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Effect of internal heat source modulations on the onset of triple diffusive convection in viscoelastic liquids

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The paper aims to study the dynamic behavior of a triple diffusive system subjected to sinusoidal (trigonometric cosine) and non-sinusoidal wave forms (square, sawtooth and triangular) of internal heat source modulation. The configuration of the system is such that a layer of viscoelastic liquid is heated and salted with two solutes from below. An Oldroyd-B type model is made use for viscoelastic liquids. In order to regulate the convection onset, internal heat source modulation is applied. This investigation is modelled using a linear stability analysis where a stationary convection is preferred. Venezian approach facilitates a solution by finding the eigen values of the problem. The influence of pertinent parameters which are varied for a wide range of values have been reported. It is captured via graphs that for small values of frequency of modulation, square wave form is more stable while sawtooth wave form is more stable for an increment in the values of frequency of modulation. Further, liquids such as Newtonian, Maxwell and Rivlin-Ericksen are analysed as the limiting cases of the problem. It seems worthwhile to discuss the results of the present study as it is the first work on linear theory of different wave forms of internal heat source modulation and thus paves a way for new theoretical and experimental endeavors.

Keywords: Convection, Internal heat source modulation, Venezian approach, Viscoelastic liquid

1 Introduction

Over the years, convective processes have been gaining quite a lot of applications, since it is one of the ways heat transfer can take place. Bénard[1] initialized the study on thermal convection experimentally and Rayleigh[2] concluded that convection onset is dependent on a non-dimensional number that characterizes the stability of the system. The process in which convection takes place due to two stratifying agents having different diffusivities is the double diffusive convection. The beginning of the concept of salting while heating is done by Stommel *et al.*[3]. Due to the fact that double diffusive convection could not overcome its major drawback of the number of diffusing components, this is when Triple Diffusive Convection (TDC) began to gain its importance. In a TDC, the existence of three stratifying agents such that all these three factors have different rate of diffusivities makes more relevance to the study. In the 19th century convection by adding a third diffusing component is initially carried out by Griffiths[4], Poulikakos[5] and Pearlstein *et al.*[6] Studies pertaining to the concept of TDC is studied by researchers Sameena[7,8], Raghunatha and Shivakumara[9], Pranesh *et al.*[10], Sameena and Pranesh[11]. TDC has an enormous number of applications which includes in the field of oceanography where there are tons of solutes that are present at the ocean bed. In order to study the behavior of these solutes towards the other influencing factors, study of multicomponent system shows its prime importance and relevance. Understanding the related changes that take place due to temperature and different levels of concentration in chemical studies, the current study shows its great necessity. Research on flora and fauna will serve a purpose in discoveries about medicines and their reaction to certain factors that have an influence on them.

On broadly classifying the liquids into newtonian and non-newtonian liquids, on the basis of whether or not the liquids obey the newtonian law, we come across a major category of liquids called the viscoelastic liquids. These liquids display the properties of viscosity and elasticity when they undergo some deformation. Examples of such liquids are polymers, oils and oil paints. Comprehensive review on viscoelastic liquids is carried out by researchers Chand[12], Siddheshwar[13], Yadav and

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Maqhusi[14], Park[15] and Vanishree and Anjana[16]. Viscoelastic liquids can further be classified into Oldroyd-B, Maxwell and Rivlin-Ericksen liquids depending upon their values of viscoelastic parameters. The wide usage of viscoelastic liquids in the industries and its importance in the field of DNA replication wherein the DNA suspensions are considered as viscoelastic liquid is a very expensive and a time taking procedure. In optics, viscoelastic liquids are very useful to print regular patterns on desired surfaces. An elaborate study on the applications of viscoelastic liquids is well explained by Pérez *et al.*[17]. Modulation being an external regulating factor can fluctuate the periodic wave types which generate the entire process of modulation. Modulation is subjected to amplitude and frequency associated with the wave form. Venezian[18] has initialized the study of a sinusoidal perturbation, termed as modulation. An extensive study on temperature modulation, gravity modulation, rotation modulation is carried out by Bhadauriaand Kiran[19], Bhadauria *et al.*[20], Siddheshwar *et al.*[21], Sun *et al.*[22], Manjula and Kiran[23], Kanchana *et al.*[24] and Kumar *et al.*[25]. Wave forms of modulation can be characterized as sinusoidal and non-sinusoidal wave types. Non-sinusoidal wave types includes square, sawtooth and triangular wave form which is studied by Siddheshwar and Kanchana[26] and Aanam *et al.*[27] on a Rayleigh–Ɓenard convection. The generation of heat in systems is a very important external regulating factor for problems involving convection. In some cases, the material itself offers its own source of heat leading to another way of convective motion. Internal heat source can generate a lot of amount of energy while the internal heat sink is an absorber of excessive heat. However, it is to be noted that internal heat sink is a negative heat source. Related studies on internal heat source are presented by Straughan and Tracey[28], Tasaka and Takeda[29], Tritton and Zarraga[30], Storesletten and Rees[31], Mahajan and Nandal[32], Nandal and Mahajan[33], Miquel[34] and Shankar *et al.*[35]. Bazylak *et al.*[36] studied internal heat source modulation in a natural convection such that a layer of fluid is distributed with heat sources. It is observed that there is a good amount of heat transfer when there is decrease in the temperature due to flux modulation. Processing and sterilization of eatables take place through internal heating. In several natural processes, internal heating gives rise to convection in the atmosphere. Its wide range of usage comprises of electronic appliances such as heaters, microwave ovens and refrigerators. Microwave heating is essential in order to get rid of the moisture from ceramics which is analyzed by Itaya *et al.*[37]. Though the applications of internal heat source are enormous, there is no literature on internal heat source modulation in a TDC, which leads to the motivation for research on this concept. Thus the main aim of this paper is to investigate the influence of sinusoidal and non-sinusoidal wave forms of internal heat source modulation on a TDC in viscoelastic liquids using a linear theory.

2 Materials and Methods

The problem considered was solved theoretically where Venezian approach was used to find the eigen value of the problem.

2.1 Mathematical Formulation

Consider a layer of viscoelastic liquid confined between two infinite parallel surfaces that are at a separation of distance d. This layer of viscoelastic liquid of thickness d is subjected to gravitational force $\vec{g}$ which is time dependent and acting vertically downwards. The density of the fluid depends upon three stratifying agents namely, temperature T and solute concentrations S_1 and S_2 such that they have different diffusivities. ΔT, ΔS_1 and ΔS_2 represents the temperature and solute concentration difference between the upper and lower surfaces. The lower surface is maintained at a higher temperature $T_0 + \Delta T$ and higher solute concentration $S_i = S_{i_0} + \Delta S_i$ $(i = 1, 2)$ at $z = 0$ when compared to the upper boundary which is maintained at a temperature T_0 and solute concentration S_{i_0} $(i = 1, 2)$ (see Fig. 1).

