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Representation of Strongly Harmonizable Periodically Correlated Processes and Their Covariances*

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This paper addresses the representation of continuous-time strongly harmonizable periodically correlated processes and their covariance functions. We show that the support of the 2-dimensional spectral measure is constrained to a set of equally spaced lines parallel to the diagonal. Our main result is that any harmonizable periodically correlated process may be represented in quadratic mean as a Fourier series whose coefficients are a family of unique jointly wide sense stationary processes; the corresponding family of cross spectral distribution functions may be simply identified from the two-dimensional spectral measure resulting from the assumption of strong harmonizability. © 1989 Academic Press, Inc.

1. INTRODUCTION

An extensive body of knowledge has been developed concerning the topic of wide sense stationary stochastic processes. Although many results are known for nonstationary processes, interpretation of the results obtained is often much less clear than for the stationary case. This paper deals with the class of periodically correlated stochastic processes which are nonstationary but in a particularly simple way and exhibit many of the properties of stationary processes. They may be said to provide a bridge between stationary and nonstationary processes.

A stochastic process $\{X(t), t \in \mathbf{I}\}$ defined on a probability space $(\Omega, \mathbf{F}, P)$

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and having finite second moments, $X(t) \in L_2(P)$ for all t, is called periodically correlated with period T if

$$E\{X(s)\, X^*(t)\} = R_X(s, t) = R_X(s+T, t+T) \tag{1a}$$

$$E\{X(t)\} = m_X(t) = m_X(t+T) \tag{1b}$$

for every $(s, t) \in \mathbf{I} \times \mathbf{I}$. If the index set $\mathbf{I}$ is taken to be the integers $\mathbf{Z}$ then the resulting process is called a periodically correlated sequence [1]. If $\mathbf{I}$ is taken to be the reals $\mathbf{R}$ then the resulting process is a continuous time periodically correlated process [2]. Except for motivating remarks we shall restrict our attention to the continuous time case; for convenience we also take $m_X(t) \equiv 0$.

Periodically correlated processes include the wide sense stationary processes, for which $R_X(s, t) = f(s-t)$ for every s, t. They also include the processes that have doubly periodic covariance functions for which there is a T satisfying $R_X(s, t) = R_X(s+mT, t+nT)$ for every s, t and m, n. Periodically correlated processes form a much larger and more interesting class than these two subclasses. For example, consider a process $X(t)$ formed by

$$X(t) = P(t)\, Z(t), \tag{2}$$

where $Z(t)$ is a wide sense stationary process and $P(t)$ is a bounded non-random periodic function with period T. Since $R_X(s, t) = P(s)\, P^*(t)\, R_Z(s-t)$, it may be concluded that $X(t)$ is periodically correlated but is not necessarily wide sense stationary, nor is $R_X(s, t)$ necessarily doubly periodic. If $Z(t)$ is continuous in quadratic mean and $P(t)$ is not continuous, then $X(t)$ will not, except in trivial cases, be continuous in quadratic mean [3].

Periodically correlated processes have also been called periodically nonstationary [4], cyclostationary [5, 6], periodically stationary [7], periodic nonstationary [8], and processes with periodic structure [9, 10]. Such processes serve for models in meteorology [10, 11], radio physics [4, 12], communication engineering [5, 6], and others [13].

The first mathematical treatment of these processes was by Gladyshev [1] although Bennett [5] discovered their characterizing property in a communication theoretic context. The representation of covariance functions of periodically correlated processes is treated by Hurd [14]. Ogura [8] introduced, using delta functions, the representation for harmonizable periodically correlated processes that is the subject of this paper. The proof of the representation, however, was left open; we provide the proof in Section 3. Gardner and Franks [6] present another representation based on the Karhunen–Loève expansion and present the solution to the Wiener–Hopf equation in terms of the functions appearing in these representations and in their covariances. Herbst [13] treats processes whose variances may be periodic in time.

Periodically correlated processes are closely related to the class of second-order processes that can be made weakly stationary by an independent uniformly distributed random time shift [15].

