Topology Vol. 30, No. 2, pp. 215–222, 1991
Printed in Great Britain.

0040-9383/91 $3.00 + .00

ELEMENTARY ZEROS OF LIE ALGEBRAS OF VECTOR FIELDS

J. F. PLANTE†

(Received 20 August 1989)

WE CONSIDER a finite dimensional Lie algebra $\mathscr{L}$ of $\mathscr{C}^k$ ($k \geq 1$) differentiable vector fields on a finite dimensional manifold M. A point $p \in M$ is a *zero* for $\mathscr{L}$ if $X(p) = 0$ for *every* $X \in \mathscr{L}$. For the classical special case of (the Lie algebra spanned by) a single vector field X, the zero p is said to be *elementary* (or *simple*) if the derivative at p of the principal part of X is invertible. The equivalent coordinate-free formulation of this definition is that the vector field, regarded as a map $X: M \to TM$, be transverse to the zero section at p [1]. From this point of view it can be seen that elementary zeros are stable, i.e., they persist under perturbation of X. In addition, such zeros can be assigned an index of $+1$ or -1 and, when all zeros are elementary and M is compact, the vector field satisfies the Poincaré–Hopf index formula. It is reasonable to ask how these notions should be formulated for zeros of an arbitrary Lie algebra. The correct definition turns out to involve the first Lie algebra cohomology of the linear part of $\mathscr{L}$ at p. That the first cohomology is relevant is suggested by results of Hirsch [5, 6] and Stowe [14], which relate stability of stationary points of group actions with the group cohomology of the linear action, and the relationship between the cohomologies of Lie groups and Lie algebras [15]. Section 1 reviews basic results concerning cohomology of Lie algebras. Section 2 applies these results to the definition and description of elementary zeros for a Lie algebra of vector fields on a manifold. Section 3 contains examples and a result, concerning commuting vector fields on a compact surface, which illustrate limitations to defining an index for a zero of a Lie algebra of vector fields.

1. PRELIMINARIES CONCERNING COHOMOLOGY OF LIE ALGEBRAS

Suppose $\mathscr{L}$ is a Lie algebra, V is a vector space, and $\rho: \mathscr{L} \to gl(V)$ is a Lie algebra homomorphism, i.e., $\rho[X, Y] = \rho(X)\rho(Y) - \rho(Y)\rho(X)$. If $X \in \mathscr{L}$, $u \in V$ we will usually write $\rho(X)u$ more simply as Xu. We define $H^0(\mathscr{L}, V)$ to be the subspace of V which is annihilated by ρ.

$$H^0(\mathscr{L}, V) = \{u \in V \mid Xu = 0 \text{ for all } X \in \mathscr{L}\}.$$

A linear map $f: \mathscr{L} \to V$ is a *1-cocycle* if, for all $X, Y \in \mathscr{L}$

$$f([X, Y]) = Xf(Y) - Yf(X).$$

If, in addition, there exists $u \in V$ such that $f(X) = Xu$ for all $X \in \mathscr{L}$ then f is said to be a *1-coboundary*. The coboundaries constitute a subspace of the linear space of cocycles and the quotient space is the cohomology space $H^1(\mathscr{L}, V)$ (ρ suppressed as usual). Sufficient background on cohomology of Lie algebras may be found in [4, 7]. The first cohomology depends both on the structure of the Lie algebra of $\mathscr{L}$ and the action ρ. For example, if ρ is

†Supported in part by a grant from the National Science Foundation.

trivial ($Xu = 0$ for all X, u), then the first cohomology is easily seen to be the linear space $\mathrm{Hom}(\mathscr{L}/[\mathscr{L}, \mathscr{L}]; V)$. When $\mathscr{L}$ is 1-dimensional, $H^1(\mathscr{L}, V)$ is isomorphic to coker $\rho(X)$ where X is a generator of $\mathscr{L}$. On the other hand, the Whitehead Lemma asserts that whenever $\mathscr{L}$ is semi-simple, $H^1(\mathscr{L}, V) = 0$ for any ρ.

