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Connectivity properties of Julia sets of Weierstrass elliptic functions

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Abstract

We discuss the connectivity properties of Julia sets of Weierstrass elliptic $\wp$ functions, accompanied by examples. We give sufficient conditions under which the Julia set is connected and show that triangular lattices satisfy this condition. We also give conditions under which the Fatou set of $\wp$ contains a toral band and provide an example of an order two elliptic function on a square lattice whose Julia set is a Cantor set.

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1. Introduction

In this paper we discuss the connectivity of Julia sets of Weierstrass elliptic $\wp$ functions. In [9,10] we studied dynamical properties of Weierstrass elliptic functions and established results about the dependence of the dynamics on the underlying lattice. The work in [9,10]

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built on earlier papers on the dynamics of meromorphic functions such as [1–3,5,6,8,12, 17] and elliptic functions in [13] and other related papers by these authors.

The existence of examples of Weierstrass elliptic $\wp$ functions with connected Julia sets is shown by giving explicit lattices Λ for which $J(\wp_\Lambda) = \mathbb{C}_\infty$, the Riemann sphere [9]. In the same paper we also give some conditions on the lattice or, equivalently its invariants g_2 and g_3, under which $J(\wp_\Lambda)$ is connected.

In this paper we give general results on connectivity of the Julia sets accompanied by examples. There remain several open questions of interest, in particular a complete characterization of the connectivity locus for Weierstrass elliptic $\wp$ functions parametrized in any of several ways studied up to now.

The paper is organized as follows. In Section 2 we give the background definitions and results for studying dynamics of elliptic functions. For details on these topics the reader is encouraged to read the earlier papers such as [6,9,10,13] and the references therein. The main results of the paper appear in Section 3; we prove that if $\wp_\Lambda$ is the Weierstrass $\wp$ function with period lattice Λ, and each critical value lies in a different Fatou component (or in the Julia set), then $J(\wp_\Lambda)$ is connected. If two critical values lie in the same Fatou component then we show the existence of a toral band, a Fatou component containing a generator of the fundamental group on the torus. If all three critical values lie in the same component of $F(\wp_\Lambda)$, then we prove that the Julia set has uncountably many components and, under the additional hypothesis of hyperbolicity, we show that $J(\wp_\Lambda)$ is a Cantor set. We also give a variety of examples of lattices whose associated Julia sets have toral bands in $F(\wp_\Lambda)$. In Section 3.2 we generalize to other hyperbolic elliptic functions and give a sufficient condition for the Julia set to be a Cantor set. One corollary of the preceding results is that if Λ is a lattice such that each Fatou component of $\wp_\Lambda$ fits entirely in one fundamental region of Λ, then $J(\wp_\Lambda)$ is connected. Another is that for any triangular lattice Λ, $J(\wp_\Lambda)$ is connected.

In Section 4 we give an example of an order two elliptic function f_Λ, which is a rational expression of $\wp_\Lambda$ for a particular square lattice Λ, such that $J(f)$ is a Cantor set. In Section 5 we establish additional topological properties of some specialized examples; we give an example of a lattice whose associated Julia set is connected but whose intersection with the real and imaginary axes (and parallel lines a fixed distance apart) is a Cantor set. We also give some lattices whose associated Julia sets contain the real and imaginary axes (and countably many parallel lines) and have a nonempty Fatou set. The paper is written with color graphics and several color references appear in the text. We attempt to make the reference to the graphics clear enough so that in black and white the meaning is also apparent.

2. The basics on Weierstrass $\wp$ dynamics

Let $\lambda_1, \lambda_2 \in \mathbb{C}\backslash\{0\}$ such that $\lambda_2/\lambda_1 \notin \mathbb{R}$. We define a lattice of points in the complex plane by $\Lambda = [\lambda_1, \lambda_2] := \{m\lambda_1 + n\lambda_2\colon\ m, n \in \mathbb{Z}\}$. Two different sets of vectors can generate the same lattice Λ; if $\Lambda = [\lambda_1, \lambda_2]$, then all other generators λ_3, λ_4 of Λ are obtained by multiplying the vector (λ_1, λ_2) by the matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

with $a, b, c, d \in \mathbb{Z}$ and $ad - bc = 1$.

We can view Λ as a group acting on $\mathbb{C}$ by translation, each $\omega \in \Lambda$ inducing the transformation of $\mathbb{C}$:

$$T_\omega : z \mapsto z + \omega.$$

Definition 2.1. A closed, connected subset Q of $\mathbb{C}$ is defined to be a *fundamental region* for Λ if

(1) for each $z \in \mathbb{C}$, Q contains at least one point in the same Λ-orbit as z;
(2) no two points in the interior of Q are in the same Λ-orbit.

If Q is any fundamental region for Λ, then for any $s \in \mathbb{C}$, the set

$$Q + s = \{z + s\colon z \in Q\}$$

is also a fundamental region. If we choose Q to be a parallelogram we call Q a *period parallelogram* for Λ.

The "appearance" of a lattice $\Lambda = [\lambda_1, \lambda_2]$ is determined by the ratio $\tau = \lambda_2/\lambda_1$. (We generally choose the generators so that $\mathrm{Im}(\tau) > 0$.) If $\Lambda = [\lambda_1, \lambda_2]$, and $k \neq 0$ is any complex number, then $k\Lambda$ is the lattice defined by taking $k\lambda$ for each $\lambda \in \Lambda$; $k\Lambda$ is said to be *similar* to Λ. For example, the lattice $\Lambda_\tau = [1, \tau]$ is similar to the lattice $\Lambda = \lambda_1 \Lambda_\tau$. Similarity is an equivalence relation between lattices, and an equivalence class of lattices is called a *shape*.

Definition 2.2.

(1) $\Lambda = [\lambda_1, \lambda_2]$ is *real rectangular* if there exist generators such that λ_1 is real and λ_2 is purely imaginary. Any lattice similar to a real rectangular lattice is *rectangular*.
(2) $\Lambda = [\lambda_1, \lambda_2]$ is *real rhombic* if there exist generators such that $\lambda_2 = \overline{\lambda_1}$. Any similar lattice is *rhombic*.
(3) A lattice Λ is *square* if $i\Lambda = \Lambda$. (Equivalently, Λ is square if it is similar to a lattice generated by $[\lambda, \lambda i]$, for some $\lambda > 0$.)
(4) A lattice Λ is *triangular* if $\Lambda = e^{2\pi i/3}\Lambda$ in which case a period parallelogram can be made from two equilateral triangles.

In each of cases (1)–(3) the period parallelogram with vertices $0, \lambda_1, \lambda_2,$ and $\lambda_3 := \lambda_1 + \lambda_2$ can be chosen to look rectangular, rhombic, or square, respectively.

Definition 2.3. An *elliptic function* is a meromorphic function in $\mathbb{C}$ which is periodic with respect to a lattice Λ.

For any $z \in \mathbb{C}$ and any lattice Λ, the *Weierstrass elliptic function* is defined by

$$\wp_\Lambda(z) = \frac{1}{z^2} + \sum_{w \in \Lambda \setminus \{0\}} \left(\frac{1}{(z-w)^2} - \frac{1}{w^2} \right).$$

Replacing every z by $-z$ in the definition we see that $\wp_\Lambda$ is an even function. The map $\wp_\Lambda$ is meromorphic, periodic with respect to Λ, and has order 2.

The derivative of the Weierstrass elliptic function is also an elliptic function which is periodic with respect to Λ defined by

$$\wp'_\Lambda(z) = -2 \sum_{w \in \Lambda} \frac{1}{(z-w)^3}.$$

The Weierstrass elliptic function and its derivative are related by the differential equation

$$\wp'_\Lambda(z)^2 = 4\wp_\Lambda(z)^3 - g_2\wp_\Lambda(z) - g_3, \tag{1}$$

where $g_2(\Lambda) = 60 \sum_{w \in \Lambda \setminus \{0\}} w^{-4}$ and $g_3(\Lambda) = 140 \sum_{w \in \Lambda \setminus \{0\}} w^{-6}$.

The numbers $g_2(\Lambda)$ and $g_3(\Lambda)$ are invariants of the lattice Λ in the following sense: if $g_2(\Lambda) = g_2(\Lambda')$ and $g_3(\Lambda) = g_3(\Lambda')$, then $\Lambda = \Lambda'$. Furthermore given any g_2 and g_3 such that $g_2^3 - 27g_3^2 \neq 0$ there exists a lattice Λ having $g_2 = g_2(\Lambda)$ and $g_3 = g_3(\Lambda)$ as its invariants [7].

We mention the following classical result [7].

Theorem 2.1. *Every elliptic function f_Λ with period lattice Λ can be written as $f_\Lambda(z) = R(\wp_\Lambda(z)) + \wp_\Lambda{}'(z)Q(\wp_\Lambda(z))$, where R and Q are rational functions with complex coefficients.*

Corollary 2.2. *Every elliptic function is n-to-one, where $n > 1$, within each fundamental region. The Weierstrass elliptic $\wp_\Lambda$ function is two-to-one in each fundamental region.*

Theorem 2.3 [7]. *For $\Lambda_\tau = [1, \tau]$, the functions $g_i(\tau) = g_i(\Lambda_\tau), i = 2, 3$, are analytic functions of τ in the open upper half plane* $\mathrm{Im}(\tau) > 0$.

We have the following homogeneity in the invariants g_2 and g_3 [10].

Lemma 2.4. *For lattices Λ and Λ', $\Lambda' = k\Lambda \iff$*

$$g_2(\Lambda') = k^{-4}g_2(\Lambda) \quad \text{and} \quad g_3(\Lambda') = k^{-6}g_3(\Lambda).$$

Theorem 2.5 [11]. *The following are equivalent*:

(1) $\wp_\Lambda(\bar{z}) = \overline{\wp_\Lambda(z)}$;
(2) *Λ is a real lattice*;
(3) $g_2, g_3 \in \mathbb{R}$.

For any lattice Λ, the Weierstrass elliptic function and its derivative satisfy the following properties: for $k \in \mathbb{C}\backslash\{0\}$,

$$\wp_{k\Lambda}(ku) = \frac{1}{k^2}\wp_\Lambda(u) \quad \text{(homogeneity of } \wp_\Lambda),$$
$$\wp'_{k\Lambda}(ku) = \frac{1}{k^3}\wp'_\Lambda(u) \quad \text{(homogeneity of } \wp'_\Lambda). \tag{2}$$

Verification of the homogeneity properties can be seen by substitution into the series definitions.

The critical values of the Weierstrass elliptic function on an arbitrary lattice $\Lambda = [\lambda_1, \lambda_2]$ are as follows. For $j = 1, 2$, notice that $\wp_\Lambda(\lambda_j - z) = \wp_\Lambda(z)$ for all z. Taking derivatives of both sides we obtain $-\wp'_\Lambda(\lambda_j - z) = \wp'_\Lambda(z)$. Substituting $z = \lambda_1/2, \lambda_2/2$, or $\lambda_3/2$, we see that $\wp'_\Lambda(z) = 0$ at these values. We use the notation

$$e_1 = \wp_\Lambda\left(\frac{\lambda_1}{2}\right), \qquad e_2 = \wp_\Lambda\left(\frac{\lambda_2}{2}\right), \qquad e_3 = \wp_\Lambda\left(\frac{\lambda_3}{2}\right)$$

to denote the critical values. Since e_1, e_2, e_3 are the distinct zeros of Eq. (1), we also write

$$\wp'_\Lambda(z)^2 = 4\big(\wp_\Lambda(z) - e_1\big)\big(\wp_\Lambda(z) - e_2\big)\big(\wp_\Lambda(z) - e_3\big). \tag{3}$$

Equating like terms in Eqs. (1) and (3), we obtain

$$e_1 + e_2 + e_3 = 0, \qquad e_1e_3 + e_2e_3 + e_1e_2 = \frac{-g_2}{4}, \qquad e_1e_2e_3 = \frac{g_3}{4}. \tag{4}$$

In the real lattice case, the homogeneity property, Eq. (2), is used to produce infinitely many lattices with real superattracting fixed points.

