Supplemental Material to
Robust Inference with Stochastic Local Unit Root Regressors in Predictive Regressions

By

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Online Supplement to 'Robust Inference with Stochastic Local Unit Root Regressors in Predictive Regressions' *

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Abstract

This supplement contains three sections. The first section collects the technical lemmas for short-horizon prediction models with QR-IVX and IVX instruments. The second section includes the technical lemmas for the long-horizon prediction model with LHIVX instruments. The third section briefly outlines the notation required for STUR coefficient heterogeneity with some additional discussion of the associated limit theory.

Throughout these proofs we use the same notation as in the main paper to which readers are referred. The technical lemmas given in the following section play central roles in the proofs of the main theorems in the paper. The IVX instrument is constructed as

$$\widetilde{z}_t := \sum_{j=1}^{t} R_{Tz}^{t-j} \Delta x_j, \ R_{Tz} := I_n + \frac{C_z}{T^\gamma}, \ C_z = c_z \cdot I_n,$$

where $c_z < 0$ and $\gamma \in (0,1)$. Therefore, $C_z^{-1} = c_z^{-1} \cdot I_n$. With a mild abuse of notation, it is often convenient to treat C_z^{-1} and $c_z^{-1} I_n$ equivalently in some of the derivations.

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A Lemmas for Short-horizon Predictive Regression

Lemma A.1 *Let Assumptions 1 and 2 hold. As $T \to \infty$,*

$$\sup_{r\in[0,1]} \left\| \eta^{(1)}_{T,\lfloor Tr\rfloor -1} \right\| = O_p\left(T^{\gamma+\frac{1}{2}}\right),$$
$$\sup_{r\in[0,1]} \left\| \eta^{(2)}_{T,\lfloor Tr\rfloor -1} \right\| = O_p\left(T^{\frac{1+\gamma}{2}}\right),$$
$$\sup_{r\in[0,1]} \left\| \eta^{(3)}_{T,\lfloor Tr\rfloor -1} \right\| = O_p\left(T^{\gamma+\frac{1}{2}}\right).$$

Proof: Write $x_{j-1} = \sum_{k=1}^{j-2} \left(\prod_{m=k+1}^{j-1} R_{Tm}\right) u_{xk} + u_{x,j-1}$ and define the autocovariance matrices $\Gamma_{ux}(h) := \mathbb{E}\left(u_{xt}u'_{x,t-h}\right)$ and $\Gamma_{ua}(h) := \mathbb{E}\left(u_{at}u'_{a,t-h}\right)$. We use the initial value $x_{-1} = \mathbf{0}_{n\times 1}$ in the following derivations although this could be relaxed at the cost of additional complications.

(i) For $\eta^{(1)}_{T,t-1}$, following the decomposition in equation (42) of the main paper in Phillips and Magdalinos (2009), we have

$$\begin{aligned}
\left\| \eta^{(1)}_{T,t-1} \right\|^2 &= \operatorname{tr}\left\{ \sum_{i,j=1}^{t-1} R_{Tz}^{t-j-1} R_{Tz}^{t-i-1} \left(x_{j-1}x'_{i-1}\right) \right\} \\
&\sim_a \sum_{i,j=1}^{t-1} \sum_{k=1}^{j-2} \sum_{l=1}^{i-2} \operatorname{tr}\left\{ R_{Tz}^{2t-2-j-i} R^{(j,k)} u_{xk}u'_{xl} R^{(i,l)} \right\} \\
&\leq \sum_{i,j=1}^{t-1} \sum_{k=1}^{j-2} \sum_{l=1}^{i-2} \left\| R_{Tz}^{2t-2-j-i} R^{(j,k)} \right\|_F \left\| u_{xk}u'_{xl} R^{(i,l)} \right\|_F \\
&\leq \sum_{i,j=1}^{t-1} \sum_{k=1}^{j-2} \sum_{l=1}^{i-2} |r_{Tz}|^{2t-2-j-i} \left\| R^{(i,l)} \right\|_F \left\| R^{(j,k)} \right\|_F \left\| u_{xk}u'_{xl} \right\|_F,
\end{aligned}$$

where $R^{(j,k)} := \prod_{m=k+1}^{j-1} R_{Tm}$, $R^{(i,l)} := \prod_{m=l+1}^{i-1} R_{Tm}$ and $r_{Tz} := 1 + c_z/T^{\gamma}$. From Phillips and Magdalinos (2009), $\sup_{1\leq t\leq T} \sum_{j=1}^{t-1} |r_{Tz}|^{t-j-1} = O\left(T^{\gamma}\right)$. From the definition of $R^{(j,k)}$, and setting $j = \lfloor Tr_1 \rfloor$ and $k = \lfloor Tr_2 \rfloor$ with $r_1,\ r_2 \in [0,1]$ we obtain

$$\begin{aligned}
&\frac{1}{T} \sup_{1\leq j\leq T} \sum_{k=1}^{j-2} \left\| R^{(j,k)} \right\|_F = \frac{1}{T} \sup_{0\leq r_1\leq 1} \sum_{k=1}^{\lfloor Tr_1\rfloor -2} \left\| R^{(\lfloor Tr_1\rfloor ,k)} \right\|_F \\
&\sim_a \frac{1}{T} \sup_{0\leq r_1\leq 1} \sum_{k=1}^{\lfloor Tr_1\rfloor -2} \left\| \exp\left(\frac{\lfloor Tr_1\rfloor -k-1}{T} C + \frac{\sum_{m=k+1}^{\lfloor Tr_1\rfloor -1} a'u_{am}}{\sqrt{T}} \cdot I_n \right) \right\|_F \\
&\leq \frac{1}{T} \sup_{0\leq r_1\leq 1} \sum_{k=1}^{\lfloor Tr_1\rfloor -2} \left\| \exp\left(\frac{\lfloor Tr_1\rfloor -k-1}{T} C - \frac{\sum_{m=1}^{k} a'u_{am}}{\sqrt{T}} \cdot I_n \right) \right\|_F \cdot \left\| \exp\left(\frac{\sum_{m=1}^{\lfloor Tr_1\rfloor -1} a'u_{am}}{\sqrt{T}} \cdot I_n \right) \right\|_F \\
&\sim_a \sup_{0\leq r_1\leq 1} \left\| \exp\left(a'B_a(r_1)\right) \cdot I_n \right\|_F \cdot \int_0^{r_1} \left\| \exp\left(\left(r_1 - r_2\right) C - \left(a'B_a\left(r_2\right)\right) \cdot I_n \right) \right\|_F dr_2 = O_p(1),
\end{aligned}$$

for both LSTUR and STUR cases. Note that

$$\sum_{h=-\infty}^{\infty} \|\Gamma_{ux}(h)\|_F < \infty, \quad \sum_{j=-\infty}^{\infty} \|\Gamma_{ua}(j)\|_F < \infty, \tag{1}$$

and

$$\begin{aligned}
\frac{1}{T}\sum_{l=1}^{i-2}\sum_{k=1}^{j-2}\left\|R^{(i,l)}\right\|_F\left\|R^{(j,k)}\right\|_F\|u_{xk}u_{xl}'\|_F &\leq \frac{1}{T}\sum_{l=1}^{i-2}\sum_{k=1}^{j-2}\|u_{xk}u_{xl}'\|_F\left(\sup_{i,l}\left\|R^{(i,l)}\right\|_F\right)\left(\sup_{j,k}\left\|R^{(j,k)}\right\|_F\right) \\
&\preceq \frac{1}{T}\sum_{l=1}^{i-2}\sum_{k=1}^{j-2}\|u_{xk}u_{xl}'\|_F \leq \frac{1}{T}\sum_{l=1}^{t-2}\sum_{k=1}^{t-2}\|u_{xk}u_{xl}'\|_F \\
&\leq \frac{1}{T}\sum_{h=-(t-2)}^{t-2}\sum_{k=1}^{t-2}\left\|u_{xk}u_{x,k-h}'\right\|_F = O_p(1), \qquad (2)
\end{aligned}$$

where $\sup_{1\leq i,l\leq T}\left\|R^{(i,l)}\right\|_F = O_p(1)$ and $\sup_{1\leq j,k\leq T}\left\|R^{(j,k)}\right\|_F = O_p(1)$. Next,

$$\begin{aligned}
\Pr\left(\frac{1}{T}\sum_{h=-(t-2)}^{t-2}\sum_{k=1}^{t-2}\left\|u_{xk}u_{x,k-h}'\right\|_F \geq b\right) &\leq \frac{1}{T}\sum_{h=-(\lfloor Tr\rfloor-2)}^{\lfloor Tr\rfloor-2}\sum_{k=1}^{\lfloor Tr\rfloor-2}\mathbb{E}\left\|u_{xk}u_{x,k-h}'\right\|_F/b \\
&\leq \frac{1}{T}\sum_{h=-(T-2)}^{T-2}\sum_{k=1}^{T-2}\mathbb{E}\left\|u_{xk}u_{x,k-h}'\right\|_F/b \\
&\to \sum_{-\infty}^{\infty}\|\Gamma_{ux}(h)\|_F/b < \breve{\varepsilon},
\end{aligned}$$

for any $\breve{\varepsilon} > 0$ and a correspondingly large enough b, as $T \to \infty$ and $t = \lfloor Tr\rfloor$ with $r \in [0,1]$. Combining the above results, as $T \to \infty$ and $t = \lfloor Tr\rfloor$ with $r \in [0,1]$,

$$\begin{aligned}
\sup_{r\in[0,1]}\left\|\eta_{T,\lfloor Tr\rfloor-1}^{(1)}\right\|^2 \preceq& \left(\sup_{r\in[0,1]}\sum_{i=1}^{\lfloor Tr\rfloor}|r_{Tz}|^{\lfloor Tr\rfloor-i}\right)^2\left(\frac{1}{T}\sum_{h=-(\lfloor Tr\rfloor-2)}^{\lfloor Tr\rfloor-2}\sum_{k=1}^{\lfloor Tr\rfloor-2}\left\|u_{xk}u_{x,k-h}'\right\|_F\right)\cdot T + o_p\left(T^{1+2\gamma}\right) \\
&= O_p\left(T^{1+2\gamma}\right).
\end{aligned}$$

(ii) For $\eta_{T,t-1}^{(2)}$, similarly we have

$$\begin{aligned}
\left\|\eta_{T,t-1}^{(2)}\right\|^2 &= \operatorname{tr}\left\{\sum_{i,j=1}^{t-1}R_{Tz}^{2t-2-j-i}\left(a'u_{ai}\right)x_{i-1}x_{j-1}'\left(u_{aj}'a\right)\right\} \\
&\leq \sum_{i,j=1}^{t-1}\sum_{k=1}^{j-2}\sum_{l=1}^{i-2}|r_{Tz}|^{2t-2-j-i}\left\|R^{(i,l)}\right\|_F\left\|R^{(j,k)}\right\|_F\left\|\left(a'u_{ai}u_{aj}'a\right)u_{xk}u_{xl}'\right\|_F \\
&\sim_a \sum_{i=1}^{t-1}\sum_{k=1}^{i-2}\sum_{l=1}^{i-2}|r_{Tz}|^{2t-2-2i}\left\|R^{(i,l)}\right\|_F\left\|R^{(i,k)}\right\|_F\left\|\left(a'F_a(1)\,\epsilon_{ai}\epsilon_{ai}'F_a'(1)a\right)u_{xk}u_{xl}'\right\|_F
\end{aligned}$$

$$\sim_a \sum_{i=1}^{t-1}\sum_{k=1}^{i-2}\sum_{l=1}^{i-2} |r_{Tz}|^{2t-2-2i} \left\| R^{(i,l)} \right\|_F \left\| R^{(i,k)} \right\|_F \left\| (a'\Omega_{aa}a)\, u_{xk}u'_{xl} \right\|_F ,$$

since $\sup_{1\le t\le T}\sum_{i=1}^{t} |r_{Tz}|^{t-i-1} = O(T^{\gamma})$, $\sup_{1\le i\le T}\sum_{k=1}^{i-2} \left\|R^{(i,k)}\right\|_F = O_p(T)$, and the BN decomposition in Phillips and Solo (1992) applies. Therefore, by equations (1) and (2), and since $\sup_{1\le i,l\le T}\left\|R^{(i,l)}\right\|_F = O_p(1)$ and $\sup_{1\le t,i\le T}|r_{Tz}|^{t-i-1} = O(1)$, we have

$$\begin{aligned}
\sup_{r\in[0,1]} \left\| \eta^{(2)}_{T,\lfloor Tr\rfloor-1} \right\|^2 &\preceq \left(\sup_{r\in[0,1]} \sum_{i=1}^{\lfloor Tr\rfloor-1} |r_{Tz}|^{2\lfloor Tr\rfloor-2i-2} \right) \cdot \left| a'\Omega_{aa}a \right| \\
&\quad \cdot \left(\frac{1}{T} \sum_{h=-(\lfloor Tr\rfloor-2)}^{\lfloor Tr\rfloor-2} \sum_{k=1}^{\lfloor Tr\rfloor-2} \left\| u_{xk}u'_{x,k-h} \right\|_F \right) \cdot T + o_p\left(T^{1+\gamma}\right) \\
&= O_p\left(T^{1+\gamma}\right).
\end{aligned}$$

(iii) The proof for the case involving $\eta^{(3)}_{T,t-1}$ follows (i), since

$$\begin{aligned}
\eta^{(3)}_{T,t-1} &= \sum_{j=1}^{t-1} R_{Tz}^{t-1-j} \left(a'u_{aj}u'_{aj}a\right) x_{j-1} \sim_a \left(a'\Sigma_{aa}a\right) \sum_{j=1}^{t-1} R_{Tz}^{t-1-j} x_{j-1} \\
&= \left(a'\Sigma_{aa}a\right) \eta^{(1)}_{T,t-1} = O_p\left(T^{\gamma+\frac{1}{2}}\right).
\end{aligned}$$

■

Remark A.1 *An alternative proof of Lemma A.1 is given in Remark A.2, showing identical stochastic orders for the IVX remainders.*

Lemma A.2 *Let Assumptions 1 and 2 hold and set $t = \lfloor Tr\rfloor$ with $r \in (0,1]$ as $T\to\infty$.*

(i) For the IVX residual term $\eta^{(2)}_{T,t-1}$

$$\begin{aligned}
\frac{1}{T^{\frac{1+\gamma}{2}}} \eta^{(2)}_{T,t-1=\lfloor Tr\rfloor-1} &\sim_a Z_a G_{a,c}(r) + O_p\left(\frac{1}{T^{\frac{1-\gamma}{2}}}\right) \\
&\rightsquigarrow \mathcal{MN}\left(\mathbf{0}_{n\times 1},\ \frac{1}{-2c_z}\left(a'\Omega_{aa}a\right) G_{a,c}(r)G_{a,c}(r)'\right), \qquad (3)
\end{aligned}$$

where $Z_a =_d \mathcal{N}\left(0,\ \frac{a'\Omega_{aa}a}{-2c_z}\right)$. Replace $G_{a,c}(r)$ by $G_a(r)$ when $C = \mathbf{0}_{n\times n}$.

(ii) For the IVX instrument $\widetilde{z}_{t-1}$

$$\frac{1}{T^{\frac{1}{2}+\gamma}} \sum_{t=1}^{T} \widetilde{z}_{t-1=\lfloor Tr\rfloor-1} = -\frac{C_z^{-1}}{\sqrt{T}} x_T + O_p\left(\frac{1}{T^{\frac{1-\gamma}{2}}}\right) \sim_a -C_z^{-1}G_{a,c}(1), \qquad (4)$$

for the LSTUR case. Replace $G_{a,c}(r)$ by $G_a(r)$ when $C = \mathbf{0}_{n\times n}$.

Proof: (i) For part (i) of Lemma A.2, in the following proof we set $t-1 = \lfloor Tr \rfloor - 1$ and use the initial condition $x_{-1} = \mathbf{0}_{n\times 1}$ and the exponential representation $R_{Tt} = \exp\left(\frac{C}{T} + I_n \frac{a' u_{at}}{\sqrt{T}}\right)$ for analytic convenience without loss of generality. Further, let $g = \gamma + \varepsilon$ for small enough $\varepsilon > 0$, and, with a mild abuse of notation, we use T^g to denote the integer part $\lfloor T^g \rfloor$, so that $\frac{T^g}{T^\gamma} + \frac{T}{T^g} \to \infty$. It follows that $R_{Tz}^{T^g} = \exp\left(\frac{T^g}{T^\gamma} C_z\right) \to \mathbf{0}_{n\times n}$ as C_z is diagonal with all elements negative by construction. Then, scaling $\eta_{T,t-1}^{(2)} = \sum_{j=1}^{t-1} R_{Tz}^{t-1-j} \left(a' u_{aj}\right) x_{j-1}$ we have

$$
\begin{aligned}
&\frac{1}{T^{\frac{1+\gamma}{2}}} \eta_{T,\lfloor Tr \rfloor}^{(2)} = \frac{1}{T^{\frac{\gamma}{2}}} \sum_{j=1}^{\lfloor Tr \rfloor} R_{Tz}^{\lfloor Tr \rfloor - j} \left(a' u_{aj}\right) \left(\frac{x_{j-1}}{\sqrt{T}}\right) \\
=&\frac{1}{T^{\frac{\gamma}{2}}} \sum_{j=\lfloor Tr \rfloor - T^g + 1}^{\lfloor Tr \rfloor} R_{Tz}^{\lfloor Tr \rfloor - j} \left(a' u_{aj}\right) \left(\frac{x_{j-1}}{\sqrt{T}}\right) + \frac{1}{T^{\frac{\gamma}{2}}} \sum_{j=1}^{\lfloor Tr \rfloor - T^g} R_{Tz}^{\lfloor Tr \rfloor - T^g - j} \left(a' u_{aj}\right) \left(R_{Tz}^{T^g} \frac{x_{j-1}}{\sqrt{T}}\right).
\end{aligned} \tag{5}
$$

Since $T^{-1/2} x_{\lfloor Tr \rfloor} \rightsquigarrow G_{a,c}(r)$ it follows by continuous mapping that $\max_{1\le j\le T} \left\| T^{-1/2} x_j \right\| \rightsquigarrow \sup_{r\in(0,1]} \left\| G_{a,c}(r) \right\| = O_p(1)$. Then

$$
R_{Tz}^{T^g} \left\| \frac{x_{j-1}}{\sqrt{T}} \right\| \le \exp\left(\frac{T^g}{T^\gamma} C_z\right) \max_{j\le T} \left\| \frac{x_{j-1}}{\sqrt{T}} \right\| = o_p(1), \tag{6}
$$

as $R_{Tz}^{T^g} = \exp\left(\frac{T^g}{T^\gamma} C_z\right) \to \mathbf{0}_{n\times n}$, and

$$
\frac{1}{T^{\frac{\gamma}{2}}} \sum_{j=1}^{\lfloor Tr \rfloor - T^g} R_{Tz}^{\lfloor Tr \rfloor - T^g - j} \left(a' u_{aj}\right) = O_p(1), \tag{7}
$$

because $\frac{\lfloor Tr \rfloor - T^g}{T^\gamma} \to \infty$ for all $r > 0$. Define the IVX coefficient $r_{Tz} := 1 + c_z / T^\gamma$. Result (7) follows because with $R_{Tz} = r_{Tz} I_n$ and for all $t_T \to \infty$ such that $\frac{t_T}{T^\gamma} \to \infty$ we have

$$
Z_{Ta,t_T} := \frac{1}{T^{\frac{\gamma}{2}}} \sum_{j=1}^{t_T} a' u_{aj} r_{Tz}^{t_T - j} \rightsquigarrow Z_a =_d \mathcal{N}\left(0, \frac{a' \Omega_{aa} a}{-2c_z}\right), \text{ (R-mixing)} \tag{8}
$$

which is established in the same way as in Phillips and Magdalinos (2007b,a). The R-mixing property of the weak convergence means that the random element Z_{Ta,t_T} is asymptotically independent of all events $E \in \mathcal{F}$, i.e., as $T \to \infty$,

$$
\Pr\left[\left(Z_{Ta,t_T} \in \cdot\right) \cap E\right] \to \Pr\left[\left(Z_{Ta,t_T} \in \cdot\right)\right] \Pr\left[E\right]. \tag{9}
$$

In this sense, the random element Z_{Ta,t_T} effectively escapes from its own probability space when R-mixing applies; see, Rényi (1963), Hall and Heyde (1980, p. 57), and Cheng and Chow (2002). Limit theorems with R-mixing such as (8) apply in very general situations, including martingale CLTs and the CLTs of McLeish et al. (1974), which include the present example. In addition to (7) we note that for all $t_T \geq 1$ such that $\frac{t_T}{T^\gamma} \to 0$ we have $\frac{1}{T^{\frac{\gamma}{2}}} \sum_{j=1}^{t_T} (a' u_{aj}) R_{Tz}^{t_T - j} = o_p(1)$. Using (6) and (7), we have, as $T \to \infty$,

$$\frac{1}{T^{\frac{\gamma}{2}}} \sum_{j=1}^{\lfloor Tr \rfloor - T^g} R_{Tz}^{\lfloor Tr \rfloor - T^g - j} (a' u_{aj}) \left(R_{Tz}^{T^g} \frac{x_{j-1}}{\sqrt{T}} \right) \to_p \mathbf{0}_{n \times 1}. \tag{10}$$

Next, since $\lfloor Tr \rfloor - T^g = \lfloor Tr \rfloor \left(1 - \frac{T^g}{\lfloor Tr \rfloor} \right) = \lfloor Tr \rfloor (1 + o(1))$, it follows that

$$T^{-1/2} x_{j-1} \rightsquigarrow G_{a,c}(r), \quad \text{for all } j \text{ satisfying } \lfloor Tr \rfloor - T^g \leq j \leq \lfloor Tr \rfloor. \tag{11}$$

Then, from (5), (6), (8), (10) and (11) we find, as $T \to \infty$,

$$\begin{aligned}
\frac{\eta_{T,\lfloor Tr \rfloor}^{(2)}}{T^{\frac{1+\gamma}{2}}} &= \frac{1}{T^{\frac{\gamma}{2}}} \sum_{j=\lfloor Tr \rfloor - T^g + 1}^{\lfloor Tr \rfloor} R_{Tz}^{\lfloor Tr \rfloor - j} (a' u_{aj}) \left(\frac{x_{j-1}}{\sqrt{T}} \right) + \frac{1}{T^{\frac{\gamma}{2}}} \sum_{j=1}^{\lfloor Tr \rfloor - T^g} R_{Tz}^{\lfloor Tr \rfloor - T^g - j} (a' u_{aj}) \left(R_{Tz}^{T^g} \frac{x_{j-1}}{\sqrt{T}} \right) \\
&= \frac{1}{T^{\frac{\gamma}{2}}} \sum_{j=\lfloor Tr \rfloor - T^g + 1}^{\lfloor Tr \rfloor} R_{Tz}^{\lfloor Tr \rfloor - j} (a' u_{aj}) \left(\frac{x_{j-1}}{\sqrt{T}} \right) + o_p(1) \qquad (12) \\
&= \left(\frac{1}{T^{\frac{\gamma}{2}}} \sum_{j=1}^{\lfloor Tr \rfloor} R_{Tz}^{\lfloor Tr \rfloor - j} (a' u_{aj}) \right) G_{a,c}(r) - \left(\frac{1}{T^{\frac{\gamma}{2}}} \sum_{j=1}^{\lfloor Tr \rfloor - T^g} R_{Tz}^{\lfloor Tr \rfloor - T^g - j} (a' u_{aj}) \right) \times R_{Tz}^{T^g} G_{a,c}(r) + o_p(1) \\
&= \left(\frac{1}{T^{\frac{\gamma}{2}}} \sum_{j=1}^{\lfloor Tr \rfloor} r_{Tz}^{\lfloor Tr \rfloor - j} (a' u_{aj}) \right) G_{a,c}(r) + o_p(1) \\
&\rightsquigarrow Z_a \times G_{a,c}(r) =_d \mathcal{MN} \left(\mathbf{0}_{n \times 1}, \ \frac{1}{-2c_z} a' \Omega_{aa} a \ G_{a,c}(r) G_{a,c}(r)' \right), \qquad (13)
\end{aligned}$$

which gives the required result. The mixed normal limit theory applies because Z_a is independent of the limit Brownian motions $(B_a'(r), B_x'(r))'$, as we show below, and thus Z_a is independent of the limit process $G_{a,c}(r)$ which depends only on $(B_a'(r), B_x'(r))'$.

