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Some factorization properties of Krull domains with infinite cyclic divisor class group

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Abstract

In this paper, we study factorization properties of Krull domains with divisor class group $\mathbb{Z}$. This continues a preliminary study of Dedekind domains with class group $\mathbb{Z}$ in Section IV of [7]. In Section 1, using the Φ-function we introduce the notion of a Φ-finite domain and then determine the relationship between these domains and BFDs and RBFDs (see [1]). In particular, we show that a Φ-finite domain need not be an RBFD. In Section 2, we obtain necessary and sufficient conditions on the set S of divisor classes of D which contain height-one prime ideals so that D is Φ-finite. This leads to the following result: if D is a Krull domain with divisor class group $\mathbb{Z}$, then D is Φ-finite if and only if D is an RBFD. We also find a bound for the elasticity, $\rho(D)$, of the domain D and show in Section 3 that, unlike the case where the divisor class group of D is finite, the elasticity of D may not be "attained" by the factorization of a single element.

0. Introduction

The study of unique factorization domains (UFDs) has been a central area of research in several branches of algebra. Only recently has much attention centered on the factorization properties of integral domains which fail to satisfy the unique factorization condition. Many of the simplest examples of integral domains which fail to be UFDs are rings of algebraic integers. For this reason, it is not surprising that much recent research in this area has centered on the study of Krull (and hence Dedekind) domains. The papers [1–3, 6–9] study factorization properties of Krull (or

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Dedekind) domains D where the divisor class group of D is finite or torsion. In this paper, we study factorization properties of Krull domains with divisor class group $\mathbb{Z}$. This continues a preliminary study of Dedekind domains with divisor class group $\mathbb{Z}$ which appeared in Section IV of [7].

After some preliminary definitions, this paper is divided into three sections. In Section 1, using the Φ-function (studied in the papers [6–9]) we introduce the notion of a Φ-finite domain and then determine the relationship between these domains and bounded factorization domains (BFDs) and rationally bounded factorization domains (RBFDs) studied in [1] and [4]. In particular, we show that a Φ-finite domain need not be an RBFD. In Section 2, we show that factorization properties of Krull domains with divisor class group $\mathbb{Z}$ are dependent (as in the finite divisor class group case) on the distribution of height-one prime ideals in the divisor class group. For such a Krull domain D we obtain necessary and sufficient conditions on the set S of divisor classes of D which contain height-one prime ideals so that D is Φ-finite. This leads to the following result: if D is a Krull domain with divisor class group $\mathbb{Z}$, then D is Φ-finite if and only if D is an RBFD. We also find a bound for the elasticity, $\rho(D)$, of the domain D (see [1–3, 9, 16, 17]) and show in Section 3 that, unlike the case where the divisor class group of D is finite, the elasticity of D may not be "attained" by the factorization of a single element.

We will use the standard notation and definitions of [1–9] throughout this paper. Let $\mathbb{Z}$, $\mathbb{Z}^+$, $\mathbb{R}$, and $\mathbb{R}^+$ represent the integers, the nonnegative integers, the real numbers, and the nonnegative real numbers, respectively. Let D be an *atomic domain* (i.e. every nonzero nonunit of D can be written as a product of irreducible elements of D) and let D^* represent the set of nonzero elements of D. Then D is a *half-factorial domain* (HFD) if for any irreducible elements $\alpha_1 \dots, \alpha_s, \beta_1, \dots, \beta_t$ of D, the equality $\alpha_1 \cdots \alpha_s = \beta_1 \cdots \beta_t$ implies that $s = t$ (see [4, 6, 7, 15, 18]). D is a *bounded factorization domain* (BFD) if for each nonzero nonunit $x \in D$ there is a positive integer n such that if $x = \alpha_1 \cdots \alpha_m$ with each α_i irreducible in D, then $m \leq n$. Krull and Noetherian domains are two classes of domains which satisfy the BFD condition (see [4]). If x is a nonzero nonunit of D, then set

$$\rho_D(x) = \sup\{m/n \mid \text{there are irreducibles } \alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_m \text{ of } D \text{ such that } x = \alpha_1 \cdots \alpha_n = \beta_1 \cdots \beta_m\}$$

and

$$\rho(D) = \sup\{\rho_D(x) \mid x \text{ is a nonzero nonunit of } D\}.$$

$\rho(D)$is called the *elasticity* of D and is studied in one form or another in the papers [1–3, 9, 16, 17]. Note that $1 \leq \rho(D) \leq \infty$ and $\rho(D) = 1$ if and only if D is an HFD. D is a *rationally bounded factorization domain* (RBFD) if $\rho(D) < \infty$. Clearly if D is an RBFD, then D is a BFD, but [1] contains many examples which show that the converse (even in the Krull domain case) does not hold. In [3], the present authors with D.D. Anderson showed that if D is a Krull domain with finite divisor class group, then

(i) $\rho(D) = r < \infty$ for some rational number $r \geq 1$, and

(ii) there exists a nonzero nonunit $x \in D$ such that $\rho_D(x) = r$.

The papers [6–9] consider factorization problems in Dedekind domains. While we concern ourselves here with the more general Krull domain setting, the proofs for factorization properties in the Dedekind domain case extend naturally to Krull domains by replacing the unique factorization of a principal ideal as a product of maximal ideals in a Dedekind domain by its unique factorization as a v-product of height-one prime ideals in a Krull domain. Because of this, we will usually refer to height-one prime ideals simply as prime ideals.

Most of the examples in this paper are given in the Dedekind domain setting and hence we will need the following notation and definitions. If for a given abelian group G and subset $S \subset G - \{0\}$ there exists a Dedekind domain D such that $\mathrm{Cl}(D) \cong G$ and $S = \{g \mid g \in G$ and g contains a nonprincipal prime ideal of $D\}$, then the pair $\{G, S\}$ is called *realizable*. Two Theorems of Grams [13, Corollaries 1.6 and 1.7] can be used to characterize realizable pairs of the form $\{G, S\}$, where $G = \mathbb{Z}$ or G is a torsion abelian group. These characterizations are as follows: (i) $\{\mathbb{Z}, S\}$ is realizable if and only if S generates $\mathbb{Z}$ and S contains both positive and negative elements of $\mathbb{Z}$ and, (ii) if G is a torsion abelian group, then $\{G, S\}$ is realizable if and only if S generates G. The concept of a realizable pair extends naturally to a Krull domain D, where in this case the set S would represent the nonzero divisor classes of D which contain height-one prime ideals. Notice the following relationship between two Krull domains with similar realizable pairs. Let G be an abelian group and R_1 and R_2 be Krull domains with realizable pairs $\{G, S_1\}$ and $\{G, S_2\}$ respectively. If $S_1 \subseteq S_2$, then $\rho(R_1) \leq \rho(R_2)$.

