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Journal of Multivariate Analysis 89 (2004) 87–96
Journal of Multivariate Analysis
http://www.elsevier.com/locate/jmva

On inadmissibility of Hotelling T^2-tests for restricted alternatives

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Received 31 July 2000

Abstract

For multinormal distributions, testing against a global shift alternative, the Hotelling T^2-test is uniformly most powerful invariant, and hence admissible. For testing against restricted alternatives this feature may no longer be true. It is shown that whenever the dispersion matrix is an M-matrix, Hotelling's T^2-test is inadmissible, though some union–intersection tests may not be so.

AMS 1991 subject classifications: 62H15

Keywords: Essentially complete class; Finite UIT; M-matrix; MUIT; UMP invariant

1. Introduction

Let $\mathbf{X}_1, \ldots, \mathbf{X}_n$ be n independent and identically distributed random vectors (i.i.d.r.v.) having a $p(\geqslant 2)$-variate normal distribution $N_p(\boldsymbol{\theta}, \boldsymbol{\Sigma})$, the covariance matrix $\boldsymbol{\Sigma}$ (though unknown) is assumed to be positive definite (p.d.). For the mean vector $\boldsymbol{\theta}$, consider the null hypothesis $\mathrm{H}_0 : \boldsymbol{\theta} = \mathbf{0}$ against (i) the global alternative $\mathrm{H}_1 : \boldsymbol{\theta} \neq \mathbf{0}$, and (ii) the positive orthant alternative;

$$\mathrm{H}_1^+ : \boldsymbol{\theta} \in \mathcal{O}_p^+; \quad \mathcal{O}_p^+ = \{\boldsymbol{\theta} \mid \boldsymbol{\theta} \geqslant \mathbf{0}, ||\boldsymbol{\theta}|| > 0\}. \tag{1.1}$$

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doi:10.1016/S0047-259X(03)00092-7

Let

$$\bar{\mathbf{X}}_n = n^{-1} \sum_{i=1}^{n} \mathbf{X}_i \quad \text{and} \quad \mathbf{S}_n = \sum_{i=1}^{n} (\mathbf{X}_i - \bar{\mathbf{X}}_n)(\mathbf{X}_i - \bar{\mathbf{X}}_n)', \tag{1.2}$$

and express the Hotelling T^2-statistic as

$$T_n^2 = n(n-1)\bar{\mathbf{X}}_n{}'\mathbf{S}_n^{-1}\bar{\mathbf{X}}_n. \tag{1.3}$$

For testing H_0 vs. H_1, the Hotelling T^2-test is uniformly most powerful invariant (UMPI), and hence, is also admissible (Simaika [13]); Stein [14] established the admissibility of the Hotelling T^2-test by using the exponential structure of the parameter space. The UMPI character, or even the admissibility of the Hotelling T^2-test may not generally hold for restricted alternatives, such as H_1^+ in (1.1). The affine-invariance structure of the parameter space $\Theta = \{\boldsymbol{\theta} \in R^p\}$ does not hold for H_1^+, and hence, when $\boldsymbol{\Sigma}$ is arbitrary p.d., restriction to invariant tests makes little sense. As such, it is conjectured, though not formally established, that possibly some other non-(affine) invariant tests dominate Hotelling's T^2-tests, and hence, the latter is inadmissible. We consider here the hypothesis testing problem H_0 vs. H_1^+ in an important class of statistical models, where it may be apriorily known that $\boldsymbol{\Sigma}$ belongs to the class of M-matrices. Note that

$$\mathbf{A} = (a_{ij}) \text{ such that } a_{ij} \leqslant 0 \text{ for all } i \neq j \text{ is an } M\text{-matrix}$$
$$\text{if and only if } \mathbf{A}^{-1} = (a^{ij}) \text{ exists and } a^{ij} \geqslant 0, \quad \forall i,j; \tag{1.4}$$

(see Tong [15], p. 78). Some statistical models where $\boldsymbol{\Sigma}$ is an M-matrix are presented in the concluding section. Our contention is to establish that for testing H_0 vs. H_1^+, $\boldsymbol{\Sigma}$ nuisance but $\boldsymbol{\Sigma}$ an M-matrix, the Hotelling T^2-test is inadmissible, whereas some other versions of the union–intersection tests (UIT) (Roy [10]) belong to Eaton's [5] essentially complete class of tests, and hence, may perform better than the Hotelling T^2-test (at least on a part of the parameter space).

