Vibrational frequencies and tuning of the African mbira

L. E. McNeil[a)]
Department of Physics and Astronomy, University of North Carolina at Chapel Hill, Chapel Hill, North Carolina 27599-3255, USA

S. Mitran[b)]
Department of Mathematics, University of North Carolina at Chapel Hill, Chapel Hill, North Carolina 27599-3250, USA

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The acoustic spectrum of the mbira, a musical instrument from Africa that produces sound by the vibration of cantilevered metal rods, has been measured. It is found that the most prominent overtones present in the spectrum have frequencies that are approximately 5 and 14 times the lowest frequency. A finite-element model of the vibration of the key that takes into account the acoustic radiation efficiency of the various normal modes reveals that the far-field power spectrum is dominated by modes involving predominately transverse motion of the key. Modes involving longitudinal motion do not radiate efficiently, and therefore contribute little to the sound produced. The high frequencies of the dominant overtones relative to the fundamental make it unlikely that the tunings of the mbira that are used by expert musicians are determined by matching the fundamental frequencies of the upper keys with the overtones of the lower keys. © *2008 Acoustical Society of America.* [DOI: 10.1121/1.2828063]

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I. INTRODUCTION

The *mbira* is a musical instrument found in multiple forms in many countries across the continent of Africa, especially in Zimbabwe, the Democratic Republic of the Congo, Mozambique, and Angola.[1] One form, known as the *mbira dzavadzimu* ("mbira of the ancestors") is particularly characteristic of the melodic music of the Shona people of Zimbabwe, Mozambique, and Zambia. The keys are put into vibration by stroking the free end with the thumb or forefinger, and each one produces a distinct musical pitch. The sound is sometimes amplified by placing the mbira inside a hollow shell such as a gourd that serves as a resonating chamber. The music played on the mbira is typically polyphonic, so that multiple keys are sounded simultaneously or alternately in sequence.

Like all melodic instruments with multiple vibrating parts, the pitches produced by the individual keys are related to one another in an organized way that is referred to as the *tuning* of the instrument. Throughout Africa many mbira tunings are used, which are related to melody and chord patterns prevalent within the local culture and also reflect the artistry of individual master players. It is reasonable to ask, however, whether those tunings are related to the musical acoustics of the instrument itself.

Because more than one mode of vibration is excited when the key is stroked, the sound produced by the vibration of the key will contain multiple frequencies. The lowest of these frequencies, which determines the pitch experienced by the listener, is called the *fundamental*. The higher frequencies, which together determine the overall characteristics of the musical sound, are referred to as *overtones*. We set out to examine the frequencies of vibration of the keys of the mbira and to explore their possible relationship to the tuning of the instrument.

II. EXPERIMENT AND CALCULATION

A. The mbira

The instrument used for these measurements is shown in Fig. 1. It was made by Josephat Mandaza in Chitungwiza, Zimbabwe in 2002 and was provided to us by Dr. Louise Meintjes of the Duke University Department of Music. The body of the instrument is made of a single piece of solid wood approximately 18.5×21.7 cm and 2.5 cm thick. A 2.5-cm-diam hole in the body near one end provides a convenient finger-hold. The 22 vibrating rods or keys range in length from 9.5 to 17.5 cm. They are clamped at one end and cantilevered from a bridge made of a metal strip embedded in the wood body, such that the freely vibrating length ranges from 3.5 to 10.5 cm. They are arranged in three groups: nine on the right-hand side of the instrument (designated R1–R9), six upper keys on the left-hand side (designated L1–L6), and seven lower keys on the left-hand side (designated B1–B7). The flattened tongue-like shapes of the freely vibrating portion of the keys are approximately 0.5–0.8 cm wide at the bridge and 0.5–1.5 cm wide at the widest point (near the free end). The cross section of each key is pentagonal, being approximately 0.09 cm thick at the edges and 0.15 cm thick in the center. The keys curve upward from the bridge (presumably to make them easier to play) such that the free end is raised above the bridge by about 10% of the key's length. The material of which the keys are made is unknown, but

[a)]Electronic mail: mcneil@physics.unc.edu.
[b)]Electronic mail: mitran@amath.unc.edu.

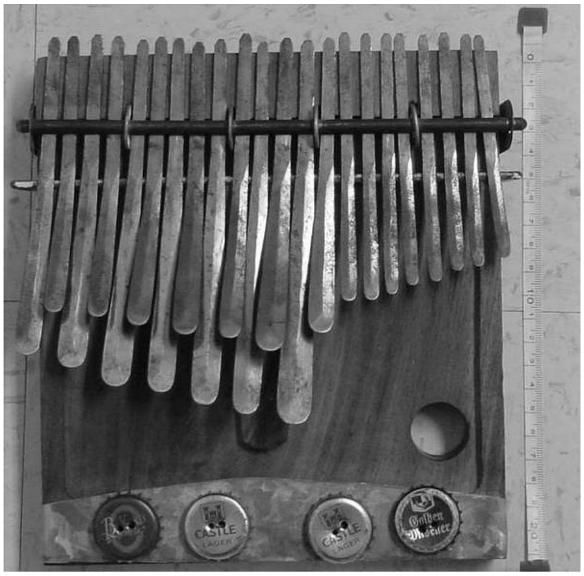

FIG. 1. Photo of the mbira used in the measurements.

from their appearance and stiffness and the presence of a small amount of rust on them they can be presumed to be some sort of mild steel.