Let G be an abelian group, S a subset of the nonzero elements of G, and $\mathscr{F}(G)$ be the multiplicative free abelian monoid with basis G. The elements of $\mathscr{F}(G)$ can be viewed as products of the form

$$F = \prod_{g \in G} g^{v_g(F)},$$

where $v_g(F) \in \mathbb{Z}^+$ and $v_g(F) = 0$ for almost all $g \in G$. Set

$$\mathscr{B}(G) = \{B \in \mathscr{F}(G) \mid \sum_{g \in G} v_g(B)g = 0\}.$$

$\mathscr{B}(G)$ is known as the *block semigroup over G*. More generally, set

$$\mathscr{B}(S) = \{B \in \mathscr{B}(G) \mid v_g(B) = 0 \text{ for } g \in G \backslash S\}.$$

Block semigroups have been studied in great detail in [10, 11, 14]. An element $B \in \mathscr{B}(S)$ is called *irreducible* if it cannot be written in the form $B = B_1 B_2$, where B_1 and B_2 are nonzero blocks of $\mathscr{B}(S)$.

While our interest in factorization problems is rooted in the study of ring theory, results about the lengths of factorizations in a Krull domain D with realizable pair

$\{G, S\}$ are combinatorial results about the block semigroup $\mathscr{B}(S)$. To see this, let H and H' be (multiplicative) atomic monoids and y a nonunit of H. Set $\mathscr{L}(y) = \{n \in \mathbb{Z}^+ \mid$ there are irreducibles $y_1, \dots, y_n \in H$ such that $y = y_1 \cdots y_n\}$. A surjective homomorphism $f: H \to H'$ is called length-preserving if $\mathscr{L}(y) = \mathscr{L}(f(y))$ for each nonunit y of H. In the obvious manner, one can define $\rho(H)$ and $\Phi(H)$ for an atomic monoid H (see [3]). If $f: H \to H'$ is a length-preserving homomorphism, then clearly $\rho(H) = \rho(H')$ and $\Phi(H) = \Phi(H')$. Let D be a Krull domain with realizable pair $\{G, S\}$ and $x \in D^*$. Define $\pi: D^* \to \mathscr{B}(S)$ by

$$\pi(x) = [P_1] \cdots [P_k],$$

where $(x) = (P_1 \cdots P_k)_v$ for $P_1 \dots, P_k$ height-one primes of D and $[P_i]$ denotes the divisor class of the ideal P_i. By [10, Proposition 1], π is length-preserving. Notice that irreducible elements $x \in D^*$ correspond to irreducible blocks $\pi(x)$ in $\mathscr{B}(S)$. Hence, the factorization properties of D are identical to those of $\mathscr{B}(S)$. In particular, $\rho(D) = \rho(D^*) = \rho(\mathscr{B}(S))$.

In Section 3, we will generalize the following concept, which is central to the study of factorization problems in Dedekind domains with torsion divisor class group. Suppose that D is such a domain and that α is a nonzero of D. Then

$$\pi(\alpha) = [P_1]^{m_1} \cdots [P_k]^{m_k},$$

where each P_i is a nonprincipal prime ideal of D. If n_i is the order of $[P_i]$, then set

$$\mathscr{Z}(\alpha) = \frac{m_1}{n_1} + \frac{m_2}{n_2} + \cdots + \frac{m_k}{n_k}.$$

In [7] the authors define $\mathscr{Z}(\alpha)$ to be the *Zaks–Skula constant* of α. Notice that if α and β are nonzero nonunits of D, then $\mathscr{Z}(\alpha\beta) = \mathscr{Z}(\alpha) + \mathscr{Z}(\beta)$. If D has torsion divisor class group, then D is an HFD if and only if $\mathscr{Z}(\alpha) = 1$ for all irreducibles α in D (see [15], [18] or [1, Corollary 2.6] for a proof).

Let D be an atomic integral domain. The study of functions $f: D^* \to \mathbb{Z}^+$ has proven to be valuable in examining factorization problems in Krull domains. Two of these functions, studied extensively in [5], are defined as follows: if $x \in D^*$, then set $l(x) = \inf\{n \mid x = x_1 \cdots x_n,\ x_i \in D$ and irreducible$\}$ and $L(x) = \sup\{n \mid x = x_1 \cdots x_n,\ x_i \in D$ and irreducible$\}$ (if u is a unit of D, then set $l(u) = L(u) = 0$). Note that $\rho_D(x) = L(x)/l(x)$. The Zaks–Skula constant is an example of what is known more generally as a semi-length function. A function $f: D^* \to \mathbb{R}^+$ is a *semi-length function* on an integral domain D if (i) $f(xy) = f(x) + f(y)$ for all $x, y \in D^*$ and (ii) $f(x) = 0$ if and only if x is a unit of D. If a semi-length function f has inf $\{f(x) \mid x$ is irreducible in $D\} > 0$ and $\sup\{f(x) \mid x$ is irreducible in $D\} < \infty$, then f is called a *bounded semi-length function* on D. If D is an atomic domain with bounded semi-length function f, then Theorem 2.1 of [1] shows that

$$\rho(D) \le \frac{\sup\{f(x) \mid x \text{ is irreducible, but not prime, in } D\}}{\inf\{f(x) \mid x \text{ is irreducible, but not prime, in } D\}}.$$

In working with abelian groups, we will later need what is known as the *Davenport constant* of a finite abelian group G. The Davenport constant of G (denoted $D(G)$) is defined as the smallest positive integer d such that for each sequence $T \subset G$ with $|T| = d$, some nonempty subsequence of T has sum 0. Notice that for an abelian group G, $D(G) \leq |G|$. The paper [12] outlines many of the known results concerning the Davenport constant of a finite abelian group G. If $G \cong \mathbb{Z}_{n_1} \oplus \cdots \oplus \mathbb{Z}_{n_k}$ is such a group written with $n_i | n_{i+1}$ for $1 \leq i \leq k-1$, then it is known that $D(G) \geq 1 + \sum_{i=1}^{k} (n_i - 1)$. Equality holds for a large class of groups (including cyclic groups, groups of rank ≤ 2, and p-groups), but not in general (see Theorem 2 in [12]). The problem of computing the Davenport constant for a general finite abelian group is still open.

