On the finiteness of the cone spectrum of certain linear transformations on Euclidean Jordan algebras

Yihui Zhou [a], M. Seetharama Gowda [b],*

[a] *Department of Biostatistics, The University of North Carolina at Chapel Hill, Chapel Hill, NC 27599, United States*

[b] *Department of Mathematics and Statistics, University of Maryland, Baltimore County, Baltimore, MD 21250, United States*

A R T I C L E   I N F O

A B S T R A C T

Let L be a linear transformation on a finite dimensional real Hilbert space H and K be a closed convex cone with dual K^* in H. The cone spectrum of L relative to K is the set of all real λ for which the linear complementarity problem

$$x \in K, \quad y = L(x) - \lambda x \in K^*, \quad \text{and} \quad \langle x, y \rangle = 0$$

admits a nonzero solution x. In the setting of a Euclidean Jordan algebra H and the corresponding symmetric cone K, we discuss the finiteness of the cone spectrum for **Z**-transformations and quadratic representations on H.

© 2009 Elsevier Inc. All rights reserved.

1. Introduction

This article deals with the eigenvalues of a linear transformation relative to a cone. Given a closed convex cone K in a finite dimensional real Hilbert space H and a linear transformation L on H, we say that a real number λ is a K-eigenvalue of L if there exists a nonzero x in H such that

$$x \in K, \quad L(x) - \lambda x \in K^*, \quad \text{and} \quad \langle x, L(x) - \lambda x \rangle = 0, \tag{1}$$

where K^* denotes the dual cone of K. While the above precise formulation appears in Seeger [23] and Lavilledieu and Seeger [16], slightly different (variational inequality) formulations have appeared much earlier in the works of Riddel [19] on nonlinear elliptic variational inequalities on a cone, Kučera [14,15], and Quittner [18] on bifurcation analysis of eigenvalues relative to a cone. Motivated by the

---

* Corresponding author.
E-mail addresses: yzhou@bios.unc.edu (Y. Zhou), gowda@math.umbc.edu (M. Seetharama Gowda).
URL: http://www.math.umbc.edu/gowda (M. Seetharama Gowda).

study of contact problems in mechanics [3], recently Queiroz et al. [17] and Judice et al. [13] studied the so-called eigenvalue complementarity problem (EiCP): Given two $n \times n$ real matrices A and B with B positive definite, find a scalar $\mu > 0$ and a nonzero vector u such that

$$v = (\mu B - A)u, \quad u \geqslant 0, \ v \geqslant 0, \quad \text{and} \quad \langle u, v \rangle = 0,$$

where $u \geqslant 0$ means that u belongs to the non-negative orthant R_+^n. These two articles describe various equivalent formulations of EiCP and computational methods for solving it. We note that when B is symmetric, EiCP reduces to (1) with $K := \sqrt{B}(R_+^n)$, $L = -(\sqrt{B})^{-1}A(\sqrt{B})^{-1}$, and $\lambda = -\mu$.

Going back to (1), when K is all of H, a K-eigenvalue of L is nothing but an eigenvalue of L and hence, in this setting, L may not have K-eigenvalues. However, when K is pointed, that is, when $K \cap (-K) = \{0\}$, Seeger [23] has shown, via Ky Fan's inequality, that for any L, there is always a K-eigenvalue. Based on this, Seeger and Torki [24] have shown that the same conclusion holds as long as K is not a subspace. In addition, when L is symmetric (=self-adjoint), Seeger and Torki [24] show that (1) is the stationary condition for the minimization problem

$$\min\{\langle L(x), x \rangle : x \in K, \ \langle x, x \rangle = 1\}.$$

For a detailed study of local minimizers of this problem, see [25]. For a given L and K, the set $\sigma(L, K)$ of all K-eigenvalues of L will be called the K-spectrum (or the cone spectrum) of L. When K is a polyhedral set generated by p vectors, Seeger [23] has shown that

$$|\sigma(L, K)| \leqslant p2^{p-1}.$$

To study the finiteness issue for nonpolyhedral cones, in [24], Seeger and Torki consider the so-called Lorentz cone $\mathcal{L}_+^n$ and show that for any symmetric transformation, the $\mathcal{L}_+^n$-spectrum is always finite and construct a 5×5 non-symmetric matrix for which the $\mathcal{L}_+^n$-spectrum is infinite; See Iusem and Seeger [12] for an example of a symmetric transformation on a nonpolyhedral cone with infinite spectrum. Influenced by this, they raise the problem of identifying cones K for which the K-spectrum of every symmetric transformation is finite. Motivated by these considerations and the observation that the Lorentz cone is a symmetric cone in the Euclidean Jordan algebra $\mathcal{L}^n$, in this article we consider some special types of transformations on Euclidean Jordan algebras and describe their cone spectra. Our focus here is on Lyapunov transformations and quadratic representations which have been extensively studied in the literature on complementarity problems [5–10]. Interestingly enough, for these special transformations, the cone spectrum is contained in the spectrum, thus proving the finiteness.