We now insist that $\mathscr{L}$ and V be finite dimensional and real. In this context the set of all actions of $\mathscr{L}$ on V is a closed subset of a finite dimensional space. The following result says that triviality of H^1 is stable under perturbations of the action.

1.1 Proposition. *If for some action ρ of $\mathscr{L}$ on V, $H^1(\mathscr{L}, V) = 0$ then this condition also holds for all actions sufficiently close to ρ. Furthermore, $H^0(\mathscr{L}, V)$ will have the same dimension for the perturbed action as it does for ρ.*

Proof. Let W be the linear space of skew symmetric bilinear maps from $V \times V$ to V ($W \cong V^{l(l+1)/2}$ where $l = \dim \mathscr{L}$). Denote by $L(\mathscr{L}, V)$ the space of linear maps from $\mathscr{L}$ to V. Define maps α: $V \to L(\mathscr{L}, V)$ and β: $L(\mathscr{L}, V) \to W$ by

$$\alpha(u)(X) = Xu \quad X \in \mathscr{L}, u \in V$$

and

$$\beta(f)(X, Y) = Xf(Y) - Yf(X) - f([X, Y])$$

where $X, Y \in \mathscr{L}$ and $f \in L(\mathscr{L}, V)$. By definitions, $H^0(\mathscr{L}, V) = \ker \alpha$ and $H^1(\mathscr{L}, V) = \ker \beta/\mathrm{image}\ \alpha$. For an action ρ' *sufficiently* close to ρ having corresponding linear maps α' and β' we have

$$\dim \mathrm{image}\ \alpha \leq \dim \mathrm{image}\ \alpha' \leq \dim \ker \beta' \leq \dim \ker \beta.$$

If $H^1(\mathscr{L}, V) = 0$ for ρ then image $\alpha = \ker \beta$. This implies that $\ker \beta'$ and image α' have the same dimension and therefore, coincide, that is, $H^1(\mathscr{L}, V) = 0$ for ρ'. The last statement of (1.1) follows since $\ker \alpha$ and $\ker \alpha'$ have the same dimension.

We now proceed toward a description of the significance of the vanishing of the first cohomology when the Lie algebra is nilpotent. Analogous results for the case of cohomology of groups may be found in Hirsch [6]. The only general tools needed are the usual long exact cohomology sequence associated with a short exact sequence of modules [4] and the following result.

1.2 Proposition. *If $\mathscr{K} \subset \mathscr{L}$ is an ideal and ρ: $\mathscr{L} \to gl(V)$ is an action then there is an exact sequence*

$$0 \to H^1(\mathscr{L}/\mathscr{K}, H^0(\mathscr{K}, V)) \xrightarrow{\text{inf}} H^1(\mathscr{L}, V) \xrightarrow{\text{res}} H^1(\mathscr{K}, V).$$

The inflation map sends a cocycle f: $\mathscr{L}/\mathscr{K} \to H^0(\mathscr{K}, V)$ to the composition $i \circ f \circ \pi$ where π is the quotient projection and i is inclusion. The restriction map sends a cocycle f: $\mathscr{L} \to V$ to its restriction to $\mathscr{K}$.

The proof of (1.2) is a straightforward adaptation of the corresponding proof for group cohomology [2]. One application of (1.2) is when $\mathscr{K} = \ker \rho$. In this case $H^0(\mathscr{K}, V) = V$ and (1.2) says that $H^1(\mathscr{L}, V)$ is non-trivial whenever H^1 is non-trivial for the effective part of ρ.

1.3 Lemma. *If there exists $Z \in \mathscr{L}$ such that $\rho(Z)$ is invertible and central in $\rho(\mathscr{L})$ then $H^1(\mathscr{L}, V) = 0$.*

Proof. The cocycle condition gives, for any $X \in \mathscr{L}$ and cocycle f,

$$0 = f([X, Z]) = Xf(Z) - Zf(X)$$

or

$$f(X) = Z^{-1}Xf(Z) = XZ^{-1}f(Z)$$

which says that f is a coboundary.

