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Nonminimal sets, their projections and integral representations of stable processes

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Abstract

New criteria are provided for determining whether an integral representation of a stable process is minimal. These criteria are based on various nonminimal sets and their projections, and have several advantages over and shed light on already available criteria. In particular, they naturally lead from a nonminimal representation to the one which is minimal. Several known examples are considered to illustrate the main results. The general approach is also adapted to show that the so-called mixed moving averages have a minimal integral representation of the mixed moving average type.

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1. Introduction

Let T be an arbitrary index set. Consider real-valued, symmetric α-stable ($S\alpha S$), $\alpha \in (0, 2)$, processes $\{X(t)\}_{t\in T}$ having an integral representation

$$\{X(t)\}_{t\in T} \stackrel{d}{=} \left\{\int_S f_t(s) M(\mathrm{d}s)\right\}_{t\in T}, \tag{1.1}$$

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where $\stackrel{d}{=}$ stands for equality in the sense of finite-dimensional distributions. Here, $(S, \mathcal{B}(S), \mu)$ is a standard Lebesgue space (see Section 2), $\{f_t\}_{t\in T} \subset L^\alpha(S,\mu)$ and M is a real-valued, independently scattered, $S\alpha S$ random measure on S with the control measure μ. The relation (1.1) means that the characteristic function of X can be expressed as

$$E \exp\left\{ \mathrm{i} \sum_{k=1}^{n} \theta_k X(t_k) \right\} = \exp\left\{ - \int_S \left| \sum_{k=1}^{n} \theta_k f_{t_k}(s) \right|^\alpha \mu(\mathrm{d}s) \right\}, \tag{1.2}$$

where $\theta_k \in \mathbb{R}$, $t_k \in T$. We also suppose that the map $f_t(s) : T \times S \mapsto \mathbb{R}$ is measurable, and that $(T, \mathcal{B}(T))$ is a standard Borel set (see Section 2). A comprehensive reference on stable processes and their integral representations is [13]. It is known, in particular, that every measurable, real-valued, $S\alpha S$ process X has an integral representation (1.1) with, for example, $S = (0, 1)$, $\mathcal{B}(S) = \mathcal{B}(0, 1)$ and $\mu =$ Lebesgue measure (see [13], Theorems 13.2.1 and 9.4.2). On the other hand, most examples of stable processes are also defined through integral representations of the type (1.1). To avoid unnecessary details, we also suppose without loss of generality that

$$\mathrm{supp}\{f_t, t \in T\} = S \quad \mu\text{-a.e.}, \tag{1.3}$$

where $\mathrm{supp}\{f_t, t \in T\}$, the support of f_t, $t \in T$, is a minimal (μ-a.e.) set $A \in \mathcal{B}(S)$ such that $\mu\{f_t(s) \neq 0, s \notin A\} = 0$ for every $t \in T$.

An $S\alpha S$ process X has many integral representations $\{f_t\}_{t\in T}$ on possibly different spaces $(S, \mathcal{B}(S), \mu)$. For example, another representation can be obtained from (1.1) by mapping $(S, \mathcal{B}(S), \mu)$ to a different space (that is, by simply making a change of variables). Among all integral representations, the so-called minimal integral representations play a fundamental role. Minimal representations were introduced by Hardin [2], and subsequently studied in depth by Rosiński [11]. As shown in [2], every separable in probability $S\alpha S$ process has a minimal integral representation. Two, commonly used, equivalent ways of defining minimal representations are as follows (see [11], Theorem 3.1).

Definition 1.1 (*Minimal Representations*). An integral representation $\{f_t\}_{t\in T}$ in (1.1) is minimal if and only if

$$\sigma\{f_u/f_v,\ u, v \in T\} = \mathcal{B}(S) \mod \mu \tag{1.4}$$

or if and only if, for every nonsingular map $\phi : S \mapsto S$ and $h : S \mapsto \mathbb{R} \setminus \{0\}$ such that, for each $t \in T$,

$$f_t(s) = h(s) f_t(\phi(s)) \quad \text{a.e. } \mu(\mathrm{d}s), \tag{1.5}$$

it follows that

$$\phi(s) = s \quad \text{a.e. } \mu(\mathrm{d}s). \tag{1.6}$$

We write $\mathcal{A} = \mathcal{B} \mod \mu$ for two σ-fields $\mathcal{A}$ and $\mathcal{B}$ when, for $A \in \mathcal{A}$, there is $B \in \mathcal{B}$ such that $\mu(A \triangle B) = 0$. A map $\phi : S \mapsto S$ is nonsingular when $\mu(A) = 0$ implies $\mu(\phi^{-1}(A)) = 0$. In other words, it cannot happen that sets of positive measure are mapped by ϕ to sets of null measure. We shall also write (1.6) as $\phi = Id$ μ-a.e. and its opposite as $\phi \neq Id$ μ-a.e.

Though Definition 1.1 and the discussion preceding it involves symmetric stable processes, the exact same definition is also taken for minimal representations in the nonsymmetric (skewed) case. See [8,11]. The case $\alpha = 1$, however, is excluded. We work with the symmetric case for simplicity.

Definition 1.1 can be easily and intuitively understood through the following example. Consider the $S\alpha S$ processes

$$X_t = \sum_{n\in\mathbb{Z}} a_{t,n}\epsilon_n = \int_{\mathbb{Z}} f_t(n)M(\mathrm{d}n), \quad t \in \mathbb{Z}, \tag{1.7}$$

where the ϵ_n are independent, $S\alpha S$ random variables such that $E\exp\{i\theta\epsilon_n\} = \exp\{-|\theta|^\alpha\}$, $\theta \in \mathbb{R}$, M has the control measure $\mu(\mathrm{d}n) = \delta_{\mathbb{Z}}(\mathrm{d}n)$ and $f_t(n) = a_{t,n}$. If (1.7) is nonminimal, then there is $\phi : \mathbb{Z} \mapsto \mathbb{Z}$ such that $\phi \neq Id$ and (1.5) holds. In particular, there is $n_0 \in \mathbb{Z}$ with $m_0 = \phi(n_0) \neq n_0$ such that

$$a_{t,n_0} = c\, a_{t,m_0} \quad (c \neq 0), \tag{1.8}$$

for all t. If (1.8) holds, the representation (1.7) can obviously be reduced to

$$X_t \overset{d}{=} \sum_{n\neq m_0} \widetilde{a}_{t,n}\epsilon_n, \tag{1.9}$$

where

$$\widetilde{a}_{t,n} = \begin{cases} a_{t,n}, & \text{if } n \neq n_0, \\ (1+|c|^{-\alpha})^{1/\alpha} a_{t,n}, & \text{if } n = n_0. \end{cases} \tag{1.10}$$

Similarly, if (1.8) does not hold, one can easily verify that the condition (1.4) holds. In view of (1.8)–(1.10), nonminimality can thus be viewed as a type of redundancy of integral representations.

In addition to the natural connections to redundancy above, minimal representations are also fundamental in the study of stable processes. In fact, minimality can be shown to be equivalent to the following condition [11]: for any $L^\alpha(Y,\nu)$, every linear isometry $U_0 : \{f_t, t \in T\} \mapsto L^\alpha(Y,\nu)$ has a unique extension to a linear isometry $U : L^\alpha(S,\mu) \mapsto L^\alpha(Y,\nu)$. According to the Banach–Lamperti theorem, linear isometries between L^α-spaces are essentially characterized by maps corresponding to a change of variables. When stable processes are invariant, for example, stationary, self-similar or other, this allows one to relate their minimal representations to the so-called nonsingular flows. One can then decompose and classify such processes into disjoint classes and study them on the basis of the nature of underlying flows, for example, by trying to relate their ergodic properties to those of underlying flows. This approach has been taken in [9,10,12,5–7], and at present seems to be the only successful way of dealing with invariant stable processes in general.

