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On Hamilton cycle decompositions of complete multipartite graphs which are both cyclic and symmetric

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ON HAMILTON CYCLE DECOMPOSITIONS OF COMPLETE MULTIPARTITE GRAPHS WHICH ARE BOTH CYCLIC AND SYMMETRIC

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APPROVAL OF THESIS/DISSERTATION

We, the faculty supervising the work of Fatima A. Akinola, affirm that the thesis, *On Hamilton cycle decompositions of complete multipartite graphs which are both cyclic and symmetric*, meets the high academic standards for original scholarship and creative work established by the Department of Mathematics and the College of Science. This work also conforms to the formatting guidelines of Marshall University. With our signatures, we approve the manuscript for publication.

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ABSTRACT

Let G be a graph with v vertices. A Hamilton cycle of a graph is a collection of edges which create a cycle using every vertex. A Hamilton cycle decomposition is cyclic if the set of cycle is invariant under a full length permutation of the vertex set. We say a decomposition is symmetric if all the cycles are invariant under an appropriate power of the full length permutation. Such decompositions are known to exist for complete graphs and families of other graphs. In this work, we show the existence of cyclic n-symmetric Hamilton cycle decompositions of a family of graphs, the complete multipartite graph $K_{m\times n}$ where the number of parts, m, is odd and the part size, n, is also odd. We classify the existence where m is prime and prove the existence in additional cases where m is a composite odd integer.

CHAPTER 1

INTRODUCTION

The study of Hamilton cycle decompositions of a graph can be traced to one of the early 1880's problem discussed by Eduoard Lucas [10] called the *probléme de ronde*. Given $2n+1$ people, can one give n dinners at a round table so that each guest sits next to each other guest exactly once? Note that the solution of the problem is equivalent to decomposing the complete graph K_{2n+1} into Hamilton cycles. We give a solution with seven guests and three meals in Figure 1 and the corresponding Hamilton cycle decomposition for K_7 in Figure 2.

The first known result in Hamilton cycle decompositions is attributed to Walecki [10].

Theorem 1.1. *The graph K_n has a Hamilton cycle decomposition if and only if n is odd.*

The necessity condition of n being odd comes from the degree of the vertices, number of edges incident to a vertex, in K_n. The sufficiency comes from a "zig-zag" construction [2] illustrated in Figure 3. Results on the existence of decompositions of the complete multipartite graphs were found by Laskar and Auerbach [9].

Theorem 1.2. *The complete multipartite graph $K_{m\times n}$ has a Hamilton cycle decomposition if and only if $(m-1)n$ is even.*

Notice that the decomposition in Figure 3 has some rotational structure to it. When a cycle is rotated, it produces another cycle in the decomposition. So, these such properties led to

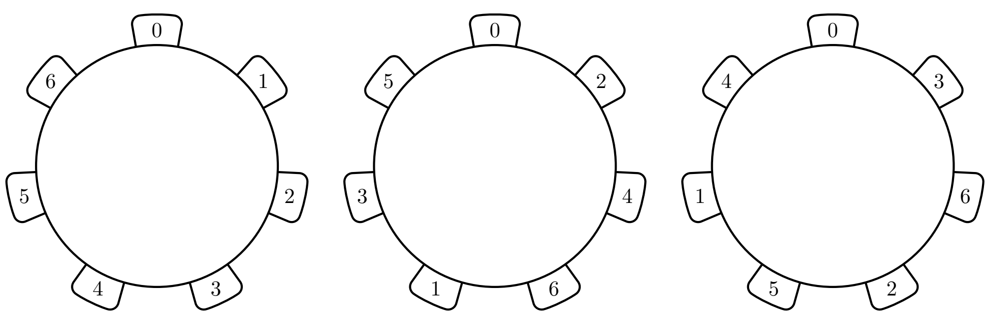

Figure 1. Solution to the Eduoard Lucas' *probléme de ronde*

In this example, three dinners are served for seven guests and the table arrangements are such that the guests each sit next to one another exactly once.

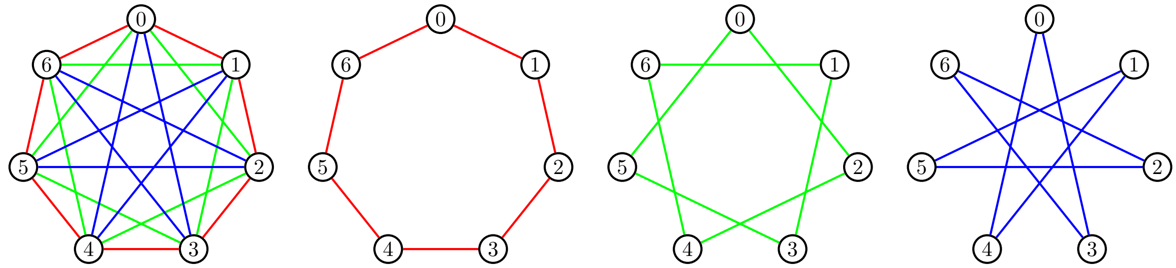

Figure 2. Hamilton cycle decomposition of K_7

The graph K_7 along with its decomposition into three Hamilton cycles. Observe that this is the Hamilton cycle decomposition corresponding to the table arrangement given in Figure 1.

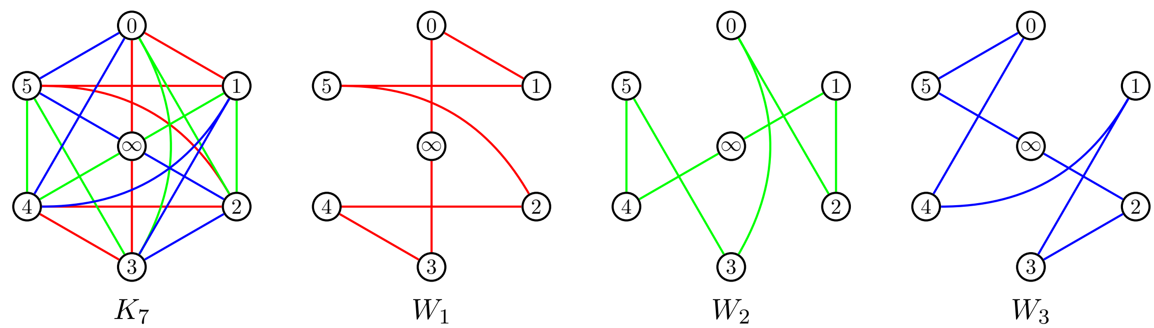

Figure 3. The Walecki construction of K_7

This decomposition is constructed by using a "zig-zag" pattern to construct one cycle, then rotate it to produce the other cycles. Observe that this decomposition is not isomorphic to that given in Figure 2, and has a different type of rotational symmetry.

the study of finding graph decompositions with additional structures to them. In this paper, we focus on decompositions which are *cyclic*, that is, the set of cycles is invariant under a full length (v-cycle) permutation and *symmetric*, that is, there is a nontrivial permutation of the vertices, such that each cycle in the decomposition is itself invariant under it.

The first major result involving cyclic Hamilton cycle decompositions of complete graphs was proven by Buratti and Del Fra [5].

Theorem 1.3. *There exists a cyclic Hamiltonian cycle decomposition of the complete graph K_n if and only if n is an odd integer but $n \neq 15$ and $n \neq p^a$, with p a prime and $a > 1$.*

Similar results involving cyclic Hamilton cycle decompositions of complete graphs minus a 1-factor, which is a complete graph with a perfect matching removed, were found by Jordon and Morris [8]. The existence of a cyclic Hamilton cycle decomposition of $K_{m\times n}$ was proven by Merola et al. [11], for the case where the number of parts m is even.

For symmetry, we have that the Walecki construction has a symmetric Hamilton cycle decomposition. Akiyama et al. [1] showed the existence of a symmetric Hamilton cycle decomposition of the complete graph which is not isomorphic to the Walecki decomposition, and Chitra and Muthusamy [7] showed an analogous result for complete multigraphs. In 2011, Brualdi and Schroeder [4] classified the existence of symmetric Hamilton cycle decompositions for the complete graph minus a 1-factor and in 2015, Schroeder [12] answered the question for complete multipartite graphs.

In 2014, Buratti and Merola [6] proved the existence of cyclic and symmetric Hamilton cycle decompositions of complete graphs. In fact, Merola et al [11] also discuss Hamilton cycle decompositions of complete multipartite graphs satisfying both properties. In this paper, we specifically investigate the existence of cyclic and symmetric Hamilton cycle decompositions of complete multipartite graphs where the number of parts is odd. Note that it follows from the necessary condition in Theorem 1.2 that the part size has to be odd.

This work is coordinated in the following order: in Chapter 2, we provide background work and definitions relevant to the work done, justify why we only need to find n-symmetry, introduce the notion of *base paths*, and prove necessary and sufficient conditions for the existence of cyclic and symmetric Hamilton cycle decompositions. In Chapter 3, we construct base paths which give rise to decompositions of $K_{m\times n}$ for cases where m is prime and n is odd. In Chapter 4, we investigate other cases of decompositions of $K_{m\times n}$ with m odd and not necessarily prime building on cyclic decompositions of the complete graph K_m. The two main results of this work are as follows:

Theorem 1.4. *Let p be an odd prime and $n \geq 3$ be an odd positive integer. Then $K_{p\times n}$ has a cyclic n-symmetric Hamilton cycle decomposition.*

Theorem 1.5. *Let m, n be positive odd integers. If $m \neq 15$ or p^a for a prime $p > 2$, $a \geq 2$ and n is a prime power which is bigger than the smallest prime divisor of m, then $K_{m\times n}$ has a cyclic n-symmetric Hamilton cycle decomposition.*

CHAPTER 2

PRELIMINARIES

We first introduce the necessary background information from algebra and number theory, and then we focus on the definition of symmetry and cyclic decompositions. Next, we introduce the notion of a base path and show a correspondence between base paths and decompositions.

2.1 BACKGROUND

In this section, we introduce some common definitions and terminologies from modern algebra and graph theory. For more information, see [3]. It is denoted $\gcd(x, y)$. Two integers a, b are *relatively prime* if their greatest common divisor is 1. The *order* of a set Γ, denoted $|\Gamma|$, is the number of elements in Γ. Let v be a positive integer. Let $\mathbb{Z}_v$ denote the ring of elements $\{0, 1, 2, \ldots, v-1\}$ inbuilt with addition and multiplication modulo v. The (additive) *order* of an element $x \in \mathbb{Z}_v$ is the smallest positive integer k such that $kx = 0$ in $\mathbb{Z}_v$. We say that an element $x \in \mathbb{Z}_v$ is a *generator* of $\mathbb{Z}_v$ if x has order v. In a ring Γ with unity, we say x is a *unit* of Γ if there exists an element $u \in \Gamma$ such that $xu = 1$. In particular, the set of units of $\mathbb{Z}_v$, denoted $\mathbb{Z}_v^\times$, are the elements in $\mathbb{Z}_v$ relatively prime to v. The set of *non-units* $\mathbb{Z}_v \setminus \mathbb{Z}_v^\times$ of $\mathbb{Z}_v$ are the elements in $\mathbb{Z}_v$ which are not units. The *Euler ϕ-function* of a positive integer v is the number of positive integers between 1 and v which are relatively prime to v. Observe that $|\mathbb{Z}_v^\times| = \phi(v)$.

Example 2.1. Let $\mathbb{Z}_5 = \{0, 1, 2, 3, 4\}$ and $x = 2$. The order of $\mathbb{Z}_5$ is 5, since it has only five elements. The order of the element $2 \in \mathbb{Z}_5$ is 5, since 5 is the smallest positive integer such that $5 \cdot 2 = 0$ in $\mathbb{Z}_5$, therefore x is a generator of $\mathbb{Z}_5$. The units of $\mathbb{Z}_5$ are $\mathbb{Z}_5^\times = \{1, 2, 3, 4\}$, and therefore $\phi(5) = 4$. The element $0 \in \mathbb{Z}_5$ is the only nonunit of $\mathbb{Z}_5$.

Now, we introduce the terms from graph theory. A *graph* $G = G(V, E)$ consists of a set of *vertices* V and a set of *edges* E; an edge is a 2-element subset of V. We also say $V(G)$ is the set of vertices of G and $E(G)$ is the set of edges of G. In this work, we focus on finite simple graphs; those are graphs with a finite number of vertices, without multiple edges or loops. For a positive integer m, the *complete graph*, denoted K_m, is a graph on m vertices containing every possible edge. See Figure 4(a) for an illustration of K_7. For a graph $H = H(V', E')$, we say H is a *subgraph*

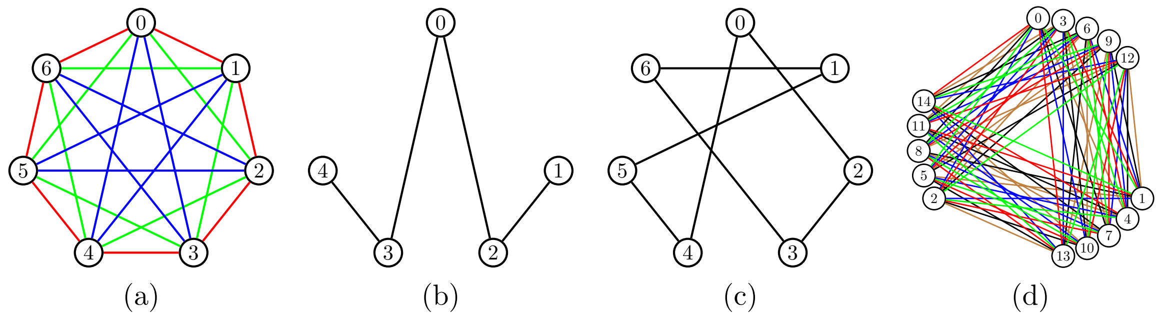

Figure 4. Examples of Graphs

In (a), we give the complete graph K_7. In (b) and (c), we have two example subgraphs of K_7, the latter of which is a Hamilton cycle. In (d), we have the complete multipartite graph $K_{3\times 5}$.

of G if $V' \subseteq V$ and $E' \subseteq E$. See Figure 4(b) and 4(c) for a subgraph of K_7. The *complete multipartite graph* $K_{m\times n}$ is a graph with mn vertices partitioned into m parts of size n where the only edges present are between vertices that are not in the same part. See Figure 4(d) for a graph of $K_{3\times 5}$. In this work, we focus mainly on complete graphs and complete multipartite graphs.

2.2 CYCLIC DECOMPOSITION AND SYMMETRY

Now, we introduce the necessary terminologies to describe the graph decompositions discussed in this paper. Let X be a finite set. First, we define a partition of X to be a set of k subsets $\{X_1, X_2, \dots, X_k\}$ for some $k \geq 1$, such that $X_1 \cup X_2 \cup \cdots \cup X_k = X$ and $X_i \cap X_j = \emptyset$ whenever $i, j \in \{1, \dots, k\}$, and $i \neq j$. Now, let G be a graph, and $H_1, H_2, \dots, H_k$ be subgraphs of G for some integer $k \geq 1$. We say $\{H_1, H_2, \dots, H_k\}$ is a *decomposition* of G if $\{E(H_1), E(H_2), \dots, E(H_k)\}$ is a partition of $E(G)$. For an integer $k \geq 3$, let $(c_0, c_1, \dots, c_{k-1})$ denote the graph with vertex set $\{c_i : i \in \mathbb{Z}_k\}$ and edge set $\{\{c_i, c_{i+1}\} : i \in \mathbb{Z}_k\}$. Call this graph a *$k$-cycle* (or simply a *cycle*), and if G contains a k-cycle with $k = |V(G)|$, then we call the cycle a *Hamilton cycle*. A *Hamilton cycle decomposition* (abbreviated HCD) of G is a decomposition of G into Hamilton cycles. See Figures 2 and 5 for Hamilton cycle decompositions of K_7 and $K_{3\times 5}$, respectively.

