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A FLUCTUATION THEORY FOR MARKOV CHAINS

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A fluctuation theory for Markov chains on an ordered countable state space is developed, using ladder processes. These are shown to be Markov renewal processes. Results are given for the joint distribution of the extremum (maximum or minimum) and the first time the extremum is achieved. Also a new classification of the states of a Markov chain is suggested. Two examples are given.

Markov chains	maximum and minimum functionals
ladder processes	Wiener–Hopf factorization

1. Introduction

In this paper we develop a fluctuation theory for discrete time Markov chains with an ordered countable state space, using ladder processes. Dinges [4] has obtained a Wiener–Hopf factorization for Markov chains. Arjas and Speed [1] and Kaspi [6] obtained this factorization for Markov-additive processes in discrete time and continuous time respectively. A factorization has also been obtained for the generator matrix of a continuous time Markov chain by Barlow, Rogers and Williams [2]. For a review of results for ladder phenomena in Levy processes, see Prabhu [7]. Our concern here is not with the Wiener–Hopf factorization *per se*, but with the distribution of maximum and minimum functionals of the Markov chain. Stone [8, 9] has dealt with the supremum of Markov and semi-Markov chains, but he does not use ladder processes. Our techniques and results can also be extended to semi-Markov chains on a continuous state space.

2. Ladder processes

Let $\{X_n, n \geqslant 0\}$ be a time homogeneous Markov chain on the state space $S = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$ with the transition probability matrix $P = (P_{jk})$ and the initial distribution $\{\pi_j, j \in S\}$. For $n \geq 0$ let

$$M_n = \max(X_0, X_1, \ldots, X_n), \qquad m_n = \min(X_0, X_1, \ldots, X_n) \tag{1}$$

be the maximum and minimum functionals of the chain. We shall denote by P_π and E_π the conditional probability distribution and expectation with the initial distribution π. For convenience we shall also write P_j and E_j for these quantities in the case where π is concentrated at $j \in S$. Let

$$N_0 \equiv 0, \qquad N_r = \min\{n : X_n > X_{N_{r-1}}\} \quad (r \geq 1), \tag{2}$$

$$H_r = X_{N_r} \quad (r \geq 0), \tag{3}$$

where we use the convention that the minimum of an empty set is $+\infty$. We shall call $\{(H_r, N_r), r \geq 0\}$ the ascending ladder process. If the inequalities are reversed in (2) we obtain the descending ladder process. We shall here consider the ascending ladder process and derive properties that connect it with the maximum functional M_n. Similar properties can be derived for the descending ladder process, but we shall illustrate these by means of an example.

Theorem 1. *The ascending ladder process is a Markov renewal process, i.e.,*

$$P\{H_{r+1} = k, N_{r+1} = n | H_r = j, N_r = m, H_{r-1}, N_{r-1}, \ldots, H_0, N_0\}$$
$$= P_j\{H_1 = k, N_1 = n - m\}. \tag{4}$$

Proof. The definition of (H_r, N_r) shows that given (H_m, N_m) $(0 \leq m \leq r)$, (H_{r+1}, N_{r+1}) depends only on (H_r, N_r) because of the Markov property. Also

$$P\{H_{r+1} = k, N_{r+1} = n | H_r = j, N_r = m\}$$
$$= P\{X_{m+1} \leq j, \ldots, X_{n-1} \leq j, X_n = k | X_m = j\}$$
$$= P\{X_1 \leq j, \ldots, X_{n-m-1} \leq j, X_{n-m} = k | X_0 = j\}$$
$$= P_j\{H_1 = k, N_1 = n - m\}$$

as desired. □

From the theory of Markov renewal processes (see, for example, Cinlar [3, p. 320]) it follows that the probabilistic behavior of the ladder process is completely described by that of the first ladder point (H_1, N_1). For $|s| < 1$ we now define the following strictly upper triangular square matrix:

$$[P_+(s)]_{j,k} = \begin{cases} E_j\{s^{N_1}; H_1 = k\} & \text{if } k > j, \\ 0 & \text{otherwise}; \end{cases} \tag{5}$$

we shall call $P_+(s)$ the (ascending) ladder matrix. For ready reference we record the following result without proof.

Theorem 2. *For* $|s|<1$ *and* $r \geqslant 0$

$$E_j\{s^{N_r}; H_r = k\} = [P_+(s)]^r_{j,k}. \tag{6}$$

For convenience let us denote the distribution of (H_1, N_1) by

$$f_{jk}^{(n)} = P_j\{H_1 = k, N_1 = n\} \quad (k > j, n \geqslant 1). \tag{7}$$

Also, let $u_{jk}^{(0)} = \delta_{jk}$,

$$u_{jk}^{(n)} = P_j\{X_m < X_n (0 \leqslant m \leqslant n-1), X_n = k\} \quad (n \geqslant 1) \tag{8}$$

and

$$R_+(s) = \left(\sum_{n=0}^{\infty} u_{jk}^{(n)} s^n\right). \tag{9}$$

Theorem 3 below establishes the renewal-theoretic relation between (7) and (8).