In this work, we provide new ways of determining whether an integral representation is minimal, and also shed light on some available minimality criteria. Our conditions begin with what we call the extended nonminimal set

$$\{(s_1,s_2) \in S^2, s_1 \neq s_2 : f_t(s_1) = a\, f_t(s_2) \text{ for suitable } t \in T, \text{ and } a \neq 0\} \tag{1.11}$$

(the descriptor "extended" is used to distinguish the set from other nonminimal sets that will be introduced). We show, for example, that an integral representation is minimal if and only if the projection onto S of the extended nonminimal set, that is, the set

$$\{s \in S : \exists s' \in S\ (s' \neq s) : f_t(s) = a\, f_t(s') \text{ for suitable } t \in T, \text{ and } a \neq 0\} \tag{1.12}$$

has zero μ-measure (see Theorem 3.1). A consequence of this is that the term 'nonsingular' can be removed from (1.5)–(1.6) in Definition 1.1 (Corollary 3.1). We are also interested in the decomposition of (1.12) into

$$C_f + D_f, \tag{1.13}$$

where C_f consists of the points s, s' that cannot be related as $f_t(s) = af_t(s')$ for suitable t, and every $s \in D_f$ can be related to $s' \in C_f$ as $f_t(s) = af_t(s')$ for suitable t (Theorem 3.3). In view of the discussion following Definition 1.1, this suggests that eliminating D_f from (1.1) could lead to a representation which is minimal. We will show that this is indeed the case, though under additional assumptions in the case of uncountable index set T (Proposition 3.1). Further main results can be found in Section 3. An advantage of working with (1.11)–(1.12), as opposed to (1.5)–(1.6), is that one no longer has to deal with nonsingular maps.

The rest of the paper is organized as follows. Section 2 consists of preliminaries. As noted above, the main results are presented in Section 3. In Section 4, we illustrate our minimality criteria with several well-known examples. In Section 5, we consider the so-called mixed moving average processes which were studied in [9,5,6]. We show that they admit a minimal representation of the mixed moving average type. Existence of such representation in the general case of $\alpha \in (0, 1]$ has been an open question, raised in [5,6]. Finally, Appendix A introduces L^p spaces of functions identified under translation which are used in Section 5.

2. Projections, projective classes and other preliminaries

We shall work throughout with measurable spaces $(S, \mathcal{B}(S))$, called standard Borel spaces, where S can be thought of as a Borel subset of a Polish space. The σ-field $\mathcal{B}(S)$ of Borel sets is defined as

$$\mathcal{B}(S) = \sigma\{A : A \subset S \text{ is open}\}.$$

When equipped with a σ-finite measure μ, $(S, \mathcal{B}(S), \mu)$ is called a standard Lebesgue space. We let

$$\overline{\mathcal{B}}(S) := \overline{\mathcal{B}}_{\mu}(S)$$

be the completion σ-field of $\mathcal{B}(S)$ under μ.

As indicated in Section 1, we shall often use the notion of projection. If S_1 and S_2 are two spaces, the projection of a set $E \in S_1 \times S_2$ onto S_1 is defined as

$$\mathrm{proj}_{S_1} E = \{s_1 \in S_1 : \exists s_2 \in S_2 : (s_1, s_2) \in E\}.$$

The projection $\mathrm{proj}_{S_2} E$ is defined in a similar way. When $S_1 = S_2 = S$, proj_S is understood as the projection onto the first variable.

When $(S_1, \mathcal{B}(S_1))$ and $(S_2, \mathcal{B}(S_2))$ are two standard Borel spaces, and $E \in \mathcal{B}(S_1) \times \mathcal{B}(S_2)$, it is well known that $\mathrm{proj}_{S_1} E$ is not necessarily in $\mathcal{B}(S_1)$. This important fact has essentially given rise to the field of the so-called (classical) descriptive set theory. See, for example, the monographs of Kechris [3], Kuratowski [4] and Srivastava [14]. Since we will work with projections and, more generally, images of maps, it is natural to use this theory in our context.

One of the key notions in descriptive set theory is that of projective classes $\Sigma_n^1(S)$, $\Pi_n^1(S)$, $\Delta_n^1(S)$, $n \in \mathbb{N}$, on a Polish space S. They can be defined recursively in n as follows. For $n = 1$,

$$\Sigma_1^1(S) = \{B : B = \mathrm{proj}_S A \text{ for } A \in \mathcal{B}(S^2)\},$$
$$\Pi_1^1(S) = \{B^c : B \in \Sigma_1^1(S)\}, \qquad \Delta_1^1(S) = \Sigma_1^1(S) \cap \Pi_1^1(S).$$

The elements of $\Sigma_1^1(S)$ and $\Pi_1^1(S)$ are called analytic and coanalytic sets, respectively, and $\Delta_1^1(S) = \mathcal{B}(S)$. Then, recursively in n,

$$\Sigma_{n+1}^1(S) = \{B : B = \mathrm{proj}_S A \text{ for } A \in \Pi_n^1(S^2)\},$$
$$\Pi_{n+1}^1(S) = \{B^c : B \in \Sigma_{n+1}^1(S)\}, \qquad \Delta_{n+1}^1(S) = \Sigma_{n+1}^1(S) \cap \Pi_{n+1}^1(S).$$

Another, more general approach is to define the class Σ_{n+1}^1 as images of sets from the projective class Π_n^1 under Borel maps (projection is one such map). We will use this standard definition without making a specific reference to it.

Much is known about the above projective classes. For example, in the following diagram, any class is a subset of every class to the right of it:

$$\begin{array}{ccccccc} & \Sigma_1^1(S) & & \Sigma_2^1(S) & \dots & \\ \mathcal{B}(S) = \Delta_1^1(S) & & \Delta_2^1(S) & \dots & & \overline{\mathcal{B}}(S) \\ & \Pi_1^1(S) & & \Pi_2^1(S) & \dots & \end{array}$$

The classes $\Sigma_n^1(S)$ and $\Pi_n^1(S)$ are closed under countable unions and intersections, and the $\Delta_n^1(S)$ are σ-fields, and so on.

Another important idea which we will often use is that of uniformization and unformizing functions defined in [3], or measurable sections defined in [14], or measurable selections defined in [15]. We are not going to describe the many related results here. When used, we will simply refer to some of these results.

Finally, in order to make the definition (1.11) precise, we need to make additional assumptions on T. Suppose that $(T, \mathcal{B}(T), \lambda)$ is also a standard Lebesgue space, equipped with a metric d and hence the corresponding convergence, and such that $\lambda(B(t, \epsilon)) > 0$, for $\epsilon > 0, t \in T$, where $B(t, \epsilon) = \{u \in T : d(t, u) < \epsilon\}$. Typical examples we have in mind are $T = \mathbb{Z}, \lambda = \delta_{\mathbb{Z}}$, $d =$ Euclidean or $T = \mathbb{R}, \lambda =$ Lebesgue, $d =$ Euclidean.

3. Main results

It will be convenient to write the functional relation appearing in (1.11) as an equivalence relation. For $s, s' \in S$, we write

$$s \sim s' \Leftrightarrow f_t(s) = a\, f_t(s') \quad \text{a.e. } \lambda(\mathrm{d}t), \text{ for } a \neq 0. \tag{3.1}$$

The equivalence relation is used in the following definition of the first nonminimal set.

Definition 3.1 (*Extended Nonminimal Set*). The extended nonminimal set of a representation $\{f_t\}_{t \in T}$ in (1.1) is defined as

$$A_f = \{(s_1, s_2) \in S^2, s_1 \neq s_2 : s_1 \sim s_2\}. \tag{3.2}$$

As mentioned in Section 1, 'extended' is used to distinguish the corresponding set from other nonminimal sets that will be introduced below.

Observe that $A_f \in \Sigma_1^1(S^2)$, which can be seen as follows. We have $A_f = \mathrm{proj}_{S^2} \widetilde{A}_f$, where

$$\widetilde{A} = \{(s_1, s_2, a) \in S^2 \times (\mathbb{R} \setminus \{0\}), s_1 \neq s_2 : f_t(s_1) = a\, f_t(s_2) \text{ a.e. } \lambda(\mathrm{d}t)\}. \tag{3.3}$$

It is enough to note that $\widetilde{A}_f \in \mathcal{B}(S^2 \times \mathbb{R})$ since $\widetilde{A}_f = \{(s_1, s_2, a) : h(s_1, s_2, a) = 0\} \cap \{s_1 \neq s_2\}$ with a $\mathcal{B}(S^2 \times \mathbb{R})$-measurable function $h(s_1, s_2, a) = \int_T 1_{\{f_t(s_1)=af_t(s_2)\}}\lambda(\mathrm{d}t)$ (Theorem A in [1], p. 143).

