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On the exceedance point process for a stationary sequence

by

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and

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Summary. It is known that the exceedance points of a high level by a stationary sequence are asymptotically Poisson as the level increases, under appropriate long range and local dependence conditions. When the local dependence conditions are relaxed, clustering of exceedances may occur, based on Poisson positions for the clusters. In this paper a detailed analysis of the exceedance point process is given, and shows that, under wide conditions, any limiting point process for exceedances is necessarily compound Poisson. Sufficient conditions are also given for the existence of such a limit. The limiting distributions of extreme order statistics are derived as corollaries.

Key words: extreme values, stochastic processes, exceedances, point processes.

This research has been supported by the Air Force Office of Scientific Research Grant No. F49620 82 C 0009.

1. Introduction

Many problems in extremal theory may be most naturally and profitably discussed in terms of certain underlying point processes. Typically one is interested in the limit of a sequence of point processes obtained from extremal considerations, and it is often the case that a Poisson convergence result can be derived. For example, Pickands [13], Resnick [14] and Shorrock [7] all consider point processes involving "record times" in i.i.d. settings - a research direction which was initiated by the works of Dwass ([2]) and Lamperti ([6]) on extremal processes. Resnick [15] further noted that many results in this setting can be derived from a "Complete Poisson Convergence Theorem" in two dimensions.

It is known that the i.i.d. assumption can often be relaxed. For example Leadbetter [8] considers the point process of exceedances of a high level u_n by a stationary sequence ξ_i (i.e. points where $\xi_i > u_n$), obtaining Poisson limits under quite weak dependence restrictions. These involve a long range dependence condition "$D(u_n)$" of mixing type, but much weaker than strong mixing, and a local dependence condition "$D'(u_n)$". Adler [1] generalizes Resnick's two dimensional result in [15] by assuming the conditions D and D'. In results of this kind, the long range dependence condition (e.g. $D(u_n)$) is used to give asymptotic independence of exceedances whereas the local restriction (e.g. $D'(u_n)$) avoids clustering of exceedances. As a result in the limit, the point process under consideration behaves just like one obtained from an i.i.d. sequence. If the local condition is weakened or omitted, then clustering of exceedances may occur. This clustering does not materially affect the asymptotic distribution of the maximum, but significantly changes those of all other extreme order statistics. Some such situations have been considered. For example, Rootzén [6] studies the exceedance point process for a class of stable processes. Leadbetter [9] considers Poisson results for cluster centers which yield the asymptotic distribution of the sequence maxima but not of other order statistics. Mori [12] characterizes the limit of a sequence of point processes in two dimensions under strong-mixing.

Our aim in this work is to study the detailed structure of the limiting forms of exceedance point processes under broad assumptions - especially when clustering may occur. The results yield, in particular, the asymptotic distributions of extreme order statistics in the more general form required by the presence of high local dependence.

In this paper we use the Laplace Transform functional to obtain the desired point process convergence results. The relevant definitions and basic theorem are cited in Section 2 along with a discussion of the dependence conditions used, and preliminary results. The main results, given in Section 3, both characterize all possible limits as compound Poisson processes, and provide sufficient conditions for the existence of such limits. The Laplace Transform approach is especially convenient for the main characterization result, and is therefore used here instead of the point process convergence criterion of Kallenberg which is often employed to give sufficient conditions for the existence of limits (cf. [9], [4]).

As noted above (cf. also [9]) the presence of exceedance clustering does not affect the asymptotic distribution of the maximum. It does, however, alter the asymptotic distributions of other order statistics, by virtue of the fact that e.g. the second largest value may now occur in the same cluster as the largest. In Section 4 we apply the results of Section 3 to obtain specific forms for the asymptotic distributions of extreme order statistics in terms of the relevant extreme value distributions of extreme value distribution for the maximum, and the cluster size distributions.

Finally we note that corresponding multi-level theorems and generalizations of the two-dimensional point process result of [11] may be found in the thesis [3].

2. Preliminaries and Framework

A point process η on [0,1] is a random element in the space of integer-valued Borel measures on [0,1] with the vague topology and Borel σ-field. The function $L_\eta(f) = E\exp(-\int_{[0,1]} f d\eta)$ defined on the set of non-negative measurable functions on [0,1] is said to be the Laplace Transform of η. As in the case of random variables, $L_\eta(f)$ completely determines the distribution of η. The following result is useful.

