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Locally best rotation-invariant rank tests for modal location

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Abstract

For a general class of unipolar, rotationally symmetric distributions on the multi-dimensional unit spherical surface, a characterization of locally best rotation-invariant test statistics is exploited in the construction of locally best rotation-invariant rank tests for modal location. Allied statistical distributional problems are appraised, and in the light of these assessments, asymptotic relative efficiency of a class of rotation-invariant rank tests (with respect to some of their parametric counterparts) is studied. Finite sample permutational distributional perspectives are also appraised.

AMS 2000 subject classification: 62H11; 62H15

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1. Introduction

For conventional univariate as well as multivariate distributions, locally most powerful (rank) tests for location have been extensively studied in the literature (Hájek et al. [5]). The situation with directional data models is far more complex and much less extensively studied, and yet such models crop up in many (often, interdisciplinary) fields of application and cater for appropriate statistical resolutions. While there is considerable interest on uniformity of distributions on spherical surfaces lacking any modal location, in real applications, one encounters more complex models where such

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modal directions are appropriate. The classical von Mises–Fisher–Langevin (vMFL) distribution and its generalizations (see Mardia and Jupp [9]) are particularly noteworthy in this respect. Yet real directional data may not generally pertain to such specific models. For example, contamination in spherical data, appraised by Ko and Guttorp [6], raises concern regarding their impact on statistical inference for spherical data models with special attention to robustness and efficiency. Parallel to conventional location-scale models, for directional data models, robust and nonparametric statistical inference procedures are needed. In this direction, reference may be made to Fisher and Hall [3] for bootstrap procedures, and Neeman and Chang [10] for rank tests for modal direction.

We confine ourselves to a general class of unipolar densities on a spherical surface with modal location $\boldsymbol{\theta}$, in a form of invariance under rotation of coordinates. For such rotation-invariant modal location models (including a majority of distributions considered for spherical data models, see Mardia and Jupp [9] and Watson [11]), we consider the hypothesis testing problem

$$\mathrm{H}_0 : \boldsymbol{\theta} = \boldsymbol{\theta}_0 \quad \text{(specified) against} \quad \mathrm{H}_1 : \boldsymbol{\theta} \neq \boldsymbol{\theta}_0, \tag{1.1}$$

As anticipated, for this hypothesis testing problem there is no uniformly most powerful rotation-invariant test (UMPRIT). Though a locally best rotation-invariant test (LBRIT) can be obtained, neither it is simple in form nor it is robust. For this reason, Neeman and Chang [10] considered some rank tests for this hypothesis testing problem, although not much is known about their optimality properties, if any. To bridge this gap, we intend to focus on locally best rotation-invariant rank tests (LBRIRT).

Section 2 is devoted to the preliminary notion. Parametric LBRIT are formulated in Section 3. For convenience of comprehension, three steps are incorporated: Step 1 explores the role of rotation-invariance (RI) and maximal invariants (MIs) in the formulation of suitable likelihood functions. In Step 2, a monotone likelihood ratio (LR) property of the MI is exploited to obtain suitable RI tests; these test statistics involve the parameter in the alternative hypothesis, and hence, cannot be the best RI test uniformly over $\boldsymbol{\theta} \neq \boldsymbol{\theta}_0$. Hence, in Step 3, locally best RI tests have been appraised. Locally best RI-rank tests (LBRIRT) are considered in Section 4. The LBRIRTs have been exploited towards the formulation of a general class of RI-rank tests in Section 5; these include some RI tests already cited in the literature. For the vMFL pdf, the empirical local powers of the LR-type test, the LBRIT and the LBRIRT are studied in Section 6. The Pitman asymptotic relative efficiency (ARE) results of spherical Wilcoxon rank test with respect to the optimal test are presented in Section 7. The last section is devoted to an illustrative example. Some of the derivations are relegated to the Appendix for smooth reading of the main results.

2. Preliminary notion

Let $S^{p-1} = \{\mathbf{x} \in [-1, 1]^p : \|\mathbf{x}\|^2 = \mathbf{x}^t\mathbf{x} = 1\}$ be the spherical surface of the unit p-sphere, $p \geqslant 2$. Let $\mathbf{X}_1, \ldots, \mathbf{X}_n$ be n independent and identically distributed (i.i.d.) random vectors (r.v.) with a probability density function (pdf) (with respect to the surface measure on S^{p-1}) $f(\mathbf{x}, \boldsymbol{\theta})$, $\mathbf{x} \in S^{p-1}$, $\boldsymbol{\theta} \in S^{p-1}$. For simplicity, we first assume that $f(\cdot)$ is unipolar with modal location $\boldsymbol{\theta}$. We also assume that $f(\mathbf{x}, \boldsymbol{\theta})$ belongs to a suitable exponential family of densities on S^{p-1}:

$$f(\mathbf{x}, \boldsymbol{\theta}) = \exp\{g(\langle \mathbf{x}, \boldsymbol{\theta} \rangle) + h(\mathbf{x}^t\mathbf{A}\mathbf{x})\}, \quad \mathbf{x} \in S^{p-1}, \ \ \boldsymbol{\theta} \in S^{p-1}, \tag{2.1}$$

where $g(\langle \mathbf{x}, \boldsymbol{\theta} \rangle)$ depends on $\mathbf{x}$ and $\boldsymbol{\theta}$ through their inner product $\langle \mathbf{x}, \boldsymbol{\theta} \rangle$, $h(\mathbf{x}^t\mathbf{A}\mathbf{x})$ on $\mathbf{x}^t\mathbf{A}\mathbf{x}$, $\mathbf{A}$ being a symmetric $p \times p$ matrix; as $\langle \mathbf{x}, \mathbf{x} \rangle = 1$, without loss of generality (WLOG), we take trace$(\mathbf{A}) = 0$

(see Mardia and Jupp [9, p. 175] and Watson [11, p. 80] for general motivation). This form of $f(\mathbf{x}, \boldsymbol{\theta})$ is invariant under rotation of coordinates, so that it would be natural to seek rotation-invariant statistical inference procedures.

To have unimodality along with RI, we assume that $g(y)$ is monotone nondecreasing (and not a constant) on $[-1, 1]$. For a bipolar density, $g(-y) = g(y), \forall y \in (-1, 1)$ so that $g(\cdot)$ cannot be monotone. Further, we assume that $g(y)$ admits continuous first and second derivatives, $g'(y)$ and $g''(y)$ a.e. on $(-1, 1)$. In addition, we assume that $g(\cdot)$ is skew-symmetric (about $y = 0$), so that $g'(-y) = g'(y), \forall y \in (-1, 1)$ and $g''(-y) = -g''(y)$; hence $g(-y) + g(y) = 2g(0)$, so that absorbing $g(0)$ in $h(\cdot)$, we could have $g(-y) = -g(y), \forall y \in [-1, 1]$. Though this skew-symmetric condition is not necessary, it holds in a majority of cases and renders simpler resolutions. In passing, we may remark that several important pdf's are special cases of (2.1). For example, the vMFL pdf corresponds to $\mathbf{A} = \mathbf{O}$ and $g(\langle \mathbf{x}, \boldsymbol{\theta} \rangle) = \kappa \langle \mathbf{x}, \boldsymbol{\theta} \rangle - c(\kappa), \kappa \geqslant 0$ and $c(\kappa)$ depends on κ alone. The Fisher–Bingham pdf corresponds to $h(u) = u, g(\langle \mathbf{x}, \boldsymbol{\theta} \rangle) = \kappa \langle \mathbf{x}, \boldsymbol{\theta} \rangle + c(\kappa, \mathbf{A})$. The Kent distribution, Fisher–Watson distribution and Bingham–Mardia distributions (viz., Mardia and Jupp [9, pp. 174–177]) all belong to this exponential family. For rotation-invariant rank tests, when not seeking local optimality properties, some of these regularity conditions may not be necessary (see Sections 3 and 5).

Let $\mathbf{Q}$ be an orthogonal matrix of order p (so that $\mathbf{Q}^t\mathbf{Q} = \mathbf{I}_p$), and let $\mathcal{O}(p)$ be the group of orthogonal transformations of $\mathbb{R}^p$ onto itself. Thus, for $\mathbf{x} \in S^{p-1}, \boldsymbol{\theta} \in S^{p-1}, \mathbf{Q} \in \mathcal{O}(p)$,

$$\mathbf{x}^t\boldsymbol{\theta} = \mathbf{x}^t\mathbf{Q}^t\mathbf{Q}\boldsymbol{\theta} = \mathbf{x}^{*t}\boldsymbol{\theta}^* \text{ where } \mathbf{x}^* \in S^{p-1}, \quad \boldsymbol{\theta}^* \in S^{p-1} \;\; \forall \mathbf{Q} \in \mathcal{O}(p), \tag{2.2}$$

exhibiting the RI of $\langle \mathbf{x}, \boldsymbol{\theta} \rangle$ on $S^{p-1} \times S^{p-1}$. Therefore, $\mathcal{O}(p)$ acts transitively on the parameter space S^{p-1} as well, that is, for every pair of elements $\boldsymbol{\theta}_1$ and $\boldsymbol{\theta}_2$ belonging to the parameter space S^{p-1}, there exists an $\mathbf{Q} \in \mathcal{O}(p)$ such that $\mathbf{Q}\boldsymbol{\theta}_1 = \boldsymbol{\theta}_2$, so that if for a $\boldsymbol{\theta} \in S^{p-1}$, we define the isotropy subgroup $\mathcal{O}_{\boldsymbol{\theta}}(p) (\subset \mathcal{O}(p))$ as

$$\mathcal{O}_{\boldsymbol{\theta}}(p) = \{\mathbf{Q} \in \mathcal{O}(p) : \mathbf{Q}\boldsymbol{\theta} = \boldsymbol{\theta}\}, \tag{2.3}$$

