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RATIONAL COHOMOLOGY OF ALGEBRAIC SOLVABLE GROUPS

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If G is an affine algebraic group over a field F, and M is a finite-dimensional F-vector space, then M is a *rational G-module* if G acts on M via a morphism of algebraic groups over $F: G \xrightarrow{P} \mathrm{Aut}_F(M)$. An infinite-dimensional F-vector space M is a rational G-module if it is the union $\bigcup_i M_i$ of finite-dimensional G-stable vector spaces M_i such that the G-action on each of them is rational. In [8], Hochschild developed the foundations of rational cohomology, i.e., cohomology $H^*_{\mathrm{rat}}(G, M)$ in the category of rational G-modules. The most recent applications of rational cohomology (e.g. [4] and [6]) seem to be mainly restricted to groups defined over fields of nonzero characteristic. In this paper we will utilize the rational cohomology groups of algebraic solvable groups defined over the rational numbers $\mathbb{Q}$. Our goal is to prove, for algebraic solvable G and for trivial $\mathbb{Q}$-coefficients, an analog (Theorem 2.23) of the following theorem of Mostow [11, 8.1] and Van Est [11, 3.6.1]:

If G is a connected simply connected real solvable Lie group and D is a discrete cocompact subgroup such that $\mathrm{Ad}_{\mathfrak{g}_{\mathbb{R}}}(G)$ and $\mathrm{Ad}_{\mathfrak{g}_{\mathbb{R}}}(D)$ have the same algebraic hulls, then the Lie algebra cohomology $H^*(\mathfrak{g}_{\mathbb{R}}, \mathbb{R})$ is isomorphic to the group cohomology $H^*(D, \mathbb{R})$ (trivial $\mathbb{R}$-coefficients in both cases).

In our situation, D will be an arithmetic subgroup that is contained and cocompact in the identity component $G^0_{\mathbb{R}}$ of the real points of G; we put the requisite conditions on G to make certain that $G^0_{\mathbb{R}}$ is contractible (see Section 1). Under our assumptions, we will obtain an isomorphism of cohomology rings (Theorem 2.23)

$$H^*(\mathfrak{g}_{\mathbb{Q}}, \mathbb{Q}) \xrightarrow[\simeq]{l_{\mathbb{Q}}} H^*(D, \mathbb{Q}).$$

As will become clear in Section 2, even though $l_{\mathbb{Q}}$ is a ring map, it is perhaps unnatural in that it involves choosing an ancillary isomorphism and so might depend upon that choice.

The paper is organized as follows. In Section 1 we list our notational conventions.

The beginning of Section 2 is a summary of results from the various cohomology theories used, while the rest of the section is devoted to the construction of $l_{\mathbb{Q}}$ and the proof of Theorem 2.23. An example is presented in Section 3; the concluding Appendix contains the rather lengthy proof of a result used in proving Theorem 2.23.

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1. Notation and conventions

We follow the notation of [2]. Throughout this paper, G will denote a solvable affine algebraic group defined over the rational numbers $\mathbb{Q}$. Since we do not need to emphasize the more sophisticated functorial aspects of algebraic groups, we will be content to think of G as the zero set of an ideal I_G of regular functions defined on $\mathrm{GL}_n(\mathbb{C})$ for some $n \geq 1$. We will assume that I_G is prime, so that G is an irreducible variety. Since we assume G to be defined over $\mathbb{Q}$, I_G can be generated by functions $f_1, \ldots, f_p$ lying in the subring $\mathbb{Q}[x_{11}, \ldots, x_{nn}, \mathrm{Det}^{-1}]$. We define the $\mathbb{Q}$-algebra $\theta_{G_{\mathbb{Q}}}$ as follows:

$$\theta_{G_{\mathbb{Q}}} = \frac{\mathbb{Q}[x_{11}, \ldots, x_{nn}, \mathrm{Det}^{-1}]}{I_G \cap \mathbb{Q}[x_{11}, \ldots, x_{nn}, \mathrm{Det}^{-1}]} . \tag{1.1}$$

Similarly, we set

$$\theta_{G_{\mathbb{R}}} = \frac{\mathbb{R}[x_{11}, \ldots, x_{nn}, \mathrm{Det}^{-1}]}{I_G \cap \mathbb{R}[x_{11}, \ldots, x_{nn}, \mathrm{Det}^{-1}]} .$$

If $A \subset \mathbb{C}$ is a subring, we set $G_A = G \cap \mathrm{GL}_n(A)$. In particular $G_{\mathbb{R}}$ is a Lie group; we denote its identity component by $G^0_{\mathbb{R}}$. It is well known that $G \simeq N \rtimes T$, when N is the maximal connected unipotent normal subgroup of G (it is defined over $\mathbb{Q}$), and T is a torus (also defined over $\mathbb{Q}$). $N_{\mathbb{R}}$ is connected, and $G^0_{\mathbb{R}} = N_{\mathbb{R}} \rtimes T^0_{\mathbb{R}}$.

G has associated to it a Lie algebra $\mathfrak{g}_{\mathbb{Q}}$ over $\mathbb{Q}$ (see [3]); the real Lie algebra $\mathfrak{g}_{\mathbb{R}} \simeq \mathfrak{g}_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{R}$ is isomorphic to the real Lie algebra of right-invariant vector fields on $G_{\mathbb{R}}$.

A (discrete) subgroup $D \subset G \cap \mathrm{GL}_n(\mathbb{Z})$ is called *arithmetic* if it is commeasurable with $G \cap \mathrm{GL}_n(\mathbb{Z})$.

(1.2) We work with the following assumptions on G and D: (i) the torus factor T in $G \simeq N \rtimes T$ is $\mathbb{Q}$-anisotropic and $\mathbb{R}$-split (see [3]). This condition implies $T_{\mathbb{R}} \simeq \mathbb{R}^* \times \cdots \times \mathbb{R}^*$, and also that $G^0_{\mathbb{R}}$ is contractible.

(ii) $D \subseteq G^0_{\mathbb{R}} \cap \mathrm{GL}_n(\mathbb{Z})$ is arithmetic and cocompact in $G^0_{\mathbb{R}}$. D is then strongly polycyclic (see [14]): $D \simeq D_N \rtimes D_T$ where $D_N = D \cap N_{\mathbb{R}}$, $D_T = D \cap T^0_{\mathbb{R}}$ and $D_T \simeq \mathbb{Z}^p$.

From [14, Theorem 2.3] and [1], D is then $\mathrm{Ad}_{\mathfrak{g}_{\mathbb{R}}}$-ample in $G^0_{\mathbb{R}}$ (since $\mathrm{Ad}_{\mathfrak{g}_{\mathbb{R}}}$ is a rational representation of G defined over $\mathbb{R}$); and so the pair $(G^0_{\mathbb{R}}, D)$ satisfies the conditions of Mostow's theorem. If M is a finite-dimensional $\mathbb{Q}$-vector space, it is a rational G-module or a $\theta_{G_{\mathbb{Q}}}$-*comodule* if it is equipped with a $\mathbb{Q}$-linear map $\Delta_M : M \to M \otimes_{\mathbb{Q}} \theta_{G_{\mathbb{Q}}}$ satisfying the axioms for counit and coassociativity (see [9]). If $v \in M$ and $\Delta_M(v) = \sum_i w_i \otimes f_i$, then if $d \in G_{\mathbb{Q}}$ the action of d on v is $d \cdot v = \sum_i f_i(d) w_i$. We note that if $G \simeq N \rtimes T$ and M is a rational N-module, then the cohomology groups $H^*_{\mathrm{rat}}(N, M)$ and $H^*(\mathfrak{n}_{\mathbb{Q}}, M)$ are rational T-modules, and so are semisimple T-modules as well as semisimple $\mathfrak{t}_{\mathbb{Q}}$-modules [8, pp. 510–511]. Finally we note that the multiplication $G \times G \to G$ induces a map $\Delta : \theta_{G_{\mathbb{Q}}} \to \theta_{G_{\mathbb{Q}}} \otimes_{\mathbb{Q}} \theta_{G_{\mathbb{Q}}}$; this map Δ defines the Hopf algebra structure on $\theta_{G_{\mathbb{Q}}}$.

2.