At the opposite end of the instrument body from the bridge, a metal strip is nailed across the width of the wood. Four metal beer-bottle caps are attached to the strip with wire such that they are free to rattle against the metal when the instrument is played. Traditionally, shells were used for this function,[2] but in more recent times bottle caps have become a convenient substitute. The resulting buzzing sound is considered to be an important characteristic of the instrument in its indigenous use in music-making, but it does not affect the pitches produced by the keys and for the purpose of the experiments described here it was undesirable. For that reason, before making the measurements we damped the vibration of the bottle caps by stuffing them with wool fleece. The vibrations of keys other than the one being measured were similarly damped.

B. Acoustic measurements

We made the acoustic measurements in a sound isolation chamber in the UNC-CH Department of Linguistics. We made sound recordings with a Shure KSM44 microphone using the omnidirectional pickup pattern and 80-Hz high-pass filter. The microphone preamplifier was a Mackie 1202-VLZ PRO mixer. We used the channel's insert-send as the signal output to minimize contributed noise and distortion. We used an Echo Digital Audio Indigo IO for the analog-to-digital signal conversion. The conversion parameters were 24-bit quantization and 22 050-Hz sample rate, and we stored the samples on a hard drive using Adobe AUDITION 1.5 software. We used the same software to perform the Fourier analysis of the audio samples. An initial baseline measurement to evaluate the environmental noise recorded that all frequencies from 100 Hz to the Nyquist frequency were at least 102 dB below full scale. We held the mbira close to the microphone with one hand and one key was set into vibration by pressing it with a finger on the other hand and allowing the finger to slide off the end of the key. Recording continued until no further vibration of the body of the mbira could be perceived with the hand holding the instrument, approximately 10 s. (This was a longer time than the sound was audible.)

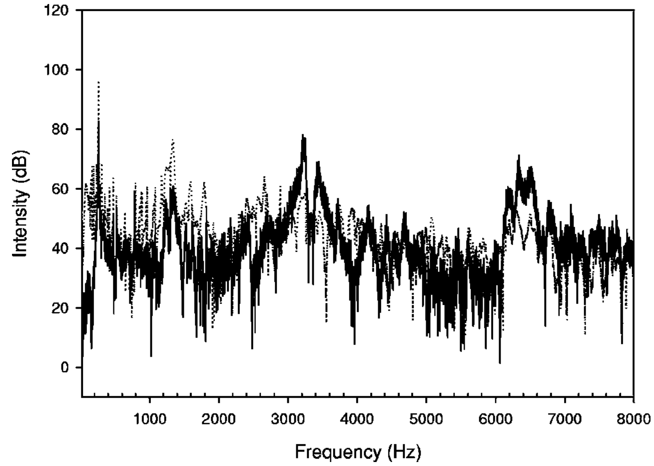

FIG. 2. Power spectrum of audio signal from key B7 (fundamental frequency 260 Hz), background subtracted (solid line); and the same spectrum recorded with all of the keys free to vibrate (dotted line).

An example of the power spectrum resulting from the Fourier transform of the audio signal is shown in Fig. 2. The spectrum contains a strong, relatively sharp peak 50–80 dB above background at the fundamental frequency of the key's vibration. This mode of vibration is responsible for the pitch the key sounds, and ranges from 117 to 948 Hz (approximately $B_2^\flat$ to $B_5^\flat$ in the American system of pitch notation). A second peak, typically 30–50 dB above background, appears at the frequency of the next higher mode of vibration. In most of the spectra a third (and sometimes a fourth) peak is also distinguishable from the background level. The frequencies recorded for each key are given in Table I. Also shown in Fig. 2 (dotted line) is the spectrum recorded from that key with all of the keys undamped, as they would be when the instrument is played. As can be seen, the damping has little if any effect on the spectrum except at the higher frequencies, where the sound is less intense.

C. Frequency calculations

We have computed the vibration frequencies of some of the mbira keys using two finite element methods. In the first method, a mbira key is treated as a one-dimensional body and discretized using curved beam (more concisely, arch) elements as described in the following. Arch elements were introduced after an initial study showed poor behavior of a model constructed using standard, straight beam elements. In the second method, a key is modeled as a three-dimensional body using standard brick elements (the C3D6 element of the CALCULIX finite element program was used). Two methods were used to provide a check against one another and to resolve a number of interesting questions that arise in comparisons of experimental and computational results.

The behavior of straight elastic beams is well understood and accurately described by the classical Euler–Bernoulli and Timoshenko beam theories. Curved beams ex-

TABLE I. Frequencies recorded for the keys of the mbira.

