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On the freeness of equisingular deformations of plane curve singularities

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Abstract

We consider surface singularities in $\mathbb{C}^3$ arising as the total space of an equisingular deformation of an isolated curve singularity of the form $f(x, y) + zg(x, y)$ with f and g weighted homogeneous. We give a criterion that such a surface is a free divisor in the sense of Saito. We deduce that the Hessian deformation defines a free divisor for nonsimple weighted homogeneous singularities, and that the failure of this property "almost" characterizes the simple singularities. The criterion also yields distinct deformations of the same curve singularity, exactly one of which is free, showing that freeness is not a topological property. © 2002 Elsevier Science B.V. All rights reserved.

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Introduction

Saito [14] introduced the notion of free divisor $V, 0 \subset \mathbb{C}^p, 0$ as a hypersurface for which the module of logarithmic vector fields $\mathrm{Derlog}(V)$ is a free $\mathcal{O}_{\mathbb{C}^p}$-module (necessarily of rank p). Most examples have concerned universal objects such as: the discriminants of the versal unfoldings of isolated hypersurface and complete intersection singularities by Saito [14] and Looijenga [12, Chapter 6]; bifurcation sets associated to the versal unfoldings of isolated hypersurface singularities, Bruce [4] and Terao [16], and more generally for $\mathcal{A}$-versal unfoldings of a well-defined class of complete intersection germs [5] and see references therein; Coxeter arrangements, by Terao [15]: the discriminant of the versal deformation of a space curve singularity, by Van Straten [18]; creating free divisors from images of stable germs by adding either adjoint divisors Mond [13] or other natural

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divisors [5]; and more generally discriminants of $\mathcal{K}_V$-versal unfoldings of sections of certain free divisors [5,6], which subsume many of the preceding.

If we seek to identify and understand free divisors which fall outside such classes of universal objects, there are only two known special results. One concerns special free divisors arising as hyperplane arrangements, using a criterion of Terao [15], including special discriminantal arrangements, Falk [10], Bayer and Brandt [2,3]. A second general result of Saito [14] shows that all isolated plane curve singularities are free divisors. Unfortunately freeness fails for all higher dimensional isolated singularities. For example, isolated surface singularities $V, 0 \subset \mathbb{C}^3, 0$, require at least four generators (see, e.g., [11]).

The purpose of this note is to consider a general class of surface singularities in $\mathbb{C}^3$ and characterize by simple conditions those which are free divisors. Consider a nonisolated surface singularity $X, 0 \subset \mathbb{C}^3, 0$ with singular set a smooth curve. If we intersect X with a plane transverse to X (as a Whitney stratified set) we obtain an isolated curve singularity $X_0, 0$, and X can be viewed as the total space of an equisingular deformation of $X_0, 0$. We consider when such equisingular deformations have a total space which is a free divisor. We shall concentrate on equisingular deformations of weighted homogeneous curve singularities of the form $F = f(x, y) + zg(x, y)$ with z the deformation parameter and g also weighted homogeneous with $\mathrm{wt}(g) \geqslant \mathrm{wt}(f)$. We shall give a necessary and sufficient condition that such a deformation defines a free divisor in terms of a homomorphism

$$\Psi : \mathrm{Derlog}(F) \to \bigl(J(f) : g\bigr),$$

where $\mathrm{Derlog}(F)$ denotes the module of logarithmic derivations which annihilate F and $J(f)$ is the Jacobian ideal of f.

We shall see that the image $\mathrm{Im}(\Psi) \subset (J(f) : g)$ represents first order information regarding the logarithmic derivations of X. Our first theorem characterizes free divisors X in terms of algebraic and numerical properties of $\mathrm{Im}(\Psi)$. An important special case occurs when all of the elements of $(J(f) : g)$ lift via Ψ to logarithmic derivations (we refer to g as being fully extendable). We give a simple numerical condition (Proposition 2) which ensures this. In this special case, Theorem 3 gives a necessary and sufficient condition in terms of the ideal $(J(f) : g)$ for X to be a free divisor. As a first consequence we deduce that Hessian deformations of nonsimple (weighted homogeneous) curve singularities always define free divisors. Second, we are able to exhibit distinct equisingular deformations of the same curve singularity, one of which is free and the other not free. As the surfaces are topologically equivalent, this shows that freeness is not a purely topological notion. This contrasts with the situation for arrangements where Terao has conjectured that whether the arrangement is free is determined by its lattice structure.

Moreover, all of these equisingular deformations have smooth singular set. This contrasts with the result of Alexandroff, Theorem 1 in [1] which characterizes the freeness of a divisor $X, 0$ for which $\mathrm{Sing}(X)$ has codimension 1 at all points in terms of $\mathrm{Sing}(X)$ being Cohen–Macaulay. Since smooth sets are Cohen–Macaulay, these results seem to contradict the theorem of Alexandroff. In fact, there is some “fine print” in Alexandroff’s theorem which asserts that $\mathrm{Sing}(X)$ being Cohen–Macaulay concerns a specific associated ideal structure rather than the intrinsic geometric structure. Our examples show that the intrinsic geometric structure does not by itself determine whether $X, 0$ is free.

Finally, from the results on Hessian deformations one might suspect the freeness of the Hessian deformation characterizes nonsimple weighted homogeneous curve singularities. This is almost true. In fact we show there are exactly two simple singularities whose Hessian deformations define free divisors: the simplest singularity A_1 (whose Hessian deformation $x^2 + y^2 + z$ defines a smooth surface which is trivially free) and the "most complicated" simple singularity E_8.