We summarize here some of the basic definitions of rational cohomology (see [5]). If V is a $\mathbb{Q}$-vector space with a $\theta_{G_{\mathbb{Q}}}$-comodule structure $\Delta_V : V \to V \otimes_{\mathbb{Q}} \theta_{G_{\mathbb{Q}}}$, we define the cochain complex

(2.1) $\qquad C^\bullet(\theta_{G_{\mathbb{Q}}}, V)$

as follows (unless noted otherwise, all tensoring is over $\mathbb{Q}$):

$$C^n(\theta_{G_{\mathbb{Q}}}, V) = V \otimes \theta_{G_{\mathbb{Q}}}^{\otimes n}, \quad n \geq 0 \qquad (\theta_{G_{\mathbb{Q}}}^{\otimes n} = \theta_{G_{\mathbb{Q}}} \otimes \overset{(n)}{\cdots} \otimes \theta_{G_{\mathbb{Q}}}),$$

$$\partial^n = \sum_{i=0}^{n+1} (-1)^i \partial^n_i,$$

where

$$\partial^n_0(v \otimes f_1 \otimes \cdots \otimes f_n) = \Delta_V(v) \otimes f_1 \otimes \cdots \otimes f_n,$$

$$\partial^n_i(v \otimes f_1 \otimes \cdots \otimes f_n) = v \otimes f_1 \otimes \cdots \otimes \Delta f_i \otimes \cdots \otimes f_n, \quad 1 \leq i \leq n,$$

$$\partial^n_{n+1}(v \otimes f_1 \otimes \cdots \otimes f_n) = v \otimes f_1 \otimes \cdots \otimes f_n \otimes 1.$$

The *rational cohomology groups* of G with coefficients in V are then the cohomology groups of the complex $C^\bullet(\theta_{G_{\mathbb{Q}}}, V)$:

$$H^n_{\mathrm{rat}}(G, V) = H^n(C^\bullet(\theta_{G_{\mathbb{Q}}}, V)), \quad n \geq 0.$$

$H^0_{\mathrm{rat}}(G, V)$ is the subspace of all $v \in V$ such that $\Delta_V(v) = v \otimes 1$, i.e.

$$H^0_{\mathrm{rat}}(G, V) = V^G.$$

The $H^i_{\mathrm{rat}}(G, V)$ for $i > 0$ are the derived functors of $V \to V^G$ in the category of $\theta_{G_{\mathbb{Q}}}$-comodules.

In particular, if $V = \mathbb{Q}$ with the trivial comodule structure, we have

$$C^0(\theta_{G_{\mathbb{Q}}}, \mathbb{Q}) = \mathbb{Q},$$

$$C^1(\theta_{G_{\mathbb{Q}}}, \mathbb{Q}) = \theta_{G_{\mathbb{Q}}},$$

$$C^n(\theta_{G_{\mathbb{Q}}}, \mathbb{Q}) = \theta_{G_{\mathbb{Q}}}^{\otimes n}, \ldots$$

We may use the complex $C^\bullet(\theta_{G_\mathbb{Q}}, \mathbb{Q})$ to define a cup product pairing

(2.2) $$H^i_{\mathrm{rat}}(G, M) \otimes H^j_{\mathrm{rat}}(G, \mathbb{Q}) \xrightarrow{\cup} H^{i+j}_{\mathrm{rat}}(G, M).$$

The pairing is defined at the cochain level as follows: if $\phi = v \otimes f_1 \otimes \cdots \otimes f_i \in C^i(\theta_{G_\mathbb{Q}}, M)$ and $\psi = g_1 \otimes \cdots \otimes g_j \in C^j(\theta_{G_\mathbb{Q}}, \mathbb{Q})$, then define

$$\phi \cup \psi = \phi \otimes \psi = v \otimes f_1 \otimes \cdots \otimes f_i \otimes g_1 \otimes \cdots \otimes g_j \in C^{i+j}(\theta_{G_\mathbb{Q}}, M).$$

In the tensor product $C^\bullet(\theta_{G_\mathbb{Q}}, M) \otimes C^\bullet(\theta_{G_\mathbb{Q}}, \mathbb{Q})$ we have the usual total differential

$$d(\phi \otimes \psi) = d\phi \otimes \psi + (-1)^{\mathrm{Deg}(\phi)} \phi \otimes d\psi;$$

it is easily checked then that $\cup$ defines a map of complexes

$$C^\bullet(\theta_{G_\mathbb{Q}}, M) \otimes C^\bullet(\theta_{G_\mathbb{Q}}, \mathbb{Q}) \xrightarrow{\cup} C^\bullet(\theta_{G_\mathbb{Q}}, M),$$

and so from the Kunneth theorem we then have cohomology pairings

$$H^i_{\mathrm{rat}}(G, M) \otimes H^j_{\mathrm{rat}}(G, \mathbb{Q}) \to H^{i+j}_{\mathrm{rat}}(G, M).$$

There is a natural evaluation or restriction map from rational cohomology to discrete group cohomology. If $D \subseteq G_\mathbb{Q}$ is a subgroup of the $\mathbb{Q}$-valued points of G (given the discrete topology), and M is a rational G-module, then we may consider M as a D-module. If we denote by $C^\bullet(D, M)$ the usual complex of nonhomogeneous M-valued cochains on D, we define the *restriction map*

(2.3) $$r_M : C^\bullet(\theta_{G_\mathbb{Q}}, M) \to C^\bullet(D, M)$$

as follows: if $\phi = v \otimes f_1 \otimes \cdots \otimes f_p \in C^p(\theta_{G_\mathbb{Q}}, M)$, then

$$r_M(\phi)(d_1, \ldots, d_p) = f_1(d_1) \cdots f_p(d_p) v \in M.$$

This makes sense since $f(d) \in \mathbb{Q}$ for all $f \in \theta_{G_\mathbb{Q}}$ and $d \in D$. It can be checked that

$$r_M(d\phi) = d(r_M(\phi)),$$

so we have

(2.4) **Lemma.** *r_M is a map of complexes; thus for all $n \geq 0$ we have a cohomology map*

$$H^n_{\mathrm{rat}}(G, M) \xrightarrow{r_M} H^n(D, M).$$

The cup product $H^i(D, M) \otimes H^j(D, \mathbb{Q}) \xrightarrow{\cup} H^{i+j}(D, M)$ is defined at the cochain level as follows: if $\phi \in C^i(D, M)$ and $\psi \in C^j(D, \mathbb{Q})$, then

$$\phi \cup \psi(d_1, \ldots, d_i, d_{i+1}, \ldots, d_{i+j}) = \psi(d_{i+1}, \ldots, d_{i+j}) \cdot \phi(d_1, \ldots, d_i).$$

The following lemma is immediate from the definitions.

(2.5) **Lemma.** *r_M preserves cup products, i.e., the following diagram commutes (where $\mathbb{Q}$-coefficients are trivial coefficients):*

$$\begin{CD}
H^i_{\mathrm{rat}}(G,M)\otimes H^j_{\mathrm{rat}}(G,\mathbb{Q}) @>{\cup}>> H^{i+j}_{\mathrm{rat}}(G,M)\\
@V{r_M\otimes r_{\mathbb{Q}}}VV @VV{r_M}V\\
H^i(D,M)\otimes H^j(D,\mathbb{Q}) @>{\cup}>> H^{i+j}(D,M)
\end{CD}$$

The third type of cohomology we consider is the cohomology of $\mathfrak{g}_{\mathbb{Q}}$-modules, where $\mathfrak{g}_{\mathbb{Q}}$ is the $\mathbb{Q}$-Lie algebra of the algebraic group G. In general, if $\mathfrak{g}$ is a finite-dimensional Lie algebra over a field F of characteristic 0, an F-vector space N is a $\mathfrak{g}$-module if we are given a Lie algebra homomorphism $\mathfrak{g}\to\mathrm{End}_F(M)$. If M is a rational G-module, we may consider M to be a $\mathfrak{g}_{\mathbb{Q}}$-module by taking the differential of the representation (see [9]). The only $\mathfrak{g}_{\mathbb{Q}}$-modules we consider will be derived from rational representations.

For convenience we present the following definitions from [10]. If $\mathfrak{h}$ is an ideal in $\mathfrak{g}_{\mathbb{Q}}$ and if we denote by $C^p_{\mathrm{alt}}(\mathfrak{h},M)$ the p-linear alternating functions on $\mathfrak{h}$ with values in M, then we may put a $\mathfrak{g}_{\mathbb{Q}}$-module structure on $C^p_{\mathrm{alt}}(\mathfrak{h},M)$ as follows. For $p=0$, $C^0_{\mathrm{alt}}(\mathfrak{h},M)=M$ is already a $\mathfrak{g}_{\mathbb{Q}}$-module. For $p>0$, $\phi\in C^p_{\mathrm{alt}}(\mathfrak{h},M)$, and $x\in\mathfrak{g}_{\mathbb{Q}}$, we define

$$(x\cdot\phi)(y_1,\dots,y_p)=x\cdot\phi(y_1,\dots,y_p)-\sum_{i=1}^{p}\phi(y_1,\dots,y_{i-1},[x,y_i],y_{i+1},\dots,y_p).$$

The differential in $C^\bullet_{\mathrm{alt}}(\mathfrak{h},M)$ is a $\mathfrak{g}_{\mathbb{Q}}$-module map:

$$d(x\cdot\phi)=x\cdot(d\phi).$$

The $\mathfrak{g}_{\mathbb{Q}}$-action on cochains induces a $\mathfrak{g}_{\mathbb{Q}}$-module structure on the cohomology groups $H^i(\mathfrak{h},M)$. It can be shown that $\mathfrak{h}$ operates trivially on all $H^i(\mathfrak{h},M)$.