Key	Fundamental (Hz)	Overtone 1 (Hz)	Overtone 2 (Hz)
B1	117	646	1705
B2	141	722	1883
B3	152	784	2033
B4	171	890	2290
B5	191	965	2530
B6	212	1080	2882
B7	260	1341	3220
L1	234	1176	3005
L2	353	1613	3990
L3	310	1502	2874
L4	392	2310	
L5	420	2000	5060
L6	475	2167	5597
R1	288	1515	3940
R2	474	2365	6116
R3	525	2480	6137
R4	587	2764	
R5	631	3022	
R6	704	2944	5200
R7	788	1576	3941
R8	852		
R9	948		

hibit more complicated behavior because of the nonlinear coupling among bending, twist, shear, and extension. Due to their importance in structural engineering, there have been many studies on the behavior of curved beams (e.g., Ref. 3). An important question is whether the scale disparity between beam length and thickness can be used to separate the three-dimensional elasticity problem into a one-dimensional problem along the beam length and a two-dimensional problem in the beam cross section. The two-dimensional problem can typically be easily approximated, and the overall complexity is reduced to solving a one-dimensional problem instead of the full three-dimensional elasticity problem. That such a splitting is possible has been established by Berdichevsky.[4] The problem of determining a specific splitting procedure is still a research subject. Specific characteristics of the problem, including amount of deformation, curvature, nature of applied forces, come into play, and some splittings are able to capture the above-mentioned effects while others fail (see Ref. 5). Even though one-dimensional curved beam theories exhibit these deficiencies, the overall economy in computational effort makes them attractive.

There are a number of formulations for curved finite elements. The basic steps in the construction of the finite element model are: (1) define an approximation function for the displacements as a function of longitudinal position; (2) construct the elemental stiffness and mass matrices using a variational formulation; (3) assemble the elemental matrices to form a global eigenproblem; and (4) solve the eigenproblem for a specific set of boundary conditions. Steps (2)–(4) are standard procedures which can be found in a number of finite element texts, e.g., Bathe.[6] The main feature which distinguishes one arch finite element formulation from an-

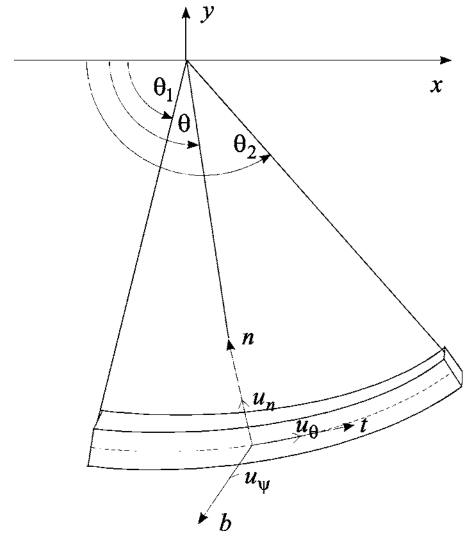

FIG. 3. Diagram showing finite element coordinate system and deformations.

other is how displacements are approximated along the beam length. A specific formulation has been introduced to study this problem.

We use a local Frenet triad (t,n,b) to describe the deformations in the finite element cross section, with the coordinates denoting the tangential, normal, and binormal direction, respectively. (See Fig. 3 for the definition of the coordinates.) We assume that the finite element is small enough to consider the radius of curvature R constant over the element's length; hence the longitudinal position can also be specified by the angle $\theta = t/R$. We assume that plane cross sections remain plane during deformation. We consider the beam element to undergo three types of deformation: (a) a tangential deformation u_θ; (b) a normal deformation u_n; (c) a cross-section rotation u_ψ. All these deformations are defined at the element mean fiber, i.e., the line along the thickness of the element that does not change in length during deformation. In general, there exists some series expansion of the displacement u in terms of the θ angle, $u=u(\theta)$. We desire for this expansion to have as few terms as possible while still providing sufficient accuracy. The distinguishing feature of the finite element approximation used here is the use of a mixed, Taylor-trigonometric expansion to approximate the displacements. The tangential deformation is approximated as

$$u_\theta = a_1 + a_2 \cos\theta + a_3 \sin\theta + a_4 \theta \cos\theta + a_5 \theta \sin\theta. \quad (1)$$

If we neglect shear deformations (relative slipping of adjacent finite element cross sections) the expansions for (u_n, u_ψ) can be written using the coefficients from the expansion for u_θ,

$$u_n = Ca_1\theta + a_2 \sin\theta - a_3 \cos\theta + a_4(\sin\theta - \theta\cos\theta) + a_5(\cos\theta + \theta\sin\theta) + a_6,$$

$$u_\psi = \frac{C}{R}a_1\theta + \frac{2}{R}a_4 \sin\theta + \frac{2}{R}a_5 \cos\theta + \frac{1}{R}a_6, \qquad (2)$$

where $C=1+I_b/AR^2$, A is the cross-sectional area, and I_b is the moment of inertia around the binormal axis. Note that twisting of the cross section is not taken into account in this model. Normal stresses are assumed to vary linearly across the cross section. The resultant forces from a cross section can be related to the displacements through

$$F_t = \frac{dF_\theta}{d\theta},$$

$$F_\theta = \frac{EA}{R}\left(\frac{du_\theta}{d\theta} - u_t\right) - \frac{M_b}{R},$$

$$M_b = \frac{EI_b}{R^2}\left(u_t + \frac{d^2u_t}{d\theta^2}\right). \qquad (3)$$