1. Freeness of surface singularities

Let $f(x, y)$ define an isolated weighted homogeneous curve singularity $X_0, 0 \subset \mathbb{C}^2, 0$. We let $\mathrm{wt}(x, y) = (a, b)$, $\mathrm{wt}(f) = d$ (so $a, b, d > 0$) and let $J(f)$ denote the Jacobian ideal of f. Also, we let $g(x, y)$ be a weighted homogeneous germ. We consider the deformation of f given by $F(x, y, z) = f(x, y) + zg(x, y)$. If $\mathrm{wt}(g) = s \geqslant \mathrm{wt}(f)$, then F defines an equisingular deformation of f. If we view F as a function of three variables, it defines, in general, a nonisolated surface singularity $X = \{(x, y, z) \in \mathbb{C}^3\colon\ F(x, y, z) = 0\}$. If F is an analytically trivial deformation, then $X \simeq X_0 \times \mathbb{C}$ is a free divisor. We determine more generally when F defines a free divisor.

First, F is weighted homogeneous if we assign the weight $\mathrm{wt}(z) = c = d - s$. If $\mathrm{wt}(g) \geqslant \mathrm{wt}(f)$, then $\mathrm{wt}(z) \leqslant 0$. Even with nonpositive weight for z, there is the Euler vector field $e = ax\partial/\partial x + by\partial/\partial y + cz\partial/\partial z$. Also, a basic object of interest is the module of vector fields annihilating F.

We let θ_p denote the module of germs of vector fields on $\mathbb{C}^p, 0$. Quite generally recall that for $X, 0 \subset \mathbb{C}^p, 0$ a hypersurface, the module of logarithmic vector fields is defined by

$$\mathrm{Derlog}(X) = \bigl\{\zeta \in \theta_p\colon\ \zeta\bigl(I(X)\bigr) \subseteq I(X)\bigr\},$$

where $I(X)$ denotes the ideal of germs vanishing on $X, 0$. Then, $X, 0$ is a *free divisor* if $\mathrm{Derlog}(X)$ is a free $\mathcal{O}_{\mathbb{C}^p,0}$-module, necessarily of rank p. If F is a reduced defining equation for $X, 0$, we also define

$$\mathrm{Derlog}(F) = \bigl\{\zeta \in \theta_p\colon\ \zeta(F) = 0\bigr\}.$$

Then, it is easily seen, e.g., by [9, Lemma 3.1],

$$\mathrm{Derlog}(X) = \mathrm{Derlog}(F) \oplus \mathcal{O}_{\mathbb{C}^p,0}\{e\}.$$

Hence, in the special case where $X, 0 \subset \mathbb{C}^3, 0$ is a surface singularity, $\mathrm{Derlog}(X)$ is a free $\mathcal{O}_{\mathbb{C}^3,0}$-module of rank 3 iff $\mathrm{Derlog}(F)$ is a free $\mathcal{O}_{\mathbb{C}^3,0}$-module of rank 2. To determine when this is true, we consider the homomorphism

$$\Psi : \mathrm{Derlog}(F) \to \bigl(J(f)\colon\ g\bigr) \tag{1.1}$$

defined by $\zeta \mapsto \zeta(z)_{|z=0}$. To see that Ψ maps to $(J(f)\colon\ g)$, let $\zeta = a_1(x, y, z)\partial/\partial x + a_2(x, y, z)\partial/\partial y + a_3(x, y, z)\partial/\partial z$. Then,

$$\zeta(F) = a_1(x, y, z)\frac{\partial F}{\partial x} + a_2(x, y, z)\frac{\partial F}{\partial y} + a_3(x, y, z)\frac{\partial F}{\partial z} = 0. \tag{1.2}$$

Since $\partial F/\partial z = g$, if we evaluate (1.2) at $z = 0$ we obtain $a_3(x, y, 0) \cdot g \in J(f)$. Also, $\zeta(z)_{|z=0} = a_3(x, y, 0)$, so the map is as defined.

We note for later use that Ψ increases weight by $s - d$ and is a module homomorphism over the ring homomorphism $i^*: \mathcal{O}_{\mathbb{C}^3,0} \to \mathcal{O}_{\mathbb{C}^2,0}$, for $i(x, y) = (x, y, 0)$. As the ring homomorphism is surjective, $\mathrm{Im}(\Psi)$ is an ideal in $\mathcal{O}_{\mathbb{C}^2,0}$. We further note that the definition of Ψ extends to $\mathrm{Derlog}(X)$; however, $\mathcal{O}_{\mathbb{C}^p,0}\{e\}$ would always be in $\ker(\Psi)$ so we instead restrict consideration to Ψ as defined in (1.1). However, in this more general form we can view Ψ as identifying first order information of a logarithmic derivation. We will see precisely how this limited information actually determines the freeness of $X, 0$.

The main criterion characterizing when F defines a free divisor is given by the following which also applies to non-equisingular deformations.

Theorem 1.1. *Suppose that $f(x, y)$ defines an isolated weighted homogeneous curve singularity in $\mathbb{C}^2, 0$. Also, let $g(x, y) \notin J(f)$ be a weighted homogeneous germ. Then the surface singularity $X, 0 \subset \mathbb{C}^3, 0$ defined by $F(x, y, z) = f(x, y) + zg(x, y)$ is a free divisor iff*:

(1) $\mathrm{Im}(\Psi)$ *is a complete intersection ideal generated by weighted homogeneous generators $\{h_1, h_2\}$ such that*

(2) $\mathrm{wt}(g) + \mathrm{wt}(h_1) + \mathrm{wt}(h_2) = 2d - a - b$.