Lemma 2.6. *Let Γ be a real lattice such that $\gamma/2$ is the smallest positive real critical point and e_r is the largest real critical value. If m is any odd integer and $k = \sqrt[3]{2e_r/(m\gamma)}$ (taking the real root) then the lattice $\Lambda = k\Gamma$ has a real superattracting fixed point at $mk\gamma/2$.*

Proof. Clearly Λ is a real lattice. Eq. (2) implies that $k\gamma/2$ is a real critical point for $\wp_\Lambda$. Since m is odd, periodicity implies that $\wp_\Lambda(mk\gamma/2) = \wp_\Lambda(k\gamma/2)$. Further, the homogeneity property implies that

$$\wp_\Lambda\left(\frac{k\gamma}{2}\right) = \wp_{k\Gamma}\left(\frac{k\gamma}{2}\right) = \frac{1}{k^2}\wp_\Gamma\left(\frac{\gamma}{2}\right) = \frac{e_r}{k^2} = \frac{mk\gamma}{2}. \quad \square$$

Fig. 1 shows the graph of $\wp_\Lambda$ on $\mathbb{R}$ for a lattice which has a superattracting fixed point.

2.1. Fatou and Julia sets for elliptic functions

We review the basic dynamical definitions and properties for meromorphic functions which appear in [1,4–6]. Let $f : \mathbb{C} \to \mathbb{C}_\infty$ be a meromorphic function where $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$ is the Riemann sphere. The *Fatou set* $F(f)$ is the set of points $z \in \mathbb{C}_\infty$ such that $\{f^n : n \in \mathbb{N}\}$ is defined and normal in some neighborhood of z. The *Julia set* is the complement of the Fatou set on the sphere, $J(f) = \mathbb{C}_\infty\backslash F(f)$. Notice that $\mathbb{C}_\infty\backslash\overline{\bigcup_{n\geqslant 0} f^{-n}(\infty)}$

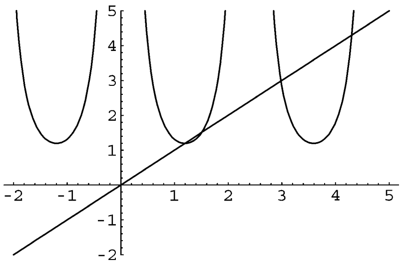

Fig. 1. Graph of $\wp_\Lambda|_{\mathbb{R}}$ with a superattracting fixed point.

is the largest open set where all iterates are defined. Since $f(\mathbb{C}_\infty\backslash\overline{\bigcup_{n\geqslant 0} f^{-n}(\infty)}) \subset \mathbb{C}_\infty\backslash\overline{\bigcup_{n\geqslant 0} f^{-n}(\infty)}$, Montel's theorem implies that

$$J(f) = \overline{\bigcup_{n\geqslant 0} f^{-n}(\infty)}.$$

Let $\mathrm{Crit}(f)$ denote the set of critical points of f, i.e.,

$$\mathrm{Crit}(f) = \{z\colon\ f'(z) = 0\}.$$

If z_0 is a critical point then $f(z_0)$ is a *critical value*. For each lattice, $\wp_\Lambda$ has three critical values and no asymptotic values. The *singular set* $\mathrm{Sing}(f)$ of f is the set of critical and finite asymptotic values of f and their limit points. A function is called *Class S* if f has only finitely many critical and asymptotic values; for each lattice Λ, every elliptic function with period lattice Λ is of Class S. The *postcritical set* of $\wp_\Lambda$ is:

$$P(\wp_\Lambda) = \overline{\bigcup_{n\geqslant 0} \wp_\Lambda^n(e_1 \cup e_2 \cup e_3)}.$$

For a meromorphic function f, a point z_0 is *periodic* of period p if there exists a $p \geqslant 1$ such that $f^p(z_0) = z_0$. We also call the set $\{z_0, f(z_0), \ldots, f^{p-1}(z_0)\}$ a *p-cycle*. The *multiplier* of a point z_0 of period p is the derivative $(f^p)'(z_0)$. A periodic point z_0 is called *attracting, repelling,* or *neutral* if $|(f^p)'(z_0)|$ is less than, greater than, or equal to 1, respectively. If $|(f^p)'(z_0)| = 0$ then z_0 is called a *superattracting* periodic point. As in the case of rational maps, the Julia set is the closure of the repelling periodic points [1].

Suppose U is a connected component of the Fatou set. We say that U is *preperiodic* if there exists $n > m \geqslant 0$ such that $f^n(U) = f^m(U)$, and the minimum of $n - m = p$ for all such n, m is the *period* of the cycle.

Since every elliptic function is of Class S the basic dynamics are similar to those of rational maps with the exception of the poles. The first result holds for all Class S functions as was shown in [4, Theorem 12] and [17].

Theorem 2.7. *For any lattice Λ, the Fatou set of an elliptic function f_Λ with period lattice Λ has no wandering domains and no Baker domains.*

In particular, Sullivan's No Wandering Domains Theorem holds in this setting so all Fatou components of f_Λ are preperiodic. Because there are only finitely many critical values, we have a bound on the number of attracting periodic points that can occur.

The next result was proved in [10]; it is only known for the Weierstrass elliptic function.

Theorem 2.8. *For any lattice Λ, $\wp_\Lambda$ has no cycle of Herman rings.*

We summarize this discussion with the following result.

Theorem 2.9. *For any lattice Λ, at most three different types of forward invariant Fatou cycles can occur for $\wp_\Lambda$, and each periodic Fatou component contains one of these*:

(1) *a linearizing neighborhood of an attracting periodic point*;
(2) *a Böttcher neighborhood of a superattracting periodic point*;
(3) *an attracting Leau petal for a periodic parabolic point. The periodic point is in $J(\wp_\Lambda)$*;
(4) *a periodic Siegel disk containing an irrationally neutral periodic point.*

The proof of Lemma 2.10 is given for $\wp_\Lambda$ in [10] but remains the same for the elliptic function f_Λ.

Lemma 2.10. *If Λ is any lattice and f_Λ is an elliptic function with period lattice Λ, then*

(1) $J(f_\Lambda) + \Lambda = J(f_\Lambda)$, *and*
(2) $F(f_\Lambda) + \Lambda = F(f_\Lambda)$.

The periodicity of f_Λ and $J(f_\Lambda)$ with respect to the lattice gives rise to different possibilities for connectivity of a component when the Fatou and Julia sets are projected to the torus. If we write

$$\pi_\Lambda : \mathbb{C} \to \mathbb{C}/\Lambda$$

for the usual toral quotient map we make the following definition.

Definition 2.4. A Fatou component F_o of the map f_Λ is called a *toral band* if F_o contains an open subset U which is simply connected in $\mathbb{C}$, but $\pi_\Lambda(U)$ is not simply connected on $\mathbb{C}/\Lambda$. In other words, U projects down to a topological band around the torus and contains a homotopically nontrivial curve.

We also introduce the notion of a double toral band.

Definition 2.5. Suppose that we have an elliptic function f_Λ with period lattice Λ. If we have a component $W \subset F(f_\Lambda)$ which contains a simple closed loop which forms the boundary of a fundamental region for Λ, then we say W is a *double toral band*.

Clearly a double toral band W contains a toral band as is shown by the next result.

Proposition 2.1 [10]. *For any lattice Λ, f_Λ has a toral band if and only if there is a component of the Fatou set which is not completely contained in the interior of one fundamental region of $\mathbb{C}/\Lambda$.*

We conclude this section with a result about Siegel disks for elliptic functions.

Proposition 2.2. *Assume f_Λ has a cycle of Siegel disks $S = \{U_1, \ldots, U_p\}$, the union of pairwise disjoint open simply connected sets, with $f_\Lambda(S) = S$, $f_\Lambda(U_j) = U_{j+1}$, $j = 1, \ldots, p-1$, $f_\Lambda(U_p) = U_1$, and $f_\Lambda^p(U_j) = U_j$ for $j = 1, \ldots, p$. Then the following hold*:

(1) *Each U_j is completely contained in one fundamental region of Λ.*
(2) *If $z \in S$, then for every $\lambda \in \Lambda, \lambda \neq 0$, $z + \lambda \notin S$.*
If $f_\Lambda = \wp_\Lambda$, then the following also holds:
(3) *If c is any critical point in the fundamental region containing U_j, for any $j = 1, \ldots, p$, and if $z \in S$ is of the form $z = c + w$, then $v = c - w$ (the point symmetric to z about the critical point) is not in S.*

Proof. Suppose there is a simply connected domain U_j as above which is not completely contained in one fundamental region for Λ. Then there exists $z \in U_j$ and $\lambda \in \Lambda$ such that $z, z + \lambda \in U_j$. Since we have a Siegel disk cycle, $f_\Lambda^p|_{U_j}$ is injective and is conformally conjugate to irrational rotation on a disk; but $f_\Lambda(z) = f_\Lambda(z + \Lambda)$, so $f_\Lambda^p(z) = f_\Lambda^p(z + \lambda)$, which cannot occur. Therefore there is no such z in U_j.

If $z \in U_j$, and $z + \lambda \in U_k$, $k \neq j$, then $f_\Lambda(z) \in U_{j+1}$, while $f_\Lambda(z + \lambda) \in U_{k+1}$ (both mod p), which contradicts the disjointness of the $U_j's$ since $f_\Lambda(z) = f_\Lambda(z + \lambda)$.

The last statement follows since $\wp_\Lambda(c + w) = \wp_\Lambda(c - w)$; this holds since $2c \in \Lambda$, and $\wp_\Lambda(w + c) = \wp_\Lambda(-w - c) = \wp_\Lambda(-w - c + 2c)$. The rest of the argument is exactly as above. □

3. Results on connectivity properties of $J(\wp_\Lambda)$

We now turn to the main connectivity results of this paper. We assume throughout this section we are considering the Weierstrass elliptic $\wp$ function $\wp_\Lambda$ with period lattice Λ. First we give sufficient conditions under which $J(\wp_\Lambda)$ is connected. The proof given here is based on a similar result for polynomials given by Milnor in [15]. Although there are infinitely many critical points for $\wp_\Lambda$, there are exactly three critical values and $\wp_\Lambda$ is locally two-to-one in each fundamental region. Therefore one can expect dynamical behavior somewhat similar to that for low degree rational maps. In particular the results in this section show that the full spectrum of connectivity possible for Fatou components of meromorphic functions, such as that shown in [2], cannot occur in this setting.