Some other basic properties of extended nonminimal sets are (just restating the properties of equivalence):

(i) (Symmetry)

$$(s_1, s_2) \in A_f \Rightarrow (s_2, s_1) \in A_f, \tag{3.4}$$

(ii) (Transitivity)

$$(s_1, s_2), (s_2, s_3) \in A_f \Rightarrow (s_2, s_3) \in A_f. \tag{3.5}$$

In view of (3.4), the latter property can also be restated as (s_1, s_2), $(s_3, s_2) \in A_f$ or (s_2, s_1), $(s_2, s_3) \in A_f \Rightarrow (s_2, s_3) \in A_f$. The two points (s_1, s_2), (s_3, s_2) or (s_2, s_1), (s_2, s_3) are easier to locate as they belong to the same cross section. The combined properties (i) and (ii) are quite stringent on the type of shapes of A_f. For example, if $S \subset \mathbb{R}$ and a line $\{s\} \times (a, b) \in A_f$, then the square $(a, b) \times (a, b) \in A_f$ or, if $S = (0, 1)$ and $\{(t, t^2) : 0 < t < 1\} \in A_f$, then $\{(t, t^{2^k}) : 0 < t < 1\} \in A_f$ for any $k \in \mathbb{Z}$. Another interesting question is whether any set A satisfying properties (i) and (ii) can be viewed as the extended nonminimal set of an integral representation. We do not pursue this question here.

Note that the definition of the extended nonminimal set involves λ-a.e. $t \in T$. As in Definition 1.1, however, we also need to make statements for every t. (This is only relevant in the case of uncountable T.) To be able to do so, we shall use the following additional assumption on $\{f_t\}_{t\in T}$.

Assumption (A). Suppose that, for every $t_0 \in T$ and $T^* \subset T$ with $\lambda(T \setminus T^*) = 0$, there are $t_k \in T^*$ such that $f_{t_k} \to f_{t_0}$ μ-a.e.

This assumption holds, for example, for stable processes continuous in probability. Stationary, stationary increments and many other stable processes are continuous in probability (see, for example, Lemma 4.3 in [7]).

(Non)minimal representations are related to extended nonminimal sets through the following result. Note that

$$\mathrm{proj}_S A_f = \{s \in S : \exists s' \neq s : s' \sim s\}. \tag{3.6}$$

Since $\mathrm{proj}_S A_f = \mathrm{proj}_S \widetilde{A}_f$ with $\widetilde{A}_f$ defined in (3.3), we have $\mathrm{proj}_S A_f \in \Sigma_1^1(S)$.

Theorem 3.1. *Under Assumption* (A)*, an integral representation* $\{f_t\}_{t\in T}$ *is minimal if and only if* $\mu(\mathrm{proj}_S A_f) = 0$*, where* A_f *is the associated extended nonminimal set.*

Proof. If $\{f_t\}_{t\in T}$ is nonminimal, there is a nonsingular map $\phi : S \mapsto S$ such that, a.e. $\mu(\mathrm{d}s)$,

$$f_t(s) = a(s) f_t(\phi(s)) \quad \text{a.e. } \lambda(\mathrm{d}t)$$

with $a(s) \neq 0$, and $\phi(s) \neq s$ on a set E with $\mu(E) > 0$. This implies that $E \subset \mathrm{proj}_S A_f$ and, since $\mu(E) > 0$, that $\mu(\mathrm{proj}_S A_f) > 0$.

Suppose now that $\mu(\mathrm{proj}_S A_f) > 0$. Then, there is $C \subset \mathrm{proj}_S A_f$ such that $\mu(C) > 0$ and $\mu(S \setminus C) > 0$. By Theorem 5.5.2 in [14], there is a $\sigma(\Sigma_1^1)$-measurable map $g_0 : C \to S$ such that $(s, g_0(s)) \in A_f$ for $s \in C$. By eliminating from C a set of zero measure, we may suppose without loss of generality that g_0 is Borel measurable. We may also suppose without loss of generality that $\mu(g_0(C)\triangle C) > 0$. (If $g_0(A) = A$ μ-a.e. for any Borel set $A \subset C$ with $\mu(A) > 0$, it would follow that $g_0 = Id$ μ-a.e. on C.) Consider now a map on S defined by

$$g(s) = \begin{cases} g_0(s), & \text{if } s \in C, \\ s, & \text{if } s \notin C. \end{cases}$$

Then, $g^{-1}(\mathcal{B}(S)) \neq \mathcal{B}(S) \mod \mu$, since $\mu(g_0(C)\triangle C) > 0$. By Proposition 5.1 in [11], there is a nonsingular map $\phi : S \mapsto S$ such that $\phi \neq Id \mu$-a.e. and

$$g(s) = g(\phi(s)) \quad \text{a.e. } \mu(\mathrm{d}s).$$

Observe next that $(s, \phi(s)) \in A_f^*$ a.e. $\mu(\mathrm{d}s)$, where $A_f^* = A_f \cup \{(s, s') : s = s'\}$. This can be seen by considering each of the cases (i) $s \in C, \phi(s) \in C$, (ii) $s \notin C, \phi(s) \notin C$, (iii) $s \in C, \phi(s) \notin C$, and (iv) $s \notin C, \phi(s) \in C$. For example, in the case (i), we have $g_0(s) = g_0(\phi(s))$ and hence, by the definitions of g_0 and C, $f_t(s) = af_t(g_0(s)) = af_t(g_0(\phi(s))) = \widetilde{a} f_t(\phi(s))$ a.e. $\lambda(\mathrm{d}t)$ for some $a, \widetilde{a} \neq 0$, which implies that $(s, \phi(s)) \in A_f^*$. In the case (iii), $g_0(s) = \phi(s)$ and hence $(s, \phi(s)) \in A_f \subset A_f^*$ by the definition of g_0. The fact $(s, \phi(s)) \in A_f^*$ a.e. $\mu(\mathrm{d}s)$ implies that, a.e. $\mu(\mathrm{d}s)$,

$$f_t(s) = a(s) f_t(\phi(s)) \quad \text{a.e. } \lambda(\mathrm{d}t).$$

The "a.e. $\mu(\mathrm{d}s)$" and "a.e. $\lambda(\mathrm{d}t)$" can be interchanged above. Using Assumption (A) and since ϕ is nonsingular, we conclude that, for $t \in T$, $f_t(s) = a(s) f_t(\phi(s))$ a.e. $\mu(\mathrm{d}s)$, which implies nonminimality. □

Theorem 3.1 has the following important and useful consequence.

Corollary 3.1. *Under Assumption* (A), *an integral representation* $\{f_t\}_{t\in T}$ *is minimal if and only if, for every* $\mathcal{B}(S)$*-measurable map* $\phi : S \mapsto S$ *and* $h : S \mapsto \mathbb{R} \setminus \{0\}$ *such that, for each* $t \in T$, $f_t(s) = h(s) f_t(\phi(s))$ *a.e.* $\mu(\mathrm{d}s)$, *it follows that* $\phi(s) = s$ *a.e.* $\mu(\mathrm{d}s)$.

Proof. Consider $\mathcal{B}(S)$-measurable maps ϕ and h as in the statement of the corollary. Arguing by contradiction, suppose $\phi \neq Id$ μ-a.e., that is, $\phi(s) \neq s$ for $s \in B$ with $\mu(B) > 0$. Since, for a.e. $\mu(\mathrm{d}s)$, $f_t(s) = h(s) f_t(\phi(s))$ a.e. $\lambda(\mathrm{d}t)$, we obtain that $B \subset \mathrm{proj}_S A_f$ μ-a.e. By Theorem 3.1, the representation is nonminimal. The converse of the corollary is elementary. □

Remark. Though it may appear surprising, Corollary 3.1 is quite intuitive. A simple example of a singular ϕ is where a set B with $\mu(B) > 0$ is mapped to a single point of zero μ-measure. By the transitivity argument, $f_t(s) = af_t(s')$ for $s, s' \in B$ and a new nonsingular map ϕ can be defined satisfying (1.6).

As extended nonminimal sets A_f are symmetric, a pair of points s and s' is included twice in A_f as (s, s') and (s', s). To consider it only once, we give the following definition. We write $E' = \{(s_1, s_2) : (s_2, s_1) \in E\}$ for $E \subset S_1 \times S_2$.

Definition 3.2 (*Reduced Nonminimal Set*). A reduced nonminimal set is defined as any B_f satisfying

$$A_f = B_f + B_f'. \tag{3.7}$$

Existence of B_f will be suggested by the example at hand. In general, it exists, for example, by mapping S into $(0, 1)$ and taking the corresponding B_f to be A_f below diagonal. For reduced nonminimal sets, we still have a result analogous to Theorem 3.1. We will not use this result in examples but have it for mathematical completeness because the decomposition (3.7) is natural to consider.