If M,N, and P are three $\mathfrak{g}_{\mathbb{Q}}$-modules, a *pairing* from M and N to P is a $\mathbb{Q}$-bilinear map $(m,n)\to m\cup n$ of $M\times N$ into P such that, for all $x\in\mathfrak{g}_{\mathbb{Q}}$,

$$x\cdot(m\cup n)=(x\cdot m)\cup n+m\cup(x\cdot n).$$

If $\mathfrak{h}$ is an ideal of $\mathfrak{g}_{\mathbb{Q}}$, we may then define a pairing of cochains

$$C^p_{\mathrm{alt}}(\mathfrak{h},M)\otimes C^q_{\mathrm{alt}}(\mathfrak{h},N)\xrightarrow{\cup}C^{p+q}_{\mathrm{alt}}(\mathfrak{h},P). \tag{2.6}$$

If $\phi\in C^p_{\mathrm{alt}}(\mathfrak{h},M)$ and $\psi\in C^q_{\mathrm{alt}}(\mathfrak{h},N)$, then

$$\phi\cup\psi(y_1,\dots,y_{p+q})=\sum_{\substack{\pi\in(p,q)\text{-}\\ \text{shuffles}}}(-1)^{\mathrm{sgn}(\pi)}\phi(y_{\pi(1)},\dots,y_{\pi(p)})\cup\psi(y_{\pi(p+1)},\dots,y_{\pi(p+q)}).$$

(A permutation $\pi\in\sum_{p+q}$ is called a (p,q)-shuffle if $\pi(1)<\pi(2)<\cdots<\pi(p)$ and $\pi(p+1)<\cdots<\pi(p+q)$.) We have

$$(2.7)\qquad \begin{aligned} &x\cdot(\phi\cup\psi)=(x\cdot\phi)\cup\psi+\phi\cup(x\cdot\psi), \quad x\in\mathfrak{g}_{\mathbb{Q}},\\ &d(\phi\cup\psi)=d\phi\cup\psi+(-1)^{\mathrm{Deg}(\phi)}\phi\cup d\psi. \end{aligned}$$

The mechanics of checking the following lemma then follow just as for rational and discrete group cohomology:

(2.8) **Lemma.** *The pairing* $C^p_{\mathrm{alt}}(\mathfrak{h},M)\otimes C^q_{\mathrm{alt}}(\mathfrak{h},N)\xrightarrow{\cup} C^{p+q}_{\mathrm{alt}}(\mathfrak{h},P)$ *induces a cohomology pairing*

$$H^p(\mathfrak{h},M)\otimes H^q(\mathfrak{h},N)\xrightarrow{\cup} H^{p+q}(\mathfrak{h},P).$$

From (2.7) we have an induced pairing on the $\mathfrak{g}_{\mathbb{Q}}$-annihilated elements

$$(2.9)\qquad H^p(\mathfrak{h},M)^{\mathfrak{g}_{\mathbb{Q}}}\otimes H^q(\mathfrak{h},N)^{\mathfrak{g}_{\mathbb{Q}}}\xrightarrow{\cup} H^{p+q}(\mathfrak{h},P)^{\mathfrak{g}_{\mathbb{Q}}}.$$

If $N=\mathbb{Q}$ with trivial $\mathfrak{g}_{\mathbb{Q}}$-action, and $P=M$ with pairing $M\times\mathbb{Q}\xrightarrow{\cup} M$ given by $(v,\alpha)\to\alpha\cdot v$, we have the usual cup product

$$H^p(\mathfrak{h},M)\otimes H^q(\mathfrak{h},\mathbb{Q})\xrightarrow{\cup} H^{p+q}(\mathfrak{h},M).$$

We are now in a position to use the following theorem of [10], which we state without proof:

(2.10) **Theorem** [10, Theorem 13]. *Let* $\mathfrak{g}$ *be a finite-dimensional Lie algebra over a field* F *of characteristic* 0, *and let* M *be a finite-dimensional* $\mathfrak{g}$*-module. Suppose* $\mathfrak{n}$ *is an ideal* $\mathfrak{g}$ *such that* $\mathfrak{g}/\mathfrak{n}\simeq\mathfrak{k}$ *is semisimple. Then for all* $n\geq 0$,

$$H^n(\mathfrak{g},M)\simeq\sum_{i+j=n} H^i(\mathfrak{n},M)^{\mathfrak{g}}\otimes_F H^j(\mathfrak{k},F),$$

by an isomorphism which is multiplicative for paired modules.

This last phrase means the following (for more details see [10, §6]). A given pairing of $\mathfrak{g}$-modules $M\times N\xrightarrow{\cup} P$ induces a cohomology pairing

$$H^i(\mathfrak{g},M)\otimes H^j(\mathfrak{g},N)\xrightarrow{\cup} H^{i+j}(\mathfrak{g},P),\quad i,j\geq 0;$$

from (2.10) we then have

$$H^i(\mathfrak{g},M)\simeq\sum_{k+l=i} H^k(\mathfrak{n},M)^{\mathfrak{g}}\otimes H^l(\mathfrak{k},F),$$

$$H^j(\mathfrak{g},N)=\sum_{p+q=j} H^p(\mathfrak{n},N)^{\mathfrak{g}}\otimes H^q(\mathfrak{k},F).$$

If we take $u\otimes v\in H^i(\mathfrak{g},M)$ and $u'\otimes v'\in H^j(\mathfrak{g},N)$, then

$$(2.11)\qquad (u\otimes v)\cup(u'\otimes v')=(-1)^{lp}(u\cup_1 u')\otimes(v\cup_2 v')\in H^{i+j}(\mathfrak{g},P),$$

where

$$H^*(\mathfrak{k},F)\otimes H^*(\mathfrak{k},F)\xrightarrow{\cup_1} H^*(\mathfrak{k},F),$$

$$H^*(\mathfrak{n},M)^{\mathfrak{g}}\otimes H^*(\mathfrak{n},N)^{\mathfrak{g}}\xrightarrow{\cup_2} H^*(\mathfrak{n},P)^{\mathfrak{g}}$$

are the other pairings used.

In particular, this gives us a decomposition of the multiplicative structure in $H^*(\mathfrak{g},F)$, F the trivial $\mathfrak{g}$-module: if we consider $\alpha\in H^i(\mathfrak{k},F)$ as $\alpha\otimes 1\in H^i(\mathfrak{k},F)\otimes H^0(\mathfrak{n},F)^{\mathfrak{g}}\simeq H^i(\mathfrak{k},F)\otimes F$, and $\beta\in H^j(\mathfrak{n},F)^{\mathfrak{g}}$ as $1\otimes\beta\in H^0(\mathfrak{k},F)\otimes H^j(\mathfrak{n},F)^{\mathfrak{g}}$, then for the pairing $F\times F\xrightarrow{\cup}F$ (multiplication in F) we have

$$(\alpha\otimes 1)\cup(1\otimes\beta)=\alpha\otimes\beta\in H^{i+j}(\mathfrak{g},F). \tag{2.12}$$

We are now prepared to consider solvable affine algebraic groups G as defined in Section 1. We recall that $G\simeq N\rtimes T$, where the torus T is presumed to be $\mathbb{Q}$-anisotropic and $\mathbb{R}$-split. $T_{\mathbb{R}}$ is then a direct product of d copies of $\mathbb{R}^*$, and $D_T\simeq\mathbb{Z}^d$ is contained in $T_{\mathbb{R}}^0$.

Recall from (2.5) that the restriction map $r_M: H^*_{\mathrm{rat}}(G,M)\to H^*(D,M)$ preserves the pairing $M\times\mathbb{Q}\xrightarrow{\cup}M$, $(v,\alpha)\to\alpha v$. Since T is a torus, its Lie algebra $\mathfrak{t}_{\mathbb{Q}}$ is abelian, so $\mathfrak{t}_{\mathbb{Q}}$ is isomorphic to the Lie algebra of the d-dimensional vector group $\mathbb{A}^d_{\mathbb{Q}}$. $\mathbb{A}^d_{\mathbb{Q}}$ is a unipotent algebraic group; if we denote its Lie algebra by $\mathfrak{a}(d)_{\mathbb{Q}}$, then from [13] we know that we have an isomorphism of cohomology rings

$$H^*(\mathfrak{a}(d)_{\mathbb{Q}},\mathbb{Q})\xrightarrow{\mathscr{S}} H^*(\mathbb{Z}^d,\mathbb{Q})\qquad\text{(trivial $\mathbb{Q}$-coeff.).}$$

If we then choose an isomorphism of $\mathbb{Q}$-vector spaces $\mathfrak{a}(d)_{\mathbb{Q}}\xrightarrow{\sim}\mathfrak{t}_{\mathbb{Q}}$ and an isomorphism of groups $D_T\xrightarrow{\sim}\mathbb{Z}^d$, we have an induced isomorphism of cohomology rings

$$H^*(\mathfrak{t}_{\mathbb{Q}},\mathbb{Q})\underset{\sim}{\xrightarrow{I_{\mathbb{Q}}}} H^*(D_T,\mathbb{Q}). \tag{2.13}$$

Moreover, $I_{\mathbb{Q}}$ is then induced by a map at the cochain level since the same is true of $\mathscr{S}$. If we denote by π the projection $D\twoheadrightarrow D_T$, then composition with $T_{\mathbb{Q}}$ induces a ring map

$$H^*(\mathfrak{t}_{\mathbb{Q}},\mathbb{Q})\xrightarrow{\pi^*\circ I_{\mathbb{Q}}} H^*(D,\mathbb{Q}).$$

Using the cup product $H^*(D,M)\otimes H^*(D,\mathbb{Q})\xrightarrow{\cup}H^*(D,M)$ we then consider the composition (which we denote by l_M)

$$\begin{tikzcd}[column sep=large] H^*_{\mathrm{rat}}(G,M)\otimes H^*(\mathfrak{t}_{\mathbb{Q}},\mathbb{Q}) \arrow[r, "r_M\otimes(\pi^*\circ I_{\mathbb{Q}})"] \arrow[rr, bend right, "l_M"'] & H^*(D,M)\otimes H^*(D,\mathbb{Q}) \arrow[r, "\cup"] & H^*(D,M) \end{tikzcd} \tag{2.14}$$

To interpret l_M we require the following two theorems from [8], which we state without proof.