To validate the mixed normality in (13) we establish that for $r \in (0,1]$

$$\begin{bmatrix} \frac{1}{T^{\frac{\gamma}{2}}} \sum_{j=1}^{\lfloor Tr \rfloor} a' u_{aj} r_{Tz}^{\lfloor Tr \rfloor - j} \\ \frac{1}{T^{\frac{1}{2}}} \sum_{j=0}^{\lfloor Tr \rfloor} u_{aj} \\ \frac{1}{T^{\frac{1}{2}}} \sum_{j=0}^{\lfloor Tr \rfloor} u_{xj} \\ \frac{1}{T^{\frac{1}{2}}} \sum_{j=0}^{\lfloor Tr \rfloor} u_{0j} \end{bmatrix} \rightsquigarrow \begin{bmatrix} Z_a \\ B_a(r) \\ B_x(r) \\ B_0(r) \end{bmatrix}. \tag{14}$$

The Gaussian limit variate Z_a is independent of the Brownian motions $(B_a'(r), B_x'(r), B_0'(r))'$ because of the joint convergence to Gaussian variates in (14) and the zero asymptotic correlation implied by

$$\mathbb{E} \left[\left(\frac{1}{T^{\frac{\gamma}{2}}} \sum_{j=1}^{\lfloor Tr \rfloor} (a' u_{aj}) r_{Tz}^{\lfloor Tr \rfloor - j} \right) \left(\frac{1}{T^{\frac{1}{2}}} \sum_{j=0}^{\lfloor Tr \rfloor} u_{aj}', \ \frac{1}{T^{\frac{1}{2}}} \sum_{j=0}^{\lfloor Tr \rfloor} u_{xj}', \ \frac{1}{T^{\frac{1}{2}}} \sum_{j=0}^{\lfloor Tr \rfloor} u_{0j} \right) \right] = O \left(\frac{1}{T^{\frac{1-\gamma}{2}}} \right). \tag{15}$$

To verify (15) we proceed as follows. First, since $R_{Tz} = r_{Tz} I_n$,

$$\mathbb{E}\left[\left(\frac{1}{T^{\frac{\gamma}{2}}}\sum_{j=1}^{\lfloor Tr\rfloor}(a'u_{aj})R_{Tz}^{\lfloor Tr\rfloor-j}\right)\left(\frac{1}{T^{\frac{1}{2}}}\sum_{j=1}^{\lfloor Tr\rfloor}u_{0j}\right)\right] = \frac{1}{T^{(1+\gamma)/2}}\sum_{j,k=1}^{\lfloor Tr\rfloor}\mathbb{E}\left((a'u_{ak})\,u_{0j}R_{Tz}^{\lfloor Tr\rfloor-k}\right)$$

$$= \frac{1}{T^{(1+\gamma)/2}}\sum_{j,k=1}^{\lfloor Tr\rfloor}a'\Gamma_{a0}(j-k)R_{Tz}^{\lfloor Tr\rfloor-k} = \frac{1}{T^{(1+\gamma)/2}}\sum_{h=-\lfloor Tr\rfloor+1}^{\lfloor Tr\rfloor-1}a'\Gamma_{a0}(h)\sum_{j,k=1}^{\lfloor Tr\rfloor}R_{Tz}^{\lfloor Tr\rfloor-k}\mathbf{1}\{j=k+h\}$$

$$= \frac{1}{T^{(1+\gamma)/2}}\sum_{h=-\lfloor Tr\rfloor+1}^{\lfloor Tr\rfloor-1}a'\Gamma_{a0}(h)\left[\sum_{k=1}^{\lfloor Tr\rfloor-h}R_{Tz}^{\lfloor Tr\rfloor-k}\mathbf{1}\{h\geq 0\}+\sum_{k=1-h}^{\lfloor Tr\rfloor}R_{Tz}^{\lfloor Tr\rfloor-k}\mathbf{1}\{h<0\}\right]$$

$$= \frac{1}{T^{(1+\gamma)/2}}\sum_{h=-\lfloor Tr\rfloor+1}^{\lfloor Tr\rfloor-1}a'\Gamma_{a0}(h)\left[\sum_{k=1}^{\lfloor Tr\rfloor-h}e^{(\lfloor Tr\rfloor-k)C_z/T^{\gamma}}\mathbf{1}\{h\geq 0\}+\sum_{k=1-h}^{\lfloor Tr\rfloor}e^{(\lfloor Tr\rfloor-k)C_z/T^{\gamma}}\mathbf{1}\{h<0\}\right]$$

$$\sim_a O\left(\frac{1}{T^{(1-\gamma)/2}}\right) = o(1). \tag{16}$$

To confirm the order of magnitude (16) note that 1-summability $\sum_{h=-\infty}^{\infty}|h|\,\|\Gamma_{a0}(h)\|<\infty$ holds by assumption, and $\sum_{k=1}^{\lfloor Tr\rfloor-h}e^{(\lfloor Tr\rfloor-k)C_z/T^{\gamma}}$ is a diagonal matrix whose diagonal elements for $h\geq 0$ are

$$\sum_{k=1}^{\lfloor Tr\rfloor-h}e^{(\lfloor Tr\rfloor-k)\frac{c_z}{T^{\gamma}}} = e^{(\lfloor Tr\rfloor-1)\frac{c_z}{T^{\gamma}}}\sum_{k=1}^{\lfloor Tr\rfloor-h}e^{-(k-1)\frac{c_z}{T^{\gamma}}} = e^{(\lfloor Tr\rfloor-1)\frac{c_z}{T^{\gamma}}}\frac{1-e^{-(\lfloor Tr\rfloor-h)\frac{c_z}{T^{\gamma}}}}{1-e^{-\frac{c_z}{T^{\gamma}}}}$$
$$= \frac{e^{(\lfloor Tr\rfloor-1)\frac{c_z}{T^{\gamma}}}-e^{(h-1)\frac{c_z}{T^{\gamma}}}}{\frac{c_z}{T^{\gamma}}+o\left(\frac{1}{T^{\gamma}}\right)} = -\frac{T^{\gamma}}{c_z}+O(1). \tag{17}$$

Next, for $h<0$ and setting $h=-m$ we have

$$\sum_{k=1+m}^{\lfloor Tr\rfloor}e^{(\lfloor Tr\rfloor-k)\frac{c_z}{T^{\gamma}}} = e^{(\lfloor Tr\rfloor-1-m)\frac{c_z}{T^{\gamma}}}\sum_{k=1+m}^{\lfloor Tr\rfloor}e^{-(k-1-m)\frac{c_z}{T^{\gamma}}} = e^{(\lfloor Tr\rfloor-1-m)\frac{c_z}{T^{\gamma}}}\frac{1-e^{-(\lfloor Tr\rfloor-m)\frac{c_z}{T^{\gamma}}}}{1-e^{-\frac{c_z}{T^{\gamma}}}}$$
$$= \frac{e^{(\lfloor Tr\rfloor-1-m)\frac{c_z}{T^{\gamma}}}-e^{-\frac{c_z}{T^{\gamma}}}}{\frac{c_z}{T^{\gamma}}+o\left(\frac{1}{T^{\gamma}}\right)} = -\frac{T^{\gamma}}{c_z}+O(1), \tag{18}$$

thereby establishing the order of magnitude given in (16). In an entirely similar fashion we obtain

$$\mathbb{E}\left[\left(\frac{1}{T^{\frac{\gamma}{2}}}\sum_{j=1}^{\lfloor Tr\rfloor}(a'u_{aj})\,r_{Tz}^{\lfloor Tr\rfloor-j}\right)\left(\frac{1}{T^{\frac{1}{2}}}\sum_{j=1}^{\lfloor Tr\rfloor}(u'_{aj},u'_{xj})\right)\right] = O\left(\frac{1}{T^{(1-\gamma)/2}}\right) = o(1), \tag{19}$$

and (15) is established. It follows that the limit variate Z_a is independent of the limit Brownian motions $(B'_a(r), B'_x(r), B'_0(r))'$ and hence the process $G_{a,c}(r)$. Hence, (13) holds and the stated result (3) is established. The proof of part (i) of Lemma A.2 is then complete.

(ii) For part (ii) of Lemma A.2, similar to Lemma B1 of Kostakis et al. (2015), we have $\sum_{t=1}^{T}\widetilde{z}_{t-1} = C_z^{-1}T^{\gamma}\left(\widetilde{z}_T - x_T + x_0\right)$. Then, from Lemma A.1 $\widetilde{z}_T = O_p\left(T^{\frac{\gamma}{2}}\right)$ and

$$\frac{1}{T^{\frac{1}{2}+\gamma}}\sum_{t=1}^{T}\widetilde{z}_{t-1} = -\frac{C_z^{-1}}{\sqrt{T}}x_T + O_p\left(T^{-\frac{1-\gamma}{2}}\right),$$

completing the proof.

■

Remark A.2 *Lemma A.2 provides an alternative proof of Lemma A.1. As Lemma A.2 (i) shows that*

$$\sup_{r\in[0,1]}\eta^{(2)}_{T,\lfloor Tr\rfloor-1} = O_p\left(T^{\frac{1+\gamma}{2}}\right), \tag{20}$$

a similar strategy can be applied to $\eta^{(1)}_{T,t}$ as

$$\begin{aligned}
&\frac{\eta^{(1)}_{T,\lfloor Tr\rfloor}}{T^{\frac{1+2\gamma}{2}}} = \frac{1}{T^{\gamma}}\sum_{j=\lfloor Tr\rfloor-T^{g}+1}^{\lfloor Tr\rfloor}R_{Tz}^{\lfloor Tr\rfloor-j}\left(\frac{x_{j-1}}{\sqrt{T}}\right) + \frac{1}{T^{\gamma}}\sum_{j=1}^{\lfloor Tr\rfloor-T^{g}}R_{Tz}^{\lfloor Tr\rfloor-T^{g}-j}\left(R_{Tz}^{T^{g}}\frac{x_{j-1}}{\sqrt{T}}\right)\\
&= \frac{1}{T^{\gamma}}\sum_{j=\lfloor Tr\rfloor-T^{g}+1}^{\lfloor Tr\rfloor}R_{Tz}^{\lfloor Tr\rfloor-j}\left(\frac{x_{t-1}}{\sqrt{T}}\right) + o_p(1)\\
&= \frac{1}{T^{\gamma}}\sum_{j=1}^{\lfloor Tr\rfloor}R_{Tz}^{\lfloor Tr\rfloor-j}\left(\frac{x_{t-1}}{\sqrt{T}}\right) - \frac{1}{T^{\gamma}}\sum_{j=1}^{\lfloor Tr\rfloor-T^{g}}R_{Tz}^{\lfloor Tr\rfloor-j}\left(\frac{x_{t-1}}{\sqrt{T}}\right) + o_p(1)\\
&= \frac{1}{T^{\gamma}}\sum_{j=1}^{\lfloor Tr\rfloor}R_{Tz}^{\lfloor Tr\rfloor-j}\left(\frac{x_{t-1}}{\sqrt{T}}\right) + o_p(1)\\
&\sim_a -C_z^{-1}G_{a,c}(r) = O_p(1),
\end{aligned} \tag{21}$$

where $t = \lfloor Tr\rfloor$ and T^{g} represents its integer value, as in the proof of Lemma A.2. Taking $\sup_{r\in[0,1]}$ *on both sides of (21) shows that*

$$\sup_{r\in[0,1]}\eta^{(1)}_{T,\lfloor Tr\rfloor-1} = O_p\left(T^{\gamma+\frac{1}{2}}\right). \tag{22}$$

Since $\eta^{(3)}_{T,t}$ and $\eta^{(1)}_{T,t}$ share the same stochastic order, it follows that

$$\sup_{r\in[0,1]}\eta^{(3)}_{T,\lfloor Tr\rfloor-1} = O_p\left(T^{\gamma+\frac{1}{2}}\right). \tag{23}$$

Lemma A.3 *Let Assumptions 1 and 2 hold. As $T\to\infty$ the following results hold:*

(i) For the numerator of the IVX estimator

$$\frac{1}{T^{\frac{1+\gamma}{2}}}\sum_{t=1}^{T}\widetilde{z}_{t-1}\underline{u}_{0t} = \frac{1}{T^{\frac{1+\gamma}{2}}}\sum_{t=1}^{T}z_{t-1}u_{0t} + \frac{1}{T^{1+\frac{\gamma}{2}}}\sum_{t=1}^{T}\eta^{(2)}_{T,t-1}u_{0t} + o_p(1).$$

(ii) The standardized sample variance of the IVX numerator satisfies

$$\frac{1}{T^{1+\gamma}}\sum_{t=1}^{T}\widetilde{z}_{t-1}\widetilde{z}_{t-1}' = \frac{1}{T^{1+\gamma}}\sum_{t=1}^{T}z_{t-1}z_{t-1}' + \frac{1}{T^{\frac{3}{2}+\gamma}}\left(\sum_{t=1}^{T}z_{t-1}\eta_{T,t-1}^{(2)\,\prime} + \sum_{t=1}^{T}\eta_{T,t-1}^{(2)}z_{t-1}'\right)$$
$$+\frac{1}{T^{2+\gamma}}\sum_{t=1}^{T}\eta_{T,t-1}^{(2)}\eta_{T,t-1}^{(2)\,\prime} + o_p\left(1\right).$$

(iii) For the denominator of the IVX estimator

$$\frac{1}{T^{1+\gamma}}\sum_{t=1}^{T}\widetilde{z}_{t-1}\underline{x}_{t-1}' = \frac{1}{T^{1+\gamma}}\sum_{t=1}^{T}z_{t-1}x_{t-1}' + \frac{1}{T^{\frac{3}{2}+\gamma}}\sum_{t=1}^{T}\eta_{T,t-1}^{(2)}x_{t-1}' + \frac{C}{T^{2+\gamma}}\sum_{t=1}^{T}\eta_{T,t-1}^{(1)}x_{t-1}'$$
$$+\frac{1}{2T^{2+\gamma}}\sum_{t=1}^{T}\eta_{T,t-1}^{(3)}x_{t-1}' + \left(\frac{1}{\sqrt{T}}x_T\right)\left(\frac{1}{T^{\frac{3}{2}}}\sum_{t=1}^{T}x_{t-1}\right)' C_z^{-1} + o_p\left(1\right).$$

Proof: For part (i) of Lemma A.3, the IVX numerator follows part (ii) of Lemma A.2 and has the following decomposition

$$\begin{aligned}
&\frac{1}{T^{\frac{1+\gamma}{2}}}\sum_{t=1}^{T}\widetilde{z}_{t-1}\underline{u}_{0t} \\
&= \frac{1}{T^{\frac{1+\gamma}{2}}}\sum_{t=1}^{T}\widetilde{z}_{t-1}u_{0t} + \frac{C_z^{-1}}{T^{\frac{1-\gamma}{2}}}\left(\frac{1}{\sqrt{T}}\sum_{t=1}^{T}u_{0t}\right)\frac{x_T}{\sqrt{T}} + o_p(1) \\
&= \frac{1}{T^{\frac{1+\gamma}{2}}}\sum_{t=1}^{T}\widetilde{z}_{t-1}u_{0t} + o_p\left(1\right) \\
&= \frac{1}{T^{\frac{1+\gamma}{2}}}\sum_{t=1}^{T}z_{t-1}u_{0t} + \frac{C}{T^{\frac{3+\gamma}{2}}}\sum_{t=1}^{T}\eta_{T,t-1}^{(1)}u_{0t} + \frac{1}{T^{\frac{2+\gamma}{2}}}\sum_{t=1}^{T}\eta_{T,t-1}^{(2)}u_{0t} + \frac{1}{2T^{\frac{3+\gamma}{2}}}\sum_{t=1}^{T}\eta_{T,t-1}^{(3)}u_{0t} + o_p\left(1\right).
\end{aligned} \tag{24}$$

The stochastic orders of the respective terms in this decomposition of the IVX numerator follow by Lemma A.1 as

$$\begin{aligned}
&\sum_{t=1}^{T}z_{t-1}u_{0t} = O_p\left(T^{\frac{1+\gamma}{2}}\right), \ \sum_{t=1}^{T}\eta_{T,t-1}^{(2)}u_{0t} = O_p\left(T^{\frac{2+\gamma}{2}}\right), \\
&\sum_{t=1}^{T}\eta_{T,t-1}^{(1)}u_{0t} = O_p\left(T^{1+\gamma}\right), \ \sum_{t=1}^{T}\eta_{T,t-1}^{(3)}u_{0t} = O_p\left(T^{1+\gamma}\right).
\end{aligned} \tag{25}$$

By (24) and (25) the asymptotic approximation to the IVX numerator then follows

$$\frac{1}{T^{\frac{1+\gamma}{2}}}\sum_{t=1}^{T}\widetilde{z}_{t-1}\underline{u}_{0t} = \frac{1}{T^{\frac{1+\gamma}{2}}}\sum_{t=1}^{T}z_{t-1}u_{0t} + \frac{1}{T^{1+\frac{\gamma}{2}}}\sum_{t=1}^{T}\eta_{T,t-1}^{(2)}u_{0t} + o_p(1). \tag{26}$$

Part (ii) of Lemma A.3 is a natural extension of part (i).

For part (iii) of Lemma A.3, the asymptotic approximation to the IVX denominator is given by

$$\begin{aligned}
&\frac{1}{T^{1+\gamma}}\sum_{t=1}^{T}\underline{x}_{t-1}\widetilde{z}_{t-1}' \\
&=\frac{1}{T^{1+\gamma}}\sum_{t=1}^{T}x_{t-1}\widetilde{z}_{t-1}'-\left(\frac{1}{T^{\frac{3}{2}}}\sum_{t=1}^{T}x_{t-1}\right)\left(\frac{1}{T^{\frac{1}{2}+\gamma}}\sum_{t=1}^{T}\widetilde{z}_{t-1}\right)' \\
&=\frac{1}{T^{1+\gamma}}\sum_{t=1}^{T}x_{t-1}\widetilde{z}_{t-1}'+\left(\frac{1}{T^{\frac{3}{2}}}\sum_{t=1}^{T}x_{t-1}\right)\left(\frac{1}{\sqrt{T}}x_T'C_z^{-1}\right)+o_p(1) \\
&=\frac{1}{T^{1+\gamma}}\sum_{t=1}^{T}x_{t-1}z_{t-1}'+\frac{1}{T^{2+\gamma}}\sum_{t=1}^{T}x_{t-1}\eta_{t-1}^{(1)}{}'C+\frac{1}{T^{\frac{3}{2}+\gamma}}\sum_{t=1}^{T}x_{t-1}\eta_{t-1}^{(2)}{}' \\
&+\frac{1}{2}\frac{1}{T^{2+\gamma}}\sum_{t=1}^{T}x_{t-1}\eta_{t-1}^{(3)}{}'+\left(\frac{1}{T^{\frac{3}{2}}}\sum_{t=1}^{T}x_{t-1}\right)\left(\frac{1}{\sqrt{T}}x_T'C_z^{-1}\right)+o_p(1)\,,
\end{aligned} \tag{27}$$

where the second equality is due to part (ii) of Lemma A.2. The stochastic orders of the five terms in equation (27) are given below.

First, it can be shown that

$$\left\|\left(\frac{1}{T^{\frac{3}{2}}}\sum_{t=1}^{T}x_{t-1}\right)\left(\frac{1}{\sqrt{T}}x_T'C_z^{-1}\right)\right\|\sim_a\left\|\left(\int_0^1 G_{a,c}(r)dr\right)G_{a,c}'(1)C_z^{-1}\right\|=O_p(1), \tag{28}$$

in the LSTUR case. When $C=\mathbf{0}_{n\times n}$, then the same approximation follows replacing $G_{a,c}(r)$ and $G_{a,c}(1)$ by $G_a(r)$ and $G_a(1)$, respectively.

Next, we show the stochastic boundedness of the other four terms of (27):

$$\begin{aligned}
\left\|\frac{1}{T^{1+\gamma}}\sum_{t=1}^{T}x_{t-1}z_{t-1}'\right\|&=O_p(1)\,; \\
\left\|\frac{1}{T^{2+\gamma}}\sum_{t=1}^{T}x_{t-1}\eta_{t-1}^{(1)}{}'\right\|&=O_p(1)\,; \\
\left\|\frac{1}{T^{\frac{3}{2}+\gamma}}\sum_{t=1}^{T}x_{t-1}\eta_{t-1}^{(2)}{}'\right\|&=O_p(1)\,; \\
\left\|\frac{1}{T^{2+\gamma}}\sum_{t=1}^{T}x_{t-1}\eta_{t-1}^{(3)}{}'\right\|&=O_p(1)\,.
\end{aligned}$$

Consider these expressions in turn, starting with $\left\|\frac{1}{T^{1+\gamma}}\sum_{t=1}^{T}x_{t-1}z_{t-1}'\right\|=O_p(1)$. Define $R_T:=I_n+\frac{C}{T}$ and apply the recursive formulae

$$x_t=R_Tx_{t-1}+u_{xt}+\frac{a'u_{at}}{\sqrt{T}}x_{t-1}+\frac{(a'u_{at})^2}{2T}x_{t-1}, \tag{29}$$

and $z_t = R_{Tz} z_{t-1} + u_{xt}$, giving

$$x_t z_t' = R_T x_{t-1} z_{t-1}' R_{Tz} + R_T x_{t-1} u_{xt}' + u_{xt} z_{t-1}' R_{Tz} + u_{xt} u_{xt}' + \frac{a' u_{at}}{\sqrt{T}} x_{t-1} z_{t-1}' R_{Tz} + \frac{a' u_{at}}{\sqrt{T}} x_{t-1} u_{xt}' + \frac{(a' u_{at})^2}{2T} x_{t-1} z_{t-1}' R_{Tz} + \frac{(a' u_{at})^2}{2T} x_{t-1} u_{xt}'. \tag{30}$$

Summing equation (30) on both sides and vectorizing the relevant arguments yields

$$\begin{aligned}
&[I_{n^2} - R_T \otimes R_{Tz}] \sum_{t=1}^{T} (x_{t-1} \otimes z_{t-1}) \\
&= (x_0 \otimes z_0) - x_T \otimes z_T + (R_T \otimes I_n) \sum_{t=1}^{T} x_{t-1} \otimes u_{xt} + (I_n \otimes R_{Tz}) \sum_{t=1}^{T} u_{xt} \otimes z_{t-1} \\
&+ \sum_{t=1}^{T} u_{xt} \otimes u_{xt} + (I_n \otimes R_{Tz}) \sum_{t=1}^{T} \left(\frac{a' u_{at}}{\sqrt{T}} x_{t-1} \right) \otimes z_{t-1} + (I_n \otimes I_n) \sum_{t=1}^{T} \left(\frac{a' u_{at}}{\sqrt{T}} x_{t-1} \right) \otimes u_{xt} \\
&+ (I_n \otimes R_{Tz}) \sum_{t=1}^{T} \left(\frac{(a' u_{at})^2}{2T} x_{t-1} \right) \otimes z_{t-1} + (I_n \otimes I_n) \sum_{t=1}^{T} \left(\frac{(a' u_{at})^2}{2T} x_{t-1} \right) \otimes u_{xt}.
\end{aligned} \tag{31}$$

By Lemma A.1, Phillips and Magdalinos (2009), and Lieberman and Phillips (2017, 2020), we have

$$\begin{aligned}
\| x_T \otimes z_T \| = O_p\left(T^{\frac{1+\gamma}{2}} \right), \quad \left\| \sum_{t=1}^{T} u_{xt} \otimes z_{t-1} \right\| = O_p(T), \\
\left\| \sum_{t=1}^{T} u_{xt} \otimes u_{xt} \right\| = O_p(T), \quad \left\| \sum_{t=1}^{T} x_{t-1} \otimes u_{xt} \right\| = O_p(T).
\end{aligned} \tag{32}$$

By Lemma 3.1 (d) of Magdalinos and Phillips (2009), Cauchy-Schwarz, and Lemma A.1, we have

$$\begin{aligned}
\left\| \sum_{t=1}^{T} \left(\frac{(a' u_{at})}{\sqrt{T}} x_{t-1} \right) \otimes z_{t-1} \right\| = O_p(T), \quad \left\| \sum_{t=1}^{T} \left(\frac{(a' u_{at})}{\sqrt{T}} x_{t-1} \right) \otimes u_{xt} \right\| = O_p(T), \\
\left\| \sum_{t=1}^{T} \frac{(a' u_{at})^2}{T} x_{t-1} \otimes z_{t-1} \right\| \preceq O_p\left(T^{\frac{1+\gamma}{2}} \right), \quad \left\| \sum_{t=1}^{T} \left(\frac{(a' u_{at})^2}{T} x_{t-1} \right) \otimes u_{xt} \right\| \preceq O_p(1).
\end{aligned} \tag{33}$$

When $\gamma \in (0, 1)$,

$$I_{n^2} - R_T \otimes R_{Tz} = -\frac{1}{T^\gamma} (I_n \otimes C_z) \left[I_{n^2} + O_p\left(\frac{1}{T^{1-\gamma}} \right) \right]. \tag{34}$$

Combining (31), (32), (33) and (34), it follows that

$$\left\| \sum_{t=1}^{T} x_{t-1} z_{t-1}' \right\| = O_p\left(T^{1+\gamma} \right). \tag{35}$$

Second, to justify $\left\|\sum_{t=1}^{T} x_{t-1}\eta_{T,t-1}^{(2)\prime}\right\| = O_p\left(T^{\frac{3}{2}+\gamma}\right)$, we apply the recursions (29) and $\eta_{T,t}^{(2)} = R_{Tz}\eta_{T,t-1}^{(2)} + (a'u_{at})\, x_{t-1}$ giving

$$\begin{aligned} x_t\eta_{T,t}^{(2)\prime} &= R_T x_{t-1}\eta_{T,t-1}^{(2)\prime} R_{Tz} + R_T x_{t-1}\left(\left(a'u_{at}\right)x_{t-1}\right)' + u_{xt}\eta_{T,t-1}^{(2)\prime} R_{Tz} + u_{xt}\left(\left(a'u_{at}\right)x_{t-1}\right)' \\ &+ \frac{a'u_{at}}{\sqrt{T}} x_{t-1}\eta_{T,t-1}^{(2)\prime} R_{Tz} + \frac{a'u_{at}}{\sqrt{T}} x_{t-1}\left(\left(a'u_{at}\right)x_{t-1}\right)' + \frac{\left(a'u_{at}\right)^2}{2T} x_{t-1}\eta_{T,t-1}^{(2)\prime} R_{Tz} \\ &+ \frac{\left(a'u_{at}\right)^2}{2T} x_{t-1}\left(\left(a'u_{at}\right)x_{t-1}\right)'. \end{aligned} \tag{36}$$

Then, summing both sides of equation (36) gives

$$\begin{aligned} &\left[I_{n^2} - R_T \otimes R_{Tz}\right]\sum_{t=1}^{T}\left(x_{t-1}\otimes\eta_{T,t-1}^{(2)}\right) \\ &= \left(x_0 \otimes \eta_{T,0}^{(2)}\right) - x_T \otimes \eta_{T,T}^{(2)} + \left(R_T \otimes I_n\right)\sum_{t=1}^{T} x_{t-1}\otimes\left(\left(a'u_{at}\right)x_{t-1}\right) \\ &+ \left(I_n \otimes I_n\right)\sum_{t=1}^{T} u_{xt}\otimes\left(\left(a'u_{at}\right)x_{t-1}\right) + \left(I_n \otimes R_{Tz}\right)\sum_{t=1}^{T}\left(\frac{a'u_{at}}{\sqrt{T}}x_{t-1}\right)\otimes\eta_{T,t-1}^{(2)} \\ &+ \left(I_n \otimes I_n\right)\sum_{t=1}^{T}\left(\frac{a'u_{at}}{\sqrt{T}}x_{t-1}\right)\otimes\left(\left(a'u_{at}\right)x_{t-1}\right) + \left(I_n \otimes R_{Tz}\right)\sum_{t=1}^{T}\left(\frac{\left(a'u_{at}\right)^2}{2T}x_{t-1}\right)\otimes\eta_{T,t-1}^{(2)} \\ &+ \left(I_n \otimes I_n\right)\sum_{t=1}^{T}\left(\frac{\left(a'u_{at}\right)^2}{2T}x_{t-1}\right)\otimes\left(\left(a'u_{at}\right)x_{t-1}\right) + \left(I_n \otimes R_{Tz}\right)\sum_{t=1}^{T} u_{xt}\otimes\eta_{T,t-1}^{(2)}. \end{aligned} \tag{37}$$

By Lemma A.1 and Lemma 10 of Lieberman and Phillips (2020), we have

$$\begin{aligned} &\left\|x_0\eta_{T,0}^{(2)\prime}\right\| = O_p(1), \quad \left\|x_T\eta_{T,T}^{(2)\prime}\right\| = O_p\left(T^{1+\frac{\gamma}{2}}\right), \quad \left\|\sum_{t=1}^{T} x_{t-1}\left(\left(a'u_{at}\right)x_{t-1}\right)'\right\| = O_p\left(T^{\frac{3}{2}}\right), \\ &\left\|\sum_{t=1}^{T}\left(\frac{a'u_{at}}{\sqrt{T}}x_{t-1}\right)\left(\left(a'u_{at}\right)x_{t-1}\right)'\right\| \sim_a \left\|\sum_{t=1}^{T} a'\Sigma_{aa}a\frac{1}{\sqrt{T}}\left(x_{t-1}'x_{t-1}'\right)\right\| = O_p\left(T^{\frac{3}{2}}\right), \\ &\left\|\sum_{t=1}^{T} u_{xt}\left(\left(a'u_{at}\right)x_{t-1}\right)'\right\| = O_p\left(T^{\frac{3}{2}}\right). \end{aligned} \tag{38}$$

By Lemma 3.1 (d) of Magdalinos and Phillips (2009), Lemma A.1, and Cauchy-Schwarz, it follows that

$$\begin{aligned} &\left\|\sum_{t=1}^{T}\left(\frac{a'u_{at}}{\sqrt{T}}x_{t-1}\right)\eta_{T,t-1}^{(2)\prime}\right\| = O_p\left(T^{\frac{3}{2}}\right), \quad \left\|\sum_{t=1}^{T}\left(\frac{\left(a'u_{at}\right)^2}{T}x_{t-1}\right)\eta_{T,t-1}^{(2)\prime}\right\| \preceq O_p\left(T^{\frac{2+\gamma}{2}}\right), \\ &\left\|\sum_{t=1}^{T} u_{xt}\eta_{T,t-1}^{(2)\prime}\right\| = O_p\left(T^{\frac{3}{2}}\right), \quad \left\|\sum_{t=1}^{T}\left(\frac{\left(a'u_{at}\right)^2}{T}x_{t-1}\right)\left(\left(a'u_{at}\right)x_{t-1}\right)'\right\| \preceq O_p\left(T^{\frac{1}{2}}\right). \end{aligned} \tag{39}$$