Finally, for any positive integer n, set

$$\mathscr{V}(n) = \{m \,|\, \text{there are irreducibles } \beta_1, \ldots, \beta_m, \alpha_1, \ldots, \alpha_n \text{ of } D \text{ with } \alpha_1 \cdots, \alpha_n = \beta_1 \cdots \beta_m\}$$

and let

$$\Phi(n) = |\mathscr{V}(n)|.$$

$\Phi(n)$ is known as the Φ-function and has been studied extensively in [6–9]. If D is a Dedekind domain with finite divisor class group G such that each nonprincipal ideal class of D contains a prime ideal, then the main result of [8] indicates that in D

$$\lim_{n \to \infty} \frac{\Phi(n)}{n} = \frac{D(G)^2 - 4}{2D(G)}.$$

1. Φ-Finite Domains

Call an atomic domain D a *Φ-finite domain* if $\Phi(n) < \infty$ for each positive integer n (we shall refer to a Φ-finite domain D as simply being *Φ-finite*). Not all Krull domains are Φ-finite (see Example 5 in [6] or Theorem 2.1 below). We immediately deduce the following relationship between domains which are RBFDs and those which are Φ-finite.

Lemma 1.1. *Let D be an RBFD. Then*

$$\Phi(n) \leq \left[\frac{\rho(D)^2 - 1}{\rho(D)}\right] \cdot n + 1.$$

Hence, if D is an RBFD, then D is Φ-finite.

Proof. The proof is similar to the proof of Corollary 1 in [8] and Theorem 2.1 in [9]. Suppose that $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_t$ are irreducible elements of D such that $\alpha_1 \cdots \alpha_n = \beta_1 \cdots \beta_t$. Then $1/\rho(D) \leq t/n \leq \rho(D)$, and hence $[1/\rho(D)] \cdot n \leq t \leq \rho(D) \cdot n$. Thus

$$\Phi(n) \leq [\rho(D) \cdot n] - [1/\rho(D)] \cdot n + 1,$$

and the inequality in the lemma follows. The second statement clearly follows from the first. □

Hence, by the main result of [3], Krull (and thus Dedekind) domains with finite divisor class group are Φ-finite. Let D be an atomic domain and suppose that x is a nonzero nonunit of D with irreducible factorization of the form $x = \alpha_1 \cdots \alpha_n$. If D is Φ-finite, then there is a bound on the length of a factorization $x = \beta_1 \cdots \beta_m$ into the product of irreducibles (namely, $m \le \max \mathscr{V}(n)$). We can thus deduce the following theorem.

Theorem 1.2. *Let D be an atomic integral domain. The following implications, and no others, hold:*

$$D \text{ is an RBFD} \quad \Rightarrow \quad D \text{ is } \Phi\text{-finite} \quad \Rightarrow \quad D \text{ is a BFD}.$$

Proof. We have already shown that the implications listed above are valid. Since a Krull domain is a BFD, Theorem 2.1 will later provide us with an example of a BFD which is not Φ-finite (see Example 5 in [6] for an alternate example). A Φ-finite domain need not be an RBFD, as Example 1.3 will show. Thus none of the above implications are reversible. □

Example 1.3. Let $G \cong \mathbb{Z}_2 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_8 \oplus \mathbb{Z}_8 \oplus \mathbb{Z}_8 \oplus \cdots \cong \sum_{i=1}^{\infty} (\sum_{j=1}^{i} \mathbb{Z}_{2^i})$,

$$v_{1,1} = (1,0,\ldots),\ v_{2,1} = (0,1,0,\ldots),\ v_{2,2} = (0,0,1,0,\ldots),\ v_{3,1} = (0,0,0,1,0,\ldots),$$
$$w_2 = (0,3,3,0,\ldots),\quad w_3 = (0,0,0,7,7,7,0,\ldots),\quad \ldots,$$

be elements of G, and

$$S = \{v_{1,1}, v_{2,1}, v_{2,2}, v_{3,1}, \ldots, w_2, w_3, \ldots\}.$$

Then $\{G, S\}$ is a realizable pair by [13]. Let D be a Dedekind domain associated with this pair and let $\{\{P_{i,j}\}_{j=1}^{i}\}_{i=1}^{\infty}$ and $\{Q_j\}_{j=1}^{\infty}$ be prime ideals of D such that the prime ideal $P_{i,j}$ comes from the ideal class $v_{i,j}$ and the prime ideal Q_j comes from the ideal class w_j. The irreducible blocks of $\mathscr{B}(S)$, along with irreducible elements from D which correspond to them, and their Zaks–Skula constants are

(1) $[P_{j,i}]^{2^j} = \pi(\alpha_{j,i})$ *with* $\mathscr{Z}(\alpha_{j,i}) = 1$ for $j \ge 1$, $1 \le i \le j$,
(2) $[Q_j]^{2^j} = \pi(\beta_j)$ with $\mathscr{Z}(\beta_j) = 1$ for $j \ge 2$, and
(3) $[Q_j][P_{j,1}][P_{j,2}] \cdots [P_{j,j}] = \pi(\gamma_j)$ with $\mathscr{Z}(\gamma_j) = (j+1)/2^j$ for $j \ge 2$.

For a given irreducible ζ of D, we will refer to the value of j which appears in its irreducible block, as listed above, as the index of ζ. Since $\mathscr{Z}(\gamma_j) \to 0$ as $j \to \infty$, by Corollary 1.7 in [9] we have that $\rho(D) = \infty$. Thus D is not an RBFD. We next argue that $\Phi(n) < \infty$ for each $n \in \mathbb{Z}^+$. Let $\theta = \delta_1 \cdots \delta_n$ be the product of n irreducibles in D. Notice that if any of the irreducibles $\delta_1, \ldots, \delta_n$ are of index $\ge n$, then the same number of irreducibles with this irreducible block appear in any factorization of θ. So without loss of generality, assume that all the irreducibles $\delta_1, \ldots, \delta_n$ are of index $< n$.

Then $1 \geq \mathscr{L}(\delta_i) \geq n/2^{n-1}$ for each i, and hence $\mathscr{L}(\delta_1 \cdots \delta_n) = \mathscr{L}(\delta_1) + \cdots + \mathscr{L}(\delta_n) \leq n$. So if $\delta_1 \cdots \delta_n = \lambda_1 \cdots \lambda_k$ for irreducibles $\lambda_1, \ldots, \lambda_k$ of D, then $\mathscr{L}(\lambda_1 \cdots \lambda_k) \leq n$. Hence

$$\frac{kn}{2^{n-1}} \leq \mathscr{L}(\lambda_1) + \cdots + \mathscr{L}(\lambda_k) = \mathscr{L}(\lambda_1 \cdots \lambda_k) \leq n,$$

which implies that $k \leq 2^{n-1}$. Thus, with the above assumption $\Phi(n) \leq 2^{n-1}$, and hence D is Φ-finite.