2. The main results

First, we appraise Eaton's [5] basic result on essentially complete class of test functions for testing against restricted alternatives, when the underlying density belongs to an exponential family, as in the present context. Let Φ be Eaton's essentially complete class of tests, that means for any test $\varphi^* \notin \Phi$ there exists a test $\varphi \in \Phi$ such that φ is at least as good as φ^*.

Theorem 1. *For testing* $H_0 : \boldsymbol{\theta} = \mathbf{0}$ *vs.* $H_1^+ : \boldsymbol{\theta} \in \mathcal{O}_p^+$, *whenever* $\boldsymbol{\Sigma}$ *is an M-matrix, the Hotelling* T^2*-test is inadmissible.*

Proof. First, note that in the current context, it follows from Theorem 2.4 of Brown and Marden [2] that the essentially complete class of tests is nonempty.

The posed hypothesis testing problem is invariant under the group of transformations of positive diagonal matrices, hence, for simplicity, $\mathbf{\Sigma}$ may be treated as the correlation matrix. Following Eaton [5], we define

$$\Omega_1 = \{\mathbf{\Sigma}^{-1}\boldsymbol{\theta} \mid \boldsymbol{\theta} \in R_p^+\} \backslash \{\mathbf{0}\}, \quad R_p^+ = \{\mathbf{x} \mid \mathbf{x} \geqslant \mathbf{0}\}. \tag{2.1}$$

Let $\mathscr{V} \subseteq R^p$ be the smallest closed convex cone containing Ω_1. Then the dual cone of $\mathscr{V}$ is defined as

$$\mathscr{V}^- = \{\mathbf{w} \mid \langle \mathbf{w}, \mathbf{x} \rangle_{\mathbf{\Sigma}} \leqslant 0, \ \forall \mathbf{x} \in \mathscr{V}\}. \tag{2.2}$$

At this stage, we make use of the fact that $\mathbf{\Sigma}$, though nuisance, is an M-matrix. Hence

$$\mathscr{V} = \{\boldsymbol{\theta} \mid \boldsymbol{\theta} \geqslant \mathbf{0}\} = R_p^+, \tag{2.3}$$

and its dual cone is

$$\begin{aligned} \mathscr{V}^- &= \{\mathbf{w} \mid \mathbf{x}'\mathbf{\Sigma}^{-1}\mathbf{w} \leqslant 0, \ \forall \mathbf{x} \in \mathscr{V}\} \\ &= \{\mathbf{w} \mid \mathbf{\Sigma}^{-1}\mathbf{w} \leqslant \mathbf{0}\} \\ &= \{\mathbf{w} \mid \mathbf{\Sigma}^{-1}\mathbf{w} \leqslant \mathbf{0}, \mathbf{w} \leqslant \mathbf{0}\}. \\ &\subseteq \{\mathbf{w} \mid \mathbf{w} \leqslant \mathbf{0}\} = R_p^-. \end{aligned} \tag{2.4}$$

The acceptance region of the Hotelling T^2-test is given by

$$\mathscr{A}_{T^2} = \{(\bar{\mathbf{X}}_n, \mathbf{S}_n) \mid T_n^2 \leqslant T_\alpha^2\}, \tag{2.5}$$

where T_α^2 is the upper $100\alpha\%$ point of the null hypothesis distribution of T_n^2 (which is linked to a F-distribution). Since $\mathscr{A}_{T^2}$ is an ellipsoidal set with origin $\mathbf{0}$, it is bounded while $\mathscr{V}^-$, as shown before, is unbounded. Therefore, Eaton's [5] condition is not tenable, and hence the Hotelling T^2-test is not a member of essentially complete class. □

Birnbaum [1] in the context of complete class type theorems noted that for testing H_0 vs. H_1, a test is admissible if and only if it is a generalized Bayes test. In the literature, there are admissible tests which are not generalized Bayes tests for some other hypothesis testing problems. For testing H_0 against restricted alternatives, due to the difficulty of integration over a restricted parameter space, explicit forms of generalized Bayes tests generally cannot be obtained. More often than not, it is hard to characterize whether or not some existing tests are generalized Bayes tests. For testing H_0 against H_1^+, the set of proper Bayes tests and their weak limits might only constitute a proper subset of essentially complete class of tests. As such when the covariance matrix is an M-matrix, there are tests which are the members of Eaton's [5] essentially complete class though they might not be generalized Bayes. In this vein, a finite union–intersection test (FUIT) (Roy et al. [11]) and a modified union–intersection test (MUIT) of Sen and Tsai [12] are presented below.