A Euclidean Jordan algebra V is a finite dimensional real Hilbert space with a Jordan product. The closed convex cone of all squares in this algebra is a symmetric cone (see Section 2 for definitions). For any element $a \in V$, we consider the corresponding Lyapunov transformation and quadratic representation, defined respectively by

$$L_a(x) := a \circ x \quad \text{and} \quad P_a(x) := 2a \circ (a \circ x) - a^2 \circ x \ (x \in V), \tag{2}$$

where for any two elements $x, y \in V$, $x \circ y$ denotes the Jordan product and $x^2 := x \circ x$. In this article, we show that with respect to the symmetric cone in V, the cone spectrum of each of these two symmetric transformations is contained in the spectrum and explicitly compute the cone spectrum. Moreover, we prove the finite cone spectrum property for any linear transformation on V that satisfies the **Z**-property:

$$x \in K, \ y \in K^* \quad \text{and} \quad \langle x, y \rangle = 0 \Rightarrow \langle L(x), y \rangle \leqslant 0.$$

Examples of such transformations include the well-known Lyapunov and Stein transformations, L_A and S_A, where for any $A \in R^{n \times n}$,

$$L_A(X) := \frac{1}{2}(AX + XA^T) \quad \text{and} \quad S_A(X) := X - AXA^T \ (X \in \mathcal{S}^n). \tag{3}$$

Here $\mathcal{S}^n$ denotes the Euclidean Jordan algebra of all $n \times n$ real symmetric matrices with Jordan product $X \circ Y = \frac{1}{2}(XY + YX)$.

An outline of the paper is as follows. In Section 2, we recall Euclidean Jordan algebra concepts and results. Section 3 deals with the finite cone spectrum result for **Z**-transformations. In

Section 4, we describe the cone spectrum of quadratic representations and of the two-sided multiplication transformation M_A.

2. Preliminaries

2.1. Euclidean Jordan algebras

In this subsection, we briefly recall concepts and results from the Euclidean Jordan algebra theory. For further details, we refer to [4,20,9].

A *Euclidean Jordan algebra* is a triple $(V, \circ, \langle \cdot, \cdot \rangle)$, where $(V, \langle \cdot, \cdot \rangle)$ is a finite dimensional space over R with inner product $\langle \cdot, \cdot \rangle$ and $(x, y) \mapsto x \circ y : V \times V \to V$ is a bilinear mapping satisfying the conditions $x \circ y = y \circ x$, $x \circ (x^2 \circ y) = x^2 \circ (x \circ y)$, and $\langle x \circ y, z \rangle = \langle y, x \circ z \rangle$ for all $x, y, z \in V$. We also assume that there exists a unit element $e \in V$ such that $x \circ e = x$ for all $x \in V$.

It is known (see Chapter V, [4]) that every Euclidean Jordan algebra is, up to an isomorphism, a product of (some) of the following: The Jordan spin algebra $\mathcal{L}^n$ ($n > 1$) whose underlying space is $R \times R^{(n-1)}$ with the usual inner product and $(x_0, \bar{x}) \circ (y_0, \bar{y}) = (x_0 y_0 + \langle \bar{x}, \bar{y} \rangle, x_0 \bar{y} + y_0 \bar{x})$, matrix algebras $\mathcal{S}^n$ of all $n \times n$ real symmetric matrices, $\mathcal{H}^n$ of all $n \times n$ complex Hermitian matrices, $\mathcal{Q}^n$ of all $n \times n$ quaternion Hermitian matrices, and $\mathcal{O}^3$ of all 3×3 octonion Hermitian matrices. In the matrix algebras, the Jordan and inner product are given respectively by $X \circ Y := \frac{1}{2}(XY + YX)$ and $\langle X, Y \rangle := Re\ trace(XY)$. (For basic properties of quaternions and octonions, see [26,1].)

In a Euclidean Jordan algebra V, the *symmetric cone* is the set of squares

$$K := \{x \circ x : x \in V\}.$$

K is a self-dual cone; in particular, $\langle x, y \rangle \geq 0$ for all $x, y \in K$. When K is a cone of matrices, we use the standard notations

$$X \succeq 0 \quad \text{for } X \in K \quad \text{and} \quad X \succ 0 \quad \text{for } X \in \text{interior}(K).$$

In V, an element $c \in V$ is an *idempotent* if $c^2 = c$; an idempotent c is a *primitive idempotent* if it is nonzero and cannot be written as a sum of two nonzero idempotents. A finite set $\{e_1, e_2, \ldots, e_m\}$ of primitive idempotents in V is a *Jordan frame* if $e_i \circ e_j = 0$ if $i \neq j$, and $\sum_1^m e_i = e$. We note that a Jordan frame is an orthogonal set.

Theorem 1 (The spectral decomposition theorem [4]). *Let V be a Euclidean Jordan algebra. Then there is a number r (called the rank of V) with the property that for every $x \in V$, there exist a Jordan frame $\{e_1, \ldots, e_r\}$ and real numbers $\lambda_1, \ldots, \lambda_r$ such that*

$$x = \lambda_1 e_1 + \cdots + \lambda_r e_r. \tag{4}$$

We refer to (4) as the *spectral decomposition (or the spectral expansion)* of x. The real numbers $\lambda_1, \lambda_2, \ldots, \lambda_r$ are called the *eigenvalues* of x. We say that an object x in V is *invertible* if all its eigenvalues are nonzero.

Let c be an idempotent in V. For $\gamma \in \{0, \frac{1}{2}, 1\}$, define the Peirce eigenspaces

$$V(c, \gamma) := \{x \in V : x \circ c = \gamma x\}.$$

We have

Theorem 2 (Peirce decomposition theorem I, Proposition IV.1.1, [4]). *V is an orthogonal direct sum of the eigenspaces $V(c, \gamma)$:*

$$V = V(c, 1) \oplus V\left(c, \frac{1}{2}\right) \oplus V(c, 0).$$

Moreover, $V(c, 1)$ and $V(c, 0)$ are subalgebras of V, and

$V(c, 1) \circ V(c, 0) = \{0\};$

$V(c, 1) \circ V\left(c, \frac{1}{2}\right) \subseteq V\left(c, \frac{1}{2}\right);$

$V(c, 0) \circ V\left(c, \frac{1}{2}\right) \subseteq V\left(c, \frac{1}{2}\right);$