1.4 Lemma. *If $\mathscr{L}$ is nilpotent and* $\dim \mathscr{L} \geq 2$ *then* $\dim(\mathscr{L}/[\mathscr{L}, \mathscr{L}]) \geq 2$.

Proof. This is true in dimension 2 since $\mathscr{L}$ is abelian in that case. Assume the lemma is true for dimension $n - 1$ ($n \geq 3$). If $\dim \mathscr{L} = n$, let $\mathscr{J} \subset \mathscr{L}$ be a central 1-dimensional ideal. Since $\mathscr{L}/\mathscr{J}$ has dimension $(n - 1)$ it has an abelian quotient of dimension at least 2 and, therefore, so does $\mathscr{L}$.

1.5. Lemma. *If $\rho: \mathscr{L} \to gl(V)$ is a non-trivial action such that every element of $\rho(\mathscr{L})$ is nilpotent then $H^1(\mathscr{L}, V) \neq 0$.*

Proof. The result is clearly true when $\dim \mathscr{L} = 1$ and V is arbitrary. It is also true, vacuously, when $\mathscr{L}$ is arbitrary and $\dim V = 1$. We assume $\dim \mathscr{L} \geq 2$ and that the result is true whenever $\dim V = n - 1$ ($n \geq 2$). In view of (1.2) we may assume ρ is effective (injective) and $\mathscr{L}$ is nilpotent. In fact, we may select a basis for V which puts every element of $\rho(\mathscr{L})$ in nilpotent upper triangular form [7]. Let V_0 be the subspace of V spanned by the first $(n - 1)$ basis vectors ($V/V_0 \simeq \mathbb{R}$) and let $\mathscr{K} \subset \mathscr{L}$ be the ideal consisting of all elements of $\mathscr{L}$ whose restrictions to V_0 are zero. Since ρ is effective $H^0(\mathscr{K}, V_0) = V_0$. If $\mathscr{K} \neq \mathscr{L}$ then (1.2) gives

$$0 \to H^1(\mathscr{L}/\mathscr{K}, V_0) \to H^1(\mathscr{L}, V)$$

and the result follows since the middle term is non-zero by the inductive hypothesis. On the other hand, if $\mathscr{K} = \mathscr{L}$ then $\mathscr{L}$ acts trivially on V_0 and the short exact sequence

$$0 \to V_0 \to V \to \mathbb{R} \to 0$$

leads to an exact sequence containing

$$H^0(\mathscr{L}, \mathbb{R}) \to H^1(\mathscr{L}, V_0) \to H^1(\mathscr{L}, V).$$

The second term is $\mathrm{Hom}(\mathscr{L}/[\mathscr{L}, \mathscr{L}], V_0)$ which has dimension ≥ 2 by (1.4). Since the first term has dimension ≤ 1, it follows that $\dim H^1(\mathscr{L}, V) \geq 1$ and the proof of (1.5) is complete.

1.6 Proposition. *If $\rho: \mathscr{L} \to gl(V)$ is an action with $[\mathscr{L}, \mathscr{L}] \neq \mathscr{L}$ and $H^1(\mathscr{L}, V) = 0$ then $H^0(\mathscr{L}, V) = 0$.*

Proof. Let V_n be the largest subspace of V on which every element of $\rho(\mathscr{L})$ is nilpotent. Since $H^0(\mathscr{L}, V) \subset V_n$ the proof will be completed by showing that $V_n = 0$. The short exact sequence

$$0 \to V_n \to V \to V/V_n \to 0$$

yields the cohomology exact sequence containing

$$H^0(\mathscr{L}, V/V_n) \to H^1(\mathscr{L}, V_n) \to H^1(\mathscr{L}, V).$$

In this case $H^0(\mathscr{L}, V/V_n') = 0$ so the hypothesis $H^1(\mathscr{L}, V) = 0$, together with (1.5), imply that $\mathscr{L}$ acts trivially on V_n' ($V_n' = H^0(\mathscr{L}, V)$) and, therefore, since

$$H^1(\mathscr{L}, V_n') \simeq \mathrm{Hom}(\mathscr{L}/[\mathscr{L}, \mathscr{L}], V_n)$$

we conclude that $V_n' = 0$.