A test which is a member of Eaton's [5] essentially complete class of tests can be established by showing the acceptance region covers R_p^- or $(\mathscr{V}^-)$ as a subspace. We formulate a FUIT based on the one-sided coordinatedwise Student t-tests. Define $\mathbf{S}_n = (S_{nij})$ as in (1.2) and let

$$t_j = \frac{\sqrt{n(n-1)}\bar{X}_{nj}}{\sqrt{S_{njj}}}, \quad j = 1, \dots, p. \tag{2.6}$$

Corresponding to a given significance level α, define

$$\alpha^* : p\alpha^* = \alpha. \tag{2.7}$$

Let then t_{n-1,α^*} be the upper $100\alpha^*\%$ point of the Student t-distribution with $n-1$ degrees of freedom. Consider the critical region $\mathscr{W}_j = \{t_j \mid t_j \geqslant t_{n-1,\alpha^*}\}$ for $j = 1, \dots, p$. Then the critical region of the FUIT is

$$\mathscr{W}^* = \bigcup_{j=1}^{p} \mathscr{W}_j \tag{2.8}$$

and the acceptance region is

$$\mathscr{A}^* = \bigcap_{j=1}^{p} \bar{\mathscr{A}}_j, \tag{2.9}$$

where $\mathscr{A}_j = R \backslash \mathscr{W}_j$ and $\bar{\mathscr{A}}_j$ denote the closure of $\mathscr{A}_j$, $j = 1, \dots, p$. Then $\mathscr{A}^*$ is a closed convex set and $R_p^- \subseteq \mathscr{A}^*$ as long as t_{n-1,α^*} is $\geqslant 0$ or $\alpha^* \leqslant \frac{1}{2}$ (or $\alpha \leqslant 1$). Therefore, the FUIT is a size-α test for H_0 vs. H_1^+ (though not an invariant one), and is a member of the essentially complete class of tests. The FUIT can be replaced by a MUIT as formulated below.

To formulate the MUIT, first we estimate $\mathbf{\Sigma}$ under the condition that $\mathbf{\Sigma}$ is an M-matrix. For the restricted alternative problem, the likelihood function depends on $\boldsymbol{\theta}, \mathbf{\Sigma}$ both in an intricate manner, though the Wishart distribution of the global maximum likelihood estimator (MLE) of $\mathbf{\Sigma}$ given below is free from $\boldsymbol{\theta}$. For this reason, we shall work with this partial likelihood function to obtain suitable estimators of $\mathbf{\Sigma}$ which belong to the M-matrices. Note that the Wishart density is

$$f(\mathbf{S}_n, \mathbf{\Sigma}) = \frac{|\,\mathbf{S}_n\,|^{\frac{(n-p-2)}{2}} \exp(-\frac{1}{2} tr \mathbf{\Sigma}^{-1} \mathbf{S}_n)}{2^{\frac{1}{2}(n-1)p} \pi^{p(p-1)/4} \Pi_{i=1}^{p} \Gamma(\frac{n-i}{2})\, |\,\mathbf{\Sigma}\,|^{\frac{(n-1)}{2}}}. \tag{2.10}$$

We maximize $f(\mathbf{S}_n, \mathbf{\Sigma})$ with respect to $\mathbf{\Sigma}$ subjecting to (i) $\mathbf{\Sigma}^{-1}$ having all nonnegative elements (ii) $\mathbf{\Sigma}$ having all non-positive off-diagonal elements. We write $\boldsymbol{\gamma} = Vec(\mathbf{\Sigma}^{-1})$ and $\boldsymbol{\gamma}_0 = Vec(\mathbf{\Sigma})$; note that both are $\binom{p+1}{2}$ vector. Partition $\boldsymbol{\gamma}_0$ as

$$\boldsymbol{\gamma}_0 = \begin{pmatrix} \boldsymbol{\gamma}_0^0 \\ \boldsymbol{\gamma}_0^* \end{pmatrix}, \tag{2.11}$$

where $\boldsymbol{\gamma}_0^*$ contains the p diagonal elements σ_{ii}, $i = 1, \dots, p$. Thus the problem is to minimize $-f(\mathbf{S}_n, \mathbf{\Sigma})$ in (2.10) with respect to $\boldsymbol{\gamma}_0$ subjecting to the conditions that $\boldsymbol{\gamma}_0^0 \leqslant \mathbf{0}$, $\bar{\boldsymbol{\gamma}}_0^* \leqslant \mathbf{0}$ and $\bar{\boldsymbol{\gamma}} \leqslant \mathbf{0}$, where $\bar{\boldsymbol{\gamma}} = -\boldsymbol{\gamma}$ and $\bar{\boldsymbol{\gamma}}_0^* = -\boldsymbol{\gamma}_0^*$. In order to apply the Kuhn–Tucker–Lagrange (KTL) point formula (Hadley [7]) for this problem, we need some further notations. Write $\mathbf{\Sigma}^{-1} = (\sigma^{ij})$, we know that $\sigma^{ij} = \frac{cofactor\ \ of\ \ \sigma_{ij}}{|\mathbf{\Sigma}|},\ \forall i,j$. In other words, σ^{ij} are the functions of $\mathbf{\Sigma}$. Let $r = \binom{p+1}{2}$ and

$$\bar{\gamma} = \begin{pmatrix} g_1(\gamma_0) \\ g_2(\gamma_0) \\ \vdots \\ g_r(\gamma_0) \end{pmatrix}. \tag{2.12}$$

