in-the-money option returns. In comparison, this sharp discrepancy is actually consistent with investors having prospect theory preferences and hence over-weighting tail events: they overweight the left tail of the market returns, so they are willing to pay a price premium for out-of-the-money puts to protect themselves against a market downturn; meanwhile, they overweight the right tail of the market returns, so they require a relatively low average return from out-of-the-money calls in the hope of a big payoff if the market shoots up. The preference revealed by the out-of-the-money option returns is fairly similar to a prospect theory agent's preference for both insurance and lotteries, and it helps us identify the degree to which a representative agent overweights tail events.

The second interesting fact from Figure 2.2 (which Baele et al. (2019) did not point out) is that put options have much lower average returns than call options. This difference is not surprising even in the rational asset pricing framework, because the market return negatively correlates with put returns, but positively so with call returns, so investors are willing to pay extra money for the hedge provided by puts. However, it is again the observed huge magnitude of the difference that contradicts the rational theory. In fact, if we rewrite (2.2) without the CPT component:

$$E(R_j - R_f) = -\frac{Cov(R_j, R_M^{-\gamma})}{E(R_M^{-\gamma})} \approx \frac{\gamma Cov(R_j, R_M)}{E(R_M^{-\gamma})}$$

then put and call options with the same absolute value of delta should lie to the two sides of the risk-free rate with approximately the same distance, as shown by the plus signs in Figure 2.2. Take the most in-the-money options as an example[4], a CRRA investor requires 2.65% expected return from a call with a delta of 0.8 and −3.69% expected return from a put with a delta of −0.8, leaving a difference of 6.34%. In comparison, the observed average return is 3.44% for the same call and −17.29% for the same put, with a difference of 20.73%. Therefore, rational asset pricing models cannot explain why puts earn returns so much lower than calls, and loss aversion is critical in explaining the huge difference: puts are valued more by investors than calls not only because of their negative correlation with the market, but also because they pay well when investors suffer a loss.

In summary, the slopes of the put and call return curves help identify the degrees of probability weighting in the loss and gain regions respectively, and the level difference between the two curves helps identify investors' loss aversion.

2.3.3 GMM Estimation: Industry Portfolios

As mentioned previously, one of the advantages of our model over Baele et al. (2019) is the flexibility in choosing test assets. While they use exclusively the market index and index options, this paper extends the set of test assets to include stock portfolios. Unlike index options, the returns of individual stocks and stock portfolios do not depend directly on the market return. However, the intuition behind the identification strategy is similar to that in the last subsection. Specifically, first rewrite the optimality condition (2.2) as:

$$E(R_j - R_f) = -\frac{Cov(R_j, m^{CRRA} + m^{CPT})}{E(m^{CRRA} + m^{CPT})}$$

4. We compare most in-the-money options because they are the least subject to probability weighting.

It then becomes clear that what determines an asset's expected return is mainly the covariance between its return and the pricing kernel. Since the pricing kernel can be understood as the marginal utility of the representative agent, the above equation says that if an asset tends to pay off well when the marginal utility is high, i.e., when the agent needs money badly, then the asset is very attractive, so the agent is willing to sacrifice a little on the expected return of this asset; on the other hand, assets that pay well in states of low marginal utility are not as desirable, so the agent requires a higher expected return from them in equilibrium.

The CPT pricing kernel in equation (2.3) shows that the representative agent's marginal utility is higher in the loss region than in the gain region due to loss aversion (if $\lambda > 1$); it is also higher in extreme states of the world, where the slope of the probability weighting function is higher. Therefore, by comparing the difference in average returns between assets that pay off well in the loss region versus the gain region, and the return difference between assets that pay well in extreme states and those that do not, the GMM allows us to estimate the degrees of loss aversion and probability weighting of a representative investor.

We estimate the CPT parameters with GMM based on the moment conditions in (2.2), using data on various industry portfolios from Kenneth French's website. Besides, following Baele et al. (2019), we set $\gamma = 1$ and $\hat{b} = 0.95$, and focus on the estimation of loss aversion (λ) and probability weighting (c_1 and c_2). We also restrict the sample period to between January 1996 and March 2016, in order to make direct comparison to the results in Baele et al. (2019). Table 2.2 presents the estimation results, with standard errors in the parentheses. Column 2 shows the results for 5 industry portfolios, which is the crudest division in the data set. Column 3 uses 30 industry portfolios, and the number of moment conditions is the closest to 27 as in Baele et al. (2019). The last column uses 49 portfolios, the largest number possible in the data.

Notice that the estimates using different numbers of industries are strikingly close to each other, showing the robustness of the results. Moreover, our estimates for probability weighting on the gain region are in line with those obtained in experimental settings. In their original study, Tversky and Kahneman (1992) estimate $c_2 = 0.61$, compared to ours between 0.60 and 0.65. An interesting fact is that even though people overweight the left tail distribution less than the right tail in labs - Tversky and Kahneman (1992) estimate $c_1 = 0.69$, investors seem to be more concerned about the left tail of market returns, with a smaller probability weighting parameter around $c_1 = 0.55$ on the loss side than a larger estimate on the gain side. The estimates for loss aversion λ range from 1.09 to 1.16, and are different than Tversky and Kahneman's $\lambda = 2.25$. However, recent studies suggest that the true level of loss aversion in the population is significantly lower. Walasek et al. (2018) find that the median estimate of λ is 1.31. Finally, in comparison to Baele et al. (2019), who estimate $\lambda = 1.32$, $c_1 = 0.62$, and $c_2 = 0.69$, our estimates are smaller for all three parameters. Thus, it seems that investors in the stock market are less averse to losses, but they also have a stronger preference for both insurance and lotteries compared to investors in the options market.

Table 2.2: GMM Estimates with Industry Portfolios Data

No. Industry Portfolios	5	30	49
Loss aversion	1.1643	1.0926	1.1441
	(0.5843)	(0.4865)	(0.5017)
Probability weighting	0.5554	0.5449	0.5463
(loss region)	(0.2227)	(0.2141)	(0.2008)
Probability weighting	0.6049	0.6549	0.6137
(gain region)	(0.2370)	(0.2379)	(0.2108)

2.3.4 GMM Estimation: Pre- and Post-Crisis

Our second analysis studies investors' loss aversion and probability weighting in different periods of history. The options data we have ranges from 1996 to 2017, and a natural cutoff point of the data is the 2007-2008 financial crisis. Intuitively, a recent loss may remind investors how painful a loss really is, and they thus become more loss averse after such a loss. The financial crisis definitely inflicted huge damage to the wealth of a lot market participants. Therefore, we expect the parameter of loss aversion to be higher after the crisis than before. Moreover, the market crash is more readily available to investors in their memory after the crisis, so they tend to overweight more on the left tail and less on the right tail of market returns.

Again, we estimate the CPT parameters based on the moment conditions in (2.2), now with the same data as in Baele et al. (2019). Specifically, we use the 13 puts and 13 calls in Figure 2.2, and the S&P 500 index itself as test assets, which amounts to 27 moment conditions in total. We obtain the GMM estimates for the full sample, pre-crisis (1996-2006), and the post-crisis (2009-2017) period, and present the results in columns 2 through 4 of Table 2.3 respectively. Standard errors are reported in parentheses.

Our estimates for the full sample in column 2 are a replication of specification (2) of Table 4 in Baele et al. (2019), using a slightly longer period of data until December 2017. Not surprisingly, the estimates in the two papers are fairly close to each other: $\lambda = 1.22$ vs. 1.32, $c_1 = 0.59$ vs. 0.62, and $c_2 = 0.75$ vs. 0.69.

More interesting is that the estimates show clearly distinct patterns before and after the crisis. First, investors are more loss averse after the crisis ($\lambda = 2.01$) than before the crisis ($\lambda = 1.08$), which is consistent with our conjecture as well as Thaler and Johnson (1990)'s experimental evidence that losses are more painful after prior losses. Notice that

Table 2.3: GMM Estimates with Options Data

	Full sample	Pre-crisis (1996-2006)	Post-crisis (2009-2017)
Loss aversion	1.2197	1.0810	2.0127
	(0.3869)	(0.4922)	(0.8475)
Probability weighting (loss region)	0.5899	0.5174	0.5258
	(0.1018)	(0.0995)	(0.1100)
Probability weighting (gain region)	0.7492	0.7105	0.9770
	(0.1197)	(0.1396)	(0.2597)

Note: This table reports the GMM estimates for the prospect theory parameters: loss aversion, and probability weighting in the loss and gain regions. We use the S&P 500 index options and the index itself as test assets. The full sample period ranges from January 1996 to December 2017.

the estimates suggest that investors treat losses and gains almost the same way before the financial crisis, whereas losses are twice as painful as gains are joyful after the crisis. Second, the probability weighting parameter on the loss side barely changes after the crisis ($c_1 = 0.53$), compared to its pre-crisis level of 0.52. On the other hand, investors weight less heavily on the right tail after the crisis ($c_2 = 0.98$) than before ($c_2 = 0.71$). To put it another way, they now care less about the states of the world where the market attains huge gains, to the extent that they weight those states almost proportional to the objective probabilities. However, investors are still as worried about a market crash as usual, and they keep putting heavy weights on the crash states in their decision making process.

Figure 2.3 plots the observed average returns of puts and calls before and after crisis in the same graph. Several interesting facts are worth noting. First, the post-crisis put returns are lower across all moneyness than pre-crisis, while all the post-crisis call returns are higher than pre-crisis, so the difference between put and call returns becomes larger. Now investors value puts even more as they pay well when the market experiences a loss, which translates into a higher estimate for loss aversion (2.01 v.s. 1.08). Second, the slope of the put return curve almost stays the same pre- and post-crisis, which explains the similar estimates for the degree of probability weighting on the loss region in these two sub-periods. However, the slope of the call return curve becomes much flatter than before, consistent with an estimate of probability weighting on the gain region that is very close to 1. Therefore, as suggested in the previous subsection, the levels and the slopes of the put and call return curves successfully identify investors' loss aversion and probability weighting.

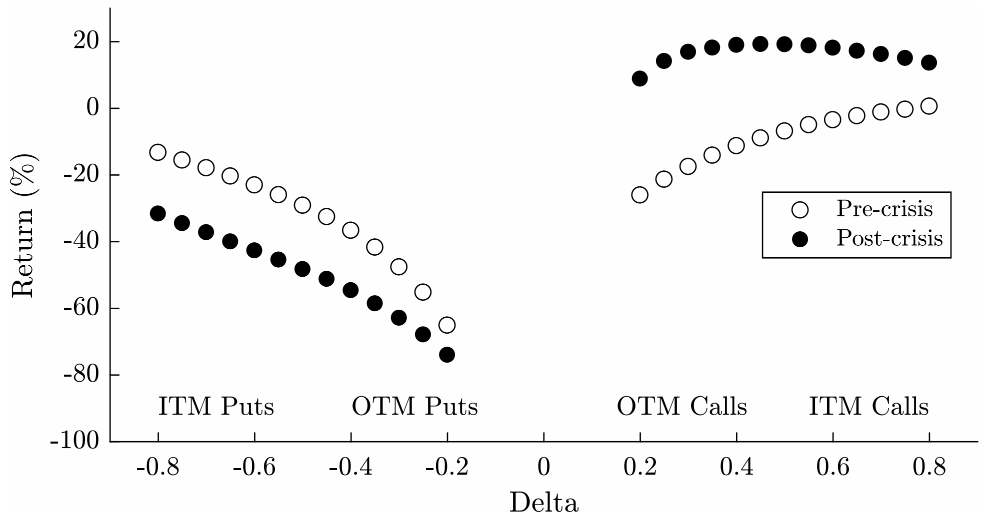


Figure 2.3: Average Returns for Puts and Calls: Pre- and Post-Crisis

Note: This graph plots the average put and call returns against their moneyness measured by delta. The left part corresponds to the puts and the right part to the calls. Pre-crisis and post-crisis returns are represented by the empty and filled circles respectively.

In previous subsection we illustrated the discrepancy between the observed average option returns and the expected returns required by a CRRA investor. In order to better compare the prospect theory and the expected utility theory in terms of explaining the financial data, we plot the observed and model-implied option returns in Figure 2.4, with the pre-crisis and post-crisis results in Figure 2.4a and Figure 2.4b respectively. Circles show the observed returns and plus signs show the CRRA-implied returns, and we include the expected returns implied by our model as asterisks. As before, the expected returns implied by a pure CRRA utility function does a poor job in fitting the actual data. Although the

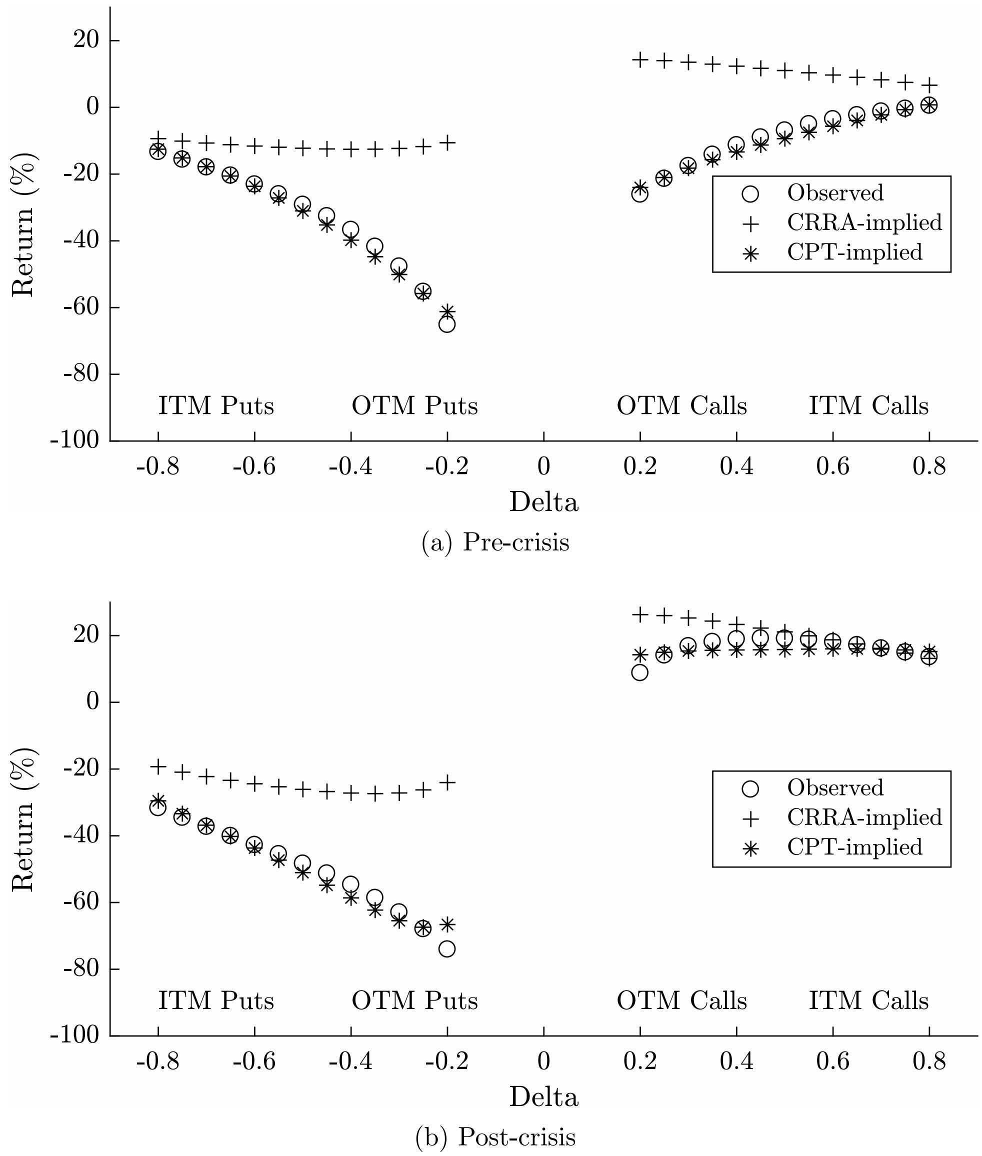


Figure 2.4: Observed and Model-implied Returns for Puts and Calls

CRRA-implied returns are close the observed returns for in-the-money call options in the post-crisis periods, as shown in the right part of Figure 2.4b, the difference between the two gets larger and larger as the calls move towards the out-of-the-money region. Therefore, even though investors have a weak tendency to overweight the right tail of the market after the financial crisis ($c_2 = 0.98$), it is still important to capture investors' preference for positively skewed assets such as the out-of-the-money calls. Compared to the pure CRRA utility, our model that combines the CRRA utility with the CPT utility fits the data amazingly well - the circles and the asterisks almost overlap with each other in both sub-periods. One exception is for the most out-of-the-money options with a delta of -0.8 or 0.8, where our model tends to give a slightly higher expected return than the observed average return. Remember that the vast majority of these deep out-of-the-money options end up valueless at maturity, with a return of -100%, which makes it very difficult to accurately estimate the correlation between these option returns and the pricing kernel, and consequently the model-implied expected returns are over-estimated.