The last example suggests the following question.

Question. Let D be a Krull domain such that the elements of the divisor class group of D have bounded order. Does D a Φ-finite domain imply that D is an RBFD?

2. On Krull domains with divisor class group $\mathbb{Z}$

We now consider Krull (and hence Dedekind) domains with divisor class group $\mathbb{Z}$. If $\{\mathbb{Z}, S\}$ is a realizable pair, then call S *bounded above* if there exists an $s \in S$ with $s' \leq s$ for all $s' \in S$. Similarly, call S *bounded below* if there exists a $t \in S$ with $t \leq t'$ for all $t' \in S$. If S is neither bounded above nor bounded below, then we will say that S is not *bounded*. We first have the following theorem.

Theorem 2.1. *Let D be a Krull domain with irreducible pair $\{\mathbb{Z}, S\}$ such that S is not bounded. Then D is not Φ-finite, and hence D is a BFD which is not an RBFD.*

Proof. We again note that by "prime ideal" we mean "height-one prime ideal". Let n_1 be the smallest positive element in S and $-m_1$ be the largest negative element in S. Let q_1 and v_1 be the positive integers such that $\mathrm{LCM}(n_1, m_1) = q_1 n_1 = v_1 m_1$. If P is a prime ideal of class n_1 and Q a prime ideal of class $-m_1$, then $[P]^{q_1}[Q]^{v_1} = \pi(\gamma)$ for some irreducible γ of D. Now let $k \geq 2$ be any positive integer. Choose $n_k, -m_k \in S$ so that for $\mathrm{LCM}(n_1, m_k) = q_2 n_1$ and $\mathrm{LCM}(m_1, n_k) = v_2 m_1$ we have that $q_2 > kq_1$ and $v_2 > kv_1$ (this is possible since S is not bounded). If $\mathrm{LCM}(n_1, m_k) = wm_k$, $\mathrm{LCM}(m_1, n_k) = xn_k$, R is a prime ideal of class $-m_k$, and T is a prime ideal of class n_k, we have that

$$[P]^{q_2}[R]^{w} = \pi(\alpha) \quad \text{and} \quad [Q]^{v_2}[T]^{x} = \pi(\beta)$$

for irreducibles α and β of D. Hence

$$([P]^{q_2}[R]^{w} \cdot ([Q]^{v_2}[T]^{x}) = ([P]^{q_1}[Q]^{v_1})^{k} \cdot B,$$

where B is some block of $\mathscr{B}(S)$. Thus 2 irreducibles factor as at least k irreducibles. Let $B = \pi(\delta)$. Then $k + L(\delta) \in \mathscr{V}(2)$, and since this argument can be repeated for any integer greater than $k + L(\delta)$, we have that $\Phi(2) = \infty$. $\square$

Corollary 2.2. *Let D be a Krull domain with realizable pair $\{\mathbb{Z}, S\}$. If D is an HFD, then S is either bounded above or bounded below.* $\square$

Making the assumption that the set S is bounded leads us to the following theorem.

Theorem 2.3. *Let D be a Krull domain with realizable pair $\{\mathbb{Z}, S\}$. If S is either bounded above or bounded below, then D is Φ-finite.*

Proof. We prove the case where S is bounded below. The case where S is bounded above follows by using an automorphisn argument (see Lemma 1.9 in [7]). Assume that $S = \{-m_1, -m_2, \ldots, -m_k, n_1, n_2, \ldots\}$ with the elements listed in ascending order so that each m_i and n_i is positive. Set $m = m_1$. If $m = 1$, then Theorem 4.9 of [7] implies that D is an HFD. Further, if S is also bounded above, then $\rho(D)$ is rational (see Theorem 10 in [3]) and the theorem follows from Lemma 1.1. So suppose that S is not bounded above and $m > 1$. The proof proceeds in three steps.

Claim 1: Let γ be an irreducible of D and R a prime ideal of D contained in class $t > 0$ such that $\pi(\gamma) = [I][R]^x$, where R does not v-divide I. Then there exists a positive constant c (which depends only on m) such that $x \leq c$.

Proof of Claim 1: Set $c = m(m+1)/2$. Notice that $c \geq m_j$ for each $1 \leq j \leq k$. Suppose that

$$\pi(\gamma) = \left[\prod_{j=1}^{k} [P_j]^{x_j}\right] \cdot [Q] \cdot [R]^w, \tag{2.1}$$

where

(1) the prime ideals P_j come from class $-m_j$,

(2) Q is a product of prime ideals taken from positive ideal classes,

(3) R does not v-divide Q, and

(4) $w > c$.

We first show that each $x_j < t$. For suppose that some $x_j \geq t$. Then $(P_j^t)_v | (\gamma)$ and $(R^{m_j})_v | (\gamma)$ implies that $[P_j]^t [R]^{m_j} = \pi(\beta)$ for some $\beta \in D^*$, contradicting the irreducibility of γ. Thus $x_j < t$ for each j.

Now, (2.1) implies that

$$\sum_{j=1}^{k} m_j x_j \geq \text{wt}.$$

By the observation that each $x_j < t$, we have

$$\text{wt} \leq \sum_{j=1}^{k} m_j x_j \leq \left[\sum_{j=1}^{k} m_j\right] \cdot t,$$

and hence

$$w \leq \sum_{j=1}^{k} m_j \leq \sum_{i=1}^{m} i = \frac{m(m+1)}{2}.$$

Claim 2: For any irreducible γ of D, let

$$\mathscr{R}_\gamma = \{z \mid z > 0, (\gamma) \text{ has a prime v-divisor from class } z\}.$$

Then there exists a positive constant d (which depends only on m) such that $|\mathscr{R}_\gamma| < d$ for each irreducible γ in D.