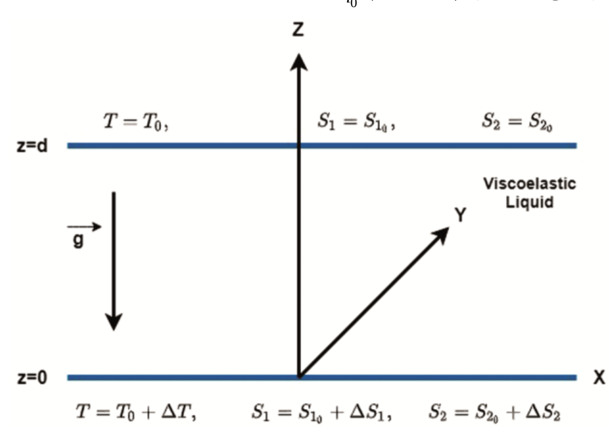

Fig. 1 — Schematic of a triple diffusive setup in a viscoelastic liquid.

The configuration that described the viscoelastic flow under Boussinesq approximation are governed by the following equations:

$$\nabla \cdot \vec{q} = 0, \quad \ldots (1)$$

$$\rho_o \left[\frac{\partial \vec{q}}{\partial t} + (\vec{q} \cdot \nabla)\vec{q} \right] = -\nabla p - \rho \vec{g} + \nabla.\tau', \quad \ldots (2)$$

$$\left[1+\lambda_1 \frac{\partial}{\partial \mathrm{t}}\right] \tau' = \mu \left[1+\lambda_1 \frac{\partial}{\partial \mathrm{t}}\right]\left[\nabla \vec{q} + \nabla \vec{q}^{tr}\right], \quad \ldots (3)$$

$$\frac{\partial T}{\partial t} + (\vec{q}.\nabla)T = \chi \nabla^2 T + \mathrm{Q}\,(1 + \varepsilon\ \mathrm{f}(\omega, \mathrm{t}))(\mathrm{T} - \mathrm{T}_0), \quad \ldots (4)$$

$$\frac{\partial S_1}{\partial t} + (\vec{q}.\nabla)S_1 = \chi_{S_1} \nabla^2 S_1, \quad \ldots (5)$$

$$\frac{\partial S_2}{\partial t} + (\vec{q}.\nabla)S_2 = \chi_{S_2} \nabla^2 S_2, \quad \ldots (6)$$

$$\rho = \rho_0 \begin{bmatrix} 1 - \alpha_t (T - T_0) + \alpha_{S_1} (S_1 - S_{1_0}) \\ +\alpha_{S_2} (S_2 - S_{2_0}) \end{bmatrix}. \quad \ldots (7)$$

where, $\vec{q}$ is the velocity, t is the time, tr is transpose, (x, y, z) are the Cartesian coordinates, p is the hydrodynamic pressure, T is the temperature, Q is the heat source, S_i is the solute concentration of the ith component $(i = 1, 2)$, λ_1 is the stress relaxation time, λ_2 is the strain retardation time, f is the wave form, $\mathrm{f}(\omega, \mathrm{t})$ is the time dependent internal heating modulation, ∇p is the pressure gradient, μ is the viscosity, ρ is the density, ε is the amplitude of modulation, ω is the frequency of modulation, α_t is the coefficient of thermal expansion, α_{S_i} is the coefficient of solutal expansion of the ith component $(i = 1, 2)$, χ is the thermal diffusivity, χ_{S_i} is the solutal diffusivity of the ith component $(i = 1, 2)$.The governing equations are solved subject to the following boundary conditions.

The thermal boundary conditions are:

$$\mathrm{T} = \mathrm{T}_0 + \Delta \mathrm{T} \text{ at } \mathrm{z} = 0;\ \mathrm{T} = \mathrm{T}_0 \text{ at } \mathrm{z} = \mathrm{d}; \quad \ldots (8)$$

The solutal boundary conditions are:

$$\mathrm{S_i} = \mathrm{S_{i_0}} + \Delta \mathrm{S_i}\ (\mathrm{i} = 1, 2) \text{ at } \mathrm{z} = 0; \quad \ldots (9)$$
$$\mathrm{S_i} = \mathrm{S_{i_0}}\ (\mathrm{i} = 1, 2) \text{ at } \mathrm{z} = \mathrm{d};$$

Combining the Eqs 2 and 3 into a single equation by using the divergence operator on Eq. 3 we get,

$$\left(1+\lambda_1 \frac{\partial}{\partial t}\right)\left[\rho_o\left(\frac{\partial \vec{q}}{\partial t} + (\vec{q}\cdot\nabla)\vec{q}\right) + \nabla p + \rho \vec{g}\right] = \mu\left(1+\lambda_2 \frac{\partial}{\partial t}\right)\left(\nabla^2 \vec{q}\right). \quad \ldots (10)$$

In the basic state, the fluids were assumed to be in a motionless state and the transfer of heat takes place through conduction described by:

$$\vec{q}_b = 0,\ p = p_b(z), T = T_b(z), \rho = \rho_b(z), \quad \ldots (11)$$
$$S_i = S_{b_i}(z)(i = 1, 2),$$

where, the subscript in the Eq. 11 represented the basic state of the fluid and these equations satisfy,

$$\frac{dp_b}{dz} + \rho_b g = 0, \quad \ldots (12)$$

$$\chi \frac{d^2 T_b}{dz^2} + Q(T - T_0) = 0, \quad \ldots (13)$$

$$\frac{d^2 S_{i_b}}{dz^2} = 0\ (i = 1, 2), \quad \ldots (14)$$

$$\rho_b = \rho_0 \begin{bmatrix} 1 - \alpha_t (T_b - T_0) + \alpha_{S_{1b}} (S_1 - S_{1_0}) \\ +\alpha_{S_{2b}} (S_2 - S_{2_0}) \end{bmatrix}, \quad \ldots (15)$$

which give rise to the steady state solution of the Eqs 13 and 14 subject to the boundary condition in Eqs 8 and 9,

$$T_b = T_0 + \Delta T \left[\frac{\mathrm{Sin}\sqrt{\frac{Qd^2}{\chi}}\left(1 - \frac{z}{d}\right)}{\mathrm{Sin}\sqrt{\frac{Qd^2}{\chi}}}\right], \quad \ldots (16)$$

$$S_{i_b} = S_{i_0} + \Delta S_i \left(1 - \frac{z}{d}\right)(i = 1, 2), \quad \ldots (17)$$