Theorem 2.1. Suppose $\eta, \eta_1, \eta_2, \ldots$ are point processes on [0,1]. Then η_n converges in distribution to η if and only if $L_{\eta_n} \to L_\eta(f)$ for each non-negative continuous function f on [0,1]. In this case $\int f d\eta_n$ converges in distribution to $\int f d\eta$ for each bounded measurable function f whose points of discontinuity constitute a set of zero η-measure a.s.

See, for example, [5] for a proof of Theorem 2.1 and a detailed account of the theory of point processes.

Throughout, $\xi_1, \xi_2, \ldots$ will be a stationary sequence of random variables. Write $M(I) = \max(\xi_i : i \in I)$ for any set I of integers, and $M_n = \max(\xi_i : 1 \leq i \leq n)$. Assume that the common distribution function F satisfies $(1-F(x))/(1-F(x-)) \to 1$ as $x \to x_F \overset{def}{=} \sup(u: F(u) < 1)$, which ensures (cf. [10], Theorem 1.1.13) the existence of a sequence $u_n^{(\tau)}$ such that

(2.1) $$1-F(u_n^{(\tau)}) \sim \tau/n \quad \text{as } n \to \infty$$

for each $\tau > 0$. Let $\chi_{n,j}^{(\tau)}$ be the indicator of the event $(\xi_j > u_n^{(\tau)}, j=1,\ldots,n)$ and $N_n^{(\tau)}$ the point process on $[0,1]$ with points $(j/n: 1 \le j \le n$ for which $\xi_j > u_n^{(\tau)})$. This is, $N_n^{(\tau)}$ is the point process (on [0,1]) of exceedances of the "level" $u_n^{(\tau)}$ by the random variables $\xi_1,\ldots\xi_n$ after "time-normalization" by the factor $1/n$. Suppose $\{u_{n,1}^{(\tau)}\}$ and $\{u_{n,2}^{(\tau)}\}$ are two different sequences satisfying (2.1), and $N_{n,1}^{(\tau)}$, $N_{n,2}^{(\tau)}$ are the corresponding point processes defined as above. Then

$$P\{N_{n,1}^{(\tau)} \neq N_{n,2}^{(\tau)}\} \le n|F(u_{n,1}^{(\tau)}) - F(u_{n,2}^{(\tau)})| \to 0 \quad \text{as } n \to \infty$$

by (2.1). Since we are only interested in weak convergence results, the choice of $\{u_n^{(\tau)}\}$ thus need not be specific, and indeed we can use any convenient $\{u_n^{(\tau)}\}$ satisfying (2.1) for our purposes.

We turn now to the type of long range dependence condition appropriate for the present context. If $\{u_n\}$ is a sequence of constants, for each n,i,j with $1 \le i \le j \le n$, define $B_i^j(u_n)$ to be the σ-field generated by the events $(\xi_s \le u_n)$, $i \le s \le j$. Also for each n and $1 \le s \le n-1$, write

$$\alpha_{n,\ell} = \max\,(|\,P(A \cap B) - P(A)P(B)|: A \in B_1^k(u_n),\ B \in B_{k+\ell}^n(u_n)).$$

$\{\xi_j\}$ is said to satisfy the condition $\Delta(u_n)$ if $\alpha_{n,\ell_n} \to 0$ as $n \to \infty$ for some sequence $\{\ell_n\}$ with $\ell_n = o(n)$. The array of constants $\alpha_{n,\ell}$, $\ell=1,2,\ldots,n-1$, will be referred to as the mixing coefficients of the condition $\Delta(u_n)$ whenever there is no danger of causing ambiguity. It is worth noting that the condition $\Delta(u_n)$ is stronger than the distributional mixing condition $D(u_n)$ (cf. [10]), but weaker than strong-mixing. For our purposes, u_n will always be $u_n^{(\tau)}$ for some $\tau > 0$. Since there are only a finite number of events involved for each n, the condition $\Lambda(u_n^{(\tau)})$ can be easily verified in some cases. Indeed, the strong mixing condition is "unnecessarily strong" for most situations in the study of extreme value theory in that it poses restrictions not just on the extremal but on the overall behavior of the underlying sequence.

The condition $\Lambda(u_n)$ can be expressed in terms of random variables as well. The following result is a special case of [18], equation (I´).

