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On Covariance Estimators of Factor Loadings in Factor Analysis

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We report a matrix expression for the covariance matrix of MLEs of factor loadings in factor analysis. We then derive the analytical formula for covariance matrix of the covariance estimators of MLEs of factor loadings by obtaining the matrix of partial derivatives, which maps the differential of sample covariance matrix (in vector form) into the differential of the covariance estimators. © 1998 Academic Press

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1. INTRODUCTION

The purpose of this paper is to report a matrix expression for the covariance matrix of MLEs of factor loadings in factor analysis and the formula for covariance matrix for the covariance estimators of MLEs of factor loadings. For the coordinatewise expression for covariance matrix of MLEs of factor loadings, see Lawley and Maxwell [6], with some corrections by Jennrich and Thayer [5].

Consider a $p \times 1$ random vector of observations x_i $(i=1, ..., n)$ with $\mathbf{E}(x_i)=0$. Let Λ be a $p \times m$ matrix of factor loadings, f_i be a $m \times 1$ vector of factor scores, and ε_i be a $p \times 1$ vector of unique factors. Then the factor analysis model is written as $x_i = \Lambda f_i + \varepsilon_i$, $i=1, ..., n$, with the assumptions $\mathbf{E}(f_i)=0$; $\mathbf{E}(\varepsilon_i)=0$; $\mathrm{Cov}(f_i, \varepsilon_i)=\mathbf{E}(f_i \varepsilon_i')=0$; and $\mathrm{Cov}(\varepsilon_i)=\mathbf{E}(\varepsilon_i \varepsilon_i')=\Psi$, where Ψ is a positive definite diagonal matrix. Assuming the orthogonal model (where the factors are uncorrelated), the covariance matrix of x_i is expressed as $\Sigma = \Lambda\Lambda' + \Psi$.

In maximum likelihood estimation, we further assume that x_i's are random samples from the normal population $N_p(0, \Sigma)$. To remove the indeterminacy regarding orthogonal rotations, the additional side condition that $\Lambda' \Psi^{-1} \Lambda$ is diagonal is typically employed. By differentiating the log likelihood function with respect to Λ and Ψ and setting them to a null matrix, with further algebra we obtain the two equations from which MLE $\hat{\Lambda}_n$ of

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Λ is computed numerically: $\Psi = \mathrm{Diag}(S_n - \Lambda\Lambda')$ and $\Lambda = \Psi^{1/2}\Omega(\tilde{\Theta} - I_m)^{1/2}$, where S_n is the sample covariance matrix with sample size n; $\mathrm{Diag}(S)$ denotes the diagonal matrix whose elements are the diagonal elements of the square matrix S; $\tilde{\Theta}$ is the diagonal matrix whose elements are the first m largest eigenvalues of $\Psi^{-1/2}S_n\Psi^{-1/2}$, and Ω is the $p \times m$ matrix whose columns are the normalized eigenvectors corresponding to the diagonal elements of $\tilde{\Theta}$.

2. COVARIANCE MATRIX OF MLEs OF FACTOR LOADINGS

Define a $pm \times pm$ matrix $V = (n')\,\mathrm{Cov}(\hat{\lambda}_n)$, where $n' = n - 1$, $\hat{\lambda}_n = \mathrm{vec}(\hat{\Lambda}_n)$, and $\mathrm{vec}(\hat{\Lambda}_n)$ denotes the pm-dimensional column vector listing m columns of $\hat{\Lambda}_n$ starting from the first column. Then the formula for V is given by

$$V = A + 2B'EB, \tag{1}$$

where the matrix A is expressed as

$$\begin{aligned} A &= A_1 - A_2 \\ &= \{M \otimes \Sigma + (M \otimes \Lambda M)(\mathrm{diag}(K_{mm}\gamma^*))(I_m \otimes \Lambda')\} - \{A_{21} \# A'_{21} \# A_{22}\}, \end{aligned} \tag{2}$$

with

$$A_{21} = (1_m \otimes \Lambda \otimes 1'_p) - (\mathrm{diag}(\lambda))(I_m \otimes 1_p 1'_p), \tag{3}$$

$$A_{22} = (\Theta(\Theta^* - I_m)\,\Theta) \otimes 1_p 1'_p \tag{4}$$

$$\gamma^* = \mathrm{vec}((\Theta - I_m)^2\,(\Theta^* - I_m) - 1_m 1'_m + (1/2)\,I_m), \qquad (= \mathrm{vec}(\Gamma^*)) \tag{5}$$

$$M = \Theta(\Theta - I_m)^{-1}, \tag{6}$$

$$\Theta^* = \mathrm{vec}^{-1}_{m \times m}((I_m \otimes \Theta - \Theta \otimes I_m + \mathrm{diag}(\mathrm{vec}(I_m)))^{-2}\, 1_{m^2}), \tag{7}$$