Proof of Claim 2: Suppose that γ is an irreducible in D with

$$\pi(\gamma) = \left[\prod_{i=1}^{k} [P_i]^{y_i}\right]\left[\prod_{j=1}^{h} [R_j]^{x_j}\right]\cdot[R_{u_1}]^{w_1}\cdot[R_{u_2}]^{w_2}\cdots[R_{u_t}]^{w_t}, \tag{2.2}$$

where

(1) each P_i is a prime ideal taken from class $-m_i$,

(2) $[\prod_{j=1}^{h} R_j^{x_j}]$ is a product of prime ideals taken from classes between 1 and $m-1$,

(3) each u_i is distinct and greater than or equal to m, and

(4) each $w_i > 0$.

Notice that (2.2) implies that

$$\sum_{i=1}^{k} m_i y_i \geq \sum_{j=1}^{t} u_j w_j. \tag{2.3}$$

We first show that $t < m(m+1)/2$. Suppose that $t \geq m(m+1)/2$. Let

$$\mathscr{C}_m = \{u_1, u_2, \ldots, u_m\},$$

$$\mathscr{C}_{m_2} = \{u_{m+1}, u_{m+2}, \ldots, u_{m+m_2}\},$$

$$\vdots$$

$$\mathscr{C}_{m_k} = \{u_{m+m_2+\ldots+m_{k-1}+1}, \ldots, u_{m+m_2+\cdots+m_k}\}.$$

Hence $|\mathscr{C}_{m_r}| = m_r$ for $1 \leq r \leq k$. Consider $\mathscr{C}_m$. Recall that the Davenport constant of $\mathbb{Z}_m$ (denoted $D(\mathbb{Z}_m)$) is m. Hence there is a subsum of the terms $u_1, \ldots, u_m$ which sums to zero modulo m. So suppose that $u_{i_1} + \cdots + u_{i_x} = vm$ for $v \geq 1$. Now $y_1 < v$, for otherwise

$$[P_m]^v[R_{u_{i_1}}]\cdots[R_{u_{i_x}}] = \pi(\beta)$$

for some $\beta \in D^*$, contradicting the irreducibility of γ (note that $(\beta) = (\gamma)$ implies that $m = 1$, a contradiction). Hence $y_1 \cdot m < \sum_{i=1}^{m} u_i$. Repeating the argument mod m_2 on $\mathscr{C}_{m_2}$, we obtain

$$y_2 \cdot m_2 < \sum_{i=m+1}^{m+m_2} u_i.$$

Repeating this argument $k-2$ times, we obtain

$$\sum_{i=1}^{k} y_i m_i < \sum_{j=1}^{m+m_2+\cdots+m_k} u_j \leq \sum_{j=1}^{t} u_j w_j,$$

which contradicts (2.3). So $|\mathscr{R}_\gamma| < (m-1) + m(m+1)/2$.

Claim 3: For each $n \in \mathbb{Z}_+$, $\Phi(n) < \infty$.

Proof of Claim 3: Let $\alpha_1 \cdots \alpha_n$ be a product of n irreducibles in D. Each principal ideal (α_i) is v-divisible by at most dc positive prime ideals. Thus $(\alpha_1 \cdots \alpha_n)$ is v-divisible by at most ndc positive prime ideals. If $\alpha_1 \cdots \alpha_n = \beta_1 \cdots \beta_t$ with each β_i irreducible, then $t \leq ndc$. The proof of the theorem is complete. □

Combining Theorems 2.1 and 2.3, we obtain the following theorem.

Theorem 2.4. *Let D be a Krull domain with realizable pair* $\{\mathbb{Z}, S\}$. *Then the following statements are equivalent.*

(1) *D is an RBFD.*

(2) *D is* Φ*-finite.*

(3) *S is either bounded above or bounded below (or both).*

Proof. (1) implies (2) by Theorem 1.2, and (2) implies (3) by contradiction using Theorem 2.1. We use Claims 1 and 2 in the proof of the previous theorem to show that (3) implies (1). We will again assume that S is bounded below, the proof of the assertion if S is bounded above is similar. Suppose that $S = \{-m_1, -m_2, \ldots, -m_t, n_1, n_2, \ldots\}$, where the integers m_i and n_j are positive. If γ is a nonzero nonunit of D, then

$$\pi(\gamma) = [Q_{m_1}]^{X_1}[Q_{m_2}]^{X_2} \cdots [Q_{m_t}]^{X_t}[P_{n_1}]^{y_1} \cdots [P_{n_k}]^{y_k}, \tag{2.4}$$

where the prime ideal Q_{m_i} comes from class $-m_i$ and the prime ideal P_{n_j} comes from class n_j. Set $\varphi(\gamma) = \sum_{j=1}^{k} y_j$. It is easy to see that $\varphi(\gamma_1\gamma_2) = \varphi(\gamma_1) + \varphi(\gamma_2)$. Claims 1 and 2 imply that φ is a bounded semi-length function on D (set $\varphi(u) = 0$ if u is a unit of D), and hence $\rho(D) < \infty$ by Theorem 2.1 of [1]. Thus D is an RBFD. □

Using Theorem 2.1 in [1], we can derive the following upper bound for $\rho(D)$ for a Krull domain with divisor class group $\mathbb{Z}$ when D is an RBFD.

Corollary 2.5. *Let D be a Krull domain with realizable pair* $\{\mathbb{Z}, S\}$, *where* $S \subseteq \{-m, -m+1, \ldots\}$ *or* $S \subseteq \{\ldots, m-1, m\}$ *for m a positive integer. Then*

$$\rho(D) \leq \frac{m(m+1)(m^2+3m-2)}{4}.$$

Proof. The theorem cited in [1] shows that if φ is a bounded semi-length function on D, then

$$\rho(D) \leq \frac{\max\{\varphi(\gamma) \mid \gamma \text{ is irreducible, but not prime, in } D\}}{\min\{\varphi(\gamma) \mid \gamma \text{ is irreducible, but not prime, in } D\}}.$$

The result follows by applying the bounds c and d derived in Claims 1 and 2 of Theorem 2.3. □

We end this section with an example which demonstrates how the elasticity of a Krull domain may behave when its divisor class group is a direct sum of copies of $\mathbb{Z}$.

Example 2.6. *Let* $G \cong \mathbb{Z}\oplus\mathbb{Z}$ and $S = \{(-1,0),(1,0),(2,0),(3,0),\ldots,\} \cup \{(0,-1),(0,1),(0,2),(0,3),\ldots\}$. The pair $\{G,S\}$ is realizable by [13]. Let D be a Dedekind domain associated to this pair.Now, let P be a prime ideal of class $(-1,0)$ and T be a prime ideal of class $(0,-1)$. For each positive integer n, let A_n be a prime ideal of class $(0,n)$ and B_n a prime ideal of class $(n,0)$. The only irreducible blocks of $\mathscr{B}(S)$ are $[T]^n[A_n] = \pi(\alpha_n)$ and $[P]^n[B_n] = \pi(\beta_n)$ for some irreducibles α_n and β_n of D (for each $n \in \mathbb{Z}^+$). A simple counting argument shows that D is a HFD, and hence $\rho(D) = 1$.