then $\mathcal{O}_{\boldsymbol{\theta}}(p)$ represents itself on the tangent space $T_{\boldsymbol{\theta}}S^{p-1}$. Therefore, $g(\mathbf{x}, \boldsymbol{\theta})$ [in (2.1)] is rotationally symmetric about its modal location $\boldsymbol{\theta}$. Further, $\mathbf{x}^t\mathbf{A}\mathbf{x}$ [in $h(\cdot)$ in (2.1)] is free from $\boldsymbol{\theta}$ and this form of exponential family is invariant under rotation of coordinates (viz., Watson [11, p. 80]). Clearly $\mathcal{O}(p)$ is a Lie group. Let $\mathbf{A}^* = \mathbf{Q}\mathbf{A}\mathbf{Q}^t$, by the group acting on the sample space and parameter space $\mathbf{Q} \cdot (\mathbf{x}, \mathbf{A}) = (\mathbf{Q}\mathbf{x}, \mathbf{Q}\mathbf{A}\mathbf{Q}^t) = (\mathbf{x}^{*t}, \mathbf{A}^*)$, then $h(\mathbf{x}^{*t}\mathbf{A}^*\mathbf{x}^*) = h(\mathbf{x}^t\mathbf{Q}^t(\mathbf{Q}\mathbf{A}\mathbf{Q}^t)\mathbf{Q}\mathbf{x}) = h(\mathbf{x}^t\mathbf{A}\mathbf{x})$, and $\text{trace}(\mathbf{A}^*) = \text{trace}(\mathbf{A}) = 0, \forall \mathbf{Q} \in \mathcal{O}(p)$. Thus, $h(\mathbf{x}^t\mathbf{A}\mathbf{x})$ remains invariant under rotation. As a result, $f(\mathbf{x}, \boldsymbol{\theta})$ in (2.1) is rotationally symmetric about its modal location $\boldsymbol{\theta}$. We therefore require our testing problem to be invariant with respect to the group $\mathcal{O}(p)$ and the isotropy subgroup $\mathcal{O}_{\boldsymbol{\theta}_0}(p)$. Note that spherically uniform pdf (on S^{p-1}) are not unimodal, and hence, we exclude them from our discussion.

3. The LBRIT

From our discussions relating to the orthogonal group $\mathcal{O}(p)$ in (2.2) and the hypothesis testing problem in (1.1), we confine ourselves to RI tests only. As such, WLOG, we take

$$\boldsymbol{\theta}_0 = (\theta_1, \boldsymbol{\theta}_{(2)}^t)^t = (1, \mathbf{0}^t)^t, \tag{3.1}$$

and set alternatives as $\boldsymbol{\theta}^t = (\theta_1, \boldsymbol{\theta}^t_{(2)})$ with $\theta_1 \neq 1$ and $\boldsymbol{\theta}_{(2)} \neq \mathbf{0}$. This reduction (under RI) allows an easy formulation of local alternatives. We set $\mathbf{X}_i = (X_{i1}, \mathbf{X}_i^{(2)t})^t$ as

$$X_{i1} = \mathbf{X}_i^t \boldsymbol{\theta}_0, \quad Y_i = \|\mathbf{X}_i - (\mathbf{X}_i^t \boldsymbol{\theta}_0)\boldsymbol{\theta}_0\| = \|\mathbf{X}_i^{(2)}\| = \left(\sum_{j=2}^{p} X_{ij}^2 \right)^{1/2}, \tag{3.2}$$

for $i = 1, \ldots, n$. Thus, if we let

$$\mathbf{U}_i^{(2)} = Y_i^{-1} \mathbf{X}_i^{(2)} \quad \text{for } i = 1, \ldots, n, \tag{3.3}$$

then $\|\mathbf{U}_i^{(2)}\| = 1, \forall i$. As mentioned above that $h(\mathbf{x}^t \mathbf{A} \mathbf{x})$ remains invariant under rotation and is free from $\boldsymbol{\theta}$. And hence, the fact that $f(\mathbf{x}, \boldsymbol{\theta})$ in (2.1) is rotationally symmetric about its modal location $\boldsymbol{\theta}$ implies that, under $\mathrm{H}_0 : \boldsymbol{\theta} = \boldsymbol{\theta}_0$, (i) $\mathbf{U}_i^{(2)}$ has the uniform density (with respect to the surface measure) on S^{p-2}, (ii) $\mathbf{U}_i^{(2)}$ and $\mathbf{X}_i^t \boldsymbol{\theta}_0$ are rotation-invariant, and (iii) $\mathbf{U}_i^{(2)}$ and X_{i1} are independent, so that $Y_i = (1 - X_{i1}^2)^{1/2}$ and $\mathbf{U}_i^{(2)}$ are independent too. On the spheres, these results are true for a general class of distributions which include the vMFL distributions as the special ones (Mardia and Jupp [9, p. 179]). These results are further extended to the two-point homogeneous spaces which include spheres as the special cases (see Theorem 3 of Chang and Tsai [1]). We have the tangent-normal decomposition (at $\boldsymbol{\theta}_0$):

$$\mathbf{X}_i = (\mathbf{X}_i^t \boldsymbol{\theta}_0)\boldsymbol{\theta}_0 + Y_i \mathbf{U}_i, \quad \mathbf{U}_i = (0, \mathbf{U}_i^{(2)t})^t, \ \ i = 1, \ldots, n, \tag{3.4}$$

where $(\mathbf{X}_i^t \boldsymbol{\theta}_0)\boldsymbol{\theta}_0$, $i = 1, \ldots, n$, capture all information in the likelihood

$$L_n(\boldsymbol{\theta}) = \prod_{i=1}^{n} f(\mathbf{X}_i, \boldsymbol{\theta}) \tag{3.5}$$

$(\boldsymbol{\theta}_0)$ while $(Y_i, \mathbf{U}_i^{(2)t})$, $i = 1, \ldots, n$, are ancillary statistics whose distribution (under H_0) does not depend on $\boldsymbol{\theta}_0$. However, when H_0 does not hold, the Y_i, $1 \leqslant i \leqslant n$ will have a distribution that would depend on $\boldsymbol{\theta}$, while the $\mathbf{U}_i^{(2)}$, $1 \leqslant i \leqslant n$, can still be made ancillary (up to a scalar factor). Thus, we work with the MI $\{(Y_i, \mathbf{U}_i^{(2)t}), 1 \leqslant i \leqslant n\}$, and invoking the invariance of the hypothesis testing (H_0 vs. H_1) problem under the group $\mathcal{O}(p)$, it seems natural to base our test statistic solely on these MI. We first present this theme in a group-theoretic framework to suit our purpose better.

Let $\mathcal{G}$ be a group that acts on a manifold $\mathcal{X}$. Then, for every $\mathbf{x} \in \mathcal{X}$, the orbit is defined as $\mathcal{G}\mathbf{x} = \{G\mathbf{x} : G \in \mathcal{G}\}$. $\mathcal{G}$ acts transitively on $\mathcal{X}$ if $\mathcal{X}$ contains only one orbit. In this case, for every $\mathbf{x}, \mathbf{y} \in \mathcal{X}$, there exists a $G \in \mathcal{G}$, such that $\mathbf{y} = G\mathbf{x}$. If a function $\phi(\mathbf{x})$ is $\mathcal{G}$-invariant, i.e., $\phi(G\mathbf{x}) = \phi(\mathbf{x}), \forall G \in \mathcal{G}, \mathbf{x} \in \mathcal{X}$, and it assumes different values on different orbits, it is called MI (Eaton [2]). Invariant hypothesis testing problems fully exploit MIs. In fact, if the probability density of the MI has monotone LR, then for one-sided alternatives, the best test can be solely based on the MI (Lehmann [8, Section 6.3]).

In our present context, for each i $(= 1, \ldots, n)$, $\mathcal{X} = S^{p-1}$, $\mathcal{G} = \{\mathbf{Q} : \mathbf{Q}^t\mathbf{Q} = \mathbf{I}_p\}$, and $\phi(\mathbf{x}_i, \boldsymbol{\theta}) = \exp\{g(\mathbf{x}_i^t \boldsymbol{\theta}) + h(\mathbf{x}_i^t \mathbf{A} \mathbf{x}_i)\}$, the pdf in (2.1) (with respect to the surface measure on S^{p-1}), where both $\mathbf{x}_i$ and $\boldsymbol{\theta} \in S^{p-1}$. By the decomposition in (3.4),

$$Y_i = \|\mathbf{X}_i^{(2)}\|^2 = 1 - X_{i1}^2 \Longrightarrow X_{i1} = \pm\sqrt{1 - Y_i{}^2}, \quad 1 \leqslant i \leqslant n, \tag{3.6}$$

with X_{i1} taking on the two values $\pm\sqrt{1-Y_i{}^2}$, given Y_i with equal probability. Note that $f(\mathbf{x}_i, \boldsymbol{\theta}) = \exp\{g(\mathbf{x}_i^t\boldsymbol{\theta}) + h(\mathbf{x}_i^t\mathbf{A}\mathbf{x}_i)\}$, where $h(\cdot)$ is free from $\boldsymbol{\theta}$. Hence, in the likelihood function $L_n(\boldsymbol{\theta})$, in (3.5), viewed as a function of $\boldsymbol{\theta} \in S^{p-1}$ (given the $\mathbf{x}_i$, $1 \leqslant i \leqslant n$), $h(\cdot)$ does not contribute to any variation (over $\boldsymbol{\theta}$). Therefore, that will drop out in a version of the LR. Thus, for simplicity of presentation and notation, in (2.1), we drop $h(\cdot)$, and take $f(\mathbf{x}, \boldsymbol{\theta}) = \exp\{g(\mathbf{x}^t\boldsymbol{\theta})\}$, $\mathbf{x}, \boldsymbol{\theta} \in S^{p-1}$. By reference to (3.1), we set $\boldsymbol{\theta}_0^t = (1, \mathbf{0}^t)$ and $\Delta = 2(1-\boldsymbol{\theta}^t\boldsymbol{\theta}_0)$, so that $0 \leqslant \Delta \leqslant 4$, for all $\boldsymbol{\theta} \in S^{p-1}$. Under $\mathrm{H}_0 : \boldsymbol{\theta} = \boldsymbol{\theta}_0$, $\Delta = 0$, and $\mathrm{H}_{1\Delta} : \boldsymbol{\theta} \neq \boldsymbol{\theta}_0$, $\Delta > 0$. Let us write

$$\mathrm{H}_{1\Delta} : \boldsymbol{\theta}^t\boldsymbol{\theta}_0 = 1 - \tfrac{1}{2}\Delta, \quad 0 \leqslant \Delta \leqslant 4, \qquad \mathrm{H}_1 = \cup_{\{0 \leqslant \Delta \leqslant 4\}} \mathrm{H}_{1\Delta}. \tag{3.7}$$