The prediction problem for periodically correlated sequences has been addressed by Gladyshev [1] who gives conditions for regularity and more recently by Pourahmadi and Salehi [16] and Miamee and Salehi [17] who develop a Wold decomposition. Pagano [18] discusses the connection between periodically correlated sequences and periodic moving average models; Tiao and Grupe [19] and Vecchia [20] examine periodic autoregressive moving average models.

This paper addresses the representation of strongly harmonizable periodically correlated processes and their covariance functions. If a process is strongly harmonizable (in the sense of Loève [21]), its covariance is a Fourier transform of a finite measure whereas the treatment of weakly harmonizable processes [22, 23] requires the use of bimeasure integration. *In all that follows, the reader should read harmonizable as strongly harmonizable and when we say a function is a Fourier transform we mean it may be represented as a Fourier integral with respect to a finite but possibly complex valued measure.*

We first show (Proposition 1) that the support of the 2-dimensional spectral measure for harmonizable periodically correlated processes is characteristically constrained to a set of equally spaced lines parallel to the diagonal. In a related result we show (Proposition 2) that for harmonizable periodically correlated processes the coefficient functions $B_k(\tau)$ appearing in (3) are determined by the restriction of the two-dimensional spectral measure to the kth off-diagonal line. We next show (Proposition 3) that any harmonizable process $X(t)$ may be written as a sum of harmonizable processes, which are unique provided their frequency support sets form a disjoint family whose union is the frequency support of $X(t)$. Our main result (Theorem 1) is that any harmonizable periodically correlated process may be represented in quadratic mean as a Fourier series whose coefficients are a family of unique jointly wide sense stationary processes; the corresponding family of cross spectral distribution functions may be simply identified from the two-dimensional spectral measure resulting from the assumption of harmonizability.

2. Notation and Background

If $X(t)$ is periodically correlated then $R_X(\sigma+\tau, \sigma)$ is periodic in σ with period T for fixed τ, and so the Fourier series representation

$$B(\sigma, \tau) = R_X(\sigma+\tau, \sigma) = \sum_{k \in \mathbf{Z}} B_k(\tau) \exp\left[i2\pi k \frac{\sigma}{T} \right] \tag{3}$$

is suggested. The convergence of (3) depends on the properties of $B(\sigma, \tau)$, which is uniformly continuous in (σ, τ) if $X(t)$ is harmonizable. The coefficient functions $B_k(\tau)$ are given by

$$B_k(\tau) = \frac{1}{T} \int_0^T B(\sigma, \tau) \exp\left[-i2\pi k \frac{\sigma}{T} \right] d\sigma. \tag{4}$$

Gladyshev [2] showed that an arbitrary function $R(s, t)$ satisfying (1a) is a covariance (i.e., non-negative definite) if and only if the functions

$$B_{jk}(\tau) = B_{k-j}(\tau) \exp\left[i2\pi j \frac{\tau}{T} \right] \tag{5}$$

are non-negative definite in the sense that any collection of integers $k_1, ..., k_n$ taking values in $\mathbf{Z}$, times $t_1, ..., t_n$ taking values in $\mathbf{R}$, and complex numbers $\alpha_1, ..., \alpha_n$ produces

$$\sum_{p=1}^{n} \sum_{q=1}^{n} B_{k_p k_q}(t_p - t_q)\, \alpha_p \alpha_q^* \geqslant 0. \tag{6}$$

This says that if a process $X(t)$ is periodically correlated, then the $\{B_k(\tau)\}$ determine (and are determined by) the cross covariance functions $\{B_{jk}(\tau)\}$ of a stationary vector process. Note that choosing the integers $k_1 = k_2 = \cdots = k_n = 0$ produces the result that $B_0(\tau)$ is a stationary covariance; further occurrences of the term stationary should be read as wide sense stationary.