Theorem 3. *Let* $R_+(s)$ *be the upper triangular matrix defined by* (8) *and* (9). *Then*

$$R_+(s) = [I - P_+(s)]^{-1}. \tag{10}$$

Proof. For $n \geqslant 1$ we have

$$\sum_{n=0}^{\infty} u_{jk}^{(n)} s^n = \sum_{r=0}^{\infty} E_j[s^{N_r}; X_{N_r} = k] = \sum_{r=0}^{\infty} [P_+(s)]^r_{jk},$$

which leads to the desired result. □

As the counterpart of (7) we define the probability

$$g_{jk}^{(n)} = P_j\{X_m < X_n (1 \leqslant m \leqslant n-1), X_n = k\} \quad \text{for } k \leqslant j, n \geqslant 1$$
$$= 0 \qquad \text{otherwise.} \tag{11}$$

Also, let

$$P_-(s) = \left(\sum_{n=1}^{\infty} g_{jk}^{(n)} s^n\right). \tag{12}$$

It is not evident that $I - P_-(s)$ has an inverse, but it turns out that $[I - P_-(s)]^{-1} = R_-(s)$, where

$$R_-(s) = \left(\sum_{n=0}^{\infty} v_{jk}^{(n)} s^n\right) \tag{13}$$

with $v_{jk}^{(0)} = \delta_{jk}$ and

$$v_{jk}^{(n)} = P_j\{X_m \leqslant X_0 (1 \leqslant m \leqslant n), X_n = k\} \quad (n \geqslant 1). \tag{14}$$

Clearly, $P_-(s)$ and $R_-(s)$ are lower triangular matrices. This result will be proved in section 4, where it will be shown that the probabilities (7), (8), (11) and (14) are special cases of certain taboo probabilites.

3. The maximum process

Let $F_n = \min\{m \le n : X_m = M_n\}$ be the index of the first maximum. We have the following:

Theorem 4. *For* $0 \le m \le n$ *and* $l \ge \max(j, k)$ *we have*

$$P_j\{F_n = m, M_n = l, X_n = k\} = u_{jl}^{(m)} v_{lk}^{(n-m)}. \tag{15}$$

Proof. We have

$$\begin{aligned} &P_j\{F_n = m, M_n = l, X_n = k\} \\ &\quad = P_j\{X_1 < l, X_2 < l, \ldots, X_{m-1} < l, X_m = l\} \\ &\qquad \cdot P\{X_{m+1} \le l, \ldots, X_{n-1} \le l, X_n = k \mid X_m = l\} \\ &\quad = u_{jl}^{(m)} P_l\{X_1 \le l, \ldots, X_{n-m-1} \le l, X_{n-m} = k\} \\ &\quad = u_{jl}^{(m)} v_{lk}^{(n-m)}. \qquad \square \end{aligned}$$

Corollary 1. *We have*

$$P_j\{F_n = m, M_n = l, X_n = k\} = P_j\{F_m = m, X_m = l\} \cdot P_l\{F_{n-m} = 0, X_{n-m} = k\}. \tag{16}$$

Proof. From (15) we find that

$$P_j\{F_n = 0, X_n = k\} = v_{jk}^{(n)} \quad (k \le j)$$

$$P_j\{F_n = n, X_n = k\} = u_{jk}^{(n)} \quad (k \ge j).$$

From these the desired result follows. $\square$

Theorem 5. *Let* $a_l = P_l\{N_1 = \infty\} \ge 0$. *Then*

$$\lim_{n \to \infty} P_j\{F_n = m, M_n = l\} = u_{jl}^{(m)} a_l. \tag{17}$$

Let $\lim_{n \to \infty} M_n = M \le \infty$. *The distribution of* M *is given by*

$$P_j\{M = l\} = \sum_{n=0}^{\infty} u_{jl}^{(n)} a_l. \tag{18}$$

Proof. From (15) we obtain

$$P_j\{F_n = m, M_n = l\} = u_{jl}^{(m)} \sum_{k \leq l} v_{lk}^{(n-m)}.$$

The desired results follow from the fact that

$$\sum_{k \leq l} v_{lk}^{(n)} = P_l\{X_m \leq X_0 \ (1 \leq m \leq n)\} = P_l\{N_1 > n\}$$
$$\to P_l\{N_1 = \infty\} \quad \text{as } n \to \infty. \qquad \square$$

Classification of states. Let

$$D = \left\{\pi = (\ldots \pi_{-2}, \pi_{-1}, \pi_0, \pi_1, \pi_2, \ldots): \pi_k \geq 0, \sum_{-\infty}^{\infty} \pi_k = 1\right\}; \tag{19}$$

D is a set of all nondefective probability mass functions on S. We shall say that a state $k \in S$ is *maximal* if there exists a $\pi \in D$ such that $P_\pi\{M = k\} > 0$.