Similar to (34), we have

$$I_{n^2} - R_T \otimes R_{Tz} = -\frac{1}{T^{\gamma}} \left(I_n \otimes C_z\right) \left[I_{n^2} + O_p\left(\frac{1}{T^{1-\gamma}}\right)\right]. \tag{40}$$

Combining (37), (38), (39), and (40) gives

$$\frac{1}{T^{\frac{3}{2}+\gamma}} \sum_{t=1}^{T} \left[x_{t-1} \otimes \eta_{T,t-1}^{(2)}\right] = -\left(I_n \otimes C_z^{-1}\right) \frac{1}{T^{\frac{3}{2}}} \sum_{t=1}^{T} x_{t-1} \otimes \left(\left(a' u_{at}\right) x_{t-1}\right)$$
$$-\left(I_n \otimes C_z^{-1}\right) \frac{1}{T^{\frac{3}{2}}} \sum_{t=1}^{T} u_{xt} \otimes \left(\left(a' u_{at}\right) x_{t-1}\right) - \left(I_n \otimes C_z^{-1}\right) \frac{1}{T^{\frac{3}{2}}} \sum_{t=1}^{T} \left(\frac{a' u_{at}}{\sqrt{T}} x_{t-1}\right) \otimes \left(\left(a' u_{at}\right) x_{t-1}\right)$$
$$-\left(I_n \otimes C_z^{-1}\right) \frac{1}{T^{\frac{3}{2}}} \sum_{t=1}^{T} \left(\frac{a' u_{at}}{\sqrt{T}} x_{t-1}\right) \otimes \eta_{T,t-1}^{(2)} - \left(I_n \otimes C_z^{-1}\right) \frac{1}{T^{\frac{3}{2}}} \sum_{t=1}^{T} u_{xt} \otimes \eta_{T,t-1}^{(2)} + o_p\left(1\right), \tag{41}$$

and

$$\left\| \frac{1}{T^{\frac{3}{2}+\gamma}} \sum_{t=1}^{T} x_{t-1} \eta_{T,t-1}^{(2)\,\prime} \right\| = O_p\left(1\right). \tag{42}$$

Third, to justify $\sum_{t=1}^{T} x_{t-1} \eta_{T,t-1}^{(1)\,\prime} = O_p\left(T^{2+\gamma}\right)$, we use the recursions (29) and $\eta_{T,t}^{(1)} = R_{Tz} \eta_{T,t-1}^{(1)} + x_{t-1}$, so that

$$x_t \eta_{T,t}^{(1)\,\prime} = R_T x_{t-1} \eta_{T,t-1}^{(1)\,\prime} R_{Tz} + R_T x_{t-1} x_{t-1}' + u_{xt} \eta_{T,t-1}^{(1)\,\prime} R_{Tz} + u_{xt} x_{t-1}' + \frac{a' u_{at}}{\sqrt{T}} x_{t-1} \eta_{T,t-1}^{(1)\,\prime} R_{Tz}$$
$$+ \frac{a' u_{at}}{\sqrt{T}} x_{t-1} x_{t-1}' + \frac{\left(a' u_{at}\right)^2}{2T} x_{t-1} \eta_{T,t-1}^{(1)\,\prime} R_{Tz} + \frac{\left(a' u_{at}\right)^2}{2T} x_{t-1} x_{t-1}',$$

giving

$$\left[I_{n^2} - R_T \otimes R_{Tz}\right] \sum_{t=1}^{T} \left(x_{t-1} \otimes \eta_{T,t-1}^{(1)}\right)$$
$$= \left(x_0 \otimes \eta_{T,0}^{(1)}\right) - x_T \otimes \eta_{T,T}^{(1)} + \left(R_T \otimes I_n\right) \sum_{t=1}^{T} x_{t-1} \otimes x_{t-1}$$
$$+ \left(I_n \otimes I_n\right) \sum_{t=1}^{T} u_{xt} \otimes \left(x_{t-1}\right) + \left(I_n \otimes R_{Tz}\right) \sum_{t=1}^{T} \left(\frac{a' u_{at}}{\sqrt{T}} x_{t-1}\right) \otimes \eta_{T,t-1}^{(1)}$$
$$+ \left(I_n \otimes I_n\right) \sum_{t=1}^{T} \left(\frac{a' u_{at}}{\sqrt{T}} x_{t-1}\right) \otimes x_{t-1} + \left(I_n \otimes R_{Tz}\right) \sum_{t=1}^{T} \left(\frac{\left(a' u_{at}\right)^2}{2T} x_{t-1}\right) \otimes \eta_{T,t-1}^{(1)}$$
$$+ \left(I_n \otimes I_n\right) \sum_{t=1}^{T} \left(\frac{\left(a' u_{at}\right)^2}{2T} x_{t-1}\right) \otimes x_{t-1} + \left(I_n \otimes R_{Tz}\right) \sum_{t=1}^{T} u_{xt} \otimes \eta_{T,t-1}^{(1)}. \tag{43}$$

By Lemma A.1 and Cauchy-Schwarz we have

$$\left\| x_0 \eta_{T,0}^{(1)\,\prime} \right\| = O_p(1), \quad \left\| x_T \eta_{T,T}^{(1)\,\prime} \right\| = O_p(T^{1+\gamma}), \quad \left\| \sum_{t=1}^{T} x_{t-1} x_{t-1}' \right\| = O_p\left(T^2\right),$$

$$\left\| \sum_{t=1}^{T} u_{xt} \eta_{T,t-1}^{(1)\,\prime} \right\| \preceq O_p\left(T^{\frac{2+2\gamma}{2}}\right), \quad \left\| \sum_{t=1}^{T} u_{xt} x_{t-1}' \right\| = O_p\left(T\right). \tag{44}$$

Again, by Lemma A.1 and Cauchy-Schwarz inequality

$$\left\| \sum_{t=1}^{T} \left(\frac{a' u_{at}}{\sqrt{T}} x_{t-1} \right) \eta_{T,t-1}^{(1)\,\prime} \right\| \preceq O_p\left(T^{1+\gamma}\right), \quad \left\| \sum_{t=1}^{T} \left(\frac{a' u_{at}}{\sqrt{T}} x_{t-1} \right) x_{t-1}' \right\| \preceq O_p\left(T\right),$$

$$\left\| \sum_{t=1}^{T} \left(\frac{\left(a' u_{at}\right)^2}{T} x_{t-1} \right) \eta_{T,t-1}^{(1)\,\prime} \right\| \preceq O_p\left(T^{1+\gamma}\right), \quad \left\| \sum_{t=1}^{T} \left(\frac{\left(a' u_{at}\right)^2}{T} x_{t-1} \right) x_{t-1}' \right\| \preceq O_p\left(T\right). \tag{45}$$

Similar to (34), we have

$$I_{n^2} - R_T \otimes R_{Tz} = -\frac{1}{T^{\gamma}} \left(I_n \otimes C_z\right) \left[I_{n^2} + O_p\left(\frac{1}{T^{1-\gamma}} \right) \right]. \tag{46}$$

Combining (43) (44) (45) and (46) then gives

$$\frac{1}{T^{2+\gamma}} \sum_{t=1}^{T} \left[x_{t-1} \otimes \eta_{T,t-1}^{(1)} \right] = -\left(I_n \otimes C_z^{-1}\right) \frac{1}{T^2} \sum_{t=1}^{T} x_{t-1} \otimes x_{t-1} + o_p(1), \tag{47}$$

and

$$\left\| \frac{1}{T^{2+\gamma}} \sum_{t=1}^{T} x_{t-1} \eta_{T,t-1}^{(1)\,\prime} \right\| = O_p\left(1\right). \tag{48}$$

Fourth, to justify $\sum_{t=1}^{T} x_{t-1} \eta_{T,t-1}^{(3)\,\prime} = O_p\left(T^{2+\gamma}\right)$, we apply similar derivations leading to equation (47), showing that

$$\begin{aligned} \frac{1}{T^{2+\gamma}} \sum_{t=1}^{T} \left[x_{t-1} \otimes \eta_{T,t-1}^{(3)} \right] &= -\left(I_n \otimes C_z^{-1}\right) \frac{1}{T^2} \sum_{t=1}^{T} x_{t-1} \otimes \left(\left(a' u_{at}\right)^2 x_{t-1} \right) + o_p(1) \\ &= O_p(1). \end{aligned} \tag{49}$$

In the end, combining (27) (28) (35) (41) (47) and (49) gives the approximation in part (iii) of Lemma A.3 as

$$\begin{aligned} \frac{1}{T^{1+\gamma}} \sum_{t=1}^{T} \widetilde{z}_{t-1} \underline{x}_{t-1}' &= \frac{1}{T^{1+\gamma}} \sum_{t=1}^{T} z_{t-1} x_{t-1}' + \frac{1}{T^{\frac{3}{2}+\gamma}} \sum_{t=1}^{T} \eta_{T,t-1}^{(2)} x_{t-1}' + \frac{C}{T^{2+\gamma}} \sum_{t=1}^{T} \eta_{T,t-1}^{(1)} x_{t-1}' \\ &\quad + \frac{1}{2T^{2+\gamma}} \sum_{t=1}^{T} \eta_{T,t-1}^{(3)} x_{t-1}' + \left(\frac{1}{\sqrt{T}} x_T \right) \left(\frac{1}{T^{\frac{3}{2}}} \sum_{t=1}^{T} x_{t-1} \right)' C_z^{-1} + o_p\left(1\right) \\ &= O_p(1). \end{aligned}$$

The proof of Lemma A.3 is then complete. ■

Lemma A.4 *Let Assumptions 1 and 2 hold and $T \to \infty$.*

(i) For the terms of the IVX numerator,

$$\frac{1}{T^{1+\gamma}} \sum_{t=1}^{T} z_{t-1} z_{t-1}' \to_p V_{zz} := -\frac{1}{2c_z} \Omega_{xx},$$

$$\frac{1}{T^{2+\gamma}} \sum_{t=1}^{T} \eta_{T,t-1}^{(2)} \eta_{T,t-1}^{(2)\prime} \rightsquigarrow V_{\eta\eta}^{(2)} := \begin{cases} -\frac{1}{2c_z} \left(a'\Omega_{aa}a\right) \int_0^1 G_{a,c}(r) G_{a,c}'(r) dr, & \text{under LSTUR} \\ -\frac{1}{2c_z} \left(a'\Omega_{aa}a\right) \int_0^1 G_a(r) G_a'(r) dr, & \text{under STUR} \end{cases},$$

$$\frac{1}{T^{\frac{3}{2}+\gamma}} \sum_{t=1}^{T} z_{t-1} \eta_{T,t-1}^{(2)\prime} \rightsquigarrow V_{z\eta}^{(2)} := \begin{cases} -\frac{1}{2c_z} \left(\Omega_{xa}a\right) \int_0^1 G_{a,c}'(r) dr, & \text{under LSTUR} \\ -\frac{1}{2c_z} \left(\Omega_{xa}a\right) \int_0^1 G_a'(r) dr, & \text{under STUR} \end{cases}.$$

(ii) For the terms of the IVX denominator,

$$\frac{1}{T^{1+\gamma}} \sum_{t=1}^{T} x_{t-1} z_{t-1}' \rightsquigarrow V_{xz}^{l} := \begin{cases} \frac{-1}{c_z} \left(\int_0^1 G_{a,c}(r) dB_x'(r) + \Omega_{xx} + \int_0^1 G_{a,c}(r) \left(a'\Omega_{ax}\right) dr \right), & \text{LSTUR} \\ \frac{-1}{c_z} \left(\int_0^1 G_a(r) dB_x'(r) + \Omega_{xx} + \int_0^1 G_a(r) \left(a'\Omega_{ax}\right) dr \right), & \text{STUR} \end{cases},$$

$$\frac{1}{T^{\frac{3}{2}+\gamma}} \sum_{t=1}^{T} x_{t-1} \eta_{T,t-1}^{(2)\prime} \rightsquigarrow V_{x\eta}^{(2)},$$

$$\frac{1}{T^{2+\gamma}} \sum_{t=1}^{T} x_{t-1} \eta_{T,t-1}^{(1)\prime} C \rightsquigarrow V_{x\eta}^{(1)} := \begin{cases} -\frac{1}{c_z} \int_0^1 G_{a,c}(r) G_{a,c}'(r) dr C, & \text{under LSTUR} \\ -\frac{1}{c_z} \int_0^1 G_a(r) G_a'(r) dr C, & \text{under STUR} \end{cases},$$

$$\frac{1}{T^{2+\gamma}} \sum_{t=1}^{T} x_{t-1} \eta_{T,t-1}^{(3)\prime} \rightsquigarrow V_{x\eta}^{(3)} := \begin{cases} -\frac{1}{c_z} \int_0^1 G_{a,c}(r) G_{a,c}'(r) \left(a'\Sigma_{aa}a\right) dr, & \text{under LSTUR} \\ -\frac{1}{c_z} \int_0^1 G_a(r) G_a'(r) \left(a'\Sigma_{aa}a\right) dr, & \text{under STUR} \end{cases},$$

where:

$$\begin{aligned} V_{x\eta}^{(2)} \quad := \quad & -\frac{1}{c_z} \left[\int_0^1 G_{a,c}(r) G_{a,c}'(r) \left(a' dB_a(r)\right) + \int_0^1 G_{a,c}(r) G_{a,c}'(r) \left(a'\Omega_{aa}a + a'\Lambda_{aa}a\right) dr \right. \\ & \left. + \int_0^1 \left(\Delta_{xa}a + \Lambda_{xa}a\right) G_{a,c}'(r) dr + \int_0^1 G_{a,c}(r) \left(a'\Lambda_{ax}^{\star}\right) dr \right], \end{aligned} \tag{50}$$

with $C \neq \mathbf{0}_{n\times n}$ in the case of the LSTUR regressors and $\Lambda_{ax}^{\star} = \sum_{h=1}^{\infty} \mathbb{E}\left(u_{at} u_{x,t+h}'\right)$; and

$$\begin{aligned} V_{x\eta}^{(2)} \quad := \quad & -\frac{1}{c_z} \left[\int_0^1 G_a(r) G_a'(r) \left(a' dB_a(r)\right) + \int_0^1 G_a(r) G_a'(r) \left(a'\Omega_{aa}a + a'\Lambda_{aa}a\right) dr \right. \\ & \left. + \int_0^1 \left(\Delta_{xa}a + \Lambda_{xa}a\right) G_a'(r) dr + \int_0^1 G_a(r) \left(a'\Lambda_{ax}^{\star}\right) dr \right], \end{aligned} \tag{51}$$

with $C = \mathbf{0}_{n\times n}$ in the case of STUR regressors and $\Lambda_{ax}^{\star} = \sum_{h=1}^{\infty} \mathbb{E}\left(u_{at} u_{x,t+h}'\right)$.

Proof: (i) For the term $\sum_{t=1}^{T} z_{t-1} z_{t-1}'$ we apply a similar decomposition to (31) as

$$\left[I_{n^2} - R_{Tz} \otimes R_{Tz}\right] \sum_{t=1}^{T} z_{t-1} \otimes z_{t-1}$$

$$
\begin{aligned}
&= z_0 \otimes z_0 - z_T \otimes z_T + (I_n \otimes R_{Tz}) \sum_{t=1}^{T} u_{xt} \otimes z_{t-1} \\
&+ (R_{Tz} \otimes I_n) \sum_{t=1}^{T} z_{t-1} \otimes u_{xt} + \sum_{t=1}^{T} u_{xt} \otimes u_{xt},
\end{aligned} \tag{52}
$$

in which $\|z_0 \otimes z_0\| = O_p(1)$, $\|z_T \otimes z_T\| = O_p(T^\gamma)$, $\left\|\sum_{t=1}^{T} z_{t-1} \otimes u_{xt}\right\| = O_p(T)$, and $\left\|\sum_{t=1}^{T} u_{xt} \otimes z_{t-1}\right\| = O_p(T)$, and $\left\|\sum_{t=1}^{T} u_{xt} \otimes u_{xt}\right\| = O_p(T)$ by Lemma A.1. Moreover,

$$
I_{n^2} - R_{Tz} \otimes R_{Tz} = -\left(\frac{C_z}{T^\gamma} \otimes I_n + I_n \otimes \frac{C_z}{T^\gamma} + O\left(\frac{1}{T^{2\gamma}}\right)\right),
$$

leading to the following approximation

$$
\begin{aligned}
&-\left(\frac{C_z}{T^\gamma} \otimes I_n + I_n \otimes \frac{C_z}{T^\gamma} + O\left(\frac{1}{T^{2\gamma}}\right)\right) \frac{1}{T} \sum_{t=1}^{T} z_{t-1} \otimes z_{t-1} \\
&= \frac{1}{T} \sum_{t=1}^{T} (u_{xt} \otimes u_{xt} + z_{t-1} \otimes u_{xt} + u_{xt} \otimes z_{t-1}) + o_p(1).
\end{aligned} \tag{53}
$$

The above equation (53) is a Lyapunov equation of the form $(I_n \otimes A + A \otimes I_n)vec(X) = -vec(Q)$ and if A is stable, the solution for X is given by $X = \int_0^\infty e^{A\tau} Q e^{A'\tau} d\tau$. Note that $\frac{1}{T}\sum_{t=1}^{T} z_{t-1} u_{xt}' \to_p \Lambda_{xx}'$ by Phillips and Magdalinos (2009), where $\Lambda_{xx} = \sum_{h=1}^{\infty} \mathbb{E}\left(u_{xt} u_{x,t-h}'\right)$ and $\Omega_{xx} = \Sigma_{xx} + \Lambda_{xx} + \Lambda_{xx}'$. Then the RHS of (52) is shown to be $\Omega_{xx} + o_p(1)$. Hence,

$$
\frac{1}{T^{1+\gamma}} \sum_{t=1}^{T} z_{t-1} z_{t-1}' \to_p \int_0^\infty e^{C_z r} \Omega_{xx} e^{C_z r} dr = -\frac{1}{2c_z} \Omega_{xx}. \tag{54}
$$

For the term $\sum_{t=1}^{T} \eta_{T,t-1}^{(2)} \eta_{T,t-1}^{(2)\prime}$, a similar decomposition as in (32) applies as

$$
\begin{aligned}
&[I_{n^2} - R_{Tz} \otimes R_{Tz}] \sum_{t=1}^{T} \eta_{T,t-1}^{(2)} \otimes \eta_{T,t-1}^{(2)} \\
=&\eta_{T,0}^{(2)} \otimes \eta_{T,0}^{(2)} - \eta_{T,T}^{(2)} \otimes \eta_{T,T}^{(2)} + (R_{Tz} \otimes I_n) \sum_{t=1}^{T} \eta_{T,t-1}^{(2)} \otimes ((a' u_{at}) x_{t-1}) \\
&+ (I_n \otimes R_{Tz}) \sum_{t=1}^{T} \left((a' u_{at})\, x_{t-1}\right) \otimes \eta_{T,t-1}^{(2)} + \sum_{t=1}^{T} \left((a' u_{at})\, x_{t-1}\right) \otimes \left((a' u_{at})\, x_{t-1}\right).
\end{aligned} \tag{55}
$$

By Lemma A.1 $\left\|\eta_{T,0}^{(2)} \otimes \eta_{T,0}^{(2)}\right\| = O_p(1)$, $\left\|\eta_{T,T}^{(2)} \otimes \eta_{T,T}^{(2)}\right\| \preceq O_p\left(T^{1+\gamma}\right)$. Following Lemma 3.1 (d) of Magdalinos and Phillips (2009), we have

$$
\left\|\sum_{t=1}^{T} \eta_{T,t-1}^{(2)} \otimes \left((a' u_{at})\, x_{t-1}\right)\right\| \preceq O_p\left(T^2\right),
$$

$$\left\| \sum_{t=1}^{T} \left(\left(a' u_{at} \right) x_{t-1} \right) \otimes \eta_{T,t-1}^{(2)} \right\| \preceq O_p \left(T^2 \right),$$
$$\left\| \sum_{t=1}^{T} \left(\left(a' u_{at} \right) x_{t-1} \right) \otimes \left(\left(a' u_{at} \right) x_{t-1} \right) \right\| = O_p \left(T^2 \right). \tag{56}$$

More specifically, we have for the first term of (56)

$$\frac{1}{T^2} \sum_{t=1}^{T} \eta_{T,t-1}^{(2)} x_{t-1}' \left(a' u_{at} \right) \sim_a \frac{1}{T^2} \sum_{t=1}^{T} \left(a' u_{at} \right) z_{a,t-1} x_{t-1} x_{t-1}' \tag{57}$$
$$= \frac{1}{T^2} \sum_{t=1}^{T} \left(a' F_a (1) \epsilon_{at} \right) z_{a,t-1} x_{t-1} x_{t-1}' - \frac{1}{T^2} \sum_{t=1}^{T} \left(a' \Delta \widetilde{\epsilon}_{at} \right) z_{a,t-1} x_{t-1} x_{t-1}' \tag{58}$$
$$= -\frac{1}{T^2} \sum_{t=1}^{T} \left(a' \Delta \widetilde{\epsilon}_{at} \right) z_{a,t-1} x_{t-1} x_{t-1}' + O_p \left(\frac{1}{T^{1-\frac{\gamma}{2}}} \right)$$
$$= -\frac{1}{T^2} \left(a' \widetilde{\epsilon}_{aT} \right) \left(z_{a,T} x_T x_T' \right) + \frac{1}{T^2} \sum_{t=1}^{T} \left(a' \widetilde{\epsilon}_{at} \right) \Delta \left(z_{a,t} x_t x_t' \right) + o_p (1) \tag{59}$$
$$\sim_a \frac{1}{T^2} \sum_{t=1}^{T} \left(a' \widetilde{\epsilon}_{at} \right) z_{a,t-1} x_{t-1} x_{t-1}' \left(\frac{C_z}{T^\gamma} + \frac{2a' u_{at}}{\sqrt{T}} I_n + \frac{2 \left(a' u_{at} \right)^2}{T} I_n \right)$$
$$+ \frac{1}{T^2} \sum_{t=1}^{T} \left(a' \widetilde{\epsilon}_{at} \right) z_{a,t-1} u_{xt} x_{t-1}' R_{Tz} R_{Tt} + \frac{1}{T^2} \sum_{t=1}^{T} \left(a' \widetilde{\epsilon}_{at} \right) z_{a,t-1} u_{xt} u_{xt}' R_{Tz}$$
$$+ \frac{1}{T^2} \sum_{t=1}^{T} \left(a' \widetilde{\epsilon}_{at} \right) \left(a' u_{at} \right) R_{Tt} x_{t-1} x_{t-1}' R_{Tt} + \frac{1}{T^2} \sum_{t=1}^{T} \left(a' \widetilde{\epsilon}_{at} \right) \left(a' u_{at} \right) R_{Tt} x_{t-1} u_{xt}'$$
$$+ \frac{1}{T^2} \sum_{t=1}^{T} \left(a' \widetilde{\epsilon}_{at} \right) \left(a' u_{at} \right) u_{xt} x_{t-1}' R_{Tt} + \frac{1}{T^2} \sum_{t=1}^{T} \left(a' \widetilde{\epsilon}_{at} \right) \left(a' u_{at} \right) u_{xt} u_{xt}'$$
$$+ \frac{1}{T^2} \sum_{t=1}^{T} \left(a' \widetilde{\epsilon}_{at} \right) z_{a,t-1} R_{Tt} x_{t-1} u_{xt}' R_{Tz} \tag{60}$$
$$= \frac{1}{T^2} \sum_{t=1}^{T} \left(a' \widetilde{\epsilon}_{at} \right) \left(a' u_{at} \right) x_{t-1} x_{t-1}' + o_p (1) \tag{61}$$
$$= \frac{1}{T^2} \sum_{t=1}^{T} \mathbb{E} \left[\left(a' \widetilde{\epsilon}_{at} \right) \left(a' u_{at} \right) \right] x_{t-1} x_{t-1}' + o_p (1)$$
$$+ \frac{1}{T^2} \sum_{t=1}^{T} \left\{ \left[\left(a' \widetilde{\epsilon}_{at} \right) \left(a' u_{at} \right) - \mathbb{E} \left[\left(a' \widetilde{\epsilon}_{at} \right) \left(a' u_{at} \right) \right] \right\} x_{t-1} x_{t-1}' + o_p (1)$$
$$\sim_a \frac{1}{T^2} \left(a' \Lambda_{aa} a \right) \sum_{t=1}^{T} x_{t-1} x_{t-1}' + o_p (1) \rightsquigarrow \left(a' \Lambda_{aa} a \right) \int_0^1 G_{a,c}(r) G_{a,c}'(r) dr, \tag{62}$$

where $C \neq \mathbf{0}_{n\times n}$, and

$$z_{a,t-1} := \sum_{j=1}^{t-1} \left(a' u_{aj}\right) r_{Tz}^{t-1-j}, \;\; r_{Tz} := 1 + \frac{c_z}{T^\gamma}. \tag{63}$$

Line (57) holds by Lemma A.2, (58) follows by the BN decomposition in Phillips and Solo (1992), (59) follows by integration-by-parts, (60) holds by Lieberman and Phillips (2020) and Phillips and Magdalinos (2009), and (62) follows by definition of the one-sided long-run covariance. Similarly,

$$\frac{1}{T^2} \sum_{t=1}^{T} \left(a' u_{at}\right) x_{t-1} \eta_{T,t-1}^{(2)\,\prime} \rightsquigarrow a' \Lambda_{aa}' a \int_0^1 G_{a,c}\left(r\right) G_{a,c}'\left(r\right) dr. \tag{64}$$

Therefore, by combining (55) and the stochastic orders of the corresponding terms, it follows that

$$\begin{aligned}
&-\left(\frac{C_z}{T^\gamma} \otimes I_n + I_n \otimes \frac{C_z}{T^\gamma} + O\left(\frac{1}{T^{2\gamma}}\right)\right) \frac{1}{T^2} \sum_{t=1}^{T} \eta_{t-1}^{(2)} \otimes \eta_{T,t-1}^{(2)} \\
&\sim_a \frac{1}{T^2} \sum_{t=1}^{T} \left(\left(a' u_{at}\right) x_{t-1}\right) \otimes \left(\left(a' u_{at}\right) x_{t-1}\right) + \frac{1}{T^2} \sum_{t=1}^{T} \left(\left(a' u_{at}\right) x_{t-1}\right) \otimes \eta_{T,t-1}^{(2)} \\
&+ \frac{1}{T^2} \sum_{t=1}^{T} \eta_{T,t-1}^{(2)} \otimes \left(x_{t-1}\left(a' u_{at}\right)\right),
\end{aligned}$$

and

$$\begin{aligned}
&\frac{1}{T^{2+\gamma}} \sum_{t=1}^{T} \eta_{T,t-1}^{(2)} \eta_{T,t-1}^{(2)\,\prime} \\
&\rightsquigarrow \begin{cases} \int_0^\infty e^{C_z s} \left[\int_0^1 G_{a,c}(r) G_{a,c}'(r) \left(a' \Omega_{aa} a\right) dr\right] e^{C_z s} ds & \text{under} \quad \text{LSTUR} \\ \int_0^\infty e^{C_z s} \left[\int_0^1 G_{a}(r) G_{a}'(r) \left(a' \Omega_{aa} a\right) dr\right] e^{C_z s} ds & \text{under} \quad \text{STUR} \end{cases} \\
&= \begin{cases} -\frac{a' \Omega_{aa} a}{2 c_z} \int_0^1 G_{a,c}(r) G_{a,c}'(r) dr & \text{under} \quad \text{LSTUR} \\ -\frac{a' \Omega_{aa} a}{2 c_z} \int_0^1 G_{a}(r) G_{a}'(r) dr & \text{under} \quad \text{STUR} \end{cases}.
\end{aligned} \tag{65}$$