The type of Lie algebra most similar to the one-dimensional Lie algebra is nilpotent. In this case there are several equivalent conditions. By "almost all" points in a real finite dimensional space we shall mean a subset that remains after removal of finitely many subspaces of lower dimension.

1.7 PROPOSITION. *For a nilpotent Lie algebra $\mathscr{L}$ and representation $\rho: \mathscr{L} \to gl(V)$ the following are equivalent.*

(i) $H^1(\mathscr{L}, V) = 0$
(ii) $H^0(\mathscr{L}, V) = 0$
(iii) *For almost all $X \in \mathscr{L}$, $\rho(X)$ is invertible.*
(iv) *For some $X \in \mathscr{L}$, $\rho(X)$ is invertible.*

Proof. That (i) implies (ii) follows from (1.6), (iii) implies (iv) is obvious, and (iv) implies (ii) follows from the definition of H^0. The proof will be completed by showing that (ii) implies (iii) and that (ii) implies (i).

Since $\mathscr{L}$ is nilpotent the complexification of ρ has a special form [7, Chapter II]. Specifically, the complexification of V is a direct sum $W_1 \oplus \ldots \oplus W_k$ of invariant subspaces such that for any $X \in \mathscr{L}$ the matrix of the restriction of $\rho_c(X)$ to W_i is the sum of a scalar matrix and a nilpotent upper triangular matrix. The map which sends X to the scalar is a Lie algebra homomorphism h_i and, assuming (ii), $\ker h_i$ is a closed subspace of lower dimension (than $\mathscr{L}$). For every $X \in \left(\mathscr{L} - \bigcup_{i=1}^{k} \ker h_i\right)$, $\rho(X)$ is invertible since all its eigenvalues are non-zero. This shows that (ii) implies (iii).

In order to complete the proof of 1.7 assume (ii) $H^0(\mathscr{L}, V) = 0$ and equivalently, (iii) $\rho(X)$ is invertible for almost all $X \in \mathscr{L}$. It must be shown that (i) $H^1(\mathscr{L}, V) = 0$. This assertion is true for abelian $\mathscr{L}$ by (1.3). In particular it is true whenever $\dim \mathscr{L} \leq 2$. Assume it is true whenever $\dim \mathscr{L} < n$ ($n \geq 3$). If $\dim \mathscr{L} = n$, then by (1.4) and (iii) there is a codimension-one ideal $\mathscr{K} \subset \mathscr{L}$ which contains an invertible element. Since $H^0(\mathscr{K}, V) = 0$ and $H^1(\mathscr{K}, V) = 0$, (1.2) implies that $H^1(\mathscr{L}, V) = 0$ and the proof of (1.7) is complete.

2. ELEMENTARY ZEROS

Consider the linear space $\mathscr{X}^r(M)$ of $\mathscr{C}^r (r \geq 1)$ differentiable vector fields on a manifold M. If $\mathscr{L}$ is a Lie algebra then an *action* of $\mathscr{L}$ on M is a linear map $A: \mathscr{L} \to \mathscr{X}^r(M)$ such that $A([X, Y]) = [A(X), A(Y)]$ whenever $X, Y \in \mathscr{L}$. A point $p \in M$ is said to be a *zero* of A if $X(p) = 0$ for all $X \in \mathscr{L}$. Note that if A were integrated to give a local Lie group action then p would be a stationary (or fixed) point of the local group action.

Suppose now that $p \in M$ is a zero for A. For each $X \in \mathscr{L}$, the derivative of $A(X)$ at p is a linear map from T_pM to $T_{0_p}(TM)$ (T denotes tangent spaces, 0_p is the zero vector at p). Identifying these spaces in the standard way we regard the derivative of $A(X)$ at p as an element of $gl(T_pM)$. The map $\mathscr{L} \to gl(T_pM)$ which sends each $X \in \mathscr{L}$ to the derivative of $A(X)$ at p is a linear action, of the type considered in the previous section, which is called the *linear part* of A at p.