Theorem 3.1. *If Λ is a lattice such that each critical value of $\wp_\Lambda$ that lies in the Fatou set is the only critical value in that component, then $J(\wp_\Lambda)$ is connected. In particular, if each Fatou component contains either* 0 *or* 1 *critical value, then $J(\wp_\Lambda)$ is connected.*

Proof. If $F(\wp_\Lambda) = \emptyset$, then $J(\wp_\Lambda) = \mathbb{C}_\infty$ is connected. Therefore we assume that there is a nonempty Fatou set, and consider a loop γ in any Fatou component W of $\wp_\Lambda$. It is enough to show that γ shrinks to a point in W since $J(\wp_\Lambda)$ is connected if and only if each component of $F(\wp_\Lambda)$ is simply connected (see, e.g., [9, Proposition 6.2(1)]). Since $\wp_\Lambda$ has only finitely many critical values, by Theorem 2.7 there are no wandering domains, so by Sullivan's nonwandering theorem some forward image $\wp_\Lambda^k(\gamma)$ lies in a simply connected region $U \subset F(\wp_\Lambda)$, and U is in a Fatou component containing one of these:

(1) a linearizing neighborhood of an attracting periodic point;
(2) a Böttcher neighborhood of a superattracting periodic point;
(3) an attracting Leau petal for a periodic parabolic point;
(4) a periodic Siegel disk (i.e., one component of a Siegel disk cycle).

We established in [10] that there are no Herman rings, and in Proposition 2.2 we established that any Siegel disk is contained in one fundamental region for the lattice Λ. Without loss of generality, we can assume that $U \subset Q_o$, a single period parallelogram (with appropriate boundaries of Q_o chosen not to intersect the boundary of U).

Furthermore since U is simply connected, we can choose it so that its boundary is a simple closed curve not passing through a critical value, and U contains either one or no critical values. Using induction on k, we show that $\wp_\Lambda^{-k}(U)$ consists of only simply connected components. Therefore γ is homotopic to a point. For the inductive argument, we first show that $\wp_\Lambda^{-1}(U)$ has simply connected components.

Case 1: If U contains no critical value, then in each fundamental region, $\wp_\Lambda^{-1}(U)$ consists of two disjoint simply connected regions.

Case 2: If U contains one critical value, then $\wp_\Lambda^{-1}(U)$ is a single simply connected set in each fundamental region, bounded by a simple closed curve, and which maps onto U by a ramified two-fold covering. This follows from the fact that $\wp_\Lambda$ is locally two-to-one near each critical point and the branched covering $\wp_\Lambda^{-1}(U) \to U$ is two-to-one in each region.

The inductive step on k is the same. By adjusting the original choice of U a little (and at most finitely many times), we can assume that each simply connected component of $\wp_\Lambda^{-1}(U)$ contains no critical value in its boundary and contains either 0 or 1 critical value in it, so we repeat the argument. □

Corollary 3.2. *If all critical values of $\wp_\Lambda$ lie in $J(\wp_\Lambda)$, then $J(\wp_\Lambda)$ is connected.*

Proof. Assume all critical values lie in $J(\wp_\Lambda)$. If $F(\wp_\Lambda)$ is empty, the $J(\wp_\Lambda) = \mathbb{C}_\infty$ and we are done. If $F(\wp_\Lambda) \neq \emptyset$, then the only possibility is that we have Siegel disk cycles. In any case by hypothesis each Fatou component contains no critical value or critical point, so we can apply Theorem 3.1. □

Corollary 3.3. *If Λ is a triangular lattice, then $J(\wp_\Lambda)$ is connected.*

Proof. We showed in [10] that for any triangular lattice the critical values e_1, e_2, and e_3 belong to disjoint components of $F(\wp_\Lambda)$ or they all belong to $J(\wp_\Lambda)$. Moreover we proved that for $j = 1, 2, 3$, $e^{2\pi i/3}e_j = e_{j+1}$ (mod 3) and there is only one "type" of critical value behavior, which is then rotated to the other two critical values. We then apply Theorem 3.1 and Corollary 3.2. □

We have defined and given examples of toral Fatou bands in [9]. We do not yet know if the Julia set associated with a toral band is connected, disconnected, or whether both can occur. However, we prove the following result which gives sufficient conditions for a toral band to exist.

Theorem 3.4. *If $\wp_\Lambda$ has a Fatou component that contains (at least) two critical values, then $\wp_\Lambda$ has a toral band.*

Proof. Writing $\Lambda = [\lambda_1, \lambda_2]$, we label three critical points $c_1 = \lambda_1/2$, $c_2 = \lambda_2/2$, and $c_3 = \lambda_3/2 = (\lambda_1 + \lambda_2)/2 = c_1 + c_2$; we treat three cases in a similar way. Suppose that $e_1 = \wp_\Lambda(c_1)$ and $e_3 = \wp_\Lambda(c_3)$ both lie in a Fatou component W. Since W is open and connected it is path connected. Let γ be a path from e_3 to e_1 in W. Let V be the component of $\wp_\Lambda^{-1}(W)$ that contains c_3. Then one component of $\wp_\Lambda^{-1}(\gamma)$ connects c_3 to c_1 or $c_1 + \lambda_2$ in V because either of these paths is contained in the period parallelogram $Q = \{s\lambda_1 + t\lambda_2\colon 0 \leqslant s, t \leqslant 1\}$. If $\wp_\Lambda^{-1}(\gamma)$ connects c_3 to c_1 we can write this path as $c_3 + z(t)$ with $z(0) = 0$ and $z(1) = c_1 - c_3 = -c_2$ (or $z(0) = 0$ and $z(1) = c_1 + \lambda_2 - c_3 = c_2$). Using Proposition 5.1 in [10], for $i = 1, 2, 3$, $c_i + z \in V$ if and only if $c_i - z \in V$; i.e., we have symmetry of a Fatou component about any critical point in it. Therefore $c_3 - z(t)$ is also a path in V connecting c_3 to $c_1 + \lambda_2$. Alternatively, V might contain c_3 and $c_1 + \lambda_2$.

In this case we repeat the argument above, using the symmetry of the component V about critical points to continue the path so that, in either case, we end up with a path in V passing through c_3 and connecting c_1 to $c_1 + \lambda_2$.

Since both c_1 and $c_1 + \lambda_2$ lie in V and V is open, the component V is not completely contained in one fundamental region of $\mathbb{C}/\Lambda$, so by [10, Proposition 5.2] we have a toral Fatou band.

If e_2 and e_3 lie in the same component, then a similar argument shows that c_2 and $c_2 + \lambda_1$ both lie in the same Fatou component.

For the third case, if e_1 and e_2 lie in the same Fatou component W, then a similar argument gives a path γ from e_2 to e_1 in W, hence a path in V, a component of $\wp_\Lambda^{-1}(W)$, written as $c_2 + z(t)$ from c_2 to c_1 or $c_1 - \lambda_1 + \lambda_2$. The path lies in a single Fatou component V. Then by symmetry we have that $c_2 - z(t)$ lies in V for all $t \in [0, 1]$, so in particular $c_2 - z(1) = -c_1 + \lambda_2 = c_1 - \lambda_1 + \lambda_2 \in V$. □

Corollary 3.5. *If every periodic Fatou component is completely contained in one fundamental period of Λ, then $J(\wp_\Lambda)$ is connected.*

Proof. We claim first that the hypothesis implies that each periodic Fatou component contains at most one critical value. If there is a Fatou component that contains 2 or more critical values, then we apply Theorem 3.4 to obtain a toral band. By Proposition 2.1, the toral band does not fit in one fundamental region, which gives a contradiction.

Therefore we have established that every Fatou component for $\wp_\Lambda$ contains either 0 or 1 critical value and we can apply Theorem 3.1. □

3.1. Toral band examples

Suppose exactly two critical values are in one Fatou component of $\wp_\Lambda$; then by Theorem 3.4 there is a toral Fatou band, and three possible cases:

(1) the third critical value is in the Julia set,
(2) the third critical value lies in a different Fatou component corresponding to a different cycle, or,
(3) there is exactly one nonrepelling cycle and the third critical value is attracted to the same cycle as the other two but lies in a separate component.

The existence of a lattice with a toral band was proved by the authors in [10]. The examples shown in this section, found numerically using Mathematica, provide illustrations of the three possibilities listed above for the critical values of Weierstrass elliptic functions with toral bands. In the first two of these examples the lattice Λ is real.

Example 3.6. $\wp_\Lambda$ has a toral band and the third critical value lies in the Julia set (because it is a pole).

We consider $g_2 \approx 26.5626$ and $g_3 \approx -26.2672$; the corresponding lattice Λ is real rectangular (see [7]). For these values $\wp_\Lambda$ has an attracting fixed point $p \approx 1.5566$, which is plotted (in orange) in Fig. 2.[1] One period parallelogram for Λ is outlined in light blue. The light points (colored yellow) in Figs. 2 and 3 iterate to p and hence belong to the Fatou set. The dark points (colored blue) lie in the Julia set. In Fig. 3 the critical values $e_1 \approx 1.5539, e_2 \approx -2.9746$, and $e_3 \approx 1.4206$ are plotted (in purple). Two of the critical values lie in the component containing p, and the third critical value is the lattice point $-\lambda_1 \approx -2.9746$, a pole, so is in the Julia set.

Example 3.7. $\wp_\Lambda$ has a toral band and the third critical value belongs to a different cycle of Fatou components than the first two critical values.

In this example $g_2 \approx 27.85$ and $g_3 \approx -28.338$ so Λ is real rhombic. The two complex critical values lie in the same Fatou component and converge to an attracting fixed point at $p \approx 1.542$; the medium shaded points (colored tan) in Figs. 4 and 5 converge to p under iteration. Also, $\wp_\Lambda$ has a superattracting fixed point at $q \approx -3.047$, and the light points (colored yellow) iterate to q. The fixed points p and q are marked (in orange) in Fig. 4. The dark points (colored navy blue) lie in the Julia set. In Fig. 5 we zoom in to a neighborhood of the superattracting fixed point q.

Example 3.8. $\wp_\Lambda$ has a toral band and the third critical value is associated with the same cycle of Fatou components as the first two, but lies in a different Fatou component.

[1] For interpretation of the references to colour in the figures, the reader is referred to the web version of this article.

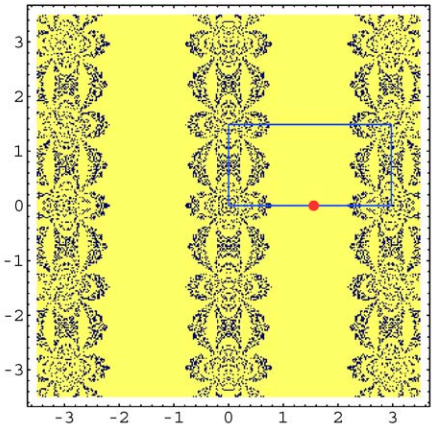

Fig. 2. The attracting fixed point of the Fatou set of Example 3.6.

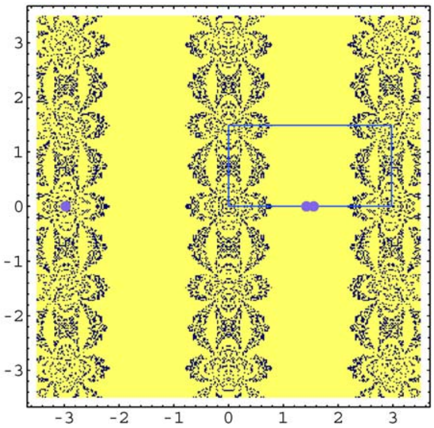

Fig. 3. Critical values of Example 3.6.