Once the longitudinal displacement functions and the forces in the cross section have been established, standard finite element procedures lead to the formation of an eigenproblem

$$\omega^2\mathbf{M}\mathbf{u} = \mathbf{K}\mathbf{u}. \qquad (4)$$

In Eq. (4), ω is the pulsation, $\mathbf{M}$ the mass matrix, $\mathbf{K}$ the stiffness matrix, and $\mathbf{u}$ the vector of displacements. In order to completely define the problem, boundary conditions must be imposed. Given the construction of the mbira, the vertical displacements at the point where the keys are clamped are set as null. The displacements in the other two directions are impeded by frictional contact between the key and the soundboard or upper clamping bar. In the computations carried out here, these displacements were also set as null. The behavior at the fret is more difficult to capture simply. Displacements along the vertical orientation (perpendicular to the soundboard) are impeded only in one direction (downwards). Lateral and longitudinal displacements are impeded due to friction between the key and the fret but not necessarily nullified. We consider the effect of a number of boundary conditions at the fret below. Solving the eigenproblem gives the vibration frequencies $\nu_i=\omega/2\pi$ and associated vibration modes $\mathbf{r}_i$.

D. Acoustic radiation from a vibrating mbira key

The sound produced by a struck mbira key is a superposition of the vibration modes from the above-presented finite element computation. Not all modes are excited by the initial striking of the key and some modes are more efficient sources of acoustic radiation than others. Furthermore, the soundboard is also excited by a vibrating key and radiates acoustic waves. All these effects have to be taken into account in order to compare a measured acoustic power spectrum with the results from a computational simulation.

1. Key playing

We adopt a simple model of how a key is played. We assume that a key is deformed from its equilibrium position by a force acting normal to the key mean fiber and then instantaneously released. The initial shape $\mathbf{u}_0$ is determined by solving the problem

$$\mathbf{K}\mathbf{u}_0 = \mathbf{f}, \qquad (5)$$

where $\mathbf{f}$ is a vector of forces applied at each node within the finite element model. For the model of key playing considered here all components are null except those corresponding to the end node of a key.

2. Vibration mode amplitudes

The initial shape $\mathbf{u}_0$ is decomposed on the vibration modes $\mathbf{R}=[\mathbf{r}_1,\mathbf{r}_2\cdots\mathbf{r}_N]$ by solving the linear system

$$\sum_{j=1}^{N}(\mathbf{r}_i\cdot\mathbf{r}_j)\mathbf{a}_j = \mathbf{r}_i\cdot\mathbf{u}_0, \quad i=1,2,\ldots,N. \qquad (6)$$

The amplitudes $\mathbf{a}_j$ specify the degree of initial excitation of each vibration mode.

3. Acoustic efficiency

Mbira keys are curved arches with significant coupling among extensional, twisting, and bending vibration modes. Only weak acoustic radiation is expected from an extensional or twisting mode, due to the small key surface area that sets the surrounding air into motion. The keys radiate directly and also set the soundboard into motion. The surface area of the soundboard is much larger than that of the keys and, conceivably, extensional modes could radiate efficiently by exciting the soundboard. However, the coupling of a key to the soundboard is accomplished by ensuring *vertical* contact at three points: the edge of the soundboard, the clamp pressing on the top of a key, and the fret pressing against the bottom of a key. The contact in the other directions is through friction forces that result from the vertical clamping forces. Longitudinal vibrations of a key produce forces which act against the frictional contact and are not expected to produce significant excitation of the soundboard. We therefore concentrate on the relative acoustic efficiency of sound radiation from the key motion itself, understanding that the intensity of acoustic waves emitted by bending modes of a key is amplified by the soundboard while those associated with extensional or twisting modes are not.

The acoustic field produced by a key is the solution of

$$\phi_{tt} - c^2\nabla^2\phi = 0, \qquad (7)$$

along with boundary conditions given by the key motion. In Eq. (7) ϕ is the acoustic velocity potential and c is the sound speed. Rather than solve this complicated problem in a domain with a moving boundary, we carry out a simple estimate. We assume that only the normal displacement associated with a vibration mode makes a significant contribution to the overall acoustic field. Assuming the vibration amplitudes to be small, the above-presented problem is replaced by $\phi_{tt}-c^2\nabla^2\phi=Ayy_t$ with the source term yy_t defined only on the mbira key, and A a scaling factor which plays no role here and shall be taken as unity. Here y is the displacement of the key normal to the mean fiber, and the source term arises from integrating the kinematic condition $v=\phi_y=y_t$

which states that the acoustic velocity equals the key velocity for positions on the key (for an extended discussion of how this equation is obtained see Refs. 7 and 8). We further assume the key to be infinitely thin. The solution to Eq. (7) can be expressed as the convolution

$$\phi(\vec{x},t) = \int_0^L \int_0^\infty G(\vec{x} - \xi\vec{e}_\xi, t - \tau) y(\xi) y_t(\xi) d\tau d\xi \tag{8}$$

with $\vec{\mathbf{e}}_\xi$ being the unit vector tangent to the key mean fiber and $G(\vec{x},t)=\delta(|\vec{x}|-ct)/(4\pi|\vec{x}|)$ the fundamental solution for the wave equation. The acoustic intensity is given by $I=pu$ with p the acoustic overpressure and u the acoustic velocity. In terms of the velocity potential we have: $I \sim \phi_t \partial\phi/\partial y$. Use of the Dirac delta allows the time integral to be evaluated and we are led to the far-field estimate

$$I(\omega) \sim \omega \int_0^L y^2(\xi,\omega) d\xi \sim \omega a^2(\omega) \mathbf{r}^n(\omega) \cdot \mathbf{r}^n(\omega), \tag{9}$$

stating that the far acoustic field is proportional to the eigenmode frequency and the squared normal-to-mean-fiber amplitude.