(2.15) **Theorem** ([8, Theorem 5.1]). *Let G be a unipotent algebraic group over a field F of characteristic 0, and let A be a rational G-module. There is an isomorphism ϕ of the rational cohomology group $H^*_{\mathrm{rat}}(G,A)$ onto the Lie algebra cohomology group $H^*(\mathfrak{g}_F, A)$ with the following property: Given any rationality injective resolution $X^\bullet$ of the rational G-module A and any $U(\mathfrak{g}_F)$-injective resolution $Y^\bullet$ of the $U(\mathfrak{g}_F)$-module A, there is a map of $X^\bullet$ regarded as an acyclic $U(\mathfrak{g}_F)$-complex, into $Y^\bullet$, and the cohomology map induced by any such map of complexes coincides with the isomorphism ϕ.*

(2.16) **Theorem** ([8, Theorem 5.2]). *Let G be a linear algebraic group over a field F of characteristic 0, let N be the maximum unipotent normal subgroup of G, and let $\mathfrak{n}_F$ be the Lie algebra of N. Let A be a rational G-module. Then $H^*_{\mathrm{rat}}(G,A)$ may be identified with $H^*_{\mathrm{rat}}(N,A)^{G/N}$, and hence is isomorphic, via the canonical isomorphism, with $H^*(\mathfrak{n}_F, A)^{G/N}$. In the case at hand, $G/N \simeq T$ is irreducible, and so $H^*(\mathfrak{n}_{\mathbb{Q}}, A)^T = H^*(\mathfrak{n}_{\mathbb{Q}}, A)^{\mathfrak{t}_{\mathbb{Q}}}$.*

We then denote by

$$\mathscr{H}_M : H^*(\mathfrak{n}_{\mathbb{Q}}, M)^{\mathfrak{t}_{\mathbb{Q}}} \xrightarrow{\sim} H^*_{\mathrm{rat}}(G,M) \tag{2.17}$$

the isomorphism provided by (2.16). From [8, pp. 507–509], $\mathscr{H}_M$ is compatible with cup products, so the following diagram commutes:

$$\begin{array}{ccc} H^*(\mathfrak{n}_{\mathbb{Q}}, M)^{\mathfrak{t}_{\mathbb{Q}}} \otimes H^*(\mathfrak{n}_{\mathbb{Q}}, \mathbb{Q})^{\mathfrak{t}_{\mathbb{Q}}} & \xrightarrow{\ \cup\ } & H^*(\mathfrak{n}_{\mathbb{Q}}, M)^{\mathfrak{t}_{\mathbb{Q}}} \\ \wr\downarrow \mathscr{H}_M \otimes \mathscr{H}_{\mathbb{Q}} & & \wr\downarrow \mathscr{H}_M \\ H^*_{\mathrm{rat}}(G,M) \otimes H^*_{\mathrm{rat}}(G,\mathbb{Q}) & \xrightarrow{\ \cup\ } & H^*_{\mathrm{rat}}(G,M) \end{array} \tag{2.18}$$

If we note that $H^*(\mathfrak{n}_{\mathbb{Q}}, M)^{\mathfrak{t}_{\mathbb{Q}}} = H^*(\mathfrak{n}_{\mathbb{Q}}, M)^{\mathfrak{g}_{\mathbb{Q}}}$ (since $H^*(\mathfrak{n}_{\mathbb{Q}}, M)^{\mathfrak{n}_{\mathbb{Q}}} = H^*(\mathfrak{n}_{\mathbb{Q}}, M)$), we may replace (2.10) with the following theorem:

(2.19) **Theorem** ([8, Theorem 5.3]). *Let G be an irreducible algebraic linear group over a field F of characteristic 0 and let A be a rational G-module. Then there is a natural isomorphism for each $n \geq 0$*

$$H^n(\mathfrak{g}_F, A) \simeq \sum_{i+j=n} H^i_{\mathrm{rat}}(G,A) \otimes_F H^j(\mathfrak{g}_F/\mathfrak{n}_F, F).$$

We now have the following.

(2.20) **Theorem.** *Under the hypotheses on G and D listed in* (1.2), *we have for each finite-dimensional rational G-module M a cohomology map* (2.14)

$$H^*(\mathfrak{g}_{\mathbb{Q}}, M) \xrightarrow{I_M} H^*(D,M).$$

The map l_M preserves cup products, i.e., the following diagram commutes:

$$\begin{array}{ccc} H^*(\mathfrak{g}_{\mathbb{Q}}, M)\otimes H^*(\mathfrak{g}_{\mathbb{Q}}, \mathbb{Q}) & \xrightarrow{\cup} & H^*(\mathfrak{g}_{\mathbb{Q}}, M) \\ \Big\downarrow{\scriptstyle l_M\otimes l_{\mathbb{Q}}} & & \Big\downarrow{\scriptstyle l_M} \\ H^*(D, M)\otimes H^*(D, \mathbb{Q}) & \xrightarrow{\cup} & H^*(D, M) \end{array}$$

where the pairing $M\times\mathbb{Q}\xrightarrow{\cup} M$ is scalar multiplication. In particular, $l_{\mathbb{Q}}$ is a ring map $H^*(\mathfrak{g}_{\mathbb{Q}}, \mathbb{Q})\to H^*(D, \mathbb{Q})$ for trivial $\mathbb{Q}$-coefficients.

Proof. The only detail that remains to be checked is the commutativity of the diagram. If $\beta\otimes\alpha\in H^i_{\mathrm{rat}}(G, \mathbb{Q})\otimes H^j(\mathfrak{t}_{\mathbb{Q}}, \mathbb{Q})$, $\upsilon\otimes u\in H^1_{\mathrm{rat}}(G, M)\otimes H^k(\mathfrak{t}_{\mathbb{Q}}, \mathbb{Q})$, we have (bearing in mind that all $\cup$-products must be understood relative to the appropriate pairing):

$$\begin{aligned} (\upsilon\otimes u)\otimes(\beta\otimes\alpha) &\xmapsto{l_M\otimes l_{\mathbb{Q}}} (r_M(\upsilon)\cup(\pi^*\circ I_{\mathbb{Q}})(u))\otimes(r_{\mathbb{Q}}(\beta)\cup(\pi^*\circ I_{\mathbb{Q}})(\alpha)) \\ &\xmapsto{\ \cup\ } [r_M(\upsilon)\cup(\pi^*\circ I_{\mathbb{Q}})(u)]\cup[r_{\mathbb{Q}}(\beta)\cup(\pi^*\circ I_{\mathbb{Q}})(\alpha)] \\ &= r_M(\upsilon)\cup[((\pi^*\circ I_{\mathbb{Q}})(u)\cup r_{\mathbb{Q}}(\beta))\cup(\pi^*\circ I_{\mathbb{Q}})(\alpha)] \\ &= (-1)^{ik} r_m(\upsilon)\cup[(r_{\mathbb{Q}}(\beta)\cup(\pi^*\circ I_{\mathbb{Q}})(u))\cup(\pi^*\circ I_{\mathbb{Q}})(\alpha)] \\ &= (-1)^{ik} r_m(\upsilon)\cup[r_{\mathbb{Q}}(\beta)\cup((\pi^*\circ I_{\mathbb{Q}})(u)\cup(\pi^*\circ I_{\mathbb{Q}})(\alpha))] \\ &= (-1)^{ik}[r_M(\upsilon)\cup r_{\mathbb{Q}}(\beta)]\cup[(\pi^*\circ I_{\mathbb{Q}})(u)\cup(\pi^*\circ I_{\mathbb{Q}})(\alpha)] \\ &= l_M((-1)^{ik}(\upsilon\cup\beta)\otimes(u\cup\alpha)) \\ &= l_M((\upsilon\otimes u)\cup(\beta\otimes\alpha)). \end{aligned}$$

This completes the proof. We note here that l_M depends upon the choice of the isomorphism $I_{\mathbb{Q}}$ (see 2.13).

Before proving anything about $l_{\mathbb{Q}}$, we require several preliminary lemmas for unipotent groups.