Replace the condition $\bar{\gamma} \leqslant \mathbf{0}$ by $g_j(\gamma_0) \leqslant 0,\ j = 1, \dots, r$ and then apply the KTL point formula to get the partial MLE of $\mathbf{\Sigma}$. This may not come out in a closed-form estimator, however it can be solved by non-linear programming from $\mathbf{S}_n$. We denote this solution by $(n-1)^{-1}\mathbf{S}_{0n}$ and note that $\mathbf{S}_{0n}$ based on the partial likelihood function $f(\mathbf{S}_n, \mathbf{\Sigma})$ is by construction an M-matrix. In that way, we may term it M-restricted partial MLE of $\mathbf{\Sigma}$.

Let $P = \{1, \dots, p\}$, and for every a: $\emptyset \subseteq a \subseteq P$, let a' be its complement and $|a|$ its cardinality. For each a, we partition $\bar{\mathbf{X}}_n$ and $\mathbf{S}_{0n}$ as

$$\bar{\mathbf{X}}_n = \begin{pmatrix} \bar{\mathbf{X}}_{na} \\ \bar{\mathbf{X}}_{na'} \end{pmatrix} \quad \text{and} \quad \mathbf{S}_{0n} = \begin{pmatrix} \mathbf{S}_{0naa} & \mathbf{S}_{0naa'} \\ \mathbf{S}_{0na'a} & \mathbf{S}_{0na'a'} \end{pmatrix}, \tag{2.13}$$

and write

$$\bar{\mathbf{X}}_{na:a'} = \bar{\mathbf{X}}_{na} - \mathbf{S}_{0naa'}\mathbf{S}^{-1}_{0na'a'}\bar{\mathbf{X}}_{na'}, \tag{2.14}$$

$$\mathbf{S}_{0naa:a'} = \mathbf{S}_{0naa} - \mathbf{S}_{0naa'}\mathbf{S}^{-1}_{0na'a'}\mathbf{S}_{0na'a}. \tag{2.15}$$

Further, let

$$I_{na} = 1\{\bar{\mathbf{X}}_{na:a'} > \mathbf{0}, \mathbf{S}^{-1}_{0na'a'}\bar{\mathbf{X}}_{na'} \leqslant 0\}, \tag{2.16}$$

for $\emptyset \subseteq a \subseteq P$, where $1\{\cdot\}$ denotes the indicator function. The classical UIT for testing H_0 vs. H_1^+ has been discussed in Sen and Tsai [12]. We proceed to adopt a suitable modification by defining U_n^0 as in (2.11) of Sen and Tsai [12] with $\mathbf{S}_n$ being replaced by $\mathbf{S}_{0n}$

$$U_n^0 = \sum_{\emptyset \subseteq a \subseteq P} \{n\bar{\mathbf{X}}'_{na:a'}\mathbf{S}^{-1}_{0naa:a'}\bar{\mathbf{X}}_{na:a'}\} I_{na}, \tag{2.17}$$

refer U_n^0 as the MUIT. Adopting the same proof as in (Perlman [8]) yields the upper bound for the MUIT:

$$\underset{\{\mathbf{\Sigma} \in \mathscr{M}_0\}}{\mathrm{Sup}} P_{\mathbf{0},\mathbf{\Sigma}}\{U_n^0 \geqslant c \mid \mathrm{H}_0\} = P_{\mathbf{0},\mathbf{I}}\{U_n^0 \geqslant c \mid \mathrm{H}_0\}, \quad \forall c \geqslant 0, \tag{2.18}$$

where $\mathscr{M}_0$ denotes the group of M-matrices. Thus the MUIT is a size-α test, the size being attained in the independent case.