Theorem 3.2. *Under Assumption* (A), *the following are equivalent:* (a) *an integral representation* $\{f_t\}_{t\in T}$ *is minimal,* (b) $\mu(\mathrm{proj}_S B_f) = 0$, *and* (c) $\mu(\mathrm{proj}_S B'_f) = 0$, *where* B_f *is the associated reduced nonminimal set.*

Proof. It is enough to show the equivalence of (b) and (c). By symmetry, it is also enough to prove that $\mu(\mathrm{proj}_S B_f) > 0$ implies $\mu(\mathrm{proj}_S B'_f) > 0$. We do so by adapting the proof of Theorem 3.1 as follows. Let $C \subset \mathrm{proj}_S B_f$ be a Borel set such that $\mu(C) > 0$ and $\mu(S \setminus C) > 0$. Arguing as in the proof of Theorem 3.1, there is a Borel measurable map $g_0 : C \mapsto B_f \subset A_f$ such that $(s, g_0(s)) \in B_f$ for any $s \in C$, and $\mu(g_0(C)\triangle C) > 0$. Define again

$$g(s) = \begin{cases} g_0(s), & \text{if } s \in C, \\ s, & \text{if } s \notin C \end{cases}$$

and note that $g^{-1}(\mathcal{B}(S)) \neq \mathcal{B}(S) \bmod \mu$. By Proposition 5.1 in [11], there is a map $\phi : S \mapsto S$ such that it is onto, one-to-one, nonsingular with nonsingular ϕ^{-1}, and

$$g(s) = g(\phi(s)) \quad \text{a.e. } \mu(\mathrm{d}s). \tag{3.8}$$

A point in C is mapped by ϕ to either C itself or its complement C^c. Two situations are then possible: (i) $\exists F \subset C$ with $\mu(F) > 0$ such that $\phi(F) \cap C = \emptyset$, or (ii) $\exists F \subset C$ with $\mu(F) > 0$ such that $\phi(F) \subset C$. In the case (i), by (3.8), $g_0(s) = \phi(s)$ a.e. $\mu(\mathrm{d}s)$ on F. Hence, $(s, \phi(s)) \in B_f$ or $(\phi(s), s) \in B'_f$ a.e. $\mu(\mathrm{d}s)$ on F. This implies that $\phi(F) \subset \mathrm{proj}_S B'_f$ μ-a.e. Since ϕ is nonsingular, we have $\mu(\phi(F)) > 0$, implying $\mu(\mathrm{proj}_S B'_f) > 0$. In the case (ii), by (3.8), $g_0(s) = g_0(\phi(s))$ a.e. $\mu(\mathrm{d}s)$ on F and, by using transitivity argument, $(s, \phi(s)) \in B_f$ or $(\phi(s), s) \in B_f$ a.e. $\mu(\mathrm{d}s)$ on F. Then, there is $F_0 \subset F$ with $\mu(F_0) > 0$ such that either $(s, \phi(s)) \in B_f$ a.e. $\mu(\mathrm{d}s)$ on F_0 or $(\phi(s), s) \in B_f$ a.e. $\mu(\mathrm{d}s)$ on F_0. Hence, either $\phi(F_0) \subset \mathrm{proj}_S B'_f$ μ-a.e. or $F_0 \subset \mathrm{proj}_S B'_f$ μ-a.e. Both of these cases lead to $\mu(\mathrm{proj}_S B'_f) > 0$. □

Another type of decomposition is

$$\mathrm{proj}_S A_f = C_f + D_f, \tag{3.9}$$

where the sets C_f and D_f are such that

(i) for any $s, s' \in C_f (s \neq s')$, $s \not\sim s'$,
(ii) for any $s \in D_f$, $\exists s' \in C_f : s \sim s'$.

We may also expect that there are maps $g : D_f \mapsto C_f$ and $a : D_f \mapsto \mathbb{R} \setminus \{0\}$ such that, for $s \in D_f$,

$$f_t(s) = a(s) f_t(g(s)) \quad \text{a.e. } \lambda(\mathrm{d}t). \tag{3.10}$$

Definition 3.3 (*Nonminimal Set*). The set D_f in the decomposition (3.9) is called a nonminimal set of a representation $\{f_t\}_{t\in T}$.

The sets D_f and C_f in the decomposition (3.9) are not unique in general. See Example 4.2 in Section 4. The decomposition (3.9) is interesting in the sense that, by construction, eliminating D_f from the underlying space S leads naturally to an integral representation which is minimal.

Theorem 3.3. *Given the decomposition (3.9) with $\overline{\mathcal{B}}(S)$-measurable sets C_f and D_f, and $\overline{\mathcal{B}}(S)$-measurable maps g and a in (3.10), we have*

$$\left\{\int_S f_t(s)M(\mathrm{d}s)\right\}_{t\in T^*} \stackrel{d}{=} \left\{\int_{S\setminus D_f} f_t(s)\widetilde{M}(\mathrm{d}s)\right\}_{t\in T^*}, \tag{3.11}$$

where $T^* \subset T$ with $\lambda(T \setminus T^*) = 0$, and the $S\alpha S$ random measure $\widetilde{M}$ has the control measure

$$\widetilde{\mu} = \begin{cases} \mu, & \text{on } S \setminus \mathrm{proj}_S A_f, \\ \mu + \mu_1, & \text{on } C_f, \end{cases} \tag{3.12}$$

with $\mu_1 = \mu_0 \circ g^{-1}$ and $\mu_0(\mathrm{d}s) = |a(s)|^\alpha \mu(\mathrm{d}s)$. The integral representation on the right-hand side of (3.11) is minimal.

Moreover, when T is uncountable, if $\{\int_S f_t(s)M(\mathrm{d}s)\}_{t\in T}$ and $\{\int_{S\setminus D_F} f_t(s)\widetilde{M}(\mathrm{d}s)\}_{t\in T}$ are continuous in probability, then (3.11) holds with $t \in T$, and the integral representation on the right-hand side of (3.11) with $t \in T$ is minimal as well.

Proof. By (3.10) and Fubini's theorem, for a.e. $\lambda(\mathrm{d}t)$, $f_t(s) = a(s)f_t(g(s))$ a.e. $\mu(\mathrm{d}s)$ on D_f. Then, by making a change of variables below, for a.e. $\lambda(\mathrm{d}t)$,

$$\begin{aligned}\int_S f_t(s)M(\mathrm{d}s) &= \int_{S\setminus D_f} f_t(s)M(\mathrm{d}s) + \int_{D_f} f_t(s)M(\mathrm{d}s) \\ &= \int_{S\setminus D_f} f_t(s)M(\mathrm{d}s) + \int_{D_f} a(s)f_t(g(s))M(\mathrm{d}s) \\ &\stackrel{d}{=} \int_{S\setminus D_f} f_t(s)M(\mathrm{d}s) + \int_{C_f} f_t(s)M_1(\mathrm{d}s),\end{aligned}$$

where M_1 is independent of M, and has the control measure $\mu_1 = \mu_0 \circ g^{-1}$ with $\mu_0(\mathrm{d}s) = |a(s)|^\alpha \mu(\mathrm{d}s)$. The result (3.11)–(3.12) is now easily deduced. The representation on the right-hand side of (3.11) is minimal by construction. The last statement of the theorem can also be easily obtained. □

When T is uncountable, we were able to obtain a minimal representation in Theorem 3.3 under the assumption of continuity in probability. In practice, we often deal with stationary, stationary increment, self-similar or other invariant processes which are continuous in probability. Hence, Theorem 3.3 is useful even in the case of uncountable T (see Example 4.2). Note also that we supposed measurable C_f, D_f, g, a in Theorem 3.3. This is the case with most of specific examples, where there are frequently obvious candidates for C_f, D_f, g, a. By Theorem 3.3, eliminating D_f from the representations of these examples leads to minimal representations. The following result also shows that $\overline{\mathcal{B}}(S)$-measurable C_f, D_f, g, a can be selected in general, though under additional assumptions in the case of uncountable set T.