The estimate (9) allows us to predict the acoustic power spectrum obtained by playing a key. When a key is played a beam eigenmode is excited with amplitude $a(\omega)$ obtained by solving Eq. (6). For each eigenmode we compute the integral (9) and thereby obtain a prediction of the acoustic wave intensity produced by that eigenmode. Note that modes for which the transverse motion $y^2(\xi,\omega)$ is small will contribute less to the overall acoustic spectrum even if their initial excitation by stroking the key $a(\omega)$ is comparable to that of other modes.

As mentioned, for a curved beam bending and extensional modes are coupled. In order to classify which modes behave more like bending modes (i.e., transverse motion is predominant) and which behave more like extensional modes (i.e., longitudinal motion is predominant) we introduce a criterion. Let z denote the deformation in the local direction of the mean fiber, and introduce the quantity

$$J(\omega) \sim \omega \int_0^L z^2(\xi,\omega) d\xi. \tag{10}$$

We define modes for which $\eta(\omega)=I(\omega)/J(\omega)>1$ as "transverse" while those for which $\eta(\omega)\leq 1$ shall be referred to as "longitudinal."

III. RESULTS

A. Measured frequencies

Figure 4 shows the ratio of the frequencies of the second and third peaks in the spectrum (hereafter referred to as the first and second overtones) recorded for each key to the fundamental frequency produced by the key, plotted versus the fundamental frequency. With the exception of the shortest key, the ratio of the frequency of the first overtone to that of the fundamental is approximately 5, ranging from 4.2 to 5.8. The second overtone is more variable, but for the longest keys (those with the lowest fundamental frequencies), its frequency is approximately 13.6 times the fundamental frequency. This suggests that the modes of vibration of the keys are all very similar.

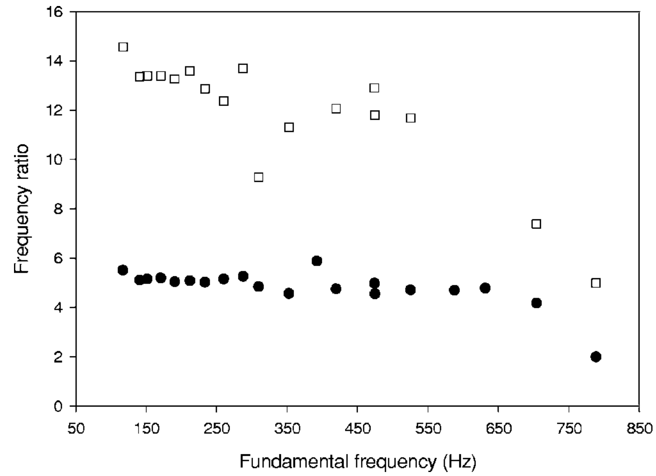

FIG. 4. Ratio of overtone frequencies to fundamental frequency. Closed symbols: First overtone. Open symbols: Second overtone.

B. Calculated frequencies

The exact material properties of the key were not known and could not be measured without damage to the instrument; hence we used the fundamental frequency to calibrate the Young's modulus (E) and density (ρ) of the material of which the key was made. Use of standard values for soft steel (E=200 GPa, ρ=7800 kg/m^3) led to a relative error of ~10% in prediction of the fundamental frequency among the keys for which the computation was carried out (B1, B7, R9). Given this calibration against the fundamental frequency, the main prediction of the model is the frequency of the overtones as well as the overall shape of the far-field acoustic spectrum. We used a smoothing spline procedure to adjust the geometric data for the keys to avoid propagating small measurement errors into the numerical procedure for computing the eigenmodes and frequencies.

The eigenmodes for key B7 computed using the three-dimensional model are shown in Fig. 5. The three-dimensional computations served as a check on the one-dimensional model and also to investigate fret boundary conditions. The computations are however quite expensive. In order to obtain convergence in the first six eigenmodes to within 1 Hz approximately 240,000 brick elements were required. Carrying out the decomposition of the initial deflection onto an eigenmode basis was not feasible. The effect of fret boundary conditions is shown in Table II. When vertical displacements are set to null the key oscillates as a shorter length beam with a consequent increase in frequency (compare mode 1, free displacement and $u_y=0$ case). Blocking further degrees of freedom changes the overtone frequencies.