Let $S' = S \cup \{(-1,-1)\}$. Again, $\{G,S'\}$ is realizable; let D' be a Dedekind domain associated to this realizable pair. Let Q be a prime ideal taken from class $(-1,-1)$. Using the same notation as above, we obtain that the blocks $[Q]^n[B_1]^n[A_n]$, $[Q]^n[B_n][A_1]^n$, $[Q][B_1][A_1]$, and $[Q]^n[A_n][B_n]$ are all irreducible (for each $n \in \mathbb{Z}^+$). Let $\{\gamma_i\}_{i=1}^{\infty}$, $\{\delta_i\}_{i=1}^{\infty}$, $\{\lambda_i\}_{i=1}^{\infty}$, and ν be irreducibles of D such that $Q^nB_1^nA_n = (\gamma_n)$, $Q^nB_nA_1^n = (\delta_n)$, $QB_1A_1 = (\nu)$, and $Q^nA_nB_n = (\lambda_n)$. Hence $\gamma_n\delta_n = \nu^n\lambda_n$ in D', and thus $\rho(D') \geq (n+1)/2$. Letting $n \to \infty$, we see that $\rho(D') = \infty$. Thus, by adding one additional element to the set S associated to an HFD, we obtain a new realizable pair and an associated Dedekind domain with infinite elasticity. Notice that by Theorem 2.4 this cannot happen when the divisor class group of D is $\mathbb{Z}$.

3. On Krull domains where S is an infinite bounded set

Let D be a Krull domain with realizable pair $\{\mathbb{Z},S\}$. Theorem 10 in [3] indicates that if S is a finite set, then $\rho(D)$ is rational and there exists some nonzero nonunit $x \in D$ such that $\rho_D(x) = \rho(D)$. Earlier in Section 2, we showed that if D is either an HFD, an RBFD, or Φ-finite, then S is either bounded above or bounded below. Because of these results, we shall center our attention in this section on the case where S is an infinite bounded set (i.e., S is either bounded above or bounded below). We will consider the following problems:

(I) If S is an infinite bounded set and D is a Krull domain with realizable pair $\{\mathbb{Z},S\}$, then is $\rho(D)$ rational?

(II) Moreover, if $\rho(D)$ is rational, then does $\rho(D) = \rho_D(x)$ for some nonzero nonunit x of D?

While we do not completely settle question I, we construct an example which gives a negative answer to question II.

For problems I and II, we can use an automorphism argument and consider only sets S which are bounded below. Thus, let

$$S = \{-m_1, -m_2, \ldots, -m_t, n_1, n_2, \ldots\} \tag{3.1}$$

with each m_i, n_j positive. In Theorem 4.1 of [7], the authors determine for the case where S is finite and $t = 1$ when D is an HFD. The same proof easily extends to the case where S is infinite. In this case, let $\alpha \in D^*$, v_i = order of n_i modulo m_1, and set

$$\mathscr{L}_{m_1}(\alpha) = \frac{y_1}{v_1} + \cdots + \frac{y_k}{v_k},$$

where

$$\pi(\alpha) = [Q]^x[P_1]^{y_1} \cdots [P_k]^{y_k} \tag{3.2}$$

for Q a prime ideal of class $-m_1$ and P_i a prime ideal of class n_i. Then D with realizable pair $\{\mathbb{Z}, S\}$ is an HFD if and only if $\mathbb{Z}_{m_1}(\alpha) = 1$ for all irreducibles $\alpha \in D$. One consequence of this characterization is the following: If D is a Krull domain with realizable pair $\{\mathbb{Z}, S\}$ and $S \subseteq \{-1, 1, 2, 3, \ldots\}$, then D is an HFD (this also follows directly from Corollary 2.5).

In the case where $t = 1$, we now answer question II in the affirmative. In this case, we will construct another Dedekind domain D' with finite divisor class group such that $\rho(D) = \rho(D')$. The following simple observation will be necessary.

Lemma 3.1. *Let D and D' be atomic integral domains with the property that for any set of irreducibles $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_m$ of D such that $\alpha_1 \cdots \alpha_n = \beta_1 \cdots \beta_m$ there exist irreducibles $\alpha'_1, \ldots, \alpha'_n, \beta'_1, \ldots, \beta'_m$ of D' such that $\alpha'_1 \cdots \alpha'_n = \beta'_1 \cdots \beta'_m$. Then $\rho(D) \le \rho(D')$.* $\square$

Theorem 3.2. *Let D be a Krull domain with realizable pair $\{\mathbb{Z}, S\}$, where $S = \{-m, n_1, n_2, \ldots\}$. Then D is an RBFD, $\rho(D)$ is rational, and there exists a nonzero nonunit x of D such that $\rho_D(x) = \rho(D)$.*

Proof. Let $\eta: \mathbb{Z} \to \mathbb{Z}_m$ be the natural map with $\eta(n) = \bar{n}$ for all $n \in \mathbb{Z}$. Let $S' = \eta(S) - \{0\} = \{c_i\}_{i=1}^{w}$. It is easy to argue that since the set S generates the group $\mathbb{Z}$, the set S' generates $\mathbb{Z}_m$. Let D' be a Dedekind domain associated to the realizable pair $\{\mathbb{Z}_m, S'\}$. Throughout this discussion, let $Q, P_1, P_2, \ldots$ be prime ideals of D as in (3.2) and $R_1, \ldots, R_w$ be prime ideals of D', where the ideal R_i is taken from the class c_i.