Definition. The zero p of the action A is *elementary* if, for the linear part of A at p, $H^1(\mathscr{L}, T_pM) = 0$.

2.1 THEOREM. *If p is an elementary zero of an action A then in some nighborhood N of p all zeros of A are elementary and the set of zeros in N is a submanifold whose tangent space at p is $H^0(\mathscr{L}, T_pM)$.*

Proof. Since the situation is local we may assume, by taking local coordinates, that M is an open subset of V and that T_pM is identified with V. Since p is elementary we have (from the proof of (1.1)) the exact sequence

$$V \xrightarrow{\alpha} L(\mathscr{L}, V) \xrightarrow{\beta} W$$

where α is determined by evaluation of the linear part of A at p. Denote by S the subspace $\alpha(V) = \ker \beta$ of $L(\mathscr{L}, V)$. Select a complementary subspace for S and denote by π the corresponding projection of $L(\mathscr{L}, V)$ onto S. Using an ordered basis $\{X_1, \ldots, X_l)$ of $\mathscr{L}$ we identify each $f \in L(\mathscr{L}, V)$ with $(f(X_1), \ldots, f(X_l)) \in V^l$. Define φ: $M \to S$ by

$$\varphi(x) = \pi(A(X_1)(x), \ldots, A(X_l)(x)).$$

We have $\varphi(p) = 0$ and $D\varphi(p) = \pi(DA(X_1)(p), \ldots DA(X_l)(p)) = \pi \circ \alpha$, where D denotes derivative. Since $D\varphi(p)$ is surjective there is a neighborhood N of p such that $\{x \in N \mid \varphi(x) = 0\}$ is a submanifold tangent at p to $\ker D\varphi(p)$ by the implicit function theorem. But $\ker D\varphi(p)$ is simply the set of vectors which are mapped to zero by every $DA(X_i)(p)$, that is, $\ker D\varphi(p) = H^0(\mathscr{L}, T_pM)$. Finally, it follows, from (1.1) and the assumption that A is $\mathscr{C}^1$, that restricting suitably the size of N will insure that every zero of A contained in N will be elementary. This completes the proof of (2.1).

2.2. COROLLARY. *Suppose A: $\mathscr{L} \to \mathscr{X}^1(M)$ is an action.*

(i) *If $[\mathscr{L}, \mathscr{L}] \neq \mathscr{L}$ then the elementary zeros of A are isolated.*

(ii) *If $\mathscr{L}$ is semi-simple then every zero of A is elementary. In particular, the set of zeros of A is a submanifold of M.*

Proof. (i) follows from (1.6) and (2.1). (ii) follows from (2.1) and the Whitehead Lemma [4, 7] which says that $H^1(\mathscr{L}, V) = 0$ whenever $\mathscr{L}$ is semi-simple.

The last statement in (ii) of (2.2) was proven by Stowe [14] in the more general context of $\mathscr{C}^1$ Lie group actions. For real analytic actions Guillemin and Sternberg [3] have shown that the semi-simple action can actually be linearized in a neighborhood of any zero.

The following result is equivalent to Theorem A of [10].

2.3 COROLLARY. *Suppose A: $\mathscr{L} \to \mathscr{X}^1(M)$ is an action where $[\mathscr{L}, \mathscr{L}] = \mathscr{L}$. If $p \in M$ is a zero for A at which the linear part is trivial, then the action A is trivial on the connected component of M which contains p.*

Proof. The hypothesis on $\mathscr{L}$ and p imply that, for the linear part of A at p, $H^0(\mathscr{L}, T_pM) = T_pM$ and $H^1(\mathscr{L}, T_pM) = 0$. Now (2.1) says that the zero set of A includes a neighborhood of p. Therefore, the set of points in M at which both A and its linear part are trivial is open in M. Since this set is also closed it must contain the connected component of p.

For later use we state the characterizations of elementary zeros for a nilpotent Lie algebra.