$V\left(c, \frac{1}{2}\right) \circ V\left(c, \frac{1}{2}\right) \subseteq V(c, 1) + V(c, 0).$

Let $\{e_1, e_2, \ldots, e_r\}$ be a Jordan frame in V. For $i, j \in \{1, 2, \ldots, r\}$, define the eigenspaces

$V_{ii} := \{x \in V : x \circ e_i = x\} = R\, e_i$

and when $i \neq j$,

$V_{ij} := \left\{ x \in V : x \circ e_i = \frac{1}{2}x = x \circ e_j \right\}.$

Then we have the following:

Theorem 3 (Peirce decomposition theorem II, Theorem IV.2.1, [4]). *The space V is the orthogonal direct sum of spaces V_{ij} $(i \leqslant j)$. Moreover,*

$V_{ij} \circ V_{ij} \subset V_{ii} + V_{jj};$

$V_{ij} \circ V_{jk} \subset V_{ik} \quad \text{if } i \neq k;$

$V_{ij} \circ V_{kl} = \{0\} \quad \text{if } \{i, j\} \cap \{k, l\} = \emptyset.$

In view of the above theorem, given any Jordan frame $\{e_1, e_2, \ldots, e_r\}$ and any $x \in V$, we have the *Peirce decomposition*

$$x = \sum_{i=1}^{r} x_i e_i + \sum_{i<j} x_{ij}, \tag{5}$$

where $x_i \in R$ and $x_{ij} \in V_{ij}$.

2.2. Lyapunov transformations and quadratic representations

Let V be a Euclidean Jordan algebra. For any element $a \in V$, we define Lyapunov transformation L_a and the quadratic representation P_a by

$L_a(x) := a \circ x \quad \text{and} \quad P_a(x) := 2a \circ (a \circ x) - a^2 \circ x.$

These transformations are linear and self-adjoint on V. Moreover, $P_a(K) \subseteq K$ for all a and $P_{P_a(x)} = P_a P_x P_a$ for all a and x, see Proposition III.2.2, [4] and Corollary II.3.2, [4].

For example, on the algebra $\mathcal{S}^n$ with $A \in \mathcal{S}^n$, the above transformations are given by

$L_A(X) = A \circ X = \frac{1}{2}(AX + XA) \quad \text{and} \quad P_A(X) = AXA.$

Since multiplication of matrices with complex or quaternion entries satisfy the associative property, we get similar expressions on $\mathcal{H}^n$ and $\mathcal{Q}^n$.

Proposition 4. *Suppose that the spectral decomposition of $a \in V$ is given by $a = a_1 e_1 + a_2 e_2 + \cdots + a_r e_r$, where $\{e_1, e_2, \ldots, e_r\}$ is a Jordan frame. Let x be given by* (5). *Then*

$$L_a(x) = \sum_{i=1}^{r} a_i x_i e_i + \sum_{i<j} \frac{a_i + a_j}{2} x_{ij} \tag{6}$$

and

$$P_a(x) = \sum_{i=1}^{r} a_i^2 x_i e_i + \sum_{i<j} a_i a_j x_{ij}. \tag{7}$$

Proof. From the properties of the Peirce spaces V_{ij}, we have $L_a(e_i) = a_i e_i$ and $L_a(x_{ij}) = \frac{a_i + a_j}{2} x_{ij}$. Also, $P_a(e_i) = a_i^2 e_i$ and $P_a(x_{ij}) = a_i a_j x_{ij}$. The stated results follow from the linearity of L_a and P_a. □

We say that two objects a and b in a Euclidean Jordan algebra V *operator commute* if their corresponding Lyapunov transformations commute, that is, $L_a L_b = L_b L_a$. It is known (see [4]) that a and b operator commute if and only if a and b have their spectral decompositions with respect to a common Jordan frame.

Proposition 5. (Proposition 6, [9]). *For a and b in V, the following are equivalent*:

(i) $a \in K, b \in K$, *and* $\langle a, b \rangle = 0$.
(ii) $a \in K, b \in K$, *and* $a \circ b = 0$.

In each case, a and b operator commute.

Remark 1. For a nonzero, non-unit idempotent c, consider the Peirce spaces $V(c, 1)$ and $V(c, 0)$. Then any $x \in V(c, 1)$ operator commutes with any $w \in V(c, 0)$. This can be seen by first noting that primitive idempotents in $V(c, 1)$ and $V(c, 0)$ are orthogonal and hence operator commute. The operator commutativity of x and w follow from their spectral expansions in $V(c, 1)$ and $V(c, 0)$.

The following results are crucially used in the proof of Theorem 12.

Proposition 6. *Let c be a nonzero idempotent in V with corresponding Peirce decomposition*

$$V = V(c, 1) \oplus V\left(c, \frac{1}{2}\right) \oplus V(c, 0).$$

Let $a = u + v + w$, where $u \in V(c, 1), v \in V\left(c, \frac{1}{2}\right)$, and $w \in V(c, 0)$. Then for any $x \in V(c, 1)$, we have the following:

(1) $P_x(v) = 0$ *and* $u \circ (x \circ v) + x \circ (u \circ v) = (u \circ x) \circ v$.
(2) $P_v(x) \in V(c, 0)$.
(3) $P_a(x) = P_u(x) + 4u \circ (v \circ x) + P_v(x)$ *with* $P_u(x) \in V(c, 1)$ *and* $4u \circ (v \circ x) \in V\left(c, \frac{1}{2}\right)$.
(4) $[P_u(x) = \lambda x, \lambda \neq 0, x$ *invertible in* $V(c, 1)] \Rightarrow u$ *invertible in* $V(c, 1)$.
(5) $[x \circ z = 0, x$ *invertible in* $V(c, 1)$ *and* $z \in V\left(c, \frac{1}{2}\right)] \Rightarrow z = 0$. *Hence,* $[u \circ (v \circ x) = 0, u$ *and* x *invertible in* $V(c, 1)$, *and* $v \in V\left(c, \frac{1}{2}\right)] \Rightarrow v = 0$.