From the definition it is obvious that the support of M is a subset of the set of maximal states. The following proposition gives one way of determining whether a given state is maximal or not.

Theorem 6. *A state k is maximal iff $a_k > 0$.*

Proof. Suppose $a_k > 0$. Then for the $\pi \in D$ which is concentrated at k we have

$$P_\pi\{M = k\} = P_k\{M = k\} = a_k > 0,$$

so that the state k is maximal.

Next suppose the state k is maximal. Then there exists a $\pi \in D$ such that $P_\pi\{M = k\} > 0$. We have

$$P_\pi\{M = k\} = \sum_{j \in S} \pi_j P_j\{M = k\} = \left(\sum_{n=0}^{\infty} \sum_{j \in S} \pi_j u_{jk}^{(n)}\right) a_k.$$

This gives $a_k > 0$, as required. $\square$

A state may be maximal or minimal or both or neither. It can be easily seen that an absorbing state is both maximal and minimal. (The converse is not true: a state which is both maximal and minimal need not be absorbing.) Also, every recurrent class (null or positive) of a Markov chain has at most one maximal state. Unfortunately, no such statement can be made about transient states.

4. Some taboo probabilities

Let $I = \{i, i+1, \ldots\}$ be a subset of S, and

$${}^I p_{jk}^{(n)} = P_j\{X_m \notin I \ (1 \leq m \leq n-1), X_n = k\} \quad (n \geq 2) \tag{20}$$

the probability of transition from j to k in n steps, avoiding the set I en route. Also, let ${}^iP_{jk}^{(1)} = P_{jk}$. As particular cases of (20) we have for $n \geq 1$,

$$\begin{aligned} {}^{j+1}P_{jk}^{(n)} &= f_{jk}^{(n)} \quad \text{for } k > j, \\ &= v_{jk}^{(n)} \quad \text{for } k \leq j \end{aligned} \tag{21}$$

and

$$\begin{aligned} {}^{k}P_{jk}^{(n)} &= u_{jk}^{(n)} \quad \text{for } k > j, \\ &= g_{jk}^{(n)} \quad \text{for } k \leq j. \end{aligned} \tag{22}$$

We have the following:

Lemma 1. *for $n \geq 2$ we have*

$${}^iP_{jk}^{(n)} = \sum_{m=1}^{n-1} \sum_{l<i} {}^lP_{jl}^{(m)} \cdot {}^{l+1}P_{lk}^{(n-m)}. \tag{23}$$

Proof. Let

$$M_{n-1}(I) = \max\{(X_1, X_2, \ldots, X_{n-1}): X_m \notin I \ (1 \leq m \leq n-1)\}$$

$$F_{n-1}(I) = \min\{m: 1 \leq m \leq n-1, X_m = M_{n-1}(I)\}.$$

Then

$$\begin{aligned} &P_j\{F_{n-1}(I) = m, M_{n-1}(I) = l, X_n = k\} \\ &= P_j\{X_1 < l, X_2 < l, \ldots, X_{m-1} < l, X_m = l\} \\ &\quad \cdot P_l\{X_1 \leq l, \ldots, X_{n-m-1} \leq l, X_{n-m} = k\} \\ &= {}^lP_{jl}^{(m)} \cdot {}^{l+1}P_{lk}^{(n-m)} \quad (1 \leq m \leq n-1, l < i). \end{aligned}$$

Adding this probability over $1 \leq m \leq n-1$ and $l < i$ we obtain the result (23). □

Theorem 7. *We have the following:*

$$f_{jk}^{(n)} = \sum_{m=1}^{n-1} \sum_{l \leq j} g_{jl}^{(m)} f_{lk}^{(n-m)} \quad (k > j, n \geq 2), \tag{24}$$

$$g_{jk}^{(n)} = \sum_{m=1}^{n-1} \sum_{l < k} g_{jl}^{(m)} f_{lk}^{(n-m)} \quad (k \leq j, n \geq 2), \tag{25}$$

$$u_{jk}^{(n)} = \sum_{m=0}^{n-1} \sum_{l=j}^{k-1} u_{jl}^{(m)} f_{lk}^{(n-m)} \quad (k > j, n \geq 1), \tag{26}$$

$$v_{jk}^{(n)} = \sum_{m=1}^{n} \sum_{l=k}^{j} g_{jl}^{(m)} v_{lk}^{(n-m)} \quad (k \leq j, n \geq 1). \tag{27}$$

Proof. The identities (24) and (25) are special cases of Lemma 1 with $i=j+1\leq k$ and $i=k\leq j$ respectively. To prove (26) we use Lemma 1 with $i=k>j$ and obtain

$$u_{jk}^{(n)}=\sum_{m=1}^{n-1}\sum_{l=j+1}^{k-1}u_{jl}^{(m)}f_{lk}^{(n-m)}+\sum_{m=1}^{n-1}\sum_{l\leq j}g_{jl}^{(m)}f_{lk}^{(n-m)}$$

where the second sum on the right side equals $f_{jk}^{(n)}$ on account of (24). This leads to (26). The remaining result (27) follows from Lemma 1 with $i=j+1>k$. □