Frequencies obtained from the one-dimensional arch element computation of the B7 key are shown in Fig. 6. Given the much simpler computational effort many more eigenmodes were computed, sufficient to accurately represent the power spectrum associated with the initial deformation imposed by stroking a key. Some of the modes are spurious

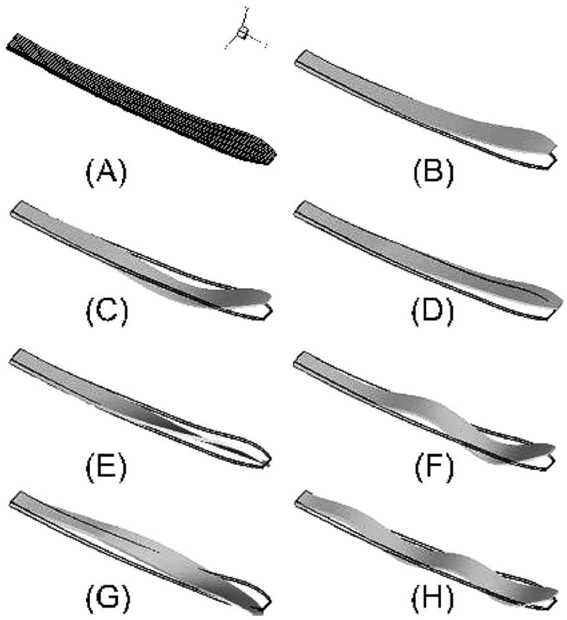

FIG. 5. (A) Three-dimensional finite element discretization. (B)–(E) Eigenmodes corresponding to seven lowest frequencies. At the fret the vertical displacements were set to null, the others were left free. Modes B,C,F,H are predominantly bending modes. Modes E,G are predominantly twisting modes. Mode D is similar to mode B but with pronounced extensional component.

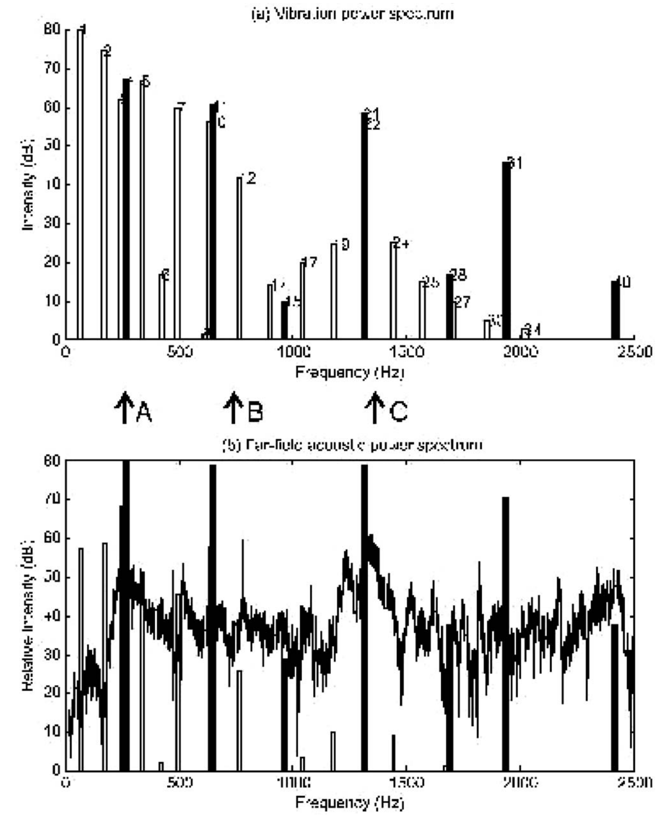

FIG. 6. Results of one-dimensional finite element analysis for the B7 key. Initial deformation produced by a force acting normal to the mean fiber at the end of the key. (a) Vibration power spectrum showing the principal transverse modes (solid bars) 4, 11, 22, 32 as well as a wide variety of longitudinal modes (open bars). (b) Far-field acoustic power spectrum predicted by the finite element analysis (bars) compared with the experimentally measured spectrum (line). Arrows A, B, and C indicate frequencies obtained from the three-dimensional model.

however and result from the simplifying assumptions of the one-dimensional model. In particular modes 1, 2 from the one-dimensional computation arise due to the rapid tapering of a key toward the end. This leads to large cross-section rotations in the finite element model. The deformation for these spurious modes corresponds to bending of the tapered key tip attached to the much stiffer main part of the key. The possibility that spurious modes appear in regions where the simplifying assumptions of one-dimensional beam theory break down is unavoidable. In particular, results should be scrutinized in region of rapid variations in geometry. Spurious modes can also arise from shear locking behavior (details on origin and elimination of spurious modes can be found in Ref. 9). Fortunately, the spurious modes have relatively small transverse displacements and contribute little to the acoustic power spectrum evaluated by Eq. (9) even though their vibration power spectrum is significant (pressing on the end of a key imposes a large initial amplitude on these modes). Hence in this work no additional modification of the finite element method was made to eliminate spurious modes.

TABLE II. Effect of boundary conditions at fret upon frequencies calculated using the three-dimensional finite element model. Displacements u_x, u_y, u_z correspond to directions along key length, perpendicular to key and vertical, perpendicular to key and horizontal.

Fret boundary condition	Mode A	Mode B	Mode C	Mode D	Mode E
Free displacements	171	728	830	2256	2348
$u_y=0$	256	749	1441	2771	3970
$u_x=0,\ u_y=0$	261	1465	1607	2797	4010
$u_x=0,\ u_y=0,\ u_z=0$	261	1465	1848	2848	4010

IV. DISCUSSION

A. Overtone frequencies

The vibrational frequencies of a simple cantilever can be calculated by standard techniques.[10] The well-known result for a straight beam of uniform cross section that is clamped at one end is that the frequencies of the first and second overtones are 6.4 and 17.5 times the fundamental frequency. However, the mbira keys are neither uniform in cross section nor straight. There is significant coupling among bending, twisting and compressional deformation.