Assumption (B). Suppose that, for every $s \in S$,

$$\lambda\{t : f_t(s) \neq 0\} > 0.$$

Assumption (C). Suppose that there are $p > 0$ and a measure γ on $\mathcal{B}(T)$, equivalent to λ, such that, for every $s \in S$,

$$\int_T |f_t(s)|^p \gamma(\mathrm{d}t) < \infty.$$

Supposing Assumption (A) and (1.3), for example, Assumption (B) is not restrictive. Indeed, consider the set $B = \{s : f_t(s) = 0 \text{ a.e. } \lambda(dt)\}$. If $\mu(B) > 0$, then, by Fubini's theorem, a.e. $\lambda(dt)$,

$$f_t(s) = 0 \quad \text{a.e. } \mu(ds), \text{ for } s \in B. \tag{3.13}$$

By Assumption (A), it is also true that the relation (3.13) holds for every $t \in T$. But this contradicts (1.3) and hence $\mu(B) = 0$. In particular, without loss of generality, the set B can be eliminated from S in the representation (1.1). Similarly, Assumption (C) is not restrictive, for example, for stationary or self-similar processes with $p = \alpha$. Indeed, considering the set $B = \{s : \int_T |f_t(s)|^\alpha \gamma(dt) < \infty\}$ and using the fact $\int_S \int_T |f_t(s)|^\alpha \mu(ds)\gamma(dt) < \infty$ for suitable γ, one can show that $\mu(B) = 0$.

Proposition 3.1. *One can choose*

(i) $C_f \in \Delta^1_2(S)$ *and* $D_f \in \Sigma^1_1(S)$, *when* T *is countable,*
(ii) $C_f \in \Sigma^1_2(S)$ *and* $D_f \in \Pi^1_2(S)$, *when* T *is uncountable and under Assumptions* (B) *and* (C),

in the decomposition (3.9), *and* $\overline{\mathcal{B}}(S)$*-measurable maps* g *and* a *in* (3.10).

Remark. Existence of minimal representations for general stable processes was established by Hardin [2] through suitable measure algebra isomorphisms (Theorem 1.1 in [2]). Proposition 3.1 and Theorem 3.3 provide an alternative, more direct and revealing way to show existence, though at the expense of making additional assumptions on processes in the case of uncountable T.

Proof. The idea of the proof is simple. If $\mathrm{proj}_S A_f$ were the set

$$\{s : \exists s' \neq s : f_t(s) = f_t(s') \text{ for all } t\}$$

(that is, $a = 1$ in nonminimal sets), then one can consider the set

$$\{\{f_t(s)\}_{t \in T} : s \in \mathrm{proj}_S A_f\} \subset \mathbb{R}^T$$

consisting of functions on T, and select measurably only one s for each point (function) in this range. This would define the set C_f, and hence D_f as its complement in $\mathrm{proj}_S A_f$. The arguments below are also best understood geometrically.

(i) For $s \in S$, consider the set $U(s) = \{t : f_t(s) \neq 0\}$ and a $\mathcal{B}(S)$-measurable map $F : S \mapsto \mathbb{R}^{T^2}$ defined by

$$F(s) = \left\{ \frac{f_{t_1}(s)}{f_{t_2}(s)} 1_{\{(t_1,t_2) \in U(s)^2\}} \right\}_{(t_1,t_2) \in T^2}.$$

Observe that

$$F(s) = F(s') \Leftrightarrow s \sim s'.$$

By Exercise 5.5.6 in [14], there is a set $C \in \Pi^1_1(S)$ such that $F|_C$ is one-to-one and $F(C) = F(S)$. One can now take

$$C_f = C \cap \mathrm{proj}_S A_f, \qquad D_f = \mathrm{proj}_S A_f \setminus C.$$

By Proposition 4.1.9 in [14], $C_f \in \Delta^1_2(S)$ and, by Proposition 4.1.2 in [14], $D_f \in \Sigma^1_1(S)$.

To show that there are $\overline{\mathcal{B}}(S)$-measurable maps g and a in (3.10), consider the set

$$B = (D_f \times C_f \times (\mathbb{R} \setminus \{0\})) \cap \{(s, s', a) : f_t(s) = a\, f_t(s') \text{ for all } t \in T\}.$$

Since $D_f \in \Sigma_1^1(S)$, $C_f \in \Delta_2^1(S)$ and $\{(s,s',a) : f_t(s) = a f_t(s') \text{ for all } t \in T\} \in \mathcal{B}(S^2 \times \mathbb{R})$, Property 1 in [4], p. 454, and basic inclusions among projective classes imply that $B \in \Delta_2^1(S^2 \times \mathbb{R})$. Then, by Exercise 38.14, i, in [3], there is a $\overline{\mathcal{B}}(S)$-measurable map $K : \text{proj}_S B \mapsto S \times \mathbb{R}$ such that $(s, K(s)) \in B$ for $s \in \text{proj}_S B$. It remains to observe that $\text{proj}_S B = D_f$.

(ii) The proof of (i) cannot be applied here because of the issues related to "almost everywhere $\lambda(\mathrm{d}t)$" and the space $\mathbb{R}^{T^2}$. Observe, however, that by Assumption (B),

$$s \sim s' \Leftrightarrow f_{t_1}(s) f_{t_2}(s') = f_{t_1}(s') f_{t_2}(s) \quad \text{a.e. } \lambda(\mathrm{d}t_1)\lambda(\mathrm{d}t_2).$$

In particular, $A_f = \{(s,s') : s \sim s'\} \in \mathcal{B}(S^2)$. Define now a $\mathcal{B}(S)$-measurable map $F_2 : A_f \mapsto L^p(T^2, \gamma^2)$ by

$$F_2(s,s') = \{f_{t_1}(s) f_{t_2}(s')\}_{(t_1,t_2)\in T^2}.$$

By Assumption (B) and since F_2 is defined on A_f, $F_2(s,s') = F_2(\bar{s},\bar{s}')$ if and only if $s \sim s' \sim \bar{s} \sim \bar{s}'$. Arguing as in part (i) and since $L^p(T^2,\gamma^2)$ is a Polish space, we can decompose A_f as

$$A_f = C^* + D^*.$$

Here, for nonequal $(s,s'), (\bar{s},\bar{s}') \in C^*$, $F_2(s,s') \neq F_2(\bar{s},\bar{s}')$ in $L^p(T^2,\gamma^2)$. In particular, $s \not\sim \bar{s}$ (we always have $s \sim s'$ and $\bar{s} \sim \bar{s}'$). Moreover, for $(\bar{s},\bar{s}') \in D^*$, there is $(s,s') \in C^*$ such that $F_2(\bar{s},\bar{s}') = F_2(s,s')$ in $L^p(T^2,\gamma^2)$. In particular, $\bar{s} \sim s$. We can also take $C^* \in \Pi_1^1(S^2)$.

Observe that, by the discussion above,

$$C_f = \text{proj}_S C^*, \qquad D_f = (\text{proj}_S A_f) \setminus C_f$$

satisfy the desired conditions. (To see that D_f has the desired property, use the fact $D_f \subset \text{proj}_S D^*$.) Observe also that $C_f \in \Sigma_2^1(S)$ and $D_f \in \Pi_2^1(S)$. Existence of $\overline{\mathcal{B}}(S)$-measurable maps g and a in (3.10) can be proved as in part (i) (using a more general Exercise 39.13, ii), in [3]. □

Remark. In this paper, we work with real-valued, $S\alpha S$ processes. Complex-valued, $S\alpha S$ processes could be considered as well. They are defined by (1.1) with complex-valued f_t, $t \in T$, and a rotationally invariant, complex-valued, $S\alpha S$ random measure M. Minimal representations for these processes are defined in the same way. The proofs of this section obviously extend to the case where f_t, $t \in T$, are complex-valued.

4. Examples

We apply here the results of Section 3 to several well-known stable processes.

Example 4.1 (*Moving Average*). A (stationary) moving average process has an integral representation (1.1) with either

$$T = \mathbb{Z},\ S = \mathbb{Z},\ \mu(\mathrm{d}s) = \delta_{\mathbb{Z}}(\mathrm{d}s) \quad \text{or} \quad T = \mathbb{R},\ S = \mathbb{R},\ \mu(\mathrm{d}s) = \mathrm{d}s \tag{4.1}$$

and

$$f_t(s) = f(t+s). \tag{4.2}$$

Letting $\lambda(\mathrm{d}t) = \delta_{\mathbb{Z}}(\mathrm{d}t)$ when $T = \mathbb{Z}$, and $\lambda(\mathrm{d}t) = \mathrm{d}t$ when $T = \mathbb{R}$, we have

$$\begin{aligned} A_f &= \{(s_1, s_2), s_1 \neq s_2 : f(t + s_1) = af(t + s_2) \text{ a.e. } \lambda(\mathrm{d}t),\ a \neq 0\} \\ &= \{(s_1, s_2), s_1 \neq s_2 : f(u) = af(u + s_2 - s_1) \text{ a.e. } \lambda(\mathrm{d}u),\ a \neq 0\} \end{aligned}$$

and

$$\mathrm{proj}_S A_f = \begin{cases} S, & \text{if } \exists u_0 \neq 0 : f(u) = af(u + u_0) \text{ a.e. } \lambda(\mathrm{d}u)\ (a \neq 0), \\ \emptyset, & \text{otherwise.} \end{cases}$$

Since $f \in L^\alpha(S, \mu)$, such u_0 does not exist and we conclude that $\mathrm{proj}_S A_f = \emptyset$. Thus $\mu(\mathrm{proj}_S A_f) = 0$ and the integral representation (4.1)–(4.2) is minimal.