Theorem 8. *Let $R_-(s)$ be the lower triangular matrix defined by* (13) *and* (14). *Then*

$$R_-(s)=[I-P_-(s)]^{-1}. \tag{28}$$

Proof. From (27) we obtain

$$\sum_{n=0}^{\infty}v_{jk}^{(n)}s^n=\delta_{jk}+\sum_{l=k}^{j}\left(\sum_{m=1}^{\infty}g_{jl}^{(m)}s^m\right)\left(\sum_{n=m}^{\infty}v_{lk}^{(n-m)}s^{n-m}\right)$$

or $R_-(s)=I+P_-(s)R_-(s)$, which leads to (28). □

Theorem 9. *For $i>\max(j,k)$ we have*

$$ {}^iP_{jk}^{(n)}=\sum_{m=0}^{n}\sum_{l=\max(j,k)}^{i-1}u_{jl}^{(m)}v_{lk}^{(n-m)}, \tag{29}$$

and

$$P_{jk}^{(n)}=\sum_{m=0}^{n}\sum_{l=\max(j,k)}^{\infty}u_{jl}^{(m)}v_{lk}^{(n-m)}. \tag{30}$$

Proof. We shall prove (29) in the case $j<k<i$, the proof being similar for $k\leq j<i$. Letting $i\to\infty$ in (29) we obtain (30).

For $j<k<i$ the right side of (23) equals

$$\sum_{m=1}^{n-1}\sum_{l=k}^{i-1}u_{jl}^{(m)}v_{lk}^{(n-m)}+\sum_{m=1}^{n-1}\sum_{l=j+1}^{k-1}u_{jl}^{(m)}f_{lk}^{(n-m)}+\sum_{m=1}^{n-1}\sum_{l\leq j}g_{jl}^{(m)}f_{lk}^{(n-m)}$$

$$=\sum_{m=1}^{n-1}\sum_{l=k}^{i-1}u_{jl}^{(m)}v_{lk}^{(n-m)}+[u_{jk}^{(n)}-f_{jk}^{(n)}]+f_{jk}^{(n)}$$

$$=\sum_{m=1}^{n}\sum_{l=k}^{i-1}u_{jl}^{(m)}v_{lk}^{(n-m)},$$

where we have used (26) and (24). This proves (29). □

Corollary 2 (Wiener–Hopf factorization). *We have*

$$I-sP=[I-P_-(s)][I-P_+(s)] \tag{31}$$

where the matrices $P_+(s)$ and $P_-(s)$ defined by (5) *and* (12) *respectively.*

Proof. Let $R(s)=(\sum_{n=0}^{\infty} P_{jk}^{(n)}s^n)$. From (30) we obtain $R(s)=R_+(s)R_-(s)$. This gives the desired result, since $R(s)^{-1}=I-sP$, and by Theorems 3 and 8, $R_+(s)^{-1}=I-P_+(s)$ and $R_-(s)^{-1}=I-P_-(s)$. □

The above factorization is equivalent to the one established by Dinges [4]. See also Arjas and Speed [1]. The identity (29) also leads to a factorization of the same type with the elements of $P_-(s)$ and $P_+(s)$ restricted to the complement of the set I. It can be proved that the factorization (31) is unique. This property can be used in principle to compute the ladder matrix $P_+(s)$. In practice, however, it is very difficult to obtain the factorization of $I-sP$. Numerical methods are useless when either s has to be left as a variable or P itself is not numerical. An alternate method, called the determinant method, is developed in the next section to overcome this difficulty. We need the following results.

Lemma 2. *We have*

$$f_{jk}^{(n)}=\sum_{l\leq j} v_{jl}^{(n-1)}P_{lk} \quad (k>j,\ n\geq 1), \tag{32}$$

and

$$g_{jk}^{(n)}=\sum_{l>k} P_{jl}u_{lk}^{(n-1)} \quad (k\leq j,\ n\geq 1). \tag{33}$$

Proof. For $n=1$ the right side of (32) reduces to $P_{jk}=f_{jk}^{(1)}$ since $v_{jl}^{(0)}=\delta_{jl}$. For $n\geq 2$ we have

$$\begin{aligned} f_{jk}^{(n)} &= P_j\{X_m\leq j\ (1\leq m\leq n-1),\ X_n=k\} \\ &= \sum_{l\leq j} P_j\{X_m\leq j\ (1\leq m\leq n-2),\ X_{n-1}=l\}P\{X_n=k\mid X_{n-1}=l\} \\ &= \sum_{l\leq j} v_{jl}^{(n-1)}P_{lk}, \end{aligned}$$

as required. The proof of (33) is similar. □

5. The determinant method

In this we assume that the state space is $S=\{0,1,2,\dots\}$. Let $A=I-sP$, and

$A(j|k)=$ submatrix of A formed by taking rows indexed by $0,1,\dots,j$ and columns indexed by $0,1,\dots,j-1,k$ if $k>j$, or
$=$ submatrix of A formed by taking rows indexed by $0,1,\dots,k-1,j$ and columns indexed by $0,1,\dots,k$ if $k\leq j$.

