

Coordinated Inventory Planning for New and Old Products under Warranty

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Abstract

This paper is motivated by the inventory planning issues faced by a manufacturer of a digital projector. The seller faces demand from two sources: new demand, and demand to replace failed items under warranty. We model this setting as a multi-period single product inventory problem where the new demand in different periods are independent and the demand for replacing failed items under warranty is proportional to the number of items under warranty. We assume linear procurement, penalty and holding costs. We consider backlogging and emergency supply cases, and study both discounted cost and average cost cases. We prove the optimality of the w -dependent base stock ordering policy where the base stock level is a function of w , the number of items currently under warranty. For the special case, where the demand for new products is stationary, we prove the optimality of a stationary w -dependent base stock policy for the finite horizon discounted and the infinite horizon discounted and average cost cases. In our computational study, we find that such an integrated policy can lead to 31% average improvement in expected costs when compared to a policy that neglects warranty repairs.

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1 Introduction

After-sale parts and services are becoming an important aspect of closed loop supply chains (see Guide and Wassenhove ([8])). In cases where the original equipment manufacturer (OEM) manages the inventory for the product under warranty, the manufacturer needs to synergize inventory planning activities across new demand and demand arising from products under warranty. This work was motivated by the inventory planning issues faced by a digital projector company. The firm had a policy to replace new any product that failed under warranty. The existing inventory policy of the firm was to plan for new demand and take care of warranty claims on a as-needed basis. Figure 1 shows the demand from new customers over a one year period and Figure 2 shows the demand for warranty claims. Note that data in both figures has been scaled to protect confidentiality. On comparison of these figures it is clear that in some periods warranty claims are a significant fraction (often greater than 15%) of the total demand. This led to rush orders at the last moment resulting in high production (overtime) and transportation costs in addition to increasing the customer waiting time for repairs (replacements). The firm was interested in evaluating the benefits of coordinated inventory planning based on new demands and failures under warranty.

In this paper we consider a discrete time multi-period inventory model that jointly manages the inventory requirements for new products and warranty claims. Throughout this paper we will restrict our attention to a single product. We assume that the demand in each period for the new product is stochastic and independent (not necessarily identical) whereas warranty claims are proportional to the number of products currently under warranty. Further, a random fraction of products under warranty in the field go out of warranty every period. The ordering, holding and penalty costs are assumed to be linear and there is no delivery lag. We study the standard backlogging case and also the emergency supply case (with no backlogging) where the demand that can not be satisfied by items in stock has to be satisfied by ordering from an emergency supplier. We theoretically prove that there exist functions $S_n(w)$ so that it is optimal to order up to $S_n(w)$ in period n if there are w items under warranty at that time. We call this a w -dependent base stock policy. The critical value $S_n(w)$ can be explicitly obtained by solving for the root of a single transcendental equation. For the special case, where the demand for new products is stationary, we prove the optimality of a stationary w -dependent base