(i.e., $\mathrm{vec}(\Theta^*) = (I_m \otimes \Theta - \Theta \otimes I_m + \mathrm{diag}(\mathrm{vec}(I_m)))^{-2}\, 1_{m^2}$), where $\otimes$ and $\#$ denote the Kronecker and the Hadamard products, respectively; K_{mm} is a $m^2 \times m^2$ matrix defined such that $K_{mm}\mathrm{vec}(G) = \mathrm{vec}(G')$ for any $m \times m$ matrix G; $\mathrm{diag}(z)$ denotes the diagonal matrix whose diagonal elements are vector z; and $\Theta = \Lambda'\Psi^{-1}\Lambda + I_m$ is the diagonal matrix whose elements are the first m largest eigenvalues of $\Psi^{-1/2}\Sigma\Psi^{-1/2}$. Next, the matrix B is given by

$$\begin{aligned} B &= -B_1 \# B_2 \\ &= -\{\Psi^{-2}\Lambda(\Theta - I_m)^{-1} \otimes 1'_p\} \\ &\qquad \# \{1'_m \otimes \Psi + (1'_m\Theta \otimes \Lambda)(\mathrm{diag}(K_{mm}\theta^{\#}))(I_m \otimes \Lambda')\}, \end{aligned} \tag{8}$$

where

$$\theta^{\#} = (I_m \otimes \Theta - \Theta \otimes I_m - 2\text{diag}(\text{vec}(\Theta(\Theta - I_m))))^{-1} 1_{m^2}$$
$$(= \Theta^{\#} 1_{m^2}). \tag{9}$$

The matrix expression for E has already been given by Lawley and Maxwell [6] and is

$$E = (\Phi \# \Phi)^{-1}, \tag{10}$$

where $\Phi = \Psi^{-1} - \Psi^{-1} \Lambda(\Theta - I_m)^{-1} \Lambda' \Psi^{-1}$.

3. ASYMPTOTIC NORMALITY OF COVARIANCE ESTIMATORS

Anderson and Rubin [1] established the asymptotic normality of MLEs of factor loadings and unique variances. Their theorem holds under the following two assumptions: (i) *$\Phi \# \Phi$ is nonsingular, where Φ is defined in* (10); *and* (ii) *Θ has ordered, distinct, diagonal elements.* We state a theorem regarding the asymptotic normality of covariance estimators of MLEs of factor loadings which holds under the same set of assumptions. Let vech(D) denote the column vector consisting of elements on and below the diagonal of the square matrix D, starting with the first column (cf., e.g., Searle [8]).

THEOREM 3.1. *Let the above assumptions* (i) *and* (ii) *hold. Let* $v = n' \cdot \text{vech}(\text{Cov}(\hat{\lambda}_n))$ *and let* $\hat{v}_n = n' \cdot \text{vech}(\widehat{\text{Cov}}(\hat{\lambda}_n))$ *be the estimator of* v. *Then, as* $n \to \infty$, $\hat{v} \to v$ a.s., *and* $\sqrt{n}(\hat{v}_n - v)$ *is asymptotically multinormal, though singular.*

Proof. By virtue of assumptions (i) and (ii), v is free from n. $\hat{v}_n$ is a continuous function of S_n, and v is the same function of Σ. S_n being a U-statistic (matrix), by virtue of the reverse martingale property, $S_n \to \Sigma$ a.s. as $n \to \infty$ (see, e.g., Sen and Singer [9]). The same U-statistic characterization of S_n leads to asymptotic multinormality of $\sqrt{n}(S_n - \Sigma)$. Furthermore, we can express $\sqrt{n}(\hat{v}_n - v)$ as

$$\sqrt{n}\,(\hat{v}_n - v) = \left(\frac{\partial v}{\partial \sigma'}\right) \text{vech}(\sqrt{n}(S_n - \Sigma)) + o_p(1), \tag{11}$$

where $\partial v / \partial \sigma'$ is the matrix of partial derivatives of $\hat{v}_n$ with respect to $s_n = \text{vech}(S_n)$ evaluated at $S_n = \Sigma$, and is a nonstochastic matrix depending only on the population parameters. Thus the asymptotic normality of $\sqrt{n}(\hat{v}_n - v)$ follows from differentiability of $\hat{v}_n$ in (11) and the asymptotic normality of $\sqrt{n}(s_n - \Sigma)$. Q.E.D