The $A(j|k)$ are square matrices. Their determinants will be denoted by $D_{jk}(s)=\det A(j|k)$. The main result we need is the following.

Theorem 10. *For* $|s|<1$ *we have*

$$[P_+(s)]_{jk} = -\frac{D_{jk}(s)}{D_{jj}(s)} \qquad (k>j\geq 0), \tag{34}$$

$$[P_-(s)]_{jk} = \delta_{jk} - \frac{D_{jk}(s)}{D_{k-1,k-1}(s)} \qquad (0\leq k\leq j). \tag{35}$$

Proof. Let $P(j)$ be the submatrix of P formed by taking rows and columns indexed by $0, 1, \ldots, j$. From Lemma 2 we obtain

$$\begin{aligned}\sum_{n=1}^{\infty} f_{jk}^{(n)} s^n &= s \sum_{l=0}^{j} \left(\sum_{n=1}^{\infty} v_{jl}^{(n-1)} s^{n-1} \right) P_{lk} \\ &= s \sum_{l=0}^{j} [I - sP(j)]_{jl}^{-1} P_{lk} = s \sum_{l=0}^{j} [A(j|j)]_{jl}^{-1} P_{lk} \\ &= -\frac{1}{D_{j,j}(s)} \sum_{l=0}^{j} [\text{adjoint of } (A(j|j)]_{l,j}(-sP_{l,k}) \\ &= -\frac{1}{D_{j,j}(s)} \sum_{l=0}^{j} ((-1)^{l+j} \text{ minor of } [A(j|j)]_{l,j})(-sP_{l,k}).\end{aligned}$$

Since the last sum is just the expansion of $\det A(j|k)$ using the last column of $A(j|k)$, it equals D_{jk}. Thus we have proved (34), and the proof of (35) is similar. □

Remark 1. Theorem 10 is a powerful computational tool for extremum processes in Markov chains. For finite chains it is possible to write computer programs to compute $D_{jk}(s)$. In most applications it is possible to compute $D_{jk}(s)$ and $D_{jk}(1)$ by using a recursion. We shall illustrate this by examples in Sections 6 and 7.2.

Remark 2. Theorem 10 gives an algebraic result for factorizing $I - sP$. Gantmacher [5] mentions this theorem for finite matrices. The result here is valid for infinite matrices, and the surprising feature that its proof is probabilistic.

6. A birth and death process

Consider a birth and death process $\{X(t), t\geq 0\}$ with birth parameters $\{\lambda_j, j\geq 0\}$ and death parameters $\{\mu_0 = 0; \mu_j; j\geq 1\}$. Let τ_n be the nth jump epoch of $\{X(t), t\geq 0\}$ and define $X_n^* = X(\tau_n +)$. Then $\{X_n^*, n\geq 0\}$ is a discrete time Markov chain on state space $\{0, 1, 2, \ldots\}$ with transition probabilties

$$P_{jk}^* = \begin{cases} \dfrac{\lambda_j}{\lambda_j+\mu_j} & (k=j+1), \\ \dfrac{\mu_j}{\lambda_j+\mu_j} & (k=j-1), \\ 0 & \text{otherwise.}\end{cases}$$

Let $X_0^* = i$ and define $T_i^* = \inf\{n \geq 0: X_n^* = 0\}$. Our aim is to study $M_i^* = \max\{X_n^*: 0 \leq n \leq T_i^*; X_0^* = i\}$. To do this we define a Markov chain $\{X_n, n \geq 0\}$ with transition probabilities

$$P_{jk} = \begin{cases} \dfrac{\lambda_j}{\lambda_j+\mu_j} & (k=j+1, j\geq 1), \\ 1 & (k=j=0), \\ \dfrac{\mu_j}{\lambda_j+\mu_j} & (k=j-1, j \geq 1), \\ 0 & \text{otherwise}. \end{cases} \tag{36}$$

Then M_i^* has the same distribution as that of $M_i = \max_{0\leq n<\infty}\{X_n\}$ with $X_0 = i$. The determinants $D_{j,k}(s)$ needed for using theorem 10 can be computed recursively as follows:

$$D_{0,0}(s) = D_{1,1}(s) = 1-s,$$
$$D_{j,j}(s) = D_{j-1,j-1}(s) - s^2\frac{\lambda_j\mu_j}{(\lambda_j+\mu_j)}D_{j-2,j-2}(s) \quad (j \geq 2). \tag{37}$$