Note that as $\boldsymbol{\theta} \in S^{p-1}$, we have $\theta_1 = 1 - \frac{1}{2}\Delta$ and $\|\boldsymbol{\theta}_{(2)}\|^2 = \Delta(1 - \frac{1}{4}\Delta)$. We take $\boldsymbol{\theta}_{(2)} = \sqrt{\Delta(1-\frac{1}{4}\Delta)}\,\boldsymbol{\xi}$, where $\boldsymbol{\xi} \in S^{p-2}$. Also note that $U_i^{(2)}$ has a spherical uniform density on S^{p-2}. Thus, the joint pdf of $(Y_i, \mathbf{U}_i^{(2)t})$ is given by

$$\begin{aligned} f^*(y, \mathbf{u}) = \tfrac{1}{2}\{&e^{g((1-\frac{1}{2}\Delta)\sqrt{1-y^2}+\sqrt{\Delta(1-\frac{1}{4}\Delta)}\; y\mathbf{u}^{(2)t}\boldsymbol{\xi})} \\ &+ e^{g(-(1-\frac{1}{2}\Delta)\sqrt{1-y^2}+\sqrt{\Delta(1-\frac{1}{4}\Delta)}\; y\mathbf{u}^{(2)t}\boldsymbol{\xi})}\}. \end{aligned} \tag{3.8}$$

Therefore, if we let $E^* = (Y_1, \mathbf{U}_1^{(2)t}; \ldots; Y_n, \mathbf{U}_n^{(2)t})$, then the joint pdf of E^* is given by

$$\begin{aligned} p(E^*, \boldsymbol{\theta}) = 2^{-n} \prod_{i=1}^n \{&e^{g((1-\frac{1}{2}\Delta)\sqrt{1-Y_i^2}+\sqrt{\Delta(1-\frac{1}{4}\Delta)}\; Y_i\mathbf{U}_i^{(2)t}\boldsymbol{\xi})} \\ &+ e^{g(-(1-\frac{1}{2}\Delta)\sqrt{1-Y_i^2}+\sqrt{\Delta(1-\frac{1}{4}\Delta)}\; Y_i\mathbf{U}_i^{(2)t}\boldsymbol{\xi})}\}. \end{aligned} \tag{3.9}$$

In order to exploit fully the $\mathcal{G}$-invariance of both H_0 and H_1, we appeal to the invariance measure $\mu(d\boldsymbol{\xi})$ on S^{p-2} (generated by the spherical uniform pdf of $\boldsymbol{\xi}$ on S^{p-2}), and construct the two integrated likelihood statistics:

$$\int_{S^{p-2}} p(E^*, \boldsymbol{\theta}_0)\mu(d\boldsymbol{\xi}) = p(E^*, \boldsymbol{\theta}_0) = 2^{-n} \prod_{i=1}^n \{e^{g(\sqrt{1-Y_i^2})} + e^{g(-\sqrt{1-Y_i^2})}\} \tag{3.10}$$

and

$$\begin{aligned} \int_{S^{p-2}} p(E^*, \boldsymbol{\theta})\mu(d\boldsymbol{\xi}) = 2^{-n} \int_{S^{p-2}} \prod_{i=1}^n \{&e^{g((1-\frac{1}{2}\Delta)\sqrt{1-Y_i^2}+\sqrt{\Delta(1-\frac{1}{4}\Delta)}\; Y_i\mathbf{U}_i^{(2)t}\boldsymbol{\xi})} \\ &+ e^{g(-(1-\frac{1}{2}\Delta)\sqrt{1-Y_i^2}+\sqrt{\Delta(1-\frac{1}{4}\Delta)}\; Y_i\mathbf{U}_i^{(2)t}\boldsymbol{\xi})}\}\mu(d\boldsymbol{\xi}) \end{aligned} \tag{3.11}$$

and consider a test statistic T_n:

$$\begin{aligned} &= \frac{\int_{S^{p-2}} \prod_{i=1}^n \{e^{g((1-\frac{1}{2}\Delta)\sqrt{1-Y_i^2}+\sqrt{\Delta(1-\frac{1}{4}\Delta)}\; Y_i\mathbf{U}_i^{(2)t}\boldsymbol{\xi})} + e^{g(-(1-\frac{1}{2}\Delta)\sqrt{1-Y_i^2}+\sqrt{\Delta(1-\frac{1}{4}\Delta)}\; Y_i\mathbf{U}_i^{(2)t}\boldsymbol{\xi})}\}\mu(d\boldsymbol{\xi})}{\prod_{i=1}^n \{e^{g(\sqrt{1-Y_i^2})} + e^{g(-\sqrt{1-Y_i^2})}\}} \\ &= T_n(E^*, \Delta), \quad \text{say}. \end{aligned} \tag{3.12}$$

The test statistic $T_n = T_n(E^*, \Delta)$ deserves further appraisal. Note that the numerator involves an integration over $\boldsymbol{\xi}$ on the surface S^{p-2}, so that it is a function of the Y_i, $\mathbf{U}_i^{(2)t}$, $1 \leqslant i \leqslant n$ as well as

$\Delta(>0)$. Thus, for each $\Delta(>0)$, T_n can be incorporated into a test for $\Delta = 0$ vs. $\Delta > 0$, using the right-hand tail of the distribution of T_n (under $\Delta = 0$). This will provide the MP RI test at the specified $\Delta(>0)$ alternative. However, in order to claim that this test is (uniformly) UMP (or best) RI test for all $\Delta > 0$, we need to show that for a given significance level α $(0 < \alpha < 1)$ and a specified $\Delta(>0)$, the critical level $c_\Delta(\alpha)$ of $T_n(E^*, \Delta)$ satisfies the following:

$$T_n(E^*, \Delta) \geqslant c_\Delta(\alpha) \Longleftrightarrow T_n^*(E^*) \geqslant c_\alpha^* \quad \forall \Delta > 0, \tag{3.13}$$

where $T_n^*(E^*)$ is free from Δ and so is c_α^*. This task is, however, not simple, and a UMPRI-test (for all $\Delta > 0$) may not generally exist. This outcome is not surprising, as even for the vMFL-distribution, the shortcomings of RI-LR tests have been discussed in the literature (Mardia and Jupp [9]). For this reason, we appraise the prospects of locally best RI tests. Towards this end, we consider the following lemma whose proof is relegated to the Appendix.

Lemma 3.1. *Under the assumed regularity conditions, as* $\Delta \to 0$,

$$T_n(E^*, \Delta) = 1 + \frac{1}{2(p-1)} \Delta L_n(E^*) + O(\Delta^{3/2}), \tag{3.14}$$

where

$$L_n(E^*) = \left\| \sum_{i=1}^{n} g'(\sqrt{1-Y_i^2}) Y_i \mathbf{U}_i^{(2)} \right\|^2 + \sum_{i=1}^{n} (g'(\sqrt{1-Y_i^2}))^2 Y_i^2$$
$$+ \sum_{i=1}^{n} \{Y_i^2 g''(\sqrt{1-Y_i^2}) - (p-1)\sqrt{1-Y_i^2}\, g'(\sqrt{1-Y_i^2})\} c(\sqrt{1-Y_i^2}) \tag{3.15}$$

with $c(\cdot)$ *defined in* (A.5).

In passing, we may note that the second and third terms on the right-hand side of (3.15) depend only on $Y_1^2, \ldots, Y_n^2$ but not on the $\mathbf{U}_i^{(2)}$, $1 \leqslant i \leqslant n$. Using the general formulation of $T_n(E^*; \Delta)$ and Lemma 3.1, we immediately arrive at the following:

Theorem 3.1. *For the density in* (2.1) *with skew-symmetric* $g(\cdot)$ *on* $(-1, 1)$, *under the assumed regularity conditions, for testing* $\mathrm{H}_0 : \boldsymbol{\theta} = \boldsymbol{\theta}_0$ *versus* $\mathrm{H}_1 : \boldsymbol{\theta} = \boldsymbol{\theta}_1$; $\boldsymbol{\theta}_1^t \boldsymbol{\theta}_0 = 1 - \frac{1}{2}\Delta$, $\Delta > 0$, *locally, for* $0 < \Delta \leqslant \Delta_0$ *(sufficiently small), at any significance level* α $(0 < \alpha < 1)$, *there exists a best RI test based on* $L_n(E^*)$, *rejecting* H_0 *in favor of* H_1 *whenever* $L_n(E^*) \geqslant l_{\alpha,n}^*$, *a critical value tuned to the level* α.

Note that a similar but more complicated test statistic exists even when $g(\cdot)$ is not skew-symmetric. As an illustration, consider the vMFL density for which $g(\langle \mathbf{x}, \boldsymbol{\theta} \rangle) = \kappa \langle \mathbf{x}, \boldsymbol{\theta} \rangle - c(\kappa)$, $\mathbf{x} \in S^{p-1}$, $\boldsymbol{\theta} \in S^{p-1}$. Thus, $g(y)$ $(= \kappa y - c(\kappa), -1 \leqslant y \leqslant 1)$ is skew-symmetric, $g'(y) = \kappa, \forall y \in (-1, 1)$ and $g''(y) = 0, \forall y \in (-1, 1)$. Thus $L_n(E^*)$ reduces to

$$\kappa^4 \left[\left(\sum_{i=1}^{n} Y_i \mathbf{U}_i^{(2)} \right)^t \left(\sum_{i=1}^{n} Y_i \mathbf{U}_i^{(2)} \right) + \sum_{i=1}^{n} Y_i^2 \right] - \kappa^2 (p-1) \sum_{i=1}^{n} \sqrt{1-Y_i^2} \tanh(\kappa \sqrt{1-Y_i^2}\,)$$

$$\Longleftrightarrow \kappa^2 \left\{ \left\| \sum_{i=1}^{n} Y_i \mathbf{U}_i^{(2)} \right\|^2 + \sum_{i=1}^{n} Y_i^2 \right\} - (p-1) \sum_{i=1}^{n} \sqrt{1-Y_i^2} \tanh(\kappa \sqrt{1-Y_i^2}\,). \tag{3.16}$$

Recall that because of RI, we choose $\boldsymbol{\theta}_0 = (1, \mathbf{0}^t)^t$, so that $\sqrt{1 - Y_i^2} = |X_{i1}|$ and $Y_i \mathbf{U}_i^{(2)} = \mathbf{X}_i^{(2)} = (X_{i2}, \ldots, X_{ip})^t$. Thus, $\| \sum_{i=1}^n Y_i \mathbf{U}_i^{(2)} \|^2 = \| \sum_{i=1}^n \mathbf{X}_i^{(2)} \|^2$ which is used in a conventional RI-LR-type testing setup. The adjustment by the second and third terms $(\kappa^2 \sum_{i=1}^n Y_i^2 - (p - 1) \sum_{i=1}^n \sqrt{1 - Y_i^2} \tanh(\kappa \sqrt{1 - Y_i^2}\,))$ relates to the locally best RI version of the former.