Corresponding to a preassigned α $(0 < \alpha < 1)$, let c_α be the critical level, obtained by equating the right hand side of (2.18) to α, and let $\mathscr{A}_0$ be the acceptance region formed by letting in (2.17) $U_n^0 \leqslant c_\alpha$. Partition the sample space R^p into $\bigcup_a I_{0na}$, and for each a, $\emptyset \subseteq a \subseteq P$, let $\mathscr{A}_{0a} = \{\bar{\mathbf{X}}_n \mid n\bar{\mathbf{X}}'_{na:a'}\mathbf{S}^{-1}_{0naa:a'}\bar{\mathbf{X}}_{na:a'} \leqslant c_\alpha\} I_{0na}$. Treating

$\mathscr{V}_n^- = \{\bar{\mathbf{X}}_n \mid \mathbf{S}_{0n}^{-1}\bar{\mathbf{X}}_n \leqslant \mathbf{0}\} = I_{0n\emptyset}$ as the skeleton (pivotal set), then by (2.17) we have that $\mathscr{A}_0 = \mathscr{V}_n^- \bigcup_{\text{attach}} \mathscr{A}_{0a}$, where $\bigcup_{\text{attach}}$ means that for each a, $\emptyset \subset a \subseteq P$, the hyperspace $\mathscr{A}_{0a}$ is attached to the boundary of $\mathscr{V}_n^-$ on the subspace I_{0na}. By the property that $\mathbf{S}_{0n}$ is an M-matrix, we have $\mathscr{V}^- \subseteq R_p^- \subseteq \mathscr{V}_n^- \subseteq \mathscr{A}_0 - \mathbf{a}^0$ for each $\mathbf{a}^0 \in \partial\mathscr{A}_0$. Thus we arrive at the following.

Theorem 2. *In the same setup of Theorem* 1, *the MUIT belongs to Eaton's essentially complete class of tests.*

Theorem 2 does not guarantee that the MUIT is admissible. However, by virtue of Theorems 1 and 2 it is interesting to see whether the Hotelling T^2-test is dominated by the MUIT. We provide an affirmative answer for it when $\mathbf{\Sigma} = \sigma^2\Delta$, where σ^2 is an unknown scalar parameter and Δ is a known M-matrix. All the examples considered in next section belong to this class. First, note that the MUIT is the same as those of UIT and LRT for the problem of testing $\mathrm{H}_0 : \boldsymbol{\theta} = \mathbf{0}$ vs. $\mathrm{H}_1^+ : \boldsymbol{\theta} \in \mathcal{O}_p^+$, when $\mathbf{\Sigma} = \sigma^2\Delta$. We make use of Theorems 1 and 2 of Tsai [16] (where the case of known $\mathbf{\Sigma}$ was treated) to cover the present case. First, we take $\Delta = \mathbf{I}$ (the identity matrix), and note that here by (2.17) and (2.6), we have the test statistics $U_n^0 = \sum_{j=0}^{p} \sum_{\{i_1,\dots,i_j\}} (t_{i_1}^2 + \cdots + t_{i_j}^2)\ I\{\bar{X}_{ni_m} > 0,\ 1 \leqslant m \leqslant j;\ \bar{X}_{ni_m} \leqslant 0,\ j+1 \leqslant m \leqslant p\}$ and $T_n^2 = \sum_{j=1}^{p} t_j^2$, respectively, where t_j is defined in (2.6) and $\{i_1,\dots,i_j\}$ is any permutation of $\{1,\dots,j\}$. Also, note that for the power comparison of these two tests it suffices enough to consider the situation that $\boldsymbol{\theta} = (0,\dots,0,\theta_p)', \theta_p \neq 0$. Let $\lambda = \frac{\theta_p}{\sigma}$, and consider a test statistic U_n^* having the power function $P_{\lambda^2}\{U_n^* \geqslant r\} = \sum_{j=0}^{p} \binom{p}{j}(\frac{1}{2})^p P\{F_{j,n-p}(n\lambda^2) \geqslant r\}$, where the $F_{j,n-p}(n\lambda^2)$ represents noncentral F random variable with noncentrality $n\lambda^2$ and degrees of freedom j and $n-p$ ($F_{0,n-p} \equiv 0$). For any α with $0 < \alpha < 1$, let $\alpha = P_0\{U_n^* \geqslant u^*\} = P_0\{T_n^2 \geqslant T_\alpha^2\}$. Then for any $\lambda^2 > 0$, by Theorem 2.1 of Dasgupta and Perlman [4], we have $P_{\lambda^2}\{U_n^* \geqslant u^*\} > P_{\lambda^2}\{T_n^2 \geqslant T_\alpha^2\}$. Next, choose u^0 such that $P_{\mathbf{0},\sigma^2\mathbf{I}}\{U_n^0 \geqslant u^0\} = \alpha$. Then for any $\lambda > 0$,