4. COVARIANCE MATRIX FOR COVARIANCE ESTIMATORS

In this section we give an explicit expression for the asymptotic covariance matrix for $\hat{v}_n$. The asymptotic covariance matrix for $\hat{v}_n$ is given by

$$\mathrm{Cov}(\hat{v}_n) = \left(\frac{\partial v}{\partial \sigma'}\right) (\mathrm{Cov}(s_n)) \left(\frac{\partial v}{\partial \sigma'}\right)', \tag{12}$$

where $\partial v / \partial \sigma'$ is a $(1/2)\, pm(pm+1) \times (1/2)\, p(p+1)$ matrix of partial derivatives connecting the differential of $\hat{v}_n$ and the differential of s_n such that $d\hat{v}_n = (\partial v/\partial \sigma')(ds_n)$. Note also that $\partial v/\partial \sigma'$ in (12) is evaluated at $S_n = \Sigma$, and we will omit it thereafter for notational simplicity. In the case of normal sampling, the formula for $\mathrm{Cov}(s_n)$ is $(1/(n'))\; H_p(I_{p^2} + K_{pp})(\Sigma \otimes \Sigma)\, H'_p$, where K_{pp} is defined such that $K_{pp}\mathrm{vec}(S) = \mathrm{vec}(S')$ for any $p \times p$ matrix S; $H_p = (G'_p G_p)^{-1}\, G'_p$ and G_p is defined such that $\mathrm{vec}(S) = G_p\, \mathrm{vech}(S)$ for any $p \times p$ matrix S.

4.1. *Matrix of Partial Derivatives of* $\hat{v}_n$

Our main task remaining is to report the expression for $\partial v/\partial \sigma'$ in (12). The actual derivation of the expression for $\partial v/\partial \sigma'$ is given in Hayashi and Sen [2]. We simply show the final results here. First let $a = \mathrm{vech}(A)$, $b = \mathrm{vec}(B)$, and $e = \mathrm{vech}(E)$. Then by the chain rule, the expression for $\partial v/\partial \sigma'$ is

$$\frac{\partial v}{\partial \sigma'} = \left(\frac{\partial v}{\partial a'}\right)\left(\frac{\partial a}{\partial \sigma'}\right) + \left(\frac{\partial v}{\partial b'}\right)\left(\frac{\partial b}{\partial \sigma'}\right) + \left(\frac{\partial v}{\partial e'}\right)\left(\frac{\partial e}{\partial \sigma'}\right), \tag{13}$$

where

$$\begin{aligned} \frac{\partial v}{\partial a'} &= I_{pm(pm+1)/2}, \\ \frac{\partial v}{\partial b'} &= 2H_{pm}(I_{p^2m^2} + K_{pm,\,pm})(I_{pm} \otimes B'E), \\ \frac{\partial v}{\partial e'} &= 2H_{pm}(B' \otimes B')\, G_p, \end{aligned} \tag{14}$$

and $K_{p,\,pm}$ is defined such that $\mathrm{vec}(B') = K_{p,\,pm}\mathrm{vec}(B)$; and H_{pm} is defined such that $H_{pm} = (G'_{pm} G_{pm})^{-1}\, G'_{pm}$, where G_{pm} is defined such that $\mathrm{vec}(R) = G_{pm}\, \mathrm{vech}(R)$ for any $pm \times pm$ matrix R.

4.2. *Matrices of Partial Derivatives of $\hat{a}$, $\hat{b}$, and $\hat{e}$*

In Section 2, we gave the matrix expressions for A, B, and E. Now, we give the expressions for the matrices of partial derivatives of these matrices. Let $a_1 = \text{vech}(A_1)$, $a_2 = \text{vech}(A_2)$, $b_1 = \text{vec}(B_1)$, $b_2 = \text{vec}(B_2)$, $\phi = \text{vech}(\Phi)$, $\mu = \text{vec}(M)$, $\theta = \text{vec}(\Theta)$, $\lambda = \text{vec}(\Lambda)$, and $\Psi = \text{vec}(\Psi)$. First, the matrix of partial derivatives of $\hat{a}$ is obtained immediately from Eq. (2) and is of the form