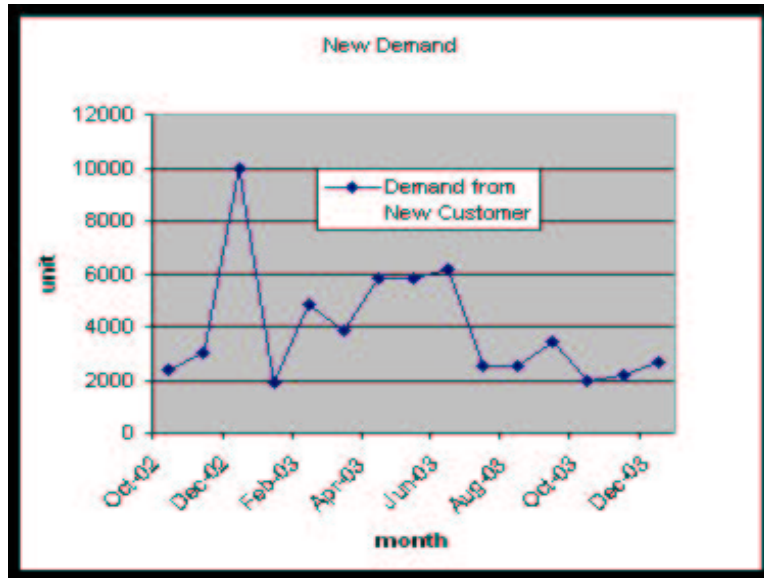


Figure 1: New Demand

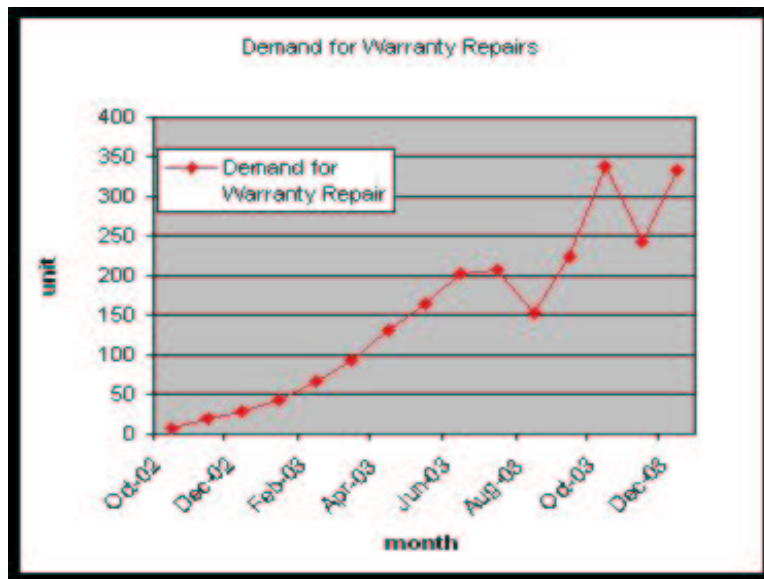


Figure 2: Demand of Warranty Repair

stock policy for the finite horizon discounted and the infinite horizon discounted and average cost cases. In our computational study, we compare the performance of such an integrated policy to a policy that only took new demand into consideration while planning inventory (reflecting the current operations at the firm). Our study indicates that on average 31% cost improvements can be obtained from using the optimal integrated policy. Among other results, we also find contrary to our intuition that the performance difference between the two policies first increases and then decreases with the failure rate.

The rest of the paper is organized as follows. We include a brief literature review of related papers in section 2. We formulate the problem in backlogging case in section 3. We study the structure of the optimal inventory policy for the finite horizon problem in section 4, infinite horizon discounted problem in section 5, and long run average cost problem in section 6. In section 7, we formulate the problem in emergency supply case and study the optimal inventory policy for finite horizon problem. In section 8, we provide computational insights. We provide extensions and conclusions in section 9.

2 Literature Review

There are three streams of research that are related to our inventory-warranty model. The first stream of research focuses on the effects of warranty under deterministic demand conditions. Porteus ([12]) considers a lot-sizing problem where the process goes out-of-control with a given probability each time it produces an item. He shows that the optimal lot size is smaller than the classical economic manufacturing quantity. Djameludin et al. ([14]) and Wang and Sheu ([13]) study other extensions of this scenario to find the optimal lot size taking into account long run production inventory and warranty costs. As opposed to this stream, our focus is on warranty systems with a periodic stochastic demand.

The second stream of literature has studied production systems with inventory dependent deterministic demand without warranty considerations. Khmelnitsky and Gerchak ([6]) study a continuous review deterministic inventory model where demand rates may vary over time, shortages are possible and the system has finite production and find the optimal production control for such a system. Baker and Urban ([1]) analyze the continuous, deterministic case of an inventory system in which the demand rate of an item is of a

polynomial functional form, dependent on the inventory level. They develop the optimal policy to maximize average profit per unit time. In our work, the future requirement of a product is not only dependent on current inventory, but also on previous sales.

The third stream of research considers inventory planning in a periodic setting under stochastic demand (see Swaminathan and Tayur ([17]) for a recent review). Within that stream of papers those that consider return or remanufacturing are related to our work. Cohen et al. ([3]) assume that a fixed fraction of the products issued in a given period is returned after a fixed sojourn time in the market and may subsequently be reused. Optimality of a periodic review order upto policy is claimed when disregarding fixed costs and procurement leadtimes. Kelle and Silver ([10]) extend this approach by allowing for fixed order costs and stochastic sojourn time in the market. They propose an approximation scheme transforming this model into a classical dynamic lotsizing problem. Yuan and Cheung ([18]) propose for this model an (s, S) -reorder policy based on the sum of the on-hand stock and the number of items in the market. The single-stage remanufacturing system was first studied by Simpson ([16]) and Inderfurth ([15]). Simpson ([16]) establishes the optimality of a three-parameter policy consisting of remanufacture-up-to, order-up-to and dispose-down-to levels. Inderfurth ([15]) extends those results to the case of positive but identical lead times for ordering and remanufacturing, and argues that if lead times are not identical then the optimal policy will be more complicated. More recently, Feinberg and Lewis ([5]) consider a single commodity inventory system in which the demand is modeled by a sequence of i.i.d random variables that can take negative values (thereby modeling some of the remanufacturing or product return settings). Multi-echelon remanufacturing system has been studied by Decroix ([4]). As opposed to this stream of research where remanufacturing or returns increase supply, in our model the demand increases when there are warranty claims. This creates additional dependence between sales in the past and demand in the future making the analysis complicated.