We now introduce some definitions which were initially given in [4], describing structures of a decomposition. Let G be a graph with v vertices, H be a subgraph of G, and $\mathcal{C}$ be a decomposition of G. Let σ be a permutation of $V(G)$. We define $\sigma(H)$ as the graph with vertex

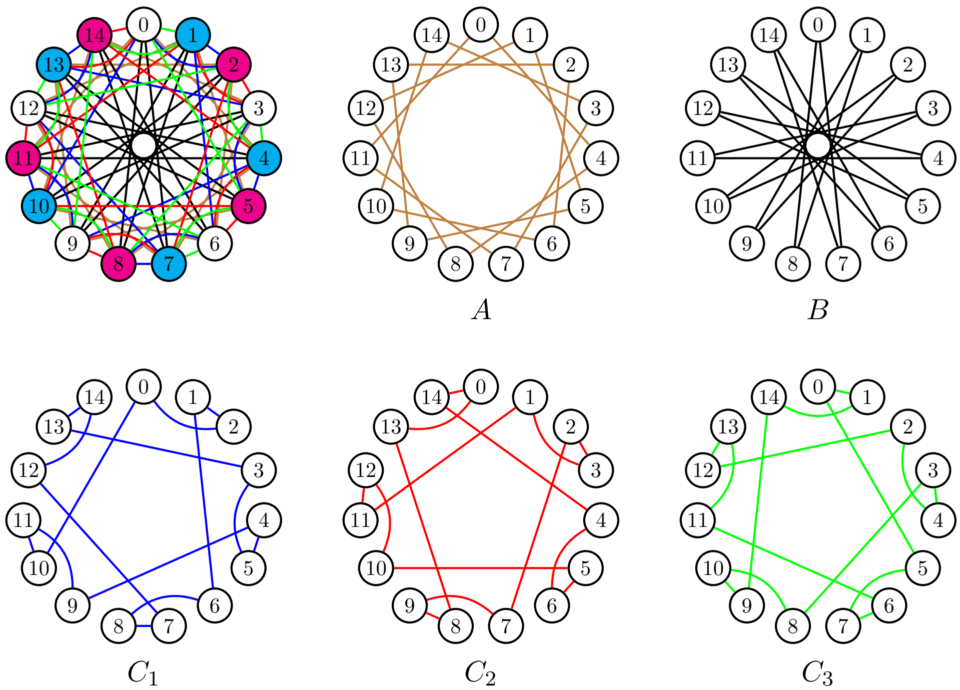

Figure 5. Cyclic 5-symmetric Hamilton cycle decomposition of the graph $K_{3\times5}$
Here, we represent the complete multipartite graph $K_{3\times5}$ from Figure 4 as the Cayley graph $X(15;\pm\{1,2,4,5,7\})$, and give a Hamilton cycle decomposition of it. Cycles A and B are each 15-symmetric, which are also 5-symmetric and are generated by the base paths $[0]_4$ and $[0]_7$, respectively. Cycles C_1, C_2 and C_3 are each 5-symmetric and are generated by the base path $[0,2,1]_6$. Altogether, $K_{3\times5}$ has a cyclic 5-symmetric HCD.

set $V(\sigma(H)) = \sigma(V(H))$ and $E(\sigma(H)) = \{\{\sigma(x),\sigma(y)\} : \{x,y\} \in E(H)\}$. We say that H is *σ-invariant* if $\sigma(H) = H$, and σ *acts* on $\mathcal{C}$ if for each $C \in \mathcal{C}, \sigma(C) \in \mathcal{C}$. If σ acts on $\mathcal{C}$ and σ is also a v-cycle, then we say $\mathcal{C}$ is a *cyclic decomposition* of G. If σ is a v-cycle on $V(G)$ and n is a divisor of v, we say H is *n-symmetric* if H is $\sigma^{\frac{v}{n}}$-invariant, and furthermore we say that $\mathcal{C}$ is *n-symmetric* if each subgraph in $\mathcal{C}$ is itself n-symmetric. In practice, for most examples, unless specified otherwise, we let G have its vertex set be labeled by $\mathbb{Z}_v$, and σ be the permutation on $\mathbb{Z}_v$ such that for all $x \in \mathbb{Z}_v$, $\sigma(x) = x + 1$.

Example 2.2. Observe that the Hamilton cycle decomposition of K_7 given in Figure 2 is 7-symmetric. In addition the Hamilton cycles A and B of $K_{3\times5}$ in Figure 5 are 15-symmetric, while the cycles C_1, C_2 and C_3 are 5-symmetric. Observe that $\sigma(A) = A$ and $\sigma(B) = B$ and for any $C \in \{C_1, C_2, C_3\}$, $\sigma^3(C) = C$. The best way to look at symmetry is to count how many non-trivial powers of σ send a cycle back to itself. For example, applying $\sigma^0, \sigma^3, \sigma^6, \sigma^9, \sigma^{12}$ to cycle C_1 send C_1 back to itself.

Since A and B are σ-invariant, they are also σ^3-invariant. So, every cycle in the decomposition of $K_{3\times 5}$, is σ^3-invariant, and thus $K_{3\times 5}$ has a 5-symmetric Hamilton cycle decomposition.

We can now generalize to these established properties of symmetric Hamilton cycle decompositions.

Lemma 2.3. *Suppose a graph G has a decomposition $\{H_1, H_2, \ldots, H_k\}$ for some $k \geq 1$, and H_i has an n_i-symmetric Hamilton cycle decomposition for each $i \in \{1, 2, \ldots, k\}$. Then G has an n-symmetric Hamilton cycle decomposition, where $n = \gcd(n_1, n_2, \ldots, n_k)$.*

We only seek n-symmetry in the Hamilton cycle decompositions of the graph $K_{m\times n}$. The justification is given as follows:

Lemma 2.4. *An n-symmetric Hamilton cycle decomposition of a graph G is also a t-symmetric Hamilton cycle decomposition for any divisor t of n.*

Proof. Suppose G has v vertices. Let $\mathcal{C}$ be an n-symmetric Hamilton cycle decomposition of G. Let t be a divisor of n and let $n = td$ for some d. Suppose $C \in \mathcal{C}$, then by symmetry of $\mathcal{C}$, we have that $\sigma^{\frac{v}{n}}(C) = C$. Then $\sigma^{\frac{v}{t}}(C) = \sigma^{\frac{vd}{td}}(C) = \sigma^{\frac{vd}{n}}(C) = (\sigma^{\frac{v}{n}})^d(C) = C$. So, C is t-symmetric, and thus $\mathcal{C}$ is t-symmetric. $\square$

The following lemma highlight how symmetry restricts the edges that can be present in a graph with a symmetric Hamilton cycle decomposition. We use that the complement of a multipartite graph is the union of disjoint complete graphs, that is $\overline{K_{m\times n}} = mK_n$.

Lemma 2.5. *Suppose a graph G has v vertices, where $n \mid v$, and suppose G has an n-symmetric Hamilton cycle decomposition. Then $G \leq K_{\frac{v}{n}\times n}$; that is, G is a subgraph of the complete multipartite graph $K_{\frac{v}{n}\times n}$.*

Proof. Let $\mathcal{C}$ be an n-symmetric Hamilton cycle decomposition of G. Then, for $C \in \mathcal{C}$, we have that $\sigma^{\frac{v}{n}}(C) = C$. Let $V(G) = \mathbb{Z}_v$. Observe that edges

$$E = \{\{i, i + kv/n\} : i \in \mathbb{Z}_v \text{ and } k \in \{0, 1, \ldots, n-1\}\}$$

form the edges of the graph $\frac{v}{n}K_n$ which is $\frac{v}{n}$ disjoint copies of K_n. So, if $\frac{v}{n}K_n \leq \overline{G}$, then, $G \leq \overline{\frac{v}{n}K_n} = K_{\frac{v}{n}\times n}$. Thus, to show that $G \leq K_{\frac{v}{n}\times n}$, it sufficient to show that for all $i \in \mathbb{Z}_v$ and $k \in \{0, 1, \ldots, n-1\}$, the edge $\{i, i + k\frac{v}{n}\} \notin E(G)$.

Suppose to the contrary that $e = \{i, i + k\frac{v}{n}\} \in E(G)$, for some $i \in \mathbb{Z}_v$ and $k \in \{0, 1, \ldots, n-1\}$. Then $e \in E(C)$ for some $C \in \mathcal{C}$. Then $\sigma^{j\frac{v}{n}}(e) \in E(C)$ for all $j \geq 0$ since $\mathcal{C}$ is n-symmetric, and in particular, we have that $\sigma^{jk\frac{v}{n}}(e) \in E(C)$. Let g be the order of k in $\mathbb{Z}_n$. So, the g-cycle $(i, i + k\frac{v}{n}, i + 2k\frac{v}{n}, \ldots, i + (g-1)k\frac{v}{n})$ is a subgraph of C, which recall is a Hamilton cycle of length v. This is a contradiction since $g < v$. □

Finally, we prove why it suffices to only find n-symmetry for the graph $K_{m\times n}$. In particular, if $K_{m\times n}$ has an n-symmetric Hamilton cycle decomposition, this classifies all types of symmetry.

Lemma 2.6. *If $K_{m\times n}$ has a t-symmetric Hamilton cycle decomposition, then t is a divisor of n.*

Proof. Let $\mathcal{C}$ be a t-symmetric Hamilton cycle decomposition of $K_{m\times n}$. By Lemma 2.5, $K_{m\times n} \leq K_{m'\times t}$ where $m't = mn$. Observe that $\overline{K_{m\times n}} = mK_n$, and $\overline{K_{m'\times t}} = m'K_t$. The only way m' disjoint copies of K_t can be a subgraph of m copies of K_n is if t divides n. □

2.3 BASE PATHS

In this section we introduce the concept of base paths [5] and how we use them to find cyclic and symmetric Hamilton cycle decompositions. Let v be a positive integer and $A \subseteq \mathbb{Z}_v \backslash \{0\}$ such that $A = -A$; that is, if $x \in A$, then $-x \in A$. We say that the subset A is *good* when it has this property. We define $\partial(G) = \{i - j \in \mathbb{Z}_v : \{i, j\} \in E(G)\}$ for a graph G with vertex set $\mathbb{Z}_v$. Note that $\partial(G)$ is a good set. For a set of graphs, $\mathcal{G} := \{G_1, G_2, \ldots, G_k\}$ each with vertex set $\mathbb{Z}_v$. Then $\partial(\mathcal{G}) = \partial(G_1) \cup \partial(G_2) \cup \cdots \cup \partial(G_k)$. We define the *Cayley graph* $G = X(v; A)$ as a graph with vertex set $V(G) = \mathbb{Z}_v$ and $\{x, y\} \in E(G)$ if and only if $y - x \in A$. Observe that the complete graphs, K_m and $K_{m\times n}$ are Cayley graphs, in particular, $K_m = X(m; \mathbb{Z}_m \backslash \{0\})$ and $K_{m\times n} = X(mn; \mathbb{Z}_{mn} \backslash m\mathbb{Z}_{mn})$. See Figure 2 for a Cayley graph of $K_7 = X(7; \mathbb{Z}_7 \backslash \{0\})$ and Figure 5 for a Cayley graph of $K_{3\times 5} = X(15; \mathbb{Z}_{15} \backslash \{0, 3, 6, 9, 12\})$.

Let d and t be positive integers such that $v = dt$. Let $a_1, \ldots, a_d$ be distinct nonzero elements of $\mathbb{Z}_v$. Let $P = [a_0 = 0, a_1, a_2, \ldots, a_{d-1}]_{a_d}$ denote the path with edges $\{a_i, a_{i+1}\}$ for each $0 \le i \le d-1$. Also, for each i, let $\Delta a_i = a_{i+1} - a_i$. We say that a path P is a *base path* if and only if it satisfies the following conditions:

- $\{a_1, \ldots, a_{d-1}\} = \{1, \ldots, d-1\} \pmod d$ and d divides a_d,
- $\Delta a_i \neq \pm \Delta a_j$ for distinct $i, j \in \{0, 1, \ldots, d-1\}$, and
- a_d generates the set $d\mathbb{Z}_v$; in other words, $|a_d| = t$ in $\mathbb{Z}_v$.

The length of a base path P is denoted $\mathrm{len}(P)$.

Let $C(P) = P \cup \sigma^{a_d}(P) \cup \sigma^{2a_d}(P) \cup \cdots \cup \sigma^{(t-1)a_d}(P)$. It follows from the above criteria that $C(P)$ is a t-symmetric Hamilton cycle. Furthermore, let $C^i(P) = \sigma^i(C(P))$. Then $\mathcal{C}(P) = \{C(P), C^1(P), \ldots, C^{d-1}(P)\}$ is a t-symmetric cyclic Hamilton cycle decomposition of $X(v; \partial(P))$.

Example 2.7. Consider the Cayley graph $X(15, \pm\{1, 2, 5\})$. Let $P = [0, 2, 1]_6$. Then $C(P) = (0, 2, 1, 6, 8, 7, 12, 14, 13, 3, 5, 4, 9, 11, 10, 0)$. Observe that $C(P)$ is the cycle C_1 given in Figure 5. Subsequently, $C^1(P)$ and $C^2(P)$ are the cycles C_2 and C_3 in Figure 5 respectively.

So, $\mathcal{C} = \{C(P), C^1(P), C^2(P)\}$ is a 5-symmetric cyclic Hamilton cycle decomposition of $X(15; \pm\{1, 2, 5\})$.

Let $G = X(v; A)$ for some good subset A of $\mathbb{Z}_v$. Suppose there exist

- integers $d_1, d_2, \ldots, d_k$, $t_1, t_2, \ldots, t_k$, and
- base paths $P_1, P_2, \ldots, P_k$ for which $\{\partial(P_1), \ldots, \partial(P_k)\}$ is a partition of A,

such that $v = d_i \cdot t_i$ for each i, $t = \gcd(t_1, t_2, \ldots, t_k)$, and each P_i has length d_i. Then $\mathcal{C}(P_1) \cup \mathcal{C}(P_2) \cup \cdots \cup \mathcal{C}(P_k)$ is a cyclic Hamilton cycle decomposition of G which is t-symmetric.

Example 2.8. Observe that $K_{3\times5} = X(15; \pm\{1,2,4,5,7\})$. Let $P_1 = [0]_4$, $P_2 = [0]_7$, and $P_3 = [0,2,1]_6$. Observe that P_1 and P_2 are base paths of length 1, and P_3 is a base path of length 3 with $\partial(P_1) = \pm\{4\}$, $\partial(P_2) = \pm\{7\}$ and $\partial(P_3) = \pm\{1,2,5\}$. We have that

$$\begin{aligned}
C(P_1) &= (0,4,8,12,1,5,9,13,2,6,10,14,3,7,11,0),\\
C(P_2) &= (0,7,14,6,13,5,12,4,11,3,10,2,9,1,8,0),\\
C(P_3) &= (0,2,1,6,8,7,12,14,13,3,5,4,9,11,10,0),\\
C^1(P_3) &= (1,3,2,7,9,8,13,0,14,4,6,5,10,12,11,1), \text{and}\\
C^2(P_3) &= (2,4,3,8,10,9,14,1,0,5,7,6,11,13,12,2).
\end{aligned}$$

Observe that $\mathcal{C}(P_1) = \{C(P_1)\}$ is a decomposition of $X(15; \pm\{4\})$, $\mathcal{C}(P_2) = \{C(P_2)\}$ is a decomposition of $X(15; \pm\{7\})$ and $\mathcal{C}(P_3) = \{C(P_3), C^1(P_3), C^2(P_3)\}$ is a decomposition of $X(15; \pm\{1,2,5\})$. Equivalently, $\{\partial(P_1), \partial(P_2), \partial(P_3)\}$ is a partition of $\mathbb{Z}_{15} \backslash 3\mathbb{Z}_{15}$. Therefore $\mathcal{C}(P_1) \cup \mathcal{C}(P_2) \cup \mathcal{C}(P_3)$ is a cyclic 5-symmetric Hamilton cycle decomposition of $K_{3\times5}$. See Figure 5.

It follows from the above argument that to find a cyclic t-symmetric Hamilton cycle decomposition of a Cayley graph $X(v; A)$, it is sufficient to find base paths $P_1, P_2, \ldots, P_k$ such that

- each P_i has a length which is a divisor of $\frac{v}{t}$, and
- $\{\partial(P_1), \ldots, \partial(P_k)\}$ is a partition of A.

Therefore, our strategy for finding a cyclic t-symmetric Hamilton cycle decomposition of $X(v; A)$ is to find such a set of base paths. We call such a set a *valid set of base paths.*

Example 2.9. The following example is attributed to Buratti [5]. Observe that $K_{21} = X(21; \mathbb{Z}_{21} \backslash \{0\})$. Let $P_1 = [0,2,1]_{15}$ and $P_2 = [0,6,15,12,16,11,3]_{14}$. Then $\partial(P_1) = \pm\{2,1,7\}$, $\partial(P_2) = \pm\{6,9,3,4,5,8,10\}$ and $\{\partial(P_1), \partial(P_2)\}$ is a partition of $\mathbb{Z}_{21} \backslash \{0\}$. Observe that $\{P_1, P_2\}$ is a valid set of base paths of K_{21}. Note that $\mathcal{C}(P_1)$ is a 3-symmetric cyclic Hamilton cycle decomposition of $X(21; \partial(P_1))$ and $\mathcal{C}(P_2)$ is a 7-symmetric Hamilton cycle decomposition of $X(21; \partial(P_2))$. So $\mathcal{C}(P_1) \cup \mathcal{C}(P_2)$ is a cyclic 1-symmetric Hamilton cycle decomposition of K_{21}. See Figure 6.