4. The LBRIRT

We formulate rank tests by replacing the Y_i by their ranks R_i, and let $\mathbf{R} = (R_1, \ldots, R_n)^t$. Since the Y_i are i.i.d. random variables, and the second and third terms on the right-hand side of (3.15) is a symmetric function $h^*(Y_1^2, \ldots, Y_n^2)$ of the Y_i^2, we have

$$\mathrm{E}[h^*(Y_1^2, \ldots, Y_n^2) \mid \mathbf{R}] = \mathrm{E}[h^*(Y_{n:1}^2, \ldots, Y_{n:n}^2)] = h_n^*, \tag{4.1}$$

which is a constant. Thus, the task is to evaluate

$$\mathrm{E}\left[\left\| \sum_{i=1}^n g'(\sqrt{1 - Y_i^2}) Y_i \mathbf{U}_i^{(2)} \right\|^2 \middle| \mathbf{R} \right] \tag{4.2}$$

and reformulate a test based on this statistic. Let us denote by $a(Y_i) = g'(\sqrt{1 - Y_i^2}) Y_i$, $i = 1, \ldots, n$, and note that $a(Y_i)$ and $\mathbf{U}_i^{(2)}$ are independent. Then we have

$$\begin{aligned}
&\mathrm{E}\left[\left\| \sum_{i=1}^n a(Y_i) \mathbf{U}_i^{(2)} \right\|^2 \middle| \mathbf{R} \right] \\
&= \sum_{i=1}^n \mathrm{E}[a^2(Y_i) \mid \mathbf{R}] + \sum_{i \neq j = 1}^n \mathrm{E}[a(Y_i) a(Y_j) \mid \mathbf{R}] \mathbf{U}_i^{(2)t} \mathbf{U}_j^{(2)} \\
&= \sum_{i=1}^n a_n(i, i) + \sum_{i \neq j = 1}^n a_n(R_i, R_j) \mathbf{U}_i^{(2)t} \mathbf{U}_j^{(2)} \\
&= \mathcal{L}_n, \quad \text{say,}
\end{aligned} \tag{4.3}$$

where $a_n(i, i) = \mathrm{E}[a^2(Y_{n:i})]$, $1 \leqslant i \leqslant n$ (so that $\sum_{i=1}^n a_n(i, i) = n\mathrm{E}[a^2(Y)] = na^*$), and

$$\begin{aligned}
a_n(R_i, R_j) &= \mathrm{E}[a(Y_{n:i}) a(Y_{n:j})] \\
&= \mathrm{E}[a(Y_{n:i})]\mathrm{E}[a(Y_{n:j})] + \mathrm{Cov}(a(Y_{n:i}), a(Y_{n:j})) \\
&= \gamma_{ni} \gamma_{nj} + \sigma_{n,ij}, \quad i \neq j = 1, \ldots, n.
\end{aligned} \tag{4.4}$$

Write $a_n(i, i) = \gamma_{ni}^2 + \sigma_{n,ii}$, $i = 1, \ldots, n$, therefore we have

$$\mathcal{L}_n = \mathrm{E}\left[\left\| \sum_{i=1}^n a(Y_i) \mathbf{U}_i^{(2)} \right\|^2 \middle| \mathbf{R} \right]$$

$$= \left\| \sum_{i=1}^{n} \gamma_{nR_i} \mathbf{U}_i^{(2)} \right\|^2 + \sum_{i=1}^{n} \sum_{j=1}^{n} \sigma_{n,R_iR_j} \mathbf{U}_i^{(2)t} \mathbf{U}_j^{(2)}. \tag{4.5}$$

Using the general theorem on LMPR tests in Hájek et al. [5, pp. 71–73], we have

Theorem 4.1. *For the density in* (2.1) *with a skew-symmetric* $g(\cdot)$ *on* $(-1, 1)$, *under the assumed regularity conditions, defining* $\mathcal{L}_n$ *as in* (4.5), *the test with critical region*

$$\mathcal{L}_n \geqslant k \tag{4.6}$$

is the locally best rotation-invariant rank test for $\mathrm{H}_0 : \boldsymbol{\theta} = \boldsymbol{\theta}_0$ *versus* $\{H_\Delta\}$, *where* $H_\Delta : \boldsymbol{\theta}^t \boldsymbol{\theta}_0 = 1 - \frac{1}{2}\Delta$; $0 < \Delta \leqslant \Delta_0$, *sufficiency small, at the respective level.*

Recall that $\mathbf{U}_i^{(2)}$ and R_i are independent, and $\mathbf{R}$ takes on each permutation of $(1, \ldots, n)$ with the same probability $(n!)^{-1}$. Under this permutation measure $(\mathcal{P}_n)$, for every $i \neq j$,

$$\mathrm{E}_{\mathcal{P}_n}[a_n(R_i, R_j)\mathbf{U}_i^{(2)t}\mathbf{U}_j^{(2)}] = \frac{1}{n(n-1)} \left(\sum_{r \neq s=1}^{n} a_n(r, s) \right) \mathbf{U}_i^{(2)t} \mathbf{U}_j^{(2)} \tag{4.7}$$

so that the second term on the right-hand side of (4.3) has the centering, under $\mathcal{P}_n$,

$$\begin{aligned} &\frac{1}{n(n-1)} \left(\sum_{r \neq s=1}^{n} a_n(r, s) \right) \sum_{i \neq j=1}^{n} \mathbf{U}_i^{(2)t} \mathbf{U}_j^{(2)} \\ &= \frac{1}{n(n-1)} \left(\sum_{r \neq s=1}^{n} a_n(r, s) \right) \left\{ \left(\sum_{i=1}^{n} \mathbf{U}_i^{(2)} \right)^t \left(\sum_{j=1}^{n} \mathbf{U}_j^{(2)} \right) - \sum_{i=1}^{n} \|\mathbf{U}_i^{(2)}\|^2 \right\} \\ &= \frac{1}{n(n-1)} \left[\sum_{r,s=1}^{n} a_n(r, s) - \sum_{r=1}^{n} a_n(r, r) \right] \left\{ \left\| \sum_{i=1}^{n} \mathbf{U}_i^{(2)} \right\|^2 - \sum_{i=1}^{n} \|\mathbf{U}_i^{(2)}\|^2 \right\}. \end{aligned} \tag{4.8}$$

Note that the permutation distribution can be generated by $\mathcal{P}_n$ and this provides a convenient conditionally distribution-free test given the $\mathbf{U}_i^{(2)}$, $1 \leqslant i \leqslant n$.

In the particular case of vMFL density, $a(Y_i) = \kappa Y_i$, $1 \leqslant i \leqslant n$. By (3.6) and the density function of X_{i1} which is of the form $f(x_1) = b(\kappa)e^{\kappa x_1}$, $-1 \leqslant x_1 \leqslant 1$, we obtain after, some standard manipulations, the density of Y_i under H_0, as

$$h(y) = \frac{\kappa}{(e^\kappa - e^{-\kappa})} \frac{y}{\sqrt{1-y^2}} (e^{\kappa\sqrt{1-y^2}} + e^{-\kappa\sqrt{1-y^2}}), \quad 0 \leqslant y \leqslant 1 \tag{4.9}$$

and thus

$$H(y) = 1 - \left(\frac{e^{\kappa\sqrt{1-y^2}} - e^{-\kappa\sqrt{1-y^2}}}{e^\kappa - e^{-\kappa}} \right), \quad 0 \leqslant y \leqslant 1. \tag{4.10}$$

Then, the expectation of the product of two order statistics $Y_{n:i}$ and $Y_{n:j}$, $1 \leqslant i < j \leqslant n$ can be approximated by

$$\mathrm{E}(Y_{n:i}Y_{n:j}) = H^{-1}\left(\frac{i}{n+1}\right)H^{-1}\left(\frac{j}{n+1}\right) + O(n^{-1}), \tag{4.11}$$

where $H^{-1}(u)$ is the inverse function of $H(u)$. Note that the pdf of Y has the support [0, 1], and hence $H^{-1}(u)$ is bounded in [0, 1]. After some simplifications, we have

$$h'(y) = \frac{1}{y(1-y^2)}h(y) + \frac{\kappa^2}{(e^{\kappa} - e^{-\kappa})}\frac{y^2}{(1-y^2)}(e^{-\kappa\sqrt{1-y^2}} - e^{\kappa\sqrt{1-y^2}}). \tag{4.12}$$

Let $p_i = H(Q_i)$, $Q(p_i) = H^{-1}(\frac{i}{n+1})$ $(p_i = \frac{i}{n+1})$. Thus,

$$Q_i' = \frac{1}{dp_i/dQ_i} = \frac{1}{h(Q_i)} = \frac{1}{h\left(H^{-1}\left(\frac{i}{n+1}\right)\right)}, \tag{4.13}$$

and

$$Q_i'' = -[h(Q_i)]^{-3}h'(Q_i) = -\left[h\left(H^{-1}\left(\frac{i}{n+1}\right)\right)\right]^{-3}h'\left(H^{-1}\left(\frac{i}{n+1}\right)\right). \tag{4.14}$$

Using these refinements, $\mathrm{E}(Y_{n:i}Y_{n:j})$ has a better approximation

$$\begin{aligned}
\mathrm{E}(Y_{n:i}Y_{n:j}) &= \mathrm{Cov}(Y_{n:i}, Y_{n:j}) + \mathrm{E}(Y_{n:i})\mathrm{E}(Y_{n:j}) \\
&= \left[H^{-1}\left(\frac{i}{n+1}\right) + \frac{i(n+1-i)}{2(n+1)^2(n+2)}Q_i''\right]\left[H^{-1}\left(\frac{j}{n+1}\right)\right. \\
&\quad \left. + \frac{j(n+1-j)}{2(n+1)^2(n+2)}Q_j''\right] + \frac{i(n+1-j)}{(n+1)^2(n+2)}Q_i'Q_j' + O(n^{-3/2}), \\
&\quad \forall\, 1 \leqslant i < j \leqslant n.
\end{aligned} \tag{4.15}$$

Thus, the entries $a_n(i, j)$ in (4.3) [or the γ_{ni}, $\sigma_{n,ij}$ in (4.5)] when substituted give us a linear function of the ranks $\mathbf{R}$ and $\mathbf{U}_i^{(2)}$, $1 \leqslant i \leqslant n$.