One question the readers may have in mind is why we find time-varying prospect theory parameters important while Baele et al. (2019) does not, even though we both use the same options data. The thing is, in the dynamic setting, they focus mainly on explaining the variance premium (see Figure 5 and Figure 6 of their paper), which is a weighted average of out-of-the-money option returns. We claim that this aggregation overlooks other detailed patterns in the option returns. In fact, the authors themselves realized that (on page 31) "a model can produce a good fit for the VP (variance premium), even if the individual option returns are not well matched, as long as the fitting errors for expected option returns cancel out across all options" in their earlier discussion about asymmetric probability weighting. And that is exactly the reason why their analysis proves time-varying parameters unnecessary. To be more specific, Figure 2.3 shows that after the crisis, out-of-the-money puts earn lower average returns while out-of-the-money calls earn higher average returns than before the crisis. Therefore, these two effects offset each other to some extent, so that their weighted average does not change much over time. In consequence, neither does the loss aversion nor the probability weighting need to change much to match the variance premium. However, because the nature of the variance premium is a weighted sum of option returns, it naturally neglects the differences in average returns across moneyness and between puts and calls. And as per our discussion above, it is exactly these return differences that help us to gauge investors' loss aversion and probability weighting. Thus, we believe time-varying prospect theory parameters are essential when it comes to explaining individual option returns.

2.3.5 Loss Aversion and Investor Sentiment

In the last subsection we demonstrated the time-varying nature of loss aversion and probability weighting, but did not investigate how they change with other economic variables. To answer that question, one way is to follow a two-step procedure: in the first step divide the full sample into finer sub-periods, perform GMM estimation for each sub-period, and then in the second step regress the estimators on other variables of interest. However, there is a tradeoff in terms of the division of the sample: we need enough sub-periods in order to see a clear pattern of time-varying CPT parameters, but we also need enough data points in each sub-period to obtain accurate parameter estimates in the first step. For example, if we want to learn how loss aversion comoves with investor sentiment, which has data of monthly frequency, then we better be able to estimate loss aversion every month as well. However, only about 20 trading days of data are available in a month, which may greatly increase the standard errors of our estimates in the first step.

We propose in this paper two procedures to address the sample division tradeoff mentioned above. The first procedure works as follows. Suppose we want to study how loss aversion λ varies with a vector of variables X, we first write their dependence structure in a linear form:

$$\lambda = X'\beta + \epsilon \tag{2.4}$$

then we estimate β using the full sample based on the following identifying assumption:

$$E[(R_j - R_f)(m^{CRRA} + \mathbf{\hat{m}^{CPT}})] = 0,\ j = 1, \ldots, J \tag{2.5}$$

where

$$\hat{m}^{CPT} = \mathbb{1}_{\{R_M \geq R_f\}}(w^+)'(1 - P(R_M - R_f)) + \hat{\lambda}\mathbb{1}_{\{R_M < R_f\}}(w^-)'(P(R_M - R_f))$$

and $\hat{\lambda} = X'\beta$. In this way, we are able to obtain monthly (or even daily, if the explanatory variables X are of daily frequency) estimates for loss aversion. Notice that because the moment condition (2.2) is linear in λ, the moment condition (2.5) is linear in β. Theoretically, we could also apply the same method to the probability weighting parameters. However, the moment condition (2.2) is highly nonlinear in c_1 and c_2, which makes the moment condition (2.5) highly nonlinear in β if we write, say, $c_1 = X'\beta$. Since GMM is notoriously not good at dealing with nonlinear parameters, our estimates for dynamic probability weighting turned out to have fairly large standard errors. Therefore, we only focus on the results for time-varying loss aversion in this procedure, by fixing the probability weighting parameters at their full sample static estimates: $c_1 = 0.59$ and $c_2 = 0.75$.

We want to emphasize here that our procedure is very different from the one used by Baele et al. (2019) in their dynamic setting. They calibrate the parameters period by period, by matching the model-implied equity and variance premium to the empirically estimated counterparts. There are two drawbacks of that method. First, since their calibration uses the conditional expectations of equity and option returns, which are not observable, these variables have to be estimated with predictive regressions. Consequently, this intermediate step introduces extra numerical errors to the calibration and makes the calibrates inaccurate, especially considering such predictive regressions for equity premium usually have a poor fit with single-digit R^2. Our procedure uses any variables as they are so that we avoid unnecessary numerical errors. Second, because the calibration involves the conditional moment conditions instead of unconditional ones, their method does not fit into the general GMM framework, so standard errors of the calibrates are not available. By contrast, our procedure still falls into the GMM framework, which allows us to draw more meaningful inference on the structural parameters based on the standard errors.

The next question is how to choose the appropriate explanatory variables X. In theory, one would like to incorporate as many relevant variables as possible to accurately estimate loss aversion, as the sample size is no longer a concern since the full sample is used. However, another drawback of GMM is that its performance decreases dramatically as the dimension of parameters increases. Thus, we need to be cautious in choosing the truly relevant variables. In light of the literature on prospect theory and its applications in finance, past stock market returns and investor sentiment are both natural candidates for our explanatory variables. Thaler and Johnson (1990) claim that people's degree of loss aversion, and hence their willingness to take gambles, depend on recent gains/losses they have experienced. The financial market is definitely a place where people learn and adapt, and past stock market returns well reflect investors' experiences in their recent investment decisions. Investor sentiment, on the other hand, has its potential to explain time-varying loss aversion as well. For example, Yu and Yuan (2011) find that investors require a higher expected return from the market when investor sentiment is low, everything else being equal, which could be due to higher loss aversion during these periods.

Besides choosing explanatory variables that make economic sense, we employ the following two-step variable selection procedure to avoid common issues in GMM estimation. First, we put one candidate variable into (2.4) every time, estimate the corresponding β using a small set of test assets to avoid severe over-identification. Second, we select all those variables that have a statistically significant slope estimate in the first step, put them all in (2.4), and jointly estimate their coefficients using a larger set of test assets to improve the efficiency of the estimators.

Within the past stock market return category, we use the mean, standard deviation, skewness, and kurtosis of the market returns over the past few years as the single explanatory variable in the first step. However, none of them predicts loss aversion with a statistically significant coefficient, no matter how long the horizon is over which we calculate the summary statistics. As for investor sentiment, we use the one proposed by Baker and Wurgler (2006, 2007). They apply a principal component analysis on five different empirical measures of investor sentiment, namely the price premium of dividend-paying stocks over non-dividend-paying stocks ($pdnd$, which the investor sentiment has a negative loading on), first day return of IPO's ($ripo$, positive loading), number of IPO's ($nipo$, positive loading), closed-end fund discount ($cefd$, negative loading), and the ratio of equity financing to debt and equity financing combined (s, positive loading). Loosely speaking, investor sentiment measures the general valuation of the market by investors. If investor sentiment is high, the market is overvalued, meaning a higher price for a fixed payoff, which pushes the distribution of asset returns to the left and makes it more likely for investors to be in the loss region. Such a high price would only possibly be sustained if investors are less loss averse, all else equal. Therefore, we expect a negative relation between investor sentiment and investors' loss aversion.

The first-step variable selection confirms our conjecture. We present the GMM estimation results in Table 2.4. Since the investor sentiment data dates back to July 1965 while we only have the options data since 1996, we remove the moment conditions that involve options and use only Kenneth French's industry portfolios as test assets. Specifications (1) through (5) of Table 2.4 use each of the components of Baker and Wurgler's investor sentiment as the single explanatory variable. We can see from the table that all but one investor sentiment components predict loss aversion statistically significantly with the correct signs. Specifically, all the components on which the investor sentiment has a positive loading negatively predicts loss aversion, and vice versa. The only exception is *ripo*, which still has the correct sign but not any statistical significance. We also test the null hypothesis that the fitted loss aversion $\hat{\lambda}$ is uncorrelated with the investor sentiment, and show the p-values in the last row. Not surprisingly, all the p-values are less than 0.0001 except for the specification with *ripo*.

When we put all the four significant components other than *ripo* into (2.4) for joint estimation, the slope coefficients become not statistically significant, which is possibly due to the multicolinearity of the explanatory variables. However, most importantly, the fitted value of loss aversion based on the four investor sentiment components is still highly correlated with investor sentiment itself, as shown by the p-value in the last column of Table 2.4. Notice that this is not simply because we use the same set of variables to construct both loss aversion and investor sentiment. In fact, Baker and Wurgler build their measure of investor sentiment by principal component analysis, or finding the direction in the component space that captures the maximal variation in these components. In our case, the loadings on the components are chosen such that they help best identify loss aversion based on the moment condition (2.5).

Table 2.4: Loss Aversion and Investor Sentiment

	(1)	(2)	(3)	(4)	(5)	(6)
$pdnd$	0.2476^{***}					0.0211
$ripo$		-0.0091				
$nipo$			-0.1260^{***}			-0.0184
$cefd$				0.4490^{**}		-0.0343
s					-27.3396^{**}	0.0104
p-value	0.0000	0.9620	0.0000	0.0000	0.0000	0.0000

Note: This table shows the GMM estimation results for the moment condition (2.5), using as test assets the industry portfolios from Kenneth French's website. The sample period ranges from July 1965 to December 2018. Columns 2 through 6 show the results for the univariate specifications using only one of the components of Baker and Wurgler's investor sentiment each time, and column 7 includes all the significant components in the estimation. The last row shows the p-value for the null hypothesis that the fitted loss aversion parameter is uncorrelated with the investor sentiment. *: $p < 0.1$, **: $p < 0.05$, ***: $p < 0.01$.

We calculate the fitted loss aversion based on our estimates in specification (6), and present its time series plot alongside with the investor sentiment in Figure 2.5. The solid line corresponds to the fitted loss aversion, and the dashed line to the investor sentiment. One can easily tell the strong negative relationship between loss aversion and the investor sentiment. For example, during the 1980s, the investor sentiment goes up and comes down several times, and our fitted loss aversion perfectly matches those moves but in the opposite direction. Besides, loss aversion drops to a low level around the 2000s tech bubble as the investor sentiment shoots high, and it leads the movement of the investor sentiment by a month or two. Intuitively, investors are more sensitive to losses after they have experienced a bubble bust.

Figure 2.6 plots the fitted loss aversion against the investor sentiment with a fitted straight line. The figure shows a clear negative relationship between loss aversion and the investor sentiment. The correlation between the two is amazingly high at -0.39, considering

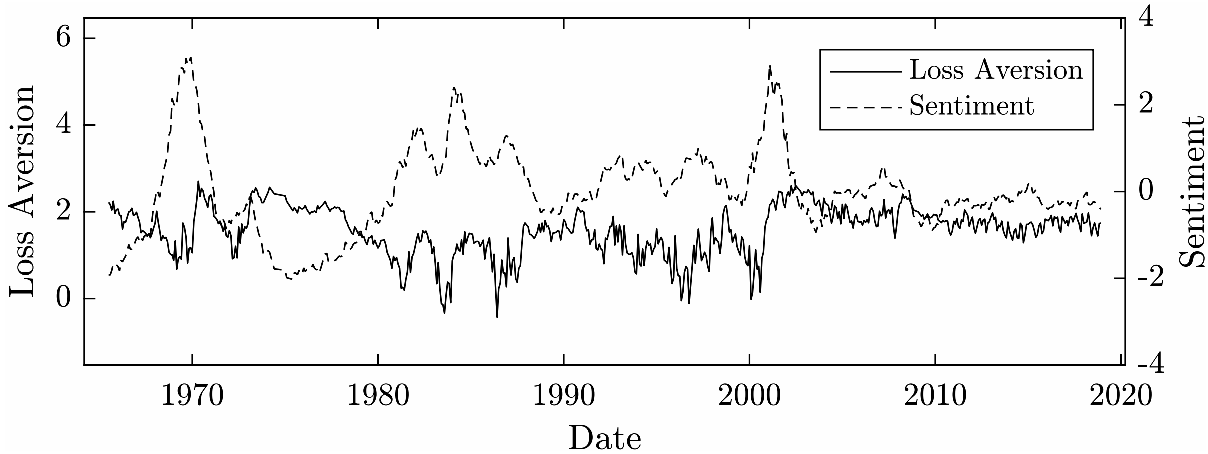


Figure 2.5: Fitted Loss Aversion and Investor Sentiment

Note: This graph plots the time series of the fitted loss aversion in solid line against the left vertical axis, and the investor sentiment in dashed line against the right vertical axis. The sample period ranges from July 1965 to December 2018.

how differently these two measures are constructed. This further corroborates our belief that these two variables are closely related to each other.

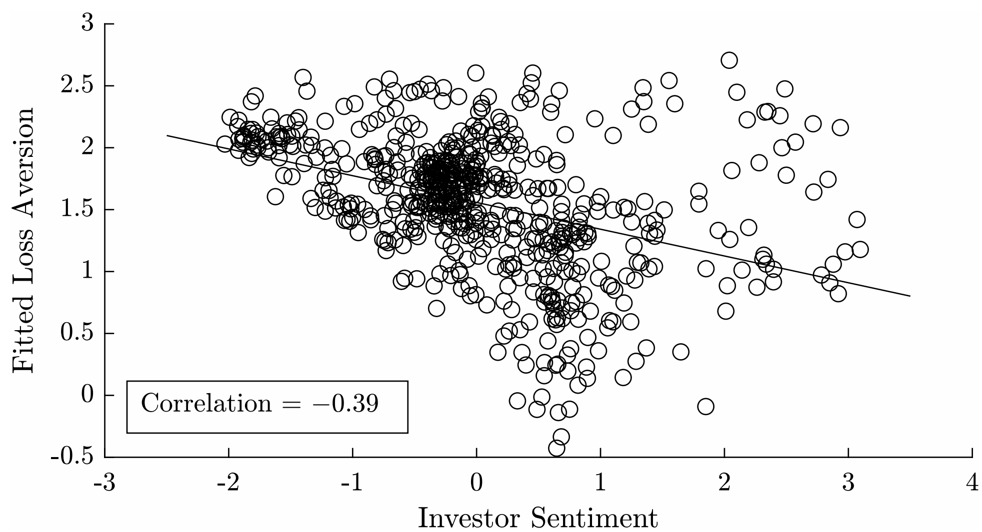


Figure 2.6: Fitted Loss Aversion against Investor Sentiment

To better understand what properties of the data on industry portfolios allow us to detect the change in investors' loss aversion, we did the following analysis. First, we divide the full sample into periods of low sentiment and high sentiment based on a threshold of 0 (the investor sentiment has a median around 0). Then for each of the 49 industry portfolios which we used in the multivariate GMM, we calculate its average return when the market is in the loss region, the average return in the gain region, and the difference between the two average returns. Finally, we plot the overall average returns of the industry portfolios against their return differences in the loss region versus gain region. Figure 2.7 and Figure 2.8 show the plots for periods of low and high sentiment respectively.