On applying disturbance to the basic state by using the perturbations given in the form:

$$\vec{q} = \vec{q}_b + \vec{q}', p = p_b + p', T = T_b + T',$$
$$\rho = \rho_b + \rho', S_i = S_{i_b} + S_i'(i = 1, 2), \quad \ldots (18)$$

where, b is the basic state value and the prime quantities represented the perturbation. Using the perturbed expressions in the governing equations and non-dimensionalizing using the following definition

$$(x^*, y^*, z^*) = \frac{1}{d}(x, y, z), \vec{q}^* = \frac{d}{\chi}\vec{q}, t^* = \frac{\chi'}{d^2}t,$$
$$p^* = \frac{d^2}{\mu_1 \chi} p', \theta = \frac{T'}{\Delta T}, \varphi_{S_i} = \frac{S_i}{\Delta S_i}(i = 1, 2), \quad \ldots (19)$$

we get the following dimensionless equations after eliminating p and ρ by usual procedure and introducing the stream function $\psi(x, z, t)$ such that $\vec{q} = \left(\frac{\partial \psi}{\partial z}, 0, -\frac{\partial \psi}{\partial x}\right)$, by restricting ourselves to 2-Dimensions we get,

$$\left(1+\Lambda_1\frac{\partial}{\partial t}\right)\begin{pmatrix} \frac{1}{\Pr}\begin{pmatrix}\frac{\partial}{\partial t}\left(\nabla^2\psi\right) \\ -J(\psi,\nabla^2\psi)\end{pmatrix} + Ra\frac{\partial\theta}{\partial x} \\ -R_{S_1}\frac{\partial\varphi_{S_1}}{\partial x} - R_{S_2}\frac{\partial\varphi_{s_2}}{\partial x}\end{pmatrix} = \left(1+\Lambda_1\frac{\partial}{\partial t}\right)\nabla^4\psi, \quad \ldots (20)$$

$$\frac{\partial\theta}{\partial t} + \frac{\partial\psi}{\partial x}g(z) = \nabla^2\theta + Ri\theta \left[1+\varepsilon f(\Omega,t)\right] + J(\psi,\theta), \quad \ldots (21)$$

$$\frac{\partial\varphi_{S_1}}{\partial t} + \frac{\partial\psi}{\partial x} = \tau_1\nabla^2\varphi_{S_1} + J(\psi,\varphi_{S_1}), \quad \ldots (22)$$

$$\frac{\partial\varphi_{S_2}}{\partial t} + \frac{\partial\psi}{\partial x} = \tau_2\nabla^2\varphi_{S_2} + J(\psi,\varphi_{S_2}), \quad \ldots (23)$$

Where $\Omega = \frac{d^2\omega}{\chi}$ is the dimensionless modulation frequency, $Ra = \frac{\rho_0\alpha_t\vec{g}d^3\Delta T}{\mu\chi}$ is the thermal Rayleigh number, $R_{S_i} = \frac{\rho_0\alpha_{S_i}\vec{g}d^3\Delta S_i}{\mu\chi}$ is the solutal Rayleigh number of the ith component $(i = 1, 2)$, $Ri = \frac{Qd^2}{\chi}$ is the internal Rayleigh number, $\Lambda_1 = \frac{\chi\lambda_1}{d^2}$ is the stress relaxation parameter, $\Lambda_2 = \frac{\chi\lambda_2}{d^2}$ is the strain retardation parameter, $\tau_i = \frac{\chi_{S_i}}{\chi}$ is the ratio of diffusivity of ith component $(i = 1, 2)$, θ is the dimensionless temperature, φ_{S_i} - Dimensionless solute of the ith component $(i = 1, 2)$, $\Pr = \frac{\mu}{\rho_0\chi}$ is the Prandtl number and $J(.,.)$ is the Jacobian with respect to x and z. To obtain the condition for the onset of convection, we neglected the Jacobian terms and on writing the equations in terms of ψ we get:

$$\begin{bmatrix} X_4X_3X_2X_1\nabla^2 - \left(1+\Lambda_1\frac{\partial}{\partial t}\right) \\ \begin{pmatrix} RaX_4X_3g(z) \\ -R_{S_1}X_4X_2 - R_{S_2}X_3X_2\end{pmatrix}\frac{\partial^2}{\partial x^2}\end{bmatrix}\psi = 0. \quad \ldots (24)$$

where,

$$X_1 = \left[\left(1+\Lambda_1\frac{\partial}{\partial t}\right)\left(\frac{1}{\Pr}\frac{\partial}{\partial t}\right) - \left(1+\Lambda_1\frac{\partial}{\partial t}\right)\nabla^2\right],$$
$$X_2 = \left[\frac{\partial}{\partial t} - \nabla^2 + Ri\theta\left(1+\varepsilon f(\Omega,t)\right)\right],$$
$$X_3 = \left[\frac{\partial}{\partial t} - \tau_1\nabla^2\right], X_4 = \left[\frac{\partial}{\partial t} - \tau_2\nabla^2\right].$$

The boundary conditions for solving the Eq. (24) is given by,

$$\psi = \frac{\partial^2\psi}{\partial z^2} = \frac{\partial^4\psi}{\partial z^4} = \frac{\partial^6\psi}{\partial z^6} = \frac{\partial^8\psi}{\partial z^8} = 0 \text{ at } z = 0, 1. \quad \ldots (25)$$

2.2 Method of Solution

The solution of the Eq. 24 is obtained by expanding the eigen value Ra and eigen function ψ in the form:

$$(Ra,\psi) = (Ra_0,\psi) + \varepsilon(Ra_1,\psi_1) + \varepsilon^2(Ra_2,\psi_2) + ..., \quad \ldots (26)$$

where, Ra_0 is the Rayleigh number for the unmodulated case and Ra_i $(i = 1, 2, ...)$ are the corrections for Ra_0. Substituting the expression in Eq. 24 and equating the coefficients of like powers of ε, yields:

$$L\psi_0 = 0, \quad \ldots (27)$$

$$L\psi_1 = \left[\begin{array}{l} X_4 X_3 X_1 Rif(\Omega,t)\nabla^2 \\ -\left(1+\Lambda_1 \frac{\partial}{\partial t}\right)\left(\begin{array}{l} Ra_1 X_4 X_3 g(z) \\ +R_{S_1} X_4 Rif(\Omega,t) + \\ R_{S_2} X_3 Rif(\Omega,t) \end{array} \right)\frac{\partial^2}{\partial x^2} \end{array} \right]\psi_0 = 0, \qquad \ldots (28)$$

$$\left. \begin{array}{l} \left(1+\Lambda_1 \frac{\partial}{\partial t}\right)\left(\begin{array}{l} Ra_1 X_4 X_3 g(z) + R_{S_1} X_4 Rif(\Omega,t) \\ +R_{S_2} X_3 Rif(\Omega,t) \end{array} \right)\frac{\partial^2 \psi_1}{\partial x^2} \\ L\psi_2 = +\left(1+\Lambda_1 \frac{\partial}{\partial t}\right) Ra_2 X_4 X_3 g(z)\frac{\partial^2 \psi_0}{\partial x^2} \\ +X_4 X_3 X_1 Rif(\Omega,t)\nabla^2 \psi_1, \end{array} \right\}, \qquad \ldots (29)$$

where, the operator L is given as,

$$L = X_5 X_4 X_3 X_1 \nabla^2 - \left(1+\Lambda_1 \frac{\partial}{\partial t}\right)$$
$$\left(\begin{array}{l} Ra_0 X_4 X_3 g(z) - R_{S_1} X_5 X_4 \\ -R_{S_2} X_5 X_3 \end{array} \right)\frac{\partial^2}{\partial x^2}, \qquad \ldots (30)$$
$$X_5 = \left[\frac{\partial}{\partial t} - \nabla^2 - Ri\right].$$