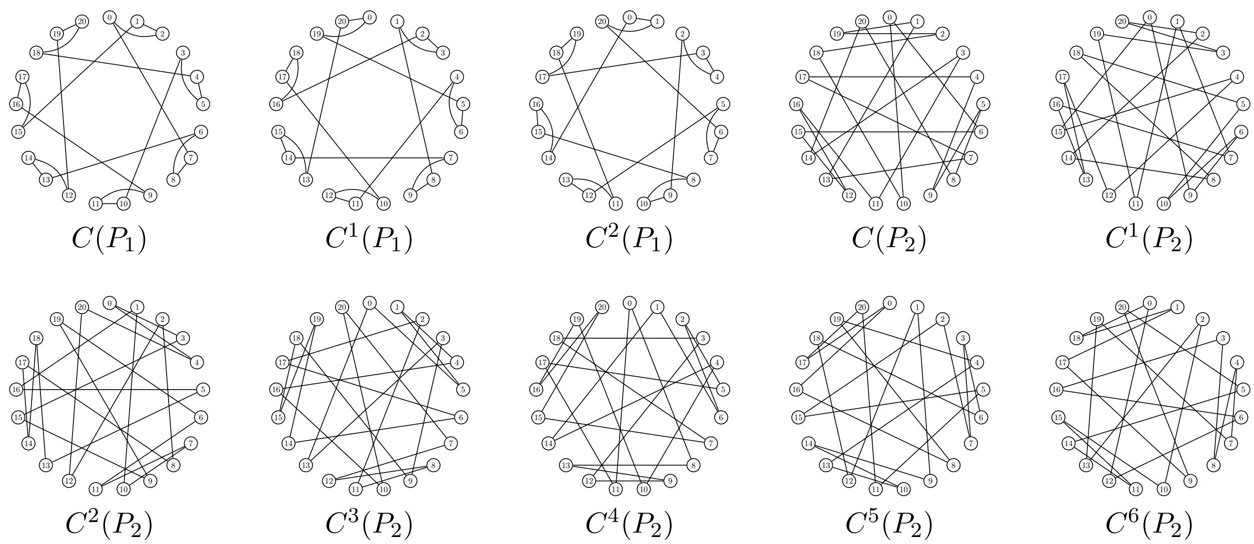

Figure 6. A cyclic decomposition of K_{21}

The first 3 cycles are generated by the base path $P_1 = [0, 2, 1]_{15}$, while the latter 7 are generated by the base path $P_2 = [0, 6, 15, 12, 16, 11, 3]_{14}$.

Not only is it sufficient to find a valid set of base paths to produce a Hamilton cycle decomposition, it is in fact necessary that such a valid set of base paths exists.

Theorem 2.10. *Let $\mathcal{C}$ be a cyclic Hamilton cycle decomposition of the Cayley graph $X(v; A)$ for a good subset $A \subseteq \mathbb{Z}_v \backslash \{0\}$. Then there exists a valid set of base paths which produces $\mathcal{C}$.*

Proof. If A is empty, then $\mathcal{C}$ is the empty set and is trivially generated from base paths, otherwise, we proceed by induction on $|A|$. Select $C \in \mathcal{C}$ and let $a_1, a_2, \ldots, a_{v-1}$ be defined so that $C = (a_0 = 0, a_1, a_2, \ldots, a_{v-1})$. Note that this is simply an ordering of $\mathbb{Z}_v$. Since $\mathcal{C}$ is cyclic, then $\sigma^i(C) \in \mathcal{C}$, for each integer i. Since $\mathcal{C}$ is finite, there exist $0 \leq i < j$ such that $\sigma^i(C) = \sigma^j(C)$. Applying σ^{-i} to both sides, we have $C = \sigma^0(C) = \sigma^{j-i}(C)$. Note that $j - i$ is positive. So, there exists a smallest positive integer d such that $\sigma^d(C) = C$. We claim that
$P = [a_0 = 0, a_1, \ldots, a_{d-1}]_{a_d}$ is a base path that generates the cycle C.

Observe that there exist integers a, b such that $\gcd(d, v) = ad + bv$. Recall that $\sigma^v(C) = C$. Then $\sigma^{\gcd(d,v)}(C) = \sigma^{ad+bv}(C) = (\sigma^d)^a[(\sigma^v)^b(C)] = C$. So, $d \leq \gcd(d, v)$, by minimality of d. But, $\gcd(d, v) \leq d$, so, $d = \gcd(d, v)$. Thus d divides v. Then $v = dt$ for some integer t. Since $d \leq \frac{v-1}{2}$, then $d \leq \frac{v}{3}$ and $t \geq 3$.

Since $\sigma^d(C) = C$, σ^d is a dihedral action (a permutation which acts on a polygon) on C. So, σ^d can either act as a rotation or a reflection. If σ^d is a reflection, then $|\sigma^d| = 2$, but $|\sigma^d| = t \geq 3$, so σ^d is a rotation. Then there exists $k \in \mathbb{Z}_v$ such that for each $i \in \mathbb{Z}_v$, $\sigma^d(a_i) = a_{i+k}$. Then, by definition of σ, $a_0 + d = a_k$. Recall that $\sigma^v = \epsilon$, the identity function. So, $a_0 = \sigma^v(a_0) = (\sigma^d)^t(a_0) = a_{tk}$. So, $tk \equiv 0 \pmod{v}$, and thus $k \equiv 0 \pmod{d}$ which implies $d \mid k$. Let $|k| = \ell$ in $\mathbb{Z}_v$, then $\ell k \equiv 0 \pmod{v}$. Observe that, in $\mathbb{Z}_v$, ℓ is a divisor of t. So, for each $i \in \mathbb{Z}_v$, $(\sigma^d)^\ell(a_i) = a_{i+\ell k} = a_i$ and thus $(\sigma^d)^\ell = \epsilon$. Since, $|\sigma| = v$, we have that $v = dt$ divides $d\ell$, which implies t divides ℓ. So, $t = \ell$ and $|k| = t$ in $\mathbb{Z}_v$. Then $t = |k| = v/\gcd(k, v) = (dt)/\gcd(k, v)$. So, $\gcd(k, v) = d$. Hence, $\langle d \rangle = \langle k \rangle$, and so they generate the same cyclic subgroup. Then

$$
\begin{aligned}
\{a_0, a_k, a_{2k}, \ldots, a_{(t-1)k}\} &= \{a_0, \sigma^d(a_0), \sigma^{2d}(a_0), \ldots, \sigma^{(t-1)}(a_0)\}, && \text{since } \sigma^d(a_0) = a_{0+k} = a_k, \\
\{a_0, a_k, a_{2k}, \ldots, a_{(t-1)k}\} &= \{0, d, 2d, \ldots, (t-1)d\}, && \text{since } \sigma^d(a_0) = a_0 + d = d, \text{ and} \\
\{a_0, a_k, a_{2k}, \ldots, a_{(t-1)k}\} &= \{a_0, a_d, a_{2d}, \ldots, a_{(t-1)d}\}, && \text{since } \langle d \rangle = \langle k \rangle \text{ in } \mathbb{Z}_v.
\end{aligned}
$$

So, a_d is a multiple of d. This gives us part of Condition 1 for proving P is a base path.

Since, $\langle d \rangle = \langle k \rangle$, there exists $u, \ell \in \{0, 1, \ldots, t-1\}$ such that $d = ku \pmod{v}$ and $k = d\ell \pmod{v}$. So, $d = ku = d\ell u \pmod{v}$. So, $\ell u = 1 \pmod{t}$. So, ℓ and u are units in $\mathbb{Z}_t$ and so ℓ and u are relatively prime to t. So, $\gcd(u, t) = 1$, and hence $\gcd(du, dt) = d$. Observe that $a_d = a_{ku} = \sigma^{du}(a_0) = \sigma^{du}(0) = du$, so $\gcd(a_d, v) = d$. This implies that $v/\gcd(a_d, v) = t$. So, $|a_d| = t$. Thus, P satisfies Condition 3 for being a base path.

Suppose $a_i \equiv a_j \pmod{d}$ for some $0 \leq i, j \leq d-1$. Then $a_j = a_i + sd$, for some integer s. So, $a_j = (\sigma^d)^s(a_i) = a_{i+sk}$. Then $j = i + sk \pmod{v}$. So, $j = i + sk + vr$, for some integer r. Thus, $j - i = sk + vr = s(d\ell) + (dt)r = d(s\ell + tr)$. So, $d \mid j - i$, and hence, $i = j$. Thus P satisfies the remaining part of Condition 1.

We have that $\{\sigma^i(C) : i \in \mathbb{Z}\} = \{C, \sigma(C), \ldots, \sigma^{d-1}(C)\}$ which then partitions a Cayley graph $X(v; \partial(P))$. Note that if C has y edges of length $\pm x$, then $\sigma(C)$ also has y edges of length $\pm x$. So, $\sigma^i(C)$ for any arbitrary i, has y edges of length $\pm x$. Since $X(v; \partial(P))$ necessarily contains exactly v edges of length $\pm x$, we have that $dy = v$, and thus $y = t$. So, every edge length in C appears t times in C. So, there are exactly d distinct edge lengths in C. Further note that, if edge $\{a_i, a_{i+1}\}$ has length $\pm x$, then, the rotation $\sigma^d(\{a_i, a_{i+1}\}) = \{a_{i+k}, a_{i+k+1}\}$ also has length $\pm x$. So, $\{a_i, a_{i+1}\}, \{a_{i+k}, a_{i+k+1}\}, \{a_{i+2k}, a_{i+2k+1}\}, \ldots, \{a_{i+(t-1)k}, a_{i+(t-1)k+1}\}$ are all distinct edges in C of length $\pm x$, (there are all t of them). Observe that these edges are equivalently $\{a_i, a_{i+1}\}, \{a_{i+d}, a_{i+d+1}\}, \{a_{i+2d}, a_{i+2d+1}\}, \ldots, \{a_{i+(t-1)d}, a_{i+(t-1)d+1}\}$. Then, edges of the same length occur at least d edges apart in C. Therefore, any path with d edges in C consists of distinct edge lengths. This shows that P satisfies Condition 2 of being a base path. Hence, P is a base path which generates the cycle C and so, $\{C, \sigma(C), \ldots, \sigma^{d-1}(C)\}$ is a cyclic Hamilton cycle decomposition of $X(v; \partial(P))$.

Consider $\mathcal{C}' = \mathcal{C} \backslash \{C, \sigma(C), \ldots, \sigma^{d-1}(C)\}$. Then, $\mathcal{C}'$ is a cyclic decomposition of the Cayley graph $X(v; A \backslash \partial(P))$. So, by induction, it follows $\mathcal{C}'$ is generated by a valid set of base paths $\{P_1, \ldots, P_k\}$ for some k and hence $\{\partial(P_1), \ldots, \partial(P_k)\}$ partitions $A \backslash \partial(P)$. Therefore $\{P, P_1, \ldots, P_k\}$ is a set of base paths for which $\partial(P), \partial(P_1), \ldots, \partial(P_k)$ is a partition of A. Hence $\{P, P_1, \ldots, P_k\}$ is a valid set of base paths of $X(v; A)$ which generates the decomposition $\mathcal{C}$. $\square$

At times it will be convenient to rewrite our vertices in terms of direct products when it comes to constructing our base paths. Now, we restate the conditions that must be satisfied for the paths to be considered base paths in direct products. First, we define direct products and give a relevant number theory result which shows isomorphisms between the vertex sets.

Let H and K be arbitrary rings. The *direct product* of H and K (denoted $H \times K$) is defined by $H \times K = \{(h, k) : h \in H, k \in K\}$ with operations done component wise. Again, let G be a graph with vertices indexed by $\mathbb{Z}_v$ and $\sigma(x) = x + 1$. First we state the *Chinese Remainder Theorem* as it allows us to go back and forth between representations of the same ring.

Lemma 2.11. *Let m and n be relatively prime, positive integers and $m, n \geq 2$. Then, there exists a ring isomorphism , $\sigma : \mathbb{Z}_{mn} \to \mathbb{Z}_m \times \mathbb{Z}_n$ given by $\sigma(x) = (x \pmod m, x \pmod n)$.*

Now, we highlight three different methods for labelling vertices indexed by $\mathbb{Z}_v$.

Let d, m and n be positive integers such that $v = mn$, $\gcd(m, n) = 1$ and d be a divisor of m. Then $\mathbb{Z}_v \cong \mathbb{Z}_m \times \mathbb{Z}_n$. Let $(x_1, y_1), (x_2, y_2), \ldots, (x_d, y_d) \in \mathbb{Z}_m \times \mathbb{Z}_n$. Let $P = [(x_0, y_0) = (0, 0), (x_1, y_1), (x_2, y_2), \ldots, (x_{d-1}, y_{d-1})]_{(x_d, y_d)}$ denote the path with edges $\{(x_i, y_i), (x_{i+1}, y_{i+1})\}$ for each $0 \leq i \leq d-1$. Also for each i, $\Delta(x_i, y_i) = (x_{i+1}, y_{i+1}) - (x_i, y_i)$.

Then $\partial(P) = \{\pm\Delta(x_i, y_i) : 0 \le i \le d-1\}$. The conditions which must be satisfied for P to be a base path are:

- $\{x_1, \ldots, x_{d-1}\} = \{1, \ldots, d-1\} \pmod d$ and $d \mid x_d$,
- $\Delta(x_i, y_i) \neq \pm\Delta(x_j, y_j)$ for distinct $i, j \in \{0, 1, \ldots, d-1\}$, and
- (x_d, y_d) has order $\frac{mn}{d}$ in $\mathbb{Z}_m \times \mathbb{Z}_n$.

Now, let $v = pn, n = n'p^t$, $\gcd(p, n') = 1$ and $t \ge 1$. Then $\mathbb{Z}_v \cong \mathbb{Z}_{p^{t+1}} \times \mathbb{Z}_{n'}$. Let $P = [(x_0, y_0) = (0, 0), (x_1, y_1), (x_2, y_2), \ldots, (x_{p-1}, y_{p-1})]_{(x_p, y_p)}$. Again for each i, $\Delta(x_i, y_i) = (x_{i+1}, y_{i+1}) - (x_i, y_i)$, and $\partial(P) = \{\pm\Delta(x_i, y_i) : 0 \le i \le p-1\}$. Then the conditions which must be satisfied for P to be a base path are:

- $\{x_1, \ldots, x_{p-1}\} = \{1, \ldots, p-1\} \pmod p$ and $p \mid x_p$,
- $\Delta(x_i, y_i) \neq \pm\Delta(x_j, y_j)$ for distinct $i, j \in \{0, 1, \ldots, p-1\}$, and
- (x_p, y_p) has order n; equivalently $p \mid x_p$, $p^2 \nmid x_p$, and y_p is a unit of $\mathbb{Z}_{n'}$.

Finally, let $v = mnq$, so that m, n, q are pairwise relatively prime. Let d be a divisor of m. Then $\mathbb{Z}_v \cong \mathbb{Z}_m \times \mathbb{Z}_n \times \mathbb{Z}_q$. Let $(x_1, y_1, z_1), (x_2, y_2, z_2), \ldots, (x_d, y_d, z_d) \in \mathbb{Z}_m \times \mathbb{Z}_n \times \mathbb{Z}_q$. Let $P = [(x_0, y_0, z_0) = (0, 0, 0), (x_1, y_1, z_1), (x_2, y_2, z_2), \ldots, (x_{d-1}, y_{d-1}, z_{d-1})]_{(x_d, y_d, z_d)}$ denote the path with edges $\{(x_i, y_i, z_i), (x_{i+1}, y_{i+1}, z_{i+1})\}$ for each $0 \le i \le d-1$. Let $\Delta(x_i, y_i, z_i) = (x_{i+1}, y_{i+1}, z_{i+1}) - (x_i, y_i, z_i)$. Then $\partial(P) = \{\pm\Delta(x_i, y_i, z_i) : 0 \le i \le d-1\}$. Then, the conditions which must be satisfied for P to be a base path are:

- $\{x_1, \ldots, x_{d-1}\} = \{1, \ldots, d-1\} \pmod d$ and $d \mid x_d$,
- $\Delta(x_i, y_i, z_i) \neq \pm\Delta(x_j, y_j, z_j)$ for distinct $i, j \in \{0, 1, \ldots, d-1\}$, and
- (x_d, y_d, z_d) has order $\frac{mnq}{d}$. That is, x_d has order $\frac{m}{d}$ and both y_d and z_d are units in $\mathbb{Z}_n$ and $\mathbb{Z}_q$ respectively.