As we did in Theorem 1.1, in all of the results that follow *we assume that $f(x, y)$ defines an isolated weighted homogeneous curve singularity in* $\mathbb{C}^2, 0$ (with weights as already given).

For the theorem to be useful we wish to identify $\mathrm{Im}(\Psi)$ without first determining $\mathrm{Derlog}(F)$. We do this in an important general case.

Definition 1.2. Given f, we shall say that g is *fully extendable* if for the deformation $F = f + zg$, the map Ψ is surjective.

Remark 1.3. If $F = f + zg$ is analytically trivial (for right equivalence), then since $J(f) = T\mathcal{R}_e \cdot f$, we conclude $g \in J(f)$. Moreover, differentiating the equation for the analytic triviality $F = f \circ \varphi$ with respect to z yields $\partial F/\partial z = \zeta'(F)$ for $\zeta' = a_1(x, y, z) \times \partial/\partial x + a_2(x, y, z)\partial/\partial y$. Hence, $\zeta = -\zeta' + \partial/\partial z \in \mathrm{Derlog}(F)$. Thus, $1 = \Psi(\zeta)$ and hence Ψ is onto.

Conversely if g is fully extendable (for f) with $g \in J(f)$, then $1 \in (J(f){:}\ g)$. Then, g being fully extendable allows us to reverse the previous argument to solve the infinitesimal equation for analytic triviality $\partial F/\partial z = \zeta'(F)$. Thus, F is analytically trivial. As the freeness of analytically trivial deformations holds, if g is fully extendable we need only consider the case $g \notin J(f)$.

A sufficient condition ensuring that a germ g is fully extendable is given by the following.

Proposition 1.4. *Suppose the curve singularity defined by $f(x, y)$ is not a simple singularity. Also, let $g(x, y)$ be a weighted homogeneous germ with* $\mathrm{wt}(g) \geqslant \mathrm{wt}(f)$.

Suppose that there is a set of weighted homogeneous generators $\{h_1, \ldots, h_k\}$ *of* $(J(f): g)$ *which satisfy*

$$2\mathrm{wt}(g) + \mathrm{wt}(h_i) > 3d - 2(a+b) \quad \textit{for } i = 1, \ldots, k \tag{1.3}$$

Then, g *is fully extendable.*

We shall prove this proposition in Section 2 after we have deduced several consequences. First, we note that in the case g is fully extendable Theorem 1.1 takes the following form.

Theorem 1.5. *Suppose that* $g(x, y)$ *is a weighted homogeneous germ which is fully extendable for* f*. If the surface singularity* $X, 0 \subset \mathbb{C}^3, 0$ *defined by* $F(x, y, z) = f(x, y) + zg(x, y)$ *is not analytically trivial, then* $X, 0$ *is a free divisor iff*
(1) $(J(f): g)$ *is a complete intersection ideal generated by weighted homogeneous generators* $\{h_1, h_2\}$ *such that*
(2) $\mathrm{wt}(g) + \mathrm{wt}(h_1) + \mathrm{wt}(h_2) = 2d - a - b$.

The first consequence of the theorem is for Hessian deformations. Consider the Hessian of f, $H(x, y) = \det(\partial^2 f/\partial x_i \partial x_j)$ with (x_1, x_2) denoting (x, y). The Hessian deformation of f is given by $F(x, y, z) = f(x, y) + zH(x, y)$. If f is not a simple singularity, then $\mathrm{wt}(H) = 2(d - a - b) \geqslant d$, so $d \geqslant 2(a + b)$. Then, $(H: J(f))$ is generated by $\{x, y\}$, and

$$\begin{aligned} 2\mathrm{wt}(H) + \min\bigl\{\mathrm{wt}(x), \mathrm{wt}(y)\bigr\} &= 3d - 2(a+b) + \bigl(d - 2(a+b) + \min\{a, b\}\bigr) \\ &> 3d - 2(a+b) \end{aligned}$$

so by Proposition 1.4, H is fully extendable for nonsimple singularities. Moreover,

$$\mathrm{wt}(H) + \mathrm{wt}(x) + \mathrm{wt}(y) = 2d - a - b.$$

Thus, by Theorem 1.5, we obtain the following corollary.

Corollary 1.6. *Suppose that* $f(x, y)$ *defines a nonsimple curve singularity. Then the Hessian deformation* $F(x, y, z) = f(x, y) + zH(x, y)$ *defines a free divisor in* $\mathbb{C}^3, 0$.

The converse of Corollary 1.6, that for simple curve singularities the Hessian deformation does not define a free surface divisor, is “almost true”.

Theorem 1.7. *The Hessian deformation of a simple curve singularity* $f(x, y)$ *in* $\mathbb{C}^2, 0$ *defines a free surface singularity in* $\mathbb{C}^3$ *only for* A_1 *and* E_8 *but in no other cases.*

For example, the cusp $f(x, y) = x^3 - y^2$ has Hessian deformation $F(x, y, z) = x^3 - y^2 - 12zx$ which is a Morse singularity, and hence not free by earlier comments. See Section 4 for the proof in the general case.