Lemma 2.2. For each n and $1 \leq \ell \leq n-1$, write

$$\beta_{n,\ell} = \sup \{|EYZ - EY \cdot EZ| : Y \text{ and } Z \text{ measurable with respect to } B_1^j(u_n) \text{ and } B_{j+\ell}^n(u_n) \text{ respectively, } 0 \leq Y, Z \leq 1, 1 \leq j \leq n-\ell.\}$$

Then $\alpha_{n,\ell} \leq \beta_{n,\ell} \leq 16\alpha_{n,\ell}$ where $\alpha_{n,\ell}$ is the mixing coefficient of the condition $\Lambda(u_n)$. In particular, ξ_j satisfies the condition $\Lambda(u_n)$ if and only if $\beta_{n,\ell_n} \to 0$ for some $\{\ell_n\}$ with $\ell_n = o(n)$.

Loynes [11] generalized the classical Extremal Types Theorem by noticing that the maxima of $\{\xi_j\}$ over appropriately chosen sets in $1,2,\ldots,n$ are asymptotically independent when $\{\xi_n\}$ is strongly mixing. The technique has been widely used in various forms since then, and the partition that we use here is similar in spirit to that in [9]. Specifically the random variables $\{\xi_i\}$ are separated into successive groups $(\xi,\ldots,\xi_{r_n})$, $(\xi_{r_{n+1}},\ldots,\xi_{2r_n})\ldots$ of r_n consecutive terms (for appropriately chosen r_n). Then all exceedances of u_n within a group are regarded as forming a cluster. The following lemma shows that the separate clusters are asymptotically independent.

Lemma 2.3 Let $\tau > 0$ be a constant and let the condition $\Lambda(u_n^{(\tau)})$ hold for the stationary sequence $\{\xi_j\}$. Suppose $\{k_n\}$ is a sequence of integers for which there exists a sequence $\{\ell_n\}$ such that

(2.2) $$k_n \ell_n / n \to 0$$

and

(2.3) $$k_n \alpha_{n,\ell_n} \to 0$$

where $\alpha_{n,\ell}$ is the mixing coefficient of the condition $\Lambda(u_n^{(\tau)})$. Then for each non-negative measurable function f on [0,1],

(2.4) $$E \exp(-\sum_{j=1}^{n} f(j/n)\chi_{n,j}^{(\tau)}) - \prod_{i=1}^{k_n} E \exp(-\sum_{j=(i-1)r_n+1}^{ir_n} f(j/n)\chi_{n,j}^{(\tau)}) \to 0$$

where $r_n = [n/k_n]$.

Proof. For simplicity of notation, write $u_n = u_n^{(\tau)}$ and $\chi_{n,j} = \chi_{n,j}^{(\tau)}$. Divide $1,2,\ldots,n$ into sets of consecutive integers $I_1, I_1^*, I_2, I_2^*, \ldots, I_{k_n}, I_{k_n}^*$ where $I_j = ((j-1)r_n+1,\ldots,jr_n-\ell_n)$, $I_j^* = (jr_n-\ell_n+1,\ldots,jr_n)$, $j=1,2,\ldots,k_n-1$, $I_{k_n} = ((k_n-1)r_n+1,\ldots,k_n r_n-\ell_n)$, $I_{k_n}^* = (k_n r_n-\ell_n+1,\ldots,n)$. Thus each set I_j contains $r_n-\ell_n$ integers, with each I_j^* except $I_{k_n}^*$ having ℓ_n integers, and $I_{k_n}^*$ having $n-k_n r_n+\ell_n \leq k_n+\ell_n$ (since $r_n=[n/k_n]$). By the non-negativity of f and since if $\chi_{nj}\neq 0$ for some $j\in I_i^*$ then $M(I_i^*) > u_n$, it is readily seen that

$$0 \leq E \exp(-\sum_{i=1}^{k_n} \sum_{j\in I_i} f(j/n)\chi_{n,j}) - E \exp(-\sum_{j=1}^{n} f(j/n)\chi_{n,j})$$

(2.5)
$$\leq (k_n-1)P\{M(I_1^*) > u_n\} + P\{M(I_{k_n}^*) > u_n\}$$
$$\leq [(k_n-1)\ell_n + (k_n+\ell_n)]\ P\{\xi_1 > u_n\}$$
$$\sim \frac{k_n(\ell_n+1)\tau}{n} \to 0 \quad \text{as} \quad n \to \infty.$$

by (2.1). It follows by Lemma 2.2 and an obvious induction that

(2.6) $$\left|E\exp\Big(-\sum_{i=1}^{k_n}\sum_{j\in I_i} f(j/n)\chi_{n,j}\Big) - \prod_{i=1}^{k_n} E\exp\Big(-\sum_{j\in I_i} f(j/n)\chi_{n,j}\Big)\right| \le 16\, k_n\alpha_{n,\ell_n}$$