We set $g_2 \approx -1.451 - 4.984i$ and $g_3 \approx -2.136 - 0.801i$, so the corresponding lattice Λ is rhombic. Then $\wp_\Lambda$ has an attracting two-cycle at $\{p_1, p_2\} \approx \{-0.422 + 0.517i, -0.537 + 2.25i\}$, which is plotted (in orange) in Fig. 6. The lightest points (colored yellow) in Figs. 6 and 7 converge to this cycle. The dark points (navy blue) lie in the Julia set, and we have shown the boundary of one period parallelogram (in light blue). In Fig. 7, we zoom in on part of Fig. 6 and plot the critical values $e_1 \approx 0.790 - 1.053i$, $e_2 \approx -0.423 + 0.517i$, and $e_3 \approx -0.367 + 0.536i$ (in purple).

3.2. Disconnected Julia sets

The fourth possibility for a toral band is that all three critical values lie in the same Fatou component of $\wp_\Lambda$. Though we do not know if this can occur for any lattice Λ for the Weierstrass elliptic function $\wp_\Lambda$, we show that the Julia set is not connected in this case.

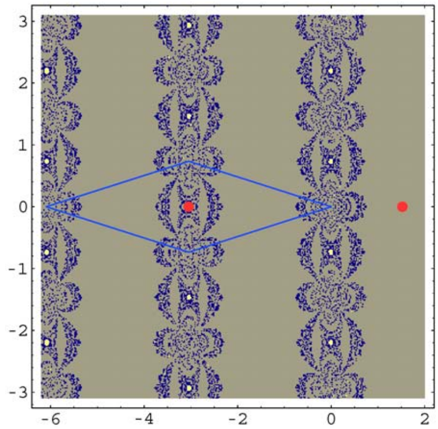

Fig. 4. Fixed points and the Fatou set of Example 3.7.

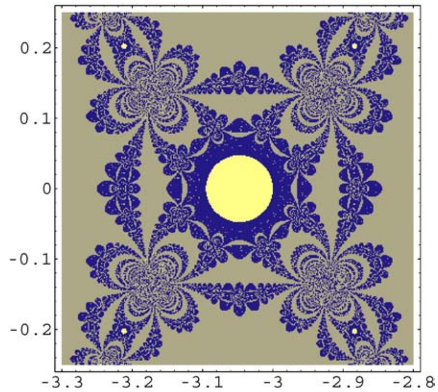

Fig. 5. Superattracting fixed point of Example 3.7.

If in addition $\wp_\Lambda$ is hyperbolic, we show that having all the critical values in one Fatou component forces $J(\wp_\Lambda)$ to be a Cantor set. We show that under these conditions there exists a double toral band component (see Definition 2.5), and show this is a sufficient condition for a Cantor Julia set. We use this result to give an example of an order two elliptic function whose Julia set is a Cantor set.

We begin by showing that in order for $J(\wp_\Lambda)$ to be totally disconnected, it is necessary that all critical values lie in the same Fatou component. Recall that if f_Λ is an elliptic function and $J(f_\Lambda)$ is totally disconnected, then $J(f_\Lambda)$ is called a *Cantor set* since it is homeomorphic to the classical middle thirds Cantor set. In this case there is exactly one Fatou component which is therefore completely invariant, so there is one nonrepelling fixed point. Meromorphic functions with Cantor Julia sets have been shown to exist in [6].

Throughout this section the notation f_Λ refers to any elliptic function with period lattice Λ; the notation $\wp_\Lambda$, as usual, refers to the Weierstrass $\wp$ function with period lattice Λ.

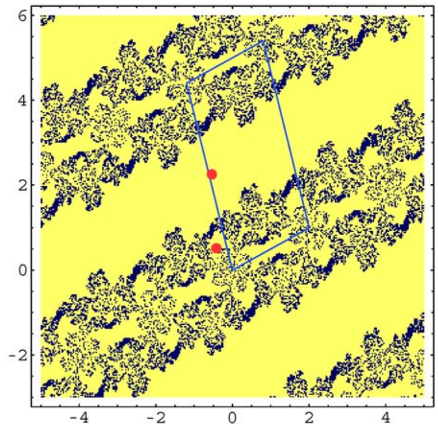

Fig. 6. Two cycle and the Fatou set of Example 3.8.

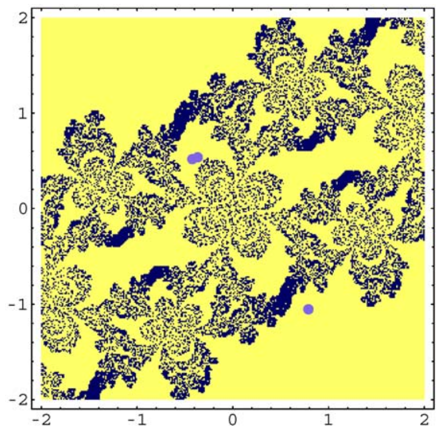

Fig. 7. Critical values of Example 3.8.

If the period lattice is understood or irrelevant, we frequently will use f for an elliptic function; all functions in this section are elliptic.

Proposition 3.1. *If $J(f)$ is totally disconnected and contains no critical point, then all critical values lie in the same component of the Fatou set, the component is completely invariant, and it corresponds to a nonrepelling fixed point.*

Proof. If $J(f)$ is totally disconnected, the Fatou set contains only one component; a totally disconnected set cannot separate the sphere. Since $J(f)$ contains no critical point, we cannot have any Siegel disks or Herman rings. Therefore the only Fatou component possible is a Böttcher domain, an attracting domain, or a Leau petal, each corresponding to a fixed point. □

We give some results in the other direction. Recall that if $J(f)$ is disconnected then it consists of uncountably many components; the proof of this is the same as for rational maps (e.g., see [16]). We turn to the definition of a hyperbolic elliptic function; the concept of hyperbolicity for a meromorphic function is similar to that for a rational map, but the equivalent notions in the rational settings are not always the same for meromorphic functions (e.g., see [17]).

Definition 3.1. We say that an elliptic function is *hyperbolic* if $J(f)$ is disjoint from $P(f)$.

We adapt the following result from [17, Theorem C] to our setting. We define the set

$$A_n(f) = \{z \in \mathbb{C}\colon\ f^n \text{ is not analytic at } z\}.$$

Since f is elliptic,

$$A_n(f) = \bigcup_{j=1}^{n} f^{-j}(\{\infty\}).$$

We therefore denote the set of prepoles for f by

$$\mathcal{A} = \bigcup_{n\geqslant 1} A_n.$$

We say that ω is an *order k prepole* if $\wp_\Lambda^k(\omega) = \infty$; so a pole is an order 1 prepole.

Theorem 3.9 [17]. *If an elliptic function f is hyperbolic then there exist $K > 1$ and $c > 0$ such that*

$$\left|(f^n)'(z)\right| > cK^n \frac{|f^n(z)| + 1}{|z| + 1},$$

for each $z \in J(f) \setminus A_n(f)$, $n \in \mathbb{N}$.

Remark 3.1. For a fixed lattice Λ, and elliptic function $f = f_\Lambda$, we have that $f(z+\lambda) = f(z)$ and $f'(z+\lambda) = f'(z)$ for every $\lambda \in \Lambda$. Therefore we can prove the following standard result about hyperbolic functions.

Theorem 3.10. *An elliptic function f is hyperbolic if and only if there exist $K > 1$ and $C > 0$, $C = C(\Lambda)$, such that*

$$\left|(f^n)'(z)\right| > CK^n, \tag{5}$$

for each $z \in J(f) \setminus A_n(f)$, $n \in \mathbb{N}$.

Proof. If f is hyperbolic, by Theorem 3.9 there is a $K > 1$ and $c > 0$ such that $|(f^n)'(z)| > cK^n \frac{|f^n(z)|+1}{|z|+1}$, for each $z \in J(f) \setminus A_n(f)$, $n \in \mathbb{N}$. In particular, for z in one period parallelogram $Q \subset B(0,R) = \{z\colon |z| < R\}$ we have that $|(f^n)'(z)| > cK^n \frac{1}{R+1}$, so we just choose $C = c/(R+1)$ for $z \in Q \cap (J(f) \setminus A_n(f))$, for each $n \in \mathbb{N}$. However from the remark above, we have that the same constant C works for all $z \in J(f) \setminus A_n(f)$, $n \in \mathbb{N}$.

The converse follows as in the case of rational maps. If the expanding condition (5) holds, there can be no critical points on $J(f)$, so $J(f) \neq \mathbb{C}_\infty$ and we have no Siegel disks or Herman rings. There cannot be parabolic cycles (i.e., periodic points of modulus 1 in $J(f)$) either. Therefore all components of $F(f)$ are mapped forward to attracting cycles so $J(f)$ is disjoint from $P(f)$. □

Corollary 3.11. *If f is hyperbolic, then there exists an $r > 1$, an $s \in \mathbb{N}$, and a neighborhood U of $J(f)$, such that*

$$\left|\left(f^s\right)'(z)\right| > r \quad \textit{for all } z \in U \setminus A_s(f).$$

We state the main result of this section.

Theorem 3.12. *If $\wp_\Lambda$ is hyperbolic and all the critical values of $\wp_\Lambda$ are contained in one Fatou component, then $J(\wp_\Lambda)$ is a Cantor set.*

Proof. Writing $\Lambda = [\lambda_1, \lambda_2]$, we label three critical points $c_1 = \lambda_1/2$, $c_2 = \lambda_2/2$, and $c_3 = \lambda_3/2 = (\lambda_1 + \lambda_2)/2 = c_1 + c_2$. By hypothesis all three critical values lie in one Fatou component, say U. Since each fundamental region for Λ maps in a two-to-one way onto $\mathbb{C}_\infty$ (except at critical points), and $e_1 = \wp_\Lambda(c_1)$ and $e_3 = \wp_\Lambda(c_3)$, we can choose a component W of $\wp_\Lambda^{-1}(U)$ that contains c_1, c_2, and c_3. We construct a path α_1 from c_1 to c_3, and a path α_2 from c_3 to c_2, both contained in W. By symmetry of $F(\wp_\Lambda)$ about critical points, we can construct the paths to lie completely in half of a fundamental domain for the lattice; i.e., in $\Omega = \{z \in \mathbb{C}\colon\ z = s\lambda_1 + t\lambda_2,\ s \in [0,1],\ t \in [0, \frac{1}{2}]\}$. Moreover we can construct the paths to be simple (no self-intersections) and such that $\alpha_1 \cap \alpha_2 = \{c_3\}$ (using a straightforward topological argument).

For any path $\alpha : [0,1] \to \mathbb{C}$, by $-\alpha$ we denote the path $-\alpha(t) = \alpha(1-t)$, which is just α traversed in reverse order. We extend the paths α_1 and α_2 in W by symmetry of each Fatou component about a critical point in such a way that we obtain a path β_1 from c_3 through c_1 to $c_3 - \lambda_2$, and a path β_2 from $c_3 - \lambda_1$ through c_2 to c_3. Again by symmetry, these paths extend to a loop q inside W which consists of

$$\beta_1 \cup -(\beta_2 - \lambda_2) \cup -(\beta_1 - \lambda_1) \cup \beta_2.$$

Furthermore, since the original paths avoid the pole at 0, q does as well. The loop q is no longer contained in Ω but it is contained completely in W. We claim that q bounds a fundamental region for Λ (the bounded component of q).