Remark 1.8. In fact, the numerical conditions in Theorems 1.1 and 1.5 and in Proposition 1.4 can be naturally rewritten in terms of the weight of the Hessian H. Condition (2) for Theorem 1.1 becomes for the set of generators $\{h_1, h_2\}$

$$\mathrm{wt}(g) + \mathrm{wt}(h_1) + \mathrm{wt}(h_2) = \mathrm{wt}(H) + \mathrm{wt}(x) + \mathrm{wt}(y)$$

and the condition (1.3) in Proposition 1.4 becomes

$$2\mathrm{wt}(g) + \mathrm{wt}(h_i) > \mathrm{wt}(f) + \mathrm{wt}(H).$$

Written in this form it is at first surprising that any g other than the Hessian satisfies the conditions. In fact, quite a few do. For example, we generally have for Pham–Brieskorn curve singularities

Corollary 1.9. *For the curve singularity defined by $f(x, y) = x^b + y^a$, suppose that $g(x, y) = x^k y^\ell$ is fully extendable and $\mathrm{wt}(g) \geqslant \mathrm{wt}(f)$. Then the monomial deformation $F(x, y, z) = x^b + y^a + zx^k y^\ell$ defines a free divisor in $\mathbb{C}^3$.*

Proof of Corollary 1.9. By Remark 1.3, we may assume $g \notin J(f)$. We have $\mathrm{wt}(x, y) = (a, b)$, $\mathrm{wt}(f) = ab$ and $\mathrm{wt}(H) = 2(ab - a - b)$. Since $J(f) = (x^{b-1}, y^{a-1})$, we see $(J(f){:}\ g) = (x^{b-1-k}, y^{a-1-\ell})$, and up to a constant factor the Hessian $H = x^{b-2}y^{a-2}$. Then, condition (2) follows from

$$\mathrm{wt}\big(x^k y^\ell\big) + \mathrm{wt}\big(x^{b-1-k}\big) + \mathrm{wt}\big(y^{a-1-\ell}\big) = 2ab - a - b.$$

By assumption, g is fully extendable, so Theorem 1.5 implies that the deformation defines a free divisor. □

Example 1.10 (*Free equisingular deformations*). For the homogeneous germ $f(x, y) = x^{10} + y^{10}$, $(a, b) = (1, 1)$ and $d = 10$. We have $J(f) = (x^9, y^9)$. Any $g = x^k y^\ell \notin J(f)$ with $\min\{k, \ell\} \geqslant 6$ satisfies (1.3), and so is fully extendable. By Corollary 1.9 such monomial deformations $x^{10} + y^{10} + zx^k y^\ell$ define free divisors.

Example 1.11 (*Non-Pham–Brieskorn free equisingular deformation*). Consider the weighted homogeneous germ $f(x, y) = x^8 + xy^5$, with $(a, b) = (5, 7)$ and $d = 40$. Since $J(f) = (8x^7 + y^5, 5xy^4)$, for $g = x^6y^2$ we have $(J(f){:}\ g) = (x^2, y^2)$. We observe that

$$2\mathrm{wt}\big(x^6y^2\big) + \min\big\{\mathrm{wt}\big(x^2\big), \mathrm{wt}\big(y^2\big)\big\} = 88 + 10 > 3 \cdot 40 - 2(5 + 7)$$

so g is fully extendable by Proposition 1.4. Also,

$$\mathrm{wt}\big(x^6y^2\big) + \mathrm{wt}\big(x^2\big) + \mathrm{wt}\big(y^2\big) = 44 + 10 + 14 = 68 = 2 \cdot 40 - 5 - 7$$

shows that condition (2) of Theorem 1.5 is satisfied. Thus, $F(x, y, z) = x^8 + xy^5 + zx^6y^2$ also defines a free divisor.

Thus, freeness can hold for non-Hessian deformations not of Pham–Brieskorn type. Both the condition that g is fully extendable and the conditions in Theorem 1 hold much more frequently then one would first expect.

Example 1.12 (*A nonfree equisingular deformation*). Consider again the homogeneous germ $f(x, y) = x^{10} + y^{10}$ from Example 1.10. This time we consider instead $g = x^7y^5 + x^5y^7$. We have $(J(f): g) = (x^4, x^2y^2, y^4)$, and (1.3) is easily seen to be satisfied so that g is fully extendable. Since $(J(f): g)$ is not a complete intersection ideal, by Theorem 1.5, $F(x, y, z) = x^{10} + y^{10} + z(x^7y^5 + x^5y^7)$ does not define a free divisor.

Remark 1.13. By considering the preceding examples, we see that condition (1) of the theorem and condition (1.3) can both fail for certain $g \notin J(f)$ with $\mathrm{wt}(g) \geqslant \mathrm{wt}(f)$. However, all examples indicate that for such g, if both g is fully extendable and $(J(f): g)$ is a complete intersection ideal, then condition (2) is satisfied. We ask whether this is always true?

2. Properties of Derlog(F) and Ψ

In this section we establish properties of Ψ, including a proof of Proposition 1.4 and an additional lemma needed for the proof of Theorem 1.1. We also establish simple weight properties of Derlog(F) needed to prove Theorem 1.7.