2.4 PROPOSITION. *If $\mathscr{L}$ is nilpotent and p is a zero of the action $A: \mathscr{L} \to \mathscr{X}^1(M)$ then the following are equivalent.*

(i) *p is an elementary zero for A.*
(ii) *If $DA(X)(p)(v) = 0$ for every $X \in \mathscr{L}$ then $v = 0$.*
(iii) *For some $X \in \mathscr{L}$, p is an elementary zero for X.*
(iv) *For almost all $X \in \mathscr{L}$, p is an elementary zero for X.*

Proof. (2.4) follows from (1.7).

The following result is an analogue for Lie algebras of the stability theorem of Stowe [14].

2.5 THEOREM. *Suppose p is an elementary zero of an action $A: \mathscr{L} \to \mathscr{X}^1(M)$. Then given a neighborhood N of p, there is a neighborhood U of A in the space of $\mathscr{C}^1$ actions such that for every $A' \in U$ there is a $p' \in N$ which is an elementary zero of A'.*

Proof. If $\{X_1, \ldots, X_l\}$ is an ordered basis for $\mathscr{L}$, the topology on the space of actions is such that a net $\{A_v\}$ of actions converges if $A_v(X_i)$ and $DA_v(X_i)$, $i = 1, \ldots, l$, converge uniformly on compact sets. We retain the notation from the proof of (2.1). For an action A' near A there will be a map φ' (determined by the original p and π) which is uniformly $\mathscr{C}^1$ close to φ on some neighborhood N of p. Since $\varphi(N)$ contains a neighborhood of 0 we may suppose φ' does as well, that is, $\varphi'(p') = 0$ for some $p' \in N$. In view of (1.1) we may also suppose that restriction of π to S' (= image α' = ker β') is an isomorphism onto S. This implies that p' is an elementary zero of A'.

2.6 COROLLARY. *Suppose $A: \mathscr{L} \to \mathscr{X}^1(M)$ is an action where $[\mathscr{L}, \mathscr{L}] = \mathscr{L}$. If $p \in M$ is a zero for A at which the linear part is trivial then there is a neighborhood U of A such that $A'(X)(p') = 0$ whenever $X \in \mathscr{L}$, $A' \in U$, and p' is in the same connected component as p.*

Proof. The hypotheses imply that $H^0(\mathscr{L}, T_pM) = T_pM$ and $H^1(\mathscr{L}, T_pM) = 0$ for the linear part of A at p, so (2.6) follows from (1.1) and (2.5) by the argument used for the proof of (2.3).

3. COMMUTING VECTOR FIELDS ON SURFACES

In [9] it is shown that when a nilpotent Lie group acts without fixed points on a compact surface M, the Euler characteristic of M is zero. This suggests the possibility of an index theorem for actions by nilpotent groups in the spirit of Poincaré–Hopf. On the other hand, it is also shown in [9] that the solvable, non-abelian, two-dimensional group acts without fixed points on every compact surface so there is no hope of this type of result for solvable groups. In this section we consider actions of abelian Lie algebras on compact surfaces without boundary.

3.1 THEOREM. *Suppose $A: \mathscr{L} \to \mathscr{X}^2(M)$ is an action where $\mathscr{L}$ is abelian and M is a compact surface. Assume that every zero of A is elementary. Then there is a set $\mathscr{S} \subset \mathscr{L}$ such that $\mathscr{L} - \mathscr{S}$ has measure zero and for $X \in \mathscr{S}$*

(i) *The set of isolated zeros of X coincides with the (finite) zero set of A.*

(ii) *If* $p_1, \ldots, p_k$ *are the zeros of A then* $\sum_{i=1}^{k} \text{index}\ X(p_i) = \chi(M)$.

Remark. The vector field $X \in \mathscr{S}$ may also have zeros which are not isolated. For example, consider the commuting vector fields on the plane

$$X = (\cos 2\pi x)\frac{\partial}{\partial y}, \qquad Y = (\sin 2\pi x)\frac{\partial}{\partial y}.$$

They induce commuting vector fields on the torus $T^2 = \mathbb{R}^2/\mathbb{Z}^2$. Every vector in the Lie algebra $\mathscr{L} \subset \mathscr{X}^1(T)$ spanned by X and Y is a scalar multiple of some $X_t = (\cos 2\pi t)X + (\sin 2\pi t)Y = \cos 2\pi(x - t)\ \partial/\partial y$. The zero set of X_t consists of the two circles $x = t + \frac{1}{4}, x = t + \frac{3}{4}$.