Proof. Fix an $x \in V(c, 1)$. As $V(c, 1)$ is a Euclidean Jordan algebra, there exists a Jordan frame $\{f_1, f_2, \ldots, f_k\}$ in $V(c, 1)$ such that the spectral decomposition of x is given by

$$x = x_1 f_1 + x_2 f_2 + \cdots + x_k f_k. \tag{8}$$

As c is the unit element in $V(c, 1)$, we have

$$f_1 + f_2 + \cdots + f_k = c. \tag{9}$$

Let $\{f_{k+1}, f_{k+2}, \ldots, f_r\}$ be a (fixed) Jordan frame in $V(c, 0) = V(e - c, 1)$. Then $\{f_1, f_2, \ldots, f_k, f_{k+1}, \ldots, f_r\}$ is a Jordan frame in V. With respect to this Jordan frame, by Peirce decomposition theorem II (or Lemma 20, [9]),

$$V(c, 1) = \sum_{\alpha} V_{ij}, \quad V\left(c, \frac{1}{2}\right) = \sum_{\beta} V_{ij}, \quad \text{and} \quad V(c, 0) = \sum_{\gamma} V_{ij}, \tag{10}$$

where

$$\alpha := \{(i,j) : 1 \leqslant i \leqslant j \leqslant k\}, \quad \beta := \{(i,j) : 1 \leqslant i \leqslant k, k+1 \leqslant j \leqslant r\},$$

and

$$\gamma := \{(i,j) : k+1 \leqslant i \leqslant j \leqslant r\}.$$

(1) As $v \in V(c, \frac{1}{2}) = \sum_{\beta} V_{ij}$, we may write $v = \sum_{\beta} v_{ij}$, where $v_{ij} \in V_{ij}$. With

$$x = x_1 f_1 + x_2 f_2 + \cdots + x_k f_k + 0 f_{k+1} + \cdots + 0 f_r,$$

we may let $x_j = 0$ for $k+1 \leqslant j \leqslant r$. Then from Proposition 4, $P_x(v) = \sum_{\beta} x_i x_j v_{ij} = 0$. For x and u in $V(c, 1)$, we have $x + u \in V(c, 1)$. Replacing x by $x + u$ in $P_x(v) = 0$, we get $P_{x+u}(v) = 0$. Now expanding and simplifying $P_{x+u}(v) = 0$, we get $u \circ (x \circ v) + x \circ (u \circ v) = (u \circ x) \circ v$.

(2) We need to show that $P_v(x) \circ c = 0$. From (9),

$$\sum_{i=1}^{k} \langle P_v(f_i), c \rangle = \langle P_v(c), c \rangle = \langle v^2 - v^2 \circ c, c \rangle = \langle v^2, c \rangle - \langle v^2, c \circ c \rangle = 0,$$

where we have used the definition of $P_v(c)$ and the facts that $\langle v^2 \circ c, c \rangle = \langle v^2, c \circ c \rangle$ and $c^2 = c$. As $P_v(K) \subseteq K$ and $\langle P_v(f_i), c \rangle \geqslant 0$, the above equality yields $\langle P_v(f_i), c \rangle = 0$ for every f_i. By Proposition 5, $P_v(f_i) \circ c = 0$. Using the linearity of P_v we get $P_v(x) \circ c = 0$.

(3) Using the properties of Peirce spaces, we get $u \circ x, u^2 \in V(c, 1), w^2 \in V(c, 0), v \circ x \in V(c, \frac{1}{2}), w \circ x = 0$, and $w^2 \circ x = 0$. Hence, $a \circ x = u \circ x + v \circ x$, $a \circ (a \circ x) = u \circ (u \circ x) + u \circ (v \circ x) + v \circ (u \circ x) + v \circ (v \circ x) + w \circ (v \circ x)$, and

$$\begin{aligned} a^2 \circ x &= (u^2 + v^2 + w^2 + 2u \circ v + 2v \circ w) \circ x \\ &= u^2 \circ x + v^2 \circ x + 2(u \circ v) \circ x + 2(v \circ w) \circ x. \end{aligned}$$

Now, using Remark 1, we have $w \circ (x \circ v) = x \circ (w \circ v)$. Hence, a direct computation leads to

$$\begin{aligned} P_a(x) &= 2a \circ (a \circ x) - a^2 \circ x \\ &= P_u(x) + 2[u \circ (v \circ x) + v \circ (u \circ x) - (u \circ v) \circ x] + P_v(x). \end{aligned}$$

Using Item (1), we have

$$P_a(x) = P_u(x) + 4u \circ (v \circ x) + P_v(x).$$

Now, u and x belong to $V(c, 1)$ (which is an algebra) and hence $P_u(x) \in V(c, 1)$. Also, from the properties of the Peirce eigenspaces, $x \circ v \in V\left(c, \frac{1}{2}\right)$ and $u \circ (x \circ v) \in V\left(c, \frac{1}{2}\right)$.

(4) As both x and u belong to $V(c, 1)$, and P_u maps $V(c, 1)$ into $V(c, 1)$, we can work within the algebra $V(c, 1)$ and prove this statement. Suppose $P_u(x) = \lambda x$. Then $P_{\lambda x} = P_u P_x P_u$ (see Corollary II.3.2, [4]). When x is invertible in $V(c, 1)$ and $\lambda \neq 0$, $P_{\lambda x}$ is invertible in $V(c, 1)$ (see Proposition II.3.1 in [4]). It follows that P_u is invertible in $V(c, 1)$. Hence, u is invertible in $V(c, 1)$ (once again, from Proposition II.3.1 in [4]).