(2.21) **Lemma.** *If $G=N$ is unipotent and $D\subset N_{\mathbb{Q}}$ is cocompact in $N_{\mathbb{R}}$, the restriction map $r_{\mathbb{Q}}: H^*_{\mathrm{rat}}(N, \mathbb{Q})\to H^*(D, \mathbb{Q})$ is an isomorphism of rings.*

Proof. As in [13], we do an induction on the length of the refined upper central series for D. If $D=\mathbb{Z}$ and $N=\mathbb{A}^1_{\mathbb{Q}}$, then $C^1(\theta_N, \mathbb{Q})\simeq\mathbb{Q}[x]$ and $Z^1(\theta_N, \mathbb{Q})$ is a one-dimensional $\mathbb{Q}$-vector space generated by x. $r_{\mathbb{Q}}(x)\in Z^1(\mathbb{Z}, \mathbb{Q})$ is the cocycle $m\to m$, $m\in\mathbb{Z}$; $r_{\mathbb{Q}}(x)$ generates $H^1(\mathbb{Z}, \mathbb{Q})$. If D is generated by a minimum of k elements, we may then assume the theorem proved for D^{k-1}, where $\mathbb{Z}\overset{i}{\hookrightarrow} D\overset{\pi}{\twoheadrightarrow} D^{k-1}$ is the last central extension in the refined upper central series for D. Denoting the algebraic hull over $\mathbb{Q}$ of D^{k-1} by N^{k-1}, we have a commutative diagram

$$\begin{tikzcd}
H^*_{\text{rat}}(\mathbb{A}^1_{\mathbb{Q}},\mathbb{Q}) \arrow[d, "r^{\text{I}}_{\mathbb{Q}}", "\wr"'] & H^*_{\text{rat}}(N,\mathbb{Q}) \arrow[l, "\alpha^*"'] \arrow[d, "r^{\text{II}}_{\mathbb{Q}}"] & H^*_{\text{rat}}(N^{k-1},\mathbb{Q}) \arrow[l, "\beta^*"'] \arrow[d, "r^{\text{III}}_{\mathbb{Q}}", "\wr"'] \\
H^*(\mathbb{Z},\mathbb{Q}) & H^*(D,\mathbb{Q}) \arrow[l, "i^*"'] & H^*(D^{k-1},\mathbb{Q}) \arrow[l, "\pi^*"']
\end{tikzcd}$$

From [13], the rings $H^*(D,\mathbb{Q})$ and $H^*(\mathfrak{n}_{\mathbb{Q}},\mathbb{Q})$ are isomorphic and satisfy Poincaré duality. From Theorem 5.1 of [8], $H^*_{\text{rat}}(N,\mathbb{Q})$ is isomorphic to $H^*(\mathfrak{n}_{\mathbb{Q}},\mathbb{Q})$ and so satisfies Poincaré duality. Since $r^{\text{III}}_{\mathbb{Q}}$ is an isomorphism by the induction hypothesis, a generator Φ'_{rat} for $H^{k-1}_{\text{rat}}(N^{k-1},\mathbb{Q})\simeq\mathbb{Q}$ maps to a generator $\Phi'\in H^{k-1}(D^{k-1},\mathbb{Q})\simeq\mathbb{Q}$. Further, from [13, 4.3] there is an element $\Psi\in H^1(D,\mathbb{Q})$ such that $\Phi=\Psi\cup\pi^*(\Phi')$ generates $H^k(D,\mathbb{Q})$, and $i^*\Psi$ generates $H^1(\mathbb{Z},\mathbb{Q})$. Similar assertions are then true for

$$H^*(\mathfrak{a}_{\mathbb{Q}},\mathbb{Q})\leftarrow H^*(\mathfrak{n}_{\mathbb{Q}},\mathbb{Q})\leftarrow H^*(\mathfrak{n}^{k-1}_{\mathbb{Q}},\mathbb{Q}),$$

and so from (2.15) and (2.18) there is an element $\omega\in H^1_{\text{rat}}(N,\mathbb{Q})$ such that $\alpha^*(\omega)$ generates $H^1_{\text{rat}}(\mathbb{A}^1_{\mathbb{Q}},\mathbb{Q})$ and $\omega\cup\beta^*(\Phi'_{\text{rat}})$ generates $H^k_{\text{rat}}(N,\mathbb{Q})$. Since all the maps are multiplicative, we may assume then that $\Psi=r^{\text{II}}_{\mathbb{Q}}(\omega)$, so that

$$r^{\text{II}}_{\mathbb{Q}}(\omega)\cup r^{\text{II}}_{\mathbb{Q}}(\beta^*(\Phi'_{\text{rat}}))=r^{\text{II}}_{\mathbb{Q}}(\omega\cup\beta^*(\Phi'_{\text{rat}}))=\Phi\neq 0.$$

Thus, $r^{\text{II}}_{\mathbb{Q}}$ is an isomorphism on H^k. Since $r^{\text{II}}_{\mathbb{Q}}$ is a ring map and both $H^*_{\text{rat}}(N,\mathbb{Q})$ and $H^*(D,\mathbb{Q})$ satisfy Poincaré duality, $r^{\text{II}}_{\mathbb{Q}}$ must be an injection; since everything is finite-dimensional over $\mathbb{Q}$, $r^{\text{II}}_{\mathbb{Q}}$ must be an isomorphism.

(2.22) **Corollary.** *If $G\simeq N\rtimes T$ is algebraic solvable and D_N is cocompact in $N_{\mathbb{R}}$, $r_{\mathbb{Q}}:H^*_{\text{rat}}(G,\mathbb{Q})\to H^*(D,\mathbb{Q})$ is an injection.*

Proof. From [8, Theorem 5.2] we have $H^*_{\text{rat}}(G,\mathbb{Q})\simeq H^*_{\text{rat}}(N,\mathbb{Q})^T$, and we may consider the natural inclusion $H^*_{\text{rat}}(G,\mathbb{Q})\subset H^*_{\text{rat}}(N,\mathbb{Q})$ to be induced by $N\hookrightarrow G$. We then have a commutative diagram

$$\begin{tikzcd}
H^*_{\text{rat}}(G,\mathbb{Q})=H^*_{\text{rat}}(N,\mathbb{Q})^T \arrow[r, hook] \arrow[d, "r_{\mathbb{Q}}"'] & H^*_{\text{rat}}(N,\mathbb{Q}) \arrow[d, "\wr"', "r_{\mathbb{Q}}"] \\
H^*(D,\mathbb{Q}) \arrow[r, "i^*"] & H^*(D_N,\mathbb{Q})
\end{tikzcd}$$

Since the composite of the upper horizontal map with the restriction from N to D_N is an injection, restriction from G to D must be an injection.

We are now prepared to prove our main theorem.

(2.23) **Theorem.** *If G and D satisfy the hypotheses of* (1.2), *then the map* $l_{\mathbb{Q}}$ *is a ring isomorphism* $l_{\mathbb{Q}}: H^*(\mathfrak{g}_{\mathbb{Q}}, \mathbb{Q}) \xrightarrow{\sim} H^*(D, \mathbb{Q})$.

Proof. If we extend scalars to $\mathbb{R}$, we see that the filtration of $C^\bullet_{\text{alt}}(\mathfrak{g}_{\mathbb{R}}, \mathbb{R})$ that produces the Hochschild–Serre spectral sequence for $\mathfrak{n}_{\mathbb{R}} \to \mathfrak{g}_{\mathbb{R}} \twoheadrightarrow \mathfrak{t}_{\mathbb{R}}$ is compatible with the filtration of the DeRham complex $\Omega^\bullet_{\text{DR}}(G^0_{\mathbb{R}}/D)$ that produces the C^∞ Leray–Serre spectral sequence for the bundle $(N_{\mathbb{R}}/D_N) \to (G^0_{\mathbb{R}}/D) \twoheadrightarrow (T^0_{\mathbb{R}}/D_T)$ (see [7, p. 52]). Considering $C^\bullet_{\text{alt}}(\mathfrak{g}_{\mathbb{R}}, \mathbb{R})$ as a complex of right-invariant forms on $G^0_{\mathbb{R}}$, it descends to a complex of global forms on $G^0_{\mathbb{R}}/D$, and so the inclusion $C^\bullet_{\text{alt}}(\mathfrak{g}_{\mathbb{R}}, \mathbb{R}) \subset \Omega^\bullet_{\text{DR}}(G^0_{\mathbb{R}}/D)$ induces a map of spectral sequences. From Mostow's theorem, the map on abutments is an isomorphism, and at the E_2-level we have

$$E_2^{pq} = H^p(\mathfrak{t}_{\mathbb{R}}, H^q(\mathfrak{n}_{\mathbb{R}}, \mathbb{R})) \to H^p(D_T, H^q(D_N, \mathbb{R})) = \bar{E}_2^{pq}.$$

We may consider $\bar{E}_2^{pq}$ to be induced from the action of D_T on BD_N by conjugation of $(BD_N)_1$; from [13], then, the D_T-module $H^q(D_N, \mathbb{R})$ is isomorphic to the D_T-module $H^q(\mathfrak{n}_{\mathbb{R}}, \mathbb{R})$ for all $q \geq 0$. Thus, $\bar{E}_2^{pq} \simeq H^p(D_T, H^q(\mathfrak{n}_{\mathbb{R}}, \mathbb{R}))$. The D_T-module structure on each $H^q(\mathfrak{n}_{\mathbb{R}}, \mathbb{R})$ is induced from a $\theta_{T_{\mathbb{R}}}$-comodule structure; since T is $\mathbb{R}$-split, the action on $H^q(\mathfrak{n}_{\mathbb{R}}, \mathbb{R})$ diagonalizes over $\mathbb{R}$ [3, p. 204]. From the Appendix, then,

$$\operatorname{Dim}_{\mathbb{R}} H^p(D_T, H^q(\mathfrak{n}_{\mathbb{R}}, \mathbb{R})) = \operatorname{Dim}_{\mathbb{R}} H^p(\mathfrak{t}_{\mathbb{R}}, H^q(\mathfrak{n}_{\mathbb{R}}, \mathbb{R})), \quad \text{all } p, q \geq 0.$$