$$
\begin{aligned}
&P_{\binom{\mathbf{0}}{\theta_p},\sigma^2\mathbf{I}}\{U_n^0 \geqslant u^0\} - P_{\lambda^2}\{U_n^* \geqslant u^*\} \\
&\quad = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \phi(x_p) \prod_{i=1}^{n} f(x_{ip}, \theta_p, \sigma^2) \prod_{i=1}^{n} dx_{ip} \\
&\qquad - \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \phi^*(x_p) \prod_{i=1}^{n} f(x_{ip}, \theta_p, \sigma^2) \prod_{i=1}^{n} dx_{ip},
\end{aligned} \tag{2.19}
$$

where

$$
\phi(x_p) = \int_{U_n^0 \geqslant u^0} \prod_{i=1}^{n} \prod_{m=1}^{p-1} f(x_{im}, 0, \sigma^2) \prod_{i=1}^{n} \prod_{m=1}^{p-1} dx_{im}, \tag{2.20}
$$

$$\phi^*(x_p) = \int_{U_n^* \geqslant u^*} \prod_{i=1}^{n} \prod_{m=1}^{p-1} f(x_{im}, 0, \sigma^2) \prod_{i=1}^{n} \prod_{m=1}^{p-1} dx_{im}, \tag{2.21}$$

and $f(x_p, \theta_p, \sigma^2)$ denotes the univariate normal density with mean θ_p and variance σ^2. Note that the facts that $f(x_p, \theta_p, \sigma^2)$ has the monotone likelihood ratio property in x_p and $\phi(x_p)$ is more righted-titled than $\phi^*(x_p)$. Hence by Theorem 2.2 of Chatterjee and De [3], we may conclude that $P_{\binom{\mathbf{0}}{\theta_p},\sigma^2\mathbf{I}}\{U_n^0 \geqslant u^0\} \geqslant P_{\lambda^2}\{U_n^* \geqslant u^*\}$. Moreover, as Δ is an M-matrix, Δ^{-1} has all nonnegative elements, and hence, proceeding as in Theorem 2 of Tsai [16] and using the arguments similar to those in (2.19)–(2.21), it follows that his Theorem 2 remains valid in this case as well. Incidentally, there is a typo in Eq. (2.9) in [16]; it should be correctly read as $T_a = \{\Delta;\ \Delta_{aa:a'}^{-1/2} \in \mathscr{T}(|a|)\}$.

As for the totally unknown $\mathbf{\Sigma}$, we study the powers of MUIT and FUIT by some simulations, and then compare them with the corresponding powers of the Hotelling T^2-test. The critical point t_{n-1,α^*} of the FUIT can be easily obtained via (2.6) and (2.7). Note that the underlying density in the right hand side of (2.18) is multinormal with mean $\mathbf{0}$ and covariance matrix $\mathbf{I}$. Thus, we may conclude that, under the null hypothesis, $\mathbf{S}_{0n}$ has the Wishart distribution $W_p(n-1, \mathbf{I})$. As such, by Theorem 2.1 of Sen and Tsai [12] and Eq. (2.18), the critical point c_α of the MUIT can be numerically obtained by

$$\alpha = \left(\frac{1}{2}\right)^p \left[\sum_{k=1}^{p-1} \binom{p}{k} \int_0^\infty P\left\{\frac{k}{n-p} F_{k,n-p} \geqslant \frac{c_\alpha}{[1 + \frac{(p-k)}{n-p+k} t]}\right\} f_{p-k,n-p+k}(t)\, dt\right] + \left(\frac{1}{2}\right)^p P\left\{\frac{p}{n-p} F_{p,n-p} \geqslant c_\alpha\right\}, \tag{2.22}$$