A typical result from the computation of the B7 key is presented in Fig. 6. The vibration power spectrum shows the initial amplitude of each eigenmode imposed by stroking the key as obtained from solving system (6). The far-field acoustic power spectrum shows the results of evaluating integral $I(\omega)$ from Eq. (9) to obtain the radiated sound intensity. Bars show the predictions of the one-dimensional model, arrows show the frequencies obtained from the three-dimensional model. The strongly excited, one-dimensional, modes 1 and 2 are spurious modes, arising from breakdown of one-dimensional simplifying hypotheses as mentioned earlier. The most efficient acoustic radiator [highest $I(\omega)$] among the low frequency modes is mode 4, which corresponds to bending of the entire key. We calibrated the observed fundamental frequency to mode 4 to account for the unknown material properties, thus obtaining a value of $E=205$ GPa for the elasticity modulus. We then observed that the next most efficient acoustic radiator is mode 21 at a frequency of 4.9 times that of the fundamental; the associated computed frequency $f=1313$ Hz is within 2% of the measured first overtone of 1341 Hz (Table I). Other vibration modes which are significant sound radiators are modes 11, 31, and 40. All of these show up as peaks within the measured acoustical spectrum [Fig. 6(c)], thereby confirming the overall accuracy of the numerical procedure.

From a modeling point of view it is interesting to compare the three-dimensional and one-dimensional approaches. The three-dimensional approach allows for accurate predictions of the low order modes, but at such a computational cost that obtaining a complete eigenbasis to compute the vibration power spectrum and the far-field acoustic power spectrum becomes prohibitive. Furthermore, contact boundary conditions at the fret introduce an additional unknown which must be explored through (expensive) numerical experimentation. The one-dimensional model sometimes predicts spurious modes, but furnishes a sufficiently accurate description of key vibrations for a correct evaluation of the far-field acoustic spectrum.

A simulation of the effect of the soundboard was also carried out. Forces from transverse modes and from longitudinal modes were applied on a rectangular slab of anisotropic, three-dimensional brick elements in the CALCULIX program. The transverse forces were applied through a dashpot element to model the frictional contact. As expected the soundboard received negligible energy from longitudinal modes by comparison to transverse modes. This is a direct consequence of the coupling between the vibrating elements and the soundboard. Hence in practice the relative attenuation of the longitudinal modes with respect to the transverse modes is even more pronounced than that computed here.

B. Tuning

The mbira is in essence a keyboard instrument (and it is sometimes called a "thumb piano" by Westerners), and therefore must have a well-defined relationship among the pitches (i.e., fundamental frequencies) of the different keys. The pitch of an individual key can be adjusted by shifting it relative to the bridge so that the vibrating length of the key is increased (to lower the pitch) or decreased (to raise the pitch). Since in its indigenous use the tuning of each key is done "by ear" (or by comparison to another such instrument) rather than by reference to an absolute standard of pitch, it is reasonable to expect that the fundamental frequencies to which the higher-pitched keys are tuned will bear some relationship to the overtone frequencies of the lower-pitched keys, so that when played together the overtones of the different keys will coincide to produce a harmonious sound.

Simple stretched strings have overtone frequencies that are in pure whole-number ratios to the fundamental frequency, and so tuning successive strings such that their fundamental frequencies are 2, 3, 4, 5, etc., times the fundamental frequency of the lowest string allows the frequencies of the overtones of the strings to overlap to produce a harmonious sound. This tuning produces musical intervals between the pitches produced by successive strings of an octave, a perfect fifth, a perfect fourth, and a pure major third. This type of tuning is known as *just* intonation, and was widely practiced in Western Europe in the 16th and early 17th century. It has been largely replaced by the modern system of *equal temperament* in which the frequency ratios for successive pitches (semitones) are fixed at $\sqrt[12]{2}$, which allows a keyboard instrument to play in different musical key signatures without retuning.

Instruments such as the mbira that produce their sound by the vibration not of simple strings but of cantilevered rods have ratios of their overtone frequencies that are different from those of strings. It is therefore reasonable to expect that the tuning of such instruments would differ from just intonation, and this is indeed what has been observed by ethnomusicologists. In a 1932 study Hugh Tracey observed[11] that in some cases mbira players tuned some keys using an overtone in preference to the fundamental to produce the desired effect. Tracey also noted[12] that the accepted tuning of the instrument varied by region, with each local group of musicians agreeing upon a "correct" local manner of tuning. In 1978 Paul Berliner made a detailed study of the tunings used by Shona mbira players in Zimbabwe.[1] The intervals between the pitches of the mbira keys in five tunings that he observed are displayed in columns 3–7 in Table III (labeled by the names of the musicians from whom he recorded them), and the intervals expected for just intonation are included in column 2 for comparison. As expected, these tunings all differ from the just intonation that would be produced in tuning simple stretched strings "by ear."

A sixth tuning included in Table III is that specified by Kevin Volans, a South African composer of classical music in the Western European tradition. In his composition *White Man Sleeps* for two harpsichords, viola da gamba and percussion, he specifies[13] that the instruments are to be tuned as listed in column 8 of Table III, which he labels "a tuning system derived from Shona Mbira music."[14] As can be seen from Table III, it deviates from the tunings recorded by Berliner from Shona musicians no more than they deviate from one another.