Thus we find that

$$D_{j,j+1}(s) = -s\frac{\lambda_j}{\lambda_j+\mu_j}D_{j-1,j-1}(s) \quad (j \geq 1), \tag{38}$$

$$D_{j,j-1}(s) = -s\frac{\mu_j}{\lambda_j+\mu_j}D_{j-2,j-2}(s) \quad (j \geq 1), \tag{39}$$

$$D_{jk}(s) = 0 \quad (|k-j| > 1). \tag{40}$$

By letting $s \to 1-$, the equations for $D_{j,k}(s)/(1-s)$ can be solved explicitly. Then $P_{\cdot}$ can be computed explicitly and Theorem 4 can be used to compute the distribution of M_i. We shall give only the final results below. Define

$$d_n = 1 + \sum_{i=1}^{n}\prod_{j=1}^{i}\frac{\mu_j}{\lambda_j} \quad \text{and} \quad d = \lim_{n\to\infty} d_n. \tag{41}$$

Then

$$P\{M_i = k\} = d_{i-1}[1/d_{k-1} - 1/d_k] \quad (k \geq i). \tag{42}$$

It is instructive to compute $P\{M_i < \infty\}$ which also is the probability that T_i^* is finite. Using (39) and (41) above we find that

$$P\{T_i^* < \infty\} = P\{M_i^* < \infty\} = P\{M_i < \infty\} = 1 - d_{i-1}/d \quad \text{if } d < \infty,$$
$$= 1 \quad \text{if } d = \infty. \tag{43}$$

But $d = \infty$ iff the birth and death process is recurrent. Thus the first passage time is finite a.s. iff the birth and death process is recurrent.

7. A Markov chain arising in queuing theory

Consider the Markov chain $\{X_n\}$ with the transition probabilities

$$P_{0k} = \lambda_k, \qquad P_{jk} = \chi_{k-j+1} \quad (k \geq j-1 \geq 0) \tag{44}$$

which arises in the imbedded chain analysis of a modified M/G/1 queue (single server queue with Poisson arrivals), where the customers who start a busy period are special in the sense that their service times have a distribution different from other (ordinary) customers. Here λ_k represents the probability that there are k arrivals during the service time of a special customer and χ_k the probability of k arrivals during the service time of an ordinary customer. Clearly, $\{\lambda_j, j \geq 0\}$ and $\{\chi_j, j \geq 0\}$ are probability distributions on $(0, 1, 2, \ldots)$. Let $L(z) = \sum_0^\infty \lambda_j z^j$ and $K(z) = \sum_0^\infty \chi_j z^j$ $(0 < z < 1)$. It is known that the equation $\xi = sK(\xi)$ has a unique continuous solution $\xi \equiv \xi(s)$ in $(0, 1)$ with $\xi(s) \to 0$ as $s \to 0+$. Also $\xi(s) \to \zeta$ as $s \to 1-$, where $\zeta < 1$ iff $\rho = K'(1) > 1$. We proceed to investigate the descending and ascending ladder processes of this Markov chain.

7.1. Descending ladder process

To investigate the descending ladder process and the minimum functional of the given Markov chain $\{X_n\}$, we start with the first descending ladder epoch $\bar{N}_1 = \min\{n : X_n < X_0\}$. Let $\bar{H}_1 = X_{\bar{N}_1}$ and

$$\bar{f}_{jk}^{(n)} = P_j\{\bar{H}_1 = k, \bar{N}_1 = n\}$$

$$= P_j\{X_m \geq X_0\,(1 \leq m \leq n-1), X_n = k\} \quad (k < j), \tag{45}$$

$$\bar{u}_{jk}^{(n)} = P_j\{X_m > X_n\ (0 \leq m \leq n-1), X_n = k\} \quad (k < j), \tag{46}$$

$$\bar{g}_{jk}^{(n)} = P_j\{X_m > X_n\ (1 \leq m \leq n-1), X_n = k\} \quad (k \geq j), \tag{47}$$

$$\bar{v}_{jk}^{(n)} = P_j\{X_m \geq X_0\,(1 \leq m \leq n), X_n = k\} \qquad (k \geq j). \tag{48}$$

The transition probabilities $j \to k$ $(j > 0, k \geq 0)$ are space-homogeneous; that is, $P_{jk} = P_{1,k-j+1}$ $(k \geq j-1 \geq 0)$. Using this fact it can be proved that (as is well known in queuing theory)

$$\bar{f}_{jk}^{(n)} = \bar{f}_{1,0}^{(n)} \quad \text{if } k = j-1,$$
$$= 0 \qquad \text{otherwise}$$

and

$$\sum_{n=1}^{\infty} \bar{f}_{10}^{(n)} s^n = \xi(s) \quad (0 < s < 1).$$

Thus

$$[\bar{P}(s)]_{jk} = \xi \delta_{j-1,k} \quad (j \geq 1, k \geq 0) \tag{49}$$

and hence

$$\sum_{n=0}^{\infty} \bar{u}_{jk}^{(n)} s^n = [I - P_-(s)]_{jk}^{-1} = \xi^{j-k} \quad (0 \leq k \leq j). \tag{50}$$