For the discrete time case, Gladyshev points out that an equivalent condition for $R(s, t)$ to be non-negative definite is that each $B_k(\tau)$ have the representation

$$B_k(\tau) = \int_0^{2\pi} \exp(i\lambda\tau)\, dG_k(\lambda), \tag{7}$$

where the $G_k(\lambda)$ are of bounded variation and the matrix $G(\lambda)$ defined by

$$G_{jk}(\lambda) = G_{k-j}((\lambda - 2\pi j)/T) \tag{8}$$

for j, $k=0, ..., T-1$ and $0 \leqslant \lambda < 2\pi$, is Hermitian and has non-negative definite increments in the sense that

$$\sum_{p=1}^{n} \sum_{q=1}^{n} \alpha_p \alpha_q^* [G_{k_p k_q}(\lambda_2) - G_{k_p k_q}(\lambda_1)] \geqslant 0 \tag{9}$$

for any collection of integers $k_1, ..., k_n$, complex numbers $\alpha_1, ..., \alpha_n$, and $0 \leqslant \lambda_1 < \lambda_2 < 2\pi/T$.

We provide a similar statement in the continuous time case under the additional assumption that $R(s, t)$ is a Fourier transform.

We next review some facts about strongly harmonizable processes. For additional information see [21–26]. A second-order stochastic process is said to be harmonizable [21] if it can be represented in quadratic mean for every $t \in \mathbf{R}$ by the integral

$$X(t) = \int_{\mathbf{R}} \exp(i\lambda t)\, Y(d\lambda), \tag{10}$$

where Y is a random measure, that is, an additive random set function from the Borel sets of $\mathbf{R}$ to $L_2(P)$, for which the set function

$$r_Y([a, b) \times [c, d)) = E\{[y(b) - y(a)][y(d) - y(c)]^*\}$$
$$(Y(\Delta) = y(b) - y(a),\ \Delta = [a, b))$$

is of bounded variation in the plane and hence defines a finite complex measure on the Borel sets of $\mathbf{R} \times \mathbf{R}$ with

$$\int_R \int_R |r_Y(d\lambda_1 \times d\lambda_2)| < \infty. \tag{11}$$

A random process $X(t)$ is harmonizable if and only if $R_X(s, t)$ is harmonizable in the sense that it may be represented by

$$R_X(s, t) = E\{X(s)\, X^*(t)\} = \int_R \int_R \exp(i\lambda_1 s - i\lambda_2 t)\, r_Y(d\lambda_1, d\lambda_2), \tag{12}$$

where r_Y satisfies (11) [21].

The random measure Y appearing in (10) is unique to the process $X(t)$ and may be obtained from $X(t)$ by inversion in the sense that $Y([a, b))$ can be given as a quadratic mean integral of $X(t)$ (see [21, p. 474]). Further, Y defined by this inversion is countably additive on bounded half-open intervals and condition (11) permits Y to be uniquely extended to bounded Borel sets [24]; we note that condition (11) also implies that it is sufficient to consider only bounded sets as any integrals of the form (10) or (12) may be approximated arbitrarily closely by integrals over bounded sets. Similarly, r_Y is unique and can be obtained by inversion of $R_X(s, t)$. Following the standard notation we shall write $r_Y(A \times B)$ as $r_Y(A, B)$ and we note that r_Y is Hermitian nonnegative definite.

The following view of harmonizable processes, due to Cramér [24], is subsequently useful. If Y is a random measure for which (11) holds, we denote $L_2(Y) = \overline{\text{sp}}\{Y(A),\ A \text{ bounded and Borel}\}$, where $\overline{\text{sp}}$ means the closure in $L_2(P)$ of the linear span of the indicated random variables, and $\Lambda_2(r_Y)$ is the set of measurable complex functions $g(\lambda)$ for which

$$\|g\|_{r_Y}^2 = \int_R \int_R g(\lambda_1)\, g^*(\lambda_2)\, r_Y(d\lambda_1, d\lambda_2) < \infty;$$

the functions $g_1, g_2 \in \Lambda_2(r_Y)$ are considered to be identical if $\|g_1 - g_2\|_{r_Y} = 0$. For any $g \in \Lambda_2(r_Y)$,

$$U(g) = \int_R g(\lambda)\, Y(d\lambda) \tag{13}$$

defines a unitary mapping from $\Lambda_2(r_Y)$ to $L_2(Y)$ and

$$(g_1, g_2)_{L_2(Y)} = \int_R \int_R g_1(\lambda_1)\, g_2^*(\lambda_2)\, r_Y(d\lambda_1, d\lambda_2),$$

for any $g_1, g_2 \in \Lambda_2(r_Y)$. The harmonizable process determined by Y is then given by $X(t) = U[\exp(i\lambda t))]$ and it follows, because the functions $\{\exp(i\lambda t),\ t \in \mathbf{R}\}$ are dense in $\Lambda_2(r_Y)$, that $L_2(Y) = L_2(X) = \overline{\text{sp}}\{X(t),\ t \in \mathbf{R}\}$. Equation (12) expresses the preservation of the inner product under the mapping U.