We denote by Q the closed bounded region in the plane with boundary q. To prove the claim we observe that by construction $Q + \lambda_1$, $Q - \lambda_1$, $Q + \lambda_2$, and $Q - \lambda_2$ are disjoint from Q except for one boundary curve in each case. By an induction argument we have that the sets $Q + m\lambda_1 + n\lambda_2$, $m, n \in \mathbb{Z}$, tile the plane and are pairwise disjoint except at the boundary curves.

Thus q is a loop lying in one component of the Fatou set, and by construction q is symmetric with respect to 0; that is, if $z \in q$ then $-z \in q$.

Let K denote the unbounded component of the complement of q. By the symmetry of q with respect to the origin, we have that 0 is not in K. But then q is a loop in the Fatou set separating $0 \in J(\wp_\Lambda)$ from $\infty \in J(\wp_\Lambda)$, so $J(\wp_\Lambda)$ is not connected. It remains to show that each component of $J(\wp_\Lambda)$ consists of just one point, which we do by showing that if C is a component of $J(\wp_\Lambda)$, then $\operatorname{diam}(C) = 0$ (using the Euclidean metric on Q).

We note that the vertices of Q are the critical points c_3, $c_3 - \lambda_2$, $-c_3$, and $c_3 - \lambda_1$, the pole (at 0) for $\wp_\Lambda$ is in the interior of Q since the boundary of Q is in the Fatou set. We label each fundamental region as follows:

$$Q = Q_{0,0},$$
$$Q_{m,n} = Q_{0,0} + m\lambda_1 + n\lambda_2, \quad m, n \in Z,$$

and note that any pair of fundamental regions corresponding to distinct (m, n) pairs are either disjoint or intersect along one boundary segment. Furthermore $\mathbb{C} = \bigcup_{m,n\in\mathbb{Z}} Q_{m,n}$.

Since $\wp_\Lambda$ is hyperbolic, $J(\wp_\Lambda)$ is compact, and we have a double toral band along the boundary of Q, using Theorem 3.10 and Corollary 3.11 we can find $s \in \mathbb{N}$ and a neighborhood U of $J = J(\wp_\Lambda) \cap Q$, with $\overline{U} \subset \operatorname{int} Q$ on which $|(\wp_\Lambda^s)'(z)| \geqslant cr$ with $r > 1$ for all $z \in U$ on which $\wp_\Lambda^s$ is defined. By compactness of J we can choose U to be simply connected and so that we have a Jordan curve $\delta = \partial U \subset F(\wp_\Lambda)$ such that J lies in the bounded component of δ. By construction of δ, all critical points and all critical values lie in the unbounded component of δ.

By symmetry, we translate the sets J and U by the elements of the lattice, and we label these as follows:

$$J_{0,0} = J; \qquad J_{m,n} = J_{0,0} + m\lambda_1 + n\lambda_2,$$

and

$$U_{0,0} = U; \qquad U_{m,n} = U_{0,0} + m\lambda_1 + n\lambda_2.$$

We also define $\delta_{m,n} = \partial U_{m,n} = \delta + m\lambda_1 + n\lambda_2$; this gives a disjoint collection of Jordan curves lying in the Fatou set.

We have that

$$J(\wp_\Lambda) = \left\{ z \in \mathbb{C}_\infty\colon\ \wp_\Lambda^k(z) \in \bigcup_{m,n\in\mathbb{Z}} U_{m,n} \text{ for all } k \text{ for which } \wp_\Lambda^k \text{ is analytic} \right\}$$

and it is clear from the construction that there are no critical points or critical values in $\bigcup_{m,n\in\mathbb{Z}} U_{m,n}$. Given any point $z \in J(\wp_\Lambda) \setminus \{\infty\}$, we have that $z \in J_{m,n}$, so $\delta_{m,n}$ gives a separation of z from ∞ (since z lies in the bounded component of its complement by construction). Therefore $\{\infty\}$ is a component of $J(\wp_\Lambda)$.

Defining $\mathcal{J} = \{z \in \mathbb{C}\colon\ \wp_\Lambda^k(z) \in \bigcup_{m,n\in\mathbb{Z}} U_{m,n}$ for all $k \geqslant 1\}$, we can write the Julia set as the following disjoint union:

$$J(\wp_\Lambda) = \mathcal{J} \cup \mathcal{A} \cup \{\infty\};$$

these are the points that stay in $\mathbb{C}$ under all iterations of $\wp_\Lambda$, the prepoles (including poles), and the point at ∞, respectively. For each $(i, j) \in \mathbb{Z}^2$, the set $\wp_\Lambda^{-1} U_{i,j}$ consists of two disjoint sets in the fundamental period $Q_{m,n}$ for every m, n, and $\wp_\Lambda$ is a holomorphic covering map of each component onto $U_{i,j}$; since $U_{i,j}$ is simply connected, the restriction of $\wp_\Lambda$ to each component is a homeomorphism of that component onto $U_{i,j}$. Therefore we can define branches:

$$p^{(i,j)}_{m,n,0}, p^{(i,j)}_{m,n,1}, \quad m, n, i, j \in \mathbb{Z},$$

of $\wp_\Lambda^{-1}$ on $\overline{U_{i,j}}$, mapping into $Q_{m,n}$, and we see that the collection of sets $p^{(i,j)}_{m,n,k}(\overline{U_{i,j}})$, $m, n, i, j \in \mathbb{Z}$, $k = 0, 1$, are pairwise disjoint compact subsets.

We proceed inductively in this way using the following notation. Write

$$U_{i,j}(m_1, n_1, k_1) = p^{(i,j)}_{m_1,n_1,k_1}\big(\overline{U}_{i,j}\big).$$

Then applying another local inverse gives:

$$U_{i,j}\big((m_1, n_1, k_1), (m_2, n_2, k_2)\big) = p^{(m_1,n_1)}_{m_2,n_2,k_2}\big(U_{i,j}(m_1, n_1, k_1)\big),$$

and so on. Letting Φ_d denote a triple (m_d, n_d, k_d), for each fixed $i, j \in \mathbb{Z}$ we obtain pairwise disjoint compact sets of the form:

$$U_{i,j}(\Phi_1, \Phi_2, \ldots, \Phi_d) = p^{(i,j)}_{\Phi_d}\big(U_{i,j}(\Phi_1, \Phi_2, \ldots, \Phi_{d-1})\big).$$

By construction, we have

$$\wp_\Lambda^{-d} U_{i,j} = \bigcup_{\Phi_1,\ldots,\Phi_d} U_{i,j}(\Phi_1, \Phi_2, \ldots, \Phi_d);$$

from this it follows that if $U^* = \bigcup_{i,j\in\mathbb{Z}} U_{i,j}$, then

$$\wp_\Lambda^{-d} U^* = \bigcup_{i,j} \bigcup_{\Phi_1,\ldots,\Phi_d} U_{i,j}(\Phi_1, \Phi_2, \ldots, \Phi_d)$$

which is still a pairwise disjoint union of compact sets.

Furthermore,

$$\mathcal{J} = \bigcap_{d=0}^{\infty} \wp_\Lambda^{-d} U^*.$$

Now let V be any component of $\mathcal{J}$; then V can lie in at most one $U_{i,j}(\Phi_1, \Phi_2, \ldots, \Phi_d)$ for each d.

By the expansion on the Julia set, we see that

$$\operatorname{diam}\big(U_{i,j}(\Phi_1, \Phi_2, \ldots, \Phi_d)\big) \leqslant r^{-[d/s]} \operatorname{diam} U_{i,j}$$

(where $[d/s]$ denotes the integer part of d/s), which goes to 0 as $d \to \infty$, so the diameter of each component in $\mathcal{J}$ is 0, making each component a single point.

It remains to consider any point $\omega \in \mathcal{A}$; our first observation is that each $U_{m,n}$ contains exactly one pole by construction, namely $\omega_{m,n} = m\lambda_1 + n\lambda_2$. For any fixed $(m, n) \in \mathbb{Z} \times \mathbb{Z}$, and any $z \in J(\wp_\Lambda)$, $z \neq \omega_{m,n}$, we show the two points lie in separate components of the Julia set. If $z \notin U_{m,n}$, then $\delta_{m,n} \subset F(\wp_\Lambda)$ separates the two points. If $z \in U_{m,n}$, then $\wp_\Lambda(z) \neq \infty$, so $\wp_\Lambda(z) \in U_{i,j}$ for some $(i, j) \in \mathbb{Z} \times \mathbb{Z}$. We apply a branch of $\wp_\Lambda^{-1}$ to the curve $\delta_{i,j}$; using the notation above we choose the branch so that the loop $\gamma = p^{(i,j)}_{m,n,k}\delta_{i,j}$ contains z in its bounded component. It then follows that $\omega_{m,n}$ is in the unbounded component of the complement of γ, so z and $\omega_{m,n}$ are not in the same component of $J(\wp_\Lambda)$.

We proceed inductively on κ. Assume that $\omega \in A_\kappa \setminus A_{\kappa-1}$ and $z \in J(\wp_\Lambda)$, $z \neq \omega$. By the inductive hypothesis, we can assume that z is not an order j prepole for any $j = 1, \ldots, \kappa - 1$. Then $\wp_\Lambda^\kappa$ is defined for z and ω; we track the sequences of $U_{i,j}$ containing the iterates $z, \wp_\Lambda(z), \ldots, \wp_\Lambda^\kappa(z)$ and $\omega, \wp_\Lambda(\omega), \ldots, \wp_\Lambda^\kappa(\omega)$ and stop when they disagree. If $z \in U_{m,n}$ and $\omega \notin U_{m,n}$, then the curve $\delta_{m,n}$ separates z and ω. If $\wp_\Lambda^j(z) \in U_{m,n}$ and $\wp_\Lambda^j(\omega) \notin U_{m,n}$, then the appropriate choice of branch for $\wp_\Lambda^{-j}$ applied to $\delta_{m,n}$ will yield a simple closed curve whose complement has z in the bounded component and ω in the unbounded component.

Finally if z, ω and the forward iterates of z and ω share the same $U_{i,j}$ itinerary under the first $\kappa - 1$ iterates of $\wp_\Lambda$, then one of two possibilities occurs. If z is also an order κ prepole then both points map after exactly $\kappa - 1$ steps to a pole $z_o \in U_{i_o,j_o}$; we can then find one of the $2^{\kappa-1}$ preimages of δ_{i_o,j_o} under $\wp_\Lambda^{\kappa-1}$, which will have to separate z from ω as they are distinct. Otherwise $\wp_\Lambda^\kappa(z) \in U_{a,b}$, for some $(a,b) \in \mathbb{Z} \times \mathbb{Z}$, so the appropriate pullback of $\delta_{a,b}$ under a branch of $\wp_\Lambda^{-\kappa}$ will yield the desired separation. Therefore we have shown that every point in $J(\wp_\Lambda)$ lies in its own component so that $J(\wp_\Lambda)$ is a Cantor set. □

Corollary 3.13. *If $\wp_\Lambda$ is hyperbolic and all the critical values of $\wp_\Lambda$ are contained in one Fatou component W, then W is the basin of attraction for an attracting fixed point and $F(\wp_\Lambda) = W$.*

Proof. We apply Theorem 3.12 and Proposition 3.1. □

Corollary 3.14. *If all three critical values are in the same Fatou component W of $F(\wp_\Lambda)$, then all critical points are in W, W is a double toral band, and $J(\wp_\Lambda)$ has infinitely many components.*

Proof. This follows from the proof of Theorem 3.12. □

Double toral bands for elliptic functions

Our main result is that the existence of a double toral band that contains all of the critical values for a hyperbolic elliptic function ensures that $J(f)$ is a Cantor set.