From the analog of Lemma 2 for descending ladder processes we obtain

$$\bar{g}_{jk}^{(n)} = \sum_{l=k+1}^{\infty} P_{jl} \bar{u}_{lk}^{(n-1)} \quad (k \geq j, n \geq 1).$$

This gives

$$[\bar{P}_+(s)]_{jk} = \sum_{n=1}^{\infty} \bar{g}_{jk}^{(n)} s^n = s \sum_{l=k}^{\infty} P_{jl} \xi^{l-k}. \tag{51}$$

In particular

$$[\bar{P}_+(s)]_{0k} = s \sum_{l=k}^{\infty} \lambda_l \xi^{l-k},$$

$$[\bar{P}_+(s)]_{1k} = s \sum_{l=k}^{\infty} \chi_l \xi^{l-k},$$

$$[\bar{P}_+(s)]_{jk} = [\bar{P}_+(s)]_{1,k-j+1} \quad (k \geq j \geq 1).$$

The space-homogeneity of transitions mentioned above gives $\bar{v}_{jk}^{(n)} = \bar{v}_{1,k-j+1}^{(n)}$ $(k \geq j \geq 1)$. From the analog of the identity (27) for descending ladder processes we obtain

$$\bar{v}_{jk}^{(n)} = \sum_{m=1}^{n} \sum_{l=j}^{k} \bar{g}_{jl}^{(m)} \bar{v}_{lk}^{(n-m)} \quad (k \geq j, n \geq 1).$$

This gives

$$\bar{v}_{1k}^{(n)} = \sum_{m=1}^{n} \sum_{l=1}^{k} \bar{g}_{1l}^{(m)} \bar{v}_{1,k-l+1}^{(n-m)}$$

and

$$\bar{v}_{0k}^{(n)} = \sum_{m=1}^{n} \bar{g}_{00}^{(m)} \bar{v}_{0k}^{(n-m)} + \sum_{m=1}^{n} \sum_{l=1}^{k} \bar{g}_{0l}^{(m)} \bar{v}_{1,k-l+1}^{(n-m)}.$$

To solve for $\bar{g}_{jk}^{(n)}$ and $\bar{v}_{jk}^{(n)}$ from the above relations it is best to introduce the generating functions $(0 < s < 1, 0 < z < 1)$

$$g_j^*(s, z) = \sum_{n=1}^{\infty} \sum_{k=j}^{\infty} \bar{g}_{jk}^{(n)} s^n z^k, \tag{52}$$

$$v_j^*(s, z) = \sum_{n=1}^{\infty} \sum_{k=j}^{\infty} v_{jk}^{(n)} s^n z^k. \tag{53}$$

We then obtain

$$g_j^*(s,z) = z^{j-1} g_1^*(s,z) \quad (j \geq 1), \qquad g_1^*(s,z) = sz \frac{K(\xi) - K(z)}{\xi - z}, \tag{54}$$

$$g_0^*(s,z) = s \frac{\xi L(\xi) - z L(z)}{\xi - z}. \tag{55}$$

Also

$$v_j^*(s,z) = z^{j-1} v_1^*(s,z) \quad (j \geq 1), \qquad v_1^*(s,z) = z \frac{z - \xi}{z - sK(z)}, \tag{56}$$

$$v_0^*(s,z) = \frac{z - sK(z) - sz[L(\xi) - L(z)]}{[z - sK(z)][1 - sL(\xi)]}. \tag{57}$$

Finally we have

$$P_j\{m_n = l, X_n = k\} = \sum_{m=0}^{n} \bar{u}_{jl}^{(m)} \bar{v}_{lk}^{(n-m)} \tag{58}$$

for $n \geq 1$, $l \leq \min(j, k)$. This gives

$$\sum_{n=0}^{\infty} E_j(z_1^{m_n} z_2^{X_n}) s^n = \sum_{l=0}^{j} \xi^{j-l} z_1^l v_l^*(s, z_2). \tag{59}$$

Substituting for $v_l^*(s, z_2)$ from (56) and (57) into (59) we find that

$$\sum_{n=0}^{\infty} E_j(z_1^{m_n} z_2^{X_n}) s^n$$
$$= \frac{z_1 z_2 (z_2 - \xi)}{z_1 z_2 - \xi} \cdot \frac{(z_1 z_2)^j - \xi^j}{z_2 - sK(z_2)} + \frac{\xi^j}{1 - sL(\xi)} \frac{z_2 - sK(z_2) - sz_2[L(\xi) - L(z_2)]}{z_2 - sK(z_2)}. \tag{60}$$

This gives in particular

$$(1-s) \sum_{0}^{\infty} s^n E_j(z^{m_n}) = \xi^j + z(1-\xi) \frac{\xi^j - z^j}{\xi - z}. \tag{61}$$