For the term $\sum_{t=1}^{T} z_{t-1} \eta_{T,t-1}^{(2)\,\prime}$, we use the fact that $\eta_{T,t}^{(2)} := \sum_{j=1}^{t} R_{Tz}^{t-j} \left(a' u_{aj}\right) x_{j-1}$ and by expansion find that the following decomposition holds approximately

$$\begin{aligned}
&\left[I_{n^2} - R_{Tz} \otimes R_{Tz}\right] \sum_{t=1}^{T} z_{t-1} \otimes \eta_{T,t-1}^{(2)} \\
=& z_0 \otimes \eta_{T,0}^{(2)} - z_T \otimes \eta_{T,T}^{(2)} + \left(R_{Tz} \otimes I_n\right) \sum_{t=1}^{T} z_{t-1} \otimes \left(\left(a' u_{at}\right) x_{t-1}\right) \\
&+ \left(I_n \otimes R_{Tz}\right) \sum_{t=1}^{T} u_{xt} \otimes \eta_{T,t-1}^{(2)} + \sum_{t=1}^{T} u_{xt} \otimes \left(\left(a' u_{at}\right) x_{t-1}\right).
\end{aligned} \tag{66}$$

By Lemma A.1, $\left\|z_0 \otimes \eta_{T,0}^{(2)}\right\| = O_p(1)$, $\left\|z_T \otimes \eta_{T,T}^{(2)}\right\| \preceq O_p\left(T^{\frac{1+2\gamma}{2}}\right)$. Further, following Lemma 3.1 (d) of Magdalinos and Phillips (2009) and derivations similar to (62), we obtain

$$\frac{1}{T^{\frac{3}{2}}}\sum_{t=1}^{T} z_{t-1}\left(\left(a' u_{at}\right) x_{t-1}\right)' \rightsquigarrow \Lambda_{xa}^{\star} a \int_0^1 G_{a,c}'(r)\, dr, \tag{67}$$

$$\frac{1}{T^{\frac{3}{2}}}\sum_{t=1}^{T} u_{xt}\eta_{T,t-1}^{(2)\,\prime} \rightsquigarrow \Lambda_{xa} a \int_0^1 G_{a,c}'(r)\, dr, \tag{68}$$

where $\Lambda_{xa}^{\star} := \sum_{h=1}^{\infty}\mathbb{E}\left(u_{xt}u_{a,t+h}'\right)$, and $\Omega_{xa} = \Lambda_{xa} + \Sigma_{xa} + \Lambda_{xa}^{\star}$. From (66) and the stochastic orders of the corresponding terms it follows that

$$\begin{aligned}
&-\left(\frac{C_z}{T^{\gamma}} \otimes I_n + I_n \otimes \frac{C_z}{T^{\gamma}} + O\left(\frac{1}{T^{2\gamma}}\right)\right)\frac{1}{T^{\frac{3}{2}}}\sum_{t=1}^{T} z_{t-1}\otimes \eta_{T,t-1}^{(2)} \\
=&\frac{1}{T^{\frac{3}{2}}}\sum_{t=1}^{T} u_{xt}\otimes\left(x_{t-1}\left(u_{at}'a\right)\right) + \frac{1}{T^{\frac{3}{2}}}\sum_{t=1}^{T} z_{t-1}\otimes\left(\left(a'u_{at}\right)x_{t-1}\right) + \frac{1}{T^{\frac{3}{2}}}\sum_{t=1}^{T} u_{xt}\otimes \eta_{T,t-1}^{(2)} + o_p(1),
\end{aligned}$$

and

$$\begin{aligned}
&\frac{1}{T^{\frac{3}{2}+\gamma}}\sum_{t=1}^{T} z_{t-1}\eta_{T,t-1}^{(2)\,\prime} \qquad (69)\\
\rightsquigarrow &\begin{cases} \int_0^{\infty} e^{C_z s}\left[\int_0^1 \Omega_{xa} a G_{a,c}'(r)dr\right] e^{C_z s} ds & \text{under LSTUR,} \\ \int_0^{\infty} e^{C_z s}\left[\int_0^1 \Omega_{xa} a G_{a}'(r)dr\right] e^{C_z s} ds & \text{under STUR,} \end{cases} \\
= &\begin{cases} -\frac{1}{2c_z}\int_0^1 \Omega_{xa} a G_{a,c}'(r)dr & \text{under LSTUR,} \\ -\frac{1}{2c_z}\int_0^1 \Omega_{xa} a G_{a}'(r)dr & \text{under STUR.} \end{cases} \qquad (70)
\end{aligned}$$

(ii) First, for the term $\sum_{t=1}^{T} x_{t-1}z_{t-1}'$, we have the following decomposition based on (31), (32), (33) and (34)

$$\begin{aligned}
&\left[I_{n^2} - R_T \otimes R_{Tz}\right]\frac{1}{T}\sum_{t=1}^{T}\left(x_{t-1}\otimes z_{t-1}\right) \\
&= \frac{1}{T}\sum_{t=1}^{T} x_{t-1}\otimes u_{xt} + \frac{1}{T}\sum_{t=1}^{T} u_{xt}\otimes z_{t-1} + \frac{1}{T}\sum_{t=1}^{T} u_{xt}\otimes u_{xt} \\
&+ \frac{1}{T}\sum_{t=1}^{T}\left(\frac{a'u_{at}}{\sqrt{T}}x_{t-1}\right)\otimes u_{xt} + \sum_{t=1}^{T}\left(\frac{\left(a'u_{at}\right)}{\sqrt{T}}x_{t-1}\right)\otimes z_{t-1} + o_p(1). \qquad (71)
\end{aligned}$$

We derive the limit theory for the elements of (71). Defining $\Lambda_{ax}^{\star} := \sum_{h=1}^{\infty}\mathbb{E}\left(u_{at}u_{x,t+h}'\right)$, the first three terms of (71) satisfy

$$\frac{1}{T}\sum_{t=1}^{T} x_{t-1}u_{xt}' + \frac{1}{T}\sum_{t=1}^{T} u_{xt}z_{t-1}' + \frac{1}{T}\sum_{t=1}^{T} u_{xt}u_{xt}'$$

$$\rightsquigarrow \int_0^1 G_{a,c}(r)dB_x'(r) + \int_0^1 G_{ac}(r)\left(a'\Lambda_{ax}^{\star}\right)dr + \Omega_{xx}, \tag{72}$$

by Lemma 3 of Lieberman and Phillips (2020) and equations (18), (19) and (20) of Phillips and Magdalinos (2009). Also, by Lemma 17 of Lieberman and Phillips (2020), it follows that

$$\frac{1}{T}\sum_{t=1}^{T}\frac{a'u_{at}}{\sqrt{T}}x_{t-1}u_{xt}' \rightsquigarrow \begin{cases} \int_0^1 G_{a,c}(r)\left(a'\Sigma_{ax}\right)adr & \text{under} \quad \text{LSTUR} \\ \int_0^1 G_a(r)\left(a'\Sigma_{ax}\right)adr & \text{under} \quad \text{STUR} \end{cases}. \tag{73}$$

Moreover,

$$\frac{1}{T}\sum_{t=1}^{T}\left(\frac{(a'u_{at})}{\sqrt{T}}x_{t-1}\right)z_{t-1}' \rightsquigarrow \begin{cases} \int_0^1 G_{a,c}(r)\left(a'\Lambda_{ax}\right)adr & \text{under} \quad \text{LSTUR} \\ \int_0^1 G_a(r)\left(a'\Lambda_{ax}\right)adr & \text{under} \quad \text{STUR} \end{cases}. \tag{74}$$

When $\gamma \in (0,1)$,

$$I_{n^2} - R_T \otimes R_{Tz} = -\frac{1}{T^{\gamma}}\left(I_n \otimes C_z\right)\left[I_{n^2} + O_p\left(\frac{1}{T^{1-\gamma}}\right)\right]. \tag{75}$$

Since $\Omega_{ax} = \Sigma_{ax} + \Lambda_{ax} + \Lambda_{ax}^{\star}$, combining the results of (71), (72), (73), (74) and (75) leads to the required results.

Second, for the term $\sum_{t=1}^{T} x_{t-1}\eta_{T,t-1}^{(2)\,\prime}$, based on (41) we need only consider the five leading terms

$$(I_n \otimes I_n)\sum_{t=1}^{T} x_{t-1} \otimes \left(\left(a'u_{at}\right)x_{t-1}\right), \tag{76}$$

$$(I_n \otimes I_n)\sum_{t=1}^{T} u_{xt} \otimes \left(\left(a'u_{at}\right)x_{t-1}\right), \tag{77}$$

$$(I_n \otimes I_n)\frac{1}{T^{\frac{3}{2}}}\sum_{t=1}^{T} u_{xt} \otimes \eta_{T,t-1}^{(2)}, \tag{78}$$

$$(I_n \otimes I_n)\sum_{t=1}^{T}\left(\frac{a'u_{at}}{\sqrt{T}}x_{t-1}\right) \otimes \left(\left(a'u_{at}\right)x_{t-1}\right), \tag{79}$$

$$(I_n \otimes I_n)\frac{1}{T^{\frac{3}{2}}}\sum_{t=1}^{T}\left(\frac{a'u_{at}}{\sqrt{T}}x_{t-1}\right) \otimes \eta_{T,t-1}^{(2)}. \tag{80}$$

Similar to Lemma 16 of Lieberman and Phillips (2020) but in the more complex matrix case here, the following functional law applies to the matrix form of the term in (76)

$$\frac{1}{T^{\frac{3}{2}}}\sum_{t=1}^{T} x_{t-1}x_{t-1}'\left(u_{at}'a\right) \rightsquigarrow \int_0^1 G_{a,c}(r)G_{a,c}'(r)\left(a'dB_a(r)\right) + \Lambda_{xa}a\int_0^1 G_{a,c}'(r)dr$$
$$+ \int_0^1 G_{a,c}(r)dra'\Lambda_{ax}^{\star} + \int_0^1 G_{a,c}(r)G_{a,c}'(r)\left[a'\left(\Lambda_{aa} + \Lambda_{aa}'\right)a\right]dr, \tag{81}$$

where $\Lambda_{ax}^{\star}$ is as defined above. The derivation of (81) is provided in Remark A.3 below. Similar results hold by replacing $G_{a,c}(r)$ by $G_a(r)$ when $C = \mathbf{0}_{n\times n}$. Moreover, for (77), we have

$$\frac{1}{T^{\frac{3}{2}}}\sum_{t=1}^{T} u_{xt}x_{t-1}'\left(u_{at}'a\right) \rightsquigarrow \begin{cases} \int_0^1 \left(\Sigma_{xa}a\right) G_{a,c}'(r)dr & \text{under} \quad \text{LSTUR}, \\ \int_0^1 \left(\Sigma_{xa}a\right) G_a'(r)dr & \text{under} \quad \text{STUR}. \end{cases} \tag{82}$$

Following a similar proof of (62) to (78) and (80) it follows that

$$\frac{1}{T^{\frac{3}{2}}}\sum_{t=1}^{T} u_{xt}\eta_{T,t-1}^{(2)\,\prime} \rightsquigarrow \begin{cases} \int_0^1 \left(\Lambda_{xa}a\right) G_{a,c}'(r)dr & \text{under} \quad \text{LSTUR} \\ \int_0^1 \left(\Lambda_{xa}a\right) G_a'(r)dr & \text{under} \quad \text{STUR} \end{cases}, \tag{83}$$

and

$$\frac{1}{T^{\frac{3}{2}}}\sum_{t=1}^{T}\left(\frac{a'u_{at}}{\sqrt{T}}x_{t-1}\right)\eta_{T,t-1}^{(2)\,\prime} \rightsquigarrow \begin{cases} \int_0^1 G_{a,c}\left(r\right) G_{a,c}'(r)\left(a'\Lambda_{aa}'a\right)dr & \text{under} \quad \text{LSTUR} \\ \int_0^1 G_a\left(r\right) G_a'(r)\left(a'\Lambda_{aa}'a\right)dr & \text{under} \quad \text{STUR} \end{cases}. \tag{84}$$

Then, applying the results of (41), (81), (82), (83), and (84), we find that

$$\begin{aligned}
&\frac{1}{T^{\frac{3}{2}+\gamma}}\sum_{t=1}^{T} x_{t-1}\eta_{T,t-1}^{(2)\,\prime}C_z \\
&\rightsquigarrow \int_0^1 G_{a,c}\left(r\right)G_{a,c}'(r)\left(a'dB_a(r)\right) + \Lambda_{xa}a\int_0^1 G_{a,c}'\left(r\right)dr + \int_0^1 G_{a,c}\left(r\right)dra'\Lambda_{ax}^{\star} \\
&\quad + \int_0^1 G_{a,c}\left(r\right)G_{a,c}'\left(r\right)\left[a'\left(\Lambda_{aa}+\Lambda_{aa}'\right)a\right]dr + \int_0^1 \left(\Sigma_{xa}a\right)G_{a,c}'(r)dr + \int_0^1 \left(\Lambda_{xa}a\right)G_{a,c}'(r)dr \\
&\quad + \int_0^1 G_{a,c}(r)G_{a,c}'(r)\left(a'\Sigma_{aa}a\right)dr + \int_0^1 G_{a,c}(r)G_{a,c}'(r)\left(a'\Lambda_{aa}'a\right)dr \\
&= \int_0^1 G_{a,c}\left(r\right)G_{a,c}'(r)\left(a'dB_a(r)\right) + \left(\Delta_{xa}a+\Lambda_{xa}a\right)\int_0^1 G_{a,c}'\left(r\right)dr + \int_0^1 G_{a,c}\left(r\right)dra'\Lambda_{ax}^{\star} \\
&\quad + \int_0^1 G_{a,c}(r)G_{a,c}'(r)\left(a'\Omega_{aa}a + a'\Lambda_{aa}a\right)dr.
\end{aligned}$$

Third, for the term $\sum_{t=1}^{T} x_{t-1}\eta_{T,t-1}^{(1)\,\prime}$, it follows that

$$I_{n^2} - R_T \otimes R_{Tz} = -\frac{1}{T^{\gamma}}\left(I_n \otimes C_z\right)\left[I_{n^2} + O_p\left(\frac{1}{T^{1-\gamma}}\right)\right]. \tag{85}$$

Combining the above result and (47), we have

$$\begin{aligned}
\frac{1}{T^{2+\gamma}}\sum_{t=1}^{T} x_{t-1}\eta_{T,t-1}^{(1)\,\prime}C &= -\frac{1}{T^2}\sum_{t=1}^{T} x_{t-1}x_{t-1}'CC_z^{-1} + o_p(1) \\
&\rightsquigarrow \begin{cases} -\int_0^1 G_{a,c}(r)G_{a,c}'(r)drCC_z^{-1}, & \text{under} \quad \text{LSTUR} \\ -\int_0^1 G_a(r)G_a'(r)drCC_z^{-1}, & \text{under} \quad \text{STUR} \end{cases}.
\end{aligned}$$

Finally, since $\eta^{(1)}_{T,t-1}$ and $\eta^{(3)}_{T,t-1}$ share the same stochastic order, we have

$$\frac{1}{T^{2+\gamma}}\sum_{t=1}^{T} x_{t-1}\eta^{(3)\,\prime}_{T,t-1} = -\frac{1}{T^{2}}\sum_{t=1}^{T} x_{t-1}x'_{t-1}\left(a'u_{at}u'_{at}a\right)C_z^{-1} + o_p(1)$$

$$\rightsquigarrow \begin{cases} -\int_0^1 G_{a,c}(r)G'_{a,c}(r)dr\,(a'\Sigma_{aa}a)\,C_z^{-1}, & \text{under} \quad \text{LSTUR} \\ -\int_0^1 G_{a}(r)G'_{a}(r)dr\,(a'\Sigma_{aa}a)\,C_z^{-1}, & \text{under} \quad \text{STUR} \end{cases}.$$

■

Remark A.3 *Lemma 16 of Lieberman and Phillips (2020) considers an LSTUR process in the scalar case. That result can be extended to the matrix case that is needed here, leading to the general formula (81). To establish the required result, we proceed by decomposing the matrix sample covariance*

$$\frac{1}{T^{\frac{3}{2}}}\sum_{t=1}^{T}\left(a'u_{at}\right)x_{t-1}x'_{t-1} = \frac{1}{T^{\frac{3}{2}}}\sum_{t=1}^{T}\left(a'\sum_{j=0}^{\infty}F_{a,j}\epsilon_{a,t-j}\right)x_{t-1}x'_{t-1}, \tag{86}$$

into constituent components for each value $j = 0,1,2,...,+\infty$. Let $B_{\epsilon_a}(\cdot) \sim \mathcal{N}\left(\mathbf{0}_{p\times 1}, \Sigma_{\epsilon,aa}\right)$ and $B_{\epsilon_x}(\cdot) \sim \mathcal{N}\left(\mathbf{0}_{n\times 1}, \Sigma_{\epsilon,xx}\right)$ be the limiting vector Brownian motions associated with partial sums of ϵ_{at} and ϵ_{xt} respectively, and define $\Sigma_{\epsilon,xa} = \mathbb{E}\left(\epsilon_{xt}\epsilon'_{at}\right) = \Sigma'_{\epsilon,ax}$.

(i) When $j = 0$,

$$\frac{1}{T^{\frac{3}{2}}}\sum_{t=1}^{T}\left(a'F_{a,0}\epsilon_{at}\right)x_{t-1}x'_{t-1} \rightsquigarrow \int_0^1 G_{a,c}(r)\,G'_{a,c}(r)\left(a'F_{a,0}dB_{\epsilon_a}(r)\right). \tag{87}$$

(ii) When $j = 1$,

$$\begin{aligned}
&\frac{1}{T^{\frac{3}{2}}}\sum_{t=1}^{T}\left(a'F_{a,1}\epsilon_{a,t-1}\right)x_{t-1}x'_{t-1} \\
&\sim_a \frac{1}{T^{\frac{3}{2}}}\sum_{t=1}^{T}\left(a'F_{a,1}\epsilon_{a,t-1}\right)\left(\exp\left(\frac{C}{T}+\frac{a'u_{a,t-1}}{\sqrt{T}}I_n\right)x_{t-2}+u_{x,t-1}\right) \\
&\times\left(\exp\left(\frac{C}{T}+\frac{a'u_{a,t-1}}{\sqrt{T}}I_n\right)x_{t-2}+u_{x,t-1}\right)' \\
&\sim_a \frac{1}{T^{\frac{3}{2}}}\sum_{t=1}^{T}\left(a'F_{a,1}\epsilon_{a,t-1}\right)x_{t-2}x'_{t-2} + \frac{1}{T^{\frac{3}{2}}}\sum_{t=1}^{T}\left(a'F_{a,1}\epsilon_{a,t-1}\right)x_{t-2}x'_{t-2}\left(\frac{2a'u_{a,t-1}}{\sqrt{T}}\right) \\
&+\frac{1}{T^{\frac{3}{2}}}\sum_{t=1}^{T}\left(a'F_{a,1}\epsilon_{a,t-1}\right)\left(x_{t-2}u'_{x,t-1}+u_{x,t-1}x'_{t-2}\right) \\
&\rightsquigarrow \int_0^1 G_{a,c}(r)\,G'_{a,c}(r)\left(a'F_{a,1}dB_{\epsilon_a}(r)\right) + 2\int_0^1\left(a'F_{a,1}\Sigma_{\epsilon,aa}F'_{a,0}a\right)G_{a,c}(r)\,G'_{a,c}(r)\,dr \\
&+\int_0^1 G_{a,c}(r)\,dr\left(a'F_{a,1}\Sigma_{\epsilon,ax}F'_{x,0}\right)+\left(F_{x,0}\Sigma_{\epsilon,xa}F'_{a,1}a\right)\int_0^1 G'_{a,c}(r)\,dr.
\end{aligned} \tag{88}$$

(iii) When $j = 2$,

$$
\begin{aligned}
&\frac{1}{T^{\frac{3}{2}}}\sum_{t=1}^{T}\left(a'F_{a,2}\epsilon_{a,t-2}\right)x_{t-1}x_{t-1}' \\
\sim_a &\frac{1}{T^{\frac{3}{2}}}\sum_{t=1}^{T}\left(a'F_{a,2}\epsilon_{a,t-2}\right)\left(\exp\left(\frac{2C}{T}+\frac{a'\left(u_{a,t-1}+u_{a,t-2}\right)}{\sqrt{T}}I_n\right)x_{t-3}+\exp\left(\frac{C}{T}+\frac{a'u_{a,t-1}}{\sqrt{T}}I_n\right)u_{x,t-2}\right. \\
&\left.+\,u_{x,t-1}\right)\cdot\left(\exp\left(\frac{2C}{T}+\frac{a'\left(u_{a,t-1}+u_{a,t-2}\right)}{\sqrt{T}}I_n\right)x_{t-3}+\exp\left(\frac{C}{T}+\frac{a'u_{a,t-1}}{\sqrt{T}}I_n\right)u_{x,t-2}+u_{x,t-1}\right)' \\
\sim_a &\frac{1}{T^{\frac{3}{2}}}\sum_{t=1}^{T}\left(a'F_{a,2}\epsilon_{a,t-2}\right)x_{t-3}x_{t-3}'+\frac{1}{T^{\frac{3}{2}}}\sum_{t=1}^{T}\left(a'F_{a,2}\epsilon_{a,t-2}\right)x_{t-3}x_{t-3}'\frac{2a'\left(u_{a,t-1}+u_{a,t-2}\right)}{\sqrt{T}} \\
&+\frac{1}{T^{\frac{3}{2}}}\sum_{t=1}^{T}\left(a'F_{a,2}\epsilon_{a,t-2}\right)x_{t-3}\left(u_{x,t-1}+u_{x,t-2}\right)'+\frac{1}{T^{\frac{3}{2}}}\sum_{t=1}^{T}\left(a'F_{a,2}\epsilon_{a,t-2}\right)\left(u_{x,t-1}+u_{x,t-2}\right)x_{t-3}' \\
&\rightsquigarrow\int_0^1 G_{a,c}\left(r\right)G_{a,c}'\left(r\right)\left(a'F_{a,2}dB_{\epsilon_a}\left(r\right)\right)+2\left[a'F_{a,2}\Sigma_{\epsilon,aa}\left(F_{a,1}+F_{a,0}\right)'a\right]\int_0^1 G_{a,c}\left(r\right)G_{a,c}'\left(r\right)dr \\
&+\int_0^1 G_{a,c}\left(r\right)dr\left(a'F_{a,2}\Sigma_{\epsilon,ax}\left(F_{x,0}+F_{x,1}\right)'\right)+\left(F_{x,0}+F_{x,1}\right)\Sigma_{\epsilon,xa}F_{a,2}'a\int_0^1 G_{a,c}'\left(r\right)dr.
\end{aligned}
\tag{89}
$$

Continuing this process from (87), (88) and (89) results in a sequence of matrix components from which upon summation we deduce that

$$
\begin{aligned}
&\frac{1}{T^{\frac{3}{2}}}\sum_{t=1}^{T}\left(a'u_{at}\right)x_{t-1}x_{t-1}'\rightsquigarrow\int_0^1 G_{a,c}\left(r\right)G_{a,c}'\left(r\right)\left(a'\sum_{j=0}^{\infty}F_{a,j}dB_{\epsilon_a}\left(r\right)\right) \\
&+\left[a'\sum_{j=1}^{\infty}F_{a,j}\Sigma_{\epsilon,aa}\left(\sum_{k=0}^{j-1}F_{a,k}'a\right)+a'\sum_{j=1}^{\infty}\left(\sum_{k=0}^{j-1}F_{a,k}\right)\Sigma_{\epsilon,aa}\left(F_{a,j}'a\right)\right]\int_0^1 G_{a,c}\left(r\right)G_{a,c}'\left(r\right)dr \\
&+\int_0^1 G_{a,c}\left(r\right)dr\left[a'\sum_{j=1}^{\infty}F_{a,j}\Sigma_{\epsilon,ax}\left(\sum_{k=0}^{j-1}F_{x,k}\right)'\right] \\
&+\sum_{j=1}^{\infty}\left(\sum_{k=0}^{j-1}F_{x,k}\right)\Sigma_{\epsilon,xa}F_{a,j}'a\int_0^1 G_{a,c}'\left(r\right)dr,
\end{aligned}
$$

which yields the limiting matrix form of (86) given in (81), viz.,

$$
\begin{aligned}
\frac{1}{T^{\frac{3}{2}}}\sum_{t=1}^{T}x_{t-1}x_{t-1}'\left(u_{at}'a\right)\rightsquigarrow&\int_0^1 G_{a,c}(r)G_{a,c}'(r)\left(a'dB_a(r)\right)+\Lambda_{xa}a\int_0^1 G_{a,c}'(r)dr \\
&+\int_0^1 G_{a,c}(r)dra'\Lambda_{ax}^{\star}+\int_0^1 G_{a,c}(r)G_{a,c}'(r)\left[a'\left(\Lambda_{aa}+\Lambda_{aa}'\right)a\right]dr.
\end{aligned}
\tag{90}
$$

Lemma A.5 *Suppose Assumptions 1 and 2 hold. As $T \to \infty$, we have the following limit theory:*

(i) For the IVX numerator,

$$\begin{pmatrix} \frac{1}{T^{\frac{1+\gamma}{2}}} \sum_{t=1}^{T} z_{t-1} u_{0t} \\ \frac{1}{T^{1+\frac{\gamma}{2}}} \sum_{t=1}^{T} \eta_{T,t-1}^{(2)} u_{0t} \end{pmatrix} \rightsquigarrow \mathcal{MN}\left(\mathbf{0}_{2n\times 1},\ \sigma_{00} \begin{bmatrix} V_{zz} & V_{z\eta}^{(2)} \\ V_{z\eta}^{(2)\prime} & V_{\eta\eta}^{(2)} \end{bmatrix}\right), \tag{91}$$

where V_{zz}, $V_{z\eta}^{(2)}$, and $V_{\eta\eta}^{(2)}$ are defined in Lemma A.4.

(ii) For the QR-IVX numerator,

$$\begin{pmatrix} \frac{1}{T^{\frac{1+\gamma}{2}}} \sum_{t=1}^{T} z_{t-1} \Psi_\tau\left(u_{0t\tau}\right) \\ \frac{1}{T^{1+\frac{\gamma}{2}}} \sum_{t=1}^{T} \eta_{T,t-1}^{(2)} \Psi_\tau\left(u_{0t\tau}\right) \end{pmatrix} \rightsquigarrow \mathcal{MN}\left(\mathbf{0}_{2n\times 1},\ \tau\left(1-\tau\right) \begin{bmatrix} V_{zz} & V_{z\eta}^{(2)} \\ V_{z\eta}^{(2)\prime} & V_{\eta\eta}^{(2)} \end{bmatrix}\right), \tag{92}$$

where V_{zz}, $V_{z\eta}^{(2)}$, and $V_{\eta\eta}^{(2)}$ are defined in Lemma A.4.