Thus, $\operatorname{Dim}_{\mathbb{R}} E_2^{pq} = \operatorname{Dim}_{\mathbb{R}} \bar{E}_2^{pq}$ for all $p, q \geq 0$. From [10, §7], $E_2 = E_3 = \cdots = E_\infty$. Since the map on abutments is an isomorphism and all E_2^{pq} are finite-dimensional, we must have each map $E_2^{pq} \to \bar{E}_2^{pq}$ an isomorphism and $\bar{E}_2 = \bar{E}_3 = \cdots = \bar{E}_\infty$ in the C^∞ Leray–Serre spectral sequence. Comparing the usual Leray–Serre spectral sequence (denoted by $\bar{\bar{E}}$) for singular $\mathbb{R}$-cohomology to the C^∞ Leray–Serre, we have

$$\operatorname{Dim}_{\mathbb{R}} \bar{\bar{E}}_2^p = \operatorname{Dim}_{\mathbb{R}} \bar{E}_2^p, \quad \text{all } p, q \geq 0.$$

All $\bar{\bar{E}}_2^{pq}$ are then finite-dimensional vector spaces, and we must have (since $\operatorname{Dim}_{\mathbb{R}} H^n(D, \mathbb{R}) = \sum_{p+q=n} \operatorname{Dim}_{\mathbb{R}} \bar{\bar{E}}_2^{pq}$ for all $n \geq 0$) $\bar{\bar{E}}_2 = \bar{\bar{E}}_3 = \cdots = \bar{\bar{E}}_\infty$ in the singular Leray–Serre spectral sequence. Thus we have a ring isomorphism

$$H^*(D, \mathbb{R}) \simeq H^*(D_N, \mathbb{R})^{D_T} \otimes_{\mathbb{R}} H^*(D_T, \mathbb{R}).$$

We now consider the inclusion of the singular $\mathbb{Q}$-cochains on $G^0_{\mathbb{R}}/D$ into the singular $\mathbb{R}$-cochains: $C^\bullet(G^0_{\mathbb{R}}/D, \mathbb{Q}) \subset C^\bullet(G^0_{\mathbb{R}}/D, \mathbb{R})$. The usual filtration then induces a map of spectral sequences (which on abutments is just extension of scalars):

$$\begin{tikzcd}
E_2^{pq} = H^p(D_T, H^q(D_N, \mathbb{Q})) \arrow[r] \arrow[d, Rightarrow] \arrow[dr, phantom, "(*)"] & H^p(D_T, H^q(D_N, \mathbb{R})) = \bar{E}_2^{pq} \arrow[d, Rightarrow] \\
H^*(D, \mathbb{Q}) \arrow[r, "-\otimes_{\mathbb{Q}} \mathbb{R}"'] & H^*(D, \mathbb{R})
\end{tikzcd}$$

We know that [7, pp. 106–107]:

$$\mathrm{Dim}_{\mathbb{Q}}\, H^q(D_N,\mathbb{Q}) = \mathrm{Dim}_{\mathbb{R}}\, H^q(D_N,\mathbb{R}), \quad \text{all } q\geq 0,$$

and that since a $\mathbb{Q}$-basis for the D_T-module $H^*(D_N,\mathbb{Q})$ gives an $\mathbb{R}$-basis for the D_T-module $H^*(D_N,\mathbb{R})\simeq H^*(D_N,\mathbb{Q})\otimes_{\mathbb{Q}}\mathbb{R}$, we have

$$H^*(D_N,\mathbb{R})^{D_T} = H^*(D_N,\mathbb{Q})^{D_T}\otimes_{\mathbb{Q}}\mathbb{R}.$$

Since the map on abutments in (*) is just extension of scalars (an injection) and $\mathrm{Dim}_{\mathbb{Q}}\, H^q(D,\mathbb{Q}) = \mathrm{Dim}_{\mathbb{R}}\, H^q(D,\mathbb{R})$ is finite for all $q\geq 0$, we must have $\mathrm{Dim}_{\mathbb{Q}}\, E_\infty^{pq} = \mathrm{Dim}_{\mathbb{R}}\, \bar{E}_\infty^{pq}$ for all $p,q\geq 0$ and $E_\infty^{pq}\to\bar{E}_\infty^{pq}$ must be an injection. Thus, all of $H^*(D_N,\mathbb{Q})^{D_T}$ must survive to E_∞, i.e.,

$$E_2^{0*} = E_\infty^{0*} = H^*(D_N,\mathbb{Q})^{D_T}.$$

Since the following square commutes,

$$\begin{tikzcd} H^*(D,\mathbb{Q}) \arrow[d] & H^*(D_T,\mathbb{Q}) \arrow[l, "\pi^*"'] \arrow[d] \\ H^*(D,\mathbb{R}) & H^*(D_T,\mathbb{R}) \arrow[l, "\pi^*"'] \end{tikzcd}$$

where the vertical injections are extension of scalars and the injectivity of the lower π^* was established earlier, the upper π^* must be an injection. Thus we have

$$E_2^{*0} = E_\infty^{*0} \simeq H^*(D_T,\mathbb{Q}).$$

$H^*(D_N,\mathbb{Q})^{D_T}$ and $H^*(D_T,\mathbb{Q})$ are then subalgebras of $\operatorname{gr} H^*(D,\mathbb{Q})$. Using the ring structure of E_∞ and $\bar{E}_\infty$ and the fact that a map of cohomology spectral sequences is multiplicative, we see that knowing

$$\operatorname{gr} H^*(D,\mathbb{R}) \simeq (H^*(D_N,\mathbb{Q})^{D_T}\otimes_{\mathbb{Q}} R)\otimes_{\mathbb{R}}(H^*(D_T,\mathbb{Q})\otimes_{\mathbb{Q}}\mathbb{R})$$

implies

$$\operatorname{gr} H^*(D,\mathbb{Q}) \simeq H^*(D_N,\mathbb{Q})^{D_T}\otimes_{\mathbb{Q}} H^*(D_T,\mathbb{Q}).$$

We already know that $H^*(D_T,\mathbb{Q})$ is a subalgebra of $H^*(D,\mathbb{Q})$; we would like to find an algebra map $H^*(D_N,\mathbb{Q})^{D_T}\to H^*(D_N,\mathbb{Q})$ such that composition with the natural pullback to $H^*(D_N,\mathbb{Q})$ is the identity.

From (2.22), $r_{\mathbb{Q}}: H^*_{\mathrm{rat}}(G,\mathbb{Q})\to H^*(D,\mathbb{Q})$ is an injection, and the composition

$$H^*_{\mathrm{rat}}(G,\mathbb{Q}) \xrightarrow{r_{\mathbb{Q}}} H^*(D,\mathbb{Q}) \xrightarrow{i^*} H^*(D_N,\mathbb{Q})$$

is an isomorphism of $H^*_{\mathrm{rat}}(G,\mathbb{Q})$ onto $H^*(D_N,\mathbb{Q})^{D_T}$:

$$H^*(D_N,\mathbb{Q})^{D_T} \simeq H^*(\mathfrak{n}_{\mathbb{Q}},\mathbb{Q})^{D_T} = H^*(\mathfrak{n}_{\mathbb{Q}},\mathbb{Q})^T \simeq H^*_{\mathrm{rat}}(G,\mathbb{Q}).$$

The first isomorphism follows from (2.16) and the equality follows from the Zariski density of D_T in $T_{\mathbb{R}}$ ([1, p. 78]). Thus $\mathrm{im}(r_{\mathbb{Q}})$ is a subalgebra of $H^*(D,\mathbb{Q})$ which maps isomorphically onto $H^*(D_N,\mathbb{Q})^{D_T}$ via i^*. From this we may deduce that $H^*(D,\mathbb{Q})$ is isomorphic to $H^*(D_N,\mathbb{Q})^{D_T}\otimes_{\mathbb{Q}} H^*(D_T,\mathbb{Q})$ and that $l_{\mathbb{Q}}$ is a ring isomorphism

$$H^*(\mathfrak{g}_{\mathbb{Q}},\mathbb{Q}) \xrightarrow[\sim]{l_{\mathbb{Q}}} H^*(D,\mathbb{Q}).$$

Remarks. (1) It seems feasible that the construction of $l_{\mathbb{Q}}$ could be generalized to irreducible solvable G whose torus factors T satisfy:

T is anisotropic over $\mathbb{Q}$; if, over $\mathbb{R}$, $T=T_{\mathrm{d}}\cdot T_{\mathrm{a}}$ with T_{a} $\mathbb{R}$-anisotropic and T_{d} $\mathbb{R}$-split, require that $D_T\cap(T_{\mathrm{a}})_{\mathbb{R}}=\{\mathrm{id}\}$ (with D again arithmetic and cocompact in $G^0_{\mathbb{R}}$).