In Figure 2.7 there is a negative relationship between average returns and the return differences, with a highly significant (p-value less than 0.01) slope coefficient in the univariate regression. This means that during periods of low sentiment, the better an industry pays off in the loss region compared to the gain region, the more attractive that industry is, and the lower expected return investors require from that industry. In comparison, Figure 2.8 shows instead a positive relationship, again with a highly significant slope coefficient estimate (p-value less than 0.01). Therefore, industries that pay off better in the loss region are not as attractive during periods of high sentiment, and hence earn higher expected returns. Although this analysis is preliminary - it does not take into account other characteristics of the industry portfolios such as their returns in extreme states, it gives us the intuition how we can learn from the industry data about investors' preferences. Remember in our earlier discussion in section 3.3, we mentioned that the GMM allows us to estimate loss aversion by comparing the difference in average returns between assets that pay off well in the loss region versus the gain region. Here, since investors have a stronger preference for industries that pay well in the loss region during periods of low sentiment, the degree of loss aversion implied by the data is thus high when the investor sentiment is low.

As mentioned before, the more relevant variables we include in the GMM estimation, the more accurate our estimates for loss aversion. Past stock market returns and investor sentiment are by no means an exhaustive set of relevant variables, but they do help us

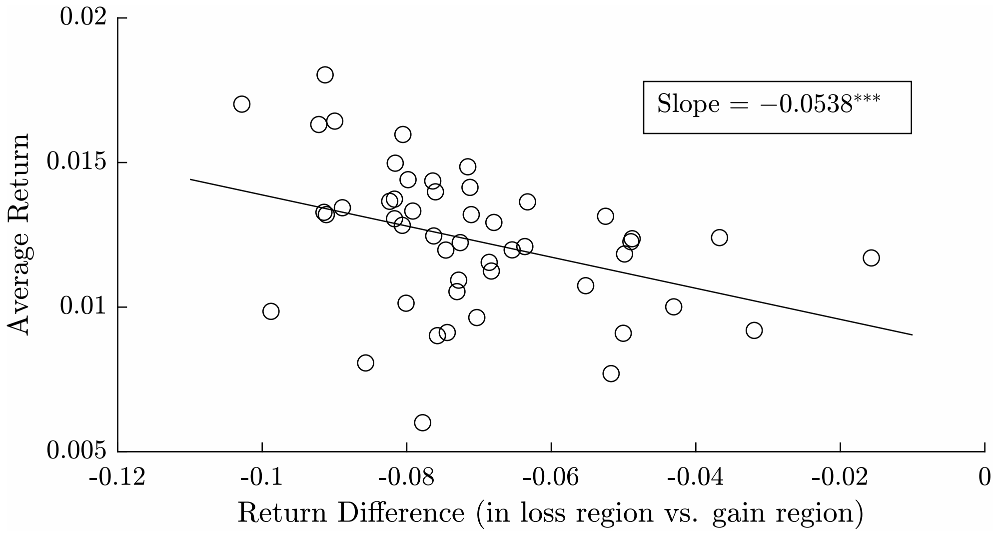


Figure 2.7: Average Industry Returns against Return Difference (Low Sentiment)

Note: This graph plots the average returns of 49 industry portfolios against the differences in their average returns when the market is in the loss region versus in the gain region, during periods of low sentiment.

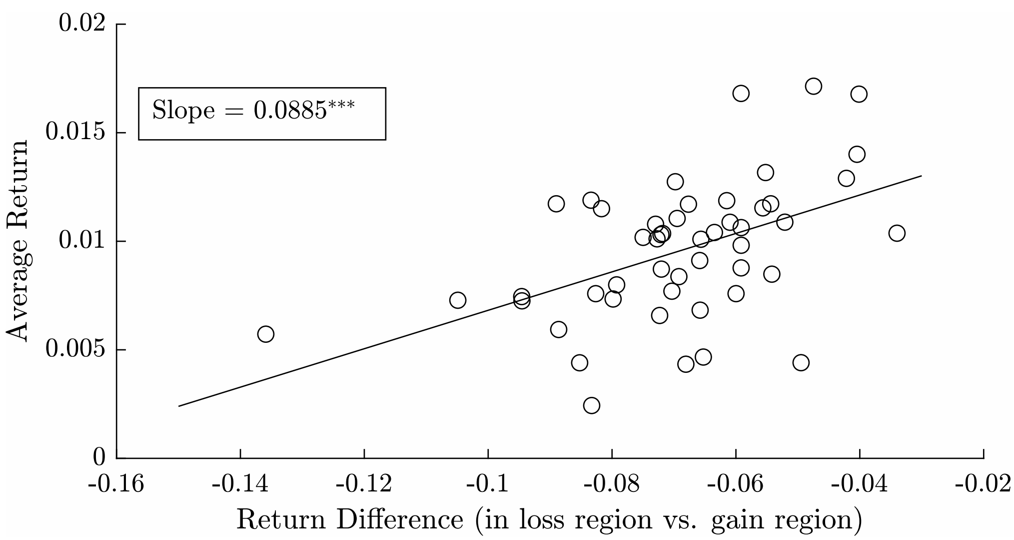


Figure 2.8: Average Industry Returns against Return Difference (High Sentiment)

Note: This graph plots the average returns of 49 industry portfolios against the differences in their average returns when the market is in the loss region versus in the gain region, during periods of high sentiment.

understand how investors' loss aversion changes over time, and our analysis in this paper is sufficient to demonstrate their close relationship with loss aversion. For future research, a tentative approach is a LASSO-type GMM. Specifically, instead of the two-step method in this paper, one can put all variables of interest into equation (2.4) and impose a L_1

penalty on β's in the GMM estimation, so that variable selection is done at the same time of estimation.

Our second method to get around the sample division tradeoff is on an ad hoc basis. Remember in subsection 2.3.2 we illustrated how we could use the difference in returns from call and put options, and the difference in returns from in-the-money and out-of-the-money options, to identify the parameters of loss aversion and probability weighting respectively. Therefore, instead of obtaining the GMM estimates for the prospect theory parameters directly, we use those return differences as proxies for the corresponding parameters of interest. Specifically, the most in-the-money options are the least subject to the impact of probability weighting, and we thus use the difference in average returns between calls with a delta of 0.8 and puts with a delta of -0.8 as a proxy for investors' loss aversion. Moreover, we use the difference in average returns between the most in-the-money puts (delta $= -0.8$) and the most out-of-the-money puts (delta $= -0.2$) to proxy for the probability weighting parameter in the loss region, and the average returns between the most in-the-money calls (delta $= 0.8$) and the most out-of-the-money calls (delta $= 0.2$) to proxy for probability weighting in the gain region. To further ensure a smaller standard error for our proxies, we take a rolling window average over six months.

As an illustration, Figure 2.9 presents the time series plot of the call-minus-put return difference alongside the investor sentiment during our sample period. Although we can still detect the tendency of our loss aversion proxy (the call-minus-put return difference) and the investor sentiment to move in opposite directions, e.g., around the 2000s tech bubble, the negative relationship is not as obvious as that in Figure 2.5. However, as we plot the call-minus-put return difference against the investor sentiment in Figure 2.10, the negative correlation between the two variables becomes clearer. In fact, the correlation coefficient

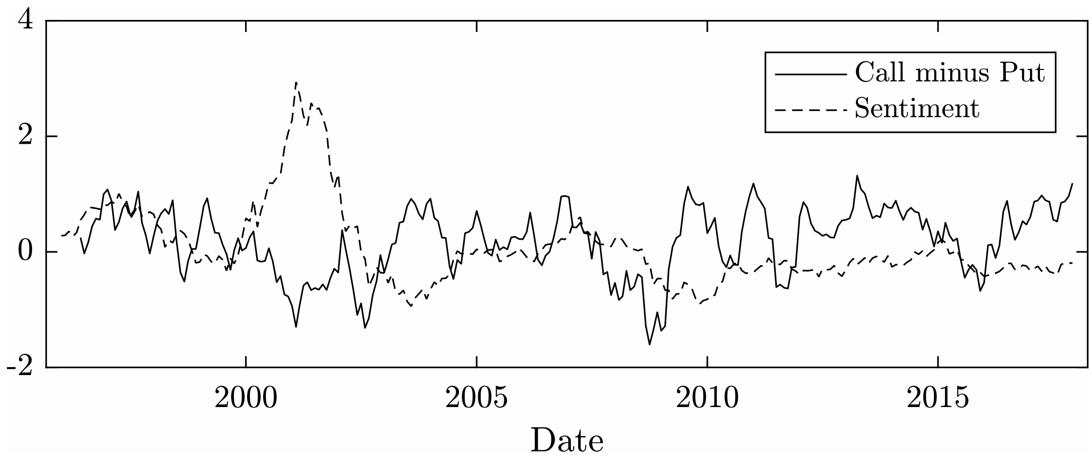


Figure 2.9: Option Return Difference and Investor Sentiment

Note: This graph plots the time series of the difference in average returns between the in-the-money calls with 0.8 delta and the in-the-money puts with -0.8 delta (the solid line), and the investor sentiment (the dashed line). The sample period ranges from January 1996 to December 2017.

-0.40 is about the same level as that in our previous analysis (-0.39), and it is highly significant with a p-value below 0.0001.

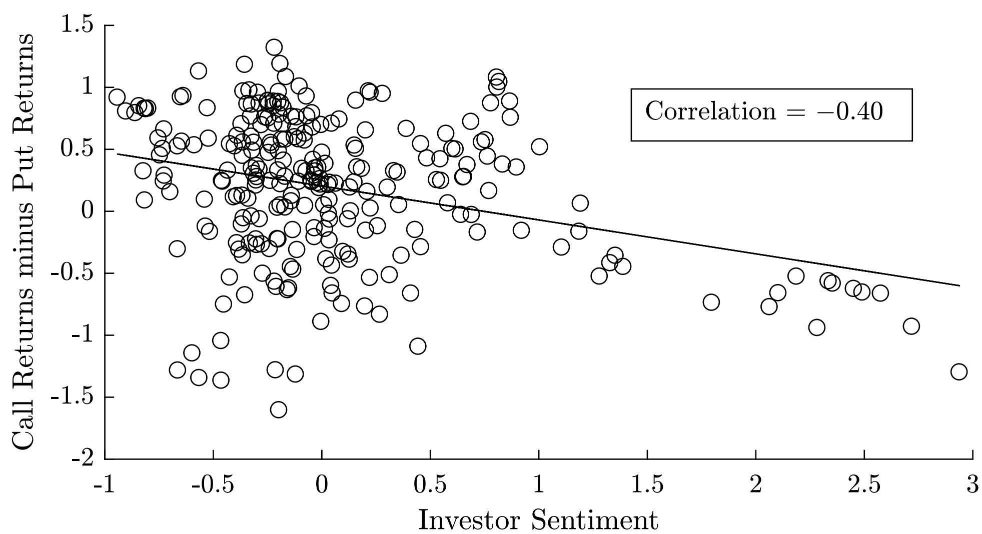


Figure 2.10: Option Return Difference against Investor Sentiment

We also study how our proxies for loss aversion and probability weighting vary with the investor sentiment and its components using regression analysis. The regression results are presented in Table 2.5, with t-statistics shown in parentheses. Specifications (1)(3)(5) correspond to the univariate regressions with the investor sentiment as the only explanatory variable, and the proxies for λ, c_1, and c_2 as the dependent variables respectively. Consistent with Figure 2.10, the proxy for loss aversion is significantly negatively correlated with the investor sentiment. On the other hand, neither of the probability weighting parameters shows a discernible relation with the investor sentiment in the statistical sense.

To further investigate which specific components of the investor sentiment explain the variation in the prospect theory parameters, we conduct multivariate regressions using all the sentiment components as regressors, and display the results in columns (2)(4)(6) for λ, c_1, and c_2 respectively. The result for loss aversion is largely consistent with our previous analysis using GMM. Specifically, three sentiment components out of five, namely *nipo*, $cefd$, and s, have highly significant coefficient estimates, and they also predict loss aversion in the correct directions we expect: loss aversion (or at least its proxy) is higher when the number of IPO's is lower, the closed-end fund discount is higher, and the equity issuing share is lower. As in our previous analysis, *ripo* still does not have significant explanatory power for loss aversion. Therefore, even though we have used different data, different sample periods, and also different estimation methods, the results from these two procedures both confirm our conjecture that investors tend to have higher degrees of loss aversion when the investor sentiment is low, and vice versa.

Now we are also able to learn more about the dynamics of probability weighting since we no longer need to worry about the high non-linearity in GMM estimation. Even though columns (3) and (5) find no significant correlation between the degrees of probability weighting and the investor sentiment itself, columns (4) and (6) tell a different story: *pdnd*, *nipo*, and s all have significant coefficient estimates for our proxy of the probability weighting in the loss region; *nipo* and $cefd$ both have significant estimates for the proxy of the probability weighting in the gain region. More importantly, these estimates also make economic sense.

Table 2.5: Option Return Differences against Investor Sentiment

	(1)	(2)	(3)	(4)	(5)	(6)
sent	-0.2736^{***}		0.0093		0.0604	
	(-5.4035)		(0.3134)		(1.4873)	
pdnd		-0.0002		0.0087^{***}		-0.0029
		(-0.0402)		(4.3588)		(-0.8483)
ripo		-0.0027		0.0005		-0.0005
		(-1.4267)		(0.5288)		(-0.3385)
nipo		0.0082^{***}		-0.0033^{***}		0.0051^{***}
		(3.8847)		(-3.2316)		(2.9177)
cefd		0.0342^{***}		0.0034		0.0313^{***}
		(3.6819)		(0.7469)		(4.0495)
s		-2.2399^{***}		1.1386^{***}		-0.7842
		(-2.7191)		(2.8485)		(-1.1452)

Note: This table shows the regression relationship between the proxies for prospect theory parameters and the investor sentiment and its components. We use the difference in average returns between ITM calls and ITM puts as the proxy for loss aversion, the difference in average returns between ITM puts and OTM puts as the proxy for probability weighting in the loss region, and the difference in average returns between ITM calls and OTM calls as the proxy for probability weighting in the gain region, respectively. The components of the investor sentiment include: the price premium of dividend-paying stocks over non-dividend-paying stocks (*pdnd*), first day return of IPO's (*ripo*), number of IPO's (*nipo*), closed-end fund discount (*cefd*), and the ratio of equity financing to debt and equity financing combined (s). t-statistics are reported in parentheses. *: $p < 0.1$, **: $p < 0.05$, ***: $p < 0.01$.