(5) Let x be invertible in $V(c, 1)$ and $z \in V\left(c, \frac{1}{2}\right)$ with $x \circ z = 0$. From (8) and (10), we can write

$$x = x_1 f_1 + x_2 f_2 + \cdots + x_k f_k + 0 f_{k+1} + \cdots + 0 f_r$$

and $z = \sum_{\beta} z_{ij}$ with $z_{ij} \in V_{ij}$. Letting $x_j = 0$ for $k+1 \leqslant j \leqslant r$, from Proposition 4,

$$0 = x \circ z = L_x(z) = \sum_{\beta} \frac{(x_i + x_j)}{2} z_{ij} = \sum_{\beta} \frac{x_i}{2} z_{ij}.$$

Due to the orthogonality of z_{ij}s and $x_i \neq 0$ for $i = 1, 2, \ldots, k$, we get $z_{ij} = 0$ and hence $z = 0$. The second statement in Item (5) follows immediately by two applications of the first statement. $\square$

Proposition 7. *For $a \in V$, suppose $P_a(x) = \lambda x + y$ with $\lambda \neq 0, x, y \in K$, and $\langle x, y \rangle = 0$. Then $y = 0$.*

Proof. Without loss of generality, let $x \neq 0$. As $\lambda \neq 0$ and $P_a(K) \subseteq K$, we have $0 \leqslant \langle P_a(x), x \rangle = \lambda ||x||^2$ and so $\lambda > 0$. Now, by Proposition 5, x and y operator commute and hence there exists a Jordan frame $\{e_1, e_2, \ldots, e_r\}$ such that $x = \lambda_1 e_1 + \lambda_2 e_2 + \cdots + \lambda_k e_k$ and $y = \mu_{k+1} e_{k+1} + \cdots + \mu_r e_r$ with $\lambda_i > 0$ for all $i = 1, 2, \ldots, k$ and $\mu_j \geqslant 0$ for $j = k + 1, \ldots, r$. We have to show that $y = 0$. Let $c := e_1 + e_2 + \cdots + e_k$. Then $x \in V(c, 1)$ and $y \in V(c, 0)$. Using Item (3) in Proposition 6, we have

$$\lambda x + y = P_a(x) = P_u(x) + 4u \circ (v \circ x) + P_v(x),$$

where $a = u + v + w$ with $u \in V(c, 1)$, $v \in V\left(c, \frac{1}{2}\right)$, and $w \in V(c, 0)$. By the orthogonality of the Peirce spaces, we have

$$P_u(x) = \lambda x, \quad 4u \circ (v \circ x) = 0, \quad \text{and} \quad P_v(x) = y.$$

As x is invertible in $V(c, 1)$, from Item (4) in Proposition 6, we have the invertibility of u in $V(c, 1)$. From Item (5), Proposition 6, we have $v = 0$. Hence, $y = P_v(x) = 0$, proving the proposition. $\square$

3. Z-transformations

A square real matrix is said to be a **Z**-matrix [2] if all its off-diagonal entries are nonpositive. Below, we consider a generalization of this concept which is nothing but the negative of a cross-positive transformation introduced in [22].

Definition 8 [10]. Let H be a finite dimensional real Hilbert space and K be a proper cone in H, i.e., $K \cap (-K) = \{0\}$ and $K - K = H$, with K^* denoting the dual. A linear transformation $L : H \longrightarrow H$ is said to have the **Z**-property with respect to K (or that L is a **Z**-transformation) if

$$x \in K, \quad y \in K^*, \quad \text{and} \quad \langle x, y \rangle = 0 \Rightarrow \langle L(x), y \rangle \leqslant 0.$$

We remark that any symmetric cone (in a Euclidean Jordan algebra) is a proper cone. For any $A \in R^{n \times n}$, the Lyapunov transformation L_A and the Stein transformation S_A defined on $\mathcal{S}^n$ by (3) are **Z**-transformations. So is the transformation L on $\mathcal{S}^n$ defined by

$$L(X) := X - A_1 X A_1^T - A_2 X A_2^T - \cdots - A_m X A_m^T,$$

where $A_i \in R^{n \times n}$ ($i = 1, 2, \ldots, m$). For properties and further examples of **Z**-transformations, we refer to [10].

We now prove our first finite cone spectrum result. In what follows, $\sigma(L)$ denotes the spectrum of the transformation L; recall that $\sigma(L, K)$ is the cone spectrum of L with respect to K.

Theorem 9. *Let H be a finite dimensional real Hilbert space and K be a proper cone. Suppose that L has the **Z**-property with respect to K. Then for any real λ,*

$$\left. \begin{array}{l} L(x) = \lambda x + y \\ x \in K, y \in K^*, \langle x, y \rangle = 0 \end{array} \right\} \Rightarrow y = 0.$$

Hence, $\sigma(L, K) \subseteq \sigma(L)$ and $|\sigma(L, K)| \leqslant \dim(H) < \infty$.

Proof. We observe that $S := L - \lambda I$ has the **Z**-property on K. Hence, from $L(x) = \lambda x + y$, $x \in K$, $y \in K^*$, $\langle x, y \rangle = 0$, we get $\langle S(x), S(x) \rangle = \langle S(x), y \rangle \leqslant 0$. This gives $S(x) = 0$, i.e., $y = L(x) - \lambda x = 0$. The containment of the cone spectrum in the spectrum and its finiteness are obvious. $\square$

We now describe the cone spectrum of the Lyapunov transformation L_a on a Euclidean Jordan algebra.