In proving (3.1), we will make use of the following result which follows immediately from Theorem 1 of [12]. By an orbit of an action by a Lie algebra we mean an orbit of the integrated Lie group action.

3.2 LEMMA. *If* $A: \mathscr{L} \to \mathscr{X}^2(M)$ *is as in (3.1) and* $\dim \mathscr{L} \geq 2$ *then for almost every 2-plane* $\mathscr{P} \subset \mathscr{L}$ *(in the sense of measure on the Grassmanian) the restriction of A to* $\mathscr{P}$ *has precisely the same orbits as does A.*

Proof of (3.1). In view of (3.2) we only need to prove (3.1) when $\dim \mathscr{L} = 2$. Assume the action is spanned by commuting vector fields Y, Z. If every orbit of the action has dimension 2 (Y, Z everywhere linearly independent) then M is a finite union of toral orbits so $\chi(M) = 0$ and (i) and (ii) are true. Now suppose the action has a non-empty (compact) subset consisting of all zero and one-dimensional orbits. Let $p \in M$ be a point having a one-dimensional orbit and suppose that $Y(p) \neq 0$. Select a local coordinate system (x, y) for a neighborhood U of p such that $Y = \frac{\partial}{\partial y}$. In U we have $Z = f\frac{\partial}{\partial x} + gY$ where f and g are $\mathscr{C}^1$ functions on U. Since Y and Z commute it follows that f and g depend only on x since $[Y, Z] = f_y\frac{\partial}{\partial x} + g_y Y$. The restriction to U of an arbitrary vector field in $\mathscr{L}$ has the form

$$aY + bZ = bf\frac{\partial}{\partial x} + (a + bg)Y.$$

For real numbers a, b we define $G(a, b, x) = a + bg(x)$. The real valued map G has 0 as a regular value, so for almost all $(a, b) \in \mathbb{R}^2$ the map $x \mapsto G(a, b, x)$ has zero as a regular value [1]. For such (a, b) the zero set of $aY + bZ \in \mathscr{L}$ in U is a subset of the 1-dimensional submanifold $\{(x, y) \in U \mid G(a, b, x) = 0\}$. Since this zero set is Y-invariant and Y is never zero in U, the zero set of $aY + bZ \in \mathscr{L}$ in U is itself a 1-dimensional submanifold. The set of 1-dimensional orbits of $\mathscr{L}$ may be covered by a countable collection $\{U_i\}$ of open subsets of $M - \{p_1, \ldots, p_k\}$. This yields a subset $\mathscr{S}_1 \subset \mathscr{L}$ such that $\mathscr{L} - \mathscr{S}_1$ has measure zero and $X \in \mathscr{S}_1$ implies that the zero set of X consists of $\{p_1, \ldots, p_k\}$ together with a 1-dimensional submanifold of $M - \{p_1, \ldots, p_k\}$. By (2.4) there is a set $\mathscr{S}_0 \subset \mathscr{L}$ such that $\mathscr{L} - \mathscr{S}_0$ has measure zero and $X \in \mathscr{S}_0$ implies that X has all of $p_1, \ldots, p_k$ as elementary zeros. In particular, $p_1, \ldots, p_k$ are isolated zeros of any $X \in \mathscr{S}_0$. For $X \in \mathscr{S} = \mathscr{S}_0 \cap \mathscr{S}_1$, the zero set of X consists of $p_1, \ldots, p_k$ together with a compact 1-dimensional submanifold of M which must be a finite union of embedded circular orbits. The circular orbits of X have pairwise disjoint tubular neighborhoods (each either a cylinder or Moebius band) such that X is transverse to the boundary of each neighborhood. Removal of the open tubular neighborhood leaves a compact surface (with boundary) having the same Euler characteristic as M such that the restriction of X is transverse to the boundary and has zero set $\{p_1, \ldots, p_k\}$. (ii) of (3.1) now follows from the Poincaré–Hopf formula [11].