Example 4.2 (*Real Part of a Harmonizable Process*). The real part of a harmonizable process (see Example 2.5 in [10]) has an integral representation (1.1) with either

$$T = \mathbb{Z},\ S = [0, 2\pi) \times [0, 2\pi), \quad \text{or} \quad T = \mathbb{R},\ S = [0, 2\pi) \times \mathbb{R}, \tag{4.3}$$

and

$$\mu(\mathrm{d}u, \mathrm{d}w) = \mathrm{d}u\widetilde{\mu}(\mathrm{d}w), \qquad f_t(u, w) = \cos(u + tw), \tag{4.4}$$

where $\widetilde{\mu}(\mathrm{d}w)$ is a finite measure.

In the case $T = \mathbb{R}$, for example,

$$\begin{aligned} \mathrm{proj}_S A_f &= \{(u, w) : \exists (u', w') \neq (u, w) : \cos(u + tw) = a\cos(u' + tw') \text{ a.e. } \mathrm{d}t,\ a \neq 0\} \\ &= [0, 2\pi) \times \mathbb{R}, \end{aligned}$$

since, for $(u, w) \in [0, \pi) \times \mathbb{R}$, we can take $(u', w') = (\pi + u, w)$ and, for $(u, w) \in [\pi, 2\pi) \times \mathbb{R}$, we can take $(u', w') = (u - \pi, w)$. Hence, $\mu(\mathrm{proj}_S A_f) > 0$ and the representation (4.3)–(4.4) is not minimal.

We can decompose $\mathrm{proj}_S A_f$ as (3.9) with

$$C_f = [0, \pi) \times \mathbb{R}, \quad D_f = [\pi, 2\pi) \times \mathbb{R}$$

and maps $g : D_f \mapsto C_f$ and $a : D_f \mapsto \mathbb{R} \setminus \{0\}$ in (3.10) defined by $g(u, w) = (u - \pi, w)$ and $a(u, w) = -1$. With these choices of g and a, the measure $\widetilde{\mu}$ in (3.12) is given by $\widetilde{\mu} = 2\mu$ on C_f. Since, being stationary processes, $\{\int_S f_t(s)M(\mathrm{d}s)\}_{t\in\mathbb{R}}$ and $\{\int_{C_f} f_t(s)\widetilde{M}(\mathrm{d}s)\}_{t\in\mathbb{R}}$ are continuous in probability, we conclude from Theorem 3.3 that

$$\int_{[0,\pi)\times\mathbb{R}} \cos(u + tw)\widetilde{M}(\mathrm{d}u, \mathrm{d}w),$$

where $\widetilde{M}$ has the control measure $\widetilde{\mu}$, is a minimal representation for a real part of a harmonizable process in the case $T = \mathbb{R}$.

Observe also that the sets C_f and D_f could be taken, for example, as $C_f = [\pi, 2\pi) \times \mathbb{R}$, $D_f = [0, \pi) \times \mathbb{R}$, with the maps g and a in (3.10) defined by $g(u, w) = (u + \pi, w)$ and $a(u, w) = -1$. Thus, as noted following Definition 3.3, these sets are not unique.

5. Minimal representations of mixed moving averages

Mixed moving averages can be defined in the context of stationary or stationary increment processes. We do so for stationary increments (and also only in continuous time) because this case is technically more difficult and since we address a question raised in [5,6]. Analogous results can be obtained in the stationary case.

Definition 5.1 (*Mixed Moving Average*). A $S\alpha S$ process X is called a (stationary increment) mixed moving average if it has an integral representation (1.1) with

$$T = \mathbb{R},\ S = X \times \mathbb{R},\ s = (x, u),\ \mu(\mathrm{d}x, \mathrm{d}u) = \nu(\mathrm{d}x)\mathrm{d}u \tag{5.1}$$

and

$$f_t(x, u) = G(x, t + u) - G(x, u), \tag{5.2}$$

where $G : X \times \mathbb{R} \mapsto \mathbb{R}$ is a Borel measurable function, and $(X, \mathcal{B}(X), \nu)$ is a standard Lebesgue space. It is supposed that f_t, $t \in \mathbb{R}$, has full support in the sense of (1.3).

We make below several nonrestrictive assumptions stated in the following proposition. The notation *Leb* stands for the Lebesgue measure on $\mathbb{R}$.

Proposition 5.1. *For a mixed moving average (5.1)–(5.2), there is an equivalent representation satisfying:*

(a) *for every* $x \in X$,

$$\int_{\mathbb{R}} |G(x, 1 + u) - G(x, u)|^\alpha \mathrm{d}u < \infty, \tag{5.3}$$

(b) *for every* $x \in X$,

$$Leb^2\{(u, v) : G(x, v + u) - G(x, u) \neq 0\} > 0, \tag{5.4}$$

(c) *there is a measure* γ *on* $\mathcal{B}(\mathbb{R})$, *equivalent to Leb, such that, for every* $x \in X$,

$$\int_{\mathbb{R}^2} |G(x, v + u) - G(x, u)|^\alpha \mathrm{d}u\gamma(\mathrm{d}v) < \infty. \tag{5.5}$$

Proof. If (a) is not satisfied, we can eliminate from X the Borel set $\{x : \int_{\mathbb{R}} |G(x, 1 + u) - G(x, u)|^\alpha \mathrm{d}u < \infty\}$ of zero ν-measure. To show that (b) is not restrictive, let $B = \{x : G(x, v + u) - G(x, u) = 0 \text{ a.e. } \mathrm{d}u\mathrm{d}v\}$ and suppose that $\nu(B) > 0$. Then, a.e. $\mathrm{d}t$,

$$G(x, t + u) - G(x, u) = 0 \quad \text{a.e. } \mathrm{d}u,$$

a.e. $\nu(\mathrm{d}x)$ for $x \in B$. Since stationary increment processes are continuous in probability, "a.e. $\mathrm{d}t$" can be replaced by "for all t" above. But this contradicts the fact that $\mathrm{supp}\{G(x, t + u) - G(x, u),\ t \in \mathbb{R}\} = X \times \mathbb{R}$ a.e. Thus, $\nu(B) = 0$ and B can be eliminated from the underlying space X. For part (c), by continuity in probability, the function $K(v) = \int_{X\times\mathbb{R}} |G(x, v + u) - G(x, u)|^\alpha \nu(\mathrm{d}x)\mathrm{d}u$ is continuous at $v = 0$ and hence, since integration is over $\mathrm{d}u$, $K(v) \le C\,|v|$, for all $v \in \mathbb{R}$. Then, there is a measure γ, equivalent to *Leb*, such that $\int_{\mathbb{R}} K(v)\gamma(\mathrm{d}v) < \infty$. In particular, the set $\{x : \int_{\mathbb{R}^2} |G(x, v + u) - G(x, u)|^\alpha \mathrm{d}u\gamma(\mathrm{d}v) = \infty\}$ has zero ν-measure and can be eliminated from X. □

In the next result, we characterize minimal representation and extended nonminimal sets of mixed moving averages.