3. Representation of Harmonizable Periodically Correlated Processes and Their Covariances

The harmonizable processes include all the wide sense stationary processes that are continuous in quadratic mean and a harmonizable process is stationary if and only if the support of r_Y is a subset of the diagonal ($\lambda_1 = \lambda_2$). We now show that a harmonizable process is periodically correlated if and only if the support of r_Y is contained in a set of equally spaced lines parallel to the main diagonal.

Proposition 1. *A harmonizable stochastic process $X(t)$ is periodically correlated with period T if and only if the support of r_Y is contained in*

$$S = \{(\lambda_1, \lambda_2) \in \mathbf{R} \times \mathbf{R} \colon \lambda_2 = \lambda_1 - 2\pi k/T,\ k \in \mathbf{Z}\}. \tag{14}$$

Proof. If $X(t)$ is harmonizable and r_Y has support in S defined by (14), then

$$R_X(s+T, t+T) = \int_S \int \exp(i\lambda_1 s - i\lambda_2 t) \exp(i\lambda_1 T - i\lambda_2 T)\, r_Y(d\lambda_1, d\lambda_2). \quad (15)$$

But for $(\lambda_1, \lambda_2) \in S$,

$$\exp(i\lambda_1 T - i\lambda_2 T) = 1 \quad (16)$$

so that (15) and (12) are identical.

Conversely, if $X(t)$ is harmonizable and periodically correlated with period T, then $R_X(s, t)$ satisfies (1a) and can be represented as (12). We use both hypotheses to write for every N,

$$\begin{aligned} R_X(s, t) &= \frac{1}{2N+1} \sum_{-N}^{N} R_X(s+kT, t+kT) \\ &= \frac{1}{2N+1} \sum_{-N}^{N} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp[i(s+kT)\,\lambda_1 - i(t+kT)\,\lambda_2]\, r_Y(d\lambda_1, d\lambda_2) \end{aligned} \quad (17)$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} D_N(\lambda_1, \lambda_2) \exp(is\lambda_1 - it\lambda_2)\, r_Y(d\lambda_1, d\lambda_2), \quad (18)$$

where

$$D_N(\lambda_1, \lambda_2) = \frac{\sin[(N+\frac{1}{2})\, T(\lambda_1 - \lambda_2)]}{2(N+\frac{1}{2}) \sin[T(\lambda_1 - \lambda_2)/2]}. \quad (19)$$

Observe that $D_N(\lambda_1, \lambda_2) = 1$ for $(\lambda_1, \lambda_2) \in S$. Further, $D_N(\lambda_1, \lambda_2)$ is bounded, continuous, and converges to $\mathbf{1}_S(\lambda_1, \lambda_2)$, the indicator function of the set S. Hence by the Lebesgue convergence theorem,

$$R_X(s, t) = \int_S \int \exp(i\lambda_1 s - i\lambda_2 t)\, r_Y(d\lambda_1, d\lambda_2). \quad (20)$$

This shows that $R_X(s, t)$ is determined by the integral over the set S rather than over the set $R \times R$. But since a Fourier transform of a finite measure uniquely determines the measure, we conclude $r_Y = \mathbf{1}_S r_Y$. ∎

We now show that the kth coefficient function $B_k(\tau)$, as defined by (4), is a Fourier transform with respect to a measure r_k that can be identified with the restriction of r_Y to the kth line

$$S_k = \{(\lambda_1, \lambda_2) \in \mathbf{R} \times \mathbf{R} : \lambda_2 = \lambda_1 - 2\pi k/T\}. \quad (21)$$