To obtain an n-symmetry, we need to find a valid set of base paths, all of whose lengths are divisors of m. In particular, its enough to find a valid set of base paths for a subgraph of $K_{m \times n}$ which is a Cayley graph that contains all the non-unit edge lengths, since we can use all the unit edge lengths to form base paths of length 1.

CHAPTER 3

DECOMPOSITIONS OF $K_{p\times n}$ FOR p PRIME

In this chapter, we prove that $K_{p\times n}$, where p is prime and n is an odd integer, has a cyclic n-symmetric Hamilton cycle decomposition. We focus on doing this with three considerations. First, we analyze the circumstances where all prime powers in the factorization of n are less than p, then expand to include all n which does not have p as a divisor, and finally we consider all odd n. We address those situations separately and, at the end, bring it all together to get the final result.

Throughout this section, we identify $V(K_{p\times n})$ by $\mathbb{Z}_p \times \mathbb{Z}_n$ and $K_{p\times n} = X(pn; (\mathbb{Z}_p \times \mathbb{Z}_n)\backslash p(\mathbb{Z}_p \times \mathbb{Z}_n))$, or in other words, $X(pn; (\mathbb{Z}_p\backslash\{0\} \times \mathbb{Z}_n))$.

3.1 CASES WHERE ALL PRIME POWERS DIVISORS OF n ARE SMALL

First, in this section, we go over some useful lemmas involving the existence of balanced functions and some properties of the Euler-totient function. We use that to prove the circumstances under which certain decompositions exist, and we demonstrate that if n has all of its prime powers in its factorization less than p, then those conditions are met.

A function $\alpha : X \to Y$ is *balanced* if the cardinalities of the inverse images of each pair of elements in Y differ by 1.

Example 3.1. Let $X = \mathbb{Z}_5$ and $Y = \mathbb{Z}_2$. Define $\alpha(x) = x \pmod 2$. Then $|\alpha^{-1}(0)| = 3$ and $|\alpha^{-1}(1)| = 2$. Thus, α is balanced.

Lemma 3.2. *Let X and Y be finite sets, then there exists a balanced function from X to Y.*

Proof. Without loss of generality, let $X = \mathbb{Z}_n$ and $Y = \mathbb{Z}_m$, for some positive integers m and n. By the division algorithm, there exist integers q and r such that $n = mq + r$ and $0 \leq r < m$. Define $\alpha : X \to Y$ so that $\alpha(x) = x \pmod m$.

Observe that $|\alpha^{-1}\{0, \ldots, r-1\}| = q + 1$ and $|\alpha^{-1}\{r, \ldots, m-1\}| = q$. So, α is balanced since the cardinality of the pullbacks of each pair of elements in X differ by at most 1. $\square$

Lemma 3.3. *Let n be a positive integer with prime factorization $p_1^{a_1} \cdots p_k^{a_k}$, where $k \geq 1$, $3 \leq p_1 < p_2 < \cdots < p_k$, and $a_i \geq 1$. Then $\phi(n) \geq 2n/p_k$.*

Proof. Since ϕ is multiplicative, we have that

$$\begin{aligned}
\frac{\phi(n)}{n} &= \frac{\phi(p_1^{a_1})}{p_1^{a_1}} \cdot \frac{\phi(p_2^{a_2})}{p_2^{a_2}} \cdots \frac{\phi(p_k^{a_k})}{p_k^{a_k}} \\
&= \frac{p_1^{a_1} - p_1^{a_1-1}}{p_1^{a_1}} \cdot \frac{p_2^{a_2} - p_2^{a_2-1}}{p_2^{a_2}} \cdots \frac{p_k^{a_k} - p_k^{a_k-1}}{p_k^{a_k}} \\
&= \left(1 - \frac{1}{p_1}\right) \cdot \left(1 - \frac{1}{p_2}\right) \cdots \left(1 - \frac{1}{p_k}\right) \\
&\geq \left(1 - \frac{1}{3}\right) \cdot \left(1 - \frac{1}{4}\right) \cdots \left(1 - \frac{1}{p_k}\right) \\
&= \frac{2}{3} \cdot \frac{3}{4} \cdots \frac{p_{k-1}}{p_k} \\
&= \frac{2}{p_k}.
\end{aligned}$$

Thus, $\phi(n) \geq 2n/p_k$. □

Using the above result, we give a method of constructing base paths that give rise to decompositions of $K_{p\times n}$ where n has all of its prime powers in its factorization less than p.

Lemma 3.4. *Let $n \geq 3$ be an odd integer. Let p be an odd prime such that $p \nmid n$ and $p\phi(n) \geq n + 2p - 3$. Then $K_{p\times n}$ has a cyclic n-symmetric Hamilton cycle decomposition.*

Proof. Let $p = 2\ell + 1$. First, we construct a base path which uses the non-unit edge lengths of the form $\{(i, 0) : i \in \mathbb{Z}_p^\times\}$. If $p = 4k + 1$ for an integer k, let P_0 be the path

$$\begin{aligned}
&[(0, 0),(1, 0),(p - 1, 0),(2, 0),(p - 2, 0), \dots ,(k, 0),(p - k, 0), \\
&\qquad (k + 1, 1),(p - k - 1, 0), \dots ,(2k, 1),(p - 2k, 0)]_{(0,1)}.
\end{aligned}$$

Then $\partial(P_0) = \{(i, 0) : i \in \mathbb{Z}_p^\times\} \cup \{\pm(1, -1), \pm(2, -1), \dots , \pm(\ell + 1, -1)\}$ and $|(0, 2)| = n$. If

$p = 4k + 3$ for an integer k, let P_0 be the path

$$[(0,0),(1,0),(-1,0),\ldots,(k,0),(-k,0),(k+1,0),(-k-1,-1),$$
$$(k+2,0),(-k-2,1),\ldots,(2k,0),(-2k,1)]_{(0,2)}.$$

Then $\partial(P_0) = \{(i,0) : i \in \mathbb{Z}_p^\times\} \cup \{\pm(x,1) : 1 \leq x \leq \ell+2, x \neq \ell-1\}$ and $|(0,2)| = n$. In both cases, P_0 is a base path which use all non-unit edge lengths in $K_{p\times n}$ with residue 0 mod n.

Let $D_n = \{x \in \mathbb{Z}_n : 0 < x < \frac{n}{2}$ and x is a nonunit$\}$. Let $\alpha : D_n \to \mathbb{Z}_p^\times$ be a balanced function. Let $Z = \{x \in \mathbb{Z}_n^\times : 1 < x < \frac{n}{2}\}$. Suppose there exists a function $\psi : D_n \to Z$ so that if $d_1 \neq d_2$ and $\alpha(d_1) = \alpha(d_2)$, then $\psi(d_1) \neq \psi(d_2)$. Then, for each $d \in D_n$, define

$$P_d = [(0,0),(\alpha(d),d),(-\alpha(d),0),(2\alpha(d),d),(-2\alpha(d),0),\ldots,(\ell\alpha(d),d),(-\ell\alpha(d),0)]_{(0,\psi(d))}.$$

Then,

$$\partial(P_d) = \{\pm(\alpha(d),d),\pm(2\alpha(d),d),\pm(3\alpha(d),d),\ldots,\pm(2\ell\alpha(d),d),(\ell\alpha(d),\psi(d))\}$$
$$= \{\pm(1,d),\pm(2,d),\ldots,\pm(2\ell,d),\pm(\ell\alpha(d),\psi(d))\}.$$

Observe that $(i,d),(i,-d) \in \partial(P_d)$ for each $i \in \mathbb{Z}_p^\times$. So, $\partial(P_0) \cup \left(\bigcup_{d\in D_n} \partial(P_d)\right)$ contains all nonunit edge lengths in $\partial(K_{p\times n})$. Furthermore, $|\partial(P_d)| = 2p$ for each $d \in D_n$, and for each $d_1, d_2, \in D_n$, $\pm(\ell\alpha(d_1),\psi(d_1)) \neq \pm(\ell\alpha(d_2),\psi(d_2))$. So, for distinct $d_1, d_2 \in D_n$, $\partial(P_{d_1})$ and $\partial(P_{d_2})$ are disjoint.

Therefore $\{P_0\} \cup \{P_d : d \in D_n\}$ is a set of base paths which are valid and uses all non-unit edge lengths in $K_{p\times n}$. So provided that such a function ψ exists, we have a cyclic n-symmetric decomposition of $K_{p\times n}$.

Assume that no such function ψ exists. That means there exists $x \in \mathbb{Z}_p^\times$ such that $|\alpha^{-1}(x)| > |Z|$. Since α is balanced we have that $|\alpha^{-1}(x)| \leq \lceil |D_n|/|\mathbb{Z}_p^\times| \rceil$. Note that $|D_n| = (n - \phi(n) - 1)/2$, $|\mathbb{Z}_p^\times| = p - 1$, and $|Z| = \phi(n)/2 - 1$. So,

$$\left\lceil \frac{n - \phi(n) - 1}{2(p-1)} \right\rceil > \frac{\phi(n)}{2} - 1.$$

Since $\frac{\phi(n)}{2} - 1 \in \mathbb{Z}$, $\frac{n-\phi(n)-1}{2(p-1)} > \frac{\phi(n)}{2} - 1$, and thus $n > p\phi(n) - 2p + 3$. This contradicts our hypothesis. Therefore, such a function ψ exists, and thus we have a cyclic, n-symmetric Hamilton cycle decomposition of $K_{p\times n}$. □

Example 3.5. Following the construction outlined in Lemma 3.4, we build base paths for a cyclic 33-symmetric Hamilton cycle decomposition of $K_{13\times 33}$. Observe that with $p = 13$ and $n = 33$, we have that $p\phi(n) \geq n + 2p - 3$. So, such a decomposition exists and we first have that $P_0 = [(0,0),(1,0),(12,0),(2,0),(11,0),(3,0),(10,0),(4,1),(9,0),(5,1),(8,0),(6,1),(7,0)]_{(0,1)}$.

The set $D_{33} = \{x \in \mathbb{Z}_{33} : 0 < x < \frac{33}{2} \text{ and } x \text{ is a nonunit}\} = \{3, 6, 9, 11, 12, 15\}$. So, we construct the balanced function α and ψ such that for any $d_1, d_2 \in D_{33}$, if $d_1 \neq d_2$ and $\alpha(d_1) = \alpha(d_2)$, then $\psi(d_1) \neq \psi(d_2)$. Using this setup, we produce the base paths P_d and identify their edge lengths as given in Table 1.

Observe that the non-units edge lengths of $K_{13\times 33}$ are contained in $\partial(P_0) \cup \partial(P_3) \cup \cdots \cup \partial(P_{15})$. So, we have that every non-unit edge length is used by one of the base paths in Table 1. Hence, $K_{13\times 33}$ has a cyclic 33-symmetric Hamilton cycle decomposition.

Lemma 3.6. *Let p be an odd prime and n has prime factorization $p_1^{a_1} \cdots p_k^{a_k}$, where $k \geq 1$ and for each $i \in \{1, \ldots, k\}$, $p_i^{a_i} < p$ and $a^i \geq 1$. Then $K_{p\times n}$ has a cyclic n-symmetric Hamilton cycle decomposition.*

Proof. It is sufficient to show that $p\phi(n) \geq n + 2p - 3$, then the result follows from Lemma 3.4.

Suppose n is prime. Then $p > n \geq 3$. So, $(p-1)(n-3) \geq 0$, and thus $p(n-1) \geq n + 2p - 3$. Hence, $p\phi(n) \geq n + 2p - 3$, since $\phi(n) = n - 1$.

Now, suppose n is not prime; then $n > 2p_k$ and recall that $p > p_k$. By Lemma 3.3, we have that $p\phi(n) \geq p(2n)/p_k$. So,

$$p\phi(n) \geq \frac{p(2n)}{p_k} = n\left(\frac{p}{p_k}\right) + 2p\left(\frac{n}{2p_k}\right) \geq n + 2p \geq n + 2p - 3.$$

So $p\phi(n) \geq n + 2p - 3$.

Therefore, in all cases, by Lemma 3.4, $K_{p\times n}$ has a cyclic n-symmetric Hamilton cycle decomposition. □

d	3	6	9	11	12	15
$\alpha(d)$	1	2	3	4	5	6
$\psi(d)$	2	2	2	2	4	4

i	P_i
3	$[(0,0),(1,3),(12,0),(2,3),(11,0),(3,3),(10,0),(4,3),(9,0),(5,3),(8,0),(6,3),(7,0)]_{(0,2)}$
6	$[(0,0),(2,6),(11,0),(4,6),(9,0),(6,6),(7,0),(8,6),(5,0),(10,6),(3,0),(12,6),(1,0)]_{(0,2)}$
9	$[(0,0),(3,9),(10,0),(6,9),(7,0),(9,9),(4,0),(12,9),(1,0),(2,9),(11,0),(5,9),(8,0)]_{(0,2)}$
11	$[(0,0),(4,11),(9,0),(8,11),(5,0),(12,11),(1,0),(3,11),(10,0),(7,11),(6,0),(11,11),(2,0)]_{(0,2)}$
12	$[(0,0),(5,12),(8,0),(10,12),(3,0),(2,12),(11,0),(7,12),(6,0),(12,12),(1,0),(4,12),(9,0)]_{(0,4)}$
15	$[(0,0),(6,15),(7,0),(12,15),(1,0),(5,15),(8,0),(11,15),(2,0),(4,15),(9,0),(10,15),(3,0)]_{(0,4)}$

i	$\partial(P_i)$
3	$\pm\{(1,3),(2,3),(3,3),(4,3),(5,3),(6,3),(7,3),(8,3),(9,3),(10,3),(11,3),(12,3),(6,2)\}$
6	$\pm\{(2,6),(4,6),(6,6),(8,6),(10,6),(12,6),(1,6),(3,6),(5,6),(7,6),(9,6),(11,6),(12,2)\}$
9	$\pm\{(3,9),(6,9),(9,9),(12,9),(2,9),(5,9),(8,9),(11,9),(1,9),(4,9),(7,9),(10,9),(5,2)\}$
11	$\pm\{(4,11),(8,11),(12,11),(3,11),(7,11),(11,11),(2,11),(6,11),(10,11),(1,11),(5,11),(9,11),(11,2)\}$
12	$\pm\{(5,12),(10,12),(2,12),(7,12),(12,12),(4,12),(9,12),(1,12),(6,12),(11,12),(3,12),(8,12),(4,4)\}$
15	$\pm\{(6,15),(12,15),(5,15),(11,15),(4,15),(10,15),(3,15),(9,15),(2,15),(8,15),(1,15),(7,15),(10,4)\}$

Table 1. Base paths for a decomposition of $K_{13\times 33}$

The first table gives functions α and ψ which are used in setting up the base paths for a cyclic 33-symmetric Hamilton cycle decomposition of $K_{13\times 33}$. The base paths are provided in the second table. The edge lengths used by the base paths are given in the last table.

Part	$\{0,1,2,3,4\}$	$\{5\}$	$\{6\}$
Sum	3	5	6

(a)

Part	P_0	P_1	P_2	P_3	P_4	P_5	P_6	$\{x\}, x \in U$
Sum	10	45	31	17	3	38	24	x

(b)

Table 2. Partition of rings of integers

In (a), we give a partition of $\mathbb{Z}_7$ into parts of size 1 and 5, each part of which has that the sum of their elements is a unit of $\mathbb{Z}_7$. Similarly, in (b), we give a partition of $\mathbb{Z}_{49}$ into parts of size 1 and 5, again each part of which has that the sum of their elements is a unit of $\mathbb{Z}_{49}$.

3.2 ANALYZING PRIME POWERS OF n WHICH ARE LARGE

In the previous section, we focus on odd integers, n whose prime power divisors in the prime factorization were all less than p. Now, we develop results that allow us to incorporate integers whose prime powers are larger than p. We begin by presenting some elementary number theory results, then discuss the existence of a particular type of partition of a ring of integers modulo q, which will be useful in our constructions and finally we iteratively construct decompositions of $K_{p\times n}$, where n is any positive odd integer relatively prime to p.