3 Model with Backlogging

In our model, inventory for a single product is managed for multiple periods. The firm offers replacement of items that fail under warranty. The demand arises from two sources: new demand, and demand to replace failed items under warranty. Let ζ_n be the new

demand in period n . Let $F_n(\cdot)$ be its cumulative distribution function (cdf), and $f_n(\cdot)$ be its probability density function (pdf). Let X_n be the inventory on hand, and W_n be the number of items under warranty at period n . In period n , we decide to order an amount A_n , and define $Y_n = X_n + A_n$. We treat $Y_n \geq X_n$ as the decision variable in period n . The delivery is assumed to be instantaneous, so that Y_n is the amount available to satisfy the demand for new and warranty claims in period n . Any demand that cannot be immediately satisfied is backlogged.

Here, we assume that the warranty is renewable, i.e., the warranty period of the replaced item starts afresh. Such warranty models have been studied in the past (see Blischke and Murthy ([2])). Also we assume that the demand to replace failed items is based on a proportional model i.e. a fixed fraction β of the items under warranty fail. This is consistent with other failure models that assume that items failure are independent and bernoulli (see Gertsbakh ([7])). Furthermore, in period n a fraction δ_n of the items under warranty remain in warranty, where $\{\delta_n, n \geq 0\}$ is a sequence of i.i.d. random variables $\in [0, 1]$ with common probability density $g(\cdot)$. This is only an approximation to the true system where the number of items remaining under warranty depends exclusively on the warranty-time and age distribution of items in the field. Under these assumptions, we get

$$\begin{aligned} W_{n+1} &= \delta_n[(1 - \beta)W_n + \min(Y_n, \beta W_n + \zeta_n)] \\ X_{n+1} &= Y_n - \zeta_n - \beta W_n. \end{aligned} \quad (1)$$

Thus $\{(W_n, X_n, Y_n), n \geq 0\}$ is a Markov decision process. Next we describe the cost structure. We assume that there is a per unit procurement cost c , holding cost h for each item remaining at the end of a period, and shortage cost p for each unit of backlogged demand in any period. Let

$$L_n(w, y) = p \int_{\zeta=y-\beta w}^{\infty} (\zeta - y + \beta w) f_n(\zeta) d\zeta + h \int_{\zeta=0}^{y-\beta w} (y - \zeta - \beta w) f_n(\zeta) d\zeta \quad (2)$$

represent the expected penalty and holding cost incurred in period n if $W_n = w, Y_n = y$. The total one period expected cost incurred for ordering up to y is given by

$$C_n(w, x, y) = c(y - x) + L_n(w, y). \quad (3)$$

Let $\alpha, 0 \leq \alpha \leq 1$ be the discount factor. Let π be any policy for choosing decision Y_n

at time n , based on the history up to time n . We consider three objective functions for choosing an optimal policy as described below.

1. Finite Horizon. The first objective of the firm is to minimize the expected total discounted cost (ETDC) over periods $0, 1, 2, \dots, N$. Let $V_N^\pi(w, x)$ be the ETDC of following the policy π over periods $0, 1, \dots, N$. Since we have a finite horizon we need to specify the terminal cost. Let $T(w, x)$ be the terminal cost at time N if $W_N = w, X_N = x$. Thus

$$V_N^\pi(w, x) = E_\pi\left(\sum_{n=0}^{N-1} \alpha^n C_n(W_n, X_n, Y_n) + \alpha^N T(W_N, X_N) | W_0 = w, X_0 = x\right) \quad (4)$$

Here E_π denotes the expectation under the assumption that the policy π is followed. Let $V_N(w, x)$ be the optimal ETDC of operating the system over period $0, \dots, N$. That is

$$V_N(w, x) = \inf_{\pi} V_N^\pi(w, x). \quad (5)$$

A policy π^* is called optimal for finite horizon ETDC if

$$V_N(w, x) = V_N^{\pi^*}(w, x) \quad \text{for all } w \text{ and } x. \quad (6)$$