Intuitively, during periods of high sentiment, investors would tend to put less weight on the left tails and more weight on the right tails when making investment decisions. And that is exactly what we have found here: among the significant variables, all those that the investor sentiment has a positive loading on also predict c_1 positively, but predict c_2 negatively; and all those that the investor sentiment has a negative loading on predict c_1 negatively and c_2 positively. Thus, the investor sentiment not only has a close relationship with loss aversion, but is also fairly relevant in explaining the time-varying pattern of probability weighting, both in the loss region and the gain region.

2.4 Conclusion

This paper studies investors' time-varying loss aversion and probability weighting. Our idea is motivated by the experimental literature on prospect theory that finds people exhibit different degrees of loss aversion and probability weighting in different situations. We build a representative agent model that captures the investor's preference over both absolute level of wealth as well as her gains and losses in the stock market. The empirical part of the paper estimates the model using data on index options and portfolios of stocks. We find that investors are more loss averse after the 2007-2008 financial crisis, and that even though they seem to be just as worried about a stock market crash as before the crisis, they definitely put less decision weight on the stock market's prospect on the right tail. Moreover, our analysis suggests that both loss aversion and probability weighting are closely related to investor sentiment and its components: when sentiment is high, investors tend to be less loss averse, and their tendency to overweight tail events is weaker in the loss region and stronger in the gain region; and vice versa.

2.A Proofs

The derivation of the CRRA pricing kernel is easy and we omit it here. For the CPT part, in equilibrium, if the investor changes marginally her portfolio weights towards asset j, she will not get a higher CPT value. That is,

$$\left.\frac{\partial V(X_T + \alpha W_0(R_j - R_f))}{\partial \alpha}\right|_{\alpha=0} = 0$$

Ai et al. (2005) show that for a rank dependent expected utility function

$$U(X) = \int u(x)\, dw(P_X(x))$$

where w is also a probability weighting function, its directional derivative with respect to a random variable Y is:

$$\frac{\partial}{\partial \alpha} U(X + \alpha Y)\bigg|_{\alpha=0} = E[u'(X) w'(P_X(X)) Y]$$

Therefore, if we rewrite the two components of $V(X_T)$ as:

$$V^+(X_T) = -\int v(X) \mathbb{1}_{\{X \geq 0\}}\, dw^+(1 - P(X))$$
$$V^-(X_T) = \int v(X) \mathbb{1}_{\{X < 0\}}\, dw^-(P(X))$$

then we have

$$\frac{\partial V^+(X_T + \alpha W_0(R_j - R_f))}{\partial \alpha}\bigg|_{\alpha=0} = E[v'(X_T) \mathbb{1}_{\{X_T \geq 0\}} (w^+)'(1 - P(X_T)) W_0(R_j - R_f)]$$
$$\frac{\partial V^-(X_T + \alpha W_0(R_j - R_f))}{\partial \alpha}\bigg|_{\alpha=0} = E[v'(X_T) \mathbb{1}_{\{X_T < 0\}} (w^-)'(P(X_T)) W_0(R_j - R_f)]$$

Combining the two expressions above together with the CRRA pricing kernel, we have thus obtained the Euler equation:

$$E[(m^{CRRA} + m^{CPT})(R_j - R_f)] = 0$$

where $m^{CPT} = v'(X_T)\mathbb{1}_{\{X_T \geq 0\}}(w^+)'(1 - P(X_T)) + v'(X_T)\mathbb{1}_{\{X_T < 0\}}(w^-)'(P(X_T))$ is the cumulative prospect theory part of the pricing kernel.

In equilibrium, $X_T = W_0(R_M - R_f)$, which is monotonic in R_M, and also $v'(X_T) = \mathbb{1}_{\{X_T \geq 0\}} + \lambda \mathbb{1}_{\{X_T < 0\}}$, so the CPT pricing kernel becomes the following:

$$m^{CPT} = \mathbb{1}_{\{R_M \geq R_f\}}(w^+)'(1 - P(R_M - R_f)) + \lambda \mathbb{1}_{\{R_M < R_f\}}(w^-)'(P(R_M - R_f))$$

Chapter 3

Efficiency of Option Pricing Models: Evidence from the Chinese Warrants Market

Abstract: Empirical evidence shows that returns on Chinese stocks have negative skewness and fat tails, which contradicts the assumptions of the Black-Scholes option pricing model. This paper improves the Black-Scholes model to fit Chinese warrants prices from two perspectives: using nonlinear GARCH (NGARCH) models to capture stochastic volatility; and introducing jumps into returns to reflect big shocks. We apply these models to fitting warrant prices and compare their pricing errors. The result shows that NGARCH models outperform the Black-Scholes model; the NGARCH-Jump model is even better at depicting the price dynamics of underlying stocks, but provides marginal improvement to warrants valuation; out-of-the-money warrants are overvalued even after model adjustment, suggesting the existence of bubbles in the Chinese

warrants market.

3.1 Introduction

Since Black and Scholes (1973) introduced the Black-Scholes option pricing model, it has been widely used to compute the theoretical prices of options. Empirical analysis has shown that there is systematic pricing error associated with the model. For example, the volatility of underlying assets implied by the observed option prices and the Black-Scholes model, namely the implied volatility, tends to vary across different strike prices and maturities, which are known as the "volatility smile" and the "term structure of the implied volatility" respectively (see Hull (2003)). Black (1975) finds that the model overvalues in-the-money options but undervalues out-of-the-money options. MacBeth and Merville (1979) find exactly the opposite. The pricing bias is also prevalent in the Chinese warrants market. Xiong and Yu (2011) treat the pricing bias as bubbles and examine a set of bubble theories, but do not consider the possible flaw in the theoretical framework of pricing models.

On the basis of the Black-Scholes model, there is a strand of literature that revises its assumptions and proposes more generalized models in order to explain and improve the pricing errors. Specifically, the original Black-Scholes model assumes the underlying stock price follows a geometric Brownian motion with a constant volatility, and these generalized models modify the assumptions in two main dimensions. First, volatility is not constant during the whole lifetime of options, but rather follows a stochastic process itself. Second, other than the diffusion process, a jump process is needed to reflect the jumpiness of stock prices. Geske and Roll (1984) point out that the assumption of the normality of returns of underlying stocks is invalid. Merton (1976) introduces jumps into the Black-Scholes model and captures big shocks to stock prices using a compound Poisson process. Ball and Torous

(1983, 1985) simplify Merton's model with Binomial approximation to the Poisson distribution, and empirically verify the existence of jumps in the dynamics of stock prices, using data on returns of NYSE common stocks, but they find that there is no significant difference between theoretical prices calculated with these two methods. Bakshi et al. (1997) derive a model that allows volatility, interest rates, and jumps to all be stochastic, and examine its performance and that of several other alternatives. There is also literature that refers to the GARCH (Generalized Autoregressive Conditional Heteroskedasticity) models. For instance, Christoffersen et al. (2010) find that the nonlinear GARCH (NGARCH) model performs best among all GARCH option pricing models with normal innovations. Duan et al. (2007) introduce jumps into GARCH models and document a better pricing performance with data on the S&P 500 index options.

Although warrants are defined slightly differently in textbooks such as Hull (2003), the Chinese warrants have exactly the same payoff structure as a typical European option. Therefore, one would expect the prices of these warrants to be largely consistent with various options pricing models mentioned above. However, to the best of our knowledge, none of the papers in the literature examines the performance of different options pricing models in the Chinese warrants market. One of the contributions of this paper is thus to fill this gap. On the other hand, it is well known that returns of Chinese stocks tend to have negative skewness and heavy tails, in contradiction to the normality assumption in the Black-Scholes model. Accordingly, stochastic volatility and jumps are very important in capturing the dynamics of underlying stock prices, which in turn is crucial to understanding the warrant prices.

In this paper, we consider several options pricing models including the canonical Black-Scholes and its generalizations, and apply them to the Chinese warrants market. First, we estimate the parameters of these models with maximum likelihood estimation (MLE), using data on the returns of the underlying stocks. Then we apply the method of Monte Carlo simulation to generate future returns of the stocks in the risk-neutral world, according to the price dynamics implied by the estimated parameters in the first step. Finally, we discount the payoffs of the warrants on expiration dates at the risk free rate to obtain their theoretical prices. By comparing the differences between theoretical and real prices, we explain a significant portion of the pricing bias from the perspective of the theoretical framework.

The rest of this paper is organized as follows: section 3.2 sets up the model, section 3.3 shows the main empirical evidence, and section 3.4 concludes.

3.2 Model Specification

Empirical evidence shows that returns on Chinese stocks exhibit negative skewness and fat tails, unlike the normality assumption made by the Black-Scholes model. In order to better depict the dynamics of stock prices, we follow Duan et al. (2007) to fix the Black-Scholes model in two respects. First, we use GARCH models to capture the stochastic volatility on returns of stock prices. Second, we incorporate jumps into the dynamics of stock returns to reflect big shocks on stock prices in the market.

3.2.1 Black-Scholes Model

The Black-Scholes model is the most widely used options pricing model. In its original form, it is specified in continuous time. Since all the alternatives we use in this paper are in discrete time, we include the discrete version of the Black-Scholes model here fore completeness. Specifically, the model assumes that the daily return of the underlying stock,

r_t, follows the stochastic process below:

$$r_t = \alpha + \sigma X_t \tag{3.1}$$

where α and σ are the mean return and the volatility of the returns during the lifetime of the option, and X_t is Gaussian white noise, with $X_t \sim N(0,1)$. Black and Scholes (1973) derive a closed-form formula for prices of options whose underlying stock follows the dynamics described above, and the formula involves only volatility but not the mean of the underlying returns. In our analysis, the analytical solutions are very close to numerical solutions given by Monte Carlo simulation, and because none of more generalized models have an analytical solution, we only display the Monte Carlo results for comparison purposes.

3.2.2 NGARCH-Normal Model

One natural generalization of the original Black-Scholes model is to allow the conditional mean and volatility of the stock returns to vary across time. In Duan et al. (2007), the authors suggest an equilibrium model that incorporates stochastic volatility to the dynamics of underlying stock prices. Specifically, their results show that under the real world measure P, daily returns on the underlying stock r_t can be described by the processes below:

$$r_t = \alpha_t + \sqrt{h_t} X_t \tag{3.2}$$

$$\alpha_t = r - \frac{h_t}{2} - \rho\sqrt{h_t} \tag{3.3}$$

$$h_t = g(h_{t-1}, X_{t-1}) \tag{3.4}$$

where α_t and h_t are the mean and variance of r_t conditional on $\mathcal{F}_{t-1}$, the information set at $t-1$, $X_t \sim N(0,1)$ is a random shock to r_t and is independent of $\mathcal{F}_{t-1}$, r is the risk free rate, and ρ is the correlation between innovations of stock returns and of the stochastic discounting factor, which for simplicity we do not specify here. Equation (3.4) states that the conditional variance at time t depends on the conditional variance at time $t-1$ and the innovation at time $t-1$, and the dependence can take any form. One common modeling device for stochastic volatility in discrete time is the GARCH-type models proposed by Engle (1982) and Bollerslev (1986). There is a whole strand of literature that studies the performance of different versions of the GARCH models in fitting asset returns, but the main purpose of this paper is to document the importance of stochastic volatility in warrants valuation instead of comparing various GARCH models. Therefore, we follow Christoffersen et al. (2010) to use a nonlinear GARCH (NGARCH) model in our analysis, because the authors of that paper show the NGARCH model performs best in terms of pricing options. Specifically, the NGARCH model specifies that the conditional variance h_t of stock returns follows the process below:

$$h_t = \beta_0 + \beta_1 h_{t-1} + \beta_2 h_{t-1} \left(X_{t-1} - c\right)^2 \tag{3.5}$$

where $\{\beta_0, \beta_1, \beta_2, c\}$ is the set of preliminary parameters. The process is no different than the canonical GARCH(1,1) model, other than the extra parameter c. This parameter captures the asymmetry in the influence of positive and negative innovations on the conditional variance next period. Usually c has a positive sign, so that a negative shock on the stock return will have a larger impact on the stock volatility the next day than a positive shock of the same magnitude, which is known as the “leverage effect”. We expect this effect to be more important in less developed financial markets such as the Chinese stock market, and that is another reason why we use this specific form of GARCH model in this paper. To ensure the conditional variance h_t is positive, we need to put restrictions on the parameters: β_0 is positive, β_1 and β_2 are nonnegative. The process is strictly stationary if $\beta_1 + \beta_2(1 + c^2) \leq 1$. The unconditional mean of h_t is finite and equals $\beta_0 / \left[1 - \beta_1 - \beta_2\left(1 + c^2\right)\right]$ if $\beta_1 + \beta_2(1 + c^2) < 1$.

3.2.3 NGARCH-Jump Model

In the NGARCH-Normal model, the innovations to the stock returns are Gaussian white noise. However, the prices of underlying stocks may also be exposed to big shocks besides small normal shocks. Such big shocks usually result from the revelation of critical information in the market. In order to reflect the effect of these big shocks, we introduce jumps into the return process of underlying stocks. In a given period, the jumps occur as often as the stock price is influenced by critical information. Duan et al. (2007) show that under this assumption the return process r_t is as follows (again for simplicity we omit the specification for the pricing kernel here):

$$r_t = \alpha_t + \sqrt{h_t} J_t \tag{3.6}$$

$$\alpha_t = r - \frac{h_t}{2} - \rho\sqrt{h_t} + \lambda(1 - K_t) \tag{3.7}$$

$$h_t = \beta_0 + \beta_1 h_{t-1} + \beta_2 h_{t-1}\left(\frac{J_{t-1} - \lambda\mu}{\sqrt{1 + \lambda\tilde{\gamma}^2}} - c\right)^2 \tag{3.8}$$

$$K_t = \exp\left(\sqrt{h_t}\mu + \frac{1}{2} h_t \gamma^2\right) \tag{3.9}$$

$$J_t = X_t^{(0)} + \sum_{j=1}^{N_t} X_t^{(j)} \tag{3.10}$$

where

$$\begin{aligned}
N_t &\sim \text{Poisson}(\lambda) \\
X_t^{(0)} &\sim N(0,1) \\
X_t^{(j)} &\sim N(\mu, \gamma^2) \text{ for } j = 1, 2, \ldots
\end{aligned}$$

and $\tilde{\gamma}^2 = \mu^2 + \gamma^2$. N_t represents the frequency of big shocks to the stock price at date t, $X_t^{(0)}$ represents the small shock, and $X_t^{(j)}$ represents big shocks. N_t and $X_t^{(j)}$ are independent and they are both independent of $\mathcal{F}_{t-1}$. Here $X_t^{(j)}$'s are "big" if γ is larger than 1, meaning that they tend to move stock prices by a bigger magnitude. Now h_t is not the conditional variance of stock returns anymore, but can be understood as a scaling factor for J_t, which is the shock to the stock return in period t that aggregates the small shock and the big shocks. Again, we require $\beta_0 > 0$, $\beta_1 \geq 0$, and $\beta_2 \geq 0$ to ensure the h_t process is positive.

Here the model differs from the NGARCH-Normal model in that the innovation term J_t involves a Poisson random sum of several normal random variables. It is easy to show that the mean and variance of J_t are

$$\begin{aligned}
E[J_t] &= E[X_t^{(0)}] + E\left[E\left[\sum_{j=1}^{N_t} X_t^{(j)} \middle| N_t\right]\right] \\
&= \mu E[N_t] \\
&= \lambda\mu
\end{aligned}$$

$$
\begin{aligned}
Var[J_t] &= Var[X_t^{(0)}] + E\left[Var\left[\sum_{j=1}^{N_t} X_t^{(j)} \middle| N_t\right]\right] + Var\left[E\left[\sum_{j=1}^{N_t} X_t^{(j)} \middle| N_t\right]\right] \\
&= 1 + \gamma^2 E[N_t] + \mu^2 E[N_t] \\
&= 1 + \lambda\tilde{\gamma}^2
\end{aligned}
$$

respectively, using the law of iterated expectations. Therefore, we standardize the innovation when specifying the GARCH model in equation (3.8) in order to make the NGARCH-Jump model comparable to the NGARCH-Normal model. Note that when $\lambda = 0$ the NGARCH-Jump model degenerates to the NGARCH-Normal model.