Letting $s \to 1-$ in (61) we find that

$$E_j(z^{m_\infty}) = 1 \qquad \text{if } \rho \leq 1,$$
$$= \zeta^j + z(1-\zeta) \frac{\zeta^j - z^j}{\zeta - z} \qquad \text{if } \rho > 1.$$

It follows that if $\rho \leq 1$ then $m_\infty \equiv 0$, while if $\rho > 1$, then

$$P_j\{m_\infty = 0\} = \zeta^j, \tag{62}$$

$$P_j\{m_\infty = k\} = (1-\zeta)\zeta^{j-k} \quad (1 \leq k \leq j). \tag{63}$$

7.2. Ascending ladder process

We next investigate the ascending ladder process and the maximum functional of the modified M/G/1 queue during a busy period. This is accomplished by setting

$$\lambda_0 = 1, \qquad \lambda_k = 0 \quad (k \geq 1) \tag{64}$$

and leaving the rest of the matrix unchanged. We derive the expressions for the matrices P_+, P_-, R_+ and R_-. The algebraic details of the derivation are somewhat tedious and are omitted. Let $D_{jk}(s)$ but the determinants defined in Section 5, and define

$$D_{-1}(s) = \frac{1}{1-s}, \qquad D_j(s) = \frac{D_{jj}(s)}{1-s} \quad (j \geq 0). \tag{65}$$

It is found that

$$D_0(s) = 1, \qquad D_1(s) = 1 - s\chi_1, \qquad D_2(s) = (1 - s\chi_1)^2 - \chi_0\chi_2 s^2$$

and generally

$$D_n(s) = \sum_{k=0}^{n} (-1)^k A_{kn} s^k \tag{66}$$

where

$$A_{kn} = \sum (-1)^{i_0} \frac{(n-i_0)!}{(n-k)!} \prod_{j=0}^{n} \frac{(\chi_j)^{i_j}}{(i_j)!}, \tag{67}$$

the sum being taken over all sequences $(i_0, i_1, \ldots, i_n)$ such that $\sum_{j=0}^{n} i_j = k$ and $\sum_{j=0}^{n} j i_j = k$. Using Theorem 10 we find that

$$\begin{aligned} [P_-(s)]_{jk} &= 1 - D_j(s)/D_{j-1}(s) && (k = j \geq 0), \\ &= s\chi_0 && (k = j-1 \geq 0), \\ &= 0 && (k < j-1), \end{aligned} \tag{68}$$

and hence

$$[R_-(s)]_{j,j-k} = (s\chi_0)^k D_{j-k-1}(s)/D_j(s) \quad (0 \leq k \leq j). \tag{69}$$

The determinants $D_{j,j+k}(s)$ $(k \geq 1)$ can be computed by using the relation

$$D_{j,j+k}(s) = -\sum_{r=1}^{j} (s\chi_0)^{j-r}(s\chi_{j+k-r})D_{r-1}(s). \tag{70}$$

Again from Theorem 10 we find that for $k \geq 1$

$$[P_+(s)]_{j,j+k} = -D_{j,j+k}(s)/D_j(s) \tag{71}$$

and hence

$$[R_+(s)]_{i,j+k} = \frac{D_k(s)D_{j+k-1}(s) - D_{k-1}(s)D_{j+k}(s)}{(s\chi_0)^k D_{j+k-1}(s)}. \tag{72}$$

We have thus obtained all the basic results necessary to compute all functionals of the ladder process and the maximum. In particular we have the following results:

(i) If $\chi_0 > 0$, then every state is maximal for the modified chain, since in that case

$$a_j = P_j\{N_1 = \infty\} = 1 - \sum_{k=1}^{\infty} [P_+(s)]_{j,j+k} = \chi_0^j / D_j(1) > 0. \tag{73}$$

(ii) $P_j\{M < \infty\} = 1$ if $\rho \leq 1$ and ζ^j if $\rho > 1$. This is clearly the probability that the busy period initiated by j customers terminates, and the result is well known in queueing theory. To show that it is a consequence of our results we note from Theorem 5 that

$$P_j\{M \leq j+k\} = \chi_0^j D_k(1)/D_{j+k}(1). \tag{74}$$

Using the relation

$$D_k(s) = (1 - s\chi_1)D_{k-1}(s) - (s\chi_0)(s\chi_2)D_{k-2}(s) - (s\chi_0)^2(s\chi_3)D_{k-3}(s) - \cdots - (s\chi_0)^{k-1}(s\chi_k)D_0(s)$$

we obtain

$$\lim_{k\to\infty} \frac{D_k(s)}{D_{k+1}(s)} = \frac{\xi(s)}{s\chi_0}$$

and hence

$$\lim_{k\to\infty} \frac{D_k(s)}{D_{k+j}(s)} = \left(\frac{\xi(s)}{s\chi_0}\right)^j. \tag{75}$$

The desired result now follows from (75).

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