which tends to zero by (2.3). Finally, using the basic inequality

(2.7) $$\left|\prod_{i=1}^{k} y_i - \prod_{i=1}^{k} x_i\right| \le \sum_{i=1}^{k} |y_i - x_i|, \quad 0 \le y_i,\ x_i \le 1, \quad i=1,2,\ldots,k,$$

we conclude that

(2.8) $$\left|\prod_{i=1}^{k_n} E\exp\Big(-\sum_{j\in I_i} f(j/n)\chi_{n,j}\Big) - \prod_{i=1}^{k_n} E\exp\Big(-\sum_{j=(i-1)r_n+1}^{ir_n} f(j/n)\chi_{n,j}\Big)\right|$$

$$\le \sum_{i=1}^{k_n}\left|E\exp\Big(-\sum_{j\in I_n} f(j/n)\chi_{n,j}\Big) - E\exp\Big(-\sum_{j=(i-1)r_n+1}^{ir_n} f(j/n)\chi_{n,j}\Big)\right|$$

$$\le k_n\ell_n P\{\xi_1 > u_n\}$$

$$\sim k_n\ell_n\tau/n \to 0 \quad \text{as } n\to\infty$$

by (2.2). The result now follows by combining (2.5), (2.6) and (2.8). □

3. Compound Poisson Convergence.

Our main purpose in this section is to characterize any distributional limit $N^{(\tau)}$ for the exceedance point processes $\{N_n^{(\tau)}\}$ when local dependence assumptions are not made and clustering of exceedances may thus occur. As noted in Section 2 (and discussed in more detail in [9]) the exceedances (if any) in each interval $(1,2,\ldots,r_n)$, $(r_n+1,\ldots,2r_n)\ldots$ may be regarded as forming the clusters, with r_n appropriately chosen. For each n the cluster size distribution may thus be regarded as the distribution of the number of exceedances in an interval which contains at least one, i.e. by stationarity

$$\pi_n\{i\} = P\{\sum_{i=1}^{r_n} X_{nj}^{(\tau)} = i \mid \sum_{j=1}^{r_n} X_{nj}^{(\tau)} > 0\}, \quad i=1,2,\ldots \quad . \tag{13.1}$$

where, as previously, $X_{nj}^{(\tau)}$ is the indicator of the event $\{\xi_j > u_n^{(\tau)}\}$. It will be shown in Theorem 3.2 that any limit in distribution for the exceedance point process $N_n^{(\tau)}$ is necessarily compound Poisson with atom sizes having distribution $\pi\{i\} = \lim_{n\to\infty} \pi_n\{i\}$. The following result is a technical lemma for use in the proof of the main theorem.

Lemma 3.1. Let $\tau > 0$ be a constant. Suppose that the condition $\Lambda(u_n^{(\tau)})$ holds for $\{\xi_j\}$ and there exists a constant $\theta \in [0,1]$ such that $\lim_{n\to\infty} P\{M_n \le u_n^{(\tau)}\} = e^{-\theta\tau}$. For a fixed continuous function f on [0,1] and a sequence $\{k_n\}$ which tends to infinity and satisfies (2.2), (2.3), define functions R_n, $\tilde{R}_n$ on [0,1] by

$$R_n(t) = \sum_{i=1}^{k_n} (1 - E\exp(-\sum_{j=(i-1)r_n+1}^{ir_n} f(j/n) X_{n,j}^{(\tau)}))\, 1_{\left(\frac{(i-1)r_n}{n}, \frac{ir_n}{n}\right]}(t) \ ,$$

$$\tilde{R}_n(t) = \sum_{i=1}^{k_n} (1 - E\exp(-f(t) \sum_{j=(i-1)r_n+1}^{ir_n} X_{n,j}^{(\tau)}))\, 1_{\left(\frac{(i-1)r_n}{n}, \frac{ir_n}{n}\right]}(t)$$

where $r_n = [n/k_n]$. Then as $n \to \infty$,

(i) $\frac{n}{r_n}(R_n(t) - \tilde{R}_n(t)) \to 0$ uniformly in t,

(ii) $\frac{n}{r_n}\tilde{R}_n(t) - \theta\tau(1 - \sum_{j=1}^{\infty} e^{-jf(t)}\pi_n\{j\})\, 1_{(0,k_n r_n/n]}(t) \to 0$ uniformly

in t, where $\pi_n\{j\}$ is defined by (3.1).

$$= \theta\tau \left(1 - \sum_{j=1}^{\infty} e^{-f(t)j}\, \pi_n\{j\}\right) 1_{(0,\frac{k_n r_n}{n}]}(t)(1 + o(1))$$

since $k_n r_n/n \to 1$, and where the o(1) term is uniform in t. The conclusion (ii) now follows at once. □

The main result is now readily obtained.