2. Infinite Horizon ETDC. The second objective function is to minimize the ETDC over the infinite horizon. In this case there is no terminal cost function. Let $V^\pi(w, x)$ be the infinite horizon ETDC of following policy π , that is

$$V^\pi(w, x) = E_\pi\left(\sum_{n=0}^{\infty} \alpha^n C_n(W_n, X_n, Y_n) | W_0 = w, X_0 = x\right) \quad (7)$$

Similarly, let $V(w, x)$ be the optimal infinite horizon ETDC, that is

$$V(w, x) = \inf_{\pi} V^\pi(w, x). \quad (8)$$

A policy π^* is called optimal for infinite horizon ETDC if

$$V(w, x) = V^{\pi^*}(w, x) \quad \text{for all } w \text{ and } x. \quad (9)$$

3. Infinite Horizon Average Cost. Let $g^\pi(w, x)$ be the expected cost per period of following policy π over infinite horizon starting from state (w, x) . Again, there is no terminal cost in this formulation. Thus assuming the limit exists,

$$g^\pi(w, x) = \lim_{N \rightarrow \infty} \frac{1}{N+1} E_\pi\left(\sum_{n=0}^N C_n(W_n, X_n, Y_n) | W_0 = w, X_0 = x\right). \quad (10)$$

Typically, this limit is independent of the starting state (w, x) . Let $g(w, x)$ be the optimal expected cost per period over infinite horizon starting in state (w, x) . That is,

$$g(w, x) = \inf_{\pi} g^{\pi}(w, x). \quad (11)$$

Again, typically this infimum does not depend on the initial state (w, x) . A policy π^* is called optimal for infinite horizon average cost if

$$V(w, x) = V^{\pi^*}(w, x) \quad \text{for all } w \text{ and } x. \quad (12)$$

We obtain optimal policies under all three objective functions, beginning with the finite horizon ETDC in the next section.

4 Finite Horizon ETDC

In this section, we study the finite horizon problem with N periods, terminal cost $T(w, x)$, and show how to compute $V_N(w, x)$ of Equation (5). First define $V_{n,N}(w, x)$ to be the optimal ETDC over periods $n, n+1, \dots, N$ starting with $W_n = w, X_n = x$. Let $G_{n,N}(w, y)$ be the ETDC over periods over $n, n+1, \dots, N$ given $W_n = w, Y_n = y$. Then the standard dynamic programming recursion yields

$$\begin{aligned} V_{N,N}(w, x) &= T(w, x) \\ G_{n,N}(w, y) &= cy + \alpha L_n(w, y) + \alpha \int_0^1 \int_0^{y-\beta w} V_{n+1,N}(\delta(w + \zeta), y - \beta w - \zeta) f_n(\zeta) d\zeta g(\delta) d\delta \\ &\quad + \alpha \int_0^1 \int_{y-\beta w}^{\infty} V_{n+1,N}(\delta((1 - \beta)w + y), y - \beta w - \zeta) f_n(\zeta) d\zeta g(\delta) d\delta \\ n &= 0, 1, \dots, N - 1 \end{aligned} \quad (13)$$

$$V_{n,N}(w, x) = \min_{y \geq x} \{G_{n,N}(w, y)\} - cx, \quad n = 0, 1, \dots, N - 1$$

where $L_n(w, y)$ is as in Equation (2). Then we have

$$V_N(w, x) = V_{0,N}(w, x) \quad (14)$$

In the ensuing analysis, we choose the following terminal cost function

$$T(w, x) = \frac{c\beta w}{1 - \alpha E[\delta]} - cx, \quad (15)$$

The first term represents the expected discounted cost of warranty claims of the w items under warranty in period N , incurred over the infinite time from then on. The term $-cx$

reflects the assumption that any leftover inventory at period can be returned at original purchase price and only backlog has to be satisfied at per unit cost c .