Proof of Proposition 1.4. We recall F is weighted homogeneous if we assign the nonpositive weight $\mathrm{wt}(z) = c = d - s$ where $\mathrm{wt}(g) = s \geqslant \mathrm{wt}(f) = d$. Let h_i be a weighted homogeneous generator satisfying

$$2\mathrm{wt}(g) + \mathrm{wt}(h_i) > 3d - 2(a + b).$$

Because Ψ is a module homomorphism over $i^*: \mathcal{O}_{\mathbb{C}^3,0} \to \mathcal{O}_{\mathbb{C}^2,0}$, it is sufficient to show that a set of generators of $(J(f): g)$ are in the image of Ψ. We shall use the notation F_x for $\partial F/\partial x$, etc. Because $h_i \in (J(f): g)$, we may solve

$$h_i \cdot g = \varphi_{i1} f_x + \varphi_{i2} f_y, \tag{2.1}$$

where we may assume φ_{ij} is weighted homogeneous. Then, by (2.1)

$$h_i \cdot F_z = \varphi_{i1} F_x + \varphi_{i2} F_y + zR_i, \tag{2.2}$$

where

$$R_i = -\varphi_{i1} g_x - \varphi_{i2} g_y. \tag{2.3}$$

We easily check from (2.3)

$$\mathrm{wt}(R_i) = 2\mathrm{wt}(g) + \mathrm{wt}(h_i) - \mathrm{wt}(f). \tag{2.4}$$

By assumption, $2\mathrm{wt}(g) + \mathrm{wt}(h_i) > 3d - 2(a + b)$. Hence, by (2.4) $\mathrm{wt}(R_i) > 3d - 2(a + b) - d$. Since the Hessian H has weight $2(d - a - b)$, we conclude $\mathrm{wt}(R_i) > \mathrm{wt}(H)$ and so $R_i \in J(f)$ for $i = 1, 2$.

We let $w^\bullet$ denote the weight filtration on $\mathcal{O}_{\mathbb{C}^2,0}$ (with w^k generated by monomials of weight $\geqslant k$). Also, let $\widetilde{w}^\bullet$ denote the induced weight filtration on $\mathcal{O}_{\mathbb{C}^3,0}$ by $\widetilde{w}^k = w^k\mathcal{O}_{\mathbb{C}^3,0}$. Likewise we have an induced weight filtration on the modules $\theta_2 = \mathcal{O}_{\mathbb{C}^2,0}\{\partial/\partial x, \partial/\partial y\}$ and $\theta(\pi_2) = \mathcal{O}_{\mathbb{C}^3,0}\{\partial/\partial x, \partial/\partial y\}$ by defining $\mathrm{wt}(\partial/\partial x) = -a$ and $\mathrm{wt}(\partial/\partial y) = -b$. With respect to this weight filtration, we have initial parts

$$in(F) \equiv f, \quad in(F_x) \equiv f_x \quad \text{and} \quad in(F_y) \equiv f_y \bmod m_z \cdot \mathcal{O}_{\mathbb{C}^3,0}.$$

Then, the map $\beta_f : \theta_2 \to \mathcal{O}_{\mathbb{C}^2,0}$ defined by $\zeta \mapsto \zeta(f)$ maps $\theta_2^{(\ell+c)}$ onto w^ℓ for all $\ell > \mathrm{wt}(H)$ (recall $c = d - s \leqslant 0$). Hence, in the terminology of [7], β_f is graded surjective in filtration $> \mathrm{wt}(H)$. Then, as F is a "deformation of f of nonnegative weight", we can apply the filtered version of the preparation theorem [7, Lemma 7.4] (or see the related filtered Nakayama's Lemma [8, Lemma 1.1]) to conclude that the corresponding map $\beta_F : \theta(\pi_2) \to \mathcal{O}_{\mathbb{C}^3,0}$ (sending $\zeta \mapsto \zeta(F)$) with the induced filtrations, maps

$$\beta_F\bigl(\theta(\pi_2)^{(\ell+c)}\bigr) = \widetilde{w}^\ell \quad \text{for all } \ell > \mathrm{wt}(H). \tag{2.5}$$

Thus, by (2.4) and (2.5) there exist $\zeta_i' \in \theta(\pi_2)$ such that $\zeta_i'(F) = R_i$. Moreover, as R_i is weighted homogeneous, we may assume that ζ_i' is weighted homogeneous (with respect to (x, y, z)). Now we define the weighted homogeneous vector fields

$$\zeta_i = \varphi_{i1}\frac{\partial}{\partial x} + \varphi_{i2}\frac{\partial}{\partial y} - h_i\frac{\partial}{\partial z} + z\zeta_i'. \tag{2.6}$$

By (2.2), (2.3), and (2.6), $\zeta_i(F) = 0$; thus, $\zeta_i \in \mathrm{Derlog}(F)$. □

For the proof of Theorem 1.1, we also need two simple properties of the image and kernel of the homomorphism Ψ. For these we consider the determinantal vector fields. If $\{u, v\}$ denote any pair of x, y, z. Then, we note that the determinantal vector field $\eta_{u,v} = F_v\partial/\partial u - F_u\partial/\partial v \in \mathrm{Derlog}(F)$.

Lemma 2.1.
(1) $J(f) \subset \mathrm{Im}(\Psi)$
(2) $\ker(\Psi) \equiv \mathcal{O}_{\mathbb{C}^3,0}\{\eta_{x,y}\} \bmod m_z \cdot \theta_3$.

Proof. For (1), $\Psi(\eta_{x,z}) = -f_x$, $\Psi(\eta_{y,z}) = -f_y$, and $\mathrm{Im}(\Psi)$ is an ideal.