First, we consider the following example highlighting a partition of rings of integers into parts which add up to units.

Example 3.7. Let $p = 5$ be prime and $\mathbb{Z}_7$, $\mathbb{Z}_{49}$ be rings of integers modulo 7 and 49, respectively. See Table 2(a) for a partition of $\mathbb{Z}_7$ into parts of sizes 1 or 5 where each part of size 1 contains a unit and each part of size 5 contains elements of $\mathbb{Z}_7$ which add up to units.
Define, for each $0 \le i \le 6$, $P_i = \{7i + j : j \in \mathbb{Z}_5\}$. Let $U = \mathbb{Z}_{49}\backslash\{P_0 \cup \cdots \cup P_6\}$. Then, $\{P_i : 0 \le i \le 6\} \cup \{\{\alpha\} : \alpha \in U\}$ is a partition of $\mathbb{Z}_{49}$ into parts of sizes 1 or 5 which adds up to units of $\mathbb{Z}_{49}$. See Table 2(b).

In what follows, we demonstrate that we can always find such partitions. We first provide two technical lemmas.

Lemma 3.8. *Let r and p both be odd and at least 3. Let a be a positive integer for which*

$r^a \geq p+2$. *Let Q and R be integers defined so that*

$$\frac{r^{a-1}-1}{2} = Q\frac{(p-1)}{2} + R, \textit{ where } \; 0 \leq R \leq \frac{p-1}{2} - 1.$$

Then $(r^a - r^{a-1})/2 - 2 \geq (p-2R-3)/2$.

Proof. It is equivalent to show that $r^a - r^{a-1} \geq p - 2R + 1$.

If $Q = 0$, then $r^{a-1} = 2R+1$. In this case,

$$r^a - r^{a-1} \geq p + 2 - r^{a-1} = p + 2 - (2R+1) = p - 2R + 1.$$

Suppose $Q > 0$. Then $(r^{a-1}-1)/2 \geq (p-1)/2$, which is equivalent to $r^{a-1} \geq p$. So,
$r^a - r^{a-1} = r^{a-1}(r-1) \geq p(r-1) \geq 2p \geq p+1 \geq p+1-2R$. □

Lemma 3.9. *Let p and r both be odd and at least 3. Let a be a positive integer. Let Q and R be integers such that*

$$\frac{r^{a-1}-1}{2} = Q\frac{(p-1)}{2} + R, \textit{ where } \; 0 \leq R \leq \frac{p-1}{2} - 1$$

and suppose $Q \geq 1$. Then $r^a - r^{a-1} \geq Q + (p - 2R - 1)$.

Proof. Note that $r^{a-1} = Q(p-1) + 2R + 1$. First, we show that $r^a \geq p(Q+1)$. We have that, since $r^a = r(r^{a-1})$,

$$\begin{aligned} r^a &= r[Q(p-1) + 2R + 1] \\ &\geq 3[Q(p-1) + 2R + 1] \\ &= pQ + Q(2p-3) + 6R + 3 \\ &\geq pQ + 2p - 3 + 6R + 3 \\ &\geq pQ + p. \end{aligned}$$

So $r^a - r^{a-1} = r^a - [Q(p-1) + 2R + 1] \geq p(Q+1) - [Q(p-1) + 2R + 1] = Q + p - 2R - 1$. □

Using these number theory results, we are now able to prove the existence of a particular type of a partition of a ring of integers which we use in our construction later.

Theorem 3.10. *For all primes p and prime powers q, if $q > p$, then there exists a partition of $\mathbb{Z}_q$ such that each part has size 1 or p, and each part adds to a unit of $\mathbb{Z}_q$.*

Proof. It is necessary to partition $\mathbb{Z}_q$ such that the non-units belong to a part of size p. Let $q = r^a$, where r is prime. Observe that the number of non-units in $\mathbb{Z}_q$ is $q/r = r^{a-1}$. So, the number of units is $\phi(q) = r^a - r^{a-1}$. There exist integers Q and R such that

$$\frac{r^{a-1}-1}{2} = Q\frac{(p-1)}{2} + R, \text{where } 0 \leq R \leq \frac{p-1}{2} - 1.$$

Represent the nonunits of $\mathbb{Z}_q$ as

$$\begin{aligned}\{0, &\pm a_{11}, \pm a_{12}, \cdots \pm a_{1(p-1)/2},\\ &\pm a_{21}, \pm a_{22}, \cdots \pm a_{2(p-1)/2},\\ &\vdots\\ &\pm a_{Q1}, \pm a_{Q2}, \ldots, \pm a_{Q(p-1)/2},\\ &\pm b_1, \pm b_2, \ldots, \pm b_R\}.\end{aligned}$$

Let $x = 2$ if $r \neq 3$; otherwise let $x = 4$. Notice that if $r = 3$, then 5 is relatively prime to q and hence, $1 + x = 5 \in \mathbb{Z}_q^\times$. If $r \neq 3$, 3 is relatively prime to q hence, $1 + x = 3 \in \mathbb{Z}_q^\times$. So, in all cases, $1 + x$ is a unit of $\mathbb{Z}_q$.

Let $u_1, u_2, \ldots, u_k$ be distinct units of $\mathbb{Z}_q$ such that $u_i \neq -u_j$ for distinct i and j, and $u_i \neq \pm 1$ or $\pm x$ for each i, with $k = (p - 2R - 3)/2$. Such a selection is possible provided that $(r^a - r^{a-1})/2 - 2 \geq (p - 2R - 3)/2$, which follws from Lemma 3.8. We define P' as $\{0, \pm b_1, \pm b_2, \ldots, \pm b_R, \pm u_1, \cdots \pm u_{(p-2R-3)/2}, 1, x\}$. Observe that

$$\sum_{\ell \in P'} \ell = 1 + x \in \mathbb{Z}_q^\times.$$

If $Q = 0$, then $\{P'\} \cup \{\{a\} : a \in \mathbb{Z}_q \backslash P'\}$ is a desired partition of $\mathbb{Z}_q$. Now, suppose $Q \geq 1$. Let $v_1, v_2, \ldots, v_Q$ be distinct units of $\mathbb{Z}_q$ such that $v_i \neq \pm u_j$ for each i and j, and $v_i \neq 1$ or x for each i. Such a selection is possible provided that $r^a - r^{a-1} \geq Q + (p - 2R - 1)$, which follows from

Lemma 3.9. For each $i \in \{1, 2, \ldots, Q\}$, define $P_i = \{\pm a_{i1}, \pm a_{i2}, \cdots \pm a_{i(p-1)}, v_i\}$. Observe that for each $i \in \{1, 2, \ldots, Q\}$,

$$\sum_{x \in P_i} x = v_i \in \mathbb{Z}_q^\times.$$

Then $\{P_1, P_2, \ldots, P_Q, P'\} \cup \{\{\alpha\} : \alpha \notin P_1 \cup P_2 \cup \cdots \cup P_Q \cup P'\}$ is a desired partition of $\mathbb{Z}_q$. $\square$

Note that the partitions of $\mathbb{Z}_7$ and $\mathbb{Z}_{49}$ given in Example 3.7 follow this construction. Note also, that the previous theorem does not require that p and q be relatively prime, however, in all applications that follow, we only apply it when p and q are relatively prime.

Using subsets of a ring of integers, we can build base paths from pre-existing base paths; the next four lemmas show the constructions.

Lemma 3.11. *Let p be an odd prime, q a prime power such that $q > p, p \nmid q$, and n an odd integer relatively prime to p and q. Let $u \in \mathbb{Z}_q^\times$. Let A be a base path of length p in $K_{p\times n}$, with $V(K_{p\times n}) = \mathbb{Z}_p \times \mathbb{Z}_n$, where*

$$A = [(x_0 = 0, y_0 = 0), (x_1, y_1), \ldots, (x_{p-1}, y_{p-1})]_{(x_p=0, y_p)}.$$

Define $A(u)$ as

$$A(u) = [(0, 0, 0), (x_1, y_1, u), \ldots, (x_{p-1}, y_{p-1}, (p-1)u)]_{(0, y_p, pu)}.$$

Then $A(u)$ is a base path of $K_{p\times nq}$.

Proof. Let $\Delta x_i = x_{i+1} - x_i$ and $\Delta y_i = y_{i+1} - y_i$ for each $i \in \{0, \ldots, p-1\}$. Then, $\partial(A) = \{\pm(\Delta x_0, \Delta y_0), \ldots, \pm(\Delta x_{p-1}, \Delta y_{p-1})\}$. Since A is a base path, we have that $\{x_1, \ldots, x_{p-1}\} = \{1, \ldots, p-1\}$, $(\Delta x_i, \Delta y_i) \neq \pm(\Delta x_j, \Delta y_j)$ whenever $i, j \in \{0, \ldots, p-1\}$ and $i \neq j$, and $|(0, y_p)| = n$.

Note that $\partial(A(u)) = \{\pm(\Delta x_i, \Delta y_i, u) : i \in \{0, 1, \ldots, p-1\}\}$. Suppose for some $i, j \in \{1, \ldots, p\}$ and $i \neq j$ that $(\Delta x_i, \Delta y_i, u) = \pm(\Delta x_j, \Delta y_j, u)$. Then $(\Delta x_i, \Delta y_i) = \pm(\Delta x_j, \Delta y_j)$, which is a contradiction. So $(\Delta x_i, \Delta y_i, u) \neq \pm(\Delta x_j, \Delta y_j, u)$ whenever $i, j \in \{0, \ldots, p-1\}$ and $i \neq j$. Since p and q are relatively prime, $p \in \mathbb{Z}_q^\times$. By closure of $(\mathbb{Z}_q^\times, \cdot)$, $pu \in \mathbb{Z}_q^\times$. Since $y_p \in \mathbb{Z}_n^\times$, then $|(0, y_p, pu)| = qn$. Therefore $A(u)$ is a base path of $K_{p\times nq}$. □

Example 3.12. Let $A = [(0,0),(1,0),(2,1)]_{(0,4)}$ be a base path of length 3 in $K_{3\times 5}$. Then $\partial(A) = \pm\{(1,0),(1,1),(1,3)\}$. Observe that $u = 6 \in \mathbb{Z}_7^\times$, so by Lemma 3.11,

$$A(6) = [(0,0,0),(1,0,6),(2,1,5)]_{(0,4,4)}$$

is a base path of $K_{3\times 35}$ with edge lengths $\partial(A(6)) = \pm\{(1,0,6),(1,1,6),(1,3,6)\}$.

Lemma 3.13. *Let p be an odd prime, q a prime power such that $q > p$, q and p are relatively prime, and n an odd integer relatively prime to p and q. Let $u \in \mathbb{Z}_q^\times$. Let $B = [(0,0)]_{(x,y)}$ be a base path of length 1 in $K_{p\times n}$. Hence (x,y) is a unit in $\mathbb{Z}_p \times \mathbb{Z}_n$. Then $B(u) = [(0,0,0)]_{(x,y,u)}$ is a base path in $K_{p\times nq}$.*

Proof. Observe that $\partial(B) = \{\pm(x,y)\}$ and $\partial(B(u)) = \{\pm(x,y,u)\}$. Since $u \in \mathbb{Z}_q^\times$, and p, n, q are pairwise relatively prime, we have that (x,y,u) is a unit of $\mathbb{Z}_p \times \mathbb{Z}_n \times \mathbb{Z}_q$. Therefore, $B(u)$ is a base path in $K_{p\times nq}$. □

Example 3.14. Let $B_1 = [(0,0)]_{(1,2)}$ be a base path of length 1 in $K_{3\times 5}$, then $\partial(B_1) = \pm\{(1,2)\}$. Again observe that $u = 6 \in \mathbb{Z}_7^\times$, so by Lemma 3.13,

$$B_1(6) = [(0,0,0)]_{(1,2,6)}$$

is a base path of $K_{3\times 35}$ with edge lengths $\partial(B_1(6)) = \pm\{(1,2,6)\}$.

Lemma 3.15. *Let p be an odd prime, q a prime power such that $q > p$, q, p relatively prime and n an odd integer relatively prime to p and q. Let $P = \{z_0, z_1, \ldots, z_{p-1}\}$ be a subset of $\mathbb{Z}_q$ which sums up to a unit of $\mathbb{Z}_q$. For each $i \in \mathbb{Z}_p$, define $Z_i = z_0 + z_1 + \cdots + z_i$. Let $B = [(0,0)]_{(x,y)}$ be a base path of length 1 in $K_{p\times n}$, again meaning (x,y) is a unit in $\mathbb{Z}_p \times \mathbb{Z}_n$. Define $B(P)$ as*

$$B(P) = [(0,0,0),(x,y,Z_0),(2x,2y,Z_1),\ldots,((p-1)x,(p-1)y,Z_{p-2})]_{(0,py,Z_{p-1})}.$$

Then $B(P)$ is a base path in $K_{p\times nq}$.

Proof. Observe that $\partial(B) = \{\pm(x, y)\}$ and $\partial(B(P)) = \{\pm(x, y, z_i) : i \in \mathbb{Z}_p\}$. Suppose there exists $i, j \in \mathbb{Z}_p$ and $i \neq j$ such that $(x, y, z_i) = \pm(x, y, z_j)$. Then $(x, y, z_i) = (x, y, z_j)$, and hence $z_i = z_j$, which is a contradiction. So $(x, y, z_i) \neq \pm(x, y, z_j)$ whenever $i, j \in \mathbb{Z}_p$ and $i \neq j$. Since p and n are relatively prime, $p \in \mathbb{Z}_n^\times$. By closure of $(\mathbb{Z}_n^\times, \cdot)$, $py \in \mathbb{Z}_n^\times$. Since P is a subset of $\mathbb{Z}_q$ which adds to a unit of $\mathbb{Z}_q$, $Z_{p-1} \in \mathbb{Z}_q^\times$. So, $|(x, py, Z_{p-1})| = qn$. Therefore $B(P)$ is a base path in $K_{p\times nq}$. $\square$

Example 3.16. Let $B_1 = [(0, 0)]_{(1,2)}$ be a base path of length 1 in $K_{3\times5}$, then $\partial(B_1) = \pm\{(1, 2)\}$. Let $P = \{0, 1, 2\}$ be a subset of $\mathbb{Z}_7$ of size 3, which adds up to a unit of $\mathbb{Z}_7$; mainly 3. Then, by Lemma 3.15,

$$B_1(P) = [(0, 0, 0), (1, 2, 0), (2, 4, 1)]_{(0,1,3)}$$

is a base path of $K_{3\times35}$ with edge length $\partial(B_1(P)) = \pm\{(1, 2, 0), (1, 2, 1), (1, 2, 2)\}$.

Lemma 3.17. *Let p be an odd prime, q an odd prime power such that $q > p$, q and p are relatively prime, and n and odd integer relatively prime to p and q. Let $P = \{z_0, z_1, \ldots, z_{p-1}\}$ be a subset of $\mathbb{Z}_q$ which sums to a unit of $\mathbb{Z}_q$ and $k \in \mathbb{Z}_p$. For each $i \in \mathbb{Z}_p$, define $Z_i^k = z_k + z_{k+1} + \cdots + z_{k+i}$. Let A be a base path in $K_{p\times n}$, with vertex set $\mathbb{Z}_p \times \mathbb{Z}_n$, given by*

$$A = [(x_0 = 0, y_0 = 0), (x_1, y_1), \ldots, (x_{p-1}, y_{p-1})]_{(x_p=0,y_p)},$$

Then $A(P)_k$, given by

$$A(P)_k = [(0, 0, 0), (x_1, y_1, Z_0^k), (x_2, y_2, Z_1^k), \ldots, (x_{p-1}, y_{p-1}, Z_{p-2}^k]_{(0,y_p,Z_{p-1}^k)},$$

is a base path of $K_{p\times nq}$.

Furthermore, for distinct $k, \ell \in \mathbb{Z}_p$, $A(P)_k$ and $A(P)_\ell$ are disjoint.

Proof. Let $\Delta x_i = x_{i+1} - x_i$ and $\Delta y_i = y_{i+1} - y_i$ for each $i \in \{0, \ldots, p-1\}$. Then $\partial(A) = \{\pm(\Delta x_0, \Delta y_0), \ldots, \pm(\Delta x_{p-1}, \Delta y_{p-1})\}$. Since A is a base path, we have that $\{x_1, \ldots, x_{p-1}\} = \{1, \ldots, p-1\}$, $(\Delta x_i, \Delta y_i) \neq \pm(\Delta x_j, \Delta y_j)$ whenever $i, j \in \{0, \ldots, p-1\}$ and $i \neq j$, and $|(0, y_p)| = n$.