(iii) For the IVX denominator, then

$$\frac{1}{T^{1+\gamma}} \sum_{t=1}^{T} \underline{x}_{t-1} \tilde{z}_{t-1}' \rightsquigarrow V_{xz} := V_{xz}^{l} + V_{x\eta}^{(1)} + V_{x\eta}^{(2)} + \frac{1}{2} V_{x\eta}^{(3)} + V_{demean}, \tag{93}$$

where V_{xz}^{l}, $V_{x\eta}^{(1)}$, $V_{x\eta}^{(2)}$ and $V_{x\eta}^{(3)}$ are defined in Lemma A.4 and

$$V_{demean} := \begin{cases} \frac{1}{c_z}\left(\int_0^1 G_{a,c}(r)dr\right) G_{a,c}'(1), & \text{under} \quad \text{LSTUR} \\ \frac{1}{c_z}\left(\int_0^1 G_{a}(r)dr\right) G_{a}'(1), & \text{under} \quad \text{STUR} \end{cases}.$$

(iv) For the QR-IVX denominator,

$$\frac{1}{T^{1+\gamma}} \sum_{t=1}^{T} p_{u0t\tau,t-1}\left(0\right) x_{t-1} \tilde{z}_{t-1}' \rightsquigarrow p_{u0\tau}\left(0\right) \cdot V_{xz}^{QRIVX}, \tag{94}$$

where $V_{xz}^{QRIVX} := V_{xz}^{l} + V_{x\eta}^{(1)} + V_{x\eta}^{(2)} + \frac{1}{2} V_{x\eta}^{(3)}$. The limiting covariance matrices V_{xz}^{l}, $V_{x\eta}^{(1)}$, $V_{x\eta}^{(2)}$ and $V_{x\eta}^{(3)}$ are defined in Lemma A.4.

Proof: (i) First, we prove the results for the IVX numerator. Define the IVX coefficient $r_{Tz} := 1 + c_z/T^\gamma$ as in earlier derivations. Based on the approximation given in equation (12) of the proof of Lemma A.2, we have

$$\begin{aligned} \frac{1}{T^{\frac{1+\gamma}{2}}} \eta_{T,t-1}^{(2)} &= \left(\frac{1}{T^{\frac{\gamma}{2}}} \sum_{j=1}^{t-1} \left(a' u_{aj}\right) R_{Tz}^{t-1-j}\right) \left(\frac{1}{\sqrt{T}} x_{t-1}\right) + O_p\left(\frac{1}{T^{\frac{1-\gamma}{2}}}\right) \\ &= Z_{Ta,t-1}\left(\frac{1}{\sqrt{T}} x_{t-1}\right) + O_p\left(\frac{1}{T^{\frac{1-\gamma}{2}}}\right), \end{aligned} \tag{95}$$

where, as earlier in (8),

$$Z_{Ta,t-1} := \frac{1}{T^{\frac{\gamma}{2}}} \sum_{j=1}^{t-1} \left(a' u_{aj}\right) r_{Tz}^{t-1-j} \rightsquigarrow Z_a :=_d \mathcal{N}\left(0,\ \frac{a'\Omega_{aa}a}{-2c_z}\right) \text{ (R-mixing)}, \tag{96}$$

for all $t \leq T$ such that $\frac{t}{T^\gamma} \to \infty$ as $T \to \infty$. Similarly, we define $Z_{Tx,t-1} := \frac{1}{T^{\frac{\gamma}{2}}} z_{t-1} = \frac{1}{T^{\frac{\gamma}{2}}} \sum_{j=1}^{t-1} \left(u_{xj}\right) r_{Tz}^{t-1-j}$ and

$$Z_{Tx,t-1} = \frac{1}{T^{\frac{\gamma}{2}}} \sum_{j=1}^{t-1} \left(u_{xj}\right) r_{Tz}^{t-1-j} \rightsquigarrow Z_x :=_d \mathcal{N}\left(\mathbf{0}_{n\times 1},\ \frac{1}{-2c_z}\Omega_{xx}\right) \text{ (R-mixing)}, \tag{97}$$

whose derivation follows part (i) of Lemma A.2, simply replacing $a'u_{aj}$ by u_{xj}. Equation (97) holds for all $t \leq T$ such that $\frac{t}{T^\gamma} \to \infty$ as $T \to \infty$. Then, setting $T^g = \lfloor T^{\gamma+\epsilon} \rfloor$ as before for some small $\epsilon > 0$ so that $\frac{T^g}{T} \to 0$, summation splitting and subsequent elimination and replacement of negligible components lead to the following simplified approximation

$$\begin{aligned}
&\begin{pmatrix} \frac{1}{T^{\frac{1+\gamma}{2}}} \sum_{t=1}^{T} z_{t-1} u_{0t} \\ \frac{1}{T^{\frac{2+\gamma}{2}}} \sum_{t=1}^{T} \eta_{T,t-1}^{(2)} u_{0t} \end{pmatrix} \sim_a \begin{pmatrix} \frac{1}{T^{\frac{1}{2}}} \sum_{t=1}^{T} Z_{Tx,t-1} u_{0t} \\ \frac{1}{T} \sum_{t=T^g+1}^{T} x_{t-1} u_{0t} Z_{Ta,t-1} \end{pmatrix} + \begin{pmatrix} \mathbf{0}_{n\times 1} \\ \frac{1}{T^{\frac{2+\gamma}{2}}} \sum_{t=1}^{T^g} \eta_{T,t-1}^{(2)} u_{0t} \end{pmatrix} \\
&= \begin{pmatrix} \frac{1}{T^{\frac{1}{2}}} \sum_{t=1}^{T} Z_{Tx,t-1} u_{0t} \\ \frac{1}{T} \sum_{t=1}^{T} x_{t-1} u_{0t} Z_{Ta,t-1} \end{pmatrix} - \begin{pmatrix} \mathbf{0}_{n\times 1} \\ \frac{1}{T} \sum_{t=1}^{T^g} x_{t-1} u_{0t} Z_{Ta,t-1} \end{pmatrix} + o_p(1), \qquad (98) \\
&= \begin{pmatrix} \frac{1}{T^{\frac{1}{2}}} \sum_{t=1}^{T} Z_{Tx,t-1} u_{0t} \\ \frac{1}{T} \sum_{t=1}^{T} x_{t-1} u_{0t} Z_{Ta,t-1} \end{pmatrix} + o_p(1), \qquad (99)
\end{aligned}$$

since $\frac{1}{T} \sum_{t=1}^{T^g} x_{t-1} u_{0t} = \frac{T^{\frac{1+g}{2}}}{T} \frac{1}{T^{\frac{1+g}{2}}} \sum_{t=1}^{T^g} x_{t-1} u_{0t} = O_p(T^{-\frac{1-g}{2}}) = o_p(1)$ justifying (99). To establish (98) we need to show that $\frac{1}{T^{\frac{2+\gamma}{2}}} \sum_{t=1}^{T^g} \eta_{T,t-1}^{(2)} u_{0t} = o_p(1)$, which follows because u_{0t} is a martingale difference. In particular, using Lemma A.1 we have

$$\sum_{t=1}^{T^g} ||\eta_{T,t-1}^{(2)}||^2 \leq \sum_{t=1}^{T^g} \sup_{1\leq t\leq T^g} ||\eta_{T,t-1}^{(2)}||^2 = O_p(T^{g+1+\gamma})$$

since $\sup_{1\leq t\leq T^g} ||\eta_{T,t-1}^{(2)}|| = O_p(T^{(1+\gamma)/2})$, which leads to $\sum_{t=1}^{T^g} \eta_{T,t-1}^{(2)} u_{0t} = O_p(T^{\frac{g+1+\gamma}{2}})$. It follows that

$$T^{-\frac{2+\gamma}{2}} \sum_{t=1}^{T^g} \eta_{T,t-1}^{(2)} u_{0t} = T^{-\frac{2+\gamma}{2}} \times O_p\left(T^{\left(\frac{g+1+\gamma}{2}\right)}\right) = o_p(1),$$

thereby justifying (98).

Next, from Phillips and Magdalinos (2007a,b, 2009) it follows directly that the first component of (99) is a sample covariance term whose limit as $T \to \infty$ is

$$\frac{1}{T^{\frac{1+\gamma}{2}}} \sum_{t=1}^{T} z_{t-1} u_{0t} \rightsquigarrow B_{0Z_x}(1) =_d \mathcal{N}\left(\mathbf{0}_{n\times 1},\ \sigma_{00} V_{zz}\right). \tag{100}$$

The second component of (99) is a sample covariance term whose limit as $T \to \infty$ is

$$\frac{1}{T}\sum_{t=1}^{T} x_{t-1}u_{0t}Z_{Ta,t-1} = \sum_{t=1}^{T}\left(\frac{x_{t-1}}{\sqrt{T}}\right)\left(\frac{u_{0t}Z_{Ta,t-1}}{\sqrt{T}}\right) \rightsquigarrow \int_0^1 G_{a,c}(r)dB_{0Z_a}(r) \tag{101}$$

$$=_d \mathcal{MN}\left(\mathbf{0}_{n\times 1},\ \frac{\sigma_{00}}{-2c_z}\left(a'\Omega_{aa}a\right)\int_0^1 G_{a,c}(r)G_{a,c}'(r)dr\right)$$

$$= \mathcal{MN}\left(\mathbf{0}_{n\times 1},\ \sigma_{00}V_{\eta\eta}^{(2)}\right). \tag{102}$$

The combined sample covariance vector has the following joint limit as $T \to \infty$

$$\begin{pmatrix} \frac{1}{T^{\frac{1}{2}}}\sum_{t=1}^{T} u_{0t}Z_{Tx,t-1} \\ \frac{1}{T}\sum_{t=1}^{T} x_{t-1}u_{0t}Z_{Ta,t-1}\end{pmatrix} \rightsquigarrow \begin{pmatrix}\int_0^1 dB_{0Z_x}(r) \\ \int_0^1 G_{a,c}(r)dB_{0Z_a}(r)\end{pmatrix} \tag{103}$$

$$=_d \mathcal{MN}\left(\mathbf{0}_{2n\times 1},\ \sigma_{00}\begin{bmatrix} V_{zz} & V_{z\eta}^{(2)} \\ V_{z\eta}^{(2)\prime} & V_{\eta\eta}^{(2)}\end{bmatrix}\right). \tag{104}$$

The asymptotic behavior (103) follows by the weak convergence (100) and the weak convergence to the limiting stochastic integral in (101) combined with the fact that for $r \in (0,1]$

$$\begin{bmatrix} T^{-\frac{1}{2}}x_{\lfloor Tr\rfloor} \\ T^{-\frac{1}{2}}\sum_{j=0}^{\lfloor Tr\rfloor} u_{0j}Z_{Ta,j-1} \\ T^{-\frac{1}{2}}\sum_{j=0}^{\lfloor Tr\rfloor} u_{0j}Z_{Tx,j-1}\end{bmatrix} \rightsquigarrow \begin{bmatrix} G_{a,c}(r) \\ B_{0Z_a}(r) \\ B_{0Z_x}(r)\end{bmatrix}, \tag{105}$$

The mixed normal joint limit in (104) follows from the independence of the limit processes $G_{a,c}(r)$ and $\left(B_{0Z_a}(r), B_{0Z_x}'(r)\right)'$, which is established below.

The first component of (105) follows from the limit behavior given in equation (7) of the main paper and by Lieberman and Phillips (2020). The second component follows because $u_{0j}Z_{Ta,j-1}$ is a martingale difference array with standardized partial sums that satisfy the central limit theorem $T^{-\frac{1}{2}}\sum_{j=0}^{\lfloor Tr\rfloor} u_{0j}Z_{Ta,j-1} \rightsquigarrow \mathcal{N}(0, \frac{r\sigma_{00}}{-2c_z}a'\Omega_{aa}a)$, as in Hall and Heyde (1980, Theorem 3.2) and Phillips and Magdalinos (2007a, Lemma 4.2), with the corresponding functional limit law $T^{-\frac{1}{2}}\sum_{j=0}^{\lfloor Tr\rfloor} u_{0j}Z_{Ta,j-1} \rightsquigarrow B_{0Z_a}(r)$. The limit process B_{0Z_a} is a scalar Brownian motion with the variance matrix $\sigma_{00}\frac{a'\Omega_{aa}a}{-2c_z}$, which follows from the stability condition

$$\frac{1}{T}\sum_{t=1}^{T} Z_{Ta,t-1}^2\mathbb{E}\left(u_{0t}^2|\mathcal{F}_{t-1}\right) \to_p \frac{\sigma_{00}}{-2c_z}a'\Omega_{aa}a, \tag{106}$$

since $\frac{1}{T}\sum_{t=1}^{T} Z_{Ta,t-1}^2 \to_p \frac{1}{-2c_z}a'\Omega_{aa}a$, as in Phillips and Magdalinos (2007a, equation (13)). The third component of (105) follows the same derivation of the second component through replacing $a'u_{aj}$ by $\kappa'u_{xj}$ where κ is any fixed n-dimensional vector. In addition, following equation (14), we have

$$\begin{bmatrix} \frac{1}{T^{\frac{\gamma}{2}}} \sum_{j=1}^{\lfloor Tr \rfloor} a' u_{aj} r_{Tz}^{\lfloor Tr \rfloor - j} \\ \frac{1}{T^{\frac{\gamma}{2}}} \sum_{j=1}^{\lfloor Tr \rfloor} u_{xj} r_{Tz}^{\lfloor Tr \rfloor - j} \\ \frac{1}{T^{\frac{1}{2}}} \sum_{j=0}^{\lfloor Tr \rfloor} u_{aj} \\ \frac{1}{T^{\frac{1}{2}}} \sum_{j=0}^{\lfloor Tr \rfloor} u_{xj} \\ \frac{1}{T^{\frac{1}{2}}} \sum_{j=0}^{\lfloor Tr \rfloor} u_{0j} \end{bmatrix} \rightsquigarrow \begin{bmatrix} Z_a \\ Z_x \\ B_a(r) \\ B_x(r) \\ B_0(r) \end{bmatrix}. \tag{107}$$

Then, the joint convergence of (105) follows by standard theory (Ibragimov and Phillips, 2008), the continuous mapping theorem, joint covergence to the Gaussian process in (107), and the individual convergence results in (105).

The Brownian motion $B_{0Z_a}(r)$ is independent of the vectorized Brownian motions $(B_0(r), B'_x(r), B'_a(r))'$ because

$$\mathbb{E}\left[\left(\frac{1}{T^{\frac{1}{2}}}\sum_{j=1}^{\lfloor Tr \rfloor} u_{0j} Z_{Ta,j-1}\right)\left(\frac{1}{T^{\frac{1}{2}}}\sum_{j=0}^{\lfloor Tr \rfloor} u'_{aj}, \quad \frac{1}{T^{\frac{1}{2}}}\sum_{j=0}^{\lfloor Tr \rfloor} u'_{xj}, \quad \frac{1}{T^{\frac{1}{2}}}\sum_{j=0}^{\lfloor Tr \rfloor} u_{0j}\right)\right] = O\left(\frac{1}{T^{\frac{1-\gamma}{2}}}\right), \tag{108}$$

and it follows that B_{0Z_a} is independent of $G_{a,c}(r)$, which depends only on $(B'_x(r), B'_a(r))'$. Similarly, the Brownian motion B_{0Z_x} can be shown to be independent of $(B'_x(r), B'_a(r))'$ using the same approach as (108). Then the mixed normality of (104) follows by equivalence in distribution and asymptotic independence between $\left(B'_{0Z_a}, B'_{0Z_x}\right)'$ and $(B'_0(r), B'_x(r), B'_a(r))'$.

To establish (108), first note that u_{0t} is a martingale difference sequence and $u_{0t}Z_{Ta,t-1}$ is a martingale difference array so that, using (96)

$$\begin{aligned} &\mathbb{E}\left[\left(\frac{1}{T^{\frac{1}{2}}}\sum_{j=1}^{\lfloor Tr \rfloor} u_{0j} Z_{Ta,j-1}\right)\left(\frac{1}{T^{\frac{1}{2}}}\sum_{j=1}^{\lfloor Tr \rfloor} u_{0j}\right)\right] \\ &= \frac{1}{T}\sum_{j=1}^{\lfloor Tr \rfloor} \mathbb{E}\left(u_{0j}^2 Z_{Ta,j-1}\right) + \frac{1}{T}\sum_{j\neq k}^{\lfloor Tr \rfloor} \mathbb{E}\left(u_{0j}u_{0k}Z_{Ta,j-1}\right) \\ &= \frac{1}{T}\sum_{j=T^g}^{\lfloor Tr \rfloor} \mathbb{E}\left(u_{0j}^2 Z_{Ta,j-1}\right) + O\left(\frac{T^g}{T}\right) \\ &\sim \frac{1}{T}\sum_{j=T^g}^{\lfloor Tr \rfloor} \mathbb{E}\left(u_{0j}^2\right)\mathbb{E}\left(Z_a\right) + O\left(\frac{T^g}{T}\right) = o(1), \end{aligned} \tag{109}$$

using (96), the R-mixing property of independence of the limit variate Z_a, and the fact that $\mathbb{E}(Z_a) = 0$. Next, using (96) again we have

$$\mathbb{E}\left[\left(\frac{1}{T^{\frac{1}{2}}}\sum_{j=1}^{\lfloor Tr \rfloor} u_{0j} Z_{Ta,j-1}\right)\left(\frac{1}{T^{\frac{1}{2}}}\sum_{j=0}^{\lfloor Tr \rfloor} u_{xj}\right)\right]$$

$$= \frac{1}{T} \sum_{j=1}^{\lfloor Tr \rfloor} \mathbb{E}\left(u_{0j} u_{xj} Z_{Ta,j-1}\right) + \frac{1}{T} \sum_{j,k=1; j \neq k}^{\lfloor Tr \rfloor} \mathbb{E}\left(u_{0j} u_{xk} Z_{Ta,j-1}\right)$$

$$\sim \frac{1}{T} \sum_{j=1}^{\lfloor Tr \rfloor} \mathbb{E}\left(u_{0j} u_{xj}\right) \mathbb{E}\left(Z_a\right) + \frac{1}{T} \sum_{j,k=1; j \neq k}^{\lfloor Tr \rfloor} \mathbb{E}\left(u_{0j} u_{xk}\right) \mathbb{E}\left(Z_a\right) = \mathbf{0}_{n \times 1}. \tag{110}$$

In a similar way, we find that

$$\mathbb{E}\left[\left(\frac{1}{T^{\frac{1}{2}}} \sum_{j=1}^{\lfloor Tr \rfloor} u_{0j} Z_{Ta,j-1}\right)\left(\frac{1}{T^{\frac{1}{2}}} \sum_{j=0}^{\lfloor Tr \rfloor} a' u_{aj}\right)\right]$$

$$= \frac{1}{T} \sum_{j=1}^{\lfloor Tr \rfloor} \mathbb{E}\left(u_{0j} a' u_{aj} Z_{Ta,j-1}\right) + \frac{1}{T} \sum_{j,k=1; j \neq k}^{\lfloor Tr \rfloor} \mathbb{E}\left(u_{0j} a' u_{ak} Z_{Ta,j-1}\right)$$

$$\sim \frac{1}{T} \sum_{j=1}^{\lfloor Tr \rfloor} \mathbb{E}\left(u_{0j} a' u_{aj}\right) \mathbb{E}\left(Z_a\right) + \frac{1}{T} \sum_{j,k=1; j \neq k}^{\lfloor Tr \rfloor} \mathbb{E}\left(u_{0j} a' u_{ak}\right) \mathbb{E}\left(Z_a\right) = 0. \tag{111}$$

The independence between $B_{0Z_a}(r)$ and $(B_0(r), B_x'(r), B_a'(r))'$ follows (109), (110) and (111). Similarly, the independence between $B_{0Z_x}(r)$ and $(B_0(r), B_x'(r), B_a'(r))'$ can be shown. Therefore, the mixed normal limit result (104) now follows from the independence of the limit processes $\left(B_{0Z_x}'(r), B_{0Z_a}'(r)\right)'$ and $G_{a,c}(r)$, thereby establishing part (i) of Lemma A.5.

For part (ii), the proof of the QR-IVX numerator follows the same strategy as part (i), replacing the prediction error u_{0t} by $\Psi_\tau(u_{0t\tau})$. Similarly, a summation split and subsequent elimination and replacement of negligible components leads to the following simplified approximation

$$\begin{pmatrix} \frac{1}{T^{\frac{1+\gamma}{2}}} \sum_{t=1}^{T} z_{t-1} \Psi_\tau(u_{0t\tau}) \\ \frac{1}{T^{\frac{2+\gamma}{2}}} \sum_{t=1}^{T} \eta_{T,t-1}^{(2)} \Psi_\tau(u_{0t\tau}) \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{T}} \sum_{t=1}^{T} Z_{Tx,t-1} \Psi_\tau(u_{0t\tau}) \\ \frac{1}{T} \sum_{t=1}^{T} x_{t-1} \Psi_\tau(u_{0t\tau}) Z_{Ta,t-1} \end{pmatrix} + o_p(1). \tag{112}$$

Next, the sample covariance term has the following limit as $T \to \infty$

$$\begin{pmatrix} \frac{1}{T^{\frac{1}{2}}} \sum_{t=1}^{T} \Psi_\tau(u_{0t\tau}) Z_{Tx,t-1} \\ \frac{1}{T} \sum_{t=1}^{T} x_{t-1} \Psi_\tau(u_{0t\tau}) Z_{Ta,t-1} \end{pmatrix} \rightsquigarrow \begin{pmatrix} \int_0^1 dB_{\Psi_\tau Z_x}(r) \\ \int_0^1 G_{a,c}(r) dB_{\Psi_\tau Z_a}(r) \end{pmatrix} \tag{113}$$

$$=_d \mathcal{MN}\left(\mathbf{0}_{2n \times 1},\ \tau(1-\tau) \begin{bmatrix} V_{zz} & V_{z\eta}^{(2)} \\ V_{z\eta}^{(2)\prime} & V_{\eta\eta}^{(2)} \end{bmatrix}\right). \tag{114}$$

The asymptotic behavior (113) follows by virtue of the joint convergence, for $r \in (0, 1]$,

$$\begin{bmatrix} T^{-\frac{1}{2}} x_{\lfloor Tr \rfloor} \\ T^{-\frac{1}{2}} \sum_{j=0}^{\lfloor Tr \rfloor} \Psi_\tau(u_{0j\tau}) Z_{Ta,j-1} \\ T^{-\frac{1}{2}} \sum_{j=0}^{\lfloor Tr \rfloor} \Psi_\tau(u_{0j\tau}) Z_{Tx,j-1} \end{bmatrix} \rightsquigarrow \begin{bmatrix} G_{a,c}(r) \\ B_{\Psi_\tau Z_a}(r) \\ B_{\Psi_\tau Z_x}(r) \end{bmatrix}, \tag{115}$$

where $B_{\Psi_\tau Z_x}(r)$ and $B_{\Psi_\tau Z_a}(r)$ are defined similarly to the mean regression case. The equivalence in distribution (114) follows by the independence of the limit processes $G_{a,c}(r)$ and $\left(B'_{\Psi_\tau Z_a}(r), B'_{\Psi_\tau Z_x}(r)\right)'$.

The weak convergence of the second component in (115) follows because $\Psi_\tau(u_{0j\tau}) Z_{Ta,j-1}$ is a martingale difference array with standardized partial sums that satisfy the central limit theorem $T^{-\frac{1}{2}} \sum_{j=0}^{\lfloor Tr \rfloor} \Psi_\tau(u_{0j\tau}) Z_{Ta,j-1} \rightsquigarrow \mathcal{N}(0, r\frac{\tau(1-\tau)}{-2c_z} a'\Omega_{aa}a)$, with the corresponding functional limit law $T^{-\frac{1}{2}} \sum_{j=0}^{\lfloor Tr \rfloor} \Psi_\tau(u_{0j\tau}) Z_{Ta,j-1} \rightsquigarrow B_{\Psi_\tau Z_a}(r)$. The limit process $B_{\Psi_\tau Z_a}$ is a scalar Brownian motion with variance $\tau(1-\tau)\frac{a'\Omega_{aa}a}{-2c_z}$, which follows from the stability condition

$$\frac{1}{T}\sum_{t=1}^{T} Z_{Ta,t-1}^2 \mathbb{E}\left(\Psi_\tau^2(u_{0t\tau}) \,|\mathcal{F}_{t-1}\right) \to_p \frac{\tau(1-\tau)}{-2c_z} a'\Omega_{aa}a. \tag{116}$$

The weak convergence of the third component in (115) follows in the same way as before replacing $a'u_{aj}$ with $\kappa' u_{xj}$ where κ is a fixed n-dimensional vector. Similar to (107), we have

$$\begin{bmatrix} \frac{1}{T^{\frac{\gamma}{2}}} \sum_{j=1}^{\lfloor Tr \rfloor} a'u_{aj} r_{Tz}^{\lfloor Tr \rfloor - j} \\ \frac{1}{T^{\frac{\gamma}{2}}} \sum_{j=1}^{\lfloor Tr \rfloor} u_{xj} r_{Tz}^{\lfloor Tr \rfloor - j} \\ \frac{1}{T^{\frac{1}{2}}} \sum_{j=0}^{\lfloor Tr \rfloor} u_{aj} \\ \frac{1}{T^{\frac{1}{2}}} \sum_{j=0}^{\lfloor Tr \rfloor} u_{xj} \\ \frac{1}{T^{\frac{1}{2}}} \sum_{j=0}^{\lfloor Tr \rfloor} \Psi_\tau(u_{0t\tau}) \end{bmatrix} \rightsquigarrow \begin{bmatrix} Z_a \\ Z_x \\ B_a(r) \\ B_x(r) \\ B_{\Psi_\tau}(r) \end{bmatrix}. \tag{117}$$

Therefore, the joint convergence of (115) follows by standard theory (Ibragimov and Phillips, 2008), continuous mapping, joint covergence to the Gaussian process in (117) and the individual convergences to the corresponding limit processes.

The Brownian motion $B_{\Psi_\tau Z_a}(r)$ is independent of each of the Brownian motions $\left(B'_{\Psi_\tau}(r), B'_x(r), B'_a(r)\right)$ because

$$\mathbb{E}\left[\left(\frac{1}{T^{\frac{1}{2}}}\sum_{j=1}^{\lfloor Tr \rfloor} \Psi_\tau(u_{0j\tau}) Z_{Ta,j-1}\right)\left(\tfrac{1}{T^{\frac{1}{2}}}\textstyle\sum_{j=0}^{\lfloor Tr \rfloor} u'_{aj}, \quad \tfrac{1}{T^{\frac{1}{2}}}\sum_{j=0}^{\lfloor Tr \rfloor} u'_{xj}, \quad \tfrac{1}{T^{\frac{1}{2}}}\sum_{j=0}^{\lfloor Tr \rfloor} \Psi_\tau(u_{0j\tau})\right)\right]$$
$$= O\left(\frac{1}{T^{\frac{1-\gamma}{2}}}\right). \tag{118}$$

It therefore follows that $B_{\Psi_\tau Z_a}(r)$ is independent of $G_{a,c}(r)$, which depends only on the vector Brownian motion $(B'_x(r), B'_a(r))$. Equation (118) is established in the same way as (108). Similarly, $B_{\Psi_\tau Z_x}(r)$ can be shown to be independent of $G_{a,c}(r)$ by following (118). Then, in view of the asymptotic independence of $\left(B'_{\Psi_\tau Z_x}(r), B'_{\Psi_\tau Z_a}(r)\right)'$ and $(B'_0(r), B'_x(r), B'_a(r))'$, the joint mixed normality of (115) follows directly.