Theorem 3.15. *If W is a double toral band for f that contains all of the critical values and f is hyperbolic, then $F(f) = W$ and $J(f)$ is a Cantor set.*

Proof. We use essentially the same proof as in Theorem 3.12. By hypothesis, we have a simple closed loop q contained in $F(f)$ such that q bounds a fundamental region for Λ (the bounded component of q).

We denote by Q the closed bounded region in the plane with boundary q. Let K denote the unbounded component of the complement of q. Since every fundamental region must contain at least one pole of order $n \geqslant 2$, we have at least one pole in Q, say p_o. Therefore $p_o \in J(f)$ is not in K. But then q is a loop in the Fatou set separating $p_o \in J(f)$ from $\infty \in J(f)$, so $J(f)$ is not connected.

We show that each component of $J(f)$ consists of just one point using hyperbolicity; as before it is enough to look at Q and show that if C is a component of $J(\wp_\Lambda) \cap Q$, then either $\operatorname{diam}(C) = 0$ or C consists of exactly one prepole.

In particular, since all critical values are contained in W we can find a loop δ in Q, bounding a neighborhood U of $J(f) \cap Q$ which lies in the bounded component of δ, and all of the critical points and critical values lie in the unbounded component of δ. By Corollary 2.2, for each $(i, j) \in \mathbb{Z} \times \mathbb{Z}$, the set $f^{-1}(U_{i,j})$ consists of n disjoint sets in the fundamental period $Q_{m,p}$ for every m, p (where n is the order of f). The rest of the proof follows exactly as in Theorem 3.12 using Theorem 3.10 and Corollary 3.11. (The notation becomes unmanageable so we omit the details.)

Since the Julia set is totally disconnected, it follows immediately that the Fatou set has exactly one component which is therefore completely invariant. □

4. An elliptic function with a Cantor Julia set

In this section we prove the existence of an elliptic function that has a Cantor Julia set. The elliptic function is of the form $f_\Gamma(z) = (\wp_\Gamma(z) + 1)/\wp_\Gamma(z)$ with a real square lattice $\Gamma = [\gamma, \gamma i]$; Γ is the period lattice for both $\wp_\Gamma$ and f_Γ. Properties of $\wp_\Gamma$ are used to control the behavior of f_Γ; in this way we construct a hyperbolic elliptic function with a double toral band. The Julia set of f_Γ is contained in the set of dark points (colored navy blue) in Fig. 8; we also show the boundary of a period parallelogram (in medium blue) and the attracting fixed point (in orange). The basic idea behind the construction is to move the poles off the real and imaginary axes just by the definition of f_Γ, and then to choose the period lattice Γ to create an attracting fixed point with the axes in the attracting basin.

For our starting point, we define the lattice $\Lambda = [\lambda, i\lambda]$, $\lambda \in \mathbb{R}$, $\lambda > 0$, to be the lattice associated with the invariants $g_2 = 4$, $g_3 = 0$. By Proposition 4.1 below the critical values

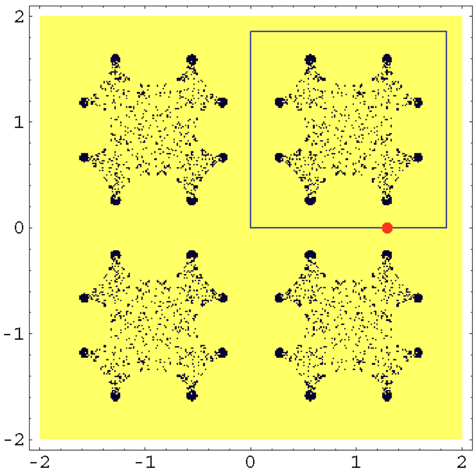

Fig. 8. A Cantor Julia set.

of $\wp_\Lambda$ are $0, 1, -1$; let $e_{1,\Lambda} = 1$. Using the tables in [14], $\lambda \approx 2.62$. This lattice is referred to as the *standard square lattice*. The first result holds for any square lattice.

Proposition 4.1. *If Γ is a square lattice then $g_3 = 0$; in this case, the critical values of $\wp_\Gamma$ are $e_3 = 0$, $e_1 = \sqrt{g_2}/2$, and $e_2 = -e_1$.*

Given any square lattice Γ and the elliptic function $f_\Gamma(z) = (\wp_\Gamma(z) + 1)/\wp_\Gamma(z)$, we prove some basic properties about the function.

Lemma 4.1. *Let Γ be a square lattice. Then the points $(\gamma + \gamma i)/2 + \Gamma$ are the poles of f_Γ. The critical points of f_Γ are the lattice points Γ, and the points $\gamma/2 + \Gamma$ and $\gamma i/2 + \Gamma$.*

Proof. Since $\wp_\Gamma(z) = 0$ if and only if $z = (\gamma + \gamma i)/2 + \Gamma$ by Proposition 4.1 $(\gamma + \gamma i)/2 + \Gamma$ are the poles of f_Γ. Since $f'_\Gamma(z) = -\wp'_\Gamma(z)/(\wp_\Gamma(z))^2$, then $f'_\Gamma(\gamma/2 + \Gamma) = 0$ and $f'_\Gamma(\gamma i/2 + \Gamma) = 0$ since $\wp'_\Gamma$ is zero at those points.

To see that the poles of $\wp_\Gamma$ are critical points of f_Γ, we observe using Eq. (1)

$$\begin{aligned} f'(0) &= \lim_{x\to 0} f'(x) = \lim_{x\to 0} \frac{\wp'_\Gamma(x)}{(\wp_\Gamma(x))^2} \\ &= \lim_{x\to 0} \frac{\sqrt{4(\wp_\Gamma(x))^3 - g_2\wp_\Gamma(x)}}{(\wp_\Gamma(x))^2} = 0. \quad \square \end{aligned}$$

The next lemma was proved in [10] for $\wp_\Gamma$, but the proof remains the same for the elliptic function f_Γ.

Lemma 4.2. *If Γ is any lattice and $f_\Gamma(z) = (\wp_\Gamma(z) + 1)/\wp_\Gamma(z)$ then*

(1) *$f_\Gamma(z) = f_\Gamma(-z)$ for all z (f is even).*
(2) *$(-1)J(f_\Gamma) = J(f_\Gamma)$ and $(-1)F(f_\Gamma) = F(f_\Gamma)$.*
(3) *If Γ is real then $\overline{J(f_\Gamma)} = J(f_\Gamma)$ and $\overline{F(f_\Gamma)} = F(f_\Gamma)$.*

Remark. Parts (1) and (2) of Lemma 4.2 are true for any elliptic function which is a rational function of $\wp_\Gamma$ for any lattice Γ. If Γ is a real lattice then (3) holds similarly.

For any real square lattice $\Gamma = [\gamma, \gamma i]$, $\wp_\Gamma$ is real on the small square formed by the half periods of Γ (see [7]). Along the real interval $[0, \gamma/2]$, $\wp_\Lambda$ decreases monotonically from ∞ to e_1. Moving vertically from $\gamma/2$ to $(\gamma + \gamma i)/2$, $\wp_\Gamma$ decreases monotonically from e_1 to 0. As we move along the other two legs of the small square formed by the half periods, $\wp_\Gamma$ decreases from 0 to $e_2 = -e_1$ to $-\infty$.

Our example is constructed using the lattice defined in the following lemma.

Lemma 4.3. *We define Γ to be the square lattice associated with the invariants $g_2(\Gamma) = 16$, $g_3(\Gamma) = 0$. Then $\Gamma = k\Lambda$, where $k = \sqrt[4]{1/4}$ and Λ is the standard square lattice, and $\wp_\Gamma(\gamma/2) = 2$. Therefore the critical values of $\wp_\Gamma$ are $2, -2$, and 0.*

Proof. Since Γ and Λ are both square, there exists a k such that $\Gamma = k\Lambda$. By Lemma 2.4

$$k^4 = \frac{g_2(\Lambda)}{g_2(\Gamma)} = \frac{1}{4}.$$

By Eq. (2)

$$\wp_\Gamma\left(\frac{\gamma}{2}\right) = \wp_{k\Lambda}\left(\frac{k\lambda}{2}\right) = \frac{1}{k^2}\wp_\Lambda\left(\frac{\lambda}{2}\right) = \frac{1}{k^2} = 2. \quad \square$$

For the rest of this section, Γ refers to the lattice defined in Lemma 4.3.

Lemma 4.4. *For all $z \in \mathbb{R}$, we have $0 \leqslant |f'_\Gamma(z)| < 1$.*

Proof. Since Γ is a real lattice, f_Γ is a real function, so $f'_\Gamma(z)$ is real if z is. For simplicity of notation, we use f and $\wp$ (and suppress the lattice Γ which is fixed throughout). Using Eq. (1), it is enough to show that

$$\left(f'\right)^2 = \left(4\wp^3 - 16\wp\right)/\wp^4 = \frac{4}{\wp} - \frac{16}{\wp^3} < 1$$

on the periodic interval $[0, \gamma]$.

Moreover, by symmetry about the critical points, we need only look at $[0, \gamma/2]$. By Lemma 4.1, $f'(z) = 0$ at the endpoints of this interval and the map f is strictly increasing on $(0, \gamma/2)$, since $\wp$ decreases from ∞ to 2 there. Since f is not constant, $f' > 0$ and has a maximum value occurring at a point $c \in (0, \gamma/2)$; the maximum for $g = (f')^2$ occurs at c as well so we solve for c

$$g'(z) = 4\wp'\left(\frac{12 - \wp^2}{\wp^4}\right),$$

which is zero in $(0, \gamma/2)$ when $\wp(z) = 2\sqrt{3}$. This occurs at precisely one point c, and at that point we have that:

$$\left(f'\right)^2(c) = \frac{4}{2\sqrt{3}} - \frac{16}{24\sqrt{3}} = \frac{4}{3\sqrt{3}} < 1. \quad \square$$

Theorem 4.5. *f_Γ is hyperbolic and has a double toral Fatou band.*

Proof. We know that $\wp_\Gamma$ is real on the small square formed by the half periods of Γ, and thus f_Γ is real there are well. We note first that f_Γ is strictly increasing on $[0, \gamma/2]$ since $\wp_\Gamma$ is strictly decreasing there. Since $f_\Gamma(0) = 1$ and $f_\Gamma(\gamma/2) = 3/2$, f_Γ increases from 1 to 3/2 on the interval $[0, \gamma/2]$. By Lemma 4.2 (1) and periodicity of f_Γ we have that $f_\Gamma : \mathbb{R} \to [1, 3/2]$.

Clearly there is at least one real fixed point. It cannot occur on the interval $[0, \gamma/2]$ since $\gamma/2 \approx 2.62/(2\sqrt{2}) < 1$. Similarly, there are no fixed points on $[\gamma, \infty)$ since $\gamma \approx 2.62/\sqrt{2} > 3/2$. Since f_Γ is even, we have that f_Γ is strictly decreasing from 3/2 to 1 on the interval $[\gamma/2, \gamma]$. Thus there can be only one fixed point on $[\gamma/2, \gamma]$. This fixed point is attracting using Lemma 4.4.