PROPOSITION 2. *If $X(t)$ is a harmonizable periodically correlated process with period T, then*

$$B_k(\tau)=\int_{\mathbf{R}} \exp(i\lambda\tau)\, r_k(d\lambda), \tag{22}$$

where r_k is a finite complex measure on the Borel sets of $\mathbf{R}$ *defined by*

$$r_k([a,b))=r_Y(S_k\cap\{(\lambda_1,\lambda_2): a\leqslant\lambda_1<b\}). \tag{23}$$

Proof. For any integer N, using (12) in (4) produces

$$B_k(\tau)=\int_R\int_R \frac{1}{2NT}\int_{-NT}^{NT} \times\exp\left[i\left(\lambda_1-\lambda_2-\frac{2\pi k}{T}\right)\sigma+i\lambda_1\tau\right]d\sigma\, r_Y(d\lambda_1,d\lambda_2), \tag{24}$$

where the interchange of the order of integration is justified by Fubini's theorem. The integral over the variable σ,

$$d_N(k,\lambda_1,\lambda_2)=\frac{1}{2NT}\int_{-NT}^{NT}\exp\left[i\left(\lambda_1-\lambda_2-\frac{2\pi k}{T}\right)\sigma\right]d\sigma =\frac{\sin[NT(\lambda_1-\lambda_2-2\pi k/T)]}{NT(\lambda_1-\lambda_2-2\pi k/T)} \tag{25}$$

is bounded by 1, continuous on $\mathbf{R}\times\mathbf{R}$, and converges as $N\to\infty$ to $\mathbf{1}_{S_k}(\lambda_1,\lambda_2)$, the characteristic function of the set S_k. Hence by the Lebesgue convergence theorem,

$$B_k(\tau)=\int_R\int_R \exp(i\lambda_1\tau)\,\mathbf{1}_{S_k}(\lambda_1,\lambda_2)\, r_Y(d\lambda_1,d\lambda_2). \tag{26}$$

This may be viewed as the integral of the function $\exp[i\tau M(\lambda_1,\lambda_2)]$ with respect to the measure on the Borel sets of $R\times R$ defined by $r_{Y_k}(A)=r_Y(S_k\cap A)$ and where M is the measurable transformation $M(\lambda_1,\lambda_2)=\lambda_1$. The integral (26) may then be expressed as

$$B_k(\tau)=\int_R \exp(i\lambda\tau)\, r_{Y_k}M^{-1}(d\lambda) \tag{27}$$

and we identify r_k in (22) with $r_{Y_k}M^{-1}$, which then produces (23) by applying the identification to the interval $[a,b)$. ∎

The functions $B_k(\tau)$ are still Fourier transforms under much weaker conditions; if $B(\cdot,\tau)\in L_1[0,T]$ for every τ, then every $B_k(\tau)$ has representation

(22) if and only if $B_0(\tau)$ is continuous at $\tau = 0$ [3]. But this is not enough to guarantee harmonizability because $\sum_{-\infty}^{\infty} \int_{-\infty}^{\infty} |r_k(d\lambda)|$ may diverge. For example, there exist continuous periodic functions $P(t)$ whose Fourier series is not absolutely convergent and so the covariance $R_X(s, t) = P(s)\, P^*(t)$ is continuous and bounded but not harmonizable [2]. So continuous time periodically correlated processes with continuous covariances are not all harmonizable; but all periodically correlated sequences are harmonizable because there are only a finite number of measures r_k [1].

The connection between periodically correlated processes and stationary vector processes was noted by Gladyshev who, in [1], presents two different avenues of analysis that illustrate this fact for periodically correlated sequences. The first is based on the observation that a periodically correlated process generates a family $\{X_s(n),\ n \in \mathbf{Z}, s \in \Gamma\}$ of jointly wide sense stationary sequences where $\Gamma = \{0, 1, ..., T-1\}$ for discrete and $\Gamma = [0, T)$ for continuous time. This observation has been subsequently applied to the problem of subordination and linear transformations of periodically correlated sequences (Pourahmadi and Salehi [16]). However, this point of view clouds the interpretation obtained when the time parameter evolves in the natural linear manner. Gladyshev gave an alternative representation for periodically correlated sequences that preserves the usual interpretation of time but also shows a connection to stationary vector processes. Precisely, if $X(t)$, $t \in \mathbf{Z}$, is a periodically correlated sequence, then