The tuning of the mbira studied in this work is also presented in Table III. While the above-described measurement procedure allows confidence in the accuracy of the frequencies listed there, their relationship to a "proper" tuning is less certain. It is possible that the tuning of this instrument may have changed since it left the hands of its musician-maker, and thus that the tuning recorded here may not accurately represent the tuning it was intended to have. However, the fact that the keys are quite firmly fixed in place so that increasing or reducing the vibrating length requires considerable force (applied with a hammer to one end of the key) suggests that the tuning has probably not changed significantly since it was last adjusted. Inspection of Table III reveals that the intervals between almost all the keys of the mbira studied here are very similar to those used by Volans and by the musicians recorded by Berliner.

The question that originally stimulated this study was

TABLE III. Mbira tuning intervals in cents (an interval of one cent corresponds to 1/100 of an equal-tempered semitone, or a frequency ratio whose logarithm equals 2.5086×10^{-4}). Column 1 labels the mbira keys, in order of increasing fundamental frequency. Column 2 ("Just") represents the intervals between the pitches sounded by adjacent keys for tuning in just intonation (see the text). Column 3–7 are tunings used by Zimbabwean musicians recorded by Paul Berliner (Ref. 1), and column 8 is a tuning used by composer Kevin Volans (Ref. 10). Column 9 contains the intervals between the pitches of adjacent keys of the mbira measured in this work.

Key	Just	Gondo	Mude	Mujuru	Kunaka	Bandambira	Volans	Mbira
B1								
	386	323	174	126	455	355	360	323
B2								
	112	157	115	117	92	199	154	130
B3								
	204	164	286	156	210	96	171	204
B4								
	182	186	37	314	178	179	175	192
B5								
	112	151	158	105	154	153	100	181
B6								
	204	198	206		171		140	171
L1								
	204	151	170		241		171	182
B7								
	182	187	180		136		189	177
R1								
	112	180	180		126		154	127
L3								
	204	191	211	263	209	118	171	225
L2								
	182	137	170		181		175	181
L4								
	112	164	139	97	164	169	100	119
L5								
	204	182	128	264	161	193	140	213
L6								
	0				15		0	4
R2								
	204	179	231	190	196	185	171	177
R3								
	182	218	180	156	181	204	189	193
R4								
	112	136	118	143	129	204	154	125

TABLE III. (*Continued.*)

Key	Just	Gondo	Mude	Mujuru	Kunaka	Bandambira	Volans	Mbira
R5								
	204	182	214	219	170	163	171	190
R6								
	182	151	172	151	201	158	175	195
R7								
	112	161	173	128	173	137	100	135
R8								
	204	192	260	240	98	251	140	185
R9								

this: *Is the tuning of the mbira determined by the overtones of the vibrations of its keys?* From the evidence here presented, it would appear that the answer is "no." Because the first overtone is approximately five times the frequency of the fundamental (corresponding to a musical interval of approximately two octaves plus a major third), the overtones of only the lowest few keys overlap with the fundamental frequencies of any of the upper keys, and the matching of overtone frequencies with fundamentals is not particularly good. The large frequency differences involved presumably make the matching of the frequencies sufficiently difficult to hear that they do not strongly influence the tuning. The lower intensity of the overtones compared to the fundamental may also make their use in tuning impractical. For the mbira the first overtone is typically 20–30 dB lower in intensity than the fundamental. In stringed instruments or Western European keyboard instruments the first and second overtones can often be comparable in intensity to (or even louder than) the fundamental.[15] However, Berliner[2] describes how master mbira maker John Kunaka deliberately constructed the lowest key (B1) on his mbira to have "two voices," i.e., the fundamental and the first overtone, with the overtone sounding two octaves plus either a fifth or a third above the fundamental. Kunaka stated that this overtone "helped the music," but that the overtones of the other keys did not and were ignored. Andrew Tracey[16] noted that in some mbiras the fundamentals of the lower-pitched keys are almost inaudible, and the maker has tuned them so that the prominent overtone, rather than the fundamental, gives the desired note. In other cases the fundamental is used, but the overtone is wildly discordant, giving the instrument a "tinkling, metallic effect." He experimented with methods of tuning the overtone to a pitch two octaves above the fundamental by removing material from the key at appropriate locations. He was successful in doing so, but remarked that he had encountered only one instrument by an African maker in which this had been done.

V. CONCLUSION

We have measured the acoustic spectrum of the keys of an African mbira. We find that the most prominent overtones present in the spectrum have frequencies that are approximately 5 and 14 times the lowest frequency. A finite-element model of the vibration of the key that takes into account the acoustic radiation efficiency of the various normal modes reveals that the far-field power spectrum is dominated by modes involving predominately transverse motion of the key. A procedure to quantify the acoustic radiation produced by each normal mode has compared favorably to experimental results.

The finding that the most prominent overtones in the sound spectrum have very high frequencies relative to the fundamental makes it unlikely that the tunings of the mbira that are used by expert musicians are determined by matching the fundamental frequencies of the upper keys with the overtones of the lower keys.

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