Theorem 3.2. Suppose $\tau > 0$ is a constant and the condition $\Lambda(u_n^{(\tau)})$ holds for the stationary sequence $\{\xi_j\}$. If $N_n^{(\tau)}$ converges in distribution to some point process $N^{(\tau)}$, then the latter must be a Compound Poisson Process with a Laplace Transform of the form

(3.2) $$\exp\left\{ \theta\tau \int_0^1 [1 - L(f(t))]\, dt\right\}$$

where L is the Laplace Transform of some probability measure π on $\{1,2,...\}$ and $\theta = -\frac{1}{\tau} \log \lim_{n\to\infty} P\{M_n \le u_n^{(\tau)}\} \in [0,1]$. If $\theta \neq 0$, then $\pi\{i\} = \lim_{n\to\infty} \pi_n\{i\}$ where π_n is defined by (3.1) for any sequence $\{k_n\}$ which tends to infinity and satisfies (2.2), (2.3).

Proof. Again we suppress the superscript τ for the simplicity of notation. By Theorem 2.1, the assumption that N_n converges in distribution implies that $N_n([0,1])$ converges in distribution since [0,1] has empty boundary (in itself). This implies, in particular, that $P\{M_n \le u_n\} = P\{N_n([0,1]) = 0\}$ converges as $n \to \infty$. It follows from [9], Theorem 2.2 that there exists a constant θ in [0,1] such that $P\{M_n \le u_n\} \to e^{-\theta\tau}$. If $\theta = 0$, the conclusion follows trivially. Assume now that $\theta > 0$, and let R_n and $\tilde{R}_n$ be as defined in Lemma 3.1 for a fixed non-negative continuous function f, and a sequence $\{k_n\}$ which tends to infinity and satisfies (2.2), (2.3). By Lemma 2.3, since the first term of (2.4) has a non zero limit the ratio of the two terms tends to one and hence

$$\log E \exp(-\int_{[0,1]} f dN_n) = \log E \exp(-\sum_{j=1}^{n} f(j/n)\chi_{nj})$$

$$= \sum_{i=1}^{k_n} \log E \exp(-\sum_{j=(i-1)r_n+1}^{ir_n} f(j/n)\chi_{n,j}) + o(1)$$

$$(3.3) \quad = (n/r_n) \sum_{i=1}^{k_n} (r_n/n) \log\{1 - [1-E \exp(-\sum_{j=(i-1)r_n+1}^{ir_n} f(j/n)\chi_{n,j})]\} + o(1)$$

$$= (n/r_n) \int_0^1 \log[1 - R_n(t)]\, dt + o(1).$$

Write $\psi(x) = -\log(1-x) - x$, $x \in [0,1)$, so that $\psi(x) \sim x^2/2$ as $x \to 0$. Hence for large n, $|\psi(R_n(t))| \leq R_n^2(t)$ for all $t \in [0,1]$ since clearly $R_n(t) \to 0$ uniformly in t by Lemma 3.1, showing that

$$(3.4) \qquad (n/r_n) \int_0^1 |\psi(R_n(t))|dt \leq \frac{r_n}{n} \int_0^1 [\frac{n}{r_n} R_n(t)]^2\, dt \to 0$$

since $((n/r_n)R_n(t)$ is uniformly bounded and $r_n/n \to 0$. Combining (3.3), (3.4) and Lemma 3.1, it follows that

$$\log E \exp(-\int_{[0,1]} f\, d\, N_n) = -(n/r_n) \int_0^1 R_n(t)dt - (n/r_n) \int_0^1 \psi(R_n(t))dt + o(1)$$

$$= -\theta \int_0^1 (1 - \sum_{j=1}^{\infty} e^{-f(t)j} \pi_n\{j\})dt + o(1)$$

This converges as $n \to \infty$ by the assumption that N_n converges in distribution. But this implies in particular that the limit $\lim_{n\to\infty} \sum_{j=1}^{\infty} e^{-sj}\pi_n\{j\}$ exists for each $s > 0$, which is equivalent to the existence of a measure π on $\{1,2,3,\ldots\}$ such that $\pi\{j\} = \lim_{n\to\infty} \pi_n\{j\}$, $j=1,2,\ldots$, and in this case

$$\lim_{n\to\infty} \sum_{j=1}^{\infty} e^{-sj}\pi_n\{j\} = \sum_{j=1}^{\infty} e^{-sj}\pi\{j\}, \qquad s > 0.$$