For (2) we may write $\xi \in \ker(\Psi)$ as

$$\xi = a_1(x, y, z)\frac{\partial}{\partial x} + a_2(x, y, z)\frac{\partial}{\partial y} + a_3(x, y, z)\frac{\partial}{\partial z}. \tag{2.7}$$

Then, arguing as in (1.1), $\Psi(\xi) = 0$ implies that $a_1(x, y, 0)f_x + a_2(x, y, 0)f_y = 0$. As $\{f_x, f_y\}$ forms a regular sequence, there exists a $\psi \in \mathcal{O}_{\mathbb{C}^3,0}$ such that

$$\bigl(a_1(x, y, 0), a_2(x, y, 0)\bigr) = \psi(f_y, -f_x) \equiv \psi(F_y, -F_x) \bmod m_z \cdot \mathcal{O}_{\mathbb{C}^3,0}$$

implying the conclusion of the lemma. □

3. Proof of Theorem 1

Sufficiency

To prove freeness we shall use one form of Saito's criterion [14] for freeness of a hypersurface. Suppose that $X, 0 \subset \mathbb{C}^p, 0$ is a hypersurface. Let $\zeta_i \in \mathrm{Derlog}(X)$ for $i = 1, \ldots, p$. If $\zeta_i = \sum a_{ij}\partial/\partial x_j$, then we let $A = (a_{ij})$ denote the matrix of coefficients. For such a situation, Saito gives the following criterion for X to be free.

Saito's criterion 3.1. If $h = \det(A)$ defines X with reduced structure, then V is a free divisor, and $\{\zeta_1, \ldots, \zeta_p\}$ generate $\mathrm{Derlog}(X)$.

To prove the theorem we shall construct the vector fields in $\mathrm{Derlog}(X)$ and prove they satisfy Saito's criterion 3.1.

Construction of generators for $\mathrm{Derlog}(X)$

We let X be the hypersurface singularity in $\mathbb{C}^3$ defined by $F = 0$. We first construct three vector fields in $\mathrm{Derlog}(X)$. We have the Euler vector field $e = ax\partial/\partial x + by\partial/\partial y + cz\partial/\partial z$. To construct the other two vector fields we use the generators of $\mathrm{Im}(\Psi)$. By assumption, it is a complete intersection ideal generated by weighted homogeneous generators $\{h_1, h_2\}$. Thus, there are vector fields $\zeta_i \in \mathrm{Derlog}(F)$ for $i = 1, 2$ of the form

$$\zeta_i = \varphi_{i1}\frac{\partial}{\partial x} + \varphi_{i2}\frac{\partial}{\partial y} - h_i\frac{\partial}{\partial z} + z\zeta_i'. \tag{3.1}$$

Furthermore, as Ψ preserves the weight decomposition (it increases weights by $s - d$), we may assume the vector fields are weighted homogeneous.

Verification that X is free

We have constructed three vector fields $e, \zeta_1, \zeta_2 \in \mathrm{Derlog}(X)$. It remains to show that they freely generate $\mathrm{Derlog}(X)$. We do this using Saito's criterion 3.1. Let A denote the matrix of coefficients, and $h = \det(A)$. As the 3 vectors are linearly dependent on X_{reg}, h vanishes on X_{reg} and hence on X. Since F is a reduced equation for $(X, 0)$, then $h = \alpha \cdot F$ for a germ $\alpha \in \mathcal{O}_{\mathbb{C}^3,0}$. It is enough to show that α is a unit. As both h and F are weighted homogeneous, then so is α. We can calculate its weight

$$\begin{aligned} \mathrm{wt}(\alpha) &= \mathrm{wt}(h) - \mathrm{wt}(F) = \mathrm{wt}(x) + \mathrm{wt}(\varphi_{12}) + \mathrm{wt}(h_2) - d \\ &= a + b + \mathrm{wt}(g) - d + \mathrm{wt}(h_1) + \mathrm{wt}(h_2) - d = 0 \end{aligned} \tag{3.2}$$

by condition (2) in the theorem. Since $\mathrm{wt}(\alpha) = 0$, to show that α is a unit it is sufficient to show that $\alpha(x, y, 0) \neq 0$.

If we set $z = 0$, the matrix A takes the form

$$\begin{pmatrix} ax & by & 0 \\ \varphi_{11} & \varphi_{12} & -h_1 \\ \varphi_{21} & \varphi_{22} & -h_2 \end{pmatrix}. \tag{3.3}$$

From (3.1), we obtain from the equations $\zeta_i(F)_{|z=0} = 0$ for $i = 1, 2$

$$h_1 \cdot g = \varphi_{11} f_x + \varphi_{12} f_y, \qquad h_2 \cdot g = \varphi_{21} f_x + \varphi_{22} f_y. \tag{3.4}$$

If we apply Cramer's rule to (3.4) we obtain

$$\Phi \cdot f_x = g(\varphi_{22} h_1 - \varphi_{12} h_2), \qquad \Phi \cdot f_y = g(-\varphi_{21} h_1 + \varphi_{11} h_2), \tag{3.5}$$

where $\Phi = \det(\varphi_{ij})$. Then, expanding (3.3) along the top row, and using (3.5) and the Euler relation, we evaluate

$$\begin{aligned} \det(A)_{|z=0} &= ax(-\varphi_{12} h_2 + \varphi_{22} h_1) - by(-\varphi_{11} h_2 + \varphi_{21} h_1) \\ &= \Phi/g(ax f_x + by f_y) = d \cdot (\Phi/g) \cdot f. \end{aligned}$$

Thus, $\alpha(x, y, 0) = d \cdot (\Phi/g)$. Finally, $\Phi \neq 0$, otherwise by (3.4) we would obtain first that $g \cdot (-\varphi_{12} h_2 + \varphi_{22} h_1) = 0$. As $g \neq 0$, this implies $-\varphi_{12} h_2 + \varphi_{22} h_1 = 0$. Since (h_1, h_2) is a complete intersection ideal, φ_{12} is divisible h_1. Using instead $g \cdot (-\varphi_{11} h_2 + \varphi_{21} h_1) = 0$ implies that φ_{11} is also divisible by h_1. By (3.4), this implies $g \in J(f)$, a contradiction. Hence, α is a unit and X is a free divisor.