5. Asymptotically optimal RI-rank tests

There is a close connection between LBRIRT tests and asymptotically optimal rotation-invariant rank tests for contiguous alternatives. In Section 4, we consider the case of a fixed but Δ small. In a conventional setup, we allow $n \to \infty$ but Δ small such that $\sqrt{n}\Delta$ has a finite positive limit as $n \to \infty$. Hence, we consider a sequence $\{H_{1n}\}$ of alternatives where

$$\mathrm{H}_0 : \boldsymbol{\theta} = \boldsymbol{\theta}_0 = (1, \mathbf{0}^t)^t \quad \text{and} \quad \mathrm{H}_{1n} : \boldsymbol{\theta} = \boldsymbol{\theta}_{(n)} : \boldsymbol{\theta}_n^t\boldsymbol{\theta}_0 = 1 - \frac{\lambda}{2n}, \tag{5.1}$$

for some fixed λ. Note that

$$\boldsymbol{\theta}_n^t\boldsymbol{\theta}_0 = 1 - \lambda/2n \Leftrightarrow \theta_{n1} = 1 - \lambda/2n \quad \text{and} \quad \|\boldsymbol{\theta}_n^{(2)}\|^2 = \frac{\lambda}{n}\left(1 - \frac{\lambda}{4n}\right) \sim \frac{\lambda}{n}. \tag{5.2}$$

Let $H(y) = P_0\{Y \leqslant y\}$, $y \in [0, 1]$. As y is bounded, we have uniformly in i $(1 \leqslant i \leqslant n)$,

$$\left|\gamma_{ni} - a\left(H^{-1}\left(\frac{i}{n}\right)\right)\right| \to 0 \quad \text{as } n \to \infty, \tag{5.3}$$

and further, uniformly in $i, j : 1 \leqslant i, j \leqslant n$,

$$\sigma_{n,ij} \to 0 \quad \text{as } n \to \infty. \tag{5.4}$$

Therefore, the LBRIRT statistic can be approximated by

$$\left\| \sum_{i=1}^{n} a\left(H^{-1}\left(\frac{i}{n}\right)\right) \mathbf{U}_i^{(2)} \right\|^2, \tag{5.5}$$

a form that has been suggested by Neeman and Chang [10] from pure asymptotic considerations. However, because of the local optimality property, we prefer to work with a general class of rank statistics of the form

$$\mathcal{L}_n^* = \left[\frac{(p-1)}{\sum_{i=1}^{n} a_n(i, i)}\right] \sum_{i=1}^{n} \sum_{j=1}^{n} a_n(R_i, R_j) \mathbf{U}_i^{(2)t} \mathbf{U}_j^{(2)}, \tag{5.6}$$

where the $a_n(i, j)$ are defined as in before, and noting that $k_{n\alpha}^*$, the α-level critical value of $\mathcal{L}_n^*$ needs to be approximated by the asymptotic distribution of $\mathcal{L}_n^*$ (under H_0). We consider next the following lemma which adds more convenience to the study of the asymptotic null distribution; its proof is outlined in the Appendix.

For the scores $a_n(i, j)$, $1 \leqslant i, j \leqslant n$, assume that there exists a sequence $\{a_n^0(i), 1 \leqslant i \leqslant n\}$ such that

$$a_n(i, j) - a_n^0(i) a_n^0(j) = b_n(i, j), \tag{5.7}$$

$$\max_{i,j} |\, b_n(i, j)| = o(1) \quad \text{as } n \to \infty. \tag{5.8}$$

Then

$$\begin{aligned} \frac{1}{n}\sum_{i=1}^{n} a_n(i, i) &= \frac{1}{n}\sum_{i=1}^{n} (a_n^0(i))^2 + \frac{1}{n}\sum_{i=1}^{n} b_n(i, i) \\ &= A_n^2 + B_n, \quad \text{say.} \end{aligned} \tag{5.9}$$

Lemma 5.1. *Define*

$$\mathcal{L}_n^{0*} = \frac{(p-1)}{nA_n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} a_n^0(R_i) a_n^0(R_j) \mathbf{U}_i^{(2)t} \mathbf{U}_j^{(2)}. \tag{5.10}$$

Then, under H_0, *as* $n \to \infty$,

$$|\mathcal{L}_n^* - \mathcal{L}_n^{0*}| \to 0 \quad \textit{in probability}. \tag{5.11}$$

We assume that the Noether condition holds:

$$\max_{1\leqslant i\leqslant n}(a_n^0(i))^2 \Big/ \sum_{i=1}^{n}[a_n^0(i)]^2 \to 0 \quad \text{as } n\to\infty. \tag{5.12}$$

So that invoking the permutational central limit theorem (on the $a_n^0(R_i)$), along with the independence of R_i and $\mathbf{U}_i^{(2)}$, and the spherical uniformity of the distribution of $\mathbf{U}_i^{(2)}$ on S^{p-2}, we obtain that under H_0,

$$\frac{1}{\sqrt{n}A_n}\sum_{i=1}^{n}a_n^0(R_i)\mathbf{U}_i^{(2)} \xrightarrow{\mathcal{D}} \mathcal{N}_{p-1}\left(\mathbf{0}, \frac{1}{p-1}\mathbf{I}_{p-1}\right), \tag{5.13}$$

and as a result

$$\mathcal{L}_n^{0*} \xrightarrow{\mathcal{D}} \chi^2_{p-1} \quad (\Rightarrow \mathcal{L}_n^{*} \xrightarrow{\mathcal{D}} \chi^2_{p-1}). \tag{5.14}$$

Next, we consider the following lemma whose proof is relegated to the Appendix.

Lemma 5.2. *Under* H_0, *as* $n\to\infty$,

$$\log(L_n(\boldsymbol{\theta}_n)/L_n(\boldsymbol{\theta}_0)) \xrightarrow{\mathcal{D}} \mathcal{N}\left(-\frac{\lambda}{2}\sigma_1^2, \lambda\sigma_1^2\right). \tag{5.15}$$

At this stage, we are in a position to make use of the celebrated Le Cam's first lemma (Le Cam [7]) and thereby arrive at the following.

Theorem 5.1. *A sequence of local alternatives* $\{\mathrm{H}_{1n}\}$ *in* (5.1) *is contiguous to* H_0.

The distribution theory of $\frac{\sqrt{p-1}}{\sqrt{n}A_n}\sum_{i=1}^{n}a_n^0(R_i)\mathbf{U}_i^{(2)}$ for contiguous alternatives then follows the standard way (as may also be found in Neeman and Chang [10]). Let $A_0^2=\int_0^1(a^0(u))^2\,du$. Then, we have the following.

Theorem 5.2. *Under a sequence of contiguous alternatives* $\{H_{1n}\}$ *in* (5.1), *as* $n\to\infty$,

$$\mathcal{L}_n^{*} \xrightarrow{\mathcal{D}} \chi^2_{p-1,\Gamma}, \tag{5.16}$$

where $\Gamma=\frac{\lambda}{(p-1)A_0^2}\langle\varphi,\varphi^0\rangle^2$ *with* φ^0 *being the optimal score function and* $\varphi=a(u)$.

6. Comparison of tests

In this section, we first investigate the empirical powers of proposed tests by Monte Carlo simulation. Independent vMFL p-dimensional variates having the concentration parameter κ are generated by the Matlab subroutine. Consider the LR-type test T_n, the proposed LBRIT L_n and the LBRIRT $\mathcal{L}_n$, where

$$T_n = \left\| \sum_{i=1}^{n} [\mathbf{X}_i - (\mathbf{X}_i^t \boldsymbol{\theta}_0)\boldsymbol{\theta}_0] \right\|^2, \tag{6.1}$$

$$L_n = \kappa^2 \left[T_n + \sum_{i=1}^{n} Y_i^2 \right] - (p-1) \sum_{i=1}^{n} \sqrt{1 - Y_i^2}\ \tanh(\kappa \sqrt{1 - Y_i^2}\ ), \tag{6.2}$$

and

$$\mathcal{L}_n = \sum_{1 \leqslant i < j \leqslant n} \mathrm{E}(Y_{n:i} Y_{n:j}) \mathbf{U}_i^{(2)t} \mathbf{U}_j^{(2)} \tag{6.3}$$

with $\mathrm{E}(Y_{n:i} Y_{n:j})$ being approximated by (4.15).

It is an interesting question: How the LR-type test (T_n) compares with the LBRIT L_n, and how the latter compares with the rank version $\mathcal{L}_n$? Since the hypothesis testing problem is RI, it is expected that L_n would perform better than T_n. On the other hand, $\mathcal{L}_n$ is a rank statistic that sacrifices some information contained in the MI Y_i^2. Therefore, $\mathcal{L}_n$ may not perform as well as L_n. This relative performance picture is presented here in a local setup; for nonlocal alternatives, $\mathcal{L}_n$ may not perform as well as the other two. For this study, we take for simplicity, $n = 50$, $p = 3$, and $\kappa = 0.5, 1, 2, 5$ at a level of significance $\alpha = 0.01$. The reported values are the empirical powers based on 10,000 replications for each combination κ and $\boldsymbol{\theta}$ under 1% level of significance. Tables 1(a)–(d) relate to the (local) power of tests for the hypothesis testing problem $\mathrm{H}_0 : \boldsymbol{\theta} = \boldsymbol{\theta}_0$ against $\mathrm{H}_1 : \boldsymbol{\theta} \neq \boldsymbol{\theta}_0$, where $\boldsymbol{\theta}_0 = (1, 0, 0)^t$. The numerical values reveal that the LBRIT L_n performs better than the LR-type test T_n, while the concentration parameter κ is small, $\mathcal{L}_n$ and L_n are close to each other, often one is better than the other. However, when the concentration parameter κ is not small ($\kappa \geqslant 1$), the test based on L_n performs better than the others. Thus, the relative advantages of the rank test has to be contrasted with potential loss of power for not so close alternative. The picture would be different for a fixed alternative and large sample sizes where all the tests would have power close to 1.