Let γ be an irreducible from D' with irreducible block $\pi(\gamma) = [R_1]^{y_1} \cdots [R_w]^{y_w}$; hence $\sum_{i=1}^{w} y_i c_i = 0$ in $\mathbb{Z}_m$. Let $n_{j_1}, \ldots, n_{j_w}$ be a fixed sequence of positive integers taken from S such that $c_i = \bar{n}_{j_i}$. Then $\sum_{i=1}^{w} y_i n_{j_i} \equiv 0 \pmod m$. Let $\delta \in D^*$ with $\pi(\delta) = [Q]^x \prod_{i=1}^{w} [P_{n_{j_i}}]^{y_i}$, where x is the positive integer such that $xm = \sum_{i=1}^{w} y_i n_{j_i}$. If δ were not irreducible in D, then there would exist nonnegative integers $x', y'_1, \ldots, y'_t$ (which are not all zero) such that $x' < x$, $y'_i \le y_i$ for each $1 \le i \le w$ with strict inequality for at least one of the y_i, and $\sum_{i=1}^{w} y_i' n_{j_i} \equiv 0 \pmod m$. This would contradict the irreducibility of γ in D'. Using the construction above, notice that if $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_r$ are irreducibles in D' and $\gamma_1, \ldots, \gamma_n, \delta_1, \ldots, \delta_r$ are the corresponding irreducible elements in D, then $\alpha_1 \cdots \alpha_n = \beta_1 \cdots \beta_r$ implies that $\gamma_1 \cdots \gamma_n = \delta_1 \cdots \delta_r$. By Lemma 3.1, $\rho(D') \le \rho(D)$.

We are able to reverse the above construction with some modifications. For D, the irreducible blocks of $\mathscr{B}(S)$ are either of the form $[Q]^x \prod_{i=1}^{k} [P_{n_i}]^{y_i}$ with $n_i \not\equiv 0 \pmod m$

for each i, or of the form $[Q]^x[P_{n_i}]$ with $n_i \equiv 0 \pmod m$. Since in any factorization in D into irreducibles of the form $x = \gamma_1 \cdots \gamma_n$ the number of irreducible blocks of the second type is uniquely determined by simply counting the number of prime ideals in the v-ideal factorization of (x) in classes which are congruent to 0 modulo m, we can associate to such irreducibles some fixed irreducible of D'. Hence, if we have irreducibles $\gamma_1, \ldots, \gamma_n, \delta_1, \ldots, \delta_r$ in D with $\gamma_1 \cdots \gamma_n = \delta_1 \cdots \delta_r$, then we are able to construct irreducibles $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_r$ of D' with $\alpha_1 \cdots \alpha_n = \beta_1 \cdots \beta_r$. Again, by Lemma 3.1, $\rho(D) \leq \rho(D')$. We now conclude $\rho(D) = \rho(D')$ and the existence of an x in D for which $\rho_D(x) = \rho(D)$. □

We can deduce the following special cases from the last theorem.

Corollary 3.3. *Let D be a Krull domain with realizable pair $\{\mathbb{Z}, S\}$.*

(1) *If $S = \{-2, n_1, n_2, \ldots\}$, then D is an* HFD.

(2) *If $m \geq 2$ and $S = \{-m, n_1, n_2, \ldots\}$, then $\rho(D) \leq m/2$. If $n_1, n_2, \ldots$ forms a complete set of residues modulo m, then $\rho(D) = m/2$.*

(3) *If $p \geq 2$ is prime and $S = \{-p, n_1, n_2, \ldots\}$, then D is an* HFD *if and only if there is a fixed integer k with $1 \leq k < p$ such that each n_i is congruent to 0 or k modulo p.*

Proof. If D' is a Dedekind domain (which is not a PID) with finite divisor class group G, then $\rho(D') \leq D(G)/2$, with equality occurring when each nonprincipal ideals class of D' contains a prime ideal (a form of this result appears in [16], [17], and as Corollary 2.3(b) in [1]). By the proof of Theorem 3.2, $\rho(D) = \rho(D')$ for some Dedekind domain D' with finite divisor class group. For (1), the divisor class group of D' is $\mathbb{Z}_2$, and thus $\rho(D) = 1$. For (2), D' has divisor class group $\mathbb{Z}_m$. Since $D(\mathbb{Z}_m) = m$, the result follows. Part (3) follows from Corollary 3.3 of [7]. □

We next present an example which helps contrast elasticity problems in Krull domains with finite divisor class group with those of Krull domains with infinite divisor class group. We have shown in Theorem 2.1 that a Krull domain with divisor class group $\mathbb{Z}$ may have infinite elasticity. Our next example shows that if $\rho(D) < \infty$ for such a domain, then there may not exist an $x \in D^*$ such that $\rho(D) = \rho_D(x)$. This shows that the $t = 2$ case is very different from the $t = 1$ case.

Example 3.4. Let D be a Dedekind domain with realizable pair $\{\mathbb{Z}, S\}$ such that $S = \{-m, -1, s_1, s_2, \ldots\}$, where the elements $m, s_1, s_2, \ldots$ are positive integers and infinitely many of the elements $\{s_i\}_{i=1}^{\infty}$ are congruent to 1 modulo m. We claim that $\rho(D) = m$, but $\rho_D(x) < m$ for every nonzero nonunit $x \in D$.

To see this, suppose that γ is an irreducible element of D whose irreducible block has representation as in (2.4) of the form

$$\pi(\gamma) = [Q_m]^{x_1}[Q_1]^{x_2}[P_{s_1}]^{y_1} \cdots [P_{s_k}]^{y_k}.$$

We first show that $y_1 + \cdots + y_k \le m$. Suppose that $y_1 + y_2 + \cdots + y_k > m$. Let $s'_1, \ldots, s'_k$ be integers such that $0 \le s'_i < m$ and $s'_i \equiv s_i \bmod m$. Notice that the set

$$T_\gamma = \{\underbrace{s'_1, \ldots, s'_1}_{y_1 \text{ times}}, \underbrace{s'_2, \ldots, s'_2}_{y_2 \text{ times}}, \ldots, \underbrace{s'_k, \ldots, s'_k}_{y_k \text{ times}}\}$$

contains more than m elements, and since the Davenport constant of $\mathbb{Z}_m$ is m, some subset of T_γ sums to zero modulo m. Hence, there exists nonnegative integers t, $y'_1, \ldots, y'_k$ with $0 \le y'_i \le y_i$ such that

$$y'_1 s_1 + y'_2 s_2 + \cdots + y'_k s_k = tm < y_1 s_1 + \cdots + y_k s_k.$$

Thus $x_1 m + x_2 > y'_1 s_1 + \cdots + y'_k s_k = tm$. If $x_1 \ge t$, then there exists a proper divisor α of γ, where

$$\pi(\alpha) = [Q_m]^t [P_{s_1}]^{y'_1} \cdots [P_{s_k}]^{y'_k},$$

contradicting the fact that γ is irreducible. If $x_1 < t$, then set $x'_2 = tm - x_1 m$. Notice that $x'_2 < x_2$ and again there exists a proper divisor β of γ, where

$$\pi(\beta) = [Q_m]^{x_1} [Q_1]^{x'_2} [P_{s_1}]^{y'_1} \cdots [P_{s_k}]^{y'_k},$$

another contradiction. Hence $y_1 + \cdots + y_k \le m$. Again, as in the proof of Corollary 2.5, $\rho(D) \le m$. To see that $\rho(D) = m$, let s_j be an element of S which is congruent to 1 modulo m. Write $s_j = mk + 1$ for $k \in \mathbb{Z}_+$. Then

$$\left[[Q_m]^{mk+1} [P_{mk+1}]^m \right]^k \left[[Q_1]^{mk+1} [P_{mk+1}] \right] = \left[[Q_m]^k [P_{mk+1}] [Q_1] \right]^{mk+1}.$$