The stability of the system was examined by introducing velocity perturbation ψ_0, expressed as:

$$\psi_0 = \mathrm{Sin}(\pi\alpha x)\,\mathrm{Sin}(\pi\alpha). \qquad \ldots (31)$$

Substituting the velocity perturbation ψ_0 in the Eq. 27, we arrived at an expression for Rayleigh number corresponding to the unmodulated case in the form:

$$Ra_0 = \frac{R_{S_1}}{\tau_1 g(z)}\left(1 - \frac{Ri}{k^2}\right) + \frac{R_{S_2}}{\tau_2 g(z)}\left(1 - \frac{Ri}{k^2}\right) + \frac{k^4(k^2 - Ri)}{\pi^2 a^2 g(z)}, \qquad \ldots (32)$$

where, $k^2 = \pi^2(1+a^2)$ and

$$g(z) = \frac{\sqrt{Ri}\,\mathrm{Cos}\left(\sqrt{Ri}(1-z)\right)}{\mathrm{Sin}(\sqrt{Ri})} \text{ such that } \int_0^1 g(z)dz = 1.$$

(See Table 1)

Eq. (28) now can be written as:

$$L\psi_1 = -\psi_0 \left[\begin{array}{l} A_1 f^{iv}(\Omega,t) + A_2 f'''(\Omega,t) + A_3 f''(\Omega,t) \\ +A_4 f'(\Omega,t) + Af(\Omega,t) \end{array} \right] \qquad \ldots (33)$$
$$-Ra_1 \pi^2 a^2 \tau_1 \tau_2 k^4 g(z).$$

Equation 33 has a solution provided that the RHS of the equation must be orthogonal to the null space operator L. Therefore the only steady term in the Eq. 33 is $Ra_1 \pi^2 a^2 \tau_1 \tau_2 k^4 g(z)$, which implies that $Ra_1 = 0$ and all odd coefficients are zero, i.e., $Ra_1 = Ra_3 = ... = 0$. Furthermore we found that,

$$L\left[\mathrm{Sin}(\pi z)\,\mathrm{Sin}(\pi\alpha x)e^{-i\Omega t}\right] = L(\Omega)\,\mathrm{Sin}(\pi z)\,\mathrm{Sin}(\pi\alpha x)e^{-i\Omega t}, \qquad \ldots (34)$$

where,

$$L(\Omega) = Y_1 + iY_2,$$

$$Y_1 = \frac{1}{\mathrm{Pr}}\left[\begin{array}{l} \mathrm{Pr}\,\pi^2 a^2 (Ra_0 g(z)(\tau_1\tau_2 k^4 + \Omega^2(\Lambda_1 k^2(\tau_1 - \tau_2 k^2) \\ -1 + R_{S_1}(\Omega^2(1+\Lambda_1(k^2(1+\tau_2) - Ri) \\ -\tau_2 k^2(k^2 - Ri)) + R_{S_2}(\Omega^2(1+\Lambda_1(k^2(1+\tau_1) \\ -Ri) - \tau_1 k^2(k^2 - Ri))) - k^2(\Omega^4 - (\tau_1 + \tau_2) \\ (k^2 + Ri)\Omega^2 k^2 - \Omega^2 \tau_1 \tau_2 k^4 (1+\Lambda_1 Ri) + \mathrm{Pr}\,k^2 \\ (k^2(-k^4\Omega^2\Lambda_2(\tau_1+\tau_2) - \Omega^2(\tau_1 + \tau_2 + \Omega^2\Lambda_2) \\ +\tau_1\tau_2 k^2(k^2 - Ri - \Omega^2\Lambda_2) + \Omega^2\Lambda_2 Ri(\tau_1 + \tau_2 k^2)) \\ +\Omega^2(-k^2 + Ri))) \end{array} \right],$$

$$Y_2 = \frac{1}{\mathrm{Pr}}\left[\begin{array}{l} \mathrm{Pr}\,\pi^2 a^2 (Ra_0 g(z)\Omega(\Omega^2\Lambda_1 - k^2(\tau_1 + \tau_2 + \Lambda_1\tau_1\tau_2 k^2) \\ +R_{S_1}\Omega(k^2(1+\Omega\tau_2 + \Lambda_1\tau_2(k^2 - Ri)) - Ri - \Omega^2\Lambda_1) \\ +R_{S_2}\Omega(k^2(1+\Omega\tau_1 + \Lambda_1\tau_1(k^2 - Ri)) - Ri - \Omega^2\Lambda_1)) \\ -k^2(\Omega^3(k^2(1 - Ri - (\tau_1+\tau_2)) + (\tau_1+\tau_2)(k^2 - Ri) \\ \Lambda_1 k^2 + (\tau_1\tau_2 k^4 - \Omega^2)\Lambda_1) \\ +\Omega^2\tau_1\tau_2 k^4(Ri - k^2) + \mathrm{Pr}\,k^2\Omega^3(1+\Lambda_2(k^2 - Ri) + \\ (\tau_1+\tau_2)\Lambda_2 k^2) + k^4\Omega((\tau_1+\tau_2)(Ri - k^2) - \tau_1\tau_2 k^2 \\ (1+\Lambda_2(k^2 - Ri)))) \end{array} \right],$$

Table 1 — Limiting cases of the present study

Limiting cases	Expression reduces to	References
$R_{S_1} = 0, R_{S_2} = 0, Ri = 0$	Classical Rayleigh Bénard problem	Rayleigh[2]
$R_{S_1} = 0, R_{S_2} = 0, Ri \neq 0$	Rayleigh Bénard convection with internal heat	Tasaka and Takeda[29], Tritton and Zarraga[30], Kanchana and Zhao[38]
$R_{S_1} \neq 0, R_{S_2} = 0, Ri = 0$	Two component convection without internal heat	Schmitt[39] and Stern[40]
$R_{S_1} \neq 0, R_{S_2} = 0, Ri \neq 0$	Two component convection with internal heat	Zhao *et al.*[41]
$R_{S_1} \neq 0, R_{S_2} \neq 0, Ri = 0$	Three component convection without internal heat	Sameenaand Pranesh[42]
$R_{S_1} \neq 0, R_{S_2} \neq 0, Ri \neq 0$	Three component convection with internal heat	Melathil*et al.*[43] and Awasthi*et al.*[44]