$(T_{\mathrm{a}})_{\mathbb{R}}$ will be compact (a product of copies of S^1); since $D_T\cap(T_{\mathrm{a}})_{\mathbb{R}}=\{\mathrm{id}\}$, D_T will still be torsion-free and isomorphic to $\mathbb{Z}^k$, where $k=\dim T_{\mathrm{d}}$. From Theorem 12 of [10], we have the following surjections ($\mathfrak{t}'_{\mathbb{Q}}$ is the Lie algebre of T_{a})

$$H^*(\mathfrak{g}_{\mathbb{Q}},\mathbb{Q}) \twoheadrightarrow H^*(\mathfrak{t}_{\mathbb{Q}},\mathbb{Q}) \twoheadrightarrow H^*(\mathfrak{t}'_{\mathbb{Q}},\mathbb{Q}).$$

From the same theorem, the relative cohomology ring $H^*(\mathfrak{g}_{\mathbb{Q}},\mathfrak{t}'_{\mathbb{Q}};\mathbb{Q})$ is a subalgebra of $H^*(\mathfrak{g}_{\mathbb{Q}},\mathbb{Q})$, and we suspect that the following composite might be an isomorphism of rings

$$H^*(\mathfrak{g}_{\mathbb{Q}},\mathfrak{t}'_{\mathbb{Q}};\mathbb{Q}) \hookrightarrow H^*(\mathfrak{g}_{\mathbb{Q}},\mathbb{Q}) \xrightarrow{l_{\mathbb{Q}}} H^*(D,\mathbb{Q}).$$

(2) Since we may think of $G^0_{\mathbb{R}}/D$ as the total space of a smooth bundle over $T^0_{\mathbb{R}}/D_T$ with fibre $N_{\mathbb{R}}/D_N$, Theorem 2.23 provides a decomposition of $H^*(G^0_{\mathbb{R}}/D,\mathbb{Q})$ that resembles the Kunneth decomposition for a product space $H^*(F\times B,\mathbb{Q})\simeq H^*(F,\mathbb{Q})\otimes_{\mathbb{Q}} H^*(B,\mathbb{Q})$. The action of $\pi_1(T^0_{\mathbb{R}}/D_T)$ on $H^*(N_{\mathbb{R}}/D_N,\mathbb{Q})$ is far from trivial, however, but $G^0_{\mathbb{R}}/D$ resembles a product space to the extent that its $\mathbb{Q}$-cohomology is the product of the base cohomology with the *invariant* cohomology of the fibre.

3. An example

(Throughout this section we drop the $\mathbb{Q}$-subscript and write $\mathfrak{g}$ rather than $\mathfrak{g}_{\mathbb{Q}}$, etc.)

Let G be the following subgroup of GL_4:

$$\begin{pmatrix} x & 3y & u & 3v \\ y & x & v & u \\ 0 & 0 & z & 3w \\ 0 & 0 & w & z \end{pmatrix} \qquad \text{with} \begin{cases} x^2-3y^2=1, \\ z^2-3w^2=1. \end{cases}$$

D is the following discrete subgroup in $G_{\mathbb{R}}$:

$$\left[\begin{matrix} \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}^m & \begin{matrix} p & 3q \\ q & p \end{matrix} \\ \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}^n \end{matrix}\right], \qquad m, n, p, q \in \mathbb{Z}.$$

$G_{\mathbb{R}}^0$ is contractible, and D is cocompact in $G_{\mathbb{R}}^0$. (This example, suggested by E. Friedlander and S. Priddy, was constructed by taking the fundamental unit $2+\sqrt{3}$ in the number field $\mathbb{Q}(\sqrt{3})$ and considering the matrix representation of its operation by multiplication on $\mathbb{Z}\oplus\mathbb{Z}(\sqrt{3})$.) D is isomorphic to the semidirect product

$$(\mathbb{Z}\oplus\mathbb{Z}) \rtimes_{\varrho} (\mathbb{Z}\oplus\mathbb{Z}),$$

where

$$\varrho(m,n) = \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}^{m-n}.$$

$\mathfrak{g}$ has as a $\mathbb{Q}$-basis the following matrices:

$$X = \left[\begin{matrix} \begin{matrix} 0 & 3 \\ 1 & 0 \end{matrix} & 0 \\ 0 & 0 \end{matrix}\right], \qquad Y = \left[\begin{matrix} 0 & 0 \\ 0 & \begin{matrix} 0 & 3 \\ 1 & 0 \end{matrix} \end{matrix}\right],$$

$$Z = \left[\begin{matrix} 0 & \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} \\ 0 & 0 \end{matrix}\right], \qquad W = \left[\begin{matrix} 0 & \begin{matrix} 0 & 3 \\ 1 & 0 \end{matrix} \\ 0 & 0 \end{matrix}\right].$$

$\mathfrak{n}$ has $\{Z, W\}$ as a $\mathbb{Q}$-basis, and is abelian; if we denote their duals by $\{Z^*, W^*\}$, we have

$$(X\cdot Z^*)(W) = -3, \qquad (Y\cdot Z^*)(W) = 3,$$

$$(X\cdot W^*)(Z) = -1, \qquad (Y\cdot W^*)(Z) = 1;$$

thus $X\cdot Z^* = -3W^*$, $Y\cdot Z^* = 3W^*$, $X\cdot W^* = -Z^*$, $Y\cdot W^* = Z^*$. Since $\mathfrak{n}$ is abelian, $H^1(\mathfrak{n},\mathbb{Q}) \simeq \operatorname{Hom}_{\mathbb{Q}}(\mathfrak{n},\mathbb{Q})$; the latter has $\mathbb{Q}$-basis $\{Z^*, W^*\}$, and it is easily shown that

$$H^1(\mathfrak{n},\mathbb{Q})^{\mathfrak{t}} = 0.$$

$H^2(\mathfrak{n},\mathbb{Q}) \simeq \mathbb{Q}$ is spanned by $Z^*\wedge W^*$; a simple calculation shows that

$$X\cdot(Z^*\wedge W^*) = Y\cdot(Z^*\wedge W^*) = 0,$$

thus,

$$H^2(\mathfrak{n},\mathbb{Q})^{\mathfrak{t}} \simeq \mathbb{Q}.$$

So we have (in the only possible non-trivial dimensions):

$$H^0(\mathfrak{n},\mathbb{Q})^{\mathfrak{t}} = H^0_{\mathrm{rat}}(G,\mathbb{Q}) \simeq \mathbb{Q},$$

$$H^1(\mathfrak{n},\mathbb{Q})^{\mathfrak{t}} = H^1_{\mathrm{rat}}(G,\mathbb{Q}) = 0,$$

$$H^2(\mathfrak{n},\mathbb{Q})^{\mathfrak{t}} = H^2_{\mathrm{rat}}(G,\mathbb{Q}) \simeq \mathbb{Q}.$$

Since $\mathfrak{t}$ is two-dimensional abelian, $H^0(\mathfrak{t},\mathbb{Q})\simeq\mathbb{Q}$, $H^1(\mathfrak{t},\mathbb{Q})\simeq\mathbb{Q}\oplus\mathbb{Q}$ and is spanned by $\{X^*, Y^*\}$, $H^2(\mathfrak{t},\mathbb{Q})\simeq\mathbb{Q}$ and is spanned by $X^*\wedge Y^*$. From (2.23) we then have

$$\begin{aligned}
&H^0(\mathfrak{g},\mathbb{Q})\simeq\mathbb{Q}, && \\
&H^1(\mathfrak{g},\mathbb{Q})\simeq\mathbb{Q}\oplus\mathbb{Q}, && \text{spanned by } \{X^*, Y^*\},\\
&H^2(\mathfrak{g},\mathbb{Q})\simeq\mathbb{Q}\oplus\mathbb{Q}, && \text{spanned by } \{X^*\wedge Y^*, Z^*\wedge W^*\},\\
&H^3(\mathfrak{g},\mathbb{Q})\simeq\mathbb{Q}\oplus\mathbb{Q}, && \text{spanned by } \{Z^*\wedge W^*\wedge X^*, Z^*\wedge W^*\wedge Y^*\},\\
&H^4(\mathfrak{g},\mathbb{Q})\simeq\mathbb{Q}, && \text{spanned by } \{X^*\wedge Y^*\wedge Z^*\wedge W^*\}.
\end{aligned}$$

We now consider $H^*(D,\mathbb{Q})$, using the Hochschild–Serre spectral sequence for

$$\begin{tikzcd}
\mathbb{Z}\oplus\mathbb{Z} \arrow[r, hook] \arrow[d, equal] & D \arrow[r, two heads] & \mathbb{Z}\oplus\mathbb{Z} \arrow[d, equal] \\
D_N & & D_T
\end{tikzcd}$$