We have that $\partial(A(P)_k) = \{\pm(\Delta x_i, \Delta y_i, z_{k+i}) : i \in \mathbb{Z}_p\}$. Suppose for some $i, j \in \mathbb{Z}_p$ that $(\Delta x_i, \Delta y_i, z_{k+i}) = \pm(\Delta x_j, \Delta y_j, z_{k+j})$ and $i \neq j$. It follows that $(\Delta x_i, \Delta y_i) = \pm(\Delta x_j, \Delta y_j)$, which is a contradiction. Observe that Z^k_{p-1} is the sum of all elements in P, so $Z^k_{p-1} \in \mathbb{Z}_q^\times$. Since $y_p \in \mathbb{Z}_n^\times$, $|(0, y_p, Z^k_{p-1})| = qn$. Therefore $A(P)_k$ is a base path of $K_{p\times nq}$.

Suppose $A(P)_k = A(P)_\ell$ for some $k, \ell \in \mathbb{Z}_p$, then $\partial(A(P)_k) = \partial(A(P)_\ell)$. So, $\{\pm(\Delta x_i, \Delta y_i, z_{k+i}) : i \in \mathbb{Z}_p\} = \{\pm(\Delta x_i, \Delta y_i, z_{\ell+i}) : i \in \mathbb{Z}_p\}$. Thus $z_{k+i} = z_{\ell+i}$ and so $k = \ell$. $\square$

Example 3.18. Let $A = [(0,0),(1,0),(2,1)]_{(0,4)}$ be a base path of length 3 in $K_{3\times 5}$, then $\partial(A) = \pm\{(1,0),(1,1),(1,3)\}$. Let $P = \{0,1,2\}$ be a subset of $\mathbb{Z}_7$ of size 3, note the elements of P add up to 3, which is a unit of $\mathbb{Z}_7$. So, by Lemma 3.17,

$$A(P)_0 = [(0,0,0),(1,0,0),(2,1,1)]_{(0,4,3)},$$
$$A(P)_1 = [(0,0,0),(1,0,1),(2,1,3)]_{(0,4,3)}, \text{ and}$$
$$A(P)_2 = [(0,0,0),(1,0,2),(2,1,2)]_{(0,4,3)}$$

are base paths of $K_{3\times 35}$ with edge lengths

$$\partial(A(P)_0) = \pm\{(1,0,0),(1,1,1),(1,3,2)\},$$
$$\partial(A(P)_1) = \pm\{(1,0,1),(1,1,2),(1,3,0)\}, \text{ and}$$
$$\partial(A(P)_2) = \pm\{(1,0,2),(1,1,0),(1,3,1)\}.$$

Here, we now provide a result giving us an iterative method for constructing a cyclic n - symmetric Hamilton cycle decomposition of $K_{p\times n}$ with n an odd positive integer such that $p \nmid n$ and n does have prime powers in its prime factorization which exceed p.

Theorem 3.19. *Suppose $K_{p\times n}$ has a cyclic n-symmetric Hamilton cycle decomposition and q is a prime power larger than p. Then $K_{p\times nq}$ has a cyclic nq-symmetric Hamilton cycle decomposition.*

Proof. If $K_{p\times n}$ has a cyclic n-symmetric Hamilton cycle decomposition, then there exists a valid set $\mathcal{S}$ of base paths which gives rise to such a decomposition. Partition $\mathcal{S}$ as $\mathcal{S}_1 \cup \mathcal{S}_p$, where $\mathcal{S}_1$ contains all base paths of length 1 and $\mathcal{S}_p$ contains all base paths of lengths p. Let $|\mathcal{S}_1| = s$

and $|\mathcal{S}_p| = r$. Then $rp + s = (p-1)n/2$. By Theorem 3.10, in $\mathbb{Z}_q$, there exists $\mathcal{P}, T$ such that $T \subseteq \mathbb{Z}_q^\times$ and $\mathcal{P}$ is a partition of $\mathbb{Z}_q \backslash T$, $|P| = p$, and $\sum_{x \in P} x \in \mathbb{Z}_q^\times$ for each $P \in \mathcal{P}$. Let $|\mathcal{P}| = \ell$ and $|T| = t$, then $q = \ell p + t$. Using the constructions from Lemmas 3.11, 3.13, 3.15, and 3.17, define

$$K_1 = \{A(u) : A \in \mathcal{S}_p, u \in T\},$$

$$K_2 = \{B(u) : B \in \mathcal{S}_1, u \in T\},$$

$$K_3 = \{B(P) : B \in \mathcal{S}_1, P \in \mathcal{P}\} \text{ and}$$

$$K_4 = \{A(P)_k : A \in \mathcal{S}_p, P \in \mathcal{P}, k \in \mathbb{Z}_p\}.$$

Let $K = K_1 \cup K_2 \cup K_3 \cup K_4$. We claim that K is a valid set of base paths of $K_{p \times nq}$.

By Lemmas 3.11, 3.13, 3.15, and 3.17, each element in K is a base path. We now need to show that each edge length in $K_{p \times nq}$ is used by exactly one base path in the set K.
We have that $|\partial(K_{p \times nq})| = (p-1)nq$. Also, $|\partial(K_1)| \leq 2prt, |\partial(K_2)| \leq 2st, |\partial(K_3)| \leq 2ps\ell$ and $|\partial(K_4)| \leq 2pr\ell p$. So,

$$2prt + 2st + 2ps\ell + 2pr\ell p = 2t(pr+s) + 2\ell p(pr+s) = 2(pr+s)(\ell p + t) = (p-1)nq.$$

Thus, there are at most $(p-1)nq$ edge lengths used by base paths in K, and hence K is a valid set of base paths of $K_{p \times nq}$ if and only if each edge length of $\partial(K_{p \times nq})$ is used by a base path in K. To that end, let $(x, y, z) \in \partial(K_{p \times nq})$. Note that $(x, y) \in \partial(K_{p \times n})$, so $(x, y) \in \partial(A)$ or $(x, y) \in \partial(B)$ for some $A \in \mathcal{S}_p$ or $B \in \mathcal{S}_1$. Similarly, $z \in P$, for some $P \in \mathcal{P}$ or $z \in T$.

If $(x, y) \in \partial(A)$ and $z \in T$, then $(x, y, z) \in \partial(A(z))$. If $(x, y) \in \partial(B)$ and $z \in T$, then $(x, y, z) \in \partial(B(z))$. If $(x, y) \in B$ and $z \in P$ for some $P \in \mathcal{P}$, then $(x, y, z) \in \partial(B(P))$. If $(x, y) \in \partial(A)$ and $z \in P$ for some $P \in \mathcal{P}$, then $(x, y, z) \in \partial(A(P)_v)$ for some $v \in \mathbb{Z}_p$. Therefore, every edge length in $\partial(K_{p \times nq})$ is used by a base path of K. So K gives rise to a cyclic nq-symmetric Hamilton cycle decomposition of $K_{p \times nq}$. $\square$

Example 3.20. Following preceding examples, let $\mathcal{C}$ be the decomposition of $K_{3 \times 5}$ given by the base paths in Table 3, and consider the partition of $\mathbb{Z}_7$ into parts of size 3 and 1 which add up to units in $\mathbb{Z}_7$, also given in Table 3. Then, we end up with the following sets K_1, K_2, K_3 and K_4,

Base paths	Edge lengths
$A = [(0,0),(1,0),(2,1)]_{(0,4)}$	$\{(1,0),(1,1),(1,3)\}$
$B_1 = [(0,0)]_{(1,2)}$	$\{(1,2)\}$
$B_2 = [(0,0)]_{(1,4)}$	$\{(1,4)\}$

(a)

Part	Sum as Units in $\mathbb{Z}_7$
$P_1 = \{0,1,2\}$	3
$P_2 = \{3,4,5\}$	5
$P_3 = \{6\}$	6

(b)

Table 3. Constructing a valid set of base paths for a decomposition of $K_{3\times 35}$

In (a), we give the base paths used by a decomposition of $K_{3\times 5}$, and in (b), we give a partition of $\mathbb{Z}_7$ into parts of size 1 and 3 which add up to a unit of $\mathbb{Z}_7$. These are used to build the cyclic 35-symmetric Hamilton cycle decomposition of $K_{3\times 35}$ given in Table 4.

given in Table 4, which give a valid set of base paths for a cyclic 35-symmetric Hamilton cycle decomposition of $K_{3\times 35}$.

3.3 MAIN RESULT

To prove $K_{p\times n}$ has a cyclic n-symmetric Hamilton cycle decomposition for any prime p and odd integer n, we need to introduce a way to build a decomposition where our part size is divisible by p. We present that, and then use all the results from this section to prove our main result. First we address a way to build a cyclic n-symmetric Hamilton cycle decomposition of $K_{p\times n}$, where p is a divisor of n.

Theorem 3.21. *Suppose $K_{p\times n}$ has a cyclic n-symmetric Hamilton cycle decomposition, where $p > 3$ is an odd prime and $p \nmid n$, then there exists a cyclic (np^t)-symmetric Hamilton cycle decomposition of $K_{p\times np^t}$ for $t \geq 1$.*

Proof. Edge lengths of $K_{p\times np^t}$ are denoted as the ordered pairs (x,y) in $\left(\mathbb{Z}^{\times}_{p^{t+1}}\right) \times \mathbb{Z}_n$. From a cyclic n-symmetric Hamilton cycle decomposition of $K_{p\times n}$, let $\mathcal{D}$ be the set of base paths of length p with non-units edge lengths. Let $C \in \mathcal{D}$, then C has the form

$$C = [(x_0,y_0) = (0,0),(x_1,y_1),\ldots,(x_{p-1},y_{p-1})]_{(0,y_p)},$$

where $\{x_1,\ldots,x_{p-1}\} = \{1,\ldots,p-1\}$, $(\Delta x_i,\Delta y_i) \neq \pm(\Delta x_j,\Delta y_j)$ whenever $i,j \in \{0,\ldots,p-1\}$, $i \neq j$, and $|(0,y_p)| = n$. For each $s \in \{0,\ldots,p^t-1\}$, define C_s as a path of length p in $K_{p\times np^t}$

Set	Definition	Base paths	Edge Lengths
K_1	$A(P_3)$	$[(0,0,0),(1,0,6),(2,1,5)]_{(0,4,4)}$	$\pm\{(1,0,6),(1,1,6),(1,3,6)\}$
K_2	$B_1(P_3)$	$[(0,0,0)]_{(1,2,6)}$	$\pm\{(1,2,6)\}$
	$B_2(P_3)$	$[(0,0,0)]_{(1,4,6)}$	$\pm\{(1,4,6)\}$
K_3	$B_1(P_1)$	$[(0,0,0),(1,2,0),(2,4,1)]_{(0,1,3)}$	$\pm\{(1,2,0),(1,2,1),(1,2,2)\}$
	$B_1(P_2)$	$[(0,0,0),(1,2,3),(2,4,0)]_{(0,1,5)}$	$\pm\{(1,2,3),(1,2,4),(1,2,5)\}$
	$B_2(P_1)$	$[(0,0,0),(1,4,0),(2,3,1)]_{(0,2,3)}$	$\pm\{(1,4,0),(1,4,1),(1,4,2)\}$
	$B_2(P_2)$	$[(0,0,0),(1,4,3),(2,3,0)]_{(0,2,5)}$	$\pm\{(1,4,3),(1,4,4),(1,4,5)\}$
K_4	$A(P_1)_0$	$[(0,0,0),(1,0,0),(2,1,1)]_{(0,4,3)}$	$\pm\{(1,0,0),(1,1,1),(1,3,2)\}$
	$A(P_1)_1$	$[(0,0,0),(1,0,1),(2,1,3)]_{(0,4,3)}$	$\pm\{(1,0,1),(1,1,2),(1,3,0)\}$
	$A(P_1)_2$	$[(0,0,0),(1,0,2),(2,1,2)]_{(0,4,3)}$	$\pm\{(1,0,2),(1,1,0),(1,3,1)\}$
	$A(P_2)_0$	$[(0,0,0),(1,0,3),(2,1,0)]_{(0,4,5)}$	$\pm\{(1,0,3),(1,1,4),(1,3,5)\}$
	$A(P_2)_1$	$[(0,0,0),(1,0,4),(2,1,2)]_{(0,4,5)}$	$\pm\{(1,0,4),(1,1,5),(1,3,3)\}$
	$A(P_2)_2$	$[(0,0,0),(1,0,5),(2,1,1)]_{(0,4,5)}$	$\pm\{(1,0,5),(1,1,3),(1,3,4)\}$

Table 4. A valid set of base paths of $\boldsymbol{K_{3\times 35}}$

The set $K = K_1 \cup K_2 \cup K_3 \cup K_4$ is a valid set of base paths of $K_{3\times 35}$ constructed using base paths of $K_{3\times 5}$ and the partition of $\mathbb{Z}_7$ given in Table 3.

given by

$$C_s = [(0,0), (x_1 + sp, y_1), (x_2 + 2sp, y_2), \ldots, (x_{p-1} + (p-1)sp, y_{p-1})]_{(p,y_p)}.$$

Let $\partial^*(C) = \partial(C_0) \cup \partial(C_1) \cup \cdots \cup \partial(C_{p^t-1})$, and let $\mathcal{C}$ be the union of all $\partial^*(C)$ for each $C \in \mathcal{D}$. Observe that for $s \in \{0, \ldots, p^t - 1\}$,

$$\partial(C_s) = \pm\{(\Delta x_\ell + sp, \Delta y_\ell) : 0 \leq \ell \leq p-2\} \cup \{(\Delta x_{p-1} + p - (p-1)sp, \Delta y_{p-1})\}.$$

We first show that C_s is a base path. We show that all the vertices are distinct, the edge lengths are all distinct, and also then show that (p, y_p) has order np^t. Suppose that $(x_a + asp, y_a) = (x_b + bsp, y_b)$ for some $a, b \in \{1, 2, \ldots, p-1\}$. Then, $x_a + asp = x_b + bsp$ and so $x_a - x_b = p(sb - sa)$. Thus, $x_a = x_b \pmod p$. This is a contradiction. So, $(x_a + asp, y_a) \neq (x_b + bsp, y_b)$ for some $a, b \in \{1, 2, \ldots, p-1\}$. Next suppose that $(x_a + asp, y_a) = (p, y_p)$ for some a, then $x_a + asp \equiv p \pmod p$. This implies that $x_a \equiv 0 \pmod p$ and this is a contradiction since $\{x_1, \ldots, x_{p-1}\} = \{1, \ldots, p-1\}$. $(0,0) \neq (p, y_p)$ since $0 \neq p$, and $(0,0) \neq (x_i + isp, y_p)$ for $1 \leq i \leq p-1$ since $x_i + isp \pmod p \neq 0$.

Now, suppose that some path C_s has a repeated edge length (x, y). Define a map $\alpha : \mathbb{Z}_{p^{t+1}} \times \mathbb{Z}_n \to \mathbb{Z}_p \times \mathbb{Z}_n$ where $\alpha(x,y) = (x \pmod p, y)$. If C_s contains the edge length (x,y) twice, then C contains $\alpha(x,y)$ as an edge length twice. Since C is a base path, this gives a contradiction. Since the order of p in $\mathbb{Z}_{p^{t+1}}$ is p^t and $y_p \in \mathbb{Z}_n^\times$, then $|(p, y_p)| = np^t$. Thus, C_s is a base path of $K_{p \times np^t}$.

Now, we show that each non-unit edge length of $K_{p\times np^t}$ is being used in our collection of iterated base paths $\mathcal{E} = \{C_s : C \in \mathcal{D}, 0 \leq s \leq p^t - 1\}$. Let (x,y) be a non-unit edge length in $K_{p\times np^t}$. Since, $\partial(K_{p\times np^t}) = \mathbb{Z}_{p^{t+1}}^\times \times \mathbb{Z}_n$, then, $p \nmid x$ and y is a non-unit of $\mathbb{Z}_n$. Consider (x', y) where $x' = x \pmod p$. Observe that (x', y) is a non-unit of $\mathbb{Z}_p \times \mathbb{Z}_n$ and since $(x', y) \in \mathbb{Z}_p \times \mathbb{Z}_n$, $(x', y) \in \partial(K_{p\times n})$. So, (x', y) must belong to a base path of length p in $\mathcal{D}$, say C. Let the iterations of C be: $C_0, C_1, \ldots, C_{p^t-1}$. Note that $\mathcal{C}$ contains every edge length of the form $(x' + sp, y)$, where $s \in \{0, \ldots, p^t - 1\}$; in particular, $(x' + sp, y)$ is an edge length of C_s. Therefore (x, y) is an edge length of some C_s. Thus, all non-unit edge lengths of $\partial(K_{p\times np^t})$ belong to $\mathcal{C}$.