Hence we arrived at a particular solution of the Eq. 33,

$$\psi_1 = -\frac{1}{\left|L(\Omega)^2\right|} Y_1 \begin{bmatrix} A_1 f'^{v}(\Omega,t) + A_2 f'''(\Omega,t) \\ +A_3 f''(\Omega,t) + A_4 f'(\Omega,t) \\ +Af(\Omega,t) \end{bmatrix} \psi_0. \quad \ldots (35)$$

where,

$$A_1 = \left[\frac{\Lambda_1 Rik^2}{\text{Pr}}\right],$$

$$A_2 = \left[\frac{\Lambda_1 Rik^4}{\text{Pr}}(\tau_2 + \tau_1) + \Lambda_2 Rik^4 + \frac{Rik^2}{\text{Pr}}\right],$$

$$A_3 = \begin{bmatrix} \Lambda_1 Ri\pi^2 a^2 (R_{S_1} + R_{S_2}) + \Lambda_2 Rik^6 (\tau_2 + \tau_1) \\ +\frac{Rik^6}{\text{Pr}}(\tau_2 + \tau_1) + \frac{\tau_2\tau_1\Lambda_1 Rik^6}{\text{Pr}} + Rik^4 \end{bmatrix},$$

$$A_4 = \begin{bmatrix} R_{S_1} Ri\pi^2 a^2 (1 + \tau_2\Lambda_1 k^2) + R_{S_2} Ri\pi^2 a^2 \\ (1 + \tau_1\Lambda_1 k^2) + Rik^6 (\tau_2 + \tau_1) \\ +\tau_2\tau_1\Lambda_2 Rik^8 + \frac{\tau_2\tau_1 Rik^6}{\text{Pr}} \end{bmatrix},$$

$$A_5 = \left[Ri\pi^2 a^2 k^2 (R_{S_1}\tau_2 + R_{S_2}\tau_1) + \tau_1\tau_2 Rik^8\right],$$

On simplifying Eq. (29), the equation for ψ_2 takes the form:

$$L\psi_2 = -\psi_1 \begin{bmatrix} A_1 f'^{v}(\Omega,t) + A_2 f'''(\Omega,t) \\ +A_3 f''(\Omega,t) + A_4 f'(\Omega,t) \\ +Af(\Omega,t) \end{bmatrix} \quad \ldots (36)$$

$$-\psi_0\left[Ra_2\pi^2 a^2 \tau_1\tau_2 k^4 g(z)\right].$$

The Eq. 36 is not completely solved, rather it is used to arrive at an expression for the correction Rayleigh number. If the equation for $L\psi_2$ is to have a solution, then the RHS of Eq. 36 must be orthogonal to $\text{Sin}(\pi z)$, or equivalently,

$$\int_0^1 \begin{pmatrix} (A_1 f'^{v}(\Omega,t) + A_2 f'''(\Omega,t) \\ +A_3 f''(\Omega,t) + A_4 f'(\Omega,t) \\ +Af(\Omega,t))\psi_1 + \\ Ra_2\pi^2 a^2 \tau_1\tau_2 k^4 g(z)\psi_0 \end{pmatrix} \text{Sin}(\pi z)dz = 0 \quad \ldots (37)$$

Finally taking the time average we arrived at an expression for correction Rayleigh number,

$$R_{2C} = \frac{\Omega Y_1 A}{2\left|L(\Omega)\right|^2 \pi^3 a^2 \tau_1\tau_2 k^4}, \quad \ldots (38)$$

where, $$A = \int_0^{\frac{2\pi}{\Omega}} \begin{bmatrix} A_1 f'^{v}(\Omega,t) + A_2 f'''(\Omega,t) \\ +A_3 f''(\Omega,t) + A_4 f'(\Omega,t) \\ +Af(\Omega,t) \end{bmatrix}^2 dt.$$

In this research paper, four types of wave forms of internal heat source modulation were considered (one sinusoidal and three non-sinusoidal). Mathematically these wave forms are defined as:

1. Sinusoidal wave form
   $f(\Omega,t) = \text{Cos}(\Omega t),$

2. Non-Sinusoidal wave forms
   (i) Square wave form
   $$f(\Omega,t) = \frac{4}{\pi}\sum_{n=1,3,5,\ldots}^{\infty} \frac{\text{Sin}(n\Omega t)}{n},$$
   (ii) Triangular wave form
   $$f(\Omega,t) = \frac{8}{\pi^2}\sum_{n=1,3,5,\ldots}^{\infty} \frac{(-1)^{\frac{n-1}{2}}}{n}\text{Sin}(n\Omega t),$$
   (iii) Sawtooth wave form
   $$f(\Omega,t) = \frac{4}{\pi^2}\sum_{n=1}^{\infty} \frac{\left[1 - \text{Cos}(n\pi)\right]\text{Cos}(n\Omega t)}{n^2}.$$

3 Results and Discussion

In this section we discuss the results with respect to the correction Rayleigh number obtained in Eq. 38 for triple diffusive convection with internal heat modulation in a viscoelastic liquid modeled by Oldroyd–B type model. The purpose of this study is to consider the internal heating modulation to regulate the onset of convection. One sinusoidal (Trigonometric Cosine TR) and three non-sinusoidal (Square SQ, Sawtooth SA and Triangular TR) wave forms were considered. From Eq. 38 it is clear that R_{2C} depends on the viscoelastic parameters Λ_1 and Λ_2, solute Rayleigh number R_{S_i} $(i = 1,2)$, diffusivity ratio τ_i $(i = 1,2)$, internal Rayleigh number Ri, Prandtl number Pr, frequency of modulation Ω and amplitude of modulation ε. The results obtained depend on the value of frequency of internal heating modulation Ω. When the values of Ω is such that Ω << 1, the modulation becomes large which in turn grows the disturbance. When the value of $\Omega \to \infty$, R_{2C} approaches zero thereby reduced the modulation effect. Therefore, moderate values of Ω were taken under consideration in the present study.

From the figures in this section, it was observed that TR wave form overlaps with SA wave form. We also observed that an increase in values of Ω decreased R_{2C} indicating that the frequency of modulation destabilized the system and reaches subcritical motion and further increase in Ω, increased R_{2C} which made the system stabilizing. Let Ω_C (around 2) be the frequency of modulation that changes the system from destabilizing to stabilizing. For $\Omega = \Omega_C$, the modulating frequency Ω obtained its maximum destabilizing effect. Also we observed that $R_{2C} > 0$ for $\Omega \le \Omega_C$ and $R_{2C} < 0$ for $\Omega > \Omega_C$.

3.1 Effect of viscoelastic parameters on the onset of convection

The influence of Λ_1 and Λ_2 on triply diffusive convection is shown in Figs 2 and 3 respectively, i.e., increasing the values of Λ_1, it was found that the system becomes unstable since the magnitude of R_{2C} decreased. This happens due to the fact that the elasticity in Λ_1 indicates the relaxation of stress in the viscoelastic liquid. When Λ_2 increased, the magnitude of R_{2C} also increased which made the system stable. This is due to the fact that Λ_2 responds to the applied stress in the liquid. It was observed that,

(i) When $\Omega \le \Omega_C$, $R_{2C|\Lambda_1=0.4} \ge R_{2C|\Lambda_1=0.6}$ and $R_{2C|\Lambda_2=0.05} \le R_{2C|\Lambda_2=0.1}$,

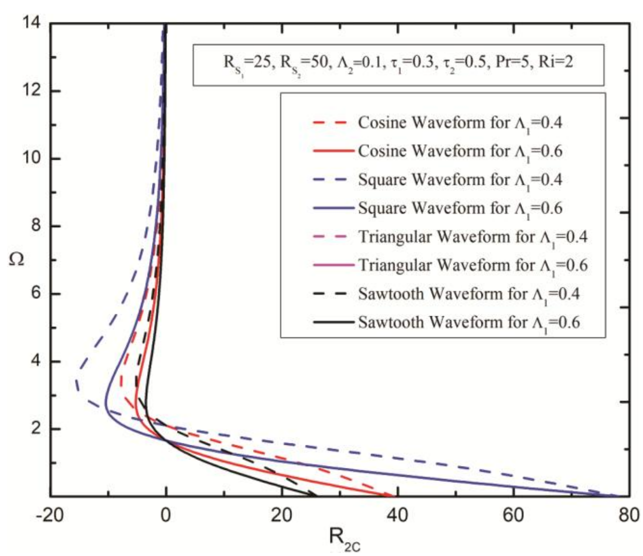

Fig. 2 — Variation of R_{2C} with Ω for different values of Λ_1 for TC, SQ, TR and SA wave forms.