Due to technical programming difficulties in generating samples from the Fisher–Bingham density function of the following form:

$$f(\mathbf{x}; \boldsymbol{\theta}, \kappa, \mathbf{A}) = \frac{1}{a(\kappa, \mathbf{A})} \exp\{\kappa \mathbf{x}^t \boldsymbol{\theta} + \mathbf{x}^t \mathbf{A} \mathbf{x}\}, \quad \|\mathbf{x}\| = 1, \tag{6.4}$$

(where $\mathbf{A}$ is symmetric with trace($\mathbf{A}$) $= 0$ and $a(\kappa, \mathbf{A})$ is a normalizing constant), additional simulation results for (6.4) are not considered.

7. Pitman ARE

Generally, the forms of LR-type test statistics, the LBRI test statistics and the LBRIR test statistics are rather complicated for the more general underlying density functions. To overcome this we adapt the asymptotic approach of Section 5, leading us to approximate the LBRIRT by a class of rank tests by Neeman and Chang [10]. As such, from simplicity and robustness points of view, we may advocate to use the spherical Wilcoxon rank test based on the

Table 1
Empirical powers of T_n, L_n and $\mathcal{L}_n$ for some $\boldsymbol{\theta}$'s

$\kappa = 0.5$	$\begin{pmatrix} 0.9000 \\ -0.0930 \\ -0.4426 \end{pmatrix}$	$\begin{pmatrix} 0.9100 \\ 0.3768 \\ 0.1730 \end{pmatrix}$	$\begin{pmatrix} 0.9200 \\ 0.1810 \\ 0.3476 \end{pmatrix}$	$\begin{pmatrix} 0.9300 \\ -0.2536 \\ -0.2661 \end{pmatrix}$	$\begin{pmatrix} 0.9400 \\ -0.0217 \\ -0.3405 \end{pmatrix}$
(a)					
T_n	0.0290	0.0290	0.0265	0.0242	0.0228
L_n	0.0320	0.0291	0.0271	0.0258	0.0236
$\mathcal{L}_n$	0.0318	0.0307	0.0262	0.0229	0.0200
$\kappa = 0.5$	$\begin{pmatrix} 0.9500 \\ -0.0049 \\ 0.3122 \end{pmatrix}$	$\begin{pmatrix} 0.9600 \\ 0.1500 \\ 0.2364 \end{pmatrix}$	$\begin{pmatrix} 0.9700 \\ 0.2145 \\ 0.1145 \end{pmatrix}$	$\begin{pmatrix} 0.9800 \\ 0.0911 \\ 0.1769 \end{pmatrix}$	$\begin{pmatrix} 0.9900 \\ -0.0165 \\ -0.1401 \end{pmatrix}$
T_n	0.0197	0.0157	0.0148	0.0137	0.0117
L_n	0.0208	0.0168	0.0169	0.0145	0.0115
$\mathcal{L}_n$	0.0226	0.2010	0.0149	0.0146	0.0139
(b)					
$\kappa = 1$	$\begin{pmatrix} 0.9000 \\ 0.3759 \\ 0.2206 \end{pmatrix}$	$\begin{pmatrix} 0.9100 \\ 0.3427 \\ -0.2333 \end{pmatrix}$	$\begin{pmatrix} 0.9200 \\ 0.3078 \\ 0.2426 \end{pmatrix}$	$\begin{pmatrix} 0.9300 \\ -0.0261 \\ 0.3666 \end{pmatrix}$	$\begin{pmatrix} 0.9400 \\ -0.1574 \\ 0.3027 \end{pmatrix}$
T_n	0.1393	0.1277	0.1070	0.0942	0.0782
L_n	0.1402	0.1314	0.1092	0.0980	0.0801
$\mathcal{L}_n$	0.1088	0.0939	0.0800	0.0719	0.0600
$\kappa = 1$	$\begin{pmatrix} 0.9500 \\ -0.2427 \\ 0.1965 \end{pmatrix}$	$\begin{pmatrix} 0.9600 \\ 0.0540 \\ -0.2748 \end{pmatrix}$	$\begin{pmatrix} 0.9700 \\ 0.2160 \\ 0.1116 \end{pmatrix}$	$\begin{pmatrix} 0.9800 \\ -0.0711 \\ -0.1859 \end{pmatrix}$	$\begin{pmatrix} 0.9900 \\ 0.0135 \\ -0.1404 \end{pmatrix}$
T_n	0.0628	0.0504	0.0413	0.0273	0.0177
L_n	0.0645	0.0511	0.0416	0.0276	0.0178
$\mathcal{L}_n$	0.0461	0.0387	0.0282	0.0249	0.0169
(c)					
$\kappa = 2$	$\begin{pmatrix} 0.9000 \\ -0.3375 \\ -0.2758 \end{pmatrix}$	$\begin{pmatrix} 0.9100 \\ 0.3783 \\ 0.1697 \end{pmatrix}$	$\begin{pmatrix} 0.9200 \\ -0.2579 \\ -0.2951 \end{pmatrix}$	$\begin{pmatrix} 0.9300 \\ 0.0694 \\ -0.3610 \end{pmatrix}$	$\begin{pmatrix} 0.9400 \\ -0.2931 \\ 0.1747 \end{pmatrix}$

Table 1 (*continued*).

T_n	0.6070	0.5434	0.4806	0.4043	0.3358
L_n	0.6187	0.5548	0.4899	0.4150	0.3438
$\mathcal{L}_n$	0.4715	0.4065	0.3541	0.3035	0.2490
$\kappa = 2$	$\begin{pmatrix} 0.9500 \\ -0.2248 \\ 0.2167 \end{pmatrix}$	$\begin{pmatrix} 0.9600 \\ -0.1123 \\ 0.2565 \end{pmatrix}$	$\begin{pmatrix} 0.9700 \\ -0.0457 \\ -0.2388 \end{pmatrix}$	$\begin{pmatrix} 0.9800 \\ 0.1838 \\ -0.0764 \end{pmatrix}$	$\begin{pmatrix} 0.9900 \\ -0.1290 \\ -0.0570 \end{pmatrix}$
T_n	0.2692	0.1914	0.1255	0.0769	0.0313
L_n	0.2746	0.1952	0.1295	0.0797	0.0314
$\mathcal{L}_n$	0.1916	0.1455	0.1011	0.0601	0.0304
(d)					
$\kappa = 5$	$\begin{pmatrix} 0.9000 \\ -0.2048 \\ -0.3848 \end{pmatrix}$	$\begin{pmatrix} 0.9100 \\ -0.2090 \\ 0.3581 \end{pmatrix}$	$\begin{pmatrix} 0.9200 \\ -0.1982 \\ -0.3381 \end{pmatrix}$	$\begin{pmatrix} 0.9300 \\ -0.1616 \\ 0.3301 \end{pmatrix}$	$\begin{pmatrix} 0.9400 \\ 0.1271 \\ -0.3166 \end{pmatrix}$
T_n	0.9997	0.9987	0.9970	0.9912	0.9757
L_n	0.9997	0.9989	0.9972	0.9920	0.9778
$\mathcal{L}_n$	0.9860	0.9755	0.9510	0.9170	0.8603
$\kappa = 5$	$\begin{pmatrix} 0.9500 \\ 0.6631 \\ 0.3051 \end{pmatrix}$	$\begin{pmatrix} 0.9600 \\ -0.2330 \\ 0.1553 \end{pmatrix}$	$\begin{pmatrix} 0.9700 \\ -0.1248 \\ 0.2086 \end{pmatrix}$	$\begin{pmatrix} 0.9800 \\ 0.1796 \\ 0.0856 \end{pmatrix}$	$\begin{pmatrix} 0.9900 \\ 0.1117 \\ -0.0862 \end{pmatrix}$
T_n	0.9415	0.8650	0.7207	0.4644	0.1963
L_n	0.9433	0.8678	0.7259	0.4690	0.2001
$\mathcal{L}_n$	0.7785	0.6497	0.4900	0.2997	0.1262

test statistic

$$\mathbf{W}_n = n^{-1/2} \sum_{i=1}^{n} R_i \mathbf{U}_i^{(2)t} \tag{7.1}$$

for the problem of testing $\mathrm{H}_0 : \boldsymbol{\theta} = \boldsymbol{\theta}_0$ against $\mathrm{H}_1 : \boldsymbol{\theta} \neq \boldsymbol{\theta}_0$.

For the vMFL distribution, after some manipulations the Pitman ARE of spherical Wilcoxon rank test with respect to the optimal test is

$$e(\mathbf{W}_n) = \frac{3}{b(p,\kappa)} \frac{[\int_{-1}^{1}\{1-\frac{1}{b(p,\kappa)}\int_{-|v|}^{|v|} e^{\kappa s}(1-s^2)^{(p-3)/2}\,ds\}\kappa e^{\kappa v}(1-v^2)^{(p-2)/2}\,dv]^2}{\int_{-1}^{1} \kappa^2 e^{\kappa v}(1-v^2)^{(p-1)/2}\,dv}, \tag{7.2}$$

where

$$b(p,\kappa) = \int_{-1}^{1} e^{\kappa v}(1-v^2)^{(p-3)/2}\,dv. \tag{7.3}$$

Table 2

	$\kappa = 0$	$\kappa = 0.01$	$\kappa = 0.1$	$\kappa = 1.0$	$\kappa = 10.0$	$\kappa = 100.0$	$\kappa = 500$
$p = 3$	0.919632	0.919634	0.919817	0.935489	0.988233	0.985074	0.984726
$p = 4$	0.877328	0.87733	0.877472	0.890296	0.971131	0.974635	0.974789
$p = 5$	0.851196	0.851197	0.851304	0.86124	0.95161	0.960729	0.961299
$p = 6$	0.833774	0.833775	0.833857	0.841622	0.933901	0.947891	0.948819
$p = 7$	0.821402	0.821402	0.821467	0.827653	0.918387	0.936675	0.937921
$p = 8$	0.812184	0.812185	0.812237	0.817261	0.904824	0.926945	0.928479
$p = 9$	0.805061	0.805061	0.805104	0.809256	0.892906	0.918459	0.920258
$p = 10$	0.799394	0.799394	0.79943	0.802915	0.882365	0.910998	0.913042
$p = 11$	0.79478	0.79478	0.794811	0.797775	0.872983	0.90438	0.906655
$p = 12$	0.790952	0.790952	0.790978	0.79353	0.864586	0.898461	0.900954
$p = 13$	0.787725	0.787726	0.787748	0.789966	0.857036	0.893127	0.895828
$p = 14$	0.784969	0.784969	0.784989	0.786934	0.850219	0.888289	0.891188
$p = 15$	0.782587	0.782587	0.782605	0.784324	0.84404	0.883873	0.886962
$p = 16$	0.780508	0.780509	0.780524	0.782055	0.838422	0.87982	0.883093
$p = 17$	0.778679	0.778679	0.778693	0.780064	0.8333	0.876083	0.879532
$p = 18$	0.777056	0.777056	0.777068	0.778304	0.828616	0.872621	0.876241
$p = 19$	0.775607	0.775607	0.775618	0.776737	0.824322	0.869402	0.873188
$p = 20$	0.774304	0.774305	0.774315	0.775333	0.820378	0.866397	0.870344
$p = 30$	0.766110	0.766110	0.766114	0.766590	0.793966	0.844299	0.849663
$p = 50$	0.759620	0.759620	0.759622	0.759800	0.772577	0.820268	0.827844
$p = 100$	0.754792	0.754792	0.754793	0.754839	0.758725	0.793306	0.804597

When $p = 3$ and $\kappa \to 0$, then $e(\mathbf{W}_n) = 9(\pi/2 - 2/3)^2/8$. We computed the efficiencies for various values of p and κ. The results appear below (Table 2).