We have $E_2^{\bullet 0}=H^\bullet(D_T,\mathbb{Q})=H^\bullet(\mathbb{Z}\oplus\mathbb{Z},\mathbb{Q})$, so $E_2^{00}=\mathbb{Q}$, $E_2^{10}=\mathbb{Q}\oplus\mathbb{Q}$, $E_2^{20}=\mathbb{Q}$. Considering $D_N=\mathbb{Z}\oplus\mathbb{Z}$, we know again that $H^2(D_N,\mathbb{Q})\simeq\mathbb{Q}$, and we may take as a generator for the cocycle $I\in C^2(D_N,\mathbb{Q})$, where $I((p,q),(p',q'))=pq'$. It can be shown directly that D_T operates trivially on the cohomology class of I, so we have

$$E_2^{02}=\mathbb{Q},\qquad E_2^{12}=\mathbb{Q}\oplus\mathbb{Q},\qquad E_2^{22}=\mathbb{Q}.$$

In $C^1(D_N,\mathbb{Q})$ we have the following cocycles Φ_1 and Φ_2:

$$\Phi_1(p,q)=p,\qquad \Phi_2(p,q)=q.$$

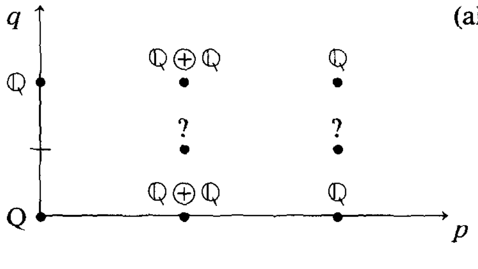
D_T acts on the Φ_i as follows: if $(s,t)\in D_T$,

$$\varrho(s,t)(\Phi_i)(p,q)=\Phi_i\left(\left(\begin{smallmatrix}2&3\\1&2\end{smallmatrix}\right)^{(s-t)}\left(\begin{smallmatrix}p\\q\end{smallmatrix}\right)\right).$$

$\{\Phi_1,\Phi_2\}$ is a basis for $H^1(D_n\mathbb{Q})$; since $\left(\begin{smallmatrix}2&3\\1&2\end{smallmatrix}\right)$ has eigenvalues $2\pm\sqrt{3}$ no nonzero $\alpha\Phi_1+\beta\Phi_2$, α and β in $\mathbb{Q}$, is fixed by $(1,0)\in D_T$. This implies

$$H^0(D_T,H^1(D_N,\mathbb{Q}))=E_2^{01}=0.$$

Thus E_2 looks like



(all other terms are 0)

Using the arguments of Section 2, the E_∞-term for $\mathbb{R}$-coefficients is

$$\begin{array}{c|ccc}
q & & & \\
\mathbb{R} & \mathbb{R}\oplus\mathbb{R} & \mathbb{R} & \\
 & & & \\
\mathbb{R} & \mathbb{R}\oplus\mathbb{R} & \mathbb{R} & \\
\hline
 & & & p
\end{array}$$

Thus the E_∞-term for $\mathbb{Q}$-coefficients must be

$$\begin{array}{c|ccc}
q & & & \\
\mathbb{Q} & \mathbb{Q}\oplus\mathbb{Q} & \mathbb{Q} & \\
 & & & \\
\mathbb{Q} & \mathbb{Q}\oplus\mathbb{Q} & \mathbb{Q} & \\
\hline
 & & & p
\end{array}$$

It is easy to see then that we must have $E_2^{11}=E_2^{21}=0$ in the $\mathbb{Q}$-coefficient spectral sequence. Thus, as a $\mathbb{Q}$-algebra, $H^*(D,\mathbb{Q})$ is generated by two elements in dimension 1 (pullbacks of a basis for $H^1(D_T,\mathbb{Q})$) and one element in dimension 2 (the restriction of a generator for $H^2_{\mathrm{rat}}(G,\mathbb{Q})$).

Appendix

We prove the following theorem.

Theorem. *If $T\subset \mathrm{GL}_m(\mathbb{C})$ is an irreducible $\mathbb{Q}$-torus of dimension n that is $\mathbb{Q}$-anisotropic and $\mathbb{R}$-split, and if $D\simeq\mathbb{Z}^n$ is an arithmetic subgroup (contained and cocompact in $T^0_{\mathbb{R}}$) and $M\simeq\mathbb{R}^c$ is a finite-dimensional $\theta_{T_{\mathbb{R}}}$-comodule, then for all $i\geq 0$*

$$H^i(\mathfrak{t}_{\mathbb{R}},M)\simeq H^i(D,M).$$

Proof. (We consider T as an algebraic group over $\mathbb{R}$.) M is a semisimple T-module, and so has a decomposition $M=M^T\oplus W$, where $W^T=0$. T irreducible implies that $M^T=M^{\mathfrak{t}_{\mathbb{R}}}$, and so $M=M^T\oplus W=M^{\mathfrak{t}_{\mathbb{R}}}\oplus W$, with $W^T=W^{\mathfrak{t}_{\mathbb{R}}}=0$. We know, since M is $\mathfrak{t}_{\mathbb{R}}$-semisimple,

$$H^i(\mathfrak{t}_{\mathbb{R}},M)=H^i(\mathfrak{t}_{\mathbb{R}},M^{\mathfrak{t}_{\mathbb{R}}}),\quad i\geq 0,$$

where $H^i(\mathfrak{t}_{\mathbb{R}},M^{\mathfrak{t}_{\mathbb{R}}})$ is a sum of copies of $H^i(\mathfrak{t}_{\mathbb{R}},\mathbb{R})$ ($\mathbb{R}$ has the trivial $\mathfrak{t}_{\mathbb{R}}$-module structure). As a D-module, $M\simeq M^T\oplus W$; from [1], D is Zariski-dense in $T_{\mathbb{R}}$, and so $W^D = 0$ and the D-action on M^T is trivial. Since $\mathfrak{t}_{\mathbb{R}}$ is an n-dimensional abelian Lie algebra, and $D = \mathbb{Z}^n$, from [12] we know

$$H^i(\mathfrak{t}_{\mathbb{R}}, M^{\mathfrak{t}_{\mathbb{R}}}) \simeq H^i(D, M^T), \quad i \geq 0.$$

We will now show that

$$H^i(D, W) = 0, \quad i \geq 0;$$

the theorem will then follow by additivity. Since M is a rational T-module and T is $\mathbb{R}$-split, we can find a basis $\{e_i\}_1^c$ for M such that the action of T is diagonalized with respect this basis [3, p. 204]. We may think of the points in $T_{\mathbb{R}}^0$ as $\mathbb{R}^n$ and D as the integer lattice $\mathbb{Z}^n$; since the action on M diagonalizes we have

$$(0, \ldots, 0, x_i, 0, \ldots, 0) \mapsto \begin{pmatrix} \alpha_{i1}^{x_i} & & 0 \\ & \ddots & \\ 0 & & \alpha_{ic}^{x_i} \end{pmatrix}, \quad i = 1, \ldots, n; \text{ all } \alpha_{ij} > 0.$$

So if $(x_1, \ldots, x_n) \in T_{\mathbb{R}}^0$, we have

$$(x_1, \ldots, x_n) \mapsto \begin{pmatrix} \beta_1^{(\sum_{i=1}^n \gamma_{1i} x_i)} & & 0 \\ & \ddots & \\ 0 & & \beta_c^{(\sum_{i=1}^n \gamma_{ci} x_i)} \end{pmatrix}, \quad \text{all } \beta_j > 0.$$

If $e_j \in M^T$ we must have $\beta_j^{(\sum_{i=1}^n \gamma_{ji} x_i)} \equiv 1$, i.e., either $\beta_j = 1$ or $\gamma_{j1} = \cdots = \gamma_{jn} = 0$. M^T is then spanned by the e_j satisfying $\beta_j^{(\sum_{i=1}^n \gamma_{ji} x_i)} = 1$. Splitting M^T off, we have left a basis for W. Since $W^T = 0$, if $e_k \in W$, then we have

(1) $\beta_k \neq 1$, and

(2) at least one of the $\gamma_{k1}, \ldots, \gamma_{kn}$ is not 0.

W is then a direct sum of one-dimensional representations $(x_1, \ldots, x_n) \mapsto \beta_j^{(\sum_{i=1}^n \gamma_{ji} x_i)}$ with $\beta_j \neq 1$ and some $\gamma_{ji} \neq 0$. If, e.g., $\gamma_{ji} \neq 0$, then we consider the ith factor of $\mathbb{Z}$ in $D \simeq \mathbb{Z}^n$; this is a normal subgroup of D. It is easy to show by direct calculation that if $\mathbb{Z}$ acts on $\mathbb{R}$ via $p \to \beta^p$ with $\beta > 0$ and $\beta \neq 1$, then $H^i(\mathbb{Z}, \mathbb{R}) = 0$ for all $i \geq 0$ (we can calculate $H^0 = H^1 = 0$ by hand; $H^i = 0$ for $i \geq 2$ follows since we may compute using the DeRham complex $\Omega_{S^1}^{\bullet}$ on S^1 with a twisted differential and $\Omega_{S^1}^i = 0$ for $i \geq 2$). From the Hochschild–Serre spectral sequence, then, all cohomology groups of D with coefficients in any of these one-dimensional representations vanish. By induction on $\mathrm{Dim}_{\mathbb{R}} W$ we then have $H^i(D, W) = 0$, $i \geq 0$. The theorem now follows.

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