(ii) When $\Omega > \Omega_C$, $R_{2C|\Lambda_1=0.4} \le R_{2C|\Lambda_1=0.6}$ and $R_{2C|\Lambda_2=0.05} \ge R_{2C|\Lambda_2=0.1}$.

3.2 Effect of solutes on the onset of convection

The influence of R_{S_1} and R_{S_2} on triply diffusive convection is shown in Figs 4 and 5 respectively. The solutes S_1 and S_2 are both added from the bottom of the triply diffusive layer. On increasing R_{S_i} $(i = 1,2)$, increased the magnitude of R_{2C}. The reason being

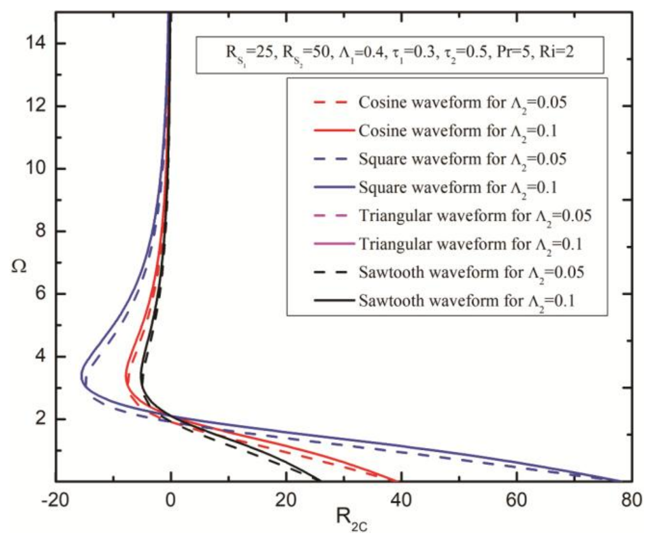

Fig. 3 — Variation of R_{2C} with Ω for different values of Λ_2 for TC, SQ, TR and SA wave forms.

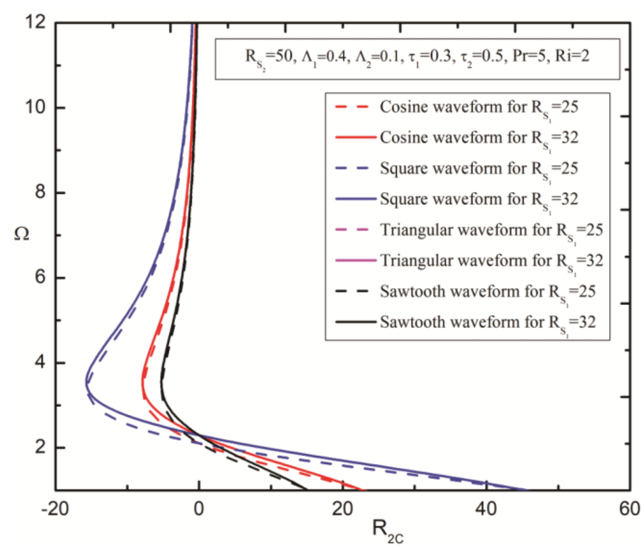

Fig. 4 — Variation of R_{2C} with Ω for different values of R_{S_1} for TC, SQ, TR and SA wave forms.

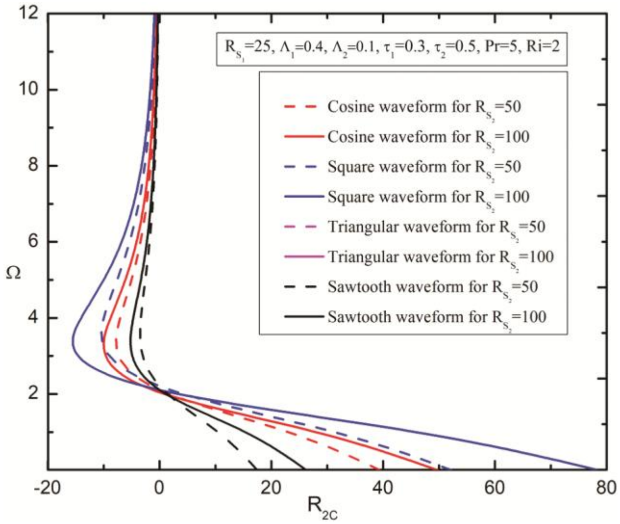

Fig. 5 — Variation of R_{2C} with Ω for different values of R_{S_2} for TC, SQ, TR and SA wave forms.

that, when the solutes are added from below, it settles at the bottom of the system without causing any disturbance in the system. As a result, it decelerated the convection onset. Further we observed that,

(i) When $\Omega \le \Omega_C$, $R_{2C|R_{S_1}=25} \le R_{2C|R_{S_1}=32}$ and $R_{2C|R_{S_2}=50} \le R_{2C|R_{S_2}=100}$,

(ii) When $\Omega > \Omega_C$, $R_{2C|R_{S_1}=25} \ge R_{2C|R_{S_1}=32}$ and $R_{2C|R_{S_2}=50} \ge R_{2C|R_{S_2}=100}$

3.3 Effect of ratio of diffusivities on the onset of convection

The influence of τ_1 and τ_2 on triply diffusive convection is shown in Figs 6 and 7 respectively. On increasing the values τ_1 and τ_2, it was found that the magnitude of R_{2C} decreased. This is due to the evident fact that the diffusivity of heat is greater than that of the solutes, thus accelerated the convection onset. We also observed that,

(i) When $\Omega \le \Omega_C$, $R_{2C|\tau_1=0.3} \ge R_{2C|\tau_1=0.6}$ and $R_{2C|\tau_2=0.4} \ge R_{2C|\tau_2=0.5,}$

(ii) When $\Omega > \Omega_C$, $R_{2C|\tau_1=0.3} \le R_{2C|\tau_1=0.6}$ and $R_{2C|\tau_2=0.4} \le R_{2C|\tau_2=0.5}$.