$$\frac{\partial a}{\partial \sigma'} = \frac{\partial a_1}{\partial \sigma'} - \frac{\partial a_2}{\partial \sigma'}, \tag{15}$$

where the matrix of partial derivatives of $\hat{a}_1$ is

$$\frac{\partial a_1}{\partial \sigma'} = A_{1(1)} \left(\frac{\partial \mu}{\partial \sigma'} \right) + A_{1(2)} \left(\frac{\partial \gamma^*}{\partial \sigma'} \right) + A_{1(3)} \left(\frac{\partial \lambda}{\partial \sigma'} \right) + A_{1(4)}, \tag{16}$$

with

$$\begin{aligned}
A_{1(1)} = H_{pm} \{ & (I_m \otimes K_{pm} \otimes I_p)(I_{m^2} \otimes \text{vec}(\Sigma)) \\
& + ((I_m \otimes \Lambda)(\text{diag}(K_{mm}\gamma^*)) \otimes I_{pm})(I_m \otimes K_{mm} \otimes I_p) \\
& \times (I_{m^2} \otimes \text{vec}(\Lambda M) + (\mu \otimes I_{pm})(I_m \otimes \Lambda)) \}, \\
A_{1(2)} = H_{pm} & (I_m \otimes \Lambda \otimes M \otimes \Lambda M)\, K^*_{m^2} K_{mm}, \\
A_{1(3)} = H_{pm} \{ & ((I_m \otimes \Lambda)(\text{diag}(K_{mm}\gamma^*)) \otimes I_{pm})(I_m \otimes K_{mm} \otimes I_p) \\
& \times (\mu \otimes I_{pm})(M \otimes I_p) \\
& + (I_{pm} \otimes (M \otimes \Lambda M)(\text{diag}(K_{mm}\gamma^*))) \\
& \times (I_m \otimes K_{pm} \otimes I_m)(\text{vec}(I_m) \otimes I_{pm})\, K_{pm} \}, \\
A_{(1(4)} = H_{pm} & (I_m \otimes K_{pm} \otimes I_p)(\mu \otimes I_{p^2})\, G_p,
\end{aligned}$$

and $K^*_{m^2} = \sum_{i=1}^{m^2} (J_{m^2,i} J'_{m^2,i} \otimes J_{m^2,i})$ with a m^2-dimensional unit vector $J_{m^2,i}$ whose ith element is 1 and the rest are 0's. Next, the matrix of partial derivatives of $\hat{a}_2$ is

$$\frac{\partial a_2}{\partial \sigma'} = A_{2(1)} A_{21(1)} \left(\frac{\partial \lambda}{\partial \sigma'} \right) + A_{2(2)} A_{22(1)} \left(\frac{\partial \theta}{\partial \sigma'} \right), \tag{17}$$

where

$$\begin{aligned}
A_{2(1)} &= (\text{diag}(H_{pm} K_{pm,pm} a_{21} \# a_{22}))\, H_{pm} \\
&\quad + (\text{diag}(H_{pm} a_{21} \# a_{22}))\, H_{pm} K_{pm,pm}, \\
A_{2(2)} &= \text{diag}(H_{pm} a_{21} \# H_{pm} K_{pm,pm} a_{21}),
\end{aligned}$$

$$A_{21(1)} = (I_m \otimes K_{p,\,pm})(I_{pm^2} \otimes 1_p)(K_{mm} \otimes I_p)(1_m \otimes I_{pm}) - (I_m \otimes 1_p 1_p' \otimes I_{pm})\, K^*_{pm},$$

$$A_{22(1)} = H_{pm}(I_m \otimes K_{pm} \otimes I_p)(I_{m^2} \otimes 1_{p^2}) \times \{\Theta(\Theta^* - I_m) \otimes I_m + I_m \otimes \Theta(\Theta^* - I_m) + (\Theta \otimes \Theta)\, T^*_{m^2}\},$$

$$T^*_{m^2} = 2(1'_{m^2}(I_m \otimes \Theta - \Theta \otimes I_m + \mathrm{diag}(\mathrm{vec}(I_m)))^{-3} \otimes I_{m^2})(I_m \otimes K_{mm} \otimes I_m) \times (I_{m^2} \otimes \mathrm{vec}(I_m) - \mathrm{vec}(I_m) \otimes I_{m^2}),$$

with $a_{21} = \mathrm{vec}(A_{21})$, $a_{22} = \mathrm{vech}(A_{22})$, and $K^*_{pm} = \sum_{i=1}^{pm} (J_{pm,\,i} J'_{pm,\,i} \otimes J_{pm,\,i})$.