Finally, for parts (iii) and (iv), the asymptotic theory of the IVX and QR-IVX denominators follows Lemmas A.2, A.4 and the continuous mapping theorem. The proof is then complete. ■

Lemma A.6 *Suppose Assumptions 1 and 2 hold.*

(i) For the short-horizon IVX numerator, as $T \to \infty$,

$$\frac{1}{T^{\frac{1+\gamma}{2}}} \sum_{t=1}^{T} \widetilde{z}_{t-1} \underline{u}_{0t} \rightsquigarrow \mathcal{MN}\left(\mathbf{0}_{n\times 1},\ \sigma_{00}\left(V_{zz} + V_{\eta\eta}^{(2)} + V_{z\eta}^{(2)} + V_{z\eta}^{(2)\,\prime}\right)\right).$$

For the short-horizon QR-IVX estimator, as $T \to \infty$,

$$\frac{1}{T^{\frac{1+\gamma}{2}}} \sum_{t=1}^{T} \widetilde{z}_{t-1} \Psi_\tau(u_{0t\tau}) \rightsquigarrow \mathcal{MN}\left(\mathbf{0}_{n\times 1},\ \tau\,(1-\tau)\left(V_{zz} + V_{\eta\eta}^{(2)} + V_{z\eta}^{(2)} + V_{z\eta}^{(2)\,\prime}\right)\right).$$

The limiting covariance matrices V_{zz}, $V_{\eta\eta}^{(2)}$, and $V_{z\eta}^{(2)}$ are defined in Lemma A.4.

(ii) For the sample variance of the short-horizon QR-IVX and IVX numerators, as $T \to \infty$,

$$\frac{1}{T^{1+\gamma}} \sum_{t=1}^{T} \widetilde{z}_{t-1} \widetilde{z}_{t-1}' \rightsquigarrow \left(V_{zz} + V_{\eta\eta}^{(2)} + V_{z\eta}^{(2)} + V_{z\eta}^{(2)\,\prime}\right),$$

where the limiting covariance matrices V_{zz}, $V_{z\eta}^{(2)}$ and $V_{\eta\eta}^{(2)}$ are defined in Lemma A.4.

(iii) For the denominators of the short-horizon IVX and QR-IVX estimators, we have, as $T \to \infty$:

$$\frac{1}{T^{1+\gamma}} \sum_{t=1}^{T} \underline{x}_{t-1} \widetilde{z}_{t-1}' \rightsquigarrow V_{xz},$$

for the short-horizon IVX estimator, where V_{xz} is defined in Lemma A.5; and

$$\frac{1}{T^{1+\gamma}} \sum_{t=1}^{T} p_{u0t\tau,t-1}(0) x_{t-1} \widetilde{z}_{t-1}' \rightsquigarrow p_{u0\tau}(0) \cdot V_{xz}^{QRIVX},$$

for the short-horizon QR-IVX estimator, where $V_{xz}^{QRIVX} := V_{xz}^{l} + V_{x\eta}^{(1)} + V_{x\eta}^{(2)} + \frac{1}{2} V_{x\eta}^{(3)}$.

Proof: For part (i) in the case of the IVX numerator, the joint mixed normality of Lemma A.5 yields

$$\begin{aligned}\frac{1}{T^{\frac{1+\gamma}{2}}} \widetilde{Z}_{-1}' \underline{u}_0 &= \frac{1}{T^{\frac{1+\gamma}{2}}} \sum_{t=1}^{T} z_{t-1} u_{0t} + \frac{1}{T^{\frac{2+\gamma}{2}}} \sum_{t=1}^{T} \eta_{T,t-1}^{(2)} u_{0t} + o_p(1) \\ &\rightsquigarrow \mathcal{MN}\left(\mathbf{0}_{n\times 1},\ \sigma_{00}\left(V_{zz} + V_{\eta\eta}^{(2)} + V_{z\eta}^{(2)} + V_{z\eta}^{(2)\,\prime}\right)\right).\end{aligned}$$

For (i) in the case of the QR-IVX estimator, using $\Psi_\tau(u_{0t\tau})$ to replace u_{0t}, $\tau(1-\tau)$ to replace σ_{00}, and B_{Ψ_τ} to replace the Brownian motion B_0, weak convergence to joint mixed normality follows from Lemma A.5. Part (ii) follows as a natural extension of (i) and the mds errors.

The limit result of the short-horizon IVX estimators in part (iii) follows directly from Lemma A.5 (iii) and the representation

$$\frac{1}{T^{1+\gamma}}\sum_{t=1}^{T}\widetilde{z}_{t-1}\underline{x}_{t-1}' = \frac{1}{T^{1+\gamma}}\sum_{t=1}^{T}z_{t-1}x_{t-1}' + \frac{1}{T^{\frac{3}{2}+\gamma}}\sum_{t=1}^{T}\eta_{T,t-1}^{(2)}x_{t-1}' + \frac{C}{T^{2+\gamma}}\sum_{t=1}^{T}\eta_{T,t-1}^{(1)}x_{t-1}' + \frac{1}{2T^{2+\gamma}}\sum_{t=1}^{T}\eta_{T,t-1}^{(3)}x_{t-1}' + \left(\frac{1}{\sqrt{T}}x_T\right)\left(\frac{1}{T^{\frac{3}{2}}}\sum_{t=1}^{T}x_{t-1}\right)' C_z^{-1} + o_p(1)\,. \tag{119}$$

The limit results of the QR-IVX denominator in part (iii) follow directly from Lemma A.5 (iii), Lemma A.2 of Lee (2016) and the approximate first-order condition given in equation (123) below. Using the fact that the QR estimator $\widehat{\beta}_{0,\tau}$ is employed in the QR-IVX estimation, the limit expression for V_{xz}^{QRIVX} is free of the term V_{demean}. ■

Lemma A.7 *For some constant $M > 0$,*

$$\sup\left\{\|H_T(\vartheta) - H_T(0)\| : \|\vartheta\| \leq T^{\frac{1+\gamma}{2}}M\right\} = o_p(1),$$

where $H_T(\vartheta) := T^{-\frac{1+\gamma}{2}}\sum_{t=1}^{T}\widetilde{z}_{t-1}\left\{\Psi_\tau\left(u_{0t\tau} - \vartheta'\check{X}_{t-1}\right) - \mathbb{E}_{t-1}\left[\Psi_\tau\left(u_{0t\tau} - \vartheta'\check{X}_{t-1}\right)\right]\right\}$.

Proof: The results follow exactly as in Lemma A.3 of Lee (2016). ■

Proof of Theorem 3.1

Proof: Since uniform convergence is confirmed in Lemma A.7, the standard result for extremum estimation with a non-smooth criterion function holds following Lee (2016). Let $\widehat{\beta}_{1,\tau} := \widehat{\beta}_{1,\tau}^{QRIVX}$ and $\widehat{\beta}_{0,\tau}$ be a preliminary QR estimator in the following argument. Define $\widehat{\vartheta}_{1,\tau} := \left(\widehat{\beta}_{1,\tau} - \beta_{1,\tau}\right)$, $\widehat{\vartheta}_{0,\tau} := \left(\widehat{\beta}_{0,\tau} - \beta_{0,\tau}\right)$ and then

$$\widehat{\beta}_{1,\tau} \sim_a \arg\min_{\beta_{1,\tau}^\star}\left(\sum_{t=1}^{T}m_t\left(\beta_{1,\tau}^\star\right)\right)'\left(\sum_{t=1}^{T}m_t\left(\beta_{1,\tau}^\star\right)\right),$$

where $m_t\left(\beta_{1,\tau}^\star\right) = \widetilde{z}_{t-1}\Psi_\tau\left(u_{0t\tau}\left(\beta_{1,\tau}^\star\right)\right) = \widetilde{z}_{t-1}\left(\tau - \mathbf{1}\left(y_{t\tau} \leq \widehat{\beta}_{0,\tau} + \beta_{1,\tau}^{\star\prime}x_{t-1}\right)\right)$, and $\mathbf{1}(\cdot)$ is the indicator function. By Theorem 1 of Xiao (2009), the QR estimator $\widehat{\beta}_{0,\tau}$ is $\sqrt{T}$-consistent. Therefore, in the QR-IVX procedure

$$\frac{1}{T^{\frac{1+\gamma}{2}}}\sum_{t=1}^{T}\widetilde{z}_{t-1}\Psi_\tau\left(y_t - \beta_{0,\tau} - \beta_{1,\tau}^{\star\prime}x_{t-1}\right) - \frac{1}{T^{\frac{1+\gamma}{2}}}\sum_{t=1}^{T}\widetilde{z}_{t-1}\Psi_\tau\left(y_t - \widehat{\beta}_{0,\tau} - \beta_{1,\tau}^{\star\prime}x_{t-1}\right)$$
$$= \frac{1}{T^{\frac{1+\gamma}{2}}}\sum_{t=1}^{T}\widetilde{z}_{t-1}\left\{\mathbf{1}\left(u_{0t\tau} \leq \left(\beta_{1,\tau}^{\star\prime} - \beta_{1,\tau}'\right)x_{t-1}\right) - \mathbf{1}\left(u_{0t\tau} \leq \left(\widehat{\beta}_{0,\tau} - \beta_{0,\tau}\right) + \left(\beta_{1,\tau}^{\star\prime} - \beta_{1,\tau}'\right)x_{t-1}\right)\right\}$$
$$= \frac{1}{T^{\frac{1+\gamma}{2}}}\sum_{t=1}^{T}\widetilde{z}_{t-1}\xi_t, \tag{120}$$

where

$$\xi_t := \left\{ \mathbf{1}\left(u_{0t\tau} \leq \left(\beta^{\star\,\prime}_{1,\tau} - \beta'_{1,\tau}\right) x_{t-1}\right) - \mathbf{1}\left(u_{0t\tau} \leq \left(\widehat{\beta}_{0,\tau} - \beta_{0,\tau}\right) + \left(\beta^{\star\,\prime}_{1,\tau} - \beta'_{1,\tau}\right) x_{t-1}\right) \right\}.$$

Taking conditional expectations $\mathbb{E}_{t-1}(\cdot)$ and for any possible value of $\beta^{\star}_{1,\tau}$ we have

$$\begin{aligned} \left\| \frac{1}{T^{\frac{1+\gamma}{2}}} \sum_{t=1}^{T} \widetilde{z}_{t-1} \cdot \mathbb{E}_{t-1}\xi_t \right\| &\sim_a \left\| p_{u0\tau}(0) \frac{1}{T^{\frac{1+\gamma}{2}}} \sum_{t=1}^{T} \widetilde{z}_{t-1} \left(\widehat{\beta}_{0,\tau} - \beta_{0,\tau}\right) \right\| \\ &= O_p\left(\frac{1}{T^{1+\frac{\gamma}{2}}}\right) \cdot \left\| \sum_{t=1}^{T} \widetilde{z}_{t-1} \right\| = O_p\left(T^{\frac{-1+\gamma}{2}}\right), \end{aligned} \tag{121}$$

where the last equality is due to Lemma A.2 and the penultimate equality results from Theorem 1 of Xiao (2009). In addition, for any possible value of $\beta^{\star}_{1,\tau}$, the conditional quadratic variation of the objective function is asymptotically diminishing as

$$\begin{aligned} \left\| \frac{1}{T^{1+\gamma}} \sum_{t=1}^{T} \mathbb{E}_{t-1}\left(\widetilde{z}_{t-1}\widetilde{z}'_{t-1} \cdot \xi_t^2\right) \right\| &\leq \left\| \frac{1}{T^{1+\gamma}} \sum_{t=1}^{T} \widetilde{z}_{t-1}\widetilde{z}'_{t-1} \right\| \cdot \left(\sup_{1\leq t\leq T} \mathbb{E}_{t-1}\xi_t^2 \right) \\ &= O_p\left(\sup_{1\leq t\leq T} \mathbb{E}_{t-1}\xi_t^2 \right) = o_p(1), \end{aligned} \tag{122}$$

under the $\sqrt{T}$-consistency of the preliminary estimator $\widehat{\beta}_{0,\tau}$. Combining (121) and (122) shows that (120) is $o_p(1)$. Thus, $\widehat{\beta}_{0,\tau}$ does not impact the QR-IVX asymptotics. Based on Lemma A.7, the QR-IVX objective function is asymptotically approximated as

$$\begin{aligned} &o_p(1) \\ &= \frac{1}{T^{\frac{1+\gamma}{2}}} \sum_{t=1}^{T} \widetilde{z}_{t-1} \left\{ \Psi_\tau\left(u_{0t\tau} - \left(\widehat{\beta}_{1,\tau} - \beta_{1,\tau}\right)' x_{t-1} - \left(\widehat{\beta}_{0,\tau} - \beta_{0,\tau}\right)\right) \right\} \\ &= \frac{1}{T^{\frac{1+\gamma}{2}}} \sum_{t=1}^{T} \widetilde{z}_{t-1} \left\{ \Psi_\tau\left(u_{0t\tau} - \widehat{\vartheta}'_{1,\tau}x_{t-1} - \widehat{\vartheta}_{0,\tau}\right) - \mathbb{E}_{t-1}\left(\Psi_\tau\left(u_{0t\tau} - \widehat{\vartheta}'_{1,\tau}x_{t-1} - \widehat{\vartheta}_{0,\tau}\right)\right) \right\} \\ &- \frac{1}{T^{\frac{1+\gamma}{2}}} \sum_{t=1}^{T} \widetilde{z}_{t-1} \left\{ \Psi_\tau(u_{0t\tau}) - \mathbb{E}_{t-1}(\Psi_\tau(u_{0t\tau})) \right\} + \frac{1}{T^{\frac{1+\gamma}{2}}} \sum_{t=1}^{T} \widetilde{z}_{t-1} \mathbb{E}_{t-1}\left(\Psi_\tau\left(u_{0t\tau} - \widehat{\vartheta}_{0,\tau} - \widehat{\vartheta}'_{1,\tau}x_{t-1}\right)\right) \\ &+ \frac{1}{T^{\frac{1+\gamma}{2}}} \sum_{t=1}^{T} \widetilde{z}_{t-1} \left\{ \Psi_\tau(u_{0t\tau}) \right\} \\ &= \frac{1}{T^{\frac{1+\gamma}{2}}} \sum_{t=1}^{T} \widetilde{z}_{t-1} \mathbb{E}_{t-1}\left(\Psi_\tau\left(u_{0t\tau} - \widehat{\vartheta}_{0,\tau} - \widehat{\vartheta}'_{1,\tau}x_{t-1}\right)\right) + \frac{1}{T^{\frac{1+\gamma}{2}}} \sum_{t=1}^{T} \widetilde{z}_{t-1} \left\{ \Psi_\tau(u_{0t\tau}) \right\} + o_p(1) \\ &\sim_a \frac{1}{T^{\frac{1+\gamma}{2}}} \sum_{t=1}^{T} \widetilde{z}_{t-1} \mathbb{E}_{t-1}\left(\Psi_\tau\left(u_{0t\tau} - \widehat{\vartheta}'_{1,\tau}x_{t-1}\right)\right) + \frac{1}{T^{\frac{1+\gamma}{2}}} \sum_{t=1}^{T} \widetilde{z}_{t-1} \left\{ \Psi_\tau(u_{0t\tau}) \right\}, \end{aligned}$$

where the third equality comes from Lemma A.7 and the last line is due to $\sqrt{T}$-consistency of the preliminary QR estimator $\widehat{\beta}_{0,\tau}$ and the uniform approximation of equation (121).

The term $\mathbb{E}_{t-1}\left(\Psi_\tau\left(u_{0t\tau}-\widehat{\vartheta}_{1,\tau}'x_{t-1}\right)\right)$ can be expanded around $\vartheta_{1,\tau}=0$ as

$$\begin{aligned}&\mathbb{E}_{t-1}\left(\Psi_\tau\left(u_{0t\tau}-\widehat{\vartheta}_{1,\tau}'x_{t-1}\right)\right)\\ =\;&\mathbb{E}_{t-1}\left[\mathbf{1}\left(u_{0t\tau}-\vartheta_{1,\tau}'x_{t-1}\right)\right]|_{\vartheta_{1,\tau}=0}+\left(\frac{\partial\mathbb{E}_{t-1}\left(\Psi_\tau\left(u_{0t\tau}-\vartheta_{1,\tau}'x_{t-1}\right)\right)}{\partial\vartheta_{1,\tau}'}|_{\vartheta_{1,\tau}=0}\right)\widehat{\vartheta}_{1,\tau}+o_p\left(\widehat{\vartheta}_{1,\tau}\right),\end{aligned}$$

where

$$\mathbb{E}_{t-1}\left(\Psi_\tau\left(u_{0t\tau}-\vartheta_{1,\tau}'x_{t-1}\right)\right)=\tau-\mathbb{E}_{t-1}\left(\mathbf{1}\left(u_{0t\tau}<\vartheta_{1,\tau}'x_{t-1}\right)\right)=\tau-\int_{-\infty}^{\vartheta_{1,\tau}'x_{t-1}}p_{u0t\tau,t-1}(s)ds,$$

and
$$\frac{\partial\mathbb{E}_{t-1}\left[\Psi_\tau(u_{0t\tau}-\vartheta_{1,\tau}'x_{t-1})\right]}{\partial\vartheta_{1,\tau}'}|_{\vartheta_{1,\tau}=0}=-x_{t-1}'p_{u0t\tau,t-1}(0).$$

So we have

$$\mathbb{E}_{t-1}\left[\Psi_\tau(u_{0t\tau}-\widehat{\vartheta}_{1,\tau}'x_{t-1})\right]=-x_{t-1}'p_{u0t\tau,t-1}(0)\widehat{\vartheta}_{1,\tau}+o_p(1).$$

The first-order condition therefore has the approximate form

$$o_p\left(1\right)=\sum_{t=1}^{T}\widetilde{z}_{t-1}\Psi_\tau\left(u_{0t\tau}\right)+\sum_{t=1}^{T}p_{u0t\tau,t-1}\left(0\right)\widetilde{z}_{t-1}x_{t-1}'\left(\widehat{\beta}_{1,\tau}-\beta_{1,\tau}\right),\tag{123}$$

and for any $\tau\in(0,1)$ the limit distributions of $\widehat{\beta}_{1,\tau}$ follows from equation (123) and Lemmas A.5 and A.6.

In a similar fashion the derivations for the short-horizon IVX estimate follow directly from the representation of the IVX estimation error and the earlier limit theory for the components. In particular,

$$T^{\frac{1+\gamma}{2}}\left(\widehat{\beta}_1^{IVX}-\beta_1\right)=\left(\frac{1}{T^{1+\gamma}}\widetilde{Z}_{-1}'\underline{X}_{-1}\right)^{-1}\left(\frac{1}{T^{\frac{1+\gamma}{2}}}\widetilde{Z}_{-1}'\underline{u}_0\right),\tag{124}$$

and, using Lemma A.6, we have

$$\begin{aligned}\frac{1}{T^{\frac{1+\gamma}{2}}}\widetilde{Z}_{-1}'\underline{u}_0&=\frac{1}{T^{\frac{1+\gamma}{2}}}\sum_{t=1}^{T}z_{t-1}u_{0t}+\frac{1}{T^{\frac{2+\gamma}{2}}}\sum_{t=1}^{T}\eta_{T,t-1}^{(2)}u_{0t}+o_p(1)\\&\rightsquigarrow\mathcal{MN}\left(\mathbf{0}_{n\times1},\ \sigma_{00}\left(V_{zz}+V_{\eta\eta}^{(2)}+V_{z\eta}^{(2)}+V_{z\eta}^{(2)\prime}\right)\right),\end{aligned}\tag{125}$$

and $\frac{1}{T^{1+\gamma}}\widetilde{Z}_{-1}'\underline{X}_{-1}\rightsquigarrow V_{zx}$. Combining these results, we have

$$T^{\frac{1+\gamma}{2}}\left(\widehat{\beta}_1^{IVX}-\beta_1\right)\rightsquigarrow\mathcal{MN}\left(\mathbf{0}_{n\times1},\ \sigma_{00}V_{zx}^{-1}\left(V_{zz}+V_{\eta\eta}^{(2)}+V_{z\eta}^{(2)}+V_{z\eta}^{(2)\prime}\right)\left(V_{zx}^{-1}\right)'\right),\tag{126}$$

giving the required limit. This completes the proof of Theorem 3.1. ■

Proof of Theorem 3.2

Proof: Based on the mixed normality of Theorem 3.1 and the asymptotic independence between the IVX numerator and the IVX denominator, the QR-IVX-Wald and IVX-Wald tests follow pivotal χ^2 distributions under the corresponding null hypotheses. ■

B Lemmas for Long-horizon Predictive Regression

We introduce several random components to facilitate analysis of long-horizon predictive regression. We define the Gaussian variates $(Z_x^{\star\prime}, Z_a^{\star\prime})'$ driven jointly by taking averages around the time point $t^\star$ across the whole prediction horizon k as

$$\left(\frac{1}{\sqrt{k}}\sum_{j=1}^{k} u'_{x,t^\star+j}, \frac{1}{\sqrt{k}}\sum_{j=1}^{k} u'_{a,t^\star+j}\right)' \rightsquigarrow \left(Z_x^{\star\prime}, Z_a^{\star\prime}\right)' =_d \mathcal{N}\left(\mathbf{0}_{2n\times 1}, \begin{pmatrix} \Omega_{xx} & \Omega_{xa} \\ \Omega_{ax} & \Omega_{aa} \end{pmatrix}\right), \tag{127}$$

for any given $t^\star$ with $t^\star = \lfloor Tr^\star \rfloor$ and $r^\star \in [0,1]$. The Brownian motion limit processes used relevant to the long-horizon predictive regression model, $(B_0(r), B'_x(r), B'_a(r))'$, are driven by partial summations of the innovations over the time horizon T as

$$\left(\frac{1}{\sqrt{T}}\sum_{t=1}^{\lfloor Tr\rfloor-k} u_{0,t+j}, \frac{1}{\sqrt{T}}\sum_{t=1}^{\lfloor Tr\rfloor-k} u'_{x,t+j-1}, \frac{1}{\sqrt{T}}\sum_{t=1}^{\lfloor Tr\rfloor-k} u'_{a,t+j-1}\right)' \rightsquigarrow \left(B_0(r), B'_x(r), B'_a(r)\right)', \tag{128}$$

with index $1 \le j \le k$. The functional limits in (127) and (128) are obtained under the rate restriction $\frac{\sqrt{T}}{T^\gamma} + \frac{T^\gamma}{k} + \frac{k}{T} \to 0$.

From (127) and (128), it follows that the two groups of limiting Gaussian variates are uncorrelated. In particular, for any $t^\star = \lfloor Tr^\star \rfloor$, $t = \lfloor Tr \rfloor$ with $r^\star \in [0,1]$ and $r \in [0,1]$,

$$\left(Z_x^{\star\prime}, Z_a^{\star\prime}\right)' \perp \left(B_0(r), B'_x(r), B'_a(r)\right)', \tag{129}$$

as the components $\left(\frac{1}{\sqrt{k}}\sum_{j=1}^{k} u'_{x,t^\star+j}, \frac{1}{\sqrt{k}}\sum_{j=1}^{k} u'_{a,t^\star+j}\right)'$ escape asymptotically from their own probability space in view of the mixing property. In addition, for any $t^\star = \lfloor Tr^\star \rfloor$ and $r^\star \in [0,1]$, we have joint weak convergence of the two groups as

$$\left(\frac{1}{\sqrt{T}}\sum_{t=1}^{\lfloor Tr\rfloor-k} u_{0,t+j}, \frac{1}{\sqrt{T}}\sum_{t=1}^{\lfloor Tr\rfloor-k} u'_{x,t+j-1}, \frac{1}{\sqrt{T}}\sum_{t=1}^{\lfloor Tr\rfloor-k} u'_{a,t+j-1}, \frac{1}{\sqrt{k}}\sum_{j=1}^{k} u'_{x,t^\star+j}, \frac{1}{\sqrt{k}}\sum_{j=1}^{k} u'_{a,t^\star+j}\right)'$$
$$\rightsquigarrow \left(B_0(r), B'_x(r), B'_a(r), Z_x^{\star\prime}, Z_a^{\star\prime}\right)', \tag{130}$$

and asymptotic independence between $(Z_x^{\star\prime}, Z_a^{\star\prime})'$ and $(B_0(r), B'_x(r), B'_a(r))'$, as in the asymptotic zero correlation (129). Based on the joint convergence (130) and asymptotic independence, the mixed normality of the LHIVX estimator and the pivotal chi-squared test statistics are established and collected together in the following results.

Lemma B.1 *Let Assumptions 1 and 2 hold. Under the rate condition* $\frac{\sqrt{T}}{T^{\gamma}} + \frac{T^{\gamma}}{k} + \frac{k}{T} \to 0$ *and since* $C_z = c_z I_n$*, we have the following:*

(i) For any $t = \lfloor Tr \rfloor$ *and* $r \in [0,1]$, $-\frac{C_z}{kT^{\frac{1}{2}+\gamma}}\eta_{Tt}^{1,k} \rightsquigarrow \begin{cases} G_{a,c}(r), & \text{under LSTUR} \\ G_a(r), & \text{under STUR} \end{cases}$.

(ii) $\sum_{t=1}^{T-k} u_{0,t+k}\eta_{T,t}^{1,k\,\prime} = O_p\left(kT^{1+\gamma}\right)$, *and* $\frac{1}{\sqrt{k}T^{\frac{3}{2}+\gamma}}\sum_{t=1}^{T-k} u_{0,t+k}\eta_{T,t}^{1,k\,\prime} = o_p(1)$.

(iii) For $t = \lfloor Tr \rfloor$ *with* $r \in [0,1]$, $-\frac{C_z}{\sqrt{k}T^{\gamma}} z_t^k \rightsquigarrow Z_x^{\star}$, *where* $Z_x^{\star}$ *is defined in* (127), *and*

$$\frac{1}{T}\sum_{t=1}^{T-k}\left(\frac{C_z}{\sqrt{k}T^{\gamma}} z_t^k\right)\left(\frac{C_z}{\sqrt{k}T^{\gamma}} z_t^k\right)' \to_p \Omega_{xx}.$$

(iv) $\frac{1}{\sqrt{k}T^{\frac{1}{2}+\gamma}}\sum_{t=1}^{T-k} u_{0,t+k} z_t^{k\,\prime} \rightsquigarrow -C_z^{-1}\int_0^1 Z_x^{\star\,\prime} dB_0(r) =_d \mathcal{N}(0,\ \sigma_{00}\Omega_{xx}C_z^{-2})$.

(v) For any $t = \lfloor Tr \rfloor$ *and* $r \in [0,1]$,

$$-\frac{C_z}{\sqrt{k}T^{\frac{1}{2}+\gamma}}\eta_{T,t}^{2,k} \rightsquigarrow \begin{cases} (a'Z_a^{\star})\, G_{a,c}(r), & \text{under LSTUR} \\ (a'Z_a^{\star})\, G_a(r), & \text{under STUR} \end{cases},$$

where $Z_a^{\star}$ *is defined in* (127).

(vi) $-\frac{C_z}{\sqrt{k}T^{1+\gamma}}\sum_{t=1}^{T-k} u_{0,t+k}\eta_{T,t}^{2,k\,\prime} \rightsquigarrow \begin{cases} \int_0^1 G_{a,c}'(r)\,(a'Z_a^{\star})\, dB_0(r), & \text{under LSTUR} \\ \int_0^1 G_a'(r)\,(a'Z_a^{\star})\, dB_0(r), & \text{under STUR} \end{cases}$.

(vii) $\frac{1}{kT^{\frac{3}{2}+2\gamma}}\sum_{t=1}^{T-k} z_t^k\eta_{T,t}^{2,k\,\prime} \rightsquigarrow \begin{cases} \int_0^1 (\Omega_{xa}a)\, G_{a,c}'(r) dr C_z^{-2}, & \text{under LSTUR} \\ \int_0^1 (\Omega_{xa}a)\, G_a'(r) dr C_z^{-2}, & \text{under STUR} \end{cases}$.