It now follows from Theorem 1.1 that

$$E \exp(-\int_{[0,1]} f dN) = \lim_{n\to\infty} E \exp(-\int_{[0,1]} f dN_n)$$

$$= \exp\{-\theta\tau \int_0^1 (1-\sum_{j=1}^{\infty} e^{-f(t)j}\pi\{j\})\, dt\}$$

where π is necessarily a probability measure. $\square$

When $\theta \neq 0$, the probability measure π in the theorem is obviously restricted to a certain class; for example, by Fatou's Lemma and stationarity,

$$\sum_{i=1}^{\infty} i\pi\{i\} = \sum_{i=1}^{\infty} i \cdot \lim_{n\to\infty} P\{\sum_{j=1}^{[n/k_n]} \chi_{n,j}^{(\tau)} = i \mid \sum_{j=1}^{[n/k_n]} \chi_{n,j}^{(\tau)} > 0\}$$

$$\leq \liminf_{n\to\infty} \sum_{i=1}^{\infty} i\, P\{\sum_{j=1}^{[n/k_n]} \chi_{n,j}^{(\tau)} = i\}/P\{\sum_{j=1}^{[n/k_n]} \chi_{n,j}^{(\tau)} > 0\}$$

$$= \liminf_{n\to\infty} (k_n/\theta\tau)\cdot E(\sum_{j=1}^{[n/k_n]} \chi_{n,j}^{(\tau)}) = \frac{1}{\theta}$$

The precise relationship between θ and π is still an open problem.

Theorem 3.2 shows that under broad conditions any limit for the exceedance point process must be compound Poisson. On the other hand, a constructive result may also be stated as follows.

Theorem 3.3. Assume that the stationary sequence $\{\xi_j\}$ satisfies the condition $\Delta(u_n^{(\tau)})$ for some $\tau > 0$ and that $\lim_{n\to\infty} P\{M_n \le u_n^{(\tau)}\} = e^{-\theta\tau}$ for some $\theta \in (0,1]$. Suppose there exists a sequence $\{k_n\}$ which tends to infinity and satisfies (2.2), (2.3), and for which the limit $\pi\{i\} = \lim_{n\to\infty} \pi_n\{i\}$ exists for each $i=1,2,\ldots$ (where $\pi_n\{i\}$ is defined by (3.1)). Then π is a probability measure, and $N_n^{(\tau)}$ converges in distribution to a Compound Poisson Process with Laplace Transform $\exp\{-\theta\tau \int_0^1 (1-\sum_{i=1}^{\infty} e^{-f(t)i}\pi\{i\})dt\}$.

Proof. The assertions follow from arguments similar to those in Theorem 3.2 provided that π is a probability, or that the family $\{\pi_n\}$ of probability measures is tight, which follows readily since $\lim_{n\to\infty} \Sigma_{i=1}^{\infty} i\pi_n\{i\} = 1/\theta < \infty$ (cf. (3.6)). □

In this result existence of the limit $\lim \pi_n(i)$ was assumed. Some particular sufficient conditions for this may be found in [4], where a result similar to Theorem 3.3 is proven by other methods.

The next result shows that when the conditions of Theorem 3.2 hold for all $\tau > 0$ then the parameter θ and the cluster size distribution for the limiting process are independent of τ.

Theorem 3.4. Suppose that for each $\tau > 0$, the stationary sequence $\{\xi_j\}$ satisfies the condition $\Delta(u_n^{(\tau)})$ and $N_n^{(\tau)}$ converges in distribution to some point process $N^{(\tau)}$. Then $N^{(\tau)}$ is a Compound Poisson Process with Laplace Transform

$\exp\{-\theta\tau\int_0^1(1 - L\circ f\,)dt\}$ where θ and L are determined as in Theorem 3.2, θ and L being independent of τ.