3.4 Effect of internal heat on the onset of convection

The influence of Ri on triply diffusive convection is shown in the Fig. 8. As the values of Ri increases,

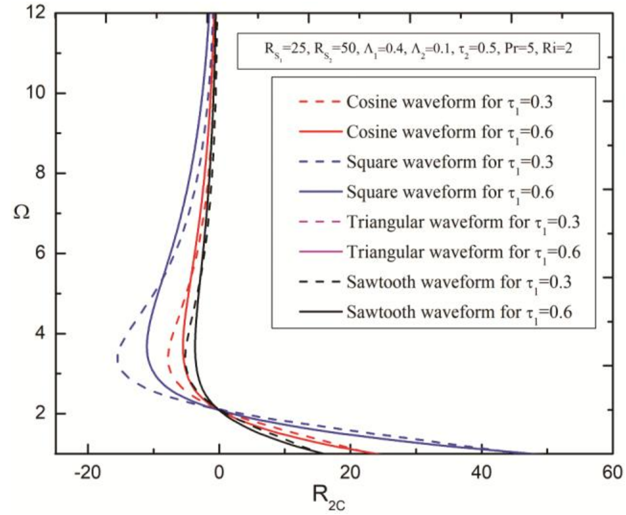

Fig. 6 — Variation of R_{2C} with Ω for different values of τ_1 for TC, SQ, TR and SA wave forms.

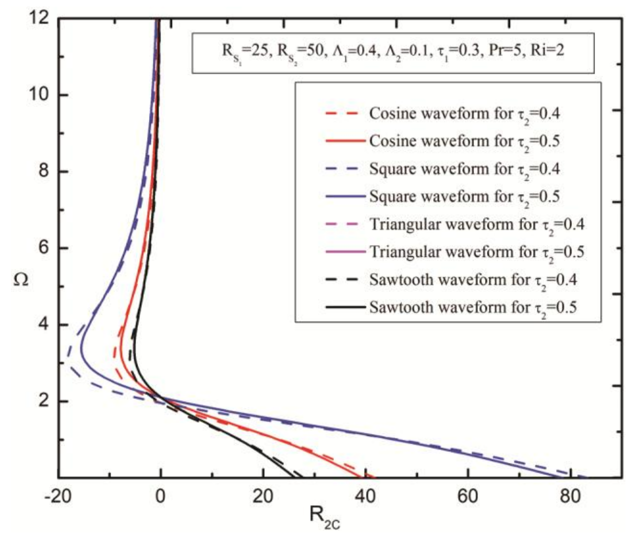

Fig. 7 — Variation of R_{2C} with Ω for different values of τ_2 for TC, SQ, TR and SA wave forms.

the magnitude of R_{2C} decreased. As Ri is increased, more and more heat is generated in the system making the system unstable and thereby accelerated the onset of convection. We also observed that,

(i) When $\Omega \le \Omega_C$, $R_{2C|Ri=-3} \ge R_{2C|Ri=2}$,

(ii) When $\Omega > \Omega_C$, $R_{2C|Ri=-3} \le R_{2C|Ri=2}$.

3.5 Effect of Prandtl number on the onset of convection

The influence of $\Pr$ on triply diffusive convection is shown in the Fig. 9. As $\Pr$ is increased, the

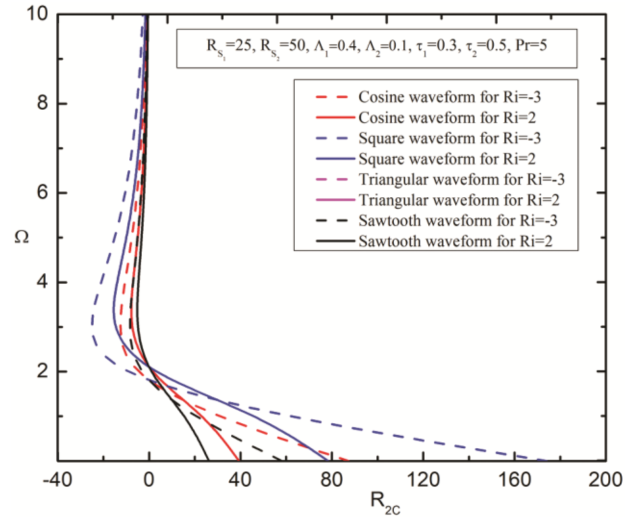

Fig. 8 — Variation of R_{2C} with Ω for different values of Ri for TC, SQ, TR and SA wave forms.

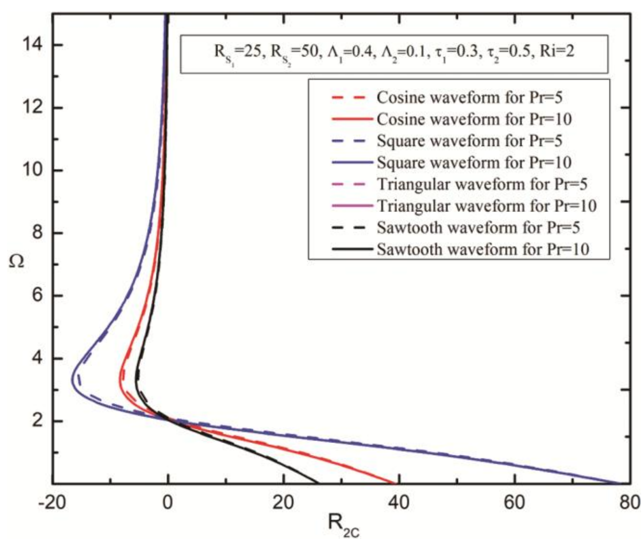

Fig. 9 — Variation of R_{2C} with Ω for different values of Pr for TC, SQ, TR and SA wave forms.

magnitude of R_{2C} also increased, indicating that Pr stabilizes the system. Further we observed that,

(i) When $\Omega \le \Omega_C$, $R_{2C|\mathrm{Pr}=5} \ge R_{2C|\mathrm{Pr}=10}$,

(ii) When $\Omega > \Omega_C$, $R_{2C|\mathrm{Pr}=5} \le R_{2C|\mathrm{Pr}=10}$.

It was noticed that for all parameters involved in the study, the following result holds true:

(i) For $\Omega \le \Omega_C$, $(R_{2C})^{TR} = (R_{2C})^{SA} < (R_{2C})^{TC} < (R_{2C})^{SQ}$,

(ii) For $\Omega > \Omega_C$, $(R_{2C})^{SQ} < (R_{2C})^{TC} < (R_{2C})^{TR} = (R_{2C})^{SA}$.

3.6 Individual effect of four types of internal heating modulations on the limiting cases of viscoelastic liquids

The values of R_{2C} for Newtonian, Oldroyd-B, Maxwell and Rivlin-Ericksen liquids for TC, SQ, SA and TR wave forms were tabulated in Table 2, Table 3, Table 4 and Table 5. The following were the results obtained:

(i) On comparison of R_{2C} for each of the fluids we observed that: $(R_{2C})_{\text{Revlin-Ericksen}} > (R_{2C})_{Newtonian} > (R_{2C})_{Maxwell} > (R_{2C})_{Oldroyd-B.}$

(ii) On comparison of the magnitude of R_{2C} for each of the fluids corresponding to each wave form: TC, SQ, SA and TR wave forms we observed that:

- $(R_{2C})^{TC}_{\text{Revlin-Ericksen}} > (R_{2C})^{TC}_{Newtonian} > (R_{2C})^{TC}_{Maxwell} > (R_{2C})^{TC}_{Oldroyd-B}$,
- $(R_{2C})^{SQ}_{\text{Revlin-Ericksen}} > (R_{2C})^{SQ}_{Newtonian} > (R_{2C})^{SQ}_{Maxwell} > (R_{2C})^{SQ}_{Oldroyd-B}$,
- $(R_{2C})^{SA}_{\text{Revlin-Ericksen}} > (R_{2C})^{SA}_{Newtonian} > (R_{2C})^{SA}_{Maxwell} > (R_{2C})^{SA}_{Oldroyd-B}$,

Table 2 — Values of correction Rayleigh number R_{2C} for Newtonian, Oldroyd-B, Maxwell and Rivlin-Ericksen liquids for trigonometric cosine wave form

R_{S_1}	R_{S_2}	(R_{2C}) for Newtonian $\Lambda_1 = 0.4, \Lambda_2 = 0.4$	(R_{2C}) for Maxwell $\Lambda_1 = 0.4, \Lambda_2 = 0$	(R_{2C}) for Oldroyd-B $\Lambda_1 = 0.4, \Lambda_2 = 0.1$	(R_{2C}) for Rivlin-Ericksen $\Lambda_1 = 0.001, \Lambda_2 = 0.05$
50	100	-0.0017	-0.0016	-0.0010	-0.0104
150		-0.0028	-0.0024	-0.0016	-0.0180
250		-0.0041	-0.0034	-0.0022	-0.0275
50	150	-0.0020	-0.0018	-0.0012	-0.0125
	200	-0.0023	-0.0021	-0.0014	-0.0147
	250	-0.0027	-0.0024	-0.0016	-0.0172

Table 3 — Values of correction Rayleigh number R_{2C} for Newtonian, Oldroyd-B, Maxwell and Rivlin-Ericksen liquids for square wave form

R_{S_1}	R_{S_2}	(R_{2C}) for Newtonian $\Lambda_1 = 0.4, \Lambda_2 = 0.4$	(R_{2C}) for Maxwell $\Lambda_1 = 0.4, \Lambda_2 = 0$	(R_{2C}) for Oldroyd-B $\Lambda_1 = 0.4, \Lambda_2 = 0.1$	(R_{2C}) for Rivlin-Ericksen $\Lambda_1 = 0.001, \Lambda_2 = 0.05$
50	100	-0.0033	-0.0030	-0.0020	-0.0200
150		-0.0054	-0.0047	-0.0032	-0.0346
250		-0.0080	-0.0065	-0.0044	-0.0529
50	150	-0.0039	-0.0036	-0.0024	-0.0240
	200	-0.0046	-0.0041	-0.0027	-0.0283
	250	-0.0053	-0.0047	-0.0031	-0.0330

Table 4 — Values of correction Rayleigh number R_{2C} for Newtonian, Oldroyd-B, Maxwell and Rivlin-Ericksen liquids for sawtooth wave form

R_{S_1}	R_{S_2}	(R_{2C}) for Newtonian $\Lambda_1 = 0.4, \Lambda_2 = 0.4$	(R_{2C}) for Maxwell $\Lambda_1 = 0.4, \Lambda_2 = 0$	(R_{2C}) for Oldroyd-B $\Lambda_1 = 0.4, \Lambda_2 = 0.1$	(R_{2C}) for Rivlin-Ericksen $\Lambda_1 = 0.001, \Lambda_2 = 0.05$
50	100	-0.0011	-0.0010	-0.0007	-0.0069
150		-0.0019	-0.0016	-0.0011	-0.0120
250		-0.0027	-0.0022	-0.0015	-0.0183
50	150	-0.0013	-0.0012	-0.0008	-0.0083
	200	-0.0015	-0.0014	-0.0009	-0.0098
	250	-0.0018	-0.0016	-0.0011	-0.0114

Table 5 — Values of correction Rayleigh number R_{2C} for Newtonian, Oldroyd-B, Maxwell and Rivlin-Ericksen liquids for Triangular wave form

R_{S_1}	R_{S_2}	(R_{2C}) for Newtonian $\Lambda_1 = 0.4, \Lambda_2 = 0.4$	(R_{2C}) for Maxwell $\Lambda_1 = 0.4, \Lambda_2 = 0$	(R_{2C}) for Oldroyd-B $\Lambda_1 = 0.4, \Lambda_2 = 0.1$	(R_{2C}) for Rivlin-Ericksen $\Lambda_1 = 0.001, \Lambda_2 = 0.05$
50	100	-0.0011	-0.0010	-0.0007	-0.0069
150		-0.0019	-0.0016	-0.0011	-0.0120
250		-0.0027	-0.0022	-0.0015	-0.0183
50	150	-0.0013	-0.0012	-0.0008	-0.0083
	200	-0.0015	-0.0014	-0.0009	-0.0098
	250	-0.0018	-0.0016	-0.0011	-0.0114

- $(R_{2C})^{TR}_{\text{Revlin-Ericksen}} > (R_{2C})^{TR}_{Newtonian} > (R_{2C})^{TR}_{Maxwell} > (R_{2C})^{TR}_{Oldroyd-B}$.

(iii) It was observed that: $(R_{2C})^{SA} = (R_{2C})^{TR}$, holds true for Newtonian, Oldroyd-B, Maxwell and Rivlin-Ericksen fluids.

4 Conclusion

A linear stability analysis for a triply diffusive convection in viscoelastic liquids heated as well as salted from below is analyzed. Additionally, the influence of sinusoidal (trigonometric cosine) and non-sinusoidal (square, sawtooth and triangular) wave forms of internal heat source modulation has shown a great impact on the system. In order to observe the stability of the system, the parameters influencing the onset of convection are analyzed and plotted graphically. The following conclusions are drawn from this research study:

(i) The parameters that influence the system in order to destabilize are Λ_1, τ_1, τ_2, Ri,

(ii) The parameters that influence the system in order to stabilize are Λ_2, R_{S_1}, R_{S_2}, Pr,

(iii) For small values of Ω, the system destabilizes and reaches a subcritical motion,

(iv) Internal heat source modulation can either have a stabilizing effect or a destabilizing effect depending on the chosen value of frequency of internal heating modulation and appropriate wave forms,

(v) Square wave form is found to be more stable for lesser values of frequency of modulation Ω and sawtooth waveform is more stable for larger values of frequency of modulation Ω on comparison with other wave forms involved in the study,

(vi) The magnitude of correction Rayleigh number that determines the onset of convection is least due to sawtooth wave form and highest due square wave form,

(vii) From the magnitude of R_{2C} it is observed that, $(R_{2C})_{\text{Revlin-Ericksen}} > (R_{2C})_{Newtonian} > (R_{2C})_{Maxwell} > (R_{2C})_{Oldroyd-B.}$ which holds true for each of the wave forms,

(viii) $(R_{2C})^{SQ} > (R_{2C})^{TC} > (R_{2C})^{SA}$, is true for each of the fluids involved in the study,

(ix) It is found that the result $(R_{2C})^{SA} = (R_{2C})^{TR}$, holds true for Newtonian, Oldroyd-B, Rivlin-Ericksen and Maxwell liquid,

(x) It has been observed that triangular wave form overlaps with sawtooth wave form.

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