8. An illustrative example

We use the data set from Fisher et al. [4, p. 279] to illustrate the use of the spherical Wilcoxon rank test based on the test statistic $\mathbf{W}_n$ defined in (7.1). The data set is the 26 measurements of magnetic remanence in specimens of Palaeozoic red-beds from Argentina. The coordinate system is (Declination, Inclination). They analyzed this data set and concluded that the data are sampled from a distribution symmetric about its mean direction (see Examples 5.2, 5.12 and 5.13 of Fisher et al. [4] for details). A question in their Example 5.14 is further appraised to see whether or not the direction (Dec. 150^0, Inc. 60^0) is acceptable as the true mean direction.

To perform the proposed tests, first the data are transformed into rectangular coordinates, as such we have $\boldsymbol{\theta}_0 = (-\frac{\sqrt{3}}{4}, -\frac{1}{4}, -\frac{\sqrt{3}}{2})^t$. By the results of Watson [9, Section 3.4], it is easy to see that under the null hypothesis, as $n \to \infty$, $2n^{-1}T_n/c_n \xrightarrow{\mathcal{D}} \chi^2_{p-1}$, where $c_n = n^{-1}\sum_{i=1}^n \mathbf{X}_i^t(\mathbf{I} - \boldsymbol{\theta}_0\boldsymbol{\theta}_0^t)\mathbf{X}_i^t$. Similarly, let $Q_n^2 = n\mathbf{W}_n^t\widehat{\boldsymbol{\Sigma}}_n^-(\boldsymbol{\theta}_0)\mathbf{W}_n$, where $\widehat{\boldsymbol{\Sigma}}_n^-(\boldsymbol{\theta}_0) = 12[n(n+1)(2n+1)]^{-1}(\mathbf{I} - \boldsymbol{\theta}_0\boldsymbol{\theta}_0^t)$. Then by Theorem 5.2, under the null hypothesis, as $n \to \infty$, $Q_n^2 \xrightarrow{\mathcal{D}} \chi^2_{p-1}$.

These lead us to the (asymptotic) observed significance level (OSL) as $Pr\{\chi_2^2 \geqslant 2n^{-1}T_n/c_n\} = 0.0028$ and $Pr\{\chi_2^2 \geqslant Q_n^2\} = 0.0027$, respectively. Thus, both the spherical T^2 test and the spherical Wilcoxon rank test are concordant regarding the untenability of the null hypothesis that the mean direction (Dec. 150^0, Inc. 60^0) as made by Fisher et al. [4, Example 5.14, p. 116]. We may remark that the asymptotic approximation for the null distributions may tend to be comparatively less precise in the tail. Hence, instead of the concluded values (0.0028 and 0.0027) the actual values could be slightly different, possibly a bit larger, yet of the same order. Hence, it would be safe to say that even at a significance level 0.005, both the OSL values convey the rejection of the null hypothesis.

Appendix

Proof of Lemma 3.1. We rewrite $T_n(E^*, \Delta)$ in (3.12) as

$$\int_{S^{p-2}} \prod_{i=1}^{n} \left\{ \frac{e^{g((1-\frac{1}{2}\Delta)\sqrt{1-Y_i^2}+\sqrt{\Delta(1-\frac{1}{4}\Delta)}\ Y_i \mathbf{U}_i^{(2)t}\boldsymbol{\xi})} + e^{g(-(1-\frac{1}{2}\Delta)\sqrt{1-Y_i^2}+\sqrt{\Delta(1-\frac{1}{4}\Delta)}\ Y_i \mathbf{U}_i^{(2)t}\boldsymbol{\xi})}}{e^{g(\sqrt{1-Y_i^2})} + e^{g(-\sqrt{1-Y_i^2})}} \right\} \times \mu(d\boldsymbol{\xi}). \tag{A.1}$$

In the next step, we write

$$\begin{aligned} &g(\pm(1-\tfrac{1}{2}\Delta)\sqrt{1-Y_i^2}+\sqrt{\Delta(1-\tfrac{1}{4}\Delta)}\ Y_i \mathbf{U}_i^{(2)t}\boldsymbol{\xi}) \\ &\quad = g(\pm\sqrt{1-Y_i^2}) + g'(\pm\sqrt{1-Y_i^2})\{\mp\tfrac{1}{2}\Delta\sqrt{1-Y_i^2}+\sqrt{\Delta(1-\tfrac{1}{4}\Delta)}\ Y_i \mathbf{U}_i^{(2)t}\boldsymbol{\xi}\} \\ &\qquad +\tfrac{1}{2}g''(\pm\sqrt{1-Y_i^2})\{\Delta(Y_i\mathbf{U}_i^{(2)t}\boldsymbol{\xi})^2 + O(\Delta^{3/2})\} + o(\Delta), \end{aligned} \tag{A.2}$$

so that after some routine manipulations, we obtain that as $\Delta \searrow 0$,

$$\begin{aligned} &e^{g(\pm(1-\frac{1}{2}\Delta)\sqrt{1-Y_i^2}+\sqrt{\Delta(1-\frac{1}{4}\Delta)}\ Y_i \mathbf{U}_i^{(2)t}\boldsymbol{\xi})} \\ &\quad = e^{g(\pm\sqrt{1-Y_i^2})}\{1+\sqrt{\Delta(1-\tfrac{1}{4}\Delta)}\ g'(\pm\sqrt{1-Y_i^2})Y_i\mathbf{U}_i^{(2)t}\boldsymbol{\xi} \\ &\qquad \mp\tfrac{1}{2}\Delta\sqrt{1-Y_i^2}\}g'(\pm\sqrt{1-Y_i^2}) + \tfrac{1}{2}\Delta[g''(\pm\sqrt{1-Y_i^2}) \\ &\qquad +(g'(\pm\sqrt{1-Y_i^2}))^2](Y_i\mathbf{U}_i^{(2)t}\boldsymbol{\xi})^2 + O(\Delta^{3/2})\}. \end{aligned} \tag{A.3}$$

As a result, the integrand in (A.1) can be expressed as

$$\begin{aligned} =\prod_{i=1}^{n} \Bigg\{ 1 + \sqrt{\Delta\left(1-\frac{1}{4}\Delta\right)} &\left[\frac{g'(\sqrt{1-Y_i^2})e^{g(\sqrt{1-Y_i^2})} + g'(-\sqrt{1-Y_i^2})e^{g(-\sqrt{1-Y_i^2})}}{e^{g(\sqrt{1-Y_i^2})}+e^{g(-\sqrt{1-Y_i^2})}} \right] \\ \times Y_i\mathbf{U}_i^{(2)t}\boldsymbol{\xi} - \frac{1}{2}\Delta &\left[\frac{\sqrt{1-Y_i^2}\{g'(\sqrt{1-Y_i^2})e^{g(\sqrt{1-Y_i^2})} - g'(-\sqrt{1-Y_i^2})e^{g(-\sqrt{1-Y_i^2})}\}}{e^{g(\sqrt{1-Y_i^2})}+e^{g(-\sqrt{1-Y_i^2})}} \right] \\ +\frac{1}{2}\Delta &\left[\frac{((g'(\sqrt{1-Y_i^2}))^2 + g''(\sqrt{1-Y_i^2}))e^{g(\sqrt{1-Y_i^2})}}{e^{g(\sqrt{1-Y_i^2})}+e^{g(-\sqrt{1-Y_i^2})}} \right. \end{aligned}$$

$$+\frac{((g'(-\sqrt{1-Y_i^2}))^2+g''(-\sqrt{1-Y_i^2}))e^{g(-\sqrt{1-Y_i^2})}}{e^{g(\sqrt{1-Y_i^2})}+e^{g(-\sqrt{1-Y_i^2})}}\Bigg](Y_i\mathbf{U}_i^{(2)t}\boldsymbol{\xi})^2+O(\Delta^{3/2})\Bigg\}$$

$$=1+\sqrt{\Delta\left(1-\frac{1}{4}\Delta\right)}\sum_{i=1}^{n}\left[\frac{g'(\sqrt{1-Y_i^2})e^{g(\sqrt{1-Y_i^2})}+g'(-\sqrt{1-Y_i^2})e^{g(-\sqrt{1-Y_i^2})}}{e^{g(\sqrt{1-Y_i^2})}+e^{g(-\sqrt{1-Y_i^2})}}\right]Y_i\mathbf{U}_i^{(2)t}\boldsymbol{\xi}$$

$$-\frac{1}{2}\Delta\sum_{i=1}^{n}\left[\frac{\sqrt{1-Y_i^2}\{g'(\sqrt{1-Y_i^2})e^{g(\sqrt{1-Y_i^2})}-g'(-\sqrt{1-Y_i^2})e^{g(-\sqrt{1-Y_i^2})}\}}{e^{g(\sqrt{1-Y_i^2})}+e^{g(-\sqrt{1-Y_i^2})}}\right]$$