Corollary 10. *Consider a Euclidean Jordan algebra V with corresponding symmetric cone K and an element $a \in V$ with its spectral decomposition $a = a_1 e_1 + a_2 e_2 + \cdots + a_r e_r$. Then*

$$\sigma(L_a, K) = \{a_1, a_2, \ldots, a_r\}.$$

Note: It is well known [20] (and follows from (6)) that

$$\sigma(L_a) \subseteq \left\{ \frac{a_i + a_j}{2} : i, j = 1, 2, \ldots, r \right\}$$

with equality holding when V is simple, that is, when V is not the direct sum of two nontrivial Euclidean Jordan algebras.

Proof. First of all, from $L_a(e_i) = a_i e_i$, we see that $\{a_1, a_2, \ldots, a_r\} \subseteq \sigma(L_a, K)$. Now suppose $\lambda \in \sigma(L_a, K)$. As L_a has the **Z**-property with respect to K, from the above theorem, we get a nonzero $x \in K$ such that $L_a(x) = \lambda x$. Let x be given by (5). In view of (6) and the orthogonality of the Peirce spaces, $L_a(x) = \lambda x$ leads to $a_i x_i = \lambda x_i$ for all $i = 1, 2, \ldots, r$ and $\frac{a_i + a_j}{2} x_{ij} = \lambda x_{ij}$ for all $i < j$. As $0 \neq x \in K$, there is at least one index i for which $x_i \neq 0$, see Exercise 7(b), Chapter IV, [4]. For such an i, $\lambda = a_i$. Thus $\lambda \in \{a_1, a_2, \ldots, a_r\}$. $\square$

Given an $n \times n$ complex matrix A, consider the (Lyapunov) transformation L_A on $\mathcal{H}^n$ (the algebra of all $n \times n$ complex Hermitian matrices) defined by

$$L_A(X) = \frac{1}{2}(AX + XA^*).$$

The cone spectrum of such a transformation is described below.

Theorem 11. *For an $n \times n$ complex matrix A, consider L_A on $\mathcal{H}^n$ defined above. Let $\sigma(A) = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ denote the spectrum of A. Then*

$$\sigma(L_A, \mathcal{H}_+^n) = \{Re(\lambda_1), Re(\lambda_2), \ldots, Re(\lambda_n)\},$$

where $\mathcal{H}_+^n$ denotes the (symmetric) cone of positive semidefinite matrices in $\mathcal{H}^n$.

Proof. Suppose w_j is an eigenvector corresponding to an eigenvalue λ_j of A. Then $0 \neq W := w_j \overline{w_j}^T \succeq 0$ satisfies $AW + WA^* = 2Re(\lambda_j)W$ proving $Re(\lambda_j) \in \sigma(L_A, \mathcal{H}_+^n)$. Now suppose that $\lambda \in \sigma(L_A, \mathcal{H}_+^n)$. Then there there exist a nonzero $X \succeq 0$ and $Y \succeq 0$ in $\mathcal{H}^n$ such that $L_A(X) = \lambda X + Y$ with $\langle X, Y \rangle = 0$. As L_A has the **Z**-property on $\mathcal{H}_+^n$, from Theorem 9, $Y = 0$ and so $AX + XA^* = 2\lambda X$. Writing $X = UDU^*$, where D is a (nonzero) nonnegative diagonal matrix and U is unitary, we get $BD + DB^* = 2\lambda D$ with $B = U^*AU$. Permuting D (if necessary), we may decompose D and B as

$$D = \begin{bmatrix} D_1 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{with } D_1 \succ 0, \quad \text{and} \quad B = \begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix}.$$

(If D is invertible, then certain blocks in D and B become vacuous.)
Putting these in $BD + DB^* = 2\lambda D$, we get

$$B_1 D_1 + D_1 B_1^* = 2\lambda D_1 \quad \text{and} \quad B_3 D_1 = 0.$$

As D_1 is positive definite, $B_3 = 0$. From the block form of B, we see that

$$B = \begin{bmatrix} B_1 & B_2 \\ 0 & B_4 \end{bmatrix}$$

and so

$$\sigma(B_1) \subseteq \sigma(B) = \sigma(A).$$

Now $B_1 D_1 + D_1 B_1^* = 2\lambda D_1$ implies that

$$\left(\sqrt{D_1}\right)^{-1} B_1 \sqrt{D_1} + \sqrt{D_1} B_1^* \left(\sqrt{D_1}\right)^{-1} = 2\lambda I_1,$$

where I_1 is the identity matrix whose size agrees with that of D_1. Putting $C_1 := (\sqrt{D_1})^{-1} B_1 \sqrt{D_1}$, we see that $C_1 + C_1^* = 2\lambda I$ and that C_1 is normal. If μ is an eigenvalue of C_1 with a corresponding eigenvector u, then $C_1 u = \mu u$ and (because C_1 is normal) $C_1^* u = \overline{\mu} u$. Thus $2\lambda u = (C_1 + C_1^*)u = 2Re(\mu)u$ giving us $\lambda = Re(\mu)$. Finally, we observe that $\mu \in \sigma(C_1) = \sigma(\sqrt{D_1})^{-1} B_1 \sqrt{D_1} = \sigma(B_1) \subseteq \sigma(A)$. This completes the proof. $\square$

4. Quadratic representations

Let V be a Euclidean Jordan algebra with the corresponding symmetric cone K. In this section we describe the cone spectrum of P_a and, in particular, show that it is finite.