(viii) $\frac{1}{kT^{2+2\gamma}}\sum_{t=1}^{T-k} \eta_{T,t}^{2,k}\eta_{T,t}^{2,k\,\prime} \rightsquigarrow \begin{cases} \int_0^1 G_{a,c}(r)G_{a,c}'(r)\,(a'\Omega_{aa}a)\, dr C_z^{-2}, & \text{under LSTUR} \\ \int_0^1 G_a(r)G_a'(r)\,(a'\Omega_{aa}a)\, dr C_z^{-2}, & \text{under STUR} \end{cases}$.

(ix) For any $t = \lfloor Tr \rfloor$ *and* $r \in [0,1]$,

$$-\frac{C_z}{kT^{\frac{1}{2}+\gamma}}\eta_{Tt}^{3,k} \rightsquigarrow \begin{cases} (a'\Sigma_{aa}a)\, G_{a,c}(r), & \text{under LSTUR} \\ (a'\Sigma_{aa}a)\, G_a(r), & \text{under STUR} \end{cases}. \tag{131}$$

(x) $\sum_{t=1}^{T-k} u_{0,t+k}\eta_{T,t}^{3,k\,\prime} = O_p\left(kT^{1+\gamma}\right)$ *and so* $\frac{1}{\sqrt{k}T^{\frac{3}{2}+\gamma}}\sum_{t=1}^{T-k} u_{0,t+k}\eta_{T,t}^{3,k\,\prime} = o_p(1)$.

Proof: (i) (ii) and (iii) correspond to the results in Lemma A.1 in Phillips and Lee (2013) simply by replacing $J_c(r)$ by $G_a(r)$ and $G_{a,c}(r)$ respectively. Result (iv) follows the same proof as their Lemma A.2, and (x) follows (ix) and the mds property of u_{0t}.

(v) We use the recursion $\eta_{T,t}^{(2)} = R_{Tz}\eta_{T,t-1}^{(2)} + (a'u_{at})\, x_{t-1}$, so that $\eta_{T,t+j-1}^{(2)} = R_{Tz}\eta_{T,t+j-2}^{(2)} + (a'u_{a,t+j-1})\, x_{t+j-2}$. Taking summation up to k gives

$$\eta_{T,t}^{2,k} = \sum_{j=1}^{k}\eta_{T,t+j-1}^{(2)} = R_{Tz}\sum_{j=1}^{k}\eta_{T,t+j-2}^{(2)} + \sum_{j=1}^{k}\left(a'u_{a,t+j-1}\right) x_{t+j-2},$$

which implies

$$(I_n - R_{Tz})\sum_{j=1}^{k}\eta_{T,t+j-1}^{(2)} = \sum_{j=1}^{k}\left(a'u_{a,t+j-1}\right)x_{t+j-2} - R_{Tz}\eta_{T,t+k-1}^{(2)} + R_{Tz}\eta_{T,t-1}^{(2)},$$

giving

$$-\frac{C_z}{\sqrt{k}T^{\frac{1+2\gamma}{2}}}\sum_{j=1}^{k}\eta_{T,t+j-1}^{(2)} = \frac{1}{\sqrt{k}T^{\frac{1}{2}}}\sum_{j=1}^{k}\left(a'u_{a,t+j-1}\right)x_{t+j-2} + o_p(1), \tag{132}$$

where $\frac{1}{\sqrt{kT}}\sup_{1\leq t\leq T}\left\|\eta_{T,t-1}^{(2)}\right\| = O_p\left(\frac{T^{\frac{\gamma}{2}}}{\sqrt{k}}\right) = o_p(1)$. Let l_T be such that $\frac{1}{l_T} + \frac{l_T}{k} \to 0$, so that

$$\frac{1}{\sqrt{k}}\sum_{j=1}^{l_T}\left(a'u_{a,t+j-1}\right)\left(x_{t+j-2}/\sqrt{T}\right) = O_p\left(\sqrt{\frac{l_T}{k}}\right) = o_p\left(1\right). \tag{133}$$

Then, setting $t = \lfloor Tr \rfloor$ with $r \in [0,1]$ and by a summation splitting argument coupled with elimination and subsequent addition of residual terms, we have the following derivations for the LSTUR case.

$$\begin{aligned}
&-\frac{C_z}{\sqrt{k}T^{\frac{1+2\gamma}{2}}}\sum_{j=1}^{k}\eta_{T,t+j-1}^{(2)} \\
&= \frac{1}{\sqrt{k}}\sum_{j=1}^{k}\left(a'u_{a,t+j-1}\right)\left(x_{t+j-2}/\sqrt{T}\right) + o_p(1) \\
&= \frac{1}{\sqrt{k}}\sum_{j=l_T}^{k}\left(a'u_{a,t+j-1}\right)\left(x_{t+j-2}/\sqrt{T}\right) + o_p(1) && (134)\\
&\sim_a \frac{1}{\sqrt{k}}\sum_{j=1}^{k}\left(a'u_{a,t+j-1}\right)\left(G_{a,c}(r)\right) - \frac{1}{\sqrt{k}}\sum_{j=1}^{l_T-1}\left(a'u_{a,t+j-1}\right)\left(G_{a,c}(r)\right) && (135)\\
&= \frac{1}{\sqrt{k}}\sum_{j=1}^{k}\left(a'u_{a,t+j-1}\right)\left(G_{a,c}(r)\right) + o_p\left(1\right) && (136)\\
&\rightsquigarrow \left(a'Z_a^{\star}\right)G_{a,c}(r), && (137)
\end{aligned}$$

where equation (134) holds due to equation (133), equation (135) holds by the FCLT developed in Lieberman and Phillips (2020), and equation (136) holds by the fact that $l_T/k \to 0$, $\frac{1}{\sqrt{kT}}\eta_{T,t}^{(2)} = O_p\left(\frac{T^{\frac{\gamma}{2}}}{\sqrt{k}}\right) = o_p(1)$ and $\frac{1}{\sqrt{k}}\sum_{j=1}^{l_T} a'u_{at} = O_p\left(\frac{\sqrt{l_T}}{\sqrt{k}}\right) = o_p\left(1\right)$. The weak convergence in (137) is established by using (127) and (128). The mixing property of the weak convergence shows that the random element $\frac{1}{\sqrt{k}}\sum_{j=1}^{\lfloor ks\rfloor} a'u_{a,t+j-1}$ is asymptotically independent of all events $E \in \mathcal{F}$, so that as $\frac{\sqrt{T}}{T^{\gamma}} + \frac{T^{\gamma}}{k} + \frac{k}{T} \to 0$,

$$\Pr\left[\left(\frac{1}{\sqrt{k}}\sum_{j=1}^{\lfloor ks\rfloor} a'u_{a,t+j-1} \in \cdot\right) \cap E\right] \to \Pr\left[\left(\frac{1}{\sqrt{k}}\sum_{j=1}^{\lfloor ks\rfloor} a'u_{a,t+j-1} \in \cdot\right)\right]\Pr\left[E\right], \tag{138}$$

where $s \in [0,1]$. In this sense, the random element $\frac{1}{\sqrt{k}} \sum_{j=1}^{\lfloor ks \rfloor} a' u_{a,t+j-1}$ effectively escapes from its own probability space when the mixing property applies (Rényi, 1963; Hall and Heyde, 1980; Cheng and Chow, 2002). As discussed in the main paper, limit theorems with the mixing property apply in very general situations, including martingale CLTs and the CLTs of McLeish et al. (1974), thereby including the present application. The weak convergence given in (137) then follows. Setting $C = \mathbf{0}_{n\times n}$ covers the STUR case.

(vi) For the LSTUR case,

$$-\frac{C_z}{\sqrt{k}T^{1+\gamma}} \sum_{t=1}^{T-k} u_{0,t+k} \eta_{T,t}^{2,k\,\prime} = \sum_{t=1}^{T-k} \left(\frac{u_{0,t+k}}{\sqrt{T}} \right) \left(\frac{-C_z \eta_{T,t}^{2,k}}{\sqrt{k} T^{\frac{1+2\gamma}{2}}} \right)' \rightsquigarrow \int_0^1 dB_0(r) G_{a,c}'(r) \left(a' Z_a^{\star}\right),$$

and for STUR, $-\frac{C_z}{\sqrt{k}T^{1+\gamma}} \sum_{t=1}^{T-k} u_{0,t+k} \eta_{T,t}^{2,k\,\prime} \rightsquigarrow \int_0^1 dB_0(r) G_a'(r) \left(a' Z_a^{\star}\right)$.

(vii) By (v) and the functional limit theory in (127)–(130), we have

$$\frac{1}{kT^{\frac{3}{2}+2\gamma}} \sum_{t=1}^{T-k} z_t^k \eta_{T,t}^{2,k\,\prime}$$

$$= C_z^{-1} \frac{1}{T} \sum_{t=1}^{T-k} \left(\frac{-C_z}{\sqrt{k}T^{\gamma}} z_t^k \right) \left(\frac{-C_z}{\sqrt{k}T^{\frac{1}{2}+\gamma}} \eta_{T,t}^{2,k} \right)' C_z^{-1}$$

$$\sim_a \frac{1}{T} \sum_{t=1}^{T-k} \left(\frac{1}{\sqrt{k}} \sum_{j=1}^{k} u_{x,t+j-1} \right) \left(\frac{1}{\sqrt{k}} \sum_{l=1}^{k} \left(a' u_{a,t+l-1}\right) \frac{x_{t+l-2}}{\sqrt{T}} \right)' C_z^{-2} \tag{139}$$

$$\sim_a \frac{1}{T} \sum_{t=1}^{T-k} \left(\frac{1}{\sqrt{k}} \sum_{j=1}^{k} F_x(1)\, \epsilon_{x,t+j-1} \right) \left(\frac{1}{\sqrt{k}} \sum_{l=1}^{k} \left(a' F_a(1)\, \epsilon_{a,t+l-1}\right) \frac{x_{t+l-2}}{\sqrt{T}} \right)' C_z^{-2} \tag{140}$$

$$\sim_a \frac{1}{T} \sum_{t=1}^{T-k} \left(\frac{1}{k} \sum_{j=1}^{k} F_x(1)\, \epsilon_{x,t+j-1} \left(\epsilon_{a,t+j-1}' F_a'(1)\, a\right) \frac{x_{t+j-2}'}{\sqrt{T}} \right) C_z^{-2}$$

$$= \frac{1}{T} \sum_{t=1}^{T-k} \left(\frac{1}{k} \sum_{j=1}^{k} F_x(1)\, \mathbb{E}\left(\epsilon_{x,t+j-1}\epsilon_{a,t+j-1}'\right) F_a'(1)\, a \frac{x_{t+j-2}'}{\sqrt{T}} \right) C_z^{-2}$$

$$+ \frac{1}{T} \sum_{t=1}^{T-k} \left(\frac{1}{k} \sum_{j=1}^{k} F_x(1) \left(\left(\epsilon_{x,t+j-1}\epsilon_{a,t+j-1}'\right) - \mathbb{E}\left(\epsilon_{x,t+j-1}\epsilon_{a,t+j-1}'\right) \right) F_a'(1)\, a \frac{x_{t+j-2}'}{\sqrt{T}} \right) C_z^{-2} \tag{141}$$

$$\rightsquigarrow \int_0^1 \left(\Omega_{xa} a\right) G_{a,c}'(r) dr C_z^{-2}. \tag{142}$$

In the above derivation, (139) holds due to (iii) and (v), (140) holds due to the Beveridge-Nelson decomposition (Phillips and Solo, 1992), (141) holds since

$$\left\| \sup_{1\le t\le T-k} \frac{1}{k} \sum_{j=1}^{k} \left(\mathbb{E}\epsilon_{x,t+j-1}\epsilon_{a,t+j-1}' - \epsilon_{x,t+j-1}\epsilon_{a,t+j-1}' \right) \right\| = o_p(1),$$

and (142) holds by a summation splitting argument coupled with elimination and subsequent addition of residual terms as in (v).

(viii) By (iii) and the functional limit theory in (127)–(130), we have

$$\begin{aligned}
&\frac{1}{kT^{2+2\gamma}}\sum_{t=1}^{T-k}\eta_{T,t}^{2,k}\eta_{T,t}^{2,k\,\prime} \\
&= C_z^{-1}\frac{1}{T}\sum_{t=1}^{T-k}\left(\frac{-C_z}{\sqrt{k}T^{\frac{1}{2}+\gamma}}\eta_{T,t}^{2,k}\right)\left(\frac{-C_z}{\sqrt{k}T^{\frac{1}{2}+\gamma}}\eta_{T,t}^{2,k}\right)' C_z^{-1} \\
&\sim_a \frac{C_z^{-1}}{T}\sum_{t=1}^{T-k}\left[\frac{1}{\sqrt{k}}\sum_{j=1}^{k}\left(a'u_{a,t+j-1}\right)\frac{x_{t+j-2}}{\sqrt{T}}\right]\left[\frac{1}{\sqrt{k}}\sum_{l=1}^{k}\left(a'u_{a,t+l-1}\right)\frac{x_{t+l-2}}{\sqrt{T}}\right]' C_z^{-1} \qquad (143) \\
&\sim_a \frac{1}{T}\sum_{t=1}^{T-k}\left[\frac{1}{\sqrt{k}}\sum_{j=1}^{k}\left(a'F_a\left(1\right)\epsilon_{a,t+j-1}\right)\frac{x_{t+j-2}}{\sqrt{T}}\right]\left[\frac{1}{\sqrt{k}}\sum_{l=1}^{k}\left(a'F_a\left(1\right)\epsilon_{a,t+l-1}\right)\frac{x_{t+l-2}}{\sqrt{T}}\right]' C_z^{-2} \qquad (144) \\
&\sim_a \frac{1}{T}\sum_{t=1}^{T-k}\left[\frac{1}{k}\sum_{j=1}^{k}\left(a'F_a\left(1\right)\epsilon_{a,t+j-1}\epsilon_{a,t+j-1}'F_a'\left(1\right)a\right)\frac{x_{t+j-2}}{\sqrt{T}}\frac{x_{t+j-2}'}{\sqrt{T}}\right] C_z^{-2} \\
&= \frac{1}{T}\sum_{t=1}^{T-k}\left[\frac{1}{k}\sum_{j=1}^{k}\left(a'F_a\left(1\right)\mathbb{E}\left(\epsilon_{a,t+j-1}\epsilon_{a,t+j-1}'\right)F_a'\left(1\right)a\right)\frac{x_{t+j-2}}{\sqrt{T}}\frac{x_{t+j-2}'}{\sqrt{T}}\right] C_z^{-2} \\
&+ \frac{C_z^{-2}}{T}\sum_{t=1}^{T-k}\left[\frac{1}{k}\sum_{j=1}^{k}\left(a'F_a\left(1\right)\left(\epsilon_{a,t+j-1}\epsilon_{a,t+j-1}' - \mathbb{E}\epsilon_{a,t+j-1}\epsilon_{a,t+j-1}'\right)F_a'\left(1\right)a\right)\frac{x_{t+j-2}}{\sqrt{T}}\frac{x_{t+j-2}'}{\sqrt{T}}\right] \\
&= \frac{1}{T}\sum_{t=1}^{T-k}\left(\frac{1}{k}\sum_{j=1}^{k}\left(a'F_a\left(1\right)\Sigma_{\epsilon,aa}F_a'\left(1\right)a\right)\frac{x_{t+j-2}}{\sqrt{T}}\frac{x_{t+j-2}'}{\sqrt{T}}\right) C_z^{-2} + o_p\left(1\right) \qquad (145) \\
&\sim_a \int_0^1 G_{a,c}(r)G_{a,c}'(r)\left(a'\Omega_{aa}a\right)dr C_z^{-2}, \qquad (146)
\end{aligned}$$

where $\Sigma_{\epsilon,aa}$ follows the defintion in Remark A.3, (143) holds due to (iii) and (v), (144) holds from the Beveridge-Nelson decomposition (Phillips and Solo, 1992), (145) holds because

$$\left\|\sup_{1\leq t\leq T-k}\frac{1}{k}\sum_{j=1}^{k}\left(\mathbb{E}\epsilon_{a,t+j-1}\epsilon_{a,t+j-1}' - \epsilon_{a,t+j-1}\epsilon_{a,t+j-1}'\right)\right\| = o_p\left(1\right),$$

and (146) holds by using a summation splitting argument coupled with elimination and subsequent addition of residual terms as in (v).

(ix) We use the same approach as in (v), employing the recursion $\eta_{T,t}^{(3)} = R_{Tz}\eta_{T,t-1}^{(3)} + (a'u_{at})^2 x_{t-1}$, and summation up to k,

$$\eta_{T,t}^{3,k} = \sum_{j=1}^{k}\eta_{T,t+j-1}^{(3)} = R_{Tz}\sum_{j=1}^{k}\eta_{T,t+j-2}^{(3)} + \sum_{j=1}^{k}(a'u_{a,t+j-1})^2 x_{t+j-2},$$

leading to

$$(I_n - R_{Tz}) \sum_{j=1}^{k} \eta_{T,t+j-1}^{(3)} = \sum_{j=1}^{k} (a' u_{a,t+j-1})^2 x_{t+j-2} - R_{Tz}\eta_{T,t+k-1}^{(3)} + R_{Tz}\eta_{T,t-1}^{(3)},$$

giving

$$-\frac{C_z}{kT^{\frac{1+2\gamma}{2}}} \sum_{j=1}^{k} \eta_{T,t+j-1}^{(3)} = \frac{1}{kT^{\frac{1}{2}}} \sum_{j=1}^{k} (a' u_{a,t+j-1})^2 x_{t+j-2} + o_p(1),$$

since $\frac{1}{k\sqrt{T}} \sup_{1\leq t\leq T} \left\| \eta_{T,t-1}^{(3)} \right\| = O_p\left(\frac{T^\gamma}{k}\right) = o_p(1)$. Again, with the facts that $\frac{1}{l_T} + \frac{l_T}{k} \to 0$ and

$$-\frac{C_z}{k} \sum_{j=1}^{l_T} \left(a' u_{a,t+j-1}\right)^2 \left(x_{t+j-2}/\sqrt{T}\right) = o_p(1), \tag{147}$$

and setting $\lfloor Tr \rfloor = t$, we have

$$\begin{aligned}
-\frac{C_z}{kT^{\frac{1+2\gamma}{2}}} \sum_{j=1}^{k} \eta_{T,t+j-1}^{(3)} &= \frac{1}{k} \sum_{j=1}^{k} \left(a' u_{a,t+j-1}\right)^2 \left(x_{t+j-2}/\sqrt{T}\right) + o_p(1) \\
&= \frac{1}{k} \sum_{j=l_T}^{k} \left(a' u_{a,t+j-1}\right)^2 \left(x_{t+j-2}/\sqrt{T}\right) + o_p(1) \qquad (148) \\
&\sim_a \frac{1}{k} \sum_{j=l_T}^{k} \left(a'\Sigma_{aa} a\right) G_{a,c}(r) + \frac{1}{k} \sum_{j=l_T}^{k} \left[\left(a' u_{a,t+j-1}\right)^2 - a'\Sigma_{aa} a\right] G_{a,c}(r) \\
&= \frac{1}{k} \sum_{j=l_T}^{k} \left(a'\Sigma_{aa} a\right) G_{a,c}(r) + o_p\left(1\right) \\
&= \frac{1}{k} \sum_{j=1}^{k} \left(a'\Sigma_{aa} a\right) G_{a,c}(r) - \frac{1}{k} \sum_{j=1}^{l_T - 1} \left(a'\Sigma_{aa} a\right) G_{a,c}(r) + o_p\left(1\right) \qquad (149) \\
&\sim_a \left(a'\Sigma_{aa} a\right) G_{a,c}(r), \qquad (150)
\end{aligned}$$

where equation (148) holds due to (147) for LSTUR and equation (150) follows because $l_T/k \to 0$. Setting $C = \mathbf{0}_{n\times n}$ delivers the results for STUR. ■

Lemma B.2 *Let Assumptions 1 and 2 hold. Under the rate condition* $\frac{\sqrt{T}}{T^\gamma} + \frac{T^\gamma}{k} + \frac{k}{T} \to 0$ *and since* $C_z = c_z I_n$*,*

$$\frac{1}{\sqrt{k}T^{\frac{1}{2}+\gamma}} \sum_{t=1}^{T-k} u_{0,t+k} \tilde{z}_t^{k\,\prime} \rightsquigarrow \mathcal{MN}\left(\mathbf{0}_{1\times n},\ \left(V_{zz}^{LH} + V_{\eta\eta}^{(2),LH} + V_{z\eta}^{(2),LH} + V_{z\eta}^{(2),LH\,\prime}\right)\sigma_{00}\right), \tag{151}$$

where

$$V_{zz}^{LH} := \Omega_{xx} C_z^{-2}, \tag{152}$$

$$V_{\eta\eta}^{(2),LH} := \begin{cases} (a'\Omega_{aa}a)\int_0^1 G_{a,c}(r)G'_{a,c}(r)drC_z^{-2}, & \text{under} \quad \text{LSTUR} \\ (a'\Omega_{aa}a)\int_0^1 G_a(r)G'_a(r)drC_z^{-2}, & \text{under} \quad \text{STUR} \end{cases}, \tag{153}$$

and

$$V_{z\eta}^{(2),LH} := \begin{cases} (\Omega_{xa}a)\int_0^1 G'_{a,c}(r)drC_z^{-2} & \text{under} \quad \text{LSTUR} \\ (\Omega_{xa}a)\int_0^1 G'_a(r)drC_z^{-2}, & \text{under} \quad \text{STUR} \end{cases}. \tag{154}$$

Proof: The LHIVX numerator has the following asymptotic approximation

$$\frac{1}{\sqrt{k}T^{\frac{1}{2}+\gamma}}\sum_{t=1}^{T-k}u_{0,t+k}\widetilde{z}_k^{k\,\prime}$$
$$= \frac{1}{\sqrt{k}T^{\frac{1}{2}+\gamma}}\sum_{t=1}^{T-k}u_{0,t+k}z_t^{k\,\prime} + \frac{1}{\sqrt{k}T^{1+\gamma}}\sum_{t=1}^{T-k}u_{0,t+k}\eta_{T,t}^{2,k\,\prime} + o_p(1) \tag{155}$$
$$\sim_a -\frac{C_z^{-1}}{\sqrt{T}}\sum_{t=1}^{T-k}u_{0,t+k}\left(\frac{1}{\sqrt{k}}\sum_{j=1}^{k}u_{x,t+j-1}\right)' - \frac{C_z^{-1}}{\sqrt{T}}\sum_{t=1}^{T-k}u_{0,t+k}\left(\frac{1}{\sqrt{k}}\sum_{j=1}^{k}a'u_{a,t+j-1}\frac{x_{t+j-2}}{\sqrt{T}}\right)' \tag{156}$$
$$\sim_a -\frac{C_z^{-1}}{\sqrt{T}}\sum_{t=1}^{T-k}u_{0,t+k}\left(\frac{1}{\sqrt{k}}\sum_{j=1}^{k}u_{x,t+j-1}\right)' - \frac{C_z^{-1}}{\sqrt{T}}\sum_{t=1}^{T-k}\frac{x'_t}{\sqrt{T}}u_{0,t+k}\left(\frac{1}{\sqrt{k}}\sum_{j=1}^{k}a'u_{a,t+j-1}\right), \tag{157}$$

where equation (155) is due to Lemma B.1 (ii), (iv), (vi) and (x), and equation (156) is due to Lemma B.1 (iii) and (v), equation (157) follows by a summation splitting argument coupled with elimination and replacement similar to that in Lemma B.1 (v) and the fact that $\frac{\lfloor Tr\rfloor+j-2}{T} = \frac{\lfloor Tr\rfloor}{T} + o(1)$ as $j \leq k$ and $\frac{1}{k} + \frac{k}{T} \to 0$. To establish mixed normality we show joint convergence of the leading terms to a stochastic integral and provide asymptotic equivalence between this stochastic integral form and the joint mixed normal distribution. For this purpose the following sufficient conditions are needed. First, we need to show the following joint weak convergence to the vector Brownian motion $\left(\widetilde{B}_{0Z_x}^{\star\,\prime}(r), \widetilde{B}_{0Z_a}^{\star\,\prime}(r)\right)'$

$$\begin{pmatrix} \frac{1}{\sqrt{T}}\sum_{t=1}^{\lfloor Tr\rfloor-k}u_{0,t+k}\left(\frac{1}{\sqrt{k}}\sum_{j=1}^{k}u_{x,t+j-1}\right) \\ \frac{1}{\sqrt{T}}\sum_{t=1}^{\lfloor Tr\rfloor-k}u_{0,t+k}\left(\frac{1}{\sqrt{k}}\sum_{j=1}^{k}a'u_{a,t+j-1}\right) \end{pmatrix}$$
$$\rightsquigarrow \begin{pmatrix} \widetilde{B}_{0Z_x}^{\star}(r) \\ \widetilde{B}_{0Z_a}^{\star}(r) \end{pmatrix} := \mathcal{N}\left(\mathbf{0}_{(n+1)\times 1},\ r\sigma_{00}\begin{pmatrix} \Omega_{xx} & \Omega_{xa}a \\ a'\Omega_{ax} & a'\Omega_{aa}a \end{pmatrix}\right) \tag{158}$$

for any $r \in [0,1]$. Then, based on the joint weak convergence to $\left(B'_0(\cdot), B'_x(\cdot), B'_a(\cdot), Z_x^{\star\,\prime}, Z_a^{\star\,\prime}\right)'$ in (130), the continuous mapping theorem is applied and the following joint convergence is shown to hold

$$\begin{pmatrix} \frac{1}{\sqrt{T}}\sum_{t=1}^{\lfloor Tr\rfloor-k} u_{0,t+k}\left(\frac{1}{\sqrt{k}}\sum_{j=1}^{k} u_{x,t+j-1}\right) \\ \frac{1}{\sqrt{T}}\sum_{t=1}^{\lfloor Tr\rfloor-k} u_{0,t+k}\left(\frac{1}{\sqrt{k}}\sum_{j=1}^{k} a' u_{a,t+j-1}\right) \\ \frac{1}{\sqrt{T}}\sum_{t=1}^{\lfloor Tr\rfloor-k} u_{0,t+j} \\ \frac{1}{\sqrt{T}}\sum_{t=1}^{\lfloor Tr\rfloor-k} u_{x,t+j-1} \\ \frac{1}{\sqrt{T}}\sum_{t=1}^{\lfloor Tr\rfloor-k} u_{a,t+j-1} \end{pmatrix} \rightsquigarrow \begin{pmatrix} \widetilde{B}^{\star}_{0Z_x}(r) \\ \widetilde{B}^{\star}_{0Z_a}(r) \\ B_0(r) \\ B_x(r) \\ B_a(r) \end{pmatrix}. \tag{159}$$

Second, we need to demonstrate the independence between $\left(\widetilde{B}^{\star\,\prime}_{0Z_x}(r), \widetilde{B}^{\star\,\prime}_{0Z_a}(r)\right)'$ and $\left(B_0(r), B_x'(r), B_a'(r)\right)'$ as defined in (158) and (128) respectively:

$$\left(\widetilde{B}^{\star\,\prime}_{0Z_x}(r), \widetilde{B}^{\star\,\prime}_{0Z_a}(r)\right)' \perp \left(B_0(r), B_x'(r), B_a'(r)\right)'. \tag{160}$$

Based on the joint convergence of (159) and the zero asymptotic covariance between the Gaussian variates, asymptotic independence follows. The joint convergence of (159) and asymptotic independence in (160) then imply the asymptotic equivalence between the limiting integral and the mixed normal representation, as in the following

$$\begin{pmatrix} \frac{1}{\sqrt{T}}\sum_{t=1}^{T-k} u_{0,t+k}\left(\frac{1}{\sqrt{k}}\sum_{j=1}^{k} u_{x,t+j-1}\right) \\ \frac{1}{\sqrt{T}}\sum_{t=1}^{T-k} u_{0,t+k}\frac{x_t}{\sqrt{T}}\left(\frac{1}{\sqrt{k}}\sum_{j=1}^{k} a' u_{a,t+j-1}\right) \end{pmatrix} \rightsquigarrow \begin{pmatrix} \int_0^1 d\widetilde{B}^{\star}_{0Z_x}(r) \\ \int_0^1 G_{a,c}(r)\, d\widetilde{B}^{\star}_{0Z_a}(r) \end{pmatrix} \tag{161}$$

$$=_d \mathcal{MN}\left(\mathbf{0}_{2n\times 1}, \begin{pmatrix} V_{zz}^{LH} & V_{z\eta}^{(2),LH} \\ V_{z\eta}^{(2),LH\,\prime} & V_{\eta\eta}^{(2),LH} \end{pmatrix}\sigma_{00}\right), \tag{162}$$

in which V_{zz}^{LH}, $V_{\eta\eta}^{(2),LH}$ and $V_{z\eta}^{(2),LH}$ are defined by (152), (153) and (154) separately. The details showing the joint convergence of (159) and the asymptotic independence of (160) are presented below.