We need the following notations:

$$\begin{aligned}
\bar{p} &= p - \frac{cE[\delta]\beta}{1 - \alpha E[\delta]} \\
S_n(w) &= \beta w + F_n^{-1}\left(\frac{\bar{p} - c(1 - \alpha)}{\bar{p} + h}\right) \\
\bar{L}_n(w, y) &= \int_0^1 [h \int_0^{y - \beta w} (y - \beta w - \zeta) f_n(\zeta) d\zeta + \bar{p} \int_{y - \beta w}^\infty (\zeta - y + \beta w) f_n(\zeta) d\zeta] g(\delta) d\delta \\
\tau_n(w, y) &= \frac{y - \beta w - \beta \delta w - F_n^{-1}\left(\frac{\bar{p} - c(1 - \alpha)}{\bar{p} + h}\right)}{1 + \delta \beta} \\
\Delta_n &= \alpha \left(\frac{c\beta}{1 - \alpha E[\delta]} E[\delta] + c \right) \mu_n + c(1 - \alpha) F_n^{-1}\left(\frac{\bar{p} - c(1 - \alpha)}{\bar{p} + h}\right) \\
&\quad + \alpha \bar{L}_n(w, S_n(w))
\end{aligned} \tag{16}$$

μ_n is the mean of the random demand from new customers at period n , and

$$\begin{aligned}
H_N(w, x) &= 0, \\
H_n(w, x) &= \begin{cases} 0 & \text{if } x \leq S_n(w) \\ G_{n,N}(w, x) - G_{n,N}(w, S_n(w)), & \text{o.w.} \end{cases} \\
&\quad \text{for } 0 \leq n \leq N - 1
\end{aligned} \tag{18}$$

Theorem 1 *Suppose the demands in each period are stochastically increasing, i.e.,*

$$F_n(x) \geq F_{n+1}(x), \text{ for } n = 0, 1, \dots, N - 2. \tag{19}$$

Then (i)

$$\begin{aligned}
G_{n,N}(w, y) &= c(1 - \alpha)y + \alpha \bar{L}_n(w, y) + (\alpha \Delta_{n+1} + \dots + \alpha^{n-1} \Delta_N) + \alpha \left(\frac{c\beta}{1 - \alpha E[\delta]} E[\delta] + c \right) w \\
&\quad + \alpha \left(\frac{c\beta}{1 - \alpha E[\delta]} E[\delta] + c \right) \mu_n \\
&\quad + \alpha \int_0^1 \int_0^{\tau_n(w, y)} H_{n+1}(\delta(w + \min(y - \beta w, \zeta)), y - \beta w - \zeta) f_n(\zeta) d\zeta g(\delta) d\delta
\end{aligned} \tag{20}$$

for $n = 0, \dots, N - 1$. $y = S_n(w)$ minimizes $G_n(w, y)$ for $n = 0, \dots, N - 1$, and $G_{n,N}(w, y)$ increases with respect to y when $y \geq S_n(w)$.

(ii)

$\int_0^{\tau_n(w, y)} H_n(\delta(w + \min(y - \beta w, \zeta)), y - \beta w - \zeta) f_{n-1}(\zeta) d\zeta$ is an increasing function of y

when $y \geq S_{n-1}(w)$ for a given w for $n = 1, \dots, N-1$

(iii)

$$V_{n,N}(w, x) = \frac{c\beta}{1 - \alpha E[\delta]} w - cx + (\Delta_n + \alpha \Delta_{n+1} + \dots + \alpha^{n-1} \Delta_N) + H_n(w, x). \quad (21)$$

for $n = 0, \dots, N-1$.

The proof follows from a series of claims using backward induction.

Claim 1: (i), (ii), and (iii) hold for $N-1$.

Proof of Claim 1:

$$\begin{aligned} & G_{N-1,N}(w, y) \\ &= cy + \alpha L_{N-1}(w, y) + \alpha \int_0^1 \int_0^{y-\beta w} \left[\frac{c\beta}{1 - \alpha E[\delta]} \delta(w + \zeta) - c(y - \beta w - \zeta) \right] f_{N-1}(\zeta) d\zeta g(\delta) d\delta \\ &+ \alpha \int_0^1 \int_{y-\beta w}^\infty \left[\frac{c\beta}{1 - \alpha E[\delta]} \delta(w + y - \beta w) - c(y - \beta w - \zeta) \right] f_{N-1}(\zeta) d\zeta g(\delta) d\delta \\ &= c(1 - \alpha)y + \alpha \left(\frac{c\beta}{1 - \alpha E[\delta]} E[\delta] + c\beta \right) w + \alpha \left(\frac{c\beta}{1 - \alpha E[\delta]} E[\delta] + c \right) \mu_{N-1} + \alpha \bar{L}_{N-1}(w, y) \end{aligned}$$

where

$$\begin{aligned} \bar{L}_{N-1}(w, y) &= \int_0^1 \left[(p - E[\delta]) \frac{c\beta}{1 - \alpha E[\delta]} \right] \int_{y-\beta w}^\infty (\zeta - y + \beta w) f_{N-1}(\zeta) d\zeta \\ &+ h \int_0^{y-\beta w} (y - \beta w - \zeta) f_{N-1}(\zeta) d\zeta g(\delta) d\delta \\ &= \int_0^1 [\bar{p} \int_{y-\beta w}^\infty (\zeta - y + \beta w) f_{N-1}(\zeta) d\zeta + h \int_0^{y-\beta w} (y - \beta w - \zeta) f_{N-1}(\zeta) d\zeta] g(\delta) d\delta \end{aligned}$$