3.3 Empirical Evidence

In this section, we first estimate the models in the previous section using time series data on returns of underlying stocks. Then we conduct Monte Carlo simulations to calculate warrant prices using risk neutral pricing, and compare the pricing performance of the Black-Scholes model, the NGARCH-Normal model, and the NGARCH-Jump model.

3.3.1 Data and Summary Statistics

Since August 22, 2005, a total of 55 warrants have been issued in the Chinese warrants market and all of them are stock warrants. We obtain the basic information and time series data of the warrants and their underlying stocks from the RESSET Database[1]. The basic information includes the types (call or put, American or European), expiration dates, and strike prices of the warrants. Table 3.1 and Table 3.2 provide a complete list of the 37 call warrants and the 18 put warrants in our sample, respectively. The time series data

1. www.resset.com/cn

includes the close prices of the warrants and their underlying stocks from the issue dates to the expirations dates. After excluding one warrant of which the data on prices of the underlying stock is missing[2], and one American warrant[3], there are 53 remaining. The data set has 16,368 observations of the 53 warrants, 36 of which are call warrants, and 19 are put warrants. The data set also provides data on the risk free rate. We use in our estimation the annual percentage rate of the three-month China's central bank bill, converted to a daily rate of return.

Table A5 and Table A6 in the Appendix show the summary statistics for the returns of the underlying stocks. For each warrant, we calculated the mean, standard deviation, skewness, kurtosis, the 5%, 25%, 50%, 75%, and 95% percentile, as well as the minimum and maximum of the underlying stock returns during the lifetime of the warrant. Several findings in the table are worth noting. First, about two thirds of the underlying stocks (35 of them) display negative skewness in their returns, which is consistent with the anecdotal evidence for Chinese stocks in general. Second, most of the stocks, actually 50 out of 53, have a kurtosis greater than 3, the kurtosis of normally distributed random variables. This is strong evidence that Chinese stocks, at least those in our sample, have fairly heavy tails in their return distributions. Third, in order to statistically test whether the returns come from normal distributions, we conduct the Jarque-Bera test for each of the stocks, and the p-values are shown in the last columns of Table A5 and Table A6. The null hypothesis of the test is that the returns are normally distributed, and a small p-value rejects the null. We can see from the table that the null hypothesis gets rejected for 41 stocks at the 5% level (35 at the 1% level). This again corroborates our belief that the underlying stock

2. Sichuan Changhong Electric Co., Ltd., listed in the Shanghai Stock Exchange.

3. Guangzhou Baiyun International Airport, listed in the Shanghai Stock Exchange.

Table 3.1: Summary Information for the 37 Call Warrants

Name (exchange)	Warrant shares (million)	Trading period Begin	Trading period End	Price at trading end Stock price	Price at trading end Stirke price	Exercise ratio	Exercise period Begin	Exercise period End
AnGang (SZ)	113.10	12/05/05	12/05/06	7.87	3.60	1	12/01/06	12/05/06
WULiang (SZ)	297.87	04/03/06	04/02/08	25.92	6.93	1	03/27/08	04/02/08
QiaoCheng (SZ)	149.52	11/24/06	11/23/07	60.10	7.00	1	11/19/07	11/23/07
GangFan (SZ)	800.00	12/12/06	12/11/08	8.80	3.95	1	11/28/08	12/11/08
ShenFa (SZ)	208.68	06/29/07	12/28/07	36.75	19.00	1	11/19/07	12/28/07
ShenFa (SZ)	104.34	06/29/07	06/27/08	21.00	19.00	1	05/16/08	06/27/08
GuoAn (SZ)	95.71	09/25/07	09/24/09	18.70	35.50	0.5	09/11/09	09/24/09
ZhongXing (SZ)	65.20	02/22/08	02/21/10	42.70	78.13	0.5	02/01/10	02/12/10
EJiao (SZ)	130.94	07/18/08	07/17/09	19.13	5.50	1	07/13/09	07/17/09
BaoGang (SH)	387.70	08/18/05	08/30/06	4.17	4.50	1	08/30/06	08/30/06
WuGang (SH)	474.00	11/23/05	11/22/06	3.35	2.90	1	11/16/06	11/22/06
BaoGang (SH)	714.91	03/31/06	03/30/07	5.70	2.00	1	03/26/07	03/30/07
GanGang (SH)	925.71	04/05/06	04/04/07	6.81	2.80	1	03/29/07	04/04/07
ShouChuang (SH)	60.00	04/24/06	04/23/07	9.97	4.55	1	04/17/07	04/23/07
WanHua (SH)	56.58	04/27/06	04/26/07	38.75	9.00	1	04/20/07	04/26/07
YaGe (SH)	90.66	05/22/06	05/21/07	26.44	3.80	1	05/17/07	05/21/07
ChangDian (SH)	1,228.01	05/25/06	05/24/07	14.49	5.50	1	05/18/07	05/24/07
GuoDian (SH)	151.07	09/05/06	09/04/07	15.24	4.80	1	08/29/07	09/04/07
YiLi (SH)	154.94	11/15/06	11/14/07	28.98	8.00	1	11/08/07	11/14/07
MAGang (SH)	1,265.00	11/29/06	11/28/08	4.10	3.40	1	11/17/08	11/28/08
ZhongHua (SH)	180.00	12/18/06	12/17/07	20.10	6.58	1	12/11/07	12/17/07
YunHua (SH)	54.00	03/08/07	03/07/09	27.72	18.23	1	02/23/09	03/06/09
WuGang (SH)	727.50	04/17/07	04/16/09	7.39	10.20	1	04/10/09	04/16/09
ShenGao (SH)	108.00	10/30/07	10/29/09	5.68	13.85	1	10/23/09	10/29/09
RIZhao (SH)	61.60	12/03/07	12/02/08	5.13	14.25	1	11/19/08	12/02/08
ShangQi (SH)	226.80	01/08/08	01/07/10	26.61	27.43	1	12/31/09	01/07/10
GanYue (SH)	56.40	02/28/08	02/27/10	7.99	20.88	1	02/08/10	02/26/10
ZhongYuan (SH)	51.45	02/26/08	08/25/09	11.71	40.38	0.5	08/19/09	08/25/09
ShiHua (SH)	3,030.00	03/04/08	03/03/10	11.15	19.68	0.5	02/25/10	03/03/10
ShangGang (SH)	291.55	03/07/08	03/06/09	3.82	8.40	1	03/02/09	03/06/09
QingPi (SH)	105.00	04/18/08	10/19/09	29.94	28.32	0.5	10/13/09	10/19/09
GuoDian (SH)	427.47	05/22/08	05/21/10	3.70	7.50	1	05/17/10	05/21/10
KangMei (SH)	166.50	05/26/08	05/25/09	8.26	10.77	0.5	05/19/09	05/25/09
BaoGang (SH)	1,600.00	07/04/08	07/03/10	6.04	12.50	0.5	06/28/10	07/03/10
GeZhou (SH)	301.63	07/11/08	01/10/10	11.71	9.19	0.5	01/04/10	01/08/10
JiangTong (SH)	1,761.20	10/10/08	10/09/10	29.49	15.44	0.25	09/27/10	10/08/10
ChangHong (SH)	573.00	08/19/09	08/18/11	-	5.23	1	08/12/11	08/18/11

Table 3.2: Summary Information for the 18 Put Warrants

Name (exchange)	Warrant shares (million)	Trading period Begin	Trading period End	Price at trading end: Stock price	Price at trading end: Stirke price	Price at trading end: Exercise ratio	Exercise period Begin	Exercise period End
GangFan (SZ)	233.34	11/04/05	05/08/07	10.72	4.85	1	05/08/07	05/08/07
WanKe (SZ)	2,140.29	12/05/05	09/04/06	6.79	3.73	1	08/29/06	09/04/06
HuaLing (SZ)	633.18	03/02/06	03/01/08	12.45	4.90	1	02/27/08	02/29/08
WULiang (SZ)	313.15	04/03/06	04/02/08	25.92	7.96	1	03/27/08	04/02/08
ShenNeng (SZ)	437.68	04/27/06	10/26/06	7.25	7.12	1	10/20/06	10/26/06
ZhongJi (SZ)	424.11	05/25/06	11/23/07	24.11	10.00	1	11/19/07	11/23/07
JiaFei (SZ)	120.00	06/30/06	06/29/07	45.21	15.10	1	06/25/07	06/29/07
NanHang (SH)	1,400.00	06/21/07	06/20/08	8.48	7.43	0.5	06/20/08	06/20/08
MaoTai (SH)	431.88	05/30/06	05/29/07	94.84	30.30	0.25	05/29/07	05/29/07
HaiEr (SH)	607.36	05/17/06	05/16/07	15.79	4.39	1	05/10/07	05/16/07
YaGe (SH)	634.63	05/22/06	05/21/07	26.44	4.25	1	05/17/07	05/21/07
WanHua (SH)	84.86	04/27/06	04/26/07	38.75	13.00	1	04/20/07	04/26/07
YuanShui (SH)	280.43	04/19/06	02/12/07	6.54	5.00	1	02/06/07	02/12/07
BaoGang (SH)	714.91	03/31/06	03/30/07	5.70	2.45	1	03/26/07	03/30/07
HuChang (SH)	567.72	03/07/06	03/06/07	25.52	13.60	1	03/06/07	03/06/07
ZhaoHang (SH)	2,241.34	03/02/06	09/01/07	39.04	5.65	1	08/27/07	08/31/07
JiChang (SH)	240.00	12/23/05	12/22/06	7.94	7.00	1	03/23/06	12/22/06
WuGang (SH)	474.00	11/23/05	11/22/06	3.35	3.13	1	11/16/06	11/22/06

returns are generally not normally distributed, in contradiction to the assumptions of the Balck-Scholes model. Therefore, introducing stochastic volatility and jumps into the model is crucial to capturing the dynamics of the stock prices as well as valuing warrants written on these stocks.

3.3.2 Model Estimation

The first step of our analysis is to estimate the model parameters that govern the dynamics of stock returns. We employ the maximum likelihood estimation (MLE) procedure here, utilizing the parametric assumption that the shock components in all the models are Gaussian white noises. Specifically, in the Black-Scholes model, the set of parameters is $\theta = \{\alpha, \sigma\}$ as in equation (3.1). The probability density function of r_t conditional on $\mathcal{F}_{t-1}$ is:

$$\phi(r_t; \alpha, \sigma) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(r_t - \alpha)^2}{2\sigma^2}\right)$$

where $\phi(\cdot;\alpha,\sigma)$ is the probability density function of a normally distributed random variable with mean α and standard deviation σ. As the daily returns are independent across time in the Black-Scholes model, the conditional log likelihood function is:

$$
\begin{aligned}
L(\theta|\mathbf{r}) &= \sum_{t=1}^{T} \log \phi\left(r_t;\alpha,\sigma\right) \\
&= -\frac{T}{2}\log(2\pi\sigma^2) - \sum_{t=1}^{T} \frac{(r_t-\alpha)^2}{2\sigma^2}
\end{aligned}
$$

where $\mathbf{r} = \{r_t\}_{t=1}^{T}$ is the time series of returns and T is the duration of the observations used for estimation. The first order conditions determine the maximum likelihood estimates:

$$
\begin{cases}
\alpha = \frac{1}{T}\sum_{t=1}^{T} r_t \\
\sigma^2 = \frac{1}{T}\sum_{t=1}^{T} (r_t-\alpha)^2
\end{cases}
$$

We therefore use the sample mean and sample standard deviation as our estimates for α and σ respectively.

As for the NGARCH-Normal model, the set of parameters is $\theta = \{\beta_0, \beta_1, \beta_2, c, \rho\}$. By equation (3.2), the probability density function of r_t conditional on $\mathcal{F}_{t-1}$ is:

$$
\phi\left(r_t;\alpha_t,h_t\right) = \frac{1}{\sqrt{2\pi h_t}}\exp\left(-\frac{(r_t-\alpha_t)^2}{2h_t}\right)
$$

Hence the conditional log likelihood function is:

$$
\begin{aligned}
L(\theta|\mathbf{r}) &= \sum_{t=1}^{T} \log \phi\left(r_t;\alpha_t,h_t\right) \\
&= -\frac{T}{2}\log(2\pi) - \frac{1}{2}\sum_{t=1}^{T}\log(h_t) - \sum_{t=1}^{T}\frac{(r_t-\alpha_t)^2}{2h_t}
\end{aligned}
\tag{3.11}
$$

Now there is no analytical solution for the maximum likelihood estimators, as the first order conditions involve polynomial terms of higher order. Therefore we compute the estimates numerically for the NGARCH-Normal model and for all the models hereafter as well. Notice that the conditional variance h_t is defined recursively based on the relationship in equation (3.5), and the time series of h_t ultimately depends on the initial value h_1. In our analysis, we use the sample variance of r_t as the initial value. Given the initial value h_1, we calculate α_1 using equation (3.3), then X_1 by equation (3.2), and finally h_2 by equation (3.5) again, and so on so forth. Therefore, we are able to compute the whole series of $\{r_t, \alpha_t, h_t\}$, and also the log likelihood function once we plug them into equation (3.11) for any given set of parameter θ. Then we maximize the likelihood function over θ to find the estimates.

In the NGARCH-Jump model, the set of parameters is:

$$\theta = \{\beta_0, \beta_1, \beta_2, c, \rho, \lambda, \mu, \gamma\}$$

By equations (3.6) and (3.10), the probability density function of r_t conditional on $\mathcal{F}_{t-1}$ and $N_t = n$ is:

$$\phi\left(r_t; \alpha_t + \sqrt{h_t} n\mu, h_t\left(1 + n\gamma^2\right)\right)$$

Since N_t is a Poisson random variable and independent of any other variables, we have $\Pr(N_t = n) = e^{-\lambda}\lambda^n/n!$. Integrate out N_t, we get the probability density function of r_t conditional on $\mathcal{F}_{t-1}$:

$$\sum_{n=0}^{\infty} \frac{e^{-\lambda}\lambda^n}{n!} \phi\left(r_t; \alpha_t + \sqrt{h_t} n\mu, h_t\left(1 + n\gamma^2\right)\right)$$

And the conditional log likelihood function is therefore:

$$L(\theta|\mathbf{r}) = \sum_{t=1}^{T} \log \left(\sum_{n=0}^{\infty} \frac{e^{-\lambda}\lambda^n}{n!} \phi \left(r_t; \alpha_t + \sqrt{h_t} n\mu, h_t \left(1 + n\gamma^2\right)\right)\right) \tag{3.12}$$

Based on the upper bound of cutoff errors given by Ball and Torous (1985), we obtain the MLE based on a finite version of the log likelihood function:

$$L_N(\theta|\mathbf{r}) = \sum_{t=1}^{T} \log \left(\sum_{n=0}^{N} \frac{e^{-\lambda}\lambda^n}{n!} \phi \left(r_t; \alpha_t + \sqrt{h_t} n\mu, h_t \left(1 + n\gamma^2\right)\right)\right) \tag{3.13}$$

with $N = 10$. The numerical procedure is very similar to that for the NGARCH-Normal model. Specifically, given initial value h_1 and a set of parameters θ, we calculate K_1 using equation (3.9), then α_1 by equation (3.7), J_1 by equation (3.6), and finally h_2 by equation (3.8) again, and so on so forth. Then we plug the series $\{r_t, \alpha_t, h_t\}$ into equation (3.13) to get the log likelihood function, and maximize it over θ to obtain the MLE.