$$X(t) = \sum_{k=0}^{T-1} a_k(t) \exp\left[\frac{i2\pi kt}{T}\right], \tag{28}$$

where the processes $\{a_k(t)\}$ are jointly stationary sequences with cross covariance functions

$$E\{a_j(s)\, a_k^*(t)\} = \int_0^{2\pi} \exp[i\lambda(s-t)]\, G_{jk}(d\lambda) \tag{29}$$

and the $G_{jk}(\lambda)$ are defined by (7) and (8). Ogura [8] presented an extension to this representation for harmonizable continuous time periodically correlated processes; we will subsequently show this follows naturally from Propoisitions 1 and 2 and the observation that any harmonizable process can be decomposed into a sum of harmonizable processes whose random spectral measures have disjoint frequently support (although the measures are not necessarily pairwise uncorrelated).

PROPOSITION 3. *Suppose* $\{E_k = [\lambda_k, \lambda_{k+1})\}$ *is any sequence of disjoint* *intervals whose union is* $\mathbf{R}$. *Any harmonizable process* $X(t)$ *may be represented in quadratic mean by*

$$X(t) = \sum_{-\infty}^{\infty} X_k(t), \tag{30}$$

where the $X_k(t)$ *are harmonizable processes defined by*

$$X_k(t) = \int_{E_k} \exp(i\lambda t)\, Y(d\lambda) = \int_{-\infty}^{\infty} \exp(i\lambda t)\, Y_k(d\lambda) \tag{31}$$

with

$$\begin{gathered} \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |r_{jk}(d\lambda_1, d\lambda_2)| < \infty, \\ r_{jk}(d\lambda_1, d\lambda_2) = E\{Y_j(d\lambda_1)\, Y_k^*(d\lambda_2)\}. \end{gathered} \tag{32}$$

For a specific sequence of disjoint intervals $\{E_k\}$, *the* $X_k(t)$ *are unique in the sense that if* $E\{|X(t)|^2\} = 0$ *for all* $t \in \mathbf{R}$, *then* $E\{|X_k(t)|^2\} = 0$ *for all* $k \in \mathbf{Z}$ *and* $t \in \mathbf{R}$.

Proof. If $\{E_k\}$ is any sequence of disjoint intervals whose union is $\mathbf{R}$, a representation of the form (30) may be obtained by writing (10) as

$$X(t) = \sum_k \int_{E_k} \exp(i\lambda t)\, Y(d\lambda) \tag{33}$$

and making the identification (31) where $Y_k = \mathbf{1}_{E_k} Y$. The definition of Y_k leads directly to

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |r_Y(d\lambda_1, d\lambda_2)| = \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |r_{jk}(d\lambda_1, d\lambda_2)|, \tag{34}$$

where $r_{jk}(d\lambda_1, d\lambda_2)$ is defined in (32).

We observe that $X_k(t)$ is the part of the process obtained by restricting the frequency variable to the set E_k. It is possible to find representations of the form (30) in which the $X_k(t)$ are not unique if the family of intervals $\{E_k\}$ is not disjoint. However, if $X(t)$ has a representation of the form (30) in which the $X_k(t)$ are given by (31) for a *disjoint* sequence $\{E_k\}$, $\bigcup_k E_k = \mathbf{R}$, then the $X_k(t)$ are unique as in the sense stated, although they depend on the sequence of sets $\{E_k\}$. The uniqueness of the $X_k(t)$ arises essentially from the fact that r_Y is uniquely determined from $R_X(s, t)$. The proof is made clearer through the use of the unitary mapping $U: \Lambda_2(r_Y) \to L_2(X) = L_2(Y)$ defined by (13). The process is given by $X(t) = U[\exp(i\lambda t)]$

and we see that $X_k(t)$ defined by (31) is $U[\exp(i\lambda t)\,\mathbf{1}_{E_k}(\lambda)]$ for $t \in \mathbf{R}$. If $X(t)$ is the null process, then