where $f_{p-k,n-p+k}(t)$ is the density function of central F random variable with degrees of freedom $p-k$ and $n-p+k$. To study the powers, we consider the following three cases: (a). $n = 10, p = 5$, $\boldsymbol{\theta} = (1,2,6,0,3)', \mathbf{\Sigma} = (\sigma_{ij})$, where $\sigma_{11} = 70$, $\sigma_{22} = 10, \sigma_{33} = 30, \sigma_{44} = 20, \sigma_{55} = 50$, and $\sigma_{ij} = -5/2,\ \forall i \neq j$. (b). $n = 13, p = 6$, $\boldsymbol{\theta} = (4,5,0,1,7,3)', \mathbf{\Sigma} = (\sigma_{ij})$, where $\sigma_{11} = 39, \sigma_{22} = 13, \sigma_{33} = 91, \sigma_{44} = 26, \sigma_{55} = 78$, $\sigma_{66} = 65$, and $\sigma_{ij} = -13/5,\ \forall i \neq j$. (c). $n = 21, p = 7$, $\boldsymbol{\theta} = (1,2,7,0,3,4,5)', \mathbf{\Sigma} = (\sigma_{ij})$, where $\sigma_{11} = 21, \sigma_{22} = 42, \sigma_{33} = 63, \sigma_{44} = 84, \sigma_{55} = 105, \sigma_{66} = 126, \sigma_{77} = 147$, and $\sigma_{ij} = -7/2,\ \forall i \neq j$. First, we generate ten thousand pairs of ($\boldsymbol{\theta}$'s, $\mathbf{\Sigma}$'s), where the components are randomly generated from uniform $(-1, 1)$ distribution so that $\mathbf{\Sigma}$ is positive definite. Then for each pair $(\boldsymbol{\theta}, \mathbf{\Sigma})$, we generate n samples from $N_p(100\boldsymbol{\theta}, 100\mathbf{\Sigma})$ to get the data set of $\bar{\mathbf{X}}_n$ and $\mathbf{S}_n$. For a given $\mathbf{S}_n$, we apply the algorithm of orthant probability (Evans and Swartz [6]) to calculate the rejection probability of FUIT under the alternative and then take the average of these ten thousand rejection probabilities as the simulation power of FUIT. As for the MUIT, we first use the optimalization algorithm mentioned in Section 2 to obtain the numerical MLE $(n-1)^{-1}\mathbf{S}_{0n}$ of $\mathbf{\Sigma}$ for a given $\mathbf{S}_n$. Repeating this procedure for each $\mathbf{S}_n$, we then can obtain the corresponding ten thousand $\mathbf{S}_{0n}$'s. However, sometime it

Table 1
The powers of three tests

$\alpha = 0.1$	Case (a)	Case (b)	Case (c)
MUIT	0.9064	0.9992	0.9977
T^2	0.8059	0.9942	0.9838
FUIT	0.9660	0.9955	0.9973
FUIT	(0.3876)	(0.6565)	(0.8997)

is quite time consuming to get a more precise $\mathbf{S}_{0n}$ for a given $\mathbf{S}_n$ by using the optimalization algorithm of Matlab package. As such it is hard to simulate the empirical distribution function of $\mathbf{U}_n^0$. To overcome the difficulty, one of feasible ways is to use $\mathbf{S}_n$ instead of $\mathbf{S}_{0n}$ and assume that the corresponding $n\bar{\mathbf{X}}'_{na:a'}\mathbf{S}_{0naa:a'}^{-1}\bar{\mathbf{X}}_{na:a'}$ and $\bar{\mathbf{X}}_{na:a'}$ are independent under alternatives, $\emptyset \subseteq a \subseteq P$. As such, we can obtain the exact distribution function of the modified statistic (say $\mathbf{U}_n$) under alternatives, which turns out to be the weighted sum of convolutions of central-F distribution and noncentral-F distribution with noncentrality $n\boldsymbol{\theta}'_{a:a'}\boldsymbol{\Sigma}_{aa:a'}^{-1}\boldsymbol{\theta}_{a:a'}$, where $\boldsymbol{\theta}_{a:a'}$ and $\boldsymbol{\Sigma}_{aa:a'}$ are defined the same as in (2.14) and (2.15) with $\boldsymbol{\theta}, \boldsymbol{\Sigma}$ replacing $\bar{\mathbf{X}}_n, \mathbf{S}_n$, respectively. By incorporating the algorithm of orthant probability, we can obtain the corresponding powers. These results along with the power of the Hotelling T^2-test are presented in Table 1. Note that the values in the bracket of last row are the powers of FUIT when the samples are generated from $N_p(1000\boldsymbol{\theta}, 1000\boldsymbol{\Sigma})$ for each pair $(\boldsymbol{\theta}, \boldsymbol{\Sigma})$.

3. Some general remarks

The problem of testing H_0 vs. H_1 is invariant under the group of invariant affine transformations. For the problem of testing H_0 vs. H_1^+, generally this hypothesis testing problem is not invariant although it is invariant under two special groups of linear transformations, positive diagonal matrices and permutation matrices. As such, a canonical reduction of the noncentrality to a single coordinate may not work out, and lacking this invariance, the usual techniques fail to provide an optimality property of the usual tests. Although the Hotelling T^2-test is invariant, it is inadmissible when the covariance matrix is an M-matrix. From the power consideration of tests, the invariance principle may not be so important for testing against restricted alternatives.