For each underlying stock, we estimate the parameters in the Black-Scholes model, the NGARCH-Normal model and the NGARCH-Jump model. Due to limitation of space, we show in Table 3.3 the estimation result for only one of these stocks, TTB[4]. The results for other stocks are available upon request. Columns 2 through 4 correspond to the three model specifications respectively, and standard errors are given in parentheses. For the Black-Scholes model, our estimates are the average daily return $\alpha = 0.0008$ and the volatility of daily returns $\sigma = 0.0287$. We plot the histogram of TTB's daily returns in Figure 3.1b together with the normal distribution implied by the estimates of α and σ. From Figure 3.1b we can see that the return distribution exhibits higher peak and fatter tails than the normal distribution, with a corresponding kurtosis of 5.1016, and it is slightly skewed to

4. Tsingtao Brewery Co. Ltd., stock code: 600600, listed in Shanghai Stock Exchange.

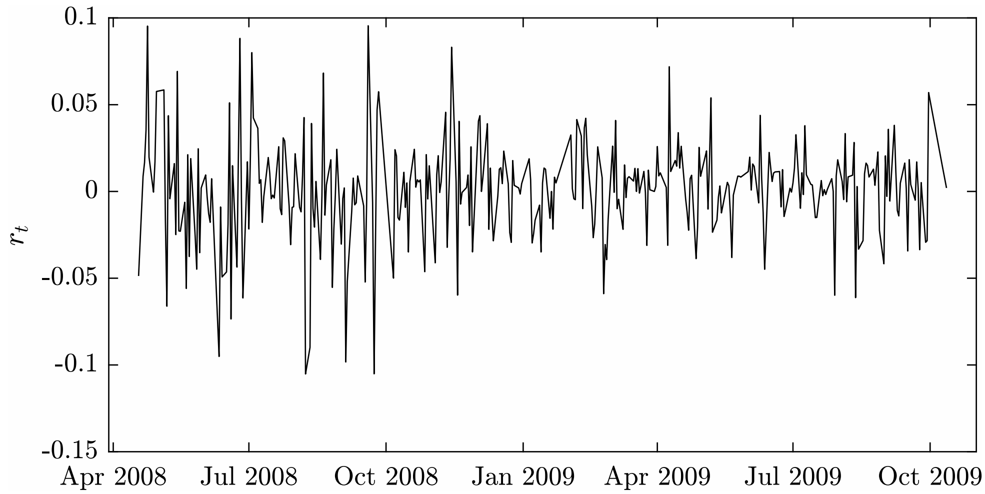
the left, with a skewness of -0.2867. Moreover, Figure 3.1a shows the time series plot of the daily returns of TTB. The figure indicates that the returns do not have a constant volatility across time, with the volatility before January 2009 obviously higher than that afterwards. Also, higher volatility is followed by higher volatility, and vice versa. In fact, as shown in Figure 3.1c, the absolute values of the stock's daily returns have significantly positive autocorrelation with lags of one, two, and four trading days, again indicative of the existence of volatility clustering. Therefore, the two parameters in the Black-Scholes model are not sufficient to fully depict the price dynamics of the TTB stock.

More interesting are the results for the NGARCH models. First, we can see from columns 3 and 4 of Table 3.3 that the estimates for β_1 and β_2 are significantly positive at the 1% level in both NGARCH models, which means that higher past volatility and return innovations of larger magnitude are associated with higher current volatility. Thus, volatility clustering is indeed present in the price dynamics of the TTB stock, and GARCH models are necessary to capture this phenomenon. Second, the estimates for the "leverage effect", c, are positive as expected, although not significantly so. This implies that for the TTB stock, positive shocks and negative shocks to the stock returns contribute similarly to future volatility. Even though this phenomenon is not common to all the underlying stocks, it is not surprising for TTB given that the unconditional distribution of its returns is only slightly negatively skewed. In contrast, we would observe a more negatively skewed distribution and a significantly positive estimate for c if the "leverage effect" exists. Third, the correlation between the innovations of stock returns and the pricing kernel has negative estimates in both models, though not significant. This result is intuitive, as most common stocks tend to pay off well when the whole economy does well and the marginal utility is low. Fourth, in the NGARCH-Jump model, λ is highly significant, meaning that there exists a jump component in the stock returns, and the estimate implies that jumps happen 1.1466 times per day on average. The mean of the jump, μ, is significantly negative, meaning the jumps have a tendency to drive the stock prices lower. Also, the standard deviation of jumps, γ, is significant and greater than one, so the jump process effectively captures the large shocks that move stock returns by a larger magnitude than the usual small shocks. Finally, the last row of Table 3.3 shows the log likelihood of the three models, and it increases monotonically as the model gets more complicated.

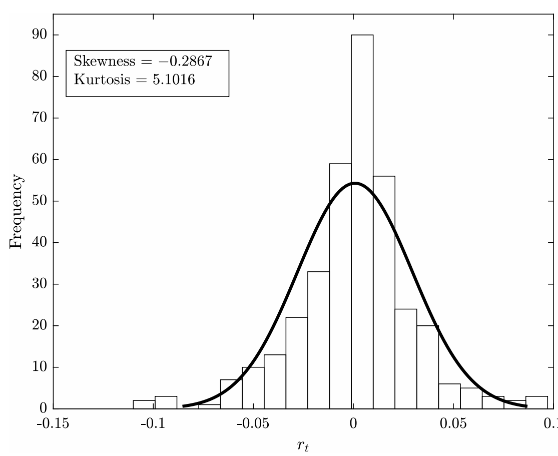
Table 3.3: Parameter Estimates for TTB

	Black-Scholes	NGARCH-Normal	NGARCH-Jump
α	0.0008		
	(0.0015)		
σ	0.0287		
	(0.0011)		
β_0		8.58×10^{-6}	1.55×10^{-6}
		(2.69×10^{-6})	(1.93×10^{-6})
β_1		0.9520	0.9390
		(0.0148)	(0.0194)
β_2		0.0218	0.0372
		(0.0046)	(0.0150)
c		0.7457	0.3624
		(0.5006)	(0.4802)
ρ		-0.0513	-0.1616
		(0.0528)	(0.1511)
λ			1.1466
			(0.2139)
μ			-0.3786
			(0.1845)
γ			2.4926
			(0.3680)
Log Likelihood	765.34	783.74	804.37

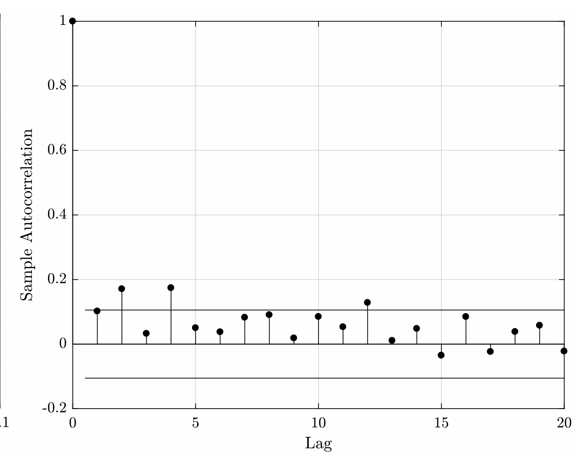
Our estimation procedure not only gives the parameter estimates, but is also able to back out the innovations to the stock returns every day based on the estimated parameters. Figure 3.2a presents the time series plot of the innovations X_t in the NGARCH-Normal model. Now the innovations do not show any clear patterns in their volatility across time,



(a) Time Series Plot



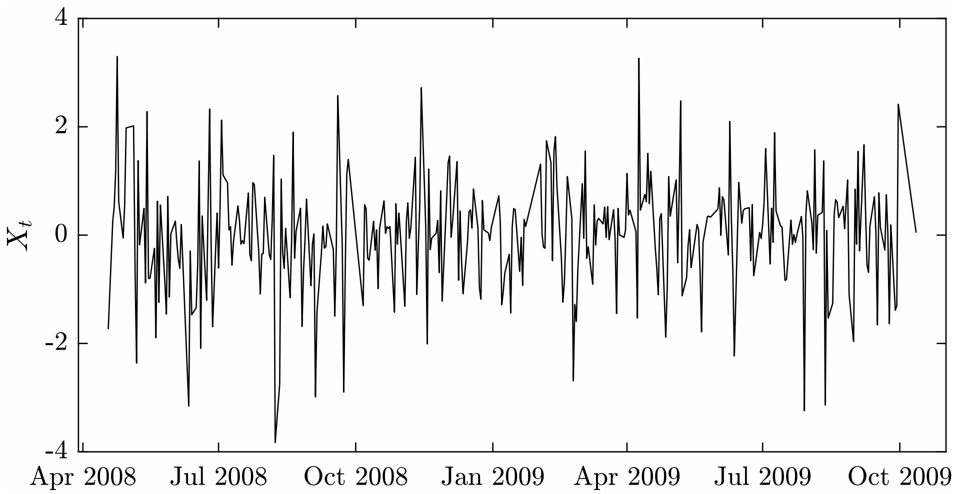
(b) Histogram



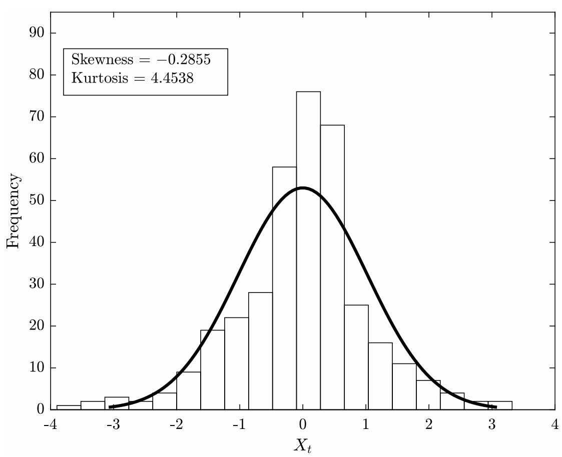
(c) Autocorrelation Plot

Figure 3.1: Daily Returns of TTB

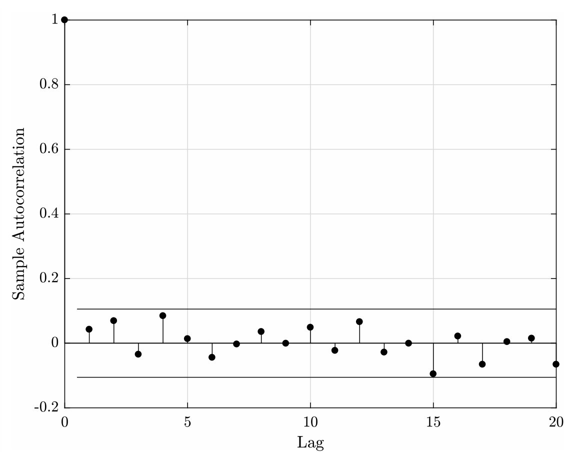
in contrast to Figure 3.1a for the Black-Scholes model. Actually, Figure 3.2c shows that the absolute values of the return innovations have an insignificant autocorrelation for lags up to 20 trading days. Therefore, the NGARCH-Normal model does fairly well in removing the volatility clustering in the daily returns of the TTB stock. On the other hand, the NGARCH-Normal model also assumes the return innovations are Gaussian white noise, $X_t \sim N(0, 1)$. As a comparison, the sample standard deviation of X_t's is 1.0265, very close to the assumed value of 1. However, the histogram of X_t displayed in Figure 3.2b shows the distribution of X_t is still quite different than normal. Although both skewness and kurtosis of X_t are smaller in magnitude than those of r_t in the Black-Scholes model, it still has a high peak around the center of its distribution, and the Jarque-Bera test rejects the null hypothesis at the 1% level. Thus, the NGARCH-Normal model is not completely consistent with the stock price dynamics of TTB, and we need further improvement in modeling.



(a) Time Series Plot



(b) Histogram



(c) Autocorrelation Plot

Figure 3.2: Return Innovations of TTB in the NGARCH-Normal Model

Besides the return innovations, our procedure also gives the scaling factors of innovations, h_t. In the NGARCH-Normal model, h_t is the same as the conditional variance. We plot the annualized conditional volatility of TTB, i.e., $\sqrt{h_t}$ in Figure 3.3. The dashed horizontal line is the unconditional volatility over the sample period, 45.63%. We can see from the figure that our estimates of conditional volatility are highly persistent, and evolve around the unconditional volatility. Also, loosely speaking, the conditional volatility after January 2009 is generally lower than before, consistent with our observation from the time series plot of the raw returns in Figure 3.1a.

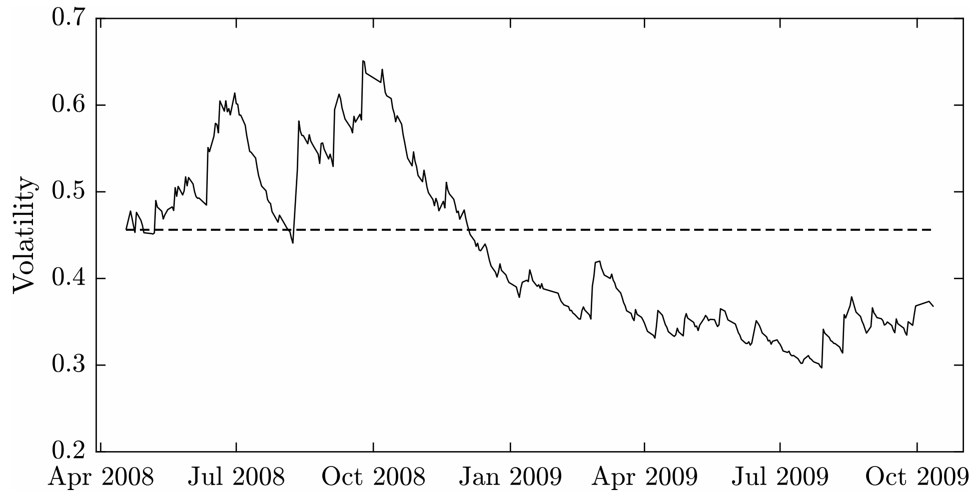
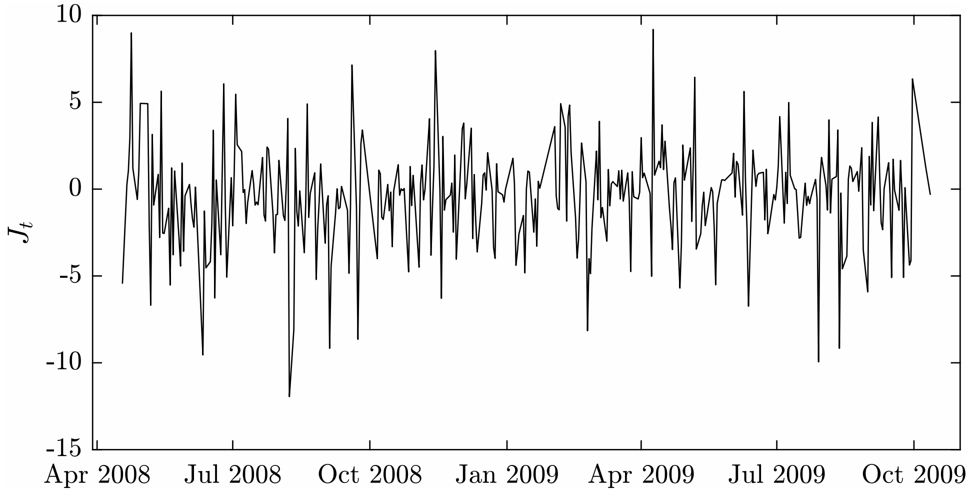