$$0 \equiv R_X(s, t) = \int_R \int_R \exp(i\lambda_1 s - i\lambda_2 t)\, r_Y(d\lambda_1, d\lambda_2) \tag{35}$$

from which we conclude that $r_Y \equiv 0$ and hence for every k and t,

$$\begin{aligned}(X_k(t), X_k(t))_{L_2(X)} &= \int_R \int_R \mathbf{1}_{E_k}(\lambda_1)\, \mathbf{1}_{E_k}(\lambda_2) \exp(i\lambda_1 t - i\lambda_2 t)\, r_Y(d\lambda_1, d\lambda_2) \\ &= 0. \quad \blacksquare \end{aligned} \tag{36}$$

Our main result, concerning the representation of periodically correlated processes, is obtained by an application of Proposition 3 to a specific sequence $\{E_k\}$ of intervals.

THEOREM 1. *A hermonizable continuous time periodically correlated process $X(t)$ with period T may be represented in quadratic mean by*

$$X(t) = \sum_{k=-\infty}^{\infty} a_k(t) \exp\left[\frac{i2\pi kt}{T}\right], \tag{37}$$

where

$$R_{jk}(s, t) = E\{a_j(s)\, a_k^*(t)\} = \int_0^{2\pi/T} \exp[i\lambda(s-t)]\, r_{k-j}\left(\frac{2\pi j}{T} + d\lambda\right). \tag{38}$$

Remark. Note in (38) that $R_{jk}(s, t)$ depends only on $s - t$.

Proof. Define the intervals $\{E_k\}$ by $E_k = [2\pi k/T,\ 2\pi(k+1)/T)$ and apply Proposition 3 to write $X(t)$ as

$$\begin{aligned} X_k(t) &= \int_{k2\pi/T}^{(k+1)2\pi/T} \exp(i\lambda t)\ Y(d\lambda) \\ &= \exp[i2\pi kt/T] \int_0^{2\pi/T} \exp(i\lambda t)\ Y(d\lambda + 2\pi k/T) \\ &= \exp[i2\pi kt/T]\ a_k(t), \end{aligned} \tag{39}$$

where we make the identification

$$a_k(t) = \int_0^{2\pi/T} \exp(i\lambda t)\ Y(d\lambda + 2\pi k/T). \tag{40}$$

Each of the processes $a_k(t)$ is harmonizable and is determined by the measure Y over the interval E_k. The cross covariance functions $R_{jk}(s, t)$ may be determined from (40) by

$$R_{jk}(s, t) = \int_0^{2\pi/T} \int_0^{2\pi/T} \exp(i\lambda_1 s - i\lambda_2 t) \times E\{Y(d\lambda_1 + 2\pi j/T)\, Y^*(d\lambda_2 + 2\pi k/T)\}. \tag{41}$$

That is, the cross covariance between $a_j(t)$ and $a_k(s)$ is determined by the integral of r_Y over the square $[2\pi j/T, 2\pi(j+1)/T) \times [2\pi k/T, 2\pi(k+1)/T)$.

To this point, we have only assumed $X(t)$ is harmonizable and applied Proposition 3 for the specific family of intervals $\{E_k\}$. If $X(t)$ is also periodically correlated with period T, then the integral (41), over the square collapses to the diagonal of the square by Proposition 1. Since by Proposition 2 we identify the measure r_{k-j} with the restriction of r_Y to the set S_{k-j}, then we may replace (41) with

$$R_{jk}(s, t) = \int_0^{2\pi/T} \exp[i\lambda(s-t)]\, r_{k-j}(d\lambda + 2\pi jT) \tag{42}$$

which depends only on $s - t$ and so the $\{a_k(t)\}$ are jointly stationary. ▌

We note that the essentials of the proofs of Propositions 1 and 2 were contained in the author's dissertation [14]; these and Proposition 3 together with the correct choice of the sequence $\{E_k\}$ yield the preceding proof of Ogura's representation.

Another statement can be deduced from the preceding. A harmonizable process is periodically correlated with period T if and only if the processes $\{a_k(t)\}$ are jointly stationary when $E_k = [2\pi k/T, 2\pi(k+1)/T)$.