$$+\frac{1}{2}\Delta\sum_{i=1}^{n}\Bigg[\frac{((g'(\sqrt{1-Y_i^2}))^2+g''(\sqrt{1-Y_i^2}))e^{g(\sqrt{1-Y_i^2})}}{e^{g(\sqrt{1-Y_i^2})}+e^{g(-\sqrt{1-Y_i^2})}}$$

$$+\frac{((g'(-\sqrt{1-Y_i^2}))^2+g''(-\sqrt{1-Y_i^2}))e^{g(-\sqrt{1-Y_i^2})}}{e^{g(\sqrt{1-Y_i^2})}+e^{g(-\sqrt{1-Y_i^2})}}\Bigg](Y_i\mathbf{U}_i^{(2)t}\boldsymbol{\xi})^2$$

$$+\frac{1}{2}\Delta\left(\sum_{i=1}^{n}\left[\frac{g'(\sqrt{1-Y_i^2})e^{g(\sqrt{1-Y_i^2})}+g'(-\sqrt{1-Y_i^2})e^{g(-\sqrt{1-Y_i^2})}}{e^{g(\sqrt{1-Y_i^2})}+e^{g(-\sqrt{1-Y_i^2})}}\right]Y_i\mathbf{U}_i^{(2)t}\boldsymbol{\xi}\right)^2$$

$$+O(\Delta^{3/2}). \tag{A.4}$$

Since $\boldsymbol{\xi}(\in S^{p-2})$ is uniform, $\int_{S^{p-2}}\boldsymbol{\xi}\,d\mu(\boldsymbol{\xi})=\mathbf{0}$ and $\int_{S^{p-2}}\boldsymbol{\xi}\boldsymbol{\xi}^t\,d\mu(\boldsymbol{\xi})=\frac{1}{p-1}\mathbf{I}_{p-1}$. Further, for skew-symmetric $g(\cdot)$ (on $[-1,1]$), we have $g'(-y)=g'(y)$ and $g''(-y)=-g''(y), \forall y\in[-1,1]$. So that writing

$$c(\sqrt{1-Y_i^2})=\frac{(e^{g(\sqrt{1-Y_i^2})}-e^{g(-\sqrt{1-Y_i^2})})}{(e^{g(\sqrt{1-Y_i^2})}+e^{g(-\sqrt{1-Y_i^2})})},\quad -1<Y_i<1, \tag{A.5}$$

we have from (A.1), (A.4) and (A.5), a formal expansion of (A.1):

$$1+\frac{1}{2(p-1)}\Delta\left\|\sum_{i=1}^{n}g'(\sqrt{1-Y_i^2})Y_i\mathbf{U}_i^{(2)}\right\|^2+\frac{1}{2(p-1)}\Delta\sum_{i=1}^{n}\{(g'(\sqrt{1-Y_i^2}))^2$$

$$+g''(\sqrt{1-Y_i^2})c(\sqrt{1-Y_i^2})\}Y_i^2$$

$$-\frac{1}{2}\Delta\sum_{i=1}^{n}\sqrt{1-Y_i^2}\,g'(\sqrt{1-Y_i^2})c(\sqrt{1-Y_i^2})+O(\Delta^{3/2})$$

$$= 1 + \frac{1}{2(p-1)}\Delta\left\{\left\|\sum_{i=1}^{n} g'(\sqrt{1-Y_i^2})Y_i\mathbf{U}_i^{(2)}\right\|^2 + \sum_{i=1}^{n}(g'(\sqrt{1-Y_i^2}))^2Y_i^2 \right.$$

$$\left. + \sum_{i=1}^{n}[Y_i^2 g''(\sqrt{1-Y_i^2}) - (p-1)\sqrt{1-Y_i^2}\, g'(\sqrt{1-Y_i^2})]c(\sqrt{1-Y_i^2})\right\}$$

$$+O(\Delta^{3/2})$$

$$= 1 + \frac{1}{2(p-1)}\Delta L_n(E^*) + O(\Delta^{3/2}), \quad \text{say,} \tag{A.6}$$

where $L_n(E^*)$ is given by (3.15). □

Proof of Lemma 5.1. If we let

$$a_n^0(u) = a_n([nu]/n), \quad 0 \leqslant u \leqslant 1, \tag{A.7}$$

and as $n \to \infty$

$$a_n(u) \to a^0(u) : \int_0^1 (a^0(u))^2\, du = A_0^2 < \infty. \tag{A.8}$$

Then, by (5.7)–(5.9), we may claim that

$$A_n^2 \to A_0^2 \quad \text{as } n \to \infty. \tag{A.9}$$

Moreover, as $n \to \infty$

$$\frac{1}{n}\mathrm{E}\left[\sum_{i=1}^{n}\sum_{j=1}^{n} b_n(R_i, R_j)\mathbf{U}_i^{(2)t}\mathbf{U}_j^{(2)}\right] = \frac{1}{n}\sum_{i=1}^{n} b_n(i,i) = B_n \to 0, \tag{A.10}$$

and

$$\mathrm{E}\left\{\frac{1}{n}\left[\sum_{i=1}^{n}\sum_{j=1}^{n} b_n(R_i, R_j)\mathbf{U}_i^{(2)t}\mathbf{U}_j^{(2)}\right]\right\}^2 = \frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n} b_n^2(i,j) \to 0. \quad \square \tag{A.11}$$

Proof of Lemma 5.2. Consider the log-likelihood ratio statistic

$$\sum_{i=1}^{n}\left\{g\left(\left(1-\frac{\lambda}{2n}\right)X_{1i} + \sqrt{\frac{\lambda}{n}\left(1-\frac{\lambda}{4n}\right)}Y_i\mathbf{U}_i^{(2)t}\boldsymbol{\xi}\right) - g(X_{1i})\right\}, \tag{A.12}$$

where $\boldsymbol{\xi} \in \Omega_{p-1}$ and $\mathbf{U}_i^{(2)} \perp X_{1i}, \forall i = 1, \ldots, n$. By routine expansion, we write the above as

$$\sqrt{\frac{\lambda}{n}}\sum_{i=1}^{n} g'(X_{1i})Y_i\mathbf{U}_i^{(2)t}\boldsymbol{\xi} - \frac{\lambda}{2n}\sum_{i=1}^{n} g'(X_{1i})X_{1i}$$

$$+\frac{\lambda}{2n}\sum_{i=1}^{n} g''(X_i)Y_i^2(\mathbf{U}_i^{(2)t}\boldsymbol{\xi})^2 + O_p(n^{-1/2}). \tag{A.13}$$

Invoking the independence of $(\mathbf{U}_i^{(2)}, Y_i)$ and X_{1i}, we notice that under H_0,

$$\sqrt{\frac{\lambda}{n}}\sum_{i=1}^{n} g'(X_{1i})Y_i\mathbf{U}_i^{(2)t}\boldsymbol{\xi} \sim \mathcal{N}(0, \lambda\sigma_1^2), \tag{A.14}$$

where

$$\begin{aligned}\sigma_1^2 &= \mathrm{E}[(g'(X_{11}))^2 Y_1^2(\mathbf{U}_1^{(2)t}\boldsymbol{\xi})^2]\\ &= \mathrm{E}[(g'(X_{11}))^2]\mathrm{E}[Y_1^2(\mathbf{U}_1^{(2)t}\boldsymbol{\xi})^2]\\ &= \mathrm{E}[(g'(X_{11}))^2]\mathrm{E}(Y_1^2)\mathrm{E}(\mathbf{U}_1^{(2)t}\boldsymbol{\xi})^2\\ &= \frac{1}{p-1}\mathrm{E}[(g'(X_{11}))^2](\mathrm{E}Y_1^2).\end{aligned} \tag{A.15}$$

Note that,

$$\begin{aligned}\frac{1}{n}\sum_{i=1}^{n} g'(X_{1i})X_{1i} &\xrightarrow{P} \mathrm{E}[g'(X_{11})X_{11}]\\ &= \int_{-1}^{1} xf'(x)(1-x^2)^{\frac{p-3}{2}}\,dx\\ &= \frac{-1}{p-1}\left[\int_{-1}^{1} f'(x)\,d(1-x^2)^{(p-1)/2}\right]\\ &= \frac{1}{p-1}\int_{-1}^{1}(1-x^2)^{(p-1)/2}f''(x)\,dx.\\ &= \frac{1}{p-1}\mathrm{E}\left[\frac{f''(X_{11})}{f(X_{11})}\right]\mathrm{E}(Y_1^2)\end{aligned} \tag{A.16}$$

and

$$\begin{aligned}\frac{1}{n}\sum_{i=1}^{n} g''(X_{1i})Y_i^2(\mathbf{U}_1^{(2)t}\boldsymbol{\xi})^2 &\xrightarrow{\mathrm{P}} \mathrm{E}[g''(X_{11})Y_1^2(\mathbf{U}_1^{(2)t}\boldsymbol{\xi})^2]\\ &= \mathrm{E}g''(X_{11})\mathrm{E}(Y_1^2)\mathrm{E}(\mathbf{U}_1^{(2)t}\boldsymbol{\xi})^2\\ &= \frac{1}{p-1}\mathrm{E}g''(X_{11})\mathrm{E}(Y_1^2),\end{aligned} \tag{A.17}$$

where

$$g'(x) = f'(x)/f(x), \quad g''(x) = \frac{f''(x)}{f(x)} - (g'(x))^2. \tag{A.18}$$

Thus,

$$\mathrm{E}g''(X_{11}) = \mathrm{E}\left[\frac{f''(X_{11})}{f(X_{11})}\right] - \mathrm{E}[g'(X_{11})]^2. \tag{A.19}$$

Therefore,

$$\frac{\lambda}{2}\left\{\frac{1}{n}\sum_{i=1}^{n}[g''(X_{1i})Y_i^2(\mathbf{U}_i^{(2)t}\boldsymbol{\xi})^2 - g'(X_{1i}X_{1i})]\right\}$$

$$\xrightarrow{P} \frac{\lambda}{2}\left[\frac{1}{p-1}\left[\mathrm{E}\left\{\frac{f''(X_{11})}{f(X_{11})}\right\} - \mathrm{E}(g'(X_{11}))^2\right]\mathrm{E}(Y_1^2) - \frac{1}{p-1}\mathrm{E}\left\{\frac{f''(X_{11})}{f(X_{11})}\right\}\mathrm{E}(Y_1^2)\right]$$

$$= -\frac{\lambda}{2}\sigma_1^2. \qquad \square \tag{A.20}$$

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