Figure 3.3: Annualized Volatility of TTB in the NGARCH-Normal Model

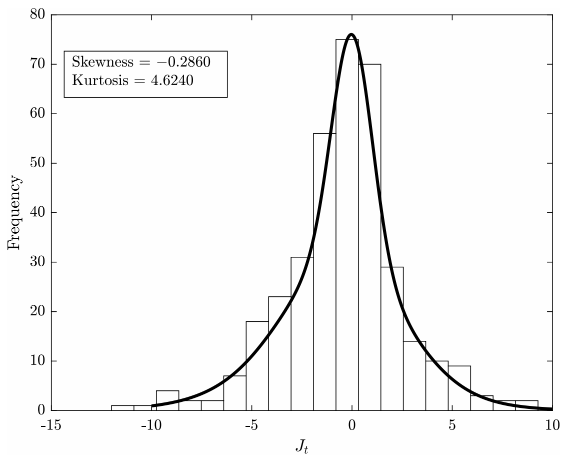
As the fitted return innovations of TTB in the NGARCH-Normal model fail to meet the assumption of normal distribution and exhibit excess kurtosis, we further investigate the property of the innovations in the NGARCH-Jump model. Figure 3.4a shows the time series plot of J_t in the NGARCH-Jump model. As in the the NGARCH-Normal model, the volatility of the innovations remains stable across time. Furthermore, the autocorrelation of the absolute values of return innovations is not significantly different than zero for all the lags up to 20 trading days as shown in Figure 3.4c. Thus, the NGARCH-Jump model does a similar job to the NGARCH-Normal model in removing the volatility clustering in the stock returns of TTB. More important is the histogram of J_t in Figure 3.4b. Remember that J_t is a random sum of several normally distributed random variables, and the number of the random variables is a Poisson variable. Therefore, J_t itself is not normal, and in fact it has a kurtosis of 4.6240. The thick solid line in Figure 3.4b represents the density function of J_t implied by our estimates of the jump component, $\{\lambda, \mu, \gamma\}$. We can see that the fitted innovations follow the implied distribution amazingly well, compared to the case of the Black-Scholes model and the NGARCH-Normal model. We note that this phenomenon is not only true for the TTB stock, but generally present for most of the stocks in our sample. Thus, the GARCH model alone is not able to fully capture the price dynamics of the underlying stocks, and jump components are necessary. Figure 3.5 shows the time series plot of the scaling factor, $\sqrt{h_t}$, in the NGARCH-Jump model. Notice that here $\sqrt{h_t}$ cannot be interpreted as conditional volatility of stock returns, and hence not comparable to $\sqrt{h_t}$ in the NGARCH-Normal model as well as the unconditional return volatility. However, it still governs how volatile the stock return is in a given day. We can see that the scaling factor is higher in the first half of the sample period than in the second half, consistent with the result in the NGARCH-Normal model.

3.3.3 Pricing Performance

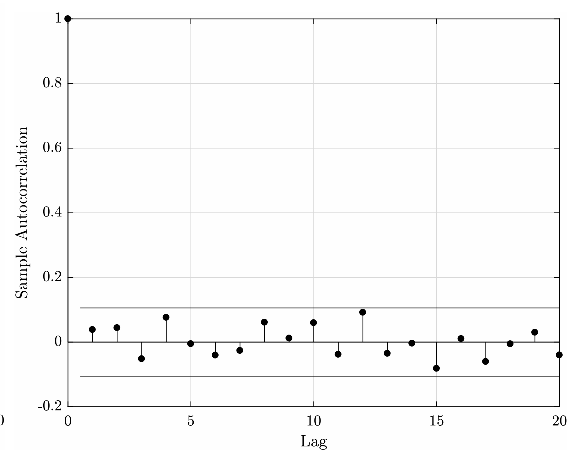
In this subsection we compare the pricing performance of the various models we estimated in the previous subsection. While the Black-Scholes model provides a closed form solution to the option prices, the NGARCH models do not, so we compute the theoretical prices implied by the NGARCH models using risk neutral pricing. First, we obtain the price dynamics of underlying stocks under the risk neutral measure Q, then conduct Monte Carlo simulations to get the expected payoffs of the warrants at maturity, and finally discount the expected payoffs at the risk-free rate to get warrant prices. According to Duan et al.



(a) Time Series Plot



(b) Histogram



(c) Autocorrelation Plot

Figure 3.4: Return Innovations of TTB in the NGARCH-Jump Model

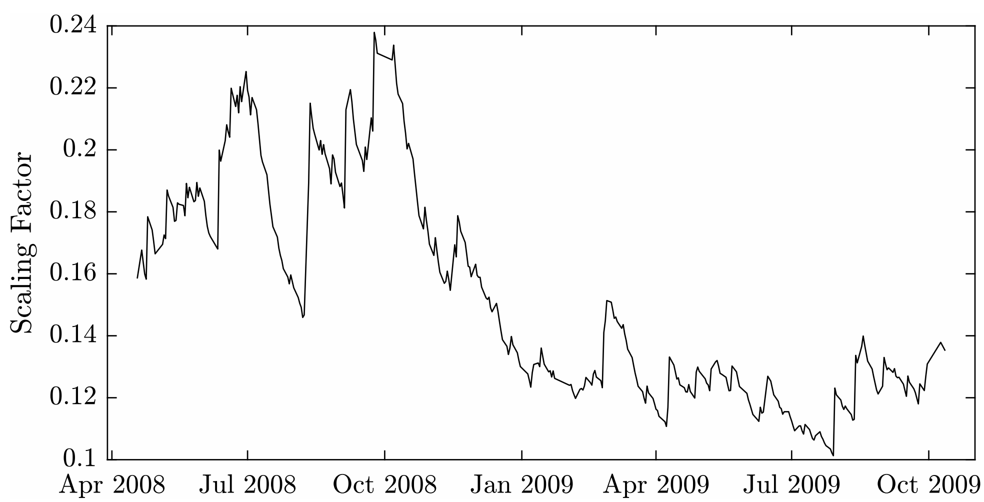


Figure 3.5: Annualized Scaling Factor of TTB in the NGARCH-Jump Model

(2007), without jumps the dynamics of underlying stock returns under measure Q are:

$$r_t = r - \frac{h_t}{2} + \sqrt{h_t} X_t \tag{3.14}$$

$$h_t = \beta_0 + \beta_1 h_{t-1} + \beta_2 h_{t-1} \left(X_{t-1} - \tilde{c} \right)^2 \tag{3.15}$$

where the parameters $\theta = \{\beta_0, \beta_1, \beta_2, c, \rho\}$ are the same as in the physical measure P, $\tilde{c} = c - \rho$, and $X_t \sim N(0, 1)$ is independent of $\mathcal{F}_{t-1}$.

With jumps, the stock return dynamics under Q is:

$$r_t = r - \frac{h_t}{2} + \lambda(1 - K_t) + \sqrt{h_t} J_t \tag{3.16}$$

$$h_t = \beta_0 + \beta_1 h_{t-1} + \beta_2 h_{t-1} \left(\frac{J_{t-1} - \lambda\mu}{\sqrt{1 + \lambda\tilde{\gamma}^2}} - \tilde{c} \right)^2 \tag{3.17}$$

$$K_t = \exp\left(\sqrt{h_t}\mu + \frac{1}{2} h_t \gamma^2 \right) \tag{3.18}$$

where

$$\tilde{c} = c - \frac{\rho}{\sqrt{1 + \lambda\tilde{\gamma}^2}}$$

Other parameters $\theta = \{\beta_0, \beta_1, \beta_2, c, \rho, \lambda, \mu, \gamma\}$, as well as the innovations J_t, are defined the same way as under measure P.

With the parameter estimates obtained in the previous subsection, we simulate the time series of underlying stock returns using the risk neutral dynamics above. For each stock, we conduct 10,000 simulations. To compare the pricing performance of different models, we define the average absolute percentage pricing error as:

$$\frac{1}{n} \sum_{i=1}^{n} \frac{\left| \widehat{C}_i - C_{i0} \right|}{C_{i0}}$$

where C_{i0} and $\widehat{C}_i$ are the real price and the theoretical price respectively, and n is the number of observations. We divide the warrants into twelve groups based on the type of the warrants (call or put), time to expiration τ, and moneyness, defined as the spot stock price over the strike price, S/K, and calculate the pricing error for each group. Table 3.4 presents the results for call warrants in Panel A, and put warrants in Panel B.

Table 3.4: Average Absolute Percentage Pricing Error

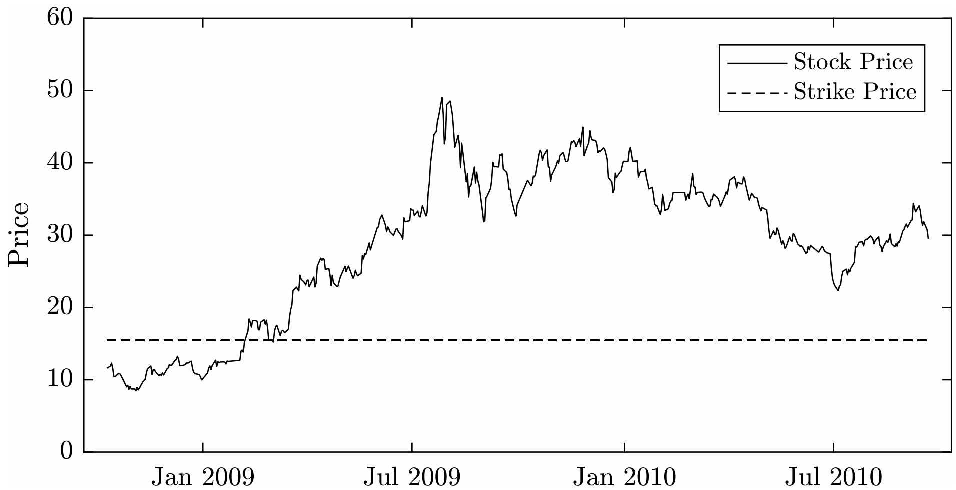
Panel A: Call Warrants				
		$S/K < 0.85$	$0.85 \leq S/K \leq 1.25$	$S/K > 1.25$
Black-Scholes	$\tau \leq 1$ year	93.28%	85.18%	70.48%
	$\tau > 1$ year	80.23%	128.82%	242.02%
NGARCH-Normal	$\tau \leq 1$ year	94.49%	67.00%	19.32%
	$\tau > 1$ year	70.97%	38.90%	13.27%
NGARCH-Jump	$\tau \leq 1$ year	94.63%	66.72%	19.38%
	$\tau > 1$ year	71.26%	37.88%	14.45%
Panel B: Put Warrants				
		$S/K < 0.85$	$0.85 \leq S/K \leq 1.25$	$S/K > 1.25$
Black-Scholes	$\tau \leq 1$ year	76.04%	88.14%	99.00%
	$\tau > 1$ year	84.57%	94.38%	99.14%
NGARCH-Normal	$\tau \leq 1$ year	31.92%	45.31%	88.81%
	$\tau > 1$ year	17.86%	67.59%	50.05%
NGARCH-Jump	$\tau \leq 1$ year	31.96%	45.96%	89.91%
	$\tau > 1$ year	18.45%	76.28%	45.88%

From the table we can draw the following conclusions. First, the Black-Scholes model does a poor job in explaining the warrant prices. The average absolute percentage pricing error is over 70% for all the categories, and it is as high as 242.02% for long term in-the-money call warrants. On the other hand, both NGARCH models achieve substantially smaller pricing errors than the Black-Scholes model in most cases, which demonstrates the importance of stochastic volatility when modeling the price dynamics of underlying stocks. Second, while the Black-Scholes model generally has a larger pricing error when the maturity is over one year than otherwise, both NGARCH models attain smaller pricing errors for long term warrants, except for the case of at-the-money put warrants. This is because the longer the time to expiration, the less realistic the constant volatility assumption is, and the more accurately the NGARCH models are able to capture the stochastic volatility. Third, the pricing errors of the NGARCH models get smaller as the warrants get more into the money. For example, the NGARCH-Normal model has a pricing error of 70.97% for long term out-of-the money call warrants, and 38.90% and 13.27% for the at-the-money and in-the-money call warrants respectively. This is largely due to the extreme over-valuation of out-of-the-money warrants as documented in Xiong and Yu (2011), and we refer interested readers to their original paper for discussion in more details. Finally, the NGARCH-Jump model has similar pricing errors to those of the NGARCH-Normal model across different categories of warrants. Therefore, even though the NGARCH-Jump model is more consistent with the true price dynamics of the underlying stocks as shown in the previous subsection, it does not perform better than the NGARCH-Normal model in terms of warrants valuation.

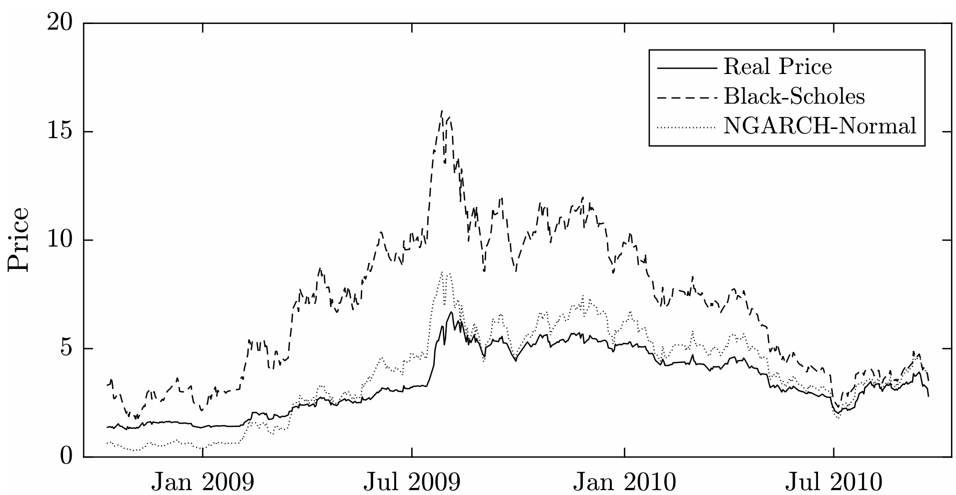
To better visualize the pricing performance of different models and how it evolves over time, we pick one typical stock, JT[5], and show the time series plots of its stock prices and warrant prices in Figure 3.6. Figure 3.6a displays JT's stock prices, and the horizontal dashed line represents the strike price. Since this is a call warrant, most of the time the warrant is in the money. Figure 3.6b shows the warrant prices, with the solid line corresponding to the observed prices, the dashed line to the theoretical prices given by the Black-Scholes model, and the dotted line to the NGARCH-normal prices. The NGARCH-Jump model gives almost the same prices to the NGARCH-Normal model because the estimated jump frequency (λ) is very small, so we do not show them in this graph. We can

5. Jiangxi Copper Co., Ltd., listed in the Shanghai Stock Exchange

see from the figure that the Black-Scholes model tends to predict systematically higher prices than the real prices. On the other hand, the NGARCH-Normal model is able to track the real prices to a close extent, which confirms our previous result that the NGARCH models perform fairly well in pricing in-the-money warrants.