Necessity

Suppose Derlog(F) is generated as an $\mathcal{O}_{\mathbb{C}^3,0}$-module by two elements $\{\zeta_1, \zeta_2\}$. Because Ψ is a module homomorphism over $i^*: \mathcal{O}_{\mathbb{C}^3,0} \to \mathcal{O}_{\mathbb{C}^2,0}$, we conclude that $\mathrm{Im}(\Psi)$ is generated as an $\mathcal{O}_{\mathbb{C}^2,0}$-module by $\{h_1, h_2\}$ where $h_i = \Psi(\zeta_i)$. As $g \notin J(f)$, $(J(f){:}\ g) \neq \mathcal{O}_{\mathbb{C}^2,0}$. Also, by Lemma 2.1, $\mathrm{Im}(\Psi)$ contains $J(f)$ and hence has finite colength. It follows that $\mathrm{Im}(\Psi)$ is a complete intersection ideal.

Even though $\mathrm{wt}(z) \leqslant 0$, we still claim, as in the case of positive weights, that the weighted homogeneous module Derlog(F) has a set of weighted homogeneous generators. Before saying more about this, we first finish the argument.

Let the weighted homogeneous generators be $\{\zeta_1, \zeta_2\}$. From these generators together with e, we may construct the matrix A as in the proof of sufficiency. Again by Saito, $\det(A)$ is a unit times F. On the other hand, we can compute the weight $\mathrm{wt}(\det(A))$ in terms of the weights $\mathrm{wt}(h_i)$ as for (2.3) to obtain

$$\mathrm{wt}\big(\det(A)\big) = a + b + \mathrm{wt}(g) - d + \mathrm{wt}(h_1) + \mathrm{wt}(h_2). \tag{3.6}$$

Since (3.6) must equal d, we obtain condition (2) in the theorem. $\square$

To justify the assertion that we may choose weighted homogeneous generators for Derlog(F), we consider generally a weighted homogeneous submodule $M \subset (\mathcal{O}_{\mathbb{C}^n,0})^p$, where we allow nonpositive weights for the coordinates $(x_1, \ldots, x_n)$ of $\mathbb{C}^n$ and a weight $\mathrm{wt}(\varepsilon_j) = c_j$ is assigned to each $\varepsilon_j = (0, \ldots, 0, 1, 0, \ldots, 0)$ (with 1 in the jth position). If all weights of the x_i were positive, there is a straightforward algebraic argument to show that M has a set of weighted homogeneous generators. To prove the result allowing nonpositive weights, we use the Artin approximation theorem.

Lemma 3.2. *Let $M \subset (\mathcal{O}_{\mathbb{C}^n,0})^p$ be a weighted homogeneous submodule, where we allow nonpositive weights for the coordinates of $\mathbb{C}^n$. Then there exist a set of weighted homogeneous generators for M.*

Proof. Let M_0 denote the submodule of M generated by all weighted homogeneous elements of M. Then, M_0 is a finitely generated submodule of M. Moreover, if $\{\zeta_1, \ldots, \zeta_r\}$ denotes the generators of M_0, then

$$\zeta_i = \sum_j g_{ij}\gamma_{ij}$$

with γ_{ij} weighted homogeneous. Replacing $\{\zeta_1, \ldots, \zeta_r\}$ by the set of γ_{ij} gives a set of weighted homogeneous generators for M_0. Thus, we may assume the ζ_i are weighted homogeneous.

We claim $M_0 = M$. Consider $\xi \in M$. If $\xi = \sum_j \xi_j$ denotes a decomposition of ξ into components ξ_j of distinct weights j, then the weighted homogeneity of M implies each $\xi_j \in M$. Hence, each $\xi_j \in M_0$. Thus, we may write $\xi_j = \sum_k \psi_{jk}\zeta_k$, where as ξ_j and the ζ_k are weighted homogeneous, we may assume the ψ_{jk} are weighted homogeneous. Then, if $\psi_k = \sum_j \psi_{jk}$ is the formal sum of terms of different weights, we have $\xi = \sum_k \psi_k\zeta_k$ in the formal power series ring. Hence, the analytic equation

$$\xi = \sum_k y_k\zeta_k \tag{3.7}$$

has the formal solution $y_k = \psi_k$. By the Artin approximation theorem, e.g., [17, Theorem 4.2], there is an analytic solution to Eq. (3.7). Thus, $\xi \in M_0$, so we have $M = M_0$. □

4. Hessian deformations of simple curve singularities

To prove Theorem 1.7, we will apply Theorem 1.1. For this, we must determine $\mathrm{Im}(\Psi)$ for each simple curve, and in the cases for which it is a complete intersection ideal, we must determine whether condition (2) is satisfied. In Table 1 we give for each simple curve, its Hessian deformation, where we absorb stray constants into z to simplify the form of the Hessian deformation. We also give a set of generators for $\mathrm{Im}(\Psi)$, and finally to determine