As previously remarked, it was shown by Gladyshev for the discrete time case that a function $R(s, t)$ satisfying (1a) is a covariance if and only if the $B_k(\tau)$ are represented by (7) and the $G_{jk}(\lambda)$ are non-negative definite in the sense of (9). A similar statement may be made in the continuous time case by assuming $R(s, t)$ is a Fourier transform. Since the results of Propositions 1 and 2 depend only on a function $R(s, t)$ satisfying (1a) and being a Fourier transform as in (12), we conclude that the support of r_Y for any such $R(s, t)$ is the set S of parallel lines and we obtain a sequence of measures $\{r_k\}$ as in (23). We may then express conditions for $R(s, t)$ to be a covariance in terms of this sequence of measures.

PROPOSITION 4. *Suppose $R(s, t)$ is a Fourier transform of the form* (12), *where* $R(s, t) = R(s+T, t+T)$ *for every* (s, t). *Then* $R(s, t)$ *is a covariance if and only if the family of distributions defined by*

$$G_{jk}(\lambda) = r_{k-j}([j2\pi/T, j2\pi/T + \lambda)) \tag{43}$$

for $0 \leqslant \lambda < 2\pi/T$ *is non-negative definite in the sense that*

$$\sum_{p=1}^{n} \sum_{q=1}^{n} a_p a_q^* [G_{k_p k_q}(\lambda_2) - G_{k_p k_q}(\lambda_1)] \geqslant 0 \tag{44}$$

for any collection of integers $k_1, ..., k_n$, *complex numbers* $a_1, ..., a_n$, *and* $0 \leqslant \lambda_1 < \lambda_2 < 2\pi/T$.

Proof. If $R(s, t) = R(s+T, t+T)$ is a Fourier transform and a covariance, then it is harmonizable [26] and we apply the preceding results to obtain the sequence of processes $\{a_j(t), t \in \mathbf{R}\}$, $j \in \mathbf{Z}$, which are jointly stationary as $R_{jk}(s, t)$ depends only on the difference $s - t$; further, by (40), the frequency support of each is an interval of length $2\pi/T$.

According to (41) and (42), for any (j, k) the restriction of r_Y to the square whose lower left corner has coordinate $(j2\pi/T, k2\pi/T)$ determines the cross spectral distributions (43) which necessarily satisfy (44) (see [27]).

Conversely, if $R(s, t) = R(s+T, t+T)$ is a Fourier transform of the form (12) and the $G_{jk}(\lambda)$ defined by (43) are non-negative definite in the sense of (44) then

$$R_{jk}(s-t) = \int_0^{2\pi/T} \exp[i\lambda(s-t)]\, G_{jk}(d\lambda) \tag{45}$$

defines the cross-covariance functions for a family of jointly stationary processes [27]. It follows that

$$\begin{aligned} R_N(s, t) &= \sum_{j=-N}^{N} \sum_{k=-N}^{N} \exp\left[\frac{i2\pi js}{T} - \frac{i2\pi kt}{T}\right] R_{jk}(s-t) \\ &= \int_{-N2\pi/T}^{(N+1)2\pi/T} \int_{-N2\pi/T}^{(N+1)2\pi/T} \exp(i\lambda_1 s - i\lambda_2 t)\, r_Y(d\lambda_1, d\lambda_2) \end{aligned} \tag{46}$$

is non-negative definite and converges pointwise to $R(s, t)$ as $N \to \infty$. ∎

This proposition partly answers a question posed by Ogura [8]. Do all bounded continuous covariance functions of periodically correlated processes necessarily have the representation

$$R_X(s, t) = \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \exp\left[\frac{i2\pi js}{T} - \frac{i2\pi kt}{T}\right] R_{jk}(s-t), \tag{47}$$

where $R_{jk}(s-t)$ is given by (45). This proposition states that the answer is yes provided $R_X(s, t)$ is also a Fourier transform of a finite measure, say r_Y, and in this event the convergence of (47) is absolute because

$$\sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} |R_{jk}(s-t)| \leqslant \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \int_0^{2\pi/T} |G_{jk}(d\lambda)|$$

$$= \int_R \int_R |r_Y(d\lambda_1, d\lambda_2)| < \infty.$$

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