(a) Stock Price



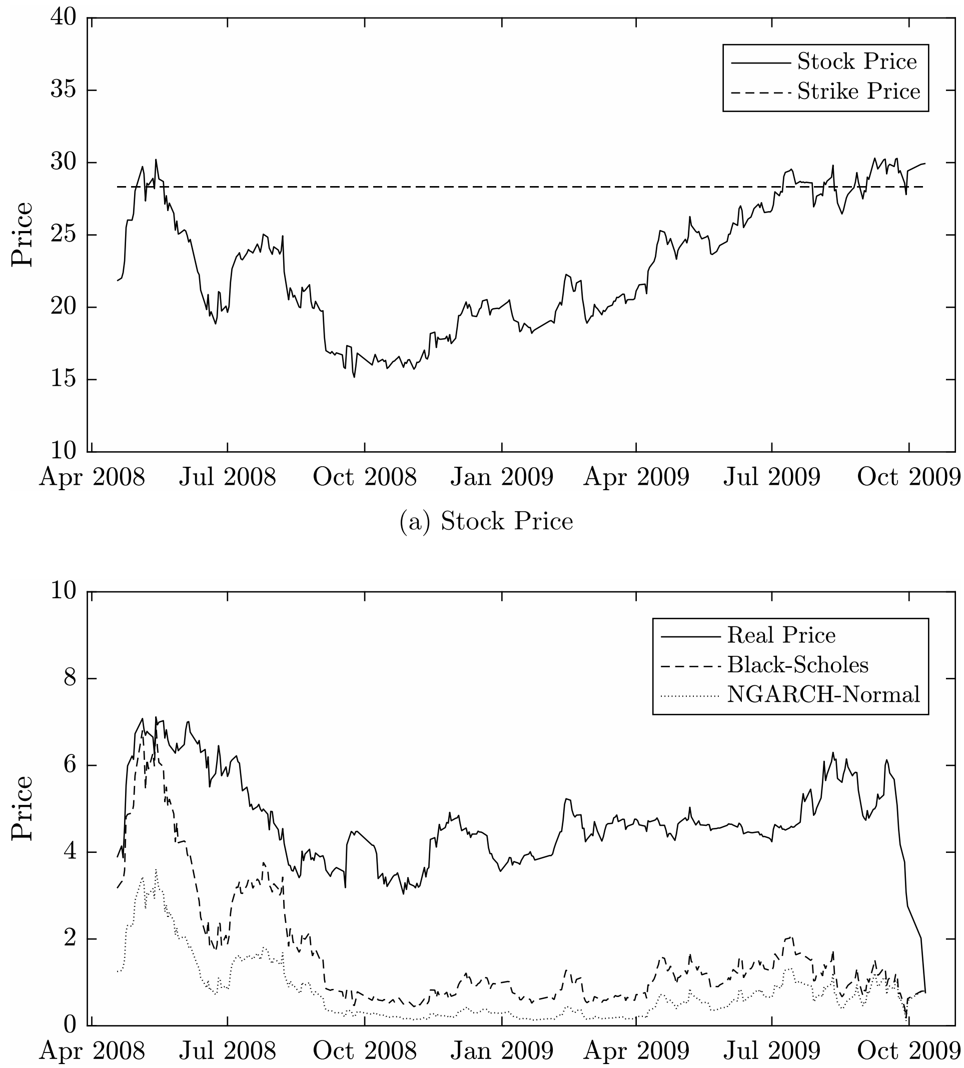
(b) Warrant Price

Figure 3.6: Time Series Plots of Stock and Warrant Prices of JT

However, these options pricing models do not always perform as well as in this case. Another typical scenario is illustrated in Figure 3.7, for the TTB stock we studied in the previous subsection. This warrant is again a call, but unlike JT, the warrant is out of the money for most of its lifetime, as shown in Figure 3.7a. We can see from Figure 3.7b that both the Black-Scholes model and the NGARCH-Normal model give substantially lower prices than the observed warrant prices, until at the expiration when the real price converges to the theoretical prices. Moreover, even though we have shown that the NGARCH-Normal model captures the dynamics of the TTB stock more accurately than Black-Scholes, it implies warrant prices that are even farther away from the real prices. Therefore, we interpret the mispricing of the TTB warrant as not due to the misspecification of our models, but rather overvaluation by the market. This is additional evidence to Xiong and Yu (2011), who demonstrate that bubbles are present in most puts in the Chinese warrants market. One explanation for this phenomenon is suggested by Barberis and Huang (2008), that investors have a tendency to overweight tail events as in the prospect theory, and they are willing to pay a lot more than the "fair" price for out-of-the-money warrants because the payoffs of these warrants have large positive skewness. We expect the overweighting tendency to be stronger in emerging markets like China, where investors tend to have gambling behaviors, but we leave that for future research.

3.4 Conclusion

This paper examines the performance of various options pricing models in the Chinese warrants market. We document that the distribution of returns on the Chinese stocks in our sample is generally negatively skewed and leptokurtic, which violates the assumption of the Black-Scholes model that returns are normally distributed. We fix this problem from two perspectives. First, we use NGARCH models to capture the stochastic volatility in stock returns. Second, we introduce jumps into the dynamics of stock returns to reflect big shocks in the market.



(a) Stock Price

(b) Warrant Price

Figure 3.7: Time Series Plots of Stock and Warrant Prices of TTB

Our results show that the NGARCH models successfully remove the volatility clustering pattern in stock returns. Although the return innovations in the NGARCH-Normal model still do not follow a normal distribution, the empirical distribution of the fitted innovations in the NGARCH-Jump model is largely consistent with the distribution implied by the model assumptions.

In order to compare the pricing performance of different models, we conduct Monte Carlo simulations to calculate theoretical prices of warrants and compare them with real prices. We find that NGARCH models substantially outperform the Black-Scholes model in most cases, and that they show greater improvement in pricing in-the-money warrants with longer time to expiration. Whereas the NGARCH-Jump model better tracks the evolution of the underlying stock prices, it does not differ much from the NGARCH-Normal model in terms of warrants valuation. Therefore, both stochastic volatility and jump components are crucial in modeling the dynamics of underlying stocks, but the former plays a more important role in matching the warrant prices in our sample. Furthermore, the Black-Scholes model as well as the NGARCH models still displays non-negligible pricing errors for out-of-the-money warrants, which provides indirect evidence to the existence of bubbles in the Chinese warrants market.

3.A Tables

Table A5: Summary Statistics for the Returns of Underlying Stocks

Code	Mean	SD	Skew	Kurt	Min	5%	25%	Median	75%	95%	Max	p-value
000898	0.30%	2.56%	0.4852	4.1802	-6.67%	-3.72%	-1.09%	0.16%	1.61%	4.96%	9.59%	0.0016
000858	0.36%	3.51%	0.0633	3.9067	-10.54%	-4.96%	-1.57%	0.20%	2.06%	7.15%	9.55%	0.0028
000069	0.55%	4.39%	-0.0277	3.0132	-10.54%	-6.90%	-2.10%	0.20%	3.31%	8.37%	9.54%	0.5000
000629	0.21%	3.85%	-0.1409	3.8907	-10.58%	-6.73%	-1.72%	0.22%	2.13%	6.96%	9.58%	0.0039
000001	0.17%	3.48%	0.3653	3.1015	-9.09%	-4.80%	-1.96%	-0.27%	2.35%	7.19%	9.34%	0.1834
000001	-0.15%	3.80%	-0.0372	3.1872	-10.54%	-6.69%	-2.51%	-0.24%	2.32%	6.40%	9.53%	0.5000
000839	0.00%	4.43%	-0.2347	3.0977	-10.58%	-8.71%	-2.59%	0.25%	2.63%	7.31%	9.56%	0.0874
000063	0.01%	3.22%	-0.1540	4.0983	-10.55%	-5.45%	-1.63%	0.00%	1.91%	5.02%	9.55%	0.0010
000423	-0.04%	2.89%	-0.2186	4.1159	-9.00%	-5.09%	-1.46%	0.06%	1.64%	4.38%	9.53%	0.0061
000629	0.48%	2.91%	0.0168	5.0514	-10.52%	-4.05%	-1.08%	0.27%	2.02%	5.38%	9.58%	0.0010
000002	0.34%	2.79%	0.3561	3.4572	-7.39%	-3.84%	-1.31%	0.31%	1.89%	5.04%	9.49%	0.0588
000932	0.27%	3.13%	-0.3331	4.8227	-10.59%	-4.65%	-1.13%	0.27%	1.99%	5.44%	9.57%	0.0010
000858	0.36%	3.51%	0.0633	3.9067	-10.54%	-4.96%	-1.57%	0.20%	2.06%	7.15%	9.55%	0.0028
000027	0.18%	2.30%	0.5673	5.3574	-5.87%	-3.27%	-1.15%	0.13%	1.42%	3.93%	9.60%	0.0013
000039	0.26%	3.31%	-0.0783	3.7801	-10.54%	-5.11%	-1.84%	0.22%	2.38%	5.92%	9.55%	0.0157
000792	0.33%	3.21%	-0.0082	4.2768	-10.54%	-4.61%	-1.48%	0.22%	2.17%	6.02%	9.51%	0.0043
600019	-0.01%	1.31%	0.5636	4.2199	-3.28%	-2.00%	-0.92%	0.00%	0.71%	2.25%	4.96%	0.0010
600005	0.14%	1.96%	0.5295	6.6344	-7.57%	-2.85%	-0.80%	0.00%	1.02%	3.75%	9.53%	0.0010
600010	0.45%	3.20%	0.4830	4.5130	-9.49%	-4.27%	-1.32%	0.18%	1.79%	6.95%	9.72%	0.0010
600001	0.34%	3.31%	0.1175	4.2222	-10.58%	-4.85%	-1.34%	0.25%	1.97%	5.72%	9.63%	0.0051
600008	0.37%	2.89%	-0.3598	4.6600	-10.56%	-4.03%	-1.48%	0.42%	2.17%	4.68%	9.51%	0.0010
600309	0.50%	3.29%	-0.1630	3.6899	-10.53%	-4.98%	-1.58%	0.51%	2.46%	5.71%	9.54%	0.0499
600177	0.63%	3.57%	-0.1058	4.1711	-10.54%	-4.89%	-1.24%	0.50%	2.58%	6.82%	9.58%	0.0063
600900	0.31%	2.42%	0.2767	5.8144	-10.49%	-3.20%	-1.10%	0.15%	1.27%	4.66%	9.50%	0.0010
600795	0.45%	3.36%	-0.1790	4.1550	-10.52%	-5.08%	-1.61%	0.59%	2.59%	5.60%	9.56%	0.0063
600887	0.20%	3.14%	-0.3132	4.0403	-10.53%	-5.11%	-1.71%	0.52%	1.98%	4.83%	9.53%	0.0057
600808	0.07%	4.13%	-0.0544	3.0871	-10.57%	-6.84%	-2.58%	0.25%	2.70%	7.13%	9.64%	0.5000

Table A6: Summary Statistics for the Returns of Underlying Stocks

Code	Mean	SD	Skew	Kurt	Min	5%	25%	Median	75%	95%	Max	p-value
600500	0.45%	4.42%	-0.2236	2.9697	-10.55%	-7.53%	-2.34%	0.77%	3.16%	8.54%	9.54%	0.3313
600096	0.08%	4.24%	-0.2622	3.5375	-10.55%	-9.23%	-1.95%	0.14%	2.58%	7.89%	9.55%	0.0292
600005	-0.06%	4.06%	-0.0792	2.9931	-10.54%	-6.93%	-2.56%	0.23%	2.50%	6.99%	9.63%	0.5000
600548	-0.15%	3.37%	-0.2483	4.1791	-10.61%	-5.91%	-1.91%	0.16%	1.78%	4.81%	9.60%	0.0010
600017	-0.11%	4.10%	-0.1848	3.2431	-10.58%	-7.78%	-2.55%	0.18%	2.47%	6.90%	9.59%	0.3501
600104	0.00%	3.84%	-0.1600	3.4469	-10.59%	-6.37%	-2.21%	0.22%	2.22%	6.67%	9.55%	0.0468
600269	0.03%	2.96%	-0.1548	4.0496	-10.57%	-5.12%	-1.77%	0.19%	1.71%	4.83%	9.58%	0.0010
600428	-0.13%	4.15%	-0.0726	3.1599	-10.60%	-7.42%	-2.46%	-0.11%	2.43%	6.96%	9.54%	0.5000
600028	-0.09%	3.14%	-0.0246	4.4512	-10.56%	-5.37%	-1.69%	0.00%	1.64%	4.80%	9.56%	0.0010
600018	-0.32%	3.49%	-0.0047	3.5643	-10.59%	-6.18%	-2.41%	-0.21%	1.88%	5.24%	9.53%	0.1664
600600	0.08%	2.87%	-0.2867	5.1016	-10.52%	-4.90%	-1.22%	0.28%	1.49%	4.36%	9.55%	0.0010
600795	0.00%	2.61%	-0.2838	4.7530	-10.54%	-4.40%	-1.47%	0.16%	1.68%	4.02%	9.51%	0.0010
600518	0.22%	3.45%	-0.2663	3.7696	-10.52%	-5.74%	-1.78%	0.46%	2.42%	5.66%	9.57%	0.0202
600019	-0.06%	2.88%	0.0186	4.4542	-10.60%	-4.95%	-1.63%	0.12%	1.38%	4.57%	9.57%	0.0010
600068	0.16%	3.15%	-0.2395	4.3416	-10.48%	-5.84%	-1.47%	0.41%	1.87%	4.69%	9.61%	0.0010
600362	0.18%	3.80%	0.1005	3.7521	-10.55%	-5.87%	-1.98%	0.06%	2.32%	7.44%	9.55%	0.0073
600029	-0.01%	5.08%	-0.0969	2.6471	-10.55%	-10.24%	-3.03%	0.38%	3.06%	9.53%	9.58%	0.3967
600519	0.33%	2.69%	0.3841	4.8170	-10.17%	-3.65%	-1.32%	0.00%	1.78%	5.14%	9.53%	0.0010
600690	0.54%	3.02%	-0.2017	4.5519	-10.55%	-3.73%	-1.11%	0.36%	2.16%	5.69%	9.50%	0.0013
600177	0.63%	3.57%	-0.1058	4.1711	-10.54%	-4.89%	-1.24%	0.50%	2.58%	6.82%	9.58%	0.0063
600309	0.50%	3.29%	-0.1630	3.6899	-10.53%	-4.98%	-1.58%	0.51%	2.46%	5.71%	9.54%	0.0499
600649	0.22%	2.38%	-0.2750	4.6165	-9.43%	-3.09%	-1.22%	0.09%	1.63%	4.09%	7.08%	0.0016
600010	0.45%	3.20%	0.4830	4.5130	-9.49%	-4.27%	-1.32%	0.18%	1.79%	6.95%	9.72%	0.0010
600009	0.33%	2.32%	-0.1776	5.5962	-10.48%	-2.68%	-0.92%	0.00%	1.72%	4.07%	8.93%	0.0010
600036	0.50%	2.80%	0.2832	3.7311	-9.39%	-3.57%	-1.28%	0.33%	2.26%	5.66%	9.53%	0.0068
600005	0.14%	1.96%	0.5295	6.6344	-7.57%	-2.85%	-0.80%	0.00%	1.02%	3.75%	9.53%	0.0010

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