Proposition 5.2. *For an integral representation of a mixed moving average given by (5.1)–(5.2) and satisfying (5.3)–(5.4), we have*

$$\mathrm{proj}_{X\times\mathbb{R}} A_f = (\mathrm{proj}_X A_G) \times \mathbb{R}, \tag{5.6}$$

where

$$A_G = \{(x, x') \in X^2, x \neq x' : x \sim x'\} \tag{5.7}$$

and, for some $a \neq 0, b, c \in \mathbb{R}$,

$$x \sim x' \Leftrightarrow G(x, u) = aG(x', u + b) + c \quad \text{a.e. } du, \tag{5.8}$$

$$\Leftrightarrow G(x, v + u) - G(x, u) = a(G(x', v + u + b) - G(x', u + b)) \quad \text{a.e. } du dv, \tag{5.9}$$

$$\Leftrightarrow (G(x, v + u) - G(x, u)) \left(G(x', y + z + b) - G(x', z + b)\right)$$
$$= (G(x, y + z) - G(x, z)) \left(G(x', v + u + b) - G(x', u + b)\right) \quad \text{a.e. } du dv dy dz. \tag{5.10}$$

In particular, the representation is minimal if and only if $\nu(\text{proj}_X A_G) = 0$. *We also have* $A_G \in \mathcal{B}(X^2)$.

Proof. If $(x, u) \in \text{proj}_{X\times\mathbb{R}} A_f$, there is $(x', u') \in X \times \mathbb{R}$, $(x', u') \neq (x, u)$ such that

$$G(x, t + u) - G(x, u) = a(G(x', t + u') - G(x', u'))$$

a.e. dt, for $a \neq 0$, or

$$G(x, t + u) = aG(x', t + u') + c$$

a.e. dt, or by making a change of variables $w = t + u$,

$$G(x, w) = aG(x', w + (u' - u)) + c = aG(x', w + b) + c$$

a.e. dw. We may have $x = x'$ only when $u \neq u'$ or $b \neq 0$. But then, with $x = x'$,

$$G(x, 1 + w) - G(x, w) = a(G(x, 1 + w + b) - G(x, w + b))$$

a.e. dw, which leads to contradiction, since the function $G(x, 1 + w) - G(x, w)$ belongs to $L^\alpha(\mathbb{R})$ for fixed x by the assumption (5.3). Thus $x \neq x'$ and $(x, x') \in A_G$. This shows that $\text{proj}_{X\times\mathbb{R}} A_f \subset (\text{proj}_X A_G) \times \mathbb{R}$, where A_G is defined by using (5.8). The converse can be shown in a similar way. The relation (5.9) is trivial, and the relation (5.10) can be shown by using the assumption (5.4).

To show that $A_G \in \mathcal{B}(X^2)$, set

$$F_{x,x'}(u, v, y, z) = (G(x, v + u) - G(x, u)) \left(G(x', y + z) - G(x', z)\right).$$

The condition (5.10) can be expressed as

$$F_{x,x'}(u, v, y, z) = F_{x,x'}(y, z - b, v, u + b) \quad \text{a.e. } du dv dy dz,$$

which is equivalent to

$$0 = \inf_{b\in\mathbb{R}} \int_{\mathbb{R}^4} |F_{x,x'}(u, v, y, z) = F_{x,x'}(y, z - b, v, u + b)|^\alpha du dz \gamma(dv)\gamma(dy)$$
$$= \inf_{b\in\mathbb{Q}} \int_{\mathbb{R}^4} |F_{x,x'}(u, v, y, z) = F_{x,x'}(y, z - b, v, u + b)|^\alpha du dz \gamma(dv)\gamma(dy) =: K(x, x')$$

(see the end of Appendix A). Since the function K is Borel measurable, the set A_G is Borel measurable as well. □

The structure of the extended nonminimal set in (5.6) suggests that, as in Section 3, we can obtain a minimal representation from a nonminimal one by eliminating a suitable nonminimal set from the underlying space X. This is achieved through the following two results.

Proposition 5.3. *For an integral representation of a mixed moving average given by (5.1)–(5.2) and satisfying (5.3)–(5.5), we have*

$$\mathrm{proj}_X A_G = C_G + D_G, \tag{5.11}$$

with $C_G \in \Sigma_1^1(X)$ *and* $D_G \in \Delta_2^1(X)$ *such that*

(i) *for* $x, x' \in C_G (x \neq x')$, $x \not\sim x'$,
(ii) *for* $x \in D_G$, $\exists x' \in C_G : x \sim x'$.

Moreover, there are $\overline{\mathcal{B}}(X)$*-measurable maps* $g : D_G \mapsto C_G$, $a : D_G \mapsto \mathbb{R} \setminus \{0\}$, $b, c : D_G \mapsto \mathbb{R}$ *such that, for* $x \in D_G$,

$$G(x, u) = a(x) G(g(x), u + b(x)) + c(x) \quad a.e.\ \mathrm{d}u. \tag{5.12}$$

Proof. Consider a map $F : X^2 \mapsto L^\alpha(\mathbb{R}^4, \mathrm{d}(Leb)^2 \mathrm{d}\gamma^2)$ defined by

$$F(x, x') = \{(G(x, v + u) - G(x, u))\,(G(x', y + z) - G(x', z))\}_{(u,z,v,y)\in\mathbb{R}^4}.$$

The space $L^\alpha(\mathbb{R}^4, \mathrm{d}(Leb)^2 \mathrm{d}\gamma^2)$ equipped with the metric

$$r_{2,\gamma^2}(f, g) = \inf_{(b_1,b_2)\in\mathbb{R}^2} \left\{ \int_{\mathbb{R}^2\times\mathbb{R}^2} |f(u, z, v, y) - g(u + b_1, z + b_2, v, y)|^\alpha \mathrm{d}u\mathrm{d}z\gamma(\mathrm{d}v)\gamma(\mathrm{d}y) \right\}^{\frac{1}{\alpha}\wedge 1},$$

is Polish, and F is a Borel map (see Appendix A). Using Exercise 5.5.6 in [14], there is a set $\widetilde{C} \in \Sigma_1^1(X^2)$ such that $F|_{\widetilde{C}}$ is one-to-one and $F(\widetilde{C}) = F(X^2)$ with respect to r_{2,γ^2}. Let $\widetilde{D} = X \setminus \widetilde{C}$ and we write

$$A_G = A_G \cap \widetilde{C} + A_G \cap \widetilde{D} =: C^* + D^*.$$

If $(x, x') \in D^*$, there is $(\bar{x}, \bar{x}') \in C^*$ such that $F(x, x') = F(\bar{x}, \bar{x}')$ with respect to r_{2,γ^2}. This yields

$$\begin{aligned}&(G(x, v + u) - G(x, u))\big(G(x', y + z) - G(x', z)\big)\\ &\quad = (G(\bar{x}, v + u + b) - G(\bar{x}, u + b))\big(G(\bar{x}', y + z + b') - G(\bar{x}', z + b')\big)\end{aligned}$$

a.e. $\mathrm{d}u\mathrm{d}v\mathrm{d}y\mathrm{d}z$, for some $b, b' \in \mathbb{R}$. By fixing u, v or y, z for which the terms are not zero (Assumption (B)), we have by (5.9) that $x \sim \bar{x}$ and $x' \sim \bar{x}'$. Since $C^*, D^* \subset A_G$, we have $x \sim x' \sim \bar{x} \sim \bar{x}'$. If nonequal $(x, x'), (\bar{x}, \bar{x}') \in C^*$, then $F(x, x') \neq F(\bar{x}, \bar{x}')$ with respect to r_{2,γ^2}. In particular, we can argue by contradiction that $x \not\sim \bar{x}$. This discussion implies that the sets

$$C_G = \mathrm{proj}_X C^*, \qquad D_G = (\mathrm{proj}_X A_G) \setminus C_G,$$

have the desired properties. Since $\widetilde{C} \in \Sigma_1^1(X^2), \widetilde{D} \in \Pi_1^1(X^2), A_G \in \mathcal{B}(X^2)$, we have $C_G \in \Sigma_1^1(X), D_G \in \Delta_2^1(X)$.

To show that there are $\overline{\mathcal{B}}(X)$-measurable maps satisfying (5.12), consider a set

$$\begin{aligned}B = (D_G \times C_G \times (\mathbb{R}\setminus\{0\}) \times \mathbb{R}^2) \cap \{(x, x', a, b, c) : G(x, u)\\ = aG(x', u + b) + c \text{ a.e. } \mathrm{d}u\}.\end{aligned}$$

Arguing as in the proof of Proposition 3.1, there is a $\overline{\mathcal{B}}(X)$-measurable map $K : \text{proj}_X B \mapsto C_G \times \mathbb{R} \times \mathbb{R}^2$ such that $(x, K(x)) \in B$ for $x \in \text{proj}_X B$. It remains to note that $\text{proj}_X B = D_G$. □

The following result is an immediate corollary of the previous decomposition.