Notice that for the problem of testing H_0 vs. H_1^+, when the covariance matrix is arbitrary p.d., the dual cone $\mathscr{V}^-$ in (2.2) becomes a lower dimensional subspace whose dimension is less than p. Thus the Lebesgue measure of the set $\mathscr{V}^-$ is null, and hence Theorem 4.1 of Eaton [5] is inapplicable.

In (possibly mixed-effects) randomized block designs, the covariance matrix is of the form $\boldsymbol{\Sigma} = \sigma^2[(1-\rho)\mathbf{I} + \rho\mathbf{1}\mathbf{1}']$, where $-1/(p-1) < \rho < 1$, $\mathbf{1} = (1, \ldots, 1)'$. For this intra-class correlation model, we have $\boldsymbol{\Sigma}^{-1} = \sigma^{-2}(1-\rho)^{-1}[\boldsymbol{I} - \frac{\rho}{1+(p-1)\rho}\mathbf{11}']$ and hence $\boldsymbol{\Sigma}$ is an M-matrix when ρ is non-positive.

With respect to the intra-class correlation model $\boldsymbol{\Sigma} = \sigma^2[(1-\rho)\mathbf{I} + \rho\mathbf{11}']$, against the null hypothesis of homogeneity of the mean, consider (i) the starshaped alternative $\mathrm{H}_1^s : \theta_1 \geqslant 2^{-1}(\theta_1+\theta_2) \geqslant \cdots \geqslant p^{-1}\sum_{i=1}^{p}\theta_i$; or (ii) the simple order alternative $\mathrm{H}_1^* : \theta_1 \leqslant \theta_2 \leqslant \cdots \leqslant \theta_p$. A linear transformation $\mathbf{Y} = \mathbf{CX}$ (where $\mathbf{C} = (c_{ij})$, for (i): $c_{ij} = [i(i+1)]^{-1}$ if $j \leqslant i$, $c_{ij} = -(i+1)^{-1}$ if $j - i = 1$ and $c_{ij} = 0$ otherwise, $\forall i = 1, \dots, p-1, j = 1, \dots, p$; for (ii): $c_{ii} = -1, c_{ij} = 1$ if $j - i = 1$ and $c_{ij} = 0$ otherwise, $\forall i = 1, \dots, p-1, j = 1, \dots, p$) transforms the original problem to the problem of positive orthant space (based on $\mathbf{Y}$) with the new covariance matrix $\boldsymbol{\Sigma}$ (for (i): $\boldsymbol{\Sigma}$ is diagonal; for (ii): $\boldsymbol{\Sigma} = (\sigma_{ij})$ with $\sigma_{ii} = 2\sigma^2(1-\rho), \sigma_{ij} = -\sigma^2(1-\rho), |j-i| = 1$, and $\sigma_{ij} = 0$ otherwise, $\forall i,\ j = 1, \dots, p-1$). In passing, we may note that the new covariance matrices in these models are all M-matrices, $\forall -1/(p-1) < \rho < 1$.

In many problems involving linear models, testing the null hypothesis that the regression parameter $\boldsymbol{\theta}$ is linear, the alternative hypothesis may be specified by inequality restraints. For example, after some linear transformations, we have $\mathrm{H}_0 : \boldsymbol{\theta} = \mathbf{0}$ against $\mathrm{H}_1^+ : \boldsymbol{\theta} \geqslant \mathbf{0}$. Consider a simple regression model $X_i = \theta_1 + \theta_2 u_i + e_i$ case, where e_i with $N(0, \sigma^2), i = 1, \dots, n$. Let $\hat{\theta}_1$ and $\hat{\theta}_2$ be the maximum likelihood estimators of θ_1 and θ_2, respectively, then we have $Cov(\hat{\theta}_1, \hat{\theta}_2) = \frac{-\bar{u}}{\sum_{i=1}^{n}(u_i - \bar{u})^2}\sigma^2$, where $\bar{u} = n^{-1}\sum_{i=1}^{n} u_i$. Thus it is easy to see that the covariance matrix of $\hat{\theta}_1$ and $\hat{\theta}_2$ is an M-matrix if $\bar{u} \geqslant 0$.

For testing H_0 against a restricted alternative which is a pointed closed convex cone, Perlman and Wu [9] downplayed the role of admissibility and other theoretical properties in favor of justifying likelihood ratio test (LRT). This note has nothing to do with the subjective review; however it provides a support for (restricted partial) LRT and UIT (see Perlman and Wu [9], p. 380) for the problems of testing H_0 against restricted alternatives.

Acknowledgments

We are grateful to the reviewers for their helpful comments and to Mr. Chien-Chih Huang for carrying out the numerical work reported in Table 1.

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