Theorem 12. *Consider $a \in V$ with its spectral decomposition*

$$a = a_1 e_1 + a_2 e_2 + \cdots + a_r e_r. \tag{11}$$

Then

$$\{a_1^2, a_2^2, \ldots, a_r^2\} \subseteq \sigma(P_a, K) \subseteq \{0, a_1^2, a_2^2, \ldots, a_r^2\}.$$

Proof. From (11), it is easily seen that $P_a(e_i) = a_i^2 e_i$ for each i. Thus $\{a_1^2, a_2^2, \ldots, a_r^2\} \subseteq \sigma(P_a, K)$. Now suppose that $0 \neq \lambda \in \sigma(P_a, K)$. Then there exist $0 \neq x \in K$ and $y \in K$ such that $P_a(x) = \lambda x + y$ with $\langle x, y \rangle = 0$. In view of Proposition 7, we have $y = 0$ and so $P_a(x) = \lambda x$. Let x be given by (5). Then from (7), we have

$$\sum_{i=1}^{r} a_i^2 x_i e_i + \sum_{i<j} a_i a_j x_{ij} = \lambda \left(\sum_{i=1}^{r} x_i e_i + \sum_{i<j} x_{ij} \right).$$

Now, the orthogonality in Peirce decomposition results in $a_i^2 x_i = \lambda x_i$ for all i and $a_i a_j x_{ij} = \lambda x_{ij}$ for all $i < j$. As $0 \neq x \in K$, there is at least one index i for which $x_i \neq 0$, see Exercise 7(b), Chapter IV, [4]. For such an i, $\lambda = a_i^2$. This proves $\sigma(P_a, K) \subseteq \{0, a_1^2, a_2^2, \ldots, a_r^2\}$ and consequently the theorem. $\square$

The following example shows that zero can be a K-eigenvalue of P_a without it being (the square of) an eigenvalue of a.

Example. Let $V = \mathcal{S}^2$ and

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Then $P_A(X) = AXA$ for all $X \in \mathcal{S}^2$. For the matrix A, the eigenvalues are -1 and 1. However, from

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

we see that zero is in the cone spectrum of P_A.

Given an $n \times n$ complex matrix A, consider the transformation M_A on $\mathcal{H}^n$ defined by

$$M_A(X) = AXA^*.$$

(Note that when $A \in \mathcal{H}^n$, this transformation reduces to a quadratic representation on $\mathcal{H}^n$.) The result below describes the cone spectrum of such a transformation.

Theorem 13. *For an $n \times n$ complex matrix A, consider M_A defined above. Then*

$$\{|\lambda_1|^2, |\lambda_2|^2, \ldots, |\lambda_n|^2\} \subseteq \sigma(M_A, \mathcal{H}_+^n) \subseteq \{0, |\lambda_1|^2, |\lambda_2|^2, \ldots, |\lambda_n|^2\},$$

where $\sigma(A) = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$.

Proof. Let w_j be an eigenvector corresponding to the eigenvector λ_j of A. Then $0 \neq W := w_j \overline{w}_j^T$ is positive semidefinite and $M_A(W) = |\lambda_j|^2 W$. Hence, we have the first inclusion in the above theorem.

Suppose $0 \neq \lambda \in \sigma(M_A, \mathcal{H}_+^n)$. We show that $\lambda = |\lambda_i|^2$ for some $\lambda_i \in \sigma(A)$ thereby proving the second inclusion of the theorem. Now there exist Y and a nonzero X in $\mathcal{H}_+^n$ such that $M_A(X) = \lambda X + Y$ and $\langle X, Y \rangle = 0$. Since operator commutativity in $\mathcal{H}^n$ is the same as matrix commutativity, X and Y commute. Hence, we can write $X = UDU^*$ and $Y = UEU^*$ for some (nonnegative) diagonal matrices $D \neq 0$ and E, with U being a unitary matrix. Then $AXA^* = \lambda X + Y$ implies $BDB^* = \lambda D + E$, where $B = U^*AU$. After a suitable rearrangement, we may write

$$D = \begin{bmatrix} D_1 & 0 \\ 0 & 0 \end{bmatrix}, \quad E = \begin{bmatrix} 0 & 0 \\ 0 & E_4 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix},$$

where D_1 is a diagonal matrix with positive entries and E_4 is a diagonal matrix with nonnegative entries. Then $BDB^* = \lambda D + E$ yields

$$B_1 D_1 B_1^* = \lambda D_1, \quad B_3 D_1 B_1^* = 0, \quad \text{and} \quad B_3 D_1 B_3^* = E_4.$$

We note that D_1 is invertible and $\lambda \neq 0$. Thus $B_1 D_1 B_1^*$ is invertible and so B_1 is invertible. From $B_3 D_1 B_1^* = 0$ we get $B_3 = 0$. Then $E_4 = B_3 D_1 B_3^* = 0$. From this we get $E = 0$, and $Y = 0$. Then,

$$B = \begin{bmatrix} B_1 & B_2 \\ 0 & B_4 \end{bmatrix}$$

and so

$$\sigma(B_1) \subseteq \sigma(B) = \sigma(A).$$

Now the equality $B_1 D_1 B_1^* = \lambda D_1$ leads to

$$\left(\sqrt{D_1}\right)^{-1} B_1 \sqrt{D_1} \sqrt{D_1} B_1^* (\sqrt{D_1})^{-1} = \lambda I_1,$$

where I_1 is the identity matrix whose size agrees with that of B_1. Putting $C_1 := \frac{1}{\sqrt{\lambda}} (\sqrt{D_1})^{-1} B_1 \sqrt{D_1}$, we see that C_1 is a unitary matrix and hence every eigenvalue of C_1 has absolute value one. If $e^{i\theta}$ is an eigenvalue of C_1, then $e^{i\theta} = \frac{1}{\sqrt{\lambda}} \lambda_j$, where $\lambda_j \in \sigma(\sqrt{D_1}^{-1} B_1 \sqrt{D_1}) = \sigma(B_1) \subseteq \sigma(A)$. Thus, $\lambda = |\lambda_j|^2$ proving the other inclusion in the stated theorem. $\quad\square$

Remarks. A cone automorphism of $\mathcal{H}^n$ is an invertible linear transformation Γ on $\mathcal{H}^n$ that satisfies $\Gamma(\mathcal{H}_+^n) = \mathcal{H}_+^n$. It is well known (see e.g., [21]) that cone automorphisms of $\mathcal{H}^n$ are given by

$$\Gamma(X) = AXA^* \quad (X \in \mathcal{H}^n)$$

for some nonsingular matrix A. In view of this, we may say that every cone automorphism of $\mathcal{H}^n$ has finite cone spectrum. A similar result holds for $\mathcal{S}^n$.