Theorem 5.1. *For an integral representation of a mixed moving average given by (5.1)–(5.2) and satisfying (5.3)–(5.5), we have*

$$\left\{\int_{X\times\mathbb{R}} (G(x,t+u) - G(x,u)) M(\mathrm{d}x, \mathrm{d}u)\right\}_{t\in\mathbb{R}} \stackrel{d}{=} \left\{\int_{(X\setminus D_G)\times\mathbb{R}} (G(x,t+u) - G(x,u)) \widetilde{M}(\mathrm{d}x, \mathrm{d}u)\right\}_{t\in\mathbb{R}}, \tag{5.13}$$

where $S\alpha S$ random measure $\widetilde{M}$ has the control measure

$$\widetilde{\nu} = \begin{cases} \nu, & \text{on } X \setminus \text{proj}_X A_G, \\ \nu + \nu_1, & \text{on } C_G, \end{cases} \tag{5.14}$$

with $\nu_1 = \nu_0 \circ g^{-1}$ and $\nu_0(\mathrm{d}x) = |a(x)|^\alpha \nu(\mathrm{d}x)$. Moreover, the integral representation on the right-hand side of (5.13) is minimal.

Remark. As the assumptions (5.3)–(5.5) are nonrestrictive, Theorem 5.1 shows that a mixed moving average has a minimal representation of a mixed moving average type. Existence of such representations in the general case of $\alpha \in (0, 1]$ has been an open question, raised in [5,6].

Proof. By (5.12), for $t \in \mathbb{R}$ and $x \in D_F$,

$$G(x,t+u) - G(x,u) = a(x)\,(G(g(x), t+u+b(x)) - G(g(x), u+b(x))) \quad \text{a.e. } \mathrm{d}u.$$

The rest of the proof is analogous to that of Theorem 3.3 and is omitted. □

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Appendix A. L^p spaces of functions identified under translation

We define here several function spaces where functions are identified up to translation. Though we expect these spaces to be known, we were not able to find them in the literature. We begin with the most basic function spaces under translation equivalence and then generalize them to the spaces that are used in the paper. We provide proofs only in the basic case for simplicity.

Let $p > 0$ and $d \in \mathbb{N}$. For $f, g \in L^p(\mathbb{R}^d)$, define

$$r(f,g) = \inf_{b\in\mathbb{R}^d} \left\{\int_{\mathbb{R}^d} |f(u) - g(u+b)|^p \mathrm{d}u\right\}^{\frac{1}{p}\wedge 1}, \tag{A.1}$$

where $a \wedge b = \min\{a, b\}$ as usual.

Proposition A.1. *r defined in (A.1) is a metric in $L^p(\mathbb{R}^d)$ with $r(f,g) = 0$ if and only if $f(u) = g(u+b)$ a.e. $\mathrm{d}u$, for some $b \in \mathbb{R}^d$.*

Proof. The only nonobvious part is that $r(f,g)=0$ implies $f(u)=g(u+b)$ a.e. du, for some $b\in\mathbb{R}^d$. If $r(f,g)=0$, there is a sequence $\{b_n\}\subset\mathbb{R}^d$ such that

$$\int_{\mathbb{R}^d} |f(u)-g(u+b_n)|^p \mathrm{d}u \to 0, \quad \text{as } n\to\infty.$$

By considering a subsequence if necessary, we may suppose without loss of generality that $b_n\to b\in\mathbb{R}^d$. (If a subsequence converges to ∞, then the convergence above is possible only with $f(u)=g(u)=0$ a.e. du.) We will show that $f(u)=g(u+b)$ a.e. du. By the triangle inequality, it is enough to show that

$$\int_{\mathbb{R}^d} |g(u)-g(u+\epsilon_n)|^p \mathrm{d}u \to 0,$$

when $\epsilon_n\to 0$, as $n\to\infty$. One can argue as in the proof of Lemma 4.3 in [7] that the function $F(v)=\int_{\mathbb{R}^d}|g(u)-g(u+v)|^p\mathrm{d}u$ is continuous at $v=0$. This yields the above convergence. □

Proposition A.2. *The metric space* $(L^p(\mathbb{R}^d),r)$ *is Polish.*

Proof. This space is separable since $L^p(\mathbb{R}^d)$ is separable with the usual metric. To show that it is complete, consider a Cauchy sequence f_n, $n\ge 1$, for which $r(f_n,f_m)\to 0$, as $n,m\to\infty$. Then, there is a sequence $\{b_{n,m}\}$ such that

$$\int_{\mathbb{R}^d} |f_n(u+b_{n,m})-f_m(u)|^p \mathrm{d}u \to 0,$$

as $n,m\to\infty$. We can select a subsequence n_k and a sequence b_k, $k\ge 1$, such that, for $k\ge 1$,

$$\left\{\int_{\mathbb{R}^d} |f_{n_k}(u+b_k)-f_{n_{k-1}}(u+b_{k-1})|^p \mathrm{d}u\right\}^{\frac{1}{p}\wedge 1} \le C\,2^{-k},$$

where $f_0(u)=0$ by convention. Then,

$$\sum_{k=1}^{\infty}\big(f_{n_k}(u+b_k)-f_{n_{k-1}}(u+b_{k-1})\big)=\lim_{k\to\infty} f_{n_k}(u+b_k)=:f(u)$$

with the convergence in the usual $L^p(\mathbb{R}^d)$ sense. In particular, $r(f_{n_k},f)\to 0$, as $k\to\infty$, which yields completeness. □

Proposition A.3. *Let* $\mathcal{B}(L^p(\mathbb{R}^d))$ *denote Borel sets under the usual* L^p *metric, and* $\mathcal{B}_r(L^p(\mathbb{R}^d))$ *denote Borel sets under the metric* r *in (A.1). Then,* $\mathcal{B}_r(L^p(\mathbb{R}^d))\subset\mathcal{B}(L^p(\mathbb{R}^d))$. *In particular, if* $(Y,\mathcal{B}(Y))$ *is a standard Borel space, a* $\mathcal{B}(Y)|\mathcal{B}(L^p(\mathbb{R}^d))$*-measurable map* $F:Y\mapsto L^p(\mathbb{R}^d)$ *is also* $\mathcal{B}(Y)|\mathcal{B}_r(L^p(\mathbb{R}^d))$*-measurable.*

Proof. This follows from the identity

$$r(f,g)=\inf_{b\in\mathbb{Q}^d}\left\{\int_{\mathbb{R}^d}|f(u)-g(u+b)|^p\mathrm{d}u\right\}^{\frac{1}{p}\wedge 1}. \quad \square$$

Denote the Lebesgue measure on $\mathcal{B}(\mathbb{R})$ by Leb. More generally, for $d\in\mathbb{N}\cup\{0\}$, a standard Lebesgue space $(S,\mathcal{B}(S),\mu)$, and functions $f,g\in L^p(\mathbb{R}^d\times S,\mathrm{d}(Leb)^d\mathrm{d}\mu)$, $p>0$, consider

$$r_{d,\mu}(f,g)=\inf_{b\in\mathbb{R}^d}\left\{\int_{\mathbb{R}^d\times S}|f(u,s)-g(u+b,s)|^p\mathrm{d}u\mu(\mathrm{d}s)\right\}^{\frac{1}{p}\wedge 1}. \tag{A.2}$$

One can show as above that $(L^p(\mathbb{R}^d \times S), r_{d,\mu})$ is a Polish space, with $r_{d,\mu}(f,g) = 0$ if and only if $f(u,s) = g(u+b,s)$ a.e. $\mathrm{d}u\mu(\mathrm{d}s)$, for some $b \in \mathbb{R}^d$. Another metric used in the paper is defined on $L^p(\mathbb{R}^d \times S, \mathrm{d}(Leb)^d \mathrm{d}\mu)$ as

$$q_{d,\mu,c}(f,g) = \inf_{b\in\mathbb{R}} \left\{ \int_{\mathbb{R}^d \times S} |f(u,s) - g(u+bc,s)|^p \mathrm{d}u\mu(\mathrm{d}s) \right\}^{\frac{1}{p}\wedge 1}, \tag{A.3}$$

where $c \in \mathbb{R}^d$ is fixed. The space $L^p(\mathbb{R}^d \times S, \mathrm{d}(Leb)^d \mathrm{d}\mu)$ is also Polish under this metric, and $\inf_{b\in\mathbb{R}}$ can be replaced by $\inf_{b\in\mathbb{Q}}$ in (A.3).

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