Proof. By the representation (3.2), it suffices to show that

$$\lim_{n\to\infty} L_{N_n^{(\tau_1)}}(f) = (\lim_{n\to\infty} L_{N_n^{(\tau_2)}}(f))^{\tau_1/\tau_2}$$ for each $\tau_1,\tau_2 > 0$ and each non-negative

continuous function f. For simplicity, we only consider the case $\tau_1=\tau<1$ and $\tau_2=1$, the proof for the other choices of τ_1,τ_2 being similar. Let f be a fixed non-negative continuous function on $[0,1]$, and

$$g(x) = \begin{cases} f(x/\tau), & 0 \le x \le \tau \\ 0 \quad , & \tau < x \le 1. \end{cases}$$

It is readily seen that

$$L_{N^{(1)}_{[n/\tau]}}(g) = E\exp(-\sum_{j=1}^{[n/\tau]} g(j/[n/\tau])X^{(1)}_{[n/\tau],j}) = E\exp(-\sum_{j=1}^{n} g(j[n/\tau])X^{(1)}_{[n/\tau],j})$$

since $(n+1)/[n/\tau] > \tau$. Thus

$$\begin{aligned} &|L_{N_n^{(\tau)}}(f) - L_{N^{(1)}_{[n/\tau]}}(g)| \\ &= \left|E\exp(-\sum_{j=1}^{n} f(j/n)X^{(\tau)}_{n,j}) - E\exp(-\sum_{j=1}^{n} g(j/[n/\tau])X^{(1)}_{[n/\tau],j})\right. \\ &\le \left|E\exp(-\sum_{j=1}^{n} f(j/n)X^{(\tau)}_{n,j}) - E\exp(-\sum_{j=1}^{n} f(j/n)X^{(1)}_{[n/\tau],j})\right| \\ &+ \left|E\exp(-\sum_{j=1}^{n} f(j/n)X^{(1)}_{[n/\tau],j}) - E\exp(-\sum_{j=1}^{n} g(j/[n/\tau])X^{(1)}_{[n/\tau],j})\right|. \end{aligned}$$

It follows from Inequality (2.7) that the two terms in the last expression are bounded by, respectively,

$$n|F(u_n^{(\tau)}) - F(u_{[n/\tau]}^{(1)})|, \qquad \sup_{1 \le j \le n} |e^{-g(j/[n/\tau])} - e^{-f(j/n)}|(\tau + o(1))$$

where both expressions tend to zero. In view of Theorem 3.2, the Laplace Transform of $N^{(1)}$ is $\exp\{-\theta \int_0^1 (1-L\circ f)\,dt\}$ for some θ and L. The above derivations imply that

$$L_{N^{(\tau)}}(f) = \lim_{n\to\infty} L_{N_n^{(\tau)}}(f) = \lim_{n\to\infty} L_{N_n^{(1)}}(g) = \exp\{-\theta \int_0^1 (1 - L\circ g)dt\}$$

$$= \exp\{-\theta\tau \int_0^1 (1 - L\circ f)dt\}$$

where the last equality holds by a change of variable. This concludes the proof. □

It is worth noting that the theorems stated in this section can be extended without further effort to considerations of joint exceedances of finitely many levels. With results of this kind, so-called "complete convergence" theorems may be obtained. A separate paper is planned on this topic.

4. Applications and Examples

First we apply our convergence results to problems that are of concern in the more traditional theory. Let $M_n^{(k)}$ be the kth largest among $\xi_1, \xi_2, \ldots, \xi_n$. It is obvious that $(M_n^{(k)} \le k-1)$ is the same event as $(N_n^{(\tau)} \le k-1)$. Using this fact, one can derive asymptotic distributions for properly normalized $M_n^{(k)}$.

Theorem 4.1. Suppose that for each $\tau > 0$, $\Lambda(u_n^{(\tau)})$ holds for $\{\xi_j\}$ and $N_n^{(\tau)}$ converges in distribution to some non-trivial point process $N^{(\tau)}$. Assume that $a_n > 0$, b_n are constants such that

$$P\{a_n(M_n - b_n) \le x\} \to G(x) \tag{4.1}$$

for some non-degenerate distribution function G_n (necessarily of extreme value type). Then for each $k = 1,2,\ldots,$

$$\lim_{n\to\infty} P\{a_n(M_n^{(k)} - b_n) \le x\}$$

$$= G(x)\left[1 + \sum_{j=1}^{k-1} \sum_{i=j}^{k-1} \frac{(-\log G(x))^j}{j!} \pi^{*j}\{i\}\right] \tag{4.2}$$

(where $G(x) > 0$, and zero where $G(x) = 0$), where for $j \ge 1$ π^{*j} is the j-fold convolution of the probability π defined by $\pi\{i\} = \lim_{n\to\infty} \pi_n\{i\}$, $i = 1,2,\ldots,$ π_n being given by (3.1) with any $\tau > 0$ and any sequence $\{k_n\}$ as described in Theorem 3.2.