It is easy to see that $y = S_{N-1}(w) = \beta w + F_{N-1}^{-1}\left(\frac{\bar{p} - c(1-\alpha)}{\bar{p} + h}\right)$ minimizes $G_{N-1,N}(w, y)$.

Hence, $G_{N-1,N}(w, y)$ increases with respect to y , when $y \geq S_{N-1}(w)$. Therefore, we get

$$V_{N-1,N}(w, x) = \begin{cases} G_{N-1,N}(w, S_{N-1}(w)) - cx & \text{if } x \leq S_{N-1}(w) \\ G_{N-1,N}(w, x) - cx & \text{o.w.} \end{cases} \quad (22)$$

Now

$$\begin{aligned} & G_{N-1,N}(w, S_{N-1}(w)) \\ &= \alpha \left(\frac{c\beta}{1 - \alpha E[\delta]} E[\delta] + c\beta \right) w + \alpha \left(\frac{c\beta}{1 - \alpha E[\delta]} E[\delta] + c \right) \mu_{N-1} + c(1 - \alpha) S_{N-1}(w) \\ &+ \alpha \bar{L}_{N-1}(w, S_{N-1}(w)) \end{aligned}$$

$$\begin{aligned}
&= \alpha \left[\frac{c\beta}{1 - \alpha E[\delta]} E[\delta] + c\beta \right] w + c(1 - \alpha)\beta w + \alpha \left(\frac{c\beta}{1 - \alpha E[\delta]} E[\delta] + c \right) \mu_{N-1} \\
&+ c(1 - \alpha) F_{N-1}^{-1} \left(\frac{\bar{p} - c(1 - \alpha)}{\bar{p} + h} \right) + \alpha \bar{L}_{N-1}(w, S_{N-1}(w)) \\
&= \frac{c\beta}{1 - \alpha E[\delta]} w + \alpha \left(\frac{c\beta}{1 - \alpha E[\delta]} E[\delta] + c \right) \mu_{N-1} + c(1 - \alpha) F_{N-1}^{-1} \left(\frac{\bar{p} - c(1 - \alpha)}{\bar{p} + h} \right) \\
&+ \alpha \bar{L}_{N-1}(w, S_{N-1}(w))
\end{aligned}$$

Using the notation Δ_{N-1} from Equation (17), we can rewrite Equation (22) as

$$\begin{aligned}
V_{N-1,N}(w, x) &= \frac{c\beta}{1 - \alpha E[\delta]} w - cx \\
&+ \Delta_{N-1} + \begin{cases} 0 & \text{if } x \leq S_{N-1}(w) \\ G_{N-1,N}(w, x) - G_{N-1,N}(w, S_{N-1}(w)) & \text{o.w.} \end{cases} \\
&= \frac{c\beta}{1 - \alpha E[\delta]} w - cx + \Delta_{N-1} + H_{N-1}(w, x)
\end{aligned}$$

The first order derivative of $\int_0^{\tau_{N-1}(w,y)} H_{N-1}(\delta(w + \min(y - \beta w, \zeta)), y - \beta w - \zeta) f_{N-2}(\zeta) d\zeta$ with respect to y is given by

$$\begin{aligned}
&\frac{\partial \int_0^{\tau_{N-1}(w,y)} H_{N-1}(\delta(w + \zeta), y - \beta w - \zeta) f_{N-2}(\zeta) d\zeta}{\partial y} \\
&= \frac{\partial \int_0^{\tau_{N-1}(w,y)} [G_{N-1,N}(\delta(w + \zeta), y - \beta w - \zeta) - G_{N-1,N}(\delta(w + \zeta), S_{N-1}(\delta(w + \zeta)))] f_{N-2}(\zeta) d\zeta}{\partial y} \\
&= \int_0^{\tau_{N-1}(w,y)} G_{N-1,N}^2(\delta(w + \zeta), y - \beta w - \zeta) f_{N-2}(\zeta) d\zeta.
\end{aligned}$$

Since $S_{N-1}(w)