Next, from (8), the matrix of partial derivatives of $\hat{b}$ is derived as

$$\frac{\partial b}{\partial \sigma'} = -(\mathrm{diag}(b_2))\left(\frac{\partial b_1}{\partial \sigma'}\right) - (\mathrm{diag}(b_1))\left(\frac{\partial b_2}{\partial \sigma'}\right), \tag{18}$$

where

$$\frac{\partial b_1}{\partial \sigma'} = B_{1(1)}\left(\frac{\partial \theta}{\partial \sigma'}\right) + B_{1(2)}\left(\frac{\partial \lambda}{\partial \sigma'}\right) + B_{1(3)}\left(\frac{\partial \psi}{\partial \sigma'}\right), \tag{19}$$

$$\frac{\partial b_2}{\partial \sigma'} = B_{2(1)}\left(\frac{\partial \theta}{\partial \sigma'}\right) + B_{2(2)}\left(\frac{\partial \lambda}{\partial \sigma'}\right) + B_{2(3)}\left(\frac{\partial \psi}{\partial \sigma'}\right), \tag{20}$$

with

$$B_{1(1)} = -(I_m \otimes K_{pp})(I_{pm} \otimes 1_p)((\Theta - I_m)^{-1} \otimes \Psi^{-2}\Lambda(\Theta - I_m)^{-1}),$$

$$B_{1(2)} = (I_m \otimes K_{pp})(I_{pm} \otimes 1_p)((\Theta - I_m)^{-1} \otimes \Psi^{-2}),$$

$$B_{1(3)} = -(I_m \otimes K_{pp})(I_{pm} \otimes 1_p)((\Theta - I_m)^{-1}\, \Lambda' \otimes I_p) \times (\Psi^{-1} \otimes \Psi^{-2} + \Psi^{-2} \otimes \Psi^{-1}),$$

$$B_{2(1)} = \{(I_m \otimes \Lambda)(\mathrm{diag}(K_{mm}\theta^{\#})) \otimes I_p\}(I_m \otimes \lambda 1'_m) + (I_m \otimes \Lambda \otimes 1'_m \Theta \otimes \Lambda)\, T_{m^2},$$

$$B_{2(2)} = \{(I_m \otimes \Lambda)(\mathrm{diag}(K_{mm}\theta^{\#})) \otimes I_p\}(\Theta 1_m \otimes I_{pm}) + \{I_{pm} \otimes ((1'_m \Theta \otimes \Lambda)(\mathrm{diag}(K_{mm}\theta^{\#})))\} \times (I_m \otimes K_{pm} \otimes I_m)(\mathrm{vec}(I_m) \otimes I_{pm})\, K_{pm},$$

$$B_{2(3)} = 1_m \otimes I_{p^2},$$

$$T_{m^2} = K^*_{m^2} K_{mm}(1'_{m^2}\Theta^{\#} \otimes \Theta^{\#}) \times \{2K^*_{m^2}(I_m \otimes (2\Theta - I_m)) + (I_m \otimes K_{mm} \otimes I_m) \times (I_{m^2} \otimes \mathrm{vec}(I_m) - \mathrm{vec}(I_m) \otimes I_{m^2})\}.$$

Finally, the matrix of partial derivatives of $\hat{e}$ is obtained from (10) and is

$$\frac{\partial e}{\partial \sigma'} = -2H_p(E \otimes E)\, G_p(\mathrm{diag}(\phi)) \left(\frac{\partial \phi}{\partial \sigma'}\right), \tag{21}$$

where

$$\frac{\partial \phi}{\partial \sigma'} = P_1 \left(\frac{\partial \theta}{\partial \sigma'}\right) + P_2 \left(\frac{\partial \lambda}{\partial \sigma'}\right) + P_3 \left(\frac{\partial \psi}{\partial \sigma'}\right), \tag{22}$$

with

$$\begin{aligned}
P_1 &= H_p(\Psi^{-1}\Lambda(\Theta - I_m)^{-1} \otimes \Psi^{-1}\Lambda(\Theta - I_m)^{-1}),\\
P_2 &= -H_p(I_{p^2} + K_{pp})(\Psi^{-1}\Lambda(\Theta - I_m)^{-1} \otimes \Psi^{-1}),\\
P_3 &= H_p(\Psi^{-1}\Lambda(\Theta - I_m)^{-1}\, \Lambda'\Psi^{-1} \otimes \Psi^{-1}\\
&\quad + \Psi^{-1} \otimes \Psi^{-1}\Lambda(\Theta - I_m)^{-1}\, \Lambda'\Psi^{-1} - \Psi^{-1} \otimes \Psi^{-1}).
\end{aligned}$$