Now we show all base paths in $\mathcal{E}$ are edge length distinct. Suppose that $\partial(C_s) \cap \partial(C'_{s'})$ is not empty and contains (x, y) for some $C, C' \in \mathcal{D}$ and $s, s' \in \{0, \ldots, p^t - 1\}$. We need to show that $C = C'$ and $s = s'$. Since $\partial(C_s) \cap \partial(C'_{s'})$ contains (x, y), $\alpha(x, y) \in \partial(C) \cap \partial(C')$, and so $C = C'$. Then, $(x, y) \in \partial(C_s) \cap \partial(C_{s'})$. So, one of the following holds;

1. $(x, y) = \pm(\Delta x_\ell + sp, \Delta y_\ell) = \pm(\Delta x_{\ell'} + s'p, \Delta y_{\ell'})$, or

2. $(x, y) = \pm(\Delta x_p + p - (p-1)sp, \Delta y_p) = \pm(\Delta x_p + p - (p-1)s'p, \Delta y_p)$, or

3. $(x, y) = \pm(\Delta x_\ell + sp, \Delta y_\ell) = \pm(\Delta x_p + p - (p-1)s'p, \Delta y_p)$.

First, suppose that $\pm(\Delta x_\ell + sp, \Delta y_\ell) = \pm(\Delta x_{\ell'} + s'p, \Delta y_{\ell'})$ for some ℓ, ℓ'. Since, C and C' have $\alpha(x, y)$ in common, $\pm(\Delta x_\ell, \Delta y_\ell) = \pm(\Delta x_{\ell'}, \Delta y_{\ell'})$ and so $\ell = \ell'$. So, $(\Delta x_\ell + sp, \Delta y_\ell) = (\Delta x_\ell + s'p, \Delta y_\ell)$. So, $sp = s'p$ in $\mathbb{Z}_{p^{t+1}}$, and thus, $s = s' \pmod{p^t}$.

Secondly, suppose that $\pm(\Delta x_p + p - (p-1)sp, \Delta y_p) = \pm(\Delta x_p + p - (p-1)s'p, \Delta y_p)$. If $\Delta y_p = 0$, then $(\Delta x_p + p - (p-1)sp, \Delta y_p) = -(\Delta x_p + p - (p-1)s'p, \Delta y_p)$, which implies $\Delta x_p \equiv -\Delta x_p \pmod{p}$. So, $\Delta x_p = 0 \pmod{p}$, which is a contradiction. Hence $\Delta y_p \neq 0$, so, $\Delta x_p + p - (p-1)sp = \Delta x_p + p - (p-1)s'p$. So, $(p-1)sp = (p-1)s'p$ in $\mathbb{Z}_{p^{t+1}}$; thus $s = s'$, modulo p^t.

Lastly, suppose that $(\Delta x_\ell + sp, \Delta y_\ell) = \pm(\Delta x_p + p - (p-1)sp, \Delta y_p)$. Since $0 \leq \ell \leq p-2$, applying α to the edge lengths, we have that $(\Delta x_\ell, \Delta y_\ell) = \pm(\Delta x_p, \Delta y_p)$. Since $\ell \neq p$, this is a contradiction. Therefore $\{C_s : 0 \leq s \leq p^t - 1, C \in \mathcal{D}\}$ is a collection of base paths with disjoint edge sets in which each non-unit edge length in $K_{p \times np^t}$ is used.

Therefore, $\mathcal{E}$ is a valid set of base paths of a cyclic (np^t)-symmetric Hamilton cycle decomposition of $K_{p \times np^t}$. $\square$

Example 3.22. Let $\mathcal{C}$ be the cyclic 5-symmetric Hamilton cycle decomposition of $K_{3\times 5}$ given in Example 3.20. Note $A = [(0,0), (1,0), (2,1)]_{(0,4)}$ with edge length $\pm\{(1,0), (1,1), (1,3)\}$ is the base path of length 3 corresponding to $\mathcal{C}$.

Following the construction in Theorem 3.21, the following are base paths of a decomposition of $K_{3\times 45}$, with vertex set $\mathbb{Z}_{27} \times \mathbb{Z}_5$, that is $t = 2$, so $p^t = 9$.

For $s \in \mathbb{Z}_9$, the base paths of the decomposition of $K_{3\times 45}$ of length 3 are given in Table 5.

i	A_i	$\partial(A_i)$
0	$[(0,0),\ (1,0),\ (2,1)]_{(3,4)}$	$\pm\{(1,0),\ (1,1),\ (1,3)\}$
1	$[(0,0),\ (4,0),\ (8,1)]_{(3,4)}$	$\pm\{(4,0),\ (4,1),(22,3)\}$
2	$[(0,0),\ (7,0),(14,1)]_{(3,4)}$	$\pm\{(7,0),\ (7,1),(16,3)\}$
3	$[(0,0),(10,0),(20,1)]_{(3,4)}$	$\pm\{(10,0),(10,1),(10,3)\}$
4	$[(0,0),(13,0),(26,1)]_{(3,4)}$	$\pm\{(13,0),(13,1),\ (4,3)\}$
5	$[(0,0),(16,0),\ (5,1)]_{(3,4)}$	$\pm\{(16,0),(16,1),(25,3)\}$
6	$[(0,0),(19,0),(11,1)]_{(3,4)}$	$\pm\{(19,0),(19,1),(19,3)\}$
7	$[(0,0),(22,0),(17,1)]_{(3,4)}$	$\pm\{(22,0),(22,1),(13,3)\}$
8	$[(0,0),(25,0),(23,1)]_{(3,4)}$	$\pm\{(25,0),(25,1),\ (7,3)\}$

Table 5. Base paths leading to a decomposition of $K_{3\times 45}$

These base paths are constructed from $A = [(0,0),(1,0),(2,1)]_{(0,4)}$ in $K_{3\times 5}$, using the method outlined in Theorem 3.21. They have pairwise disjoint edge lengths and contain all non-unit edge lengths of $K_{3\times 45}$. This extends to a valid set of base paths of $K_{3\times 45}$, by including base paths of length 1 corresponding to unused unit edge lengths.

Now, we bring it all together with the main result of this chapter.

Theorem 3.23. *Let p be an odd prime and n be an odd positive integer at least 3. Then $K_{p\times n}$ has a cyclic n-symmetric Hamilton cycle decomposition.*

Proof. Let $n = p^a p_1^{a_1} \cdots p_k^{a_k} q_1^{b_1} \cdots q_\ell^{b_\ell}$, where $p, p_1, \ldots, p_k, q_1, \ldots, q_\ell$ are distinct primes, $k, \ell, a \geq 0$, $k + \ell + a \geq 1$ and for each $i \in \{1, \ldots, k\}$ and $j \in \{1, \ldots, \ell\}$, $a_i \geq 1, b_j \geq 1$, $p_i^{a_i} < p$, $q_j^{b_j} > p$. Let $P = p_1^{a_1} \cdots p_k^{a_k}$ and $Q = q_1^{b_1} \cdots q_\ell^{b_\ell}$. From Lemma 3.6, $K_{p\times P}$ has a cyclic P-symmetric Hamilton cycle decomposition. By Theorem 3.19, $K_{p\times Pq_1^{b_1}}$ has a cyclic $Pq_1^{b_1}$-symmetric Hamilton cycle decomposition and consequently, $K_{p\times PQ}$ has a cyclic PQ-symmetric Hamilton cycle decomposition. By Theorem 3.21, $K_{p\times p^a PQ}$ has a cyclic $p^a PQ$-symmetric Hamilton cycle decomposition and hence, $K_{p\times n}$ has a cyclic n-symmetric Hamilton cycle decomposition. $\square$

CHAPTER 4

DECOMPOSITIONS OF $K_{m\times n}$ FOR GENERAL m

In this chapter, we highlight all other cyclic n-symmetric Hamilton cycle decomposition that we found with an odd number of parts not necessarily prime. We now leverage the result in Theorem 3.23 to get some other decomposition of complete multipartite graphs and prove similar results to those in Section 3.2. We begin by building base paths from pre-existing base paths of a cyclic decomposition of K_m.

Lemma 4.1. *Let m be an odd integer with smallest prime divisor p and divisor d of m where $p \mid d$. Let n be an integer relatively prime to m, with $n > p$. Suppose A is a base path of length d in K_m, with vertex set $\mathbb{Z}_m$, where $A = [x_0 = 0, x_1, x_2, \ldots, x_{d-1}]_{x_d}$. Let $P = \{z_0, z_1, \ldots, z_{p-1}\}$ be a subset of $\mathbb{Z}_n$ which sums up to a unit of $\mathbb{Z}_n$. Let $k \in \mathbb{Z}_p$, for each $i \in \mathbb{Z}_p$, define $Z_i^k = z_k + z_{k+1} + \cdots + z_{k+i}$. Then,*

$$A(P)_k = [(0,0), (x_1, Z_0^k), (x_2, Z_1^k), \ldots, (x_{d-1}, Z_{p-2}^k)]_{(x_d, Z_{p-1}^k)}$$

is a base path of $K_{m\times n}$.

Proof. Let $\Delta x_i = x_{i+1} - x_i$ for each $i \in \{0, \ldots, d-1\}$. Then $\partial(A) = \{\pm\Delta x_0, \ldots, \pm\Delta x_{d-1}\}$. Since A is a base path, we have that $\{x_1, \ldots, x_{d-1}\} = \{1, \ldots, d-1\}$, $d \mid x_d$, $\Delta x_i \neq \pm\Delta x_j$ whenever $i, j \in \{0, \ldots, d-1\}$ and $i \neq j$, and $|x_d| = m/d$.

We have that $\partial(A(P)_k) = \{\pm(\Delta x_i, z_{k+i}) : i \in \mathbb{Z}_p\}$. Suppose for some $i, j \in \mathbb{Z}_p$ that $(\Delta x_i, z_{k+i}) = \pm(\Delta x_j, z_{k+j})$ and $i \neq j$. It follows that $\Delta x_i = \pm\Delta x_j$, which is a contradiction. Since Z_{p-1}^k is the sum of all elements in P, so $Z_{p-1}^k \in \mathbb{Z}_n^\times$. So, $|(x_d, Z_{p-1}^k)| = (m/d)n$. Therefore $A(P)_k$ is a base path of $K_{m\times n}$. □

Example 4.2. Observe that 3 is the smallest prime divisor of 21. Let $A = [0, 2, 1]_{15}$ be a base path of length 3 in K_{21}, then $\partial(A) = \pm\{2, 20, 14\}$. Let $P = \{0, 1, 2\}$ be a subset of $\mathbb{Z}_{17}$ of size 3;

note the elements of P add up to a unit of $\mathbb{Z}_{17}$. Then, by Lemma 4.1,

$$A(P)_0 = [(0,0),(2,0),(1,1)]_{(15,3)},$$
$$A(P)_1 = [(0,0),(2,1),(1,3)]_{(15,3)}, \text{ and}$$
$$A(P)_2 = [(0,0),(2,2),(1,2)]_{(15,3)}$$

are base paths of $K_{21\times 17}$ with edge lengths

$$\partial(A(P)_0) = \pm\{(2,0),(20,1),(14,2)\},$$
$$\partial(A(P)_1) = \pm\{(2,1),(20,2),(14,0)\}, \text{ and}$$
$$\partial(A(P)_2) = \pm\{(2,2),(20,0),(14,1)\}.$$

Lemma 4.3. *Let m be an odd integer with the smallest prime divisor p, divisor d of m where $p \nmid d$. Let n be relatively prime to m, with $n > p$. Suppose A is a base path of length d in K_m, where $A = [x_0 = 0, x_1, x_2, \dots, x_{d-1}]_{x_d}$. Let $P = \{z_0, z_1, \dots, z_{p-1}\}$ be a subset of $\mathbb{Z}_n$ which sums up to a unit of $\mathbb{Z}_n$. For each $i \in \mathbb{Z}_p$, define $Z_i = z_0 + z_1 + \cdots + z_i$. If $p \nmid d$, let $k \in \mathbb{Z}_{pd}$. Define q_k, q'_k, r_k, r'_k so that $k = q_k d + r_k, 1 \le r_k \le d$ and $k = q'_k p + r'_k, 1 \le r'_k \le p$. Let $A(P)$ be given by*

$$A(P) = [(0,0),(w_1,y_1),(w_2,y_2),\dots,(w_{pd-1},y_{pd-1})]_{(w_{pd},y_{pd})},$$

where $w_k = x_{r_k} + q_k x_d$ and $y_k = Z_{r'_k} + q'_k Z_{p-1}$. Then, $A(P)$ is a base path in $K_{m\times n}$.

Proof. Let $\Delta x_i = x_{i+1} - x_i$ for each $i \in \{0,\dots,d-1\}$. Then $\partial(A) = \{\pm\Delta x_0, \dots, \pm\Delta x_{d-1}\}$. Since A is a base path, we have that $x_i \ne x_j \pmod d$ for each $i,j \in \{1,\dots,d-1\}$, $i \ne j$, $d \mid x_d$, $\Delta x_i \ne \pm\Delta x_j$ whenever $i,j \in \{0,\dots,d-1\}$ and $i \ne j$, and $|x_d| = m/d$.

Now we show that $\{w_1, w_2, \dots, w_{pd-1}\} = \{1,2,\dots,pd-1\} \pmod{pd}$ and $pd \mid w_{pd}$. Suppose $w_k = w_s$ for some $k,s \in \{0,\dots,pd-1\}, k \ne s$. Then, $x_{r_k} + q_k x_d = x_{r_s} + q_s x_d$. So, $x_{r_k} - x_{r_s} = x_d(q_s - q_k)$. But, $d \mid x_d$, so $d \mid x_{r_k} - x_{r_s}$. We have that $x_{r_k} = x_{r_s} \pmod d$, but the x_is are all different modulo d. So $r_k = r_s$ and thus $x_{r_k} - x_{r_s} = 0$. Hence, $x_d(q_s - q_k) = 0 \pmod{pd}$ and $(x_d/d)(q_s - q_k) = 0 \pmod p$. But since $p \nmid x_d/d$, $q_s = q_k \pmod p$ where $0 \le q_i \le p-1$. This implies $q_k = q_s$.

We have that $\Delta w_k = \Delta x_{r_k}$ and $\Delta y_k = z_{r'_k}$ for each k. So, $\partial(A(P)) = \{\pm(\Delta x_{r_k}, z_{r'_k}) : k \in \{0, 1, \ldots, pd-1\}\}$. Suppose for some $k, s \in \{0, \ldots, pd-1\}$ that $(\Delta x_{r_k}, z_{r'_k}) = \pm(\Delta x_{r_s}, z_{r'_s})$. It follows that $\Delta x_{r_k} = \pm\Delta x_{r_s}$, $r_k = r_s$ and so $k = s \pmod{d}$. Similarly, $z_{r'_k} = z_{r'_s}$, so $r'_k = r'_s$ and hence $k = s \pmod{p}$. Thus, $k = s \pmod{pd}$. So, all edge lengths in $\partial(A(P))$ are distinct.