Table 1
Simple curve singularities and their Hessian deformations

Simple curves	Hessian deformations	$\mathrm{Im}(\Psi)$	Δ
A_1	$x^2 + y^2 + z$	(x, y)	0
A_{n-1}, $n \geqslant 3$	$x^n + y^2 + zx^{n-2}$	(x^2, y)	2
D_{n+1}, $n \geqslant 3$	$x^n + xy^2 + z\left(\binom{n}{2}x^{n-1} - y^2\right)$	(x^2, xy, y^2)	
E_6	$x^3 + xy^3 + z(4x^2y - y^4)$	(x, y^2)	2
E_7	$x^3 + y^4 + zxy^2$	(x, y^2)	3
E_8	$x^3 + y^5 + zxy^3$	(x, y)	0

whether condition (2) is satisfied, we list Δ which is the difference of the two sides of the equation for condition (2)

$$\Delta = \mathrm{wt}(g) + \mathrm{wt}(h_1) + \mathrm{wt}(h_2) - (2d - a - b). \tag{4.1}$$

Once we have justified the results in this table, we can apply Theorem 1.1 to complete the proof of Theorem 1.7. For D_{n+1}, $\mathrm{Im}(\Psi)$ is not a complete intersection ideal so the Hessian deformation is not free. For $A_{n-1},\ n \geqslant 3$, E_6, and E_7, $\mathrm{Im}(\Psi)$ is a complete intersection ideal, but $\Delta \neq 0$ so for none of these is the Hessian deformation free. Finally, for A_1 and E_8, $\mathrm{Im}(\Psi)$ is a complete intersection ideal and $\Delta = 0$ so the Hessian deformations are free.

To establish the results in the table for $\mathrm{Im}(\Psi)$, we use either determinantal vector fields or vector fields found with the assistance of the program Macaulay to show the image is attained. To show that we have not missed any generators, we use the map

$$\beta'_F : \theta_3 \to \mathcal{O}_{\mathbb{C}^3,0}$$

which sends $\zeta \mapsto \zeta(F)$. It increases weight by $\mathrm{wt}(F)$, and hence preserves the weight decomposition. Also, $\mathrm{Derlog}(F) = \ker(\beta'_F)$. Hence, if $M_{(k)}$ denotes the weight k part of M, then $\mathrm{Derlog}(F)_{(k)} = \ker(\beta'_F|\theta_{3(k)})$.

$\mathbf{A}_{n-1},\ n \geqslant 3$: First, $\Psi(\eta_{yz}) = 2y$ and $\zeta = x\partial/\partial x - (nx^2 + (n-2)z)\partial/\partial z \in \mathrm{Derlog}(F)$ with $\Psi(\zeta) = -nx^2$. Moreover, a calculation of $\ker(\beta'_F)$ shows $\mathrm{Derlog}(F)_{(-2)} = \ker(\beta'_F|\theta_{3(-2)}) = 0$, so $\mathrm{Im}(\Psi)_{(2)} = 0$. Hence, $x \notin \mathrm{Im}(\Psi)$; and it is as claimed.

$\mathbf{D}_{n+1},\ n \geqslant 3$: To begin, $\Psi(\eta_{yz}) = 2xy$ and $\Psi(\eta_{xz}) \equiv y^2 \bmod m^3_{x,y}$. In addition, let

$$\begin{aligned}\zeta \;=\; & -2(n-1)(x-z)x\frac{\partial}{\partial x} + \big((n+1)x + (n-1)^2 z\big)y\frac{\partial}{\partial y} \\ & + \big(4x + 2(n-1)^2 z\big)(x-z)\frac{\partial}{\partial z}.\end{aligned}$$

Then, it is easily checked that $\zeta \in \mathrm{Derlog}(F)$, and we see $\Psi(\zeta) = 4x^2$. Moreover, a calculation using $\ker(\beta'_F)$ in weights 0 and $n-3$ shows that $x, y \notin \mathrm{Im}(\Psi)$. Hence, $\mathrm{Im}(\Psi) = (x^2, xy, y^2)$.

$\mathbf{E}_6$ and $\mathbf{E}_7$: For both of these, a computation of $\ker(\beta'_F)$ using Macaulay yields three generators for each, which under Ψ map to x, xy, and y^2. Thus, $\mathrm{Im}(\Psi) = (x, y^2)$.

$\mathbf{A}_1$: $\Psi(\eta_{yz}) = 2y$ and $\Psi(\eta_{xz}) = 2x$, and it is easily checked $\Delta = 0$.

$\mathbf{E}_8$: Let

$$\begin{aligned}\zeta_1 \;&=\; y^2 z\frac{\partial}{\partial x} + \big(-x - 1/5yz^2\big)\frac{\partial}{\partial y} + \big(5y + 3/5z^3\big)\frac{\partial}{\partial z}, \\ \zeta_2 \;&=\; y^3\frac{\partial}{\partial x} + \big(-1/5y^2 z\big)\frac{\partial}{\partial y} + \big(-3x + 3/5yz^2\big)\frac{\partial}{\partial z}.\end{aligned}$$

It is straightforward to check $\zeta_1, \zeta_2 \in \mathrm{Derlog}(F)$. Then, $\Psi(\zeta_1) = 5y$ and $\Psi(\zeta_2) = -3x$. Hence, $\mathrm{Im}(\Psi) = (x, y)$ and a calculation shows $\Delta = 0$. This completes the verification of the table, so Theorem 1.7 follows.

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