First, equation (158) follows by the Cramér-Wold device and the functional limit law that accompanies the MGCLT in Hall and Heyde (1980, Theorem 3.2) and Pollard (1984, Theorem VIII.1). Since $\left(u_{0,t+k}\left(\frac{1}{\sqrt{T}}\sum_{j=1}^{k} u'_{x,t+j-1}\right), u_{0,t+k}\left(\frac{1}{\sqrt{T}}\sum_{j=1}^{k} u'_{a,t+j-1}a\right)\right)'$ is a martingale difference array, we choose any n-dimensional real-valued vector $b_1^\star > \mathbf{0}_{n\times 1}$ and any positive real number $b_2^\star$ and apply the MGCLT to the sequence

$$Z_{t+k-1}^{LH} u_{0,t+k} := \left[b_1^{\star\,\prime}\left(\frac{1}{\sqrt{k}}\sum_{j=1}^{k} u_{x,t+j-1}\right) + b_2^\star\left(\frac{1}{\sqrt{k}}\sum_{j=1}^{k} a' u_{a,t+j-1}\right)\right] u_{0,t+k}.$$

Then, using the MGCLT as in Lemma A.2 of Phillips and Lee (2013) and Kostakis et al. (2015) it follows that

$$\frac{1}{\sqrt{T}}\sum_{t=1}^{T-k} Z_{t+k-1}^{LH} u_{0,t+k} \rightsquigarrow \mathcal{N}\left(0,\ V_Z^{LH}\sigma_{00}\right),$$

where $V_Z^{LH} := b_1^{\star\prime}\Omega_{xx}b_1^\star + b_1^{\star\prime}\Omega_{xa}ab_2^\star + b_2^\star a'\Omega_{ax}b_1^\star + b_2^\star a'\Omega_{aa}ab_2^\star$. Parallel to the short horizion case, the corresponding functional law (158) holds due to the stability condition for $\left\{Z_{t+k-1}^{LH}\right\}_{t=1}^{T-k}$, viz.,

$$\frac{1}{T}\sum_{t=1}^{T-k}\left(Z_{t+k-1}^{LH}\right)^2 \mathbb{E}_{t+k-1}\left(u_{0,t+k}^2\right) \to_p \sigma_{00}V_Z^{LH},$$

where $\frac{1}{T}\sum_{t=1}^{T-k}\left(Z_{t+k-1}^{LH}\right)^2 \to_p V_Z^{LH}$, as shown in Lemma A.2 of Phillips and Lee (2013), and

$$\begin{aligned}\frac{1}{T}\sum_{t=1}^{T-k}\left(Z_{t+k-1}^{LH}\right)^2 H_{t+k} =& \frac{1}{T}\sum_{t=1}^{T-k}\left(Z_{t+k-1}^{LH}\right)^2\left(H_{t+k}-\sigma_{00}\right) + \frac{\sigma_{00}}{T}\sum_{t=1}^{T-k}\left(Z_{t+k-1}^{LH}\right)^2 \\ \sim_a& \frac{\sigma_{00}}{T}\sum_{t=1}^{T-k}\left(Z_{t+k-1}^{LH}\right)^2 \to_p \sigma_{00}V_Z^{LH},\end{aligned}$$

since $\left\|\frac{1}{T}\sum_{t=1}^{T-k}\left(Z_{t+k-1}^{LH}\right)^2\left(H_{t+k}-\sigma_{00}\right)\right\| \preceq \sup_{t,k}\left\|Z_{t+k-1}^{LH}\right\|^2 \cdot \left|\frac{1}{T}\sum_{t=1}^{T-k} H_{t+k}-\sigma_{00}\right| = o_p(1)$.

Second, the Brownian motion $\widetilde{B}_{0Z_a}^\star(r)$ is independent of each of the Brownian motions $\left(B_0(r), B_x'(r), B_a'(r)\right)'$ because

$$\begin{aligned}&\mathbb{E}\left\|\left(\frac{1}{\sqrt{kT}}\sum_{t=1}^{\lfloor Tr\rfloor-k} u_{0,t+k}\sum_{j=1}^{k}\begin{pmatrix} u_{x,t+j-1} \\ a'u_{a,t+j-1}\end{pmatrix}\right)\cdot\frac{1}{\sqrt{T}}\sum_{l=1}^{\lfloor Tr\rfloor-k}\begin{pmatrix} a'u_{a,l+k}, & u_{x,l+k}', & u_{0,l+k}\end{pmatrix}\right\| \\ &\sim \mathbb{E}\left\|\left(\frac{1}{\sqrt{kT}}\sum_{t=1}^{\lfloor Tr\rfloor-k} u_{0,t+k}\sum_{j=1}^{k}\begin{pmatrix} u_{x,t+j-1} \\ a'u_{a,t+j-1}\end{pmatrix}\right)\cdot\frac{1}{\sqrt{T}}\sum_{l=1}^{\lfloor Tr\rfloor-k}\begin{pmatrix} a'F_a(1)\epsilon_{a,l+k}, & \epsilon_{x,l+k}'F_x'(1), & u_{0,l+k}\end{pmatrix}\right\| \quad (163) \\ &\sim \left\|\left(\frac{1}{T}\sum_{t=1}^{\lfloor Tr\rfloor-k}\mathbb{E}\begin{pmatrix} Z_x^\star \\ a'Z_a^\star\end{pmatrix}\mathbb{E}\begin{pmatrix} a'F_a(1)\epsilon_{a,t+k}u_{0,t+k}, & \epsilon_{x,t+k}'F_x'(1)u_{0,t+k}, & u_{0,t+k}^2\end{pmatrix}\right)\right\| = 0, \quad (164)\end{aligned}$$

where (163) is due to the Beveridge-Nelson decomposition (Phillips and Solo, 1992), and (164) is due to the zero asymptotic covariance shown in (129) and the mixing property of the limit variates $(Z_x^{\star\prime}, Z_a^{\star\prime})'$ given in equation (127). It follows that $\left(\widetilde{B}_{0Z_x}^{\star\prime}(r), \widetilde{B}_{0Z_a}^{\star\prime}(r)\right)'$ are independent of $G_{a,c}(r)$, which depends only on $\left(B_x'(r), B_a'(r)\right)'$. Weak convergence to the limiting stochastic integral in equation (161) then follows directly from standard theory (Ibragimov and Phillips, 2008). The equivalence in distribution between the stochastic integral and the mixed normality in equation (162) follows by the asymptotic independence in equation (164). ■

Lemma B.3 *Suppose Assumptions 1 and 2 hold. Under the rate condition,* $\frac{\sqrt{T}}{T^\gamma} + \frac{T^\gamma}{k} + \frac{k}{T} \to 0$,

(i) $\frac{1}{T^{1+\gamma}k^2}\sum_{t=1}^{T-k} x_t^k z_t^{k\prime} \to_p -\frac{1}{2}\Omega_{xx}C_z^{-1}$.

(ii) $\frac{1}{T^{2+\gamma}k^2}\sum_{t=1}^{T-k}x_t^k\eta_{T,t}^{1,k\prime} \rightsquigarrow \begin{cases} -\int_0^1 G_{a,c}(r)G'_{a,c}(r)drC_z^{-1}, & \text{under LSTUR}, \\ -\int_0^1 G_a(r)G'_a(r)drC_z^{-1}, & \text{under STUR}. \end{cases}$

(iii) $\frac{1}{T^{\frac{3}{2}+\gamma}k^2}\sum_{t=1}^{T-k}x_t^k\eta_{T,t}^{2,k\prime} = o_p(1).$

(iv) $\frac{1}{T^{2+\gamma}k^2}\sum_{t=1}^{T-k}x_t^k\eta_{T,t}^{3,k\prime} \rightsquigarrow \begin{cases} -\int_0^1 G_{a,c}(r)G'_{a,c}(r)\left(a'\Sigma_{aa}a\right)drC_z^{-1}, & \text{under LSTUR}, \\ -\int_0^1 G_a(r)G'_a(r)\left(a'\Sigma_{aa}a\right)drC_z^{-1}, & \text{under STUR}, \end{cases}$

Proof: (i) To simplify derivations it is assumed that the matrix C is proportional to the identity matrix and it is sometimes convenient to employ $\frac{1}{C}$ and $\frac{1}{C_z}$ to denote C^{-1} and C_z^{-1}. Let $R^{(t+j,s)} := \prod_{m=s+1}^{t+j-1}R_{Ts}$, and $R_T := I_n + C/T$ and then

$$\begin{aligned}\sum_{t=1}^{T-k}x_t^k z_t^{k\prime} &= \sum_{t=1}^{T-k}\left(\sum_{j=1}^{k}x_{t+j-1}\right)\left(\sum_{i=1}^{k}z_{t+i-1}\right)' \\ &= \sum_{t=1}^{T-k}\left(\sum_{j=1}^{k}\left(\sum_{s=1}^{t+j-2}R^{(t+j,s)}u_{xs} + u_{x,t+j-1}\right)\right)\left(\sum_{i=1}^{k}\sum_{l=1}^{t+i-1}R_{Tz}^{t+i-1-l}u_{xl}\right)'.\end{aligned}$$

Using LL to represent terms of lower order compared to the leading term, we have

$$\begin{aligned}\mathbb{E}\left[x_t^k z_t^{k\prime}\right] =& \mathbb{E}\left[\left(\sum_{j=1}^{k}\left(\sum_{s=1}^{t+j-2}R^{(t+j,s)}u_{xs} + u_{x,t+j-1}\right)\right)\left(\sum_{i=1}^{k}\sum_{l=1}^{t+i-1}R_{Tz}^{t+i-1-l}u_{xl}\right)'\right] \\ &\sim \mathbb{E}\left[\left(\sum_{j=1}^{k}\sum_{s=1}^{t+j-1}R_T^{t+j-1-s}u_{xs}\right)\left(\sum_{i=1}^{k}\sum_{l=1}^{t+i-1}R_{Tz}^{t+i-1-l}u_{xl}\right)'\right] \\ &+ \mathbb{E}\left[\left(\sum_{j=1}^{k}\sum_{s=1}^{t+j-2}R_T^{t+j-1-s}\left(\sum_{m=s+1}^{t+j-1}\frac{a'u_{am}}{\sqrt{T}}\right)u_{xs}\right)\left(\sum_{i=1}^{k}\sum_{l=1}^{t+i-1}R_{Tz}^{t+i-1-l}u_{xl}\right)'\right] \\ &+ \mathbb{E}\left[\left(\sum_{j=1}^{k}\sum_{s=1}^{t+j-2}R_T^{t+j-1-s}\left(\sum_{m=s+1}^{t+j-1}\frac{\left(a'F_a(1)\epsilon_{am}\right)^2}{2T}\right)u_{xs}\right)\left(\sum_{i=1}^{k}\sum_{l=1}^{t+i-1}R_{Tz}^{t+i-1-l}u_{xl}\right)'\right] \\ &= \mathbb{E}\left[\left(\sum_{j=1}^{k}\sum_{s=1}^{t+j-1}R_T^{t+j-1-s}u_{xs}\right)\left(\sum_{i=1}^{k}\sum_{l=1}^{t+i-1}R_{Tz}^{t+i-1-l}u_{xl}\right)'\right] + LL \\ &= \Omega_{xx}\frac{\left(1-R_T^k\right)\left(1-R_{Tz}^k\right)}{\left(1-R_T\right)\left(1-R_{Tz}\right)}\left\{\left(R_TR_{Tz}\right)^0 + \left(R_TR_{Tz}\right)^1 + ... + \left(R_TR_{Tz}\right)^{t-1}\right\} \\ &+ \Omega_{xx}\frac{1}{\left(1-R_T\right)\left(1-R_{Tz}\right)}\left\{\begin{array}{c}\left(1-R_T\right)\left(1-R_{Tz}\right) + \left(1-R_T^2\right)\left(1-R_{Tz}^2\right) \\ +... + \left(1-R_T^{k-1}\right)\left(1-R_{Tz}^{k-1}\right)\end{array}\right\} + LL \\ &=: AA + BB + LL,\end{aligned}$$

since $R^{(t+j,s)} \sim_a R_T^{t+j-1-s}\prod_{m=s+1}^{t+j-1}\exp\left(\frac{a'u_{am}}{\sqrt{T}}\right) \sim_a R_T^{t+j-1-s} + R_T^{t+j-1-s}\sum_{m=s+1}^{t+j-1}\frac{a'u_{am}}{\sqrt{T}} + R_T^{t+j-1-s}\sum_{m=s+1}^{t+j-1}\frac{\left(a'F_a(1)\epsilon_{am}\right)^2}{2T}$ where $R_T = I_n + \frac{C}{T}$. First, for term AA, since $\frac{\sqrt{T}}{k} + \frac{k}{T} +$

$\frac{T^\gamma}{k} \to 0$, $R_{Tz}^k \to \mathbf{0}_{n\times n}$ and $R_T^k \sim \exp\left(C\frac{k}{T}\right)$, thus

$$AA = \left(\frac{\Omega_{xx}}{CC_z}\right) T^{1+\gamma}\left(1-R_T^k\right)\left(1-R_{Tz}^k\right)\left\{\frac{1-\left(R_T R_{Tz}\right)^t}{1-R_T R_{Tz}}\right\}$$
$$= \left(\frac{\Omega_{xx}}{CC_z}\right) T^{1+\gamma}\left(1-R_T^k\right)\left\{\frac{1-\left(R_T R_{Tz}\right)^t}{1-R_T R_{Tz}}\right\}(1+o(1)).$$

Note that $T^\gamma\left(1-R_T R_{Tz}\right) = -C_z - \frac{C}{T^{1-\gamma}} + \frac{CC_z}{T} = -C_z + o(1)$ and $\left(R_T R_{Tz}\right)^k \to \mathbf{0}_{n\times n}$ where t is sufficiently large since $R_{Tz}^t \to \mathbf{0}_{n\times n}$. Therefore

$$AA = \left(\frac{\Omega_{xx}}{CC_z}\right) T^{1+2\gamma}\left(1-R_T^k\right)\left\{\frac{1+o(1)}{-C_z+o(1)}\right\}(1+o(1)),$$

with $1-R_T^k \sim -\frac{k}{T}C$. As $T\to\infty$,

$$\frac{AA}{kT^{2\gamma}} = \left(\frac{\Omega_{xx}}{CC_z}\right)(-C+o(1))\left\{\frac{1+o(1)}{-C_z+o(1)}\right\}(1+o(1)) \to \frac{\Omega_{xx}}{C_z^2},$$

and $AA = O_p\left(kT^{2\gamma}\right)$. Next, for the term BB, we have

$$BB = \left(\frac{\Omega_{xx}}{CC_z}\right) T^{1+\gamma}\left\{(k-1) - \frac{1-R_T^{k-1}}{1-R_T}R_T - \frac{1-R_{Tz}^{k-1}}{1-R_{Tz}}R_{Tz} + \frac{1-\left(R_T R_{Tz}\right)^{k-1}}{1-R_T R_{Tz}}R_T R_{Tz}\right\}$$
$$\sim \left(\frac{\Omega_{xx}}{CC_z}\right) T^{1+\gamma}\left\{k - \frac{T}{-C}\left(1-\exp\left(C\frac{k}{T}\right)\right) - \frac{T^\gamma}{-C_z}\left(1-\exp\left(C_z\frac{k}{T^\gamma}\right)\right) + T^\gamma\frac{1+o(1)}{(-C_z+o(1))}\right\},$$

since $\frac{T}{k^2}\left\{k - \frac{T}{-C}\left(1-\exp\left(C\frac{k}{T}\right)\right)\right\} = \frac{T}{C}\left\{-C\frac{k}{T} - \frac{C^2}{2}\frac{k^2}{T^2} - \frac{C^3}{6}\frac{k^3}{T^3} + \ldots\right\} = -k - \frac{C}{2}\frac{k^2}{T} - \frac{C^2}{6}\frac{k^3}{T^2} + \ldots = -k - \frac{C}{2}\frac{k^2}{T} + o\left(\frac{k^2}{T}\right)$. If $\frac{T^{1+\gamma}}{k^2} \to 0$, it follows that

$$BB = \left(\frac{\Omega_{xx}}{CC_z}\right) T^{1+\gamma}\frac{k^2}{T}\left\{-\frac{C}{2} + o(1) + \frac{T^{1+\gamma}}{k^2}O_p(1)\right\} \Rightarrow \frac{BB}{k^2T^\gamma} = -\left(\frac{\Omega_{xx}}{2C_z}\right) + o_p(1).$$

Since γ is a choice parameter in constructing the IVX instrument and γ is selected so that $\frac{T^{1+\gamma}}{k^2} \to 0$, the term BB dominates the term AA, and the lower-order term LL can be shown to have the stochastic order $O_p\left(kT^{2\gamma} + k^3T^{\gamma-1}\right)$. Therefore,

$$\frac{1}{k^2T^{1+\gamma}}\sum_{t=1}^{T-k} x_t^k z_t^{k\,\prime} = -\frac{1}{2}\Omega_{xx}C_z^{-1} + o_p(1).$$

(ii) By standard functional limit theory we have, as $T\to\infty$,

$$\frac{1}{k^2T^{2+\gamma}}\sum_{t=1}^{T-k} x_t^k \eta_{T,t}^{1,t\,\prime} = \frac{1}{T}\sum_{t=1}^{T-k}\left(\frac{1}{k\sqrt{T}}x_t^k\right)\left(\frac{C_z}{kT^{\frac{1}{2}+\gamma}}\eta_{T,t}^{1,k}\right)' C_z^{-1}$$
$$\rightsquigarrow \begin{cases} -\int_0^1 G_{a,c}(r)G_{a,c}'(r)dr\cdot C_z^{-1} & \text{under} \quad \text{LSTUR} \\ -\int_0^1 G_a(r)G_a'(r)dr\cdot C_z^{-1} & \text{under} \quad \text{STUR} \end{cases}.$$

as required.

(iii) The proof follows as in Lemma B.1, showing that

$$\left\| \frac{1}{k^2 T^{\frac{3}{2}+\gamma}} \sum_{t=1}^{T-k} x_t^k \eta_{T,t}^{2,k\,\prime} \right\| = \frac{1}{\sqrt{k}} \left\| \frac{1}{\sqrt{T}} \sum_{t=1}^{T-k} \left(\frac{x_t^k}{k\sqrt{T}} \right) \left(\frac{\eta_{T,t}^{2,k\,\prime}}{\sqrt{k} T^{\frac{1}{2}+\gamma}} \right) \right\| \preceq O_p\left(\frac{1}{\sqrt{k}}\right).$$

(iv) Similar to (ii), by the functional limit theory in (127) and (128) we have, as $T \to \infty$,

$$\begin{aligned} \frac{1}{k^2 T^{2+\gamma}} \sum_{t=1}^{T-k} x_t^k \eta_{T,t}^{3,t\,\prime} =& \frac{1}{T} \sum_{t=1}^{T-k} \left(\frac{1}{k\sqrt{T}} x_t^k \right) \left(\frac{C_z}{k T^{\frac{1}{2}+\gamma}} \eta_{T,t}^{3,k} \right)' C_z^{-1} \\ & \rightsquigarrow \begin{cases} -\left(a'\Sigma_{aa}a\right) \int_0^1 G_{a,c}(r) G_{a,c}'(r) dr \cdot C_z^{-1} & \text{under LSTUR,} \\ -\left(a'\Sigma_{aa}a\right) \int_0^1 G_a(r) G_a'(r) dr \cdot C_z^{-1} & \text{under STUR.} \end{cases} \end{aligned}$$

■

Lemma B.4 *Suppose Assumptions 1 and 2 hold. Under the rate condition that* $\frac{1}{T} + \frac{\sqrt{T}}{T^\gamma} + \frac{T^\gamma}{k} + \frac{k}{T} \to 0$, *we have:*

$$\frac{1}{T^{1+\gamma} k^2} \sum_{t=1}^{T-k} x_t^k \tilde{z}_t^{k\,\prime} \rightsquigarrow \Upsilon, \tag{165}$$

where

$$\Upsilon := \left(-\frac{1}{2}\Omega_{xx} - \int_0^1 G_{a,c}(r) G_{a,c}'(r) dr \cdot C - \frac{1}{2} \int_0^1 G_{a,c}(r) G_{a,c}'(r) \left(a'\Sigma_{aa}a\right) dr \right) \cdot C_z^{-1}, \tag{166}$$

in the LSTUR case; and

$$\Upsilon := \left(-\frac{1}{2}\Omega_{xx} - \frac{1}{2} \int_0^1 G_a(r) G_a'(r) \left(a'\Sigma_{aa}a\right) dr \right) \cdot C_z^{-1}, \tag{167}$$

in the STUR case.

Proof: The proof follows from Lemma B.3 and the continuous mapping theorem. ■

Proofs of Theorems 3.3 and 3.4

Proof: The proofs follow directly from the continuous mapping theorem, the mixed normality in Lemma B.2, the asymptotic independence in Lemma B.2, and the weak convergence of the denominator in Lemma B.4. ■

C Notation for STUR Coefficient Heterogeneity

For notational ease the predictive regression model in the main paper uses a homogeneous coefficient vector ($a_i = a$ for all i) for the stochastic STUR regressors. As indicated in the main paper, this formulation can be generalized to allow for heterogeneous coefficient vectors at the cost of some additional notational complexity. In such cases we have vectors $a_i \neq a_j$ for at least two indexes (i, j). A more convenient matrix notation is introduced to accommodate this extension. Upon doing so, the proofs of the results in the paper proceed as before.

The extended data generating process (DGP) for the STUR/LSTUR regressors has the following explicit form

$$x_t = R_{Tt} x_{t-1} + u_{xt},$$

where

$$R_{Tt} := \begin{cases} I_n + \frac{C}{T} + \frac{\breve{D}_{at}}{\sqrt{T}} + \frac{\breve{D}_{at}^2}{2T} & \text{for} \quad \text{LSTUR}, \\ I_n + \frac{\breve{D}_{at}}{\sqrt{T}} + \frac{\breve{D}_{at}^2}{2T} & \text{for} \quad \text{STUR}, \end{cases}$$

and $\breve{D}_{at} := \operatorname{diag}\{a_1' u_{at}, ..., a_n' u_{at}\}$ with $\{a_i\}_{i=1}^n$ a collection of p-dimensional real-valued vectors. Define $\mathbf{1}_{n\times 1}$ and $\mathbf{1}_{1\times n}$ as the n-dimensional column and row vectors of ones. Allowing for weak dependence in the same manner as the main paper we define the instantaneous and long-run covariance matrices related to full set of innovations $(u_{0t}, u_{xt}^{'}, u_{at}^{'})'$ as follows:

$$\sigma_{00} = \mathbb{E}\left(u_{0t}^2\right), \ \Sigma_{xx} = \mathbb{E}\left(u_{xt} u_{xt}'\right), \ \overline{\Sigma}_{aa} = \mathbb{E}\left(\breve{D}_{at} \mathbf{1}_{n\times 1} \mathbf{1}_{1\times n} \breve{D}_{at}\right),$$

$$\Omega_{xx} = \sum_{h=-\infty}^{\infty} \mathbb{E}\left(u_{xt} u_{xt}'\right), \ \overline{\Omega}_{aa} = \sum_{h=-\infty}^{\infty} \mathbb{E}\left(\breve{D}_{at} \mathbf{1}_{n\times 1} \mathbf{1}_{1\times n} \breve{D}_{a,t-h}\right),$$

$$\Lambda_{xx} = \sum_{h=1}^{\infty} \mathbb{E}\left(u_{xt} u_{xt}'\right), \ \overline{\Lambda}_{aa} = \sum_{h=1}^{\infty} \mathbb{E}\left(\breve{D}_{at} \mathbf{1}_{n\times 1} \mathbf{1}_{1\times n} \breve{D}_{a,t-h}\right),$$

$$\Omega_{xx} = \Lambda_{xx} + \Lambda_{xx}' + \Sigma_{xx}, \ \overline{\Omega}_{aa} = \overline{\Lambda}_{aa} + \overline{\Lambda}_{aa}' + \overline{\Sigma}_{xx}, \ \overline{\Omega}_{aa}^{\star} = \overline{\Lambda}_{aa}^{\star} + \overline{\Lambda}_{aa}^{\star\,\prime} + \overline{\Sigma}_{aa}^{\star},$$

$$\overline{\Sigma}_{aa}^{\star} = \mathbb{E}\left(\breve{D}_{at} \breve{D}_{at}\right), \ \overline{\Omega}_{aa}^{\star} = \sum_{h=-\infty}^{\infty} \mathbb{E}\left(\breve{D}_{at} \breve{D}_{a,t-h}\right), \ \overline{\Lambda}_{aa}^{\star} = \sum_{h=1}^{\infty} \mathbb{E}\left(\breve{D}_{at} \breve{D}_{a,t-h}\right).$$

Under Assumption 1 and employing the Beveridge-Nelson decomposition and functional limit theory in the usual way Phillips and Solo (1992) we have

$$\frac{1}{\sqrt{T}} \sum_{j=1}^{[Ts]} \begin{bmatrix} u_{0j} \\ u_{xj} \\ \breve{D}_{aj} \cdot \mathbf{1}_{n\times 1} \end{bmatrix} \rightsquigarrow \begin{bmatrix} B_0(s) \\ B_x(s) \\ \breve{D}_{B_a}(s) \cdot \mathbf{1}_{n\times 1} \end{bmatrix} = BM \begin{bmatrix} \sigma_{00}^2 & \Delta_{x0}' & \overline{\Delta}_{0a}' \\ \Delta_{x0} & \Omega_{xx} & \overline{\Omega}_{xa} \\ \overline{\Delta}_{a0} & \overline{\Omega}_{xa}' & \overline{\Omega}_{aa} \end{bmatrix},$$

where BM signifies vector Brownian motion, $\Delta_{x0} := \sum_{h=0}^{\infty} \mathbb{E}\left(u_{0t} u_{x,t-h}'\right)$, $\overline{\Delta}_{a0} := \sum_{h=0}^{\infty} \mathbb{E}\left(\breve{D}_{at} \mathbf{1}_{n\times 1} u_{x,t-h}'\right)$, $\overline{\Omega}_{xa} := \sum_{j=-\infty}^{\infty} \mathbb{E}\left(u_{xt} \cdot \mathbf{1}_{1\times n} \breve{D}_{a,t-j}'\right)$. With this result and notation, Theorems 3.1 and

3.2 still hold, leading to mixed normal asymptotics for the short-horizon IVX and QR-IVX estimators and delivering pivotal chi-squared limit theory for the given test statistics under the null of no predictability.

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