4.3. *Matrices of Partial Derivatives of* $\hat{\gamma}^*$, $\hat{\mu}$, $\hat{\theta}$, $\hat{\lambda}_n$, *and* $\hat{\psi}_n$

The matrices of partial derivatives in Section 4.2 are expressed in terms of the matrices of partial derivatives of $\hat{\gamma}^*$, $\hat{\mu}$, $\hat{\theta}$, $\hat{\lambda}_n$, and $\hat{\psi}_n$. Thus we need to obtain the expressions for these. First, the matrices of partial derivatives of $\hat{\gamma}^*$, $\hat{\mu}$, and $\hat{\theta}$ are obtained from (5), (6), and $\Theta = \Lambda'\Psi^{-1}\Lambda + I_m$, respectively, and are as follows:

$$\frac{\partial \gamma^*}{\partial \sigma'} = (2(\Theta^* - I_m)' \otimes (\Theta - I_m) + (I_m \otimes (\Theta - I_m)^2)\, T^*_{m^2}) \left(\frac{\partial \theta}{\partial \sigma'}\right), \tag{23}$$

$$\frac{\partial \mu}{\partial \sigma'} = ((\Theta - I_m)^{-1} \otimes I_m - (\Theta - I_m)^{-1} \otimes \Theta(\Theta - I_m)^{-1}) \left(\frac{\partial \theta}{\partial \sigma'}\right), \tag{24}$$

$$\frac{\partial \theta}{\partial \sigma'} = (I_{m^2} + K_{mm})(I_m \otimes \Lambda'\Psi^{-1}) \left(\frac{\partial \lambda}{\partial \sigma'}\right) - (\Lambda'\Psi^{-1} \otimes \Lambda'\Psi^{-1}) \left(\frac{\partial \psi}{\partial \sigma'}\right). \tag{25}$$

Jennrich and Clarkson's [4] Eqs. (23), (24), (26), and (27) give the formulas that connect the differential of $\hat{\Lambda}_n$ with the differentials of $\hat{\Psi}_n$ and S_n, and the differential of $\mathrm{vdg}(\hat{\Psi}_n)$ with the differential of S_n, where $\mathrm{vdg}(\hat{\Psi}_n)$ denotes the diagonal elements of $\hat{\Psi}_n$ arranged as a vector. From these equations, we derive the matrices of partial derivatives of $\hat{\lambda}_n$ and $\hat{\psi}_n$, which are

$$\frac{\partial \lambda}{\partial \sigma'} = (W^{-1}Z \otimes I_p) \left(G_p - \left(\frac{\partial \psi}{\partial \sigma'}\right)\right) - (W^{-1} \otimes \Lambda)(Y_1 + Y_2), \tag{26}$$

$$\frac{\partial \psi}{\partial \sigma'} = K_p^*(Q \# Q)^{-1}\, K_p^{*\prime}(Q \otimes Q)\, G_p, \tag{27}$$

where

$$W = \Lambda' \Sigma^{-1} \Lambda, \quad Z = \Lambda' \Sigma^{-1}, \quad Q = I_p - \Lambda W^{-1} Z, \quad K_p^* = \sum_{i=1}^{p} (J_{p,i} J'_{p,i} \otimes J_{p,i}),$$

and

$$Y_1 = \left(\frac{1}{2}\right) (diag(vec(W^{-1})))(Z \otimes Z) \left(G_p - \left(\frac{\partial \psi}{\partial \sigma'}\right)\right), \tag{28}$$

$$Y_2 = (I_m \otimes W - W \otimes I_m)^+ \left\{((I_m - W)\, Z \otimes Z)\, G_p - (Z \otimes Z) \left(\frac{\partial \psi}{\partial \sigma'}\right)\right\}, \tag{29}$$

with the Moore–Penrose inverse $+$. Essentially the identical expression to (27), using Φ, instead of Q, is obtained by Ihara and Kano [3]. [Note that an alternative matrix formula for $Cov(\hat{\lambda}_n)$ is given by $(\partial\lambda/\partial\sigma')(Cov(s_n))(\partial\lambda/\partial\sigma')'$.]

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