The transversality argument in the proof of (3.1) was used by Simen [13] in proving that any compact manifold of dimension $2n$ which admits an abelian action with every orbit of dimension at least n must have Euler characteristic zero. That result was apparently rediscovered in [8].

One might hope that it would be possible to generalize the notion of index to a zero of an action by a nilpotent Lie algebra $\mathscr{L}$. It is clear from the following example that one cannot take the index of a generic vector field. Define commuting vector fields on $T^2 = \mathbb{R}^2/\mathbb{Z}^2$ by

$$X = \cos 2\pi x \frac{\partial}{\partial x} + \cos 2\pi y \frac{\partial}{\partial y}$$

$$Y = \cos 2\pi x \frac{\partial}{\partial x} - \cos 2\pi y \frac{\partial}{\partial y}.$$

The common zeros of X and Y are at the four points $(\frac{1}{4}, \frac{1}{4})$, $(\frac{1}{4}, \frac{3}{4})$, $(\frac{3}{4}, \frac{1}{4})$, $(\frac{3}{4}, \frac{3}{4})$. At each of these points either X or Y has a saddle point and the other is a source or sink. Therefore, index $(X, p) = -$ index $(Y, p) = \pm 1$ for each zero p of the action spanned by X and Y. It is also clear that the restrictions of the action to a neighborhood of each zero are equivalent, so if an index formula were possible, the index of the action would have to be zero at each p.

REFERENCES

1. R. ABRAHAM and J. ROBBIN: *Transversal mappings and flows*, Benjamin, New York (1967).
2. M. ATIYAH and C. T. C. WALL: *Cohomology of groups, in Algebraic Number Theory*, pp. 94–115, edited by Cassels and Frohlich, Thompson, Washington, D.C. (1967).
3. V. GUILLEMIN and S. STERNBERG: Remarks on a paper of Hermann, *Trans. A.M.S.* **130** (1968), 110–116.
4. P. HILTON and U. STAMMBACH: *A course in homological algebra*, Springer, New York (1971).
5. M. HIRSCH: Stability of stationary points of group actions, *Ann. Math.* **109** (1979), 537–544.
6. M. HIRSCH: Stability of stationary points and cohomology of groups, *Proc. A.M.S.* **79**(2) (1980), 191–196.
7. N. JACOBSON: *Lie algebras*, Dover, New York (1979).
8. P. MOLINO and F. TURIEL: Une observation sur les actions de $\mathbb{R}^p$ sur les variétés compactes de caracteristique non nulle, *Comm. Math. Helv.* **61** (1986), 370–375.
9. J. PLANTE: Fixed points of Lie group actions on surfaces, *Ergodic Th. & Dynam. Systs.* **6** (1986), 149–161.
10. J. PLANTE: Lie algebras of vector fields which vanish at a point, *J. London Math. Soc.* (2) **38** (1988), 379–384.
11. C. PUGH: A generalized Poincaré index formula, *Topology* 7(2) (1968), 217–226.
12. R. SACKSTEDER: Degeneracy of orbits of actions of $\mathbb{R}^m$ on a manifold, *Comm. Math. Helv.* **41**(1) (1966–7), 1–9.
13. D. SIMEN: $\mathbb{R}^k$ actions on manifolds, *Am. J. Math.* **104**(1) (1982), 1–7.
14. D. STOWE: The stationary set of a group action, *Proc. A.M.S.* **79**(1) (1980), 139–146.
15. W. VAN EST: Group cohomology and Lie algebra cohomology in Lie groups, *Proc. Neder. Akad. Weten.* Amsterdam (Series A) **56**(5) (1953), 484–504.

Mathematics Department
The University of North Carolina at Chapel Hill,
Phillips Hall CB #3250,
Chapel Hill, NC 27599.
U.S.A.