Hence, there exist irreducibles α, β, and γ of D with $\alpha^k \beta = \gamma^{mk+1}$. Thus the product of $k+1$ irreducibles factors as $mk+1$ irreducibles, and hence $(mk+1)/(k+1) \le \rho(D)$. Since there are infinitely many s_j which are congruent to 1 modulo m,

$$\rho(D) \ge \lim_{k \to \infty} \frac{mk+1}{k+1} = m.$$

Consequently, $\rho(D) = m$.

To prove the second assertion of the claim we will need the following lemma.

Lemma 3.5. *If α is irreducible in D with*

$$(\alpha) = Q_m^{x_1} Q_1^{x_2} P_{s_1}^{y_1} \cdots P_{s_k}^{y_k}$$

and $y_1 + \cdots + y_k = m$, then $x_2 = 0$. Furthermore, for such an α, $s_i \not\equiv 0 \pmod m$ for any $1 \le i \le k$.

Proof. We first show that

$$y_1 s_1 + y_2 s_2 + \cdots + y_k s_k = tm \tag{3.3}$$

for some $t > 0$. Suppose that $y_1s_1 + \cdots + y_ks_k \neq tm$ for any $t \in \mathbb{Z}_+$. Again, some subsum of the elements in T_α sums to zero modulo m. As before, suppose that

$$y_1{}'s_1 + \cdots + y_k{}'s_k = qm$$

for $q \in \mathbb{Z}_+$. If $x_1 \geq q$, then there exists a proper divisor β of α with $\pi(\beta) = [Q_m]^q[P_{s_1}]^{y_1'} \cdots [P_{s_k}]^{y_k'}$, a contradiction. Thus $x_1 < q$ and $x_1m + x_2 > qm$. Let $x_2' = qm - x_1m$. Then there exists a proper divisor γ of α with

$$\pi(\gamma) = [Q_m]^{x_1}[Q_1]^{x_2'}[P_{s_1}]^{y_1'} \cdots [P_{s_k}]^{y_k'},$$

another contradiction. Thus (3.3) holds. Now suppose that

$$\pi(\alpha) = [Q_m]^{x_1}[Q_1]^{x_2}[P_{s_1}]^{y_1} \cdots [P_{s_k}]^{y_k}$$

is irreducible with $y_1 + \cdots + y_k = m$ and $x_2 \neq 0$. If $y_1s_1 + \cdots + y_ks_k = tm$, then notice that $x_1m + x_2 = tm$ implies that $x_2 = m(t - x_1)$ and hence $x_2 \geq m$. Now, T_α has exactly m elements; suppose $T_\alpha = \{t_im + r_i\}_{i=1}^m$, where each $0 \leq r_i < m$. Then

$$x_1m + x_2 = (t_1m + r_1) + \cdots + (t_m m + r_m).$$

Since $x_2 \geq m$, each $t_i > x_1$; for if not, then for some $x_1' \leq x_1$ and $x_2' \leq x_2$ we would have

$$x_1'm + x_2' = t_jm + r_j.$$

Hence, there exists a divisor β of α with $\pi(\beta) = [Q_m]^{x_1'}[Q_1]^{x_2'}[P_{s_j}]$ (for some $1 \leq j \leq m$) which contradicts the irreducibility of α. Now, choose any i with $1 \leq i \leq m$. For $x_2' = (t_im + r_i) - x_1m$ we have that $x_1m + x_2' = t_im + r_i$ and there exists another divisor δ of α with $\pi(\delta) = [Q_m]^{x_1}[Q_1]^{x_2'}[P_{s_i}]$, again contradicting the irreducibility of α. Thus $x_2 = 0$.

Now, suppose that $\pi(\alpha) = [Q_m]^{x_1}[P_{s_1}]^{y_1} \cdots [P_{s_k}]^{y_k}$ is irreducible with $s_j \equiv 0 \pmod m$ for some $1 \leq j \leq k$ and $y_1 + \cdots + y_k = m$. If $s_j = wm$, then notice that $x_1m \geq wm$. Hence, $[Q_m]^w[P_{s_j}]$ is an irreducible block in $\mathscr{B}(S)$ and there exists an irreducible β of D with $\pi(\beta) = [Q_m]^w[P_{s_j}]$ which divides α. This contradicts the irreducibility of α. Thus $s_j \not\equiv 0 \pmod m$ for each j, and the proof of the lemma is complete. □

We now show that $\rho_D(x) \neq m$ for any x in D. Suppose that $\rho_D(x) = m$ for some x, where the prime ideal factorization of x is given by $(x) = Q_m^{x_1} Q_1^{x_2} P_{s_1}^{y_1} \cdots P_{s_k}^{y_k}$. Using the semi-length function discussed earlier, we have

$$\frac{1}{m}(y_1 + \cdots + y_k) \leq l(x) \leq L(x) \leq y_1 + \cdots + y_k.$$

Therefore, $\rho_D(x) = L(x)/l(x) = m$ implies for this x that $l(x) = \frac{1}{m}(y_1 + \cdots + y_k)$ and $L(x) = y_1 + \cdots + y_k$. However, $l(x) = \frac{1}{m}(y_1 + \cdots + y_k)$ occurs only when $x = \alpha_1 \cdots \alpha_{l(x)}$, where each of the α_i's is of the form discribed in the preceding lemma.

On the other hand, $L(x) = y_1 + \cdots + y_k$ occurs only when $x = \beta_1 \cdots \beta_{L(x)}$, where each of the β_j's is of the form $Q_m{}^{x_j} P_{s_j}$. In the first case, Lemma 3.5 implies that $s_j \not\equiv 0$ (mod m) for all j. In the second case, we have $s_j \equiv 0 \pmod{m}$ for all j. Therefore, such an x cannot exist.

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