Thus all points in $\mathbb{R}$ are attracted to this real fixed point. This implies that the entire real axis is contained in the Fatou set. Since $\wp_\Gamma$ is real on the vertical line l from 0 to $\gamma i/2$, we have that f_Γ is real there as well. The invariance of the Fatou set implies that l lies in the Fatou set, and thus the entire imaginary axis lies in the Fatou set by Lemmas 2.10 and 4.2(1). By Lemma 2.10 we have that the square with vertices $0, \gamma, \gamma + \gamma i, \gamma i$ lies in a component W of the Fatou set, hence we have a double toral band. Using Lemma 4.1, all critical points lie in W and f is hyperbolic. □

Theorem 4.6. *Let Γ be the square lattice associated with the invariants $g_2(\Gamma) = 16$, $g_3(\Gamma) = 0$. Then $J(f_\Gamma)$ is a Cantor set.*

Proof. We apply Theorems 4.5 and 3.15. □

5. Special lattices and connectivity

We return in this section to the study of Julia sets of the Weierstrass $\wp$ function. We showed in Corollary 3.3 of Section 3 that triangular lattices give rise to connected Julia sets for $\wp$. Here we collect results on other special lattice shapes and give some examples of Julia sets which contain the entire real and imaginary axes. We also give examples of Λ such that $J(\wp_\Lambda)$ is connected but the intersection of $J(\wp_\Lambda)$ with $\mathbb{R}$ (and other lines) is a Cantor set.

In [9] we proved that every Weierstrass elliptic function on a square lattice with a superattracting fixed point is connected, and in [9,10] we constructed examples of square lattices with this property. Lemma 2.6 provides a formula for obtaining these examples. In Section 5.3, we prove that we obtain Cantor sets for the intersection of the Julia set with the axes, lattice boundaries, and lattice diagonals.

5.1. Real rectangular lattice

We prove the existence of a real rectangular (nonsquare) lattice with a connected Julia set. In this example, two critical values lie in different components of the Fatou set but are associated to the same superattracting fixed point, and one critical value lies in the Julia set.

Theorem 5.1. *Let $a = 3$, and let $\Gamma = [\gamma_1, \gamma_2]$, $\gamma_1 > 0$ be the real rectangular lattice with invariants $g_2(\Gamma) = 4(a^2 + a + 1)/a^2$ and $g_3(\Gamma) = -4(a + 1)/a^2$. If $k = \sqrt[3]{2/(3\gamma_1)}$ (taking the real root) and $\Lambda = k\Gamma$ then the Julia set of $\wp_\Lambda$ is connected.*

Proof. Using Eq. (4) we have that

$$e_{1,\Gamma} = 1, \qquad e_{2,\Gamma} = \frac{-1-a}{a} = \frac{-4}{3}, \qquad e_{3,\Gamma} = \frac{1}{a} = \frac{1}{3}.$$

Using Eq. (2) we have

$$e_{1,\Lambda} = \wp_\Lambda\left(\frac{\lambda_1}{2}\right) = \wp_{k\Gamma}\left(\frac{k\gamma_1}{2}\right) = \frac{1}{k^2}\wp_\Gamma\left(\frac{\gamma_1}{2}\right) = \frac{3\lambda_1}{2},$$

and thus $e_{1,\Lambda} = 3\lambda_1/2$ is a superattracting fixed point. A similar argument gives $e_{3,\Lambda} = \lambda_1/2$, and thus $\wp_\Lambda(e_{3,\Lambda}) = e_{1,\Lambda}$.

We claim that e_1 and e_3 cannot lie in the same Fatou component. Let U denote the Fatou component containing e_1. Since U is a superattracting invariant Fatou component, we have that there is a conformal change of coordinates from U onto the open unit disk $\mathbb{D}$ which conjugates $\wp_\Lambda|_U$ to the $z \to z^2$ power map on $\mathbb{D}$ (cf. [18, Section 68, Theorem 4]). If $e_{3,\Lambda} \in U$ then the local coordinate change extends throughout a region $V \subset U$ with $e_{3,\Lambda} \in \partial V$. However, the local coordinate change cannot extend over to a neighboring period parallelogram since $\wp_\Lambda$ is periodic but the power map is not. Therefore $e_{3,\Lambda} \notin U$.

Using Eq. (4), we have $e_{2,\Lambda} = -2\lambda_1 \in J(\wp_\Lambda)$. Thus no Fatou component contains more than one critical value and $J(\wp_\Lambda)$ is connected by Theorem 3.1. □

5.2. *Real rhombic lattices*

We turn to a discussion of some possibilities for Fatou sets for rhombic period lattices. The lattices in this section are all real rhombic lattices and have the form $\Lambda = [\lambda_1, \overline{\lambda_1}]$. We begin with two propositions about the critical orbits and possible Fatou components for real rhombic lattices.

Proposition 5.1. *If Λ is a real rhombic lattice then*

(1) *There is one real critical value e_3; $e_3 < 0$ if $g_3 < 0$ and $e_3 > 0$ if $g_3 > 0$.*
(2) *There are two complex critical values satisfying $e_1 = \overline{e_2}$.*

Proof. In [9] we showed that e_3 has the same sign as g_3. Theorem 2.5(1) implies that $e_1 = \overline{e_2}$. □

Proposition 5.2. *For any real rhombic lattice Λ one of the following must occur*:

(1) $J(\wp_\Lambda) = \mathbb{C}_\infty$;
(2) *There exist one real postcritical orbit and two conjugate postcritical orbits*; *therefore there are at most two different types of periodic Fatou components. If the nonreal critical values lie in the Fatou set, then they are associated with cycles with the same period and multiplier.*

Proof. Suppose the Fatou set in nonempty. Since Λ is real, Theorem 2.5(1) implies that $\wp_\Lambda$ maps the real line to the real line, and thus the postcritical orbit of the real critical point never moves off of the real axis. By Proposition 5.1 $e_1 = \overline{e_2}$. Applying Theorem 2.5(1), we have that $\wp^n(e_1) = \wp^n(\overline{e_2}) = \overline{\wp^n(e_2)}$ for all $n \geqslant 0$. Since $\wp'_\Lambda(\bar{z}) = \overline{\wp'_\Lambda(z)}$, the nonreal critical points are associated with cycles of the same period and multiplier. Thus there can be at most two different types of Fatou components. □

Theorem 5.2. *If Λ is a real rhombic lattice such that the complex critical values are associated with nonrepelling complex cycles and the real critical value is in $J(\wp_\Lambda)$ then the real and imaginary axes are contained in $J(\wp_\Lambda)$.*

Proof. Since Λ is real, Theorem 2.5(1) implies that $\wp_\Lambda$ maps the real line to the real line, and thus no point on $\mathbb{R}$ can leave the real axis.

Since the real critical value is in $J(\wp_\Lambda)$, then the real postcritical orbit is either associated with a Siegel disk or is not associated to any Fatou cycle by Theorems 2.7 and 2.8. No interval in $\mathbb{R}$ can lie within a Siegel disk component because $\wp_\Lambda : \mathbb{R} \to \mathbb{R}$, which would contradict that $\wp_\Lambda^n$ is conjugate to an irrational rotation of the unit disk.

If the complex cycle is attracting or parabolic then no $z \in \mathbb{R}$ can approach the complex cycle because it cannot leave the real axis. If the complex cycle is associated with a Siegel disk, then again no point on the real axis can lie within the cycle.

Therefore, the entire real axis must lie in $J(\wp_\Lambda)$. Since Λ is real, we have that $\wp_\Lambda(iy) \in \mathbb{R}$ for $y \in \mathbb{R}$ (see [9]), and thus the imaginary axis must lie in $J(\wp_\Lambda)$ by the invariance of the Julia set. □

We next give some experimental examples of the Julia and Fatou sets of Weierstrass elliptic functions on real rhombic period lattices.

Example 5.3. A real rhombic lattice with a connected Julia set.

If $g_2 \approx 14.4676$ and $g_3 \approx -13.0598$ then Λ is real rhombic. Here, $\wp_\Lambda$ has a superattracting real fixed point at $p \approx -2.25$, which is plotted (in orange) in Fig. 9, and the lightest points (colored yellow) in Figs. 9 and 10 iterate to this fixed point. Further, the complex critical values head to an attracting two-cycle at $\approx \{1.2554 + 0.428441i, 1.2554 - 0.428441i\}$, also plotted (in orange) in Fig. 9; the medium shaded points (colored tan) iterate to this cycle. The Julia set is the set of dark points (navy blue). In Fig. 10, we zoom in on the component containing the superattracting fixed point. Since each critical value is contained in a distinct Fatou component, Theorem 3.1 implies that the Julia set is connected.

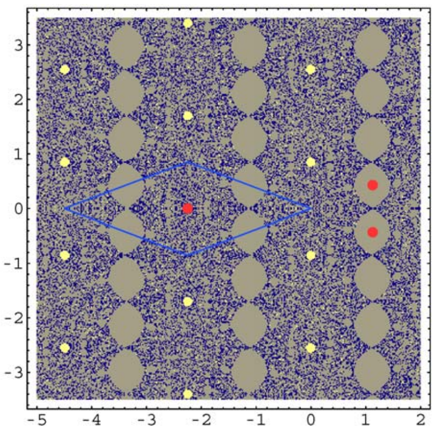

Fig. 9. Fixed points and the Fatou set of Example 5.3.

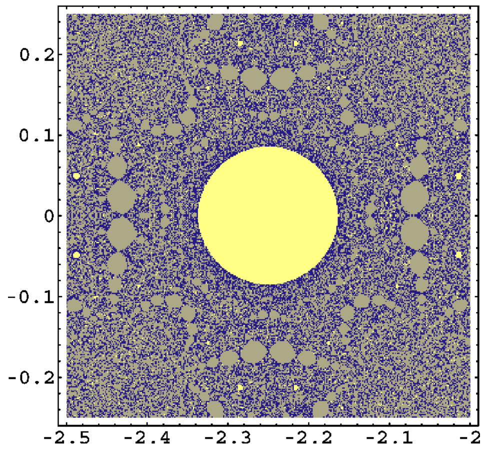

Fig. 10. Superattracting fixed point of Example 5.3.

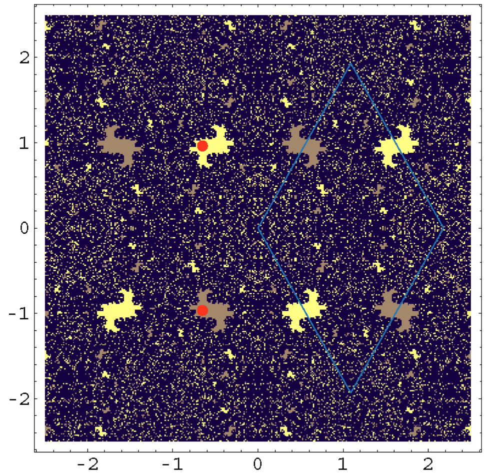

Fig. 11. Fixed points of Example 5.4.

Example 5.4. A real rhombic lattice with a nonempty Fatou set where the real and imaginary axes are in $J(\wp_\Lambda)$.