Since $w_{pd} = px_d$, we have that $|w_{pd}| = m/pd$. Similarly, $y_{pd} = dz_p$, so, $|y_{pd}| = n$. So, $|(w_{pd}, y_{pd})| =$lcm$(m/(pd), n) = mn/(pd)$. Therefore, $A(P)$ is a base path of $K_{m\times n}$. □

Example 4.4. Again observe that 3 is the smallest prime divisor of 21, and $B = [0, 6, 15, 12, 16, 11, 3]_{14}$ be a base path of length 7 in K_{21}. Then $\partial(B) = \pm\{6, 9, 18, 4, 16, 13, 11\}$. Let $P = \{0, 1, 2\}$ be a subset of $\mathbb{Z}_{17}$ of size 3, which adds up to a unit of $\mathbb{Z}_{17}$, namely 3. Since $3 \nmid 7$, we have by Lemma 4.3 that,

$$\begin{aligned} B(P) = [&(0,0),(6,0),(15,1),(12,3),(16,3),(11,4),(3,6),\\ &(14,6),(20,7),(8,9),(5,9),(9,10),(4,12),(17,12),\\ &(7,13),(13,15),(1,15),(19,16),(2,1),(18,1),(10,2)]_{(0,4)} \end{aligned}$$

is a base path of $K_{21\times 17}$ with edge lengths

$$\begin{aligned} \partial(B(P)) = \pm\{&(6,0),(9,1),(18,2),(4,0),(16,1),(13,2),(11,0),\\ &(6,1),(9,2),(18,0),(4,1),(16,2),(13,0),(11,1),(6,2),\\ &(9,0),(18,1),(4,2),(16,0),(13,1),(11,2)\}. \end{aligned}$$

Lemma 4.5. *Let m be an odd integer with smallest prime divisor p. Let n be relatively prime to m, with $m > p$. Suppose A is a base path of length d in K_m, with vertex set $\mathbb{Z}_m$ and let $A = [0, x_1, x_2, \ldots, x_{d-1}]_{x_d}$. Let $u \in \mathbb{Z}_n^\times$. Define $A(u)$ as*

$$A(u) = [(0,0),(x_1,u),\ldots,(x_{d-1},(d-1)u)]_{(x_d,du)}.$$

Then $A(u)$ is a base path in $K_{m\times n}$.

Proof. Let $\Delta x_i = x_{i+1} - x_i$ for each $i \in \{0,\ldots,d-1\}$. Then $\partial(A) = \{\pm\Delta x_1,\ldots,\pm\Delta x_d\}$. Since A is a base path, we have that $x_i \neq x_j (\mathrm{mod}\ d)$ for each $i,j \in \{1,\ldots,d-1\}$, $d \mid x_d$, $\Delta x_i \neq \pm\Delta x_j$ whenever $i,j \in \{0,\ldots,d-1\}$ and $i \neq j$, and $|x_d| = m/d$.

Note that $\partial(A(u)) = \{\pm(\Delta x_i, u) : i \in \{0,1,\ldots,d-1\}\}$. Suppose for some $i,j \in \{0,\ldots,d-1\}$ and $i \neq j$ that $(\Delta x_i, u) = \pm(\Delta x_j, u)$. Then $\Delta x_i = \Delta x_j$, which is a contradiction. So $(\Delta x_i, u) \neq \pm(\Delta x_j, u)$ whenever $i,j \in \{0,\ldots,d-1\}$ and $i \neq j$. Since m and n are relatively prime, and $d \mid m$, $d \in \mathbb{Z}_n^\times$. By closure of $(\mathbb{Z}_n^\times, \cdot)$, $du \in \mathbb{Z}_n^\times$ and $|(x_d, du)| = \mathrm{lcm}(m/d, n) = mn/d$. Therefore $A(u)$ is a base path in $K_{m\times n}$. $\square$

Example 4.6. Again note that 3 is the smallest prime divisor of 21. Let $A = [0,2,1]_{15}$ be a base path of length 3 in K_{21}, then $\partial(A) = \pm\{2,20,14\}$. Note that 4 is a unit of $\mathbb{Z}_{17}$, so by Lemma 4.5, $A(4) = [(0,0),(2,4),(1,8)]_{(15,12)}$ is a base path of $K_{21\times 17}$ with edge length $\partial(A(u)) = \pm\{(2,4),(20,4),(14,4)\}$.

With Lemmas 4.1, 4.3 and 4.5, we now get a similar result to Theorem 3.19 at the end of Section 3.2.

Theorem 4.7. *Let m and n be positive odd integers, and let p be a prime divisor of m. If n is a prime power bigger than p, and K_m has a cyclic Hamilton cycle decomposition, then $K_{m\times n}$ has a cyclic n-symmetric Hamilton cycle decomposition.*

Proof. Let $\mathcal{C}$ be a valid set of base paths which give rise to such a cyclic Hamilton cycle decomposition of K_m. Let D be the set of all positive divisors of m. For each divisor $d \in D$, let r_d denote the number of base paths of length d in $\mathcal{C}$. Then, by counting the total number of edge lengths in K_m, we have that $\sum_{d\in D} d \cdot r_d = (m-1)/2$. Furthermore, let $D_1 = \{d \in D : p \mid d\}$ and $D_2 = \{d \in D : p \nmid d\}$. Let $\mathcal{C}_1, \mathcal{C}_2$ be subset of $\mathcal{C}$ whose lengths belong to D_1 and D_2, respectively.

By Theorem 3.10, there exist $\mathcal{P}, T$ such that $T \subseteq \mathbb{Z}_n^\times$, $\mathcal{P}$ is a partition of $\mathbb{Z}_n \setminus T$ and for each $P \in \mathcal{P}$, both $|P| = p$ and $\sum_{x\in P} x \in \mathbb{Z}_n^\times$. Let $|\mathcal{P}| = \ell$ and $|T| = t$, then $n = \ell p + t$. Using the constructions defined in Lemmas 4.1, 4.3 and 4.5, define

$$K_1 = \{A(P)_k : A \in \mathcal{C}_1, P \in \mathcal{P}, k \in \mathbb{Z}_p\},$$
$$K_2 = \{A(P) : A \in \mathcal{C}_2, P \in \mathcal{P}\}, \text{ and}$$
$$K_3 = \{A(u) : A \in \mathcal{C},\ u \in T\}.$$

Let $K = K_1 \cup K_2 \cup K_3$. We claim that K is a valid set of base paths of $K_{m\times n}$.

By Lemmas 4.1, 4.3, and 4.5, each element in K is a base path. We now need to show that each edge length in $K_{m\times n}$ is used by exactly one base path in the set K. We have that $|\partial(K_{m\times n})| = n(m-1)$. So,

$$\begin{aligned}
|\partial(K_1)| &\leq \sum_{A\in\mathcal{C}_1}\sum_{P\in\mathcal{P}}\sum_{k=0}^{p-1}|\partial(A(P)_k)| = \sum_{A\in\mathcal{C}_1}\sum_{P\in\mathcal{P}}\sum_{k=0}^{p-1} 2\cdot(\text{len}(A)) = \sum_{A\in\mathcal{C}_1}\sum_{P\in\mathcal{P}} 2p\cdot(\text{len}(A)) \\
&= \sum_{A\in\mathcal{C}_1} 2p\ell\cdot(\text{len}(A)) = \sum_{d\in D_1} 2p\ell d\cdot r_d = 2\ell p\sum_{d\in D_1} d\cdot r_d, \\
|\partial(K_2)| &\leq \sum_{A\in\mathcal{C}_2}\sum_{P\in\mathcal{P}}|\partial(A(P))| = \sum_{A\in\mathcal{C}_2}\sum_{P\in\mathcal{P}} 2\cdot\text{len}(A) = \sum_{A\in\mathcal{C}_2} 2\ell\cdot(\text{len}(A)) = \sum_{d\in D_2} 2\ell pd\cdot r_d \\
&= 2\ell\sum_{d\in D_2} pd\cdot r_d, \text{ and} \\
|\partial(K_3)| &\leq \sum_{A\in\mathcal{C}}\sum_{u\in T}|\partial(A(u))| = \sum_{A\in\mathcal{C}}\sum_{u\in T} 2\cdot(\text{len}(A)) = \sum_{A\in\mathcal{C}} 2t\cdot\text{len}(A)) = \sum_{d\in D} 2td\cdot r_d \\
&= 2t\sum_{d\in D} d\cdot r_d.
\end{aligned}$$

Furthermore,

$$\begin{aligned}
|\partial(K_1)| + |\partial(K_2)| + |\partial(K_3)| &\leq \left(2\ell p\sum_{d\in D_1} d\cdot r_d\right) + \left(2\ell\sum_{d\in D_2} pd\cdot r_d\right) + \left(2t\sum_{d\in D} d\cdot r_d\right) \\
&= 2\left[\ell p\left(\sum_{d\in D_1} d\cdot r_d + \sum_{d\in D_2} d\cdot r_d\right) + t\sum_{d\in D} d\cdot r_d\right] \\
&= 2\left[\ell p\sum_{d\in D_1\cup D_2} d\cdot r_d + t\sum_{d\in D} d\cdot r_d\right].
\end{aligned}$$

Base paths	Edge lengths
$A = [0, 2, 1]_{15}$	$\{2,\ 20,\ 14\}$
$B = [0, 6, 15, 12, 16, 11, 3]_{14}$	$\{6, 9, 18, 4, 16, 13, 11\}$

Table 6. Valid set of base paths for a cyclic decomposition of K_{21}

There are 10 distinct edge lengths in K_{21} which are accounted for in the base paths of length 3 and 7 given here. This particular example is attributed to Buratti [5].

Continuing with our manipulation, we have that

$$\begin{aligned}
|\partial(K_1)| + |\partial(K_2)| + |\partial(K_3)| &\leq 2\left[\ell p \sum_{d\in D} d\cdot r_d + t\sum_{d\in D} d\cdot r_d\right] \\
&= 2\left[(\ell p + t)\sum_{d\in D} d\cdot r_d\right] \\
&= 2n\sum_{d\in D} d\cdot r_d \\
&= 2n\left(\frac{m-1}{2}\right) \\
&= n(m-1).
\end{aligned}$$

Thus, there are at most $n(m-1)$ distinct edge lengths used by base paths in K. Note that K has exactly $n(m-1)$ distinct edge lengths, and is therefore a valid set of base paths of $K_{m\times n}$ if and only if each edge length of $\partial(K_{m\times n})$ appears in K.

Now, we show that every edge length in $\partial(K_{m\times n})$ is used by some base path in K. Let $(x, y) \in \partial(K_{m\times n})$. Then, $x \in \partial(K_m)$, and so, $x \in \partial(A)$ for some $A \in \mathcal{C}$, and $y \in T$ or $y \in P$ for some $P \in \mathcal{P}$. Observe that $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2$ and $\mathcal{C}_1 \cap \mathcal{C}_2 = \emptyset$.

If $A \in \mathcal{C}_1$ and $y \in P$ for some $P \in \mathcal{P}$, then $(x, y) \in \partial(A(P)_v)$ for some $v \in \mathbb{Z}_p$. If $A \in \mathcal{C}_2$ and $y \in P$ for some $P \in \mathcal{P}$, then $(x, y) \in \partial(A(P))$. If $y \in T$, then $(x, y) \in \partial(A(y))$. Thus, all edge lengths in $\partial(K_{m\times n})$ are used by a base path of K. So K is a valid set of bath paths of $K_{p\times n}$. $\square$

Example 4.8. As demonstrated in preceding examples, we have a cyclic decomposition of K_{21} with the base paths in Table 6. Observe that $\mathrm{len}(A) = 3$ and $\mathrm{len}(B) = 7$. Let $P = \{0, 1, 2\}$ and consider the partition $\{P \cup \{\{u\} : u \in \mathbb{Z}_{17} \backslash P\}\}$ of $\mathbb{Z}_{17}$. Note that the elements of P add to a unit of $\mathbb{Z}_{17}$, namely 3. Then, we end up with the following sets K_1, K_2 and K_3, given in Table 7, which give a valid set of base paths for a cyclic 17-symmetric Hamilton cycle decomposition of $K_{21\times 17}$.

With the result from the previous theorem, we can now leverage the following theorem by Buratti [5] in order to reach our main conclusion.

Theorem 4.9. *There exists a cyclic decomposition of K_m if and only if m is an odd integer and $m \neq 15$ or p^a for a prime $p > 2$, $a \geq 2$.*

The following corollary immediately follows from Theorem 4.9 and Theorem 4.7.

Corollary 4.10. *Let m, n be positive odd integers. If $m \neq 15$ or p^a for a prime $p > 2$, $a \geq 2$ and n is a prime power bigger than the smallest prime divisor of m, then $K_{m\times n}$ has a cyclic n-symmetric Hamilton cycle decomposition.*

Set	Name	Base paths	Edge Lengths
K_1	$A(P)_0$	$[(0,0),(2,0),(1,1)]_{(15,3)}$	$\pm\{(2,0),(20,1),(14,2)\}$
	$A(P)_1$	$[(0,0),(2,1),(1,3)]_{(15,3)}$	$\pm\{(2,1),(20,2),(14,0$
	$A(P)_2$	$[(0,0),(2,2),(1,2)]_{(15,3)}$	$\pm\{(2,2),(20,0),(14,1)\}$
K_2	$B(P)$	$[(0,0),(6,0),(15,1),(12,3),(16,3),(11,4),(3,6),$ $(14,6),(20,7),(8,9),(5,9),(9,10),(4,12),(17,12),$ $(7,13),(13,15),(1,15),(19,16),(2,1),(18,1),(10,2)]_{(0,4)}$	$\pm\{(6,0),(9,1),(18,2),(4,0),(16,1),(13,2),(11,0),$ $(6,1),(9,2),(18,0),(4,1),(16,2),(13,0),(11,1),$ $(6,2),(9,0),(18,1),(4,2),(16,0),(13,1)]_{(11,2)}\}$
K_3	$A(u)$	$[(0,0),(2,u),(1,2u)]_{(15,3u)}$	$\pm\{(2,u),(20,u),(14,u)\}$
	$B(u)$	$[(0,0),(6,u),(15,2u),(12,3u),(16,4u),(11,5u),(3,6u)]_{(14,7u)}$	$\pm\{(6,u),(9,u),(18,u),(4,u),(16,u),(13,u),(11,u)\}$

Table 7. Valid set of base paths of $K_{21\times 17}$

We use the cyclic decomposition of K_{21} given in Table 6 and the partition $\{P \cup \{\{u\} : u \in \mathbb{Z}_{17} \setminus P\}\}$ of $\mathbb{Z}_{17}$, where $P = \{0,1,2\}$ to produce a valid set of base paths leading to a cyclic 17-symmetric Hamilton cycle decomposition of $K_{21\times 17}$.

CHAPTER 5

FUTURE DIRECTIONS

In the course of this work, we were able to find cyclic n-symmetric Hamilton cycle decompositions of the graph $K_{m\times n}$ in cases where m is odd, the part size n is odd and

- m is an odd prime,
- $m \neq 15$,
- m is not a nontrivial prime power, and
- the smallest prime divisor of m is less than n.

This leaves a lot of cases unresolved. What is left to be investigated are cases when m is not prime (composite) and in particular when m and n have a factor in common. In an attempt to build a decomposition of $K_{pq\times n}$, we derive the following lemma, taking a decomposition of $K_{p\times n}$, but could not completely get a valid decomposition of the graph $K_{pq\times n}$. The proof for the following lemma is similar to the proof shown in Lemma where the ordinates are switched.

Lemma 5.1. *Let p be an odd prime, q a prime power such that $q > p$, q, p relatively prime and n an odd integer relatively prime to p and q.*

Suppose A is a base path of length p in $K_{p\times n}$, where
$A = [(x_0, z_0) = (0, 0), (x_1, z_1), (x_2, z_2), \ldots, (x_{p-1}, z_{p-1})]_{(0,z_p)}$. *Define*
$A_q = [w_0, w_1, w_2, \ldots, w_{pq-1}]_{w_{pq}}$, *where* $w_i = (x_{r_i}, q_i(r_i + 1), z_{r_i} + q_i z_p)$ *for* $i = q_i p + r_i$, $0 \leq r_i < p$. *Then A_q is a base path of length pq in a decomposition of $K_{pq\times n}$. Observe that*
$\partial(A_q) = \pm\{(x, y, z) : y \in \mathbb{Z}_q, (x, z) \in \partial(A)\}$.

Despite our best attempts, we could not leverage this to get any substantial results. But we are hopeful for the future.

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APPENDIX A

APPROVAL LETTER

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Office of Research Integrity

October 26, 2020

Fatima Akinola
Department of Mathematics
523 Smith Hall
Marshall University

Dear Ms. Akinola:

This letter is in response to the submitted thesis abstract entitled *"On the Hamilton cycle decomposition of complete multipartite graphs which are both cyclic and symmetric."* After assessing the abstract, it has been deemed not to be human subject research and therefore exempt from oversight of the Marshall University Institutional Review Board (IRB). The Code of Federal Regulations (45CFR46) has set forth the criteria utilized in making this determination. Since the information in this study does not involve human subjects as defined in the above referenced instruction, it is not considered human subject research. If there are any changes to the abstract you provided then you would need to resubmit that information to the Office of Research Integrity for review and a determination.

I appreciate your willingness to submit the abstract for determination. Please feel free to contact the Office of Research Integrity if you have any questions regarding future protocols that may require IRB review.

Sincerely,

Bruce F. Day

Bruce F. Day, ThD, CIP
Director

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