5. Conclusions

In this paper, motivated by a problem posed by Seeger and Torki [24] we showed that for **Z**-transformations on proper cones and quadratic representations on symmetric cones (in a Euclidean Jordan algebra), the cone spectrum is finite. A different proof of Theorem 12 based on case-by-case analysis on simple algebras appears in the technical report [27]. Yet another proof based on inertia formula appears in [11].

References

[1] J.C. Baez, The octonions, Bull. Amer. Math. Soc. 39 (2002) 145–205.
[2] A. Berman, R.J. Plemmons, Nonnegative Matrices in the Mathematical Sciences, SIAM, Philadelphia, 1994.
[3] A.P. Costa, I.N. Figueiredo, J. Judice, J.A.C. Martins, A complementarity eigen problem in the stability analysis of finite dimensional elastic systems with frictional contact, in: M. Ferris, O.L. Mangasarian, J.-S. Pang (Eds.), Complementarity: Applications, Algorithms, and Extensions, Kluwer, New York, 2001, pp. 67–83.
[4] J. Faraut, A. Korányi, Analysis on Symmetric Cones, Oxford University Press, Oxford, 1994.
[5] M.S. Gowda, T. Parthasarathy, Complementarity forms of the theorems of Lyapunov and Stein, and related results, Linear Algebra Appl. 320 (2000) 131–144.
[6] M.S. Gowda, Y. Song, On semidefinite linear complementarity problems, Math. Program. 88 (2000) 575–587.
[7] M.S. Gowda, Y. Song, Some new results for the semidefinite linear complementarity problem, SIAM J. Matrix Anal. Appl. 24 (2002) 25–39.
[8] M.S. Gowda, R. Sznajder, Some global uniqueness and solvability results for complementarity problems over symmetric cones, SIAM J. Optim. 18 (2007) 461–481.
[9] M.S. Gowda, R. Sznajder, J. Tao, Some **P**-properties for linear transformations on Euclidean Jordan algebras, Linear Algebra Appl. 393 (2004) 203–232.
[10] M.S. Gowda, J. Tao, **Z**-transformations on proper and symmetric cones, Math. Program., Series B 117 (2009) 195–221.
[11] M.S. Gowda, J. Tao, M. Moldovan, Some inertia theorems in Euclidean Jordan algebras, Linear Algebra Appl. 430 (2009) 1992–2011.
[12] A. Iusem, A. Seeger, On convex cones with infinitely many critical angles, Optimization 56 (2007) 115–128.
[13] J.J. Judice, H.D. Sherali, I.M. Ribeiro, The eigenvalue complementarity problem, Comput. Optim. Appl. 37 (2007) 139–156.
[14] M. Kučera, A new method for obtaining eigenvalues of variational inequalities: operators with multiple eigenvalues, Czechoslovak Math. J. 32 (1982) 197–207.
[15] M. Kučera, Bifurcation points of variational inequalities, Czechoslovak Math. J. 32 (1982) 208–227.
[16] P. Lavilledieu, A. Seeger, Existence de valeurs propres pour les systémes multivoques: résultats anciens et nouveaux, Ann. Sci. Math. Québec 25 (2001) 47–70.
[17] M. Queiroz, J.J. Judice, C. Humes, The symmetric eigenvalue complementarity problem, Math. Comput. 73 (2003) 1849–1863.
[18] P. Quittner, Spectral analysis of variational inequalities, Comment. Math. Univ. Carolin. 27 (1986) 605–629.
[19] R.C. Riddel, Eigenvalue problems for nonlinear elliptic variational inequalities on a cone, J. Funct. Anal. 26 (1977) 333–355.
[20] S.H. Schmieta, F. Alizadeh, Extension of primal-dual interior point algorithms to symmetric cones, Math. Program., Series A 96 (2003) 409–438.
[21] H. Schneider, Positive operators and an inertia theorems, Numer. Math. 7 (1965) 11–17.
[22] H. Schneider, M. Vidyasagar, Cross-positive matrices, SIAM J. Numer. Anal. 7 (1970) 508–519.
[23] A. Seeger, Eigenvalue analysis of equilibrium processes defined by linear complementarity conditions, Linear Algebra Appl. 292 (1999) 1–14.
[24] A. Seeger, M. Torki, On eigenvalues induced by a cone constraint, Linear Algebra Appl. 372 (2003) 181–206.
[25] A. Seeger, M. Torki, Local minima of quadratic forms on convex cones, Research Report, Department of Mathematics, University of Avignon, Avignon, France, J. Global Optim. 44 (2009) 1–28.
[26] F. Zhang, Quaternions and matrices of quaternions, Linear Algebra Appl. 251 (1997) 21–57.
[27] Y. Zhou, M.S. Gowda, On the finiteness of cone spectrum of certain linear transformations on Euclidean Jordan algebras, Technical Report, Department of Mathematics and Statistics, University of Maryland, Baltimore County, Baltimore, MD 21250, USA, October 2007.