If $g_2 \approx 0.7$ and $g_3 \approx 7.15124$ then Λ is real rhombic. The real critical point $\lambda_3/2 = (\lambda_1 + \overline{\lambda_1})/2$ satisfies $\wp_\Lambda^4(\lambda_3/2) = 3\lambda_3 \in J(\wp_\Lambda)$, and hence the real and imaginary axes lie in the Julia set by Theorem 5.2. There are attracting fixed points at $p \approx -0.652 + 0.964i$ and $\overline{p} \approx -0.652 - 0.964i$, marked (in orange) in Fig. 11; the lightest points (colored yellow) iterate to the fixed point at p, and the medium shaded points (colored tan) iterate to the fixed point at $\overline{p}$. The Julia set is the set of dark points (colored navy blue). Fig. 12 is a zoom in to the real critical point $\lambda_3/2 \approx 1.088$.

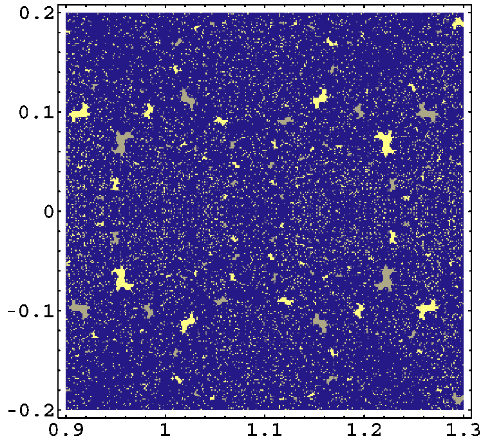

Fig. 12. Critical point of Example 5.4.

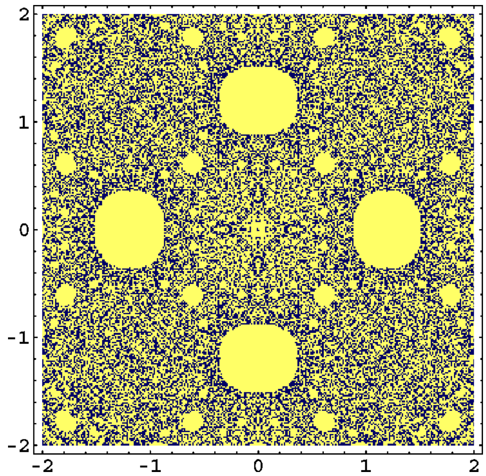

Fig. 13. $J(\wp_\Lambda)$ for a Γ with a superattracting fixed point.

5.3. Cantor intersections

We show first that for any real rectangular square lattice Γ with an attracting fixed point (for $\wp_\Gamma$), the intersection of the Julia set with the real axis is a Cantor set. As a consequence we obtain Cantor intersections of $J(\wp_\Gamma)$ with other lines through the origin as well. By symmetry the result holds for certain lines parallel to these. We fix a real square lattice $\Gamma = [\gamma, \gamma i]$, $\gamma > 0$ for which there is an attracting fixed point. It was shown in [9] that there are infinitely many distinct values of $g_2 > 0$, or equivalently γ, for which $\wp_\Gamma$ has a superattracting fixed point (see Fig. 1 for an example). Fig. 13 shows the Julia set for the lattice in Fig. 1 restricted to one period parallelogram centered at the origin. The lattice can be generated by $\gamma \approx 2.395$, and the superattracting fixed point occurs at $e_1 = \gamma/2 \approx 1.197$.

By Eq. (1) it is clear that both the fixed point and its derivative vary analytically in $g_2 > 0$ (and γ), and therefore nearby in g_2 parameter space there are attracting fixed points as well. The next two propositions are an amalgamation of known results from [9] and [10].

We label the periodic intervals for $\wp_\Gamma|_{\mathbb{R}}$ by

$$I_k = \left[k\gamma, (k+1)\gamma\right], \quad k \in \mathbb{Z};$$

the endpoints of I_k are the poles of $\wp_\Gamma$ on $\mathbb{R}$.

We describe the possibilities for $J(\wp_\Gamma)$.

Proposition 5.3. *For any real rectangular square lattice Γ one of the following must occur*:

(1) $J(\wp_\Gamma) = \mathbb{C}_\infty$;
(2) *There exists exactly one (super)attracting cycle for $\wp_\Gamma$, and the immediate basin contains e_1; this is the only periodic cycle for $F(\wp_\Gamma)$;*
(3) *There exists exactly one rationally neutral cycle for $\wp_\Gamma$, and the immediate basin contains e_1; this is the only periodic cycle for $F(\wp_\Gamma)$;*
(4) *In the case of* (2) *or* (3), *the following also hold*:
   (a) *The nonrepelling periodic orbit lies completely in $[e_1, \infty)$.*
   (b) *Suppose $k_o\lambda \leqslant e_1 < (k_o+1)\lambda$; if $A = \{p_1, \ldots, p_n\}$ denotes the nonrepelling cycle, then $A \cap I_{k_o} \neq \emptyset$.*

We add to this in the case of an attracting fixed point.

Proposition 5.4. *Assume that Γ is a real rectangular square lattice such that t_o is an attracting fixed point. Then $t_o \in \mathbb{R}$, and*:

(1) *We have real critical values $e_1 > 0$, $e_2 = -e_1$, and $e_3 = 0$.*
(2) *$\wp_\Gamma : \mathbb{R} \to \mathbb{R} \cup \infty$; moreover $\wp_\Gamma([e_1, \infty]) = [e_1, \infty]$.*
(3) *If I_{k_o} is the interval containing e_1 we let c_{k_o} denote the critical point in I_{k_o}. Then $t_o \in I_{k_o}$ and there is a repelling fixed point $p \in I_{k_o}$ such that, writing $p = c_{k_o} + q$, $0 < q < \gamma/2$, the real immediate basin of attraction for t_o is $B = (c_{k_o} - q, c_{k_o} + q)$;*
(4) *$\wp_\Gamma(B) = B$.*

In the next proposition we prove the existence of Cantor intersections for $J(\wp_\Gamma)$; we carry over all of the notation from above.

Proposition 5.5. *If Γ is a real rectangular square lattice with an attracting fixed point, we have the following*:

(1) *$J(\wp_\Gamma) \cap \mathbb{R}$ is a Cantor set*;
(2) *$J(\wp_\Gamma) \cap \{z\colon z = iy,\ y \in \mathbb{R}\}$ is a Cantor set*;
(3) *$J(\wp_\Gamma) \cap \{z\colon z = a + ia,\ a \in \mathbb{R}\}$ is a Cantor set;*
(4) *$J(\wp_\Gamma) \cap \{z\colon z = a - ia,\ a \in \mathbb{R}\}$ is a Cantor set.*

Proof. Because of the symmetry of the Julia and Fatou sets with respect to each periodic interval, it follows from Propositions 5.3 and 5.4 that $\wp_\Gamma(I_j) = [e_1, \infty]$ for all j, and that $\wp_\Gamma^{-1}(B)$ consists of infinitely many disjoint open intervals, one in each I_j. We label those intervals $B_j \subset I_j$; proceeding as in [5, Proposition 2 for the tangent family], we label the intervals complementary to the B_j's using C_j such that C_o contains the pole at the origin; then C_{k_o+1} contains the repelling fixed point p.

It follows from our construction and the strict monotonicity of $\wp'_\Gamma$ for rectangular square lattices that $\wp_\Gamma : C_j \to [e_1, \infty] \setminus (B \cap [e_1, \infty])$ for each j and $|\wp'_\Gamma(x)| > 1$ for each $x \in C_j$.

Defining

$$JR = \left\{ x \in \mathbb{R} \mid \wp_\Gamma^n(x) \in \bigcup_{j \geqslant k_o} C_j \text{ for all } n \text{ for which } \wp_\Gamma^n \text{ is analytic} \right\},$$

we have that

$$z \in JR \Longleftrightarrow z \in J(\wp_\Gamma) \cap \mathbb{R}.$$

We show that JR is a Cantor set by showing that $\operatorname{diam}(\wp_\Gamma^{-n} C_j) \to 0$ as $n \to \infty$, and that each real prepole is a component of JR as in the proof of Theorem 3.12 and following the method outlined in [5].

For any point $x \in (-\infty, -e_1)$, $\wp_\Gamma^{-1}(x)$ is purely imaginary and consists of infinitely many pairs of points of the form $\pm ia$, $a \in \mathbb{R}$. More precisely, in each imaginary interval iI_k, $k \in \mathbb{Z}$, there are exactly two preimages. Therefore for each k, the set

$$\wp_\Gamma^{-1}(JR) \cap \big[ik\gamma, i(k\gamma + \gamma/2)\big] = J(\wp_\Gamma) \cap \big[ik\gamma, i(k\gamma + \gamma/2)\big]$$

is a homeomorphic image of JR and hence a Cantor set. If we denote by JR_i the set $J(\wp_\Gamma) \cap \{z = iy\}$, then we have shown that JR_i is a Cantor set.

We now consider the set $\wp_\Gamma^{-1}(JR_i)$. First, we know that the preimage of 0 consists of all critical points of the form $\frac{1}{2}(k\gamma + ik\gamma)$; i.e., 0 is a critical value so there is only one preimage in each period parallelogram for Γ. We define the central diagonal intervals for each $k \in \mathbb{Z}$: $D_k = \{x + ix \colon x \in I_k\}$ and $d_k = \{x - ix \colon x \in I_k\}$. If $w \in JR_i$ is of the form $w = iy$, $y < 0$, then we have two preimages of w in each D_k; if $w = iy$, $y > 0$, then we have two preimages in each d_k. As above, taking the countable union of homeomorphic copies of the Cantor set JR_i gives the result. □

By symmetry of $J(\wp_\Gamma)$ with respect to the lattice, we do not need to restrict the previous result to lines through the origin. If $\gamma \in \Gamma$, and A is a set in $\mathbb{C}$, then

$$A + \gamma = \{z + \gamma,\ z \in A\}.$$

Corollary 5.5. *If Γ is a real rectangular square lattice with an attracting fixed point, we have the following for any $\gamma \in \Gamma$:*

(1) $J(\wp_\Gamma) \cap (\mathbb{R} + \gamma)$ *is a Cantor set*;
(2) $J(\wp_\Gamma) \cap (\{z\colon z = iy, y \in \mathbb{R}\} + \gamma)$ *is a Cantor set*;
(3) $J(\wp_\Gamma) \cap (\{z\colon z = a + ia, a \in \mathbb{R}\} + \gamma)$ *is a Cantor set*;
(4) $J(\wp_\Gamma) \cap (\{z\colon z = a - ia, a \in \mathbb{R}\} + \gamma)$ *is a Cantor set.*

Corollary 3.3 states that every triangular lattice Γ gives rise to a connected Julia set for $\wp_\Gamma$; however the following results also holds.

Proposition 5.6. *If Γ is a real triangular lattice with an attracting fixed point, we have the following for any $\gamma \in \Gamma$:*

(1) $J(\wp_\Gamma) \cap (\mathbb{R} + \gamma)$ *is a Cantor set*;
(2) $J(\wp_\Gamma) \cap (e^{2\pi i/3}\mathbb{R} + \gamma)$ *is a Cantor set*;
(3) $J(\wp_\Gamma) \cap (e^{4\pi i/3}\mathbb{R} + \gamma)$ *is a Cantor set.*

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