Proof: According to Theorem 3.4, the Laplace Transform of $N^{(\tau)}$, $\tau > 0$, is given by (3.2) where $\theta \in (0,1]$ is such that $P\{M_n \le u_n^{(\tau)}\} \to e^{-\theta\tau}$ and L the Laplace Transform of the probability measure π stated in Theorem 3.4, θ and L being independent of τ. Since $P\{a_n(M_n - b_n) \le x\} \xrightarrow{d} G(x)$, Theorem 2.5 of [9] implies that G is one of the three extreme value type distributions, and $\lim_{n\to\infty} P\{a_n(\hat{M}_n - b_n) \le x\} = G^{1/\theta}(x)$ where $\hat{M}_n$ is the maximum of n independent random variables all having the same distributions as ξ_1. Thus

$$\lim_{n\to\infty} P\{\hat{M}_n \le a_n^{-1} G^{-1}(e^{-\theta\tau}) + b_n\} = G^{1/\theta}(G^{-1}(e^{-\theta\tau})) = e^{-\tau},$$

which shows by Theorem 1.5.1 of [10] that

$$1 - F(a_n^{-1}G^{-1}(e^{-\theta\tau}) + b_n) \sim \tau/n \quad \text{as } n \to \infty.$$

Writing $\tau(x) = -\log G^{1/\theta}(x)$, we thus have

(4.3) $$1 - F(a_n^{-1}x + b_n) \sim \tau(x)/n.$$

Now it follows from (3.2), (4.3) and the fact that $N_n^{(\tau)}([0,1]) \xrightarrow{d} N^{(\tau)}([0,1])$ (cf. Theorem 2.1) that

$$\lim_{n\to\infty} P\{a_n(M_n^{(k)} - b_n) \le x\} = \lim_{n\to\infty} P\{M_n^{(k)} \le u_n^{(\tau(x))}\}$$

$$= \lim_{n\to\infty} P\{N_n^{(\tau(x))}([0,1]) \le k-1\} = P\{N^{(\tau(x))}([0,1]) \le k-1\}$$

$$= e^{-\theta\tau(x)}[1 + \sum_{j=1}^{k-1} \frac{(\theta\tau(x))^j}{j!} \sum_{i=j}^{k-1} \pi^{*j}(i)]$$

which gives (4.2) since $e^{-\theta\tau(x)} = G(x)$. □

We end with two examples which illustrate the theory.

Example 4.2. A trivial example of a case where clustering occurs is given by $\xi_j = \max(\eta_j, \eta_{j+1})$ where $\{\eta_j\}$ is an i.i.d. sequence. In this case $\theta = 1/2$, clusters have size 2 (in the limit) and the limiting distribution (4.2) for $M_n^{(k)}$ becomes

$$\lim P\{a_n(M_n^{(k)} - b_n) \le x\} = G(x)[1 + \sum_{j=1}^{[(k-1)/2]} \frac{(-\log G(x))^j}{j!}]$$

where G, a_n, b_n are as in (4.1). This is an obvious modification of the classical result and simply reflects the fact that exceedances occur (predominantly) in pairs.

A more interesting example, with stochastic cluster sizes, is the following:

Example 4.3. Consider the sequence

$$\xi_j = \max_{k \geq 0} \rho^k Z_{j-k}$$

where $0<\rho<1$ and $\{Z_j\}$ is an i.i.d. sequence with common d.f. $\exp(-1/x)$, $x > 0$. This example was due to L. de Haan who showed that $\{\xi_j\}$ has extremal index $\theta = 1 - \rho$ ((cf. [9]), which can be any value between zero and one. It can be shown by some calculation (cf. [3], Chapter Five) that the limits (3.4) exist and are given by $\pi\{i\} = \rho^{i-1}(1-\rho)$. It then follows from Theorem 3.3 that $N_n^{(\tau)}$ converges in distribution to a Compound Poisson Process with Laplace Transform $\exp\{-(1 - \rho)\tau \int_0^1 (1- \sum_{j=1}^{\infty}\pi\{j\}e^{-jf(t)})dt\}$.

In particular the limiting cluster sizes follow a geometric distribution.

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