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The large contraction principle and existence of periodic solutions for infinite delay Volterra difference equations

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Abstract: In this article, we establish sufficient conditions for the existence of periodic solutions of a nonlinear infinite delay Volterra difference equation:

$$\Delta x(n) = p(n) + b(n)h(x(n)) + \sum_{k=-\infty}^{n} B(n,k)g(x(k)).$$

We employ a Krasnosel'skiĭ type fixed point theorem, originally proved by Burton. The primary sufficient condition is not verifiable in terms of the parameters of the difference equation, and so we provide three applications in which the primary sufficient condition is verified.

Key words: Large contraction, Volterra difference equation, infinite delay, periodic solution, fixed point

1. Introduction

Krasnosel'skiĭ [11, 19] is credited with a fixed point theorem in Banach spaces in which the fixed point operator is expressed as the sum of a compact operator and a contraction. This theorem has been generalized in different ways [2, 3, 5, 13, 14, 20]. In [5], Burton introduced the concept of a large contraction and proved an extension of the Krasnosel'skiĭ fixed point theorem to the case in which the fixed point operator is expressed as the sum of a compact operator and a large contraction. Burton's theorem has proved to be quite useful in the study of both delay differential equations and delay Volterra difference equations [1, 4–6, 10, 15–18], as well as other functional or fractional equations [7–9, 12].

In this paper we shall apply Burton's extended Krasnosel'skiĭ fixed point theorem to

$$\Delta x(n) = p(n) + b(n)h(x(n)) + \sum_{k=-\infty}^{n} B(n,k)g(x(k)), \quad n \in \mathbb{Z}, \tag{1.1}$$

and obtain sufficient conditions on the terms p, b, h, B, and g such that there exists a nontrivial periodic solution of (1.1). The sufficient conditions are not verifiable for general p, b, h, B, and g and so the primary purpose of this article is to consider three applications in which the sufficient conditions are verifiable.

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In Section 2, we shall provide the definition of a large contraction and state without proof Burton's extended Krasnosel'skiĭ fixed point theorem. In Section 3, we shall employ Burton's theorem and obtain sufficient conditions for the existence of a nontrivial periodic solution of (1.1). We shall close in Section 4 with three specific applications in which the primary sufficient condition is realized. The three applications are motivated by two specific large contractions, one observed by Burton [5] and one observed by Raffoul [17, 18].

2. Preliminaries

Denote the set of all integers and real numbers by $\mathbb{Z}$ and $\mathbb{R}$, respectively. Define $\mathbb{N}_a = \{a, a+1, a+2, \cdots\}$ for any $a \in \mathbb{R}$. Assume that empty sums and products are taken to be 0 and 1, respectively. First, we introduce the concept of large contraction.

Definition 2.1 *[5] Let $(\mathcal{M}, d)$ be a metric space and $B : \mathcal{M} \to \mathcal{M}$. B is said to be a large contraction mapping (or a large contraction) if φ, $\psi \in \mathcal{M}$, with $\varphi \neq \psi$, then $d\big(B\varphi, B\psi\big) \leq d(\varphi, \psi)$, and if for all $\varepsilon > 0$, there exists a $\delta < 1$ such that*

$$\big[\varphi, \psi \in \mathcal{M}, d(\varphi, \psi) \geq \varepsilon\big] \Rightarrow d\big(B\varphi, B\psi\big) \leq \delta d(\varphi, \psi).$$

Remark 2.2 *It is clear from the definition that if δ serves as a large contraction coefficient for $\varepsilon_1 > 0$ and $\varepsilon_2 > \varepsilon_1$, then δ serves as a large contraction coefficient for $\varepsilon_2 > 0$. This comment is made since throughout Section 4, it is assumed that $0 < \varepsilon_1 < 1$.*

The following theorem is Burton's extended Krasnosel'skiĭ fixed point theorem.

Theorem 2.3 *[5] Let $\mathcal{M}$ be a bounded convex nonempty subset of a Banach space $(\mathbb{B}, \|.\|)$. Suppose that A and B map $\mathcal{M}$ into $\mathbb{B}$ such that:*

1. *x, $y \in \mathcal{M}$, implies $Ax + By \in \mathcal{M}$;*
2. *A is compact and continuous;*
3. *B is a large contraction mapping.*

Then there exists $z \in \mathcal{M}$ with $z = Az + Bz$.

3. Existence of periodic solutions

In this section, we establish sufficient conditions for the existence of periodic solutions of the Volterra difference equation (1.1). We assume that there exists a least positive real number T such that

$$p(n+T) = p(n), \text{ for all } n \in \mathbb{Z},$$

$$b(n+T) = b(n), \text{ for all } n \in \mathbb{Z},$$

and

$$B(n+T, k+T) = B(n,k), \text{ for all } (n,k) \in \mathbb{Z}^2.$$

We assume that h and g are real valued mappings.

Let P_T be the set of all T-periodic sequences $\{x(n)\}$, periodic in n. Then $(P_T, \|.\|)$ is a Banach space with respect to the maximum norm

$$\|x\| = \max_{n\in\{0,1,2,\cdots,T-1\}} |x(n)|.$$

Define $a(n) = 1 + b(n)$ and assume throughout that

$$1 - \prod_{k=n-T}^{n-1} a(k) \neq 0, \quad n \in \mathbb{Z}. \tag{3.1}$$

Define the mapping H by

$$H(x) = h(x) - x, \quad x \in \mathbb{R}. \tag{3.2}$$

We begin with the following lemma.

Lemma 3.1 *Assume $x \in P_T$. Then $x(n)$ is a solution of (1.1) if and only if*

$$x(n) = \Big(1 - \prod_{k=n-T}^{n-1} a(k)\Big)^{-1} \sum_{l=n-T}^{n-1} \Big[p(l) + b(l)H(x(l)) + \sum_{m=-\infty}^{l} B(l,m)g(x(m))\Big] \prod_{s=l+1}^{n-1} a(s). \tag{3.3}$$

Proof Let $x \in P_T$. Rewrite (1.1) in the form

$$x(n+1) - a(n)x(n) = p(n) + b(n)H(x(n)) + \sum_{k=-\infty}^{n} B(n,k)g(x(k)). \tag{3.4}$$

Thus, x is a solution of (1.1) if and only if x is a solution of (3.4). For each n multiply (3.4) by $\Big(\prod_{k=-T}^{n} a(k)\Big)^{-1}$ and sum the equations from $l = (n-T)$ to $l = (n-1)$ to obtain

$$x(n)\Big(\prod_{k=-T}^{n-1} a(k)\Big)^{-1} - x(n-T)\Big(\prod_{k=-T}^{n-T-1} a(k)\Big)^{-1} = \sum_{l=n-T}^{n-1} \Big[p(l) + b(l)H(x(l)) + \sum_{m=-\infty}^{l} B(l,m)g(x(m))\Big]\Big(\prod_{s=-T}^{l} a(s)\Big)^{-1}. \tag{3.5}$$

Thus, x is a solution of (1.1) if and only if x is a solution of (3.5). To obtain the representation in (3.3), note that $x \in P_T$, which implies $x(n-T) = x(n)$,

$$\Big(\prod_{k=-T}^{n-1} a(k)\Big)^{-1} - \Big(\prod_{k=-T}^{n-T-1} a(k)\Big)^{-1} = \Big(\prod_{k=-T}^{n-1} a(k)\Big)^{-1}\Big(1 - \prod_{k=n-T}^{n-1} a(k)\Big),$$

and

$$\Big(\prod_{k=-T}^{n-1} a(k)\Big)\Big(\prod_{s=-T}^{l} a(s)\Big)^{-1} = \Big(\prod_{s=l+1}^{n-1} a(s)\Big).$$

□

For simplicity, set

$$\eta = \Big|\Big(1 - \prod_{k=n-T}^{n-1} a(k)\Big)^{-1}\Big|$$

and note that η is independent of n due to the periodicity of $a(n)$. Set

$$M = \eta \prod_{k=0}^{T-1} (1 + |b(k)|).$$

Then for $n \in \{0, 1, 2, \cdots, T-1\}$ and $l \in \{n-T, n-T+1, \cdots, n-1\}$, we have

$$\begin{aligned} \Big|\Big(1 - \prod_{k=n-T}^{n-1} a(k)\Big)^{-1} \prod_{s=l+1}^{n-1} a(s)\Big| &= \eta \prod_{s=l+1}^{n-1} |a(s)| \le \eta \prod_{s=l+1}^{n-1} (1 + |b(s)|) \\ &\le \eta \prod_{k=n-T+1}^{n-1} (1 + |b(k)|) = M. \end{aligned} \tag{3.6}$$

Let J be a positive constant. This constant J will be carefully chosen in the applications. Define the set

$$\mathbb{M}_J = \{\varphi \in P_T : \|\varphi\| \le J\} \tag{3.7}$$

and note that $\mathbb{M}_J$ is a bounded and convex subset of the Banach space P_T. Let the mapping $A : \mathbb{M}_J \to P_T$ be defined by

$$\big(A\varphi\big)(n) = \Big(1 - \prod_{k=n-T}^{n-1} a(k)\Big)^{-1} \sum_{l=n-T}^{n-1} \Big[p(l) + \sum_{m=-\infty}^{l} B(l,m) g(\varphi(m))\Big] \prod_{s=l+1}^{n-1} a(s), \tag{3.8}$$

for $n \in \mathbb{Z}$. Similarly, we set the map $B : \mathbb{M}_J \to P_T$ by

$$\big(B\psi\big)(n) = \Big(1 - \prod_{k=n-T}^{n-1} a(k)\Big)^{-1} \sum_{l=n-T}^{n-1} \Big[b(l) H(\psi(l))\Big] \prod_{s=l+1}^{n-1} a(s), \quad n \in \mathbb{Z}. \tag{3.9}$$

Assume that $g(x)$ satisfies a local Lipschitz condition in x; in particular, assume there exists a positive constant L such that

$$|g(z) - g(w)| \le L\|z - w\|, \text{ for } |z|, |w| \le J. \tag{3.10}$$

Then, for $\varphi \in \mathbb{M}_J$, we obtain

$$\begin{aligned} |g(\varphi(n))| &= |g(\varphi(n)) - g(0) + g(0)| \le |g(\varphi(n)) - g(0)| + |g(0)| \\ &\le L|\varphi(n)| + |g(0)| \le LJ + |g(0)|. \end{aligned} \tag{3.11}$$

Lemma 3.2 *Assume that g satisfies the Lipschitz condition given in (3.10). Suppose that there exists a positive constant Λ such that*

$$\sum_{k=-\infty}^{n} |B(n,k)| \le \Lambda, \tag{3.12}$$

for all n. Then the mapping $A : \mathbb{M}_J \to P_T$ is continuous.

Proof Let $\varphi \in \mathbb{M}_J$ and assume $\{\varphi_j\}$ is a sequence of functions in $\mathbb{M}_J$ with $\|\varphi_j - \varphi\| \to 0$ as $j \to \infty$. Then

$$\begin{aligned}
&\big|\big(A\varphi_j\big)(n) - \big(A\varphi\big)(n)\big| \\
&\qquad \leq \Big|\Big(1 - \prod_{k=n-T}^{n-1} a(k)\Big)^{-1} \sum_{l=n-T}^{n-1} \Big[\sum_{m=-\infty}^{l} B(l,m)g(\varphi_j(m))\Big] \prod_{s=l+1}^{n-1} a(s) \\
&\qquad - \Big(1 - \prod_{k=n-T}^{n-1} a(k)\Big)^{-1} \sum_{l=n-T}^{n-1} \Big[\sum_{m=--\infty}^{l} B(l,m)g(\varphi(m))\Big] \prod_{s=l+1}^{n-1} a(s)\Big| \\
&\qquad \leq \eta \sum_{l=n-T}^{n-1} \sum_{m=-\infty}^{l} |B(l,m)||g(\varphi_j(m)) - g(\varphi(m))| \prod_{s=l+1}^{n-1} |a(s)| \\
&\qquad \leq M\Lambda LT\|\varphi_j - \varphi\|,
\end{aligned}$$

which implies $\big\|\big(A\varphi_j\big) - \big(A\varphi\big)\big\| \to 0$ as $j \to \infty$. □

Define a parameter $\Theta > 0$ by

$$\Theta = \max_{x\in[-J,J]} \sum_{l=n-T}^{n-1} \Big[|p(l)| + |b(l)||H(x)|\Big], \tag{3.13}$$

for all $n \in \mathbb{Z}$.

Theorem 3.3 *Assume that (3.10), (3.12), and (3.13) hold. Then* $A(\mathbb{M}_J) \to P_T$ *is relatively compact.*

Proof Define the map $\bar{A} : \mathbb{M}_J \to \mathbb{R}^T$ by

$$\bar{A}\varphi = \{A\varphi(0), \ldots, A\varphi(T-1)\}.$$

Since $A(\mathbb{M}_J) \to P_T$, it is sufficient to show that $\bar{A} : \mathbb{M}_J \to \mathbb{R}^T$ is relatively compact. In particular, it is sufficient to show that $\bar{A}(\mathbb{M}_J)$ is uniformly bounded. Using (3.10), (3.12), and (3.13), we obtain

$$\begin{aligned}
\big|\big(\bar{A}\varphi\big)(n)\big| &= \Big|\Big(1 - \prod_{k=n-T}^{n-1} a(k)\Big)^{-1} \sum_{l=n-T}^{n-1} \Big[p(l) + \sum_{m=-\infty}^{l} B(l,m)g(\varphi(m))\Big] \prod_{s=l+1}^{n-1} a(s)\Big| \\
&\leq \eta \sum_{l=n-T}^{n-1} \Big[|p(l)| + \sum_{m=-\infty}^{l} |B(l,m)||g(\varphi(m))|\Big] \prod_{s=l+1}^{n-1} |a(s)| \\
&\leq M\Big[\Theta + \Lambda T\big(LJ + |g(0)|\big)\Big],
\end{aligned}$$

which implies that $\bar{A}(\mathbb{M}_J)$ is uniformly bounded. □

The following lemma gives a relationship between the mappings H and B, where B is the mapping defined in (3.9), in the sense of a large contraction.

Lemma 3.4 *Assume that* $b(n) \geq 0$ *for* $n \in \mathbb{Z}$. *If* H *is a large contraction on* $\mathbb{M}_J$, *then so is the mapping* B.

Proof If H is a large contraction on $\mathbb{M}_J$ then, for $x,\ y \in \mathbb{M}_J$, we have

$$\|Hx - Hy\| \le \|x - y\|.$$

We first note that under the assumption, $b(n) \ge 0$ for $n \in \mathbb{Z}$, it follows that

$$\Big|\Big(1 - \prod_{k=n-T}^{n-1} a(k)\Big)^{-1}\Big|\Big|\sum_{l=n-T}^{n-1} b(l) \prod_{s=l+1}^{n-1} a(s)\Big| = 1.$$

Let $c(l) = \prod_{s=l+1}^{n-1} a(s)$. Then,

$$\big(\nabla c\big)(l) = \prod_{s=l+1}^{n-1} a(s) - \prod_{s=l}^{n-1} a(s) = -b(l) \prod_{s=l+1}^{n-1} a(s).$$

Consequently,

$$\begin{aligned}\Big|\Big(1 - \prod_{k=n-T}^{n-1} a(k)\Big)^{-1}\Big| \sum_{l=n-T}^{n-1} b(l) \prod_{s=l+1}^{n-1} a(s) &= -\Big|\Big(1 - \prod_{k=n-T}^{n-1} a(k)\Big)^{-1}\Big| \sum_{l=n-T}^{n-1} \big(\nabla c\big)(l)\\ &= -\Big|\Big(1 - \prod_{k=n-T}^{n-1} a(k)\Big)^{-1}\Big|\big[c(n-1) - c(n-T-1)\big]\\ &= -\Big|\Big(1 - \prod_{k=n-T}^{n-1} a(k)\Big)^{-1}\Big|\Big(1 - \prod_{k=n-T}^{n-1} a(k)\Big).\end{aligned}$$

Now, consider

$$\begin{aligned}\big|\big(Bx\big)(n) - \big(By\big)(n)\big| &= \Big|\Big(1 - \prod_{k=n-T}^{n-1} a(k)\Big)^{-1} \sum_{l=n-T}^{n-1} \Big[b(l)H(x(l))\Big] \prod_{s=l+1}^{n-1} a(s)\\ &- \Big(1 - \prod_{k=n-T}^{n-1} a(k)\Big)^{-1} \sum_{l=n-T}^{n-1} \Big[b(l)H(y(l))\Big] \prod_{s=l+1}^{n-1} a(s)\Big|\\ &\le \Big|\Big(1 - \prod_{k=n-T}^{n-1} a(k)\Big)^{-1}\Big| \sum_{l=n-T}^{n-1} b(l)|H(x(l)) - H(y(l))| \prod_{s=l+1}^{n-1} |a(s)|\\ &\le \|x - y\|\Big|\Big(1 - \prod_{k=n-T}^{n-1} a(k)\Big)^{-1}\Big| \sum_{l=n-T}^{n-1} b(l) \prod_{s=l+1}^{n-1} |a(s)| = \|x - y\|. \qquad (3.14)\end{aligned}$$

Thus, we have

$$\|Bx - By\| \le \|x - y\|.$$

Let $\varepsilon > 0$ and assume $\delta < 1$ is such that if $\|x - y\| \ge \varepsilon$ then $\|Hx - Hy\| < \delta\|x - y\|$. Since

$$\Big|\Big(1 - \prod_{k=n-T}^{n-1} a(k)\Big)^{-1}\Big|\Big|\sum_{l=n-T}^{n-1} b(l) \prod_{s=l+1}^{n-1} a(s)\Big| = 1,$$

(3.14) is readily modified to show

$$\|Bx - By\| < \delta\|x - y\|;$$

in particular, B is a large contraction. □

We state the main theorem, which provides sufficient conditions for the existence of a T periodic solution of (1.1).

Theorem 3.5 *Assume that there exists a least positive real number T such that*

$$p(n+T) = p(n), \text{ for all } n \in \mathbb{Z},$$

$$b(n+T) = b(n), \text{ for all } n \in \mathbb{Z},$$

and

$$B(n+T, k+T) = B(n,k), \text{ for all } (n,k) \in \mathbb{Z}^2.$$

We assume that h and g are real valued mappings and assume g satisfies the Lipschitz condition (3.10). Assume the kernel $B(n,k)$ satisfies (3.12) and define Θ by (3.13). Assume that $b(n) \geq 0$ for $n \in \mathbb{Z}$ and assume there exists $J > 0$ such that $H(x) = h(x) - x$ is a large contraction on $[-J, J]$. Finally, assume

$$M[\Theta + \Lambda T(LJ + |g(0)|)] \leq J. \tag{3.15}$$

Then (1.1) has a periodic solution.

Proof Define the operators A and B by (3.8) and (3.9). We have shown that A is compact and continuous, and that B is a large contraction on $\mathbb{M}_J$. Moreover, (3.15) implies that if $\varphi_1, \varphi_2 \in \mathbb{M}_J$ then

$$|A\varphi_1(n) + B\varphi_2)| \leq M[\Theta + \Lambda T(LJ + |g(0)|)] \leq J.$$

In particular,

$$A\varphi_1 + B\varphi_2 \in \mathbb{M}_J.$$

Thus, Theorem 2.3 applies and the theorem is proved. □

4. Applications

We have seen that (3.15) is the primary sufficient condition. In this section we present three specific applications of (1.1) and verify (3.15) in each application. Throughout this section it is assumed that $0 < \varepsilon_1 < 1$.

Burton [5] defined the concept of large contraction and showed that

$$x - x^3$$

is a large contraction in a neighborhood of $x = 0$. Later, Raffoul [18] extended Burton's arguments and showed that

$$x - x^5$$

is a large contraction in a neighborhood of $x = 0$. In particular, Raffoul [18] established the following lemma.

Lemma 4.1 *Let* $J = 5^{-1/4}$. *If*

$$\mathbb{M}_J = \{\varphi : \mathbb{Z} \to \mathbb{R} \mid \|\varphi\| \leq J\},$$

then the mapping H *defined by*

$$H(x(n)) = -x(n) + (x(n))^5$$

is a large contraction on the set $\mathbb{M}_J$.

Example 1. In [18], Raffoul first showed that for $x, y \in [-J, J]$,

$$|H(x) - H(y)| \leq |x - y|[1 - \frac{(x^4 + y^4)}{2}], \tag{4.1}$$

and then he showed that H is a large contraction on $\mathbb{M}_J$.

We employ Lemma 4.1 and Theorem 3.5 and find conditions on $c > 0$ such that the following Volterra difference equation has a 4-periodic solution. Let $c > 0$ and consider the Volterra difference equation

$$\Delta x(n) = c\Big(\sin\Big(\frac{n\pi}{2}\Big) + \big(1 + \cos n\pi\big)h(x) + \sum_{k=-\infty}^{n} 2^{k-n} x^2(k)\Big), \quad n \in \mathbb{Z}, \tag{4.2}$$

where $h(x) = x^5$. We calculate the estimates required to apply Theorem 3.5 for (4.2). Here $p(n) = c\sin\left(\frac{n\pi}{2}\right)$, $b(n) = c(1 + \cos n\pi)$, $B(n,k) = c2^{k-n}$, and $g(x) = x^2$. We observe that 4 is the least positive real number such that $p(n+4) = p(n)$, $b(n+4) = b(n)$, and $B(n+4, k+4) = B(n,k)$ for all n, $k \in \mathbb{Z}$. Set $h(x) = x^5$ and define $a(n) = ((1+c) + c\cos n\pi)$. Note that

$$1 - \prod_{k=n-4}^{n-1} a(k) = 1 - \prod_{k=1}^{4}[(1+c) + c\cos(n-k)\pi)] = 1 - (1+2c)^2 \neq 0, \quad n \in \mathbb{Z};$$

in particular, (3.1) is satisfied. Using Lemma 4.1, the mapping

$$H(x(n)) = h(x(n)) - x(n)$$

is a large contraction on the set $\mathbb{M}_J$ for $J = 5^{-1/4}$. If z, $w \in [-J, J]$,

$$|g(z) - g(w)| = \left|z^2 - w^2\right| \leq \big[|z| + |w|\big]|z - w| \leq 2J|z - w|,$$

which implies that (3.10) is satisfied with $L = 2J$. Since $b(l) \geq 0$ for all l, $M = 1$. Note that

$$\sum_{k=-\infty}^{n} |B(n,k)| = c\sum_{k=-\infty}^{n} 2^{k-n} = \Lambda < \infty$$

implies (3.12) is satisfied with $\Lambda = 2c$. Let $\psi \in \mathbb{M}_J$. For all $n \in \mathbb{Z}$,

$$\sum_{l=n-4}^{n-1}\Big[|p(l)| + |b(l)||H(\psi(l))|\Big] = c\sum_{l=n-4}^{n-1}\Big[|\sin(l\frac{\pi}{2})| + (1 + \cos n\pi)|\psi(l) - \psi^5(l)|\Big]$$
$$\leq c(2 + 4|J||1 - J^4|) \leq c(2 + 4J). \tag{4.3}$$

Set $\Theta = c(2 + 4J)$.

Now we choose $c > 0$ such that (3.15) is satisfied. In particular, we require that

$$c[2 + 4J + 2(4(2J)J)] \leq J.$$

Thus, define

$$f(c) = J - c[2 + 4J + 2(4(2J)J)].$$

Then $f(c) > 0$ on $[0, \frac{J}{2+4J+16J^2})$; in particular, if $c \in (0, \frac{J}{2+4J+16J^2}]$, then (3.15) is satisfied and we have the following result.

Theorem 4.2 *Let $h(x) = x^5$ and $J = 5^{\frac{-1}{4}}$, and set $c_0 = \frac{J}{2+4J+16J^2}$. Then for each $c \in (0, c_0]$, (4.2) has a nontrivial 4-periodic solution.*

Example 2. We again consider (4.2) and we consider $h(x) = f(x^5)$. Set $J = 5^{-1/4}$ and assume that f is differentiable and increasing on $[-J, J]$. Further, assume there exists $0 < \alpha < 1$ such that $\alpha \leq f'(u) \leq 1$ for $u \in [-J, J]$. In [18], Raffoul first showed that for $x, y \in [-J, J]$,

$$\begin{aligned} |(x - x^5) - (y - y^5)| &= |(x - y) - (x - y)(x^4 + x^3y + x^2y^2 + xy^3 + y^4)| \\ &\leq |x - y|[1 - \frac{(x^4 + y^4)}{2}], \end{aligned} \tag{4.4}$$

and then showed that $x - x^5$ is a large contraction on $\mathbb{M}_J$. Now for $H(x) = f(x^5) - x$, let $x, y \in [-J, J]$. Then there exists c between x^5 and y^5 such that

$$\begin{aligned} |(x - y) - (f(x^5) - f(y^5))| &= |(x - y) - f'(c)((x^5) - (y^5))| \\ &= |(x - y) - f'(c)(x - y)(x^4 + x^3y + x^2y^2 + xy^3 + y^4)| \\ &= |x - y||(1 - f'(c)(x^4 + x^3y + x^2y^2 + xy^3 + y^4))| \\ &\leq |x - y|[1 - \alpha\frac{(x^4 + y^4)}{2}]. \end{aligned}$$

Thus, it follows that $H(x) = x - f(x^5)$ is a large contraction on $[-J, J]$; the estimates in Example 1 remain valid and so the following result is obtained.

Theorem 4.3 *Let $h(x) = f(x^5)$. Set $J = 5^{\frac{-1}{4}}$ and assume f is differentiable and increasing on $[-J, J]$. Further, assume there exists $0 < \alpha < 1$ such that $\alpha \leq f'(u) \leq 1$ for $u \in [-J, J]$. Set $c_0 = \frac{J}{2+4J+16J^2}$. Then for each $c \in (0, c_0]$, (4.2) has a nontrivial 4-periodic solution.*

Remark 4.4 *Note that $f(u) = \sin(u)$ satisfies the hypotheses of Theorem 4.3.*

Example 3. In this example, we consider $H(x) = x - x^{2n+1}$ where $1 \leq n \leq 70$ is an integer. In [5], Burton showed that $x - x^3$ is a large contraction on $[-J, J]$ for $J = \frac{1}{\sqrt{3}}$. Therefore, the following theorem is valid.

Theorem 4.5 *Let $h(x) = x^3$ and $J = 3^{\frac{-1}{2}}$, and set $c_0 = \frac{J}{2+4J+16J^2}$. Then for each $c \in (0, c_0]$, (4.2) has a nontrivial 4-periodic solution.*

For his results, Burton [5] derived the calculation, for $x, y \in [-J, J]$,

$$|x - x^3 - y + y^3| \le |x - y| \left(1 - \frac{x^2 + y^2}{2}\right).$$

In [18], Raffoul derived the analogous (4.4). At this time, it is not clear to us how to generalize these derivations. Instead, we produce an alternate argument to show that if $1 \le n \le 70$ is an integer, then $x - x^{2n+1}$ is a large contraction on $[-J_n, J_n]$ for $J_n = (\frac{1}{2n+1})^{\frac{1}{2n}}$. Note that for $H(x) = x - x^{2n+1}$, $|H'(x)| \le 1$ for $x \in [-J_n, J_n]$. Thus, for each $\varepsilon > 0$, we only need to exhibit $0 \le \delta < 1$ such that if $x, y \in [-J_n, J_n]$ and $|x - y| \ge \varepsilon$, then

$$|(x - x^{2n+1}) - (y - y^{2n+1})| \le \delta |x - y|.$$

Let $0 < \varepsilon < 1$. First, assume $0 \le x < y \le J_n$ and $y - x \ge \varepsilon$. In particular, $0 \le x$ and $y \ge \varepsilon$. Moreover, $0 < y \le J = (\frac{1}{2n+1})^{\frac{1}{2n}}$. Then

$$\varepsilon^{2n} \le y^{2n} \le \sum_{l=0}^{2n} x^{2n-l} y^l < (2n+1) y^{2n} \le (2n+1) J^{2n} = 1. \tag{4.5}$$

In particular,

$$0 < 1 - \sum_{l=0}^{2n} x^{2n-l} y^l \le 1 - \varepsilon^{2n}$$

and

$$|(x - y) - (x^{(2n+1)} - y^{(2n+1)})| = |x - y|(1 - \sum_{l=0}^{2n} (x^{2n-l} y^l)) \le |x - y|(1 - \varepsilon^{2n}).$$

Thus, $\delta = 1 - \varepsilon^{2n}$ is a candidate in the definition of large contraction.

By the oddness of $H(x) = x - x^{2n+1}$, if $0 \ge x > y \ge -J_n$, and $x - y \ge \varepsilon$, then

$$|(x - y) - (x^{(2n+1)} - y^{(2n+1)})| \le |x - y|(1 - \varepsilon^{2n}).$$

Now consider the case $x \le 0 \le y$ and $y - x \ge \varepsilon$. Note that $\max\{|x|, |y|\} \ge \frac{\varepsilon}{2}$ and by the oddness of H we can assume without loss of generality that $y = \max\{|x|, |y|\}$ and $y \ge \frac{\varepsilon}{2}$. We seek an inequality analogous to (4.5). The upper bound is straightforward since

$$\sum_{l=0}^{2n} x^{2n-l} y^l \le \sum_{l=0}^{2n} |x|^{2n-l} y^l \le (2n+1) y^{2n} \le (2n+1) J^{2n} = 1.$$

To obtain a lower bound, write

$$\sum_{l=0}^{2n} x^{2n-l} y^l = \sum_{l=0}^{n-1} (x^{2n-2l} y^{2l} + x^{2n-2l-1} y^{2l+1}) + y^{2n}.$$

Each term $(x^{2n-2l} y^{2l} + x^{2n-2l-1} y^{2l+1})$ satisfies

$$-(\frac{2n-2l-1}{2n-2l})^{2n-2l-1} \frac{1}{2n-2l} y^{2n} \le (x^{2n-2l} y^{2l} + x^{2n-2l-1} y^{2l+1}) \le 0$$

for $-y \le x \le 0$ as shown by usual optimization techniques of calculus. Thus,

$$\Big((\sum_{l=0}^{n-1} -(\frac{2n-2l-1}{2n-2l})^{2n-2l-1}(\frac{1}{2n-2l})) + 1\Big) y^{2n} \le \sum_{l=0}^{2n} x^{2n-l} y^{l}.$$

Now,

$$\sum_{l=0}^{\infty} -(\frac{2n-2l-1}{2n-2l})^{2n-2l-1}(\frac{1}{2n-2l})$$

diverges. However, numerical calculations verify that

$$0 < \Big(\sum_{l=0}^{n-1} (\frac{2n-2l-1}{2n-2l})^{2n-2l-1}(\frac{1}{2n-2l})\Big) < 1$$

for $n = 1, \dots, 70$. For $n \in \{1, \dots, 70\}$, set

$$\delta_n = (\sum_{l=0}^{n-1} (\frac{2n-2l-1}{2n-2l})^{2n-2l-1}(\frac{1}{2n-2l})).$$

Then, in analogue to (4.5),

$$0 < \frac{(1-\delta_n)}{2^{2n}} \varepsilon^{2n} \le (1-\delta_n) y^{2n} \le \sum_{l=0}^{2n} x^{2n-l} y^{l} \le (2n+1) y^{2n} \le (2n+1) J^{2n} = 1, \tag{4.6}$$

if $x \le 0 < y$, $y - x \ge \varepsilon$, $y \ge \frac{\varepsilon}{2}$. Thus, for each $n \in \{1, \dots, 70\}$, $H(x) = x - x^{2n+1}$ is a large contraction on $[-J_n, J_n]$ since for $x, y \in [-J_n, J_n]$, $|y - x| \ge \varepsilon$,

$$|(x-y) - (x^{(2n+1)} - y^{(2n+1)})| \le |x-y|(1 - \frac{(1-\delta_n)}{2^{2n}} \varepsilon^{2n}).$$

We now state analogues to Theorems 4.2 and 4.3. Note that the only estimate H impacts in the calculations of Example 1 is in the calculation of (4.3). Now (4.3) reads as

$$\sum_{l=n-4}^{n-1} \Big[|p(l)| + |b(l)||H(\psi(l))|\Big] \le c(2 + 4J_n|1 - J_n^{2n}|) \le c(2 + 4J_n),$$

and so

$$f(c) = J_n - c[2 + 4J_n + 2(4(2J_n)J_n)],$$

as in Example 1.

Theorem 4.6 *Let $1 \le n \le 70$ denote an integer and set $h(x) = x^{2n+1}$, $J_n = (\frac{1}{2n+1})^{\frac{1}{2n}}$, and $c_0 = \frac{J_n}{2+4J_n+16J_n^2}$. Then for each $c \in (0, c_0]$, (4.2) has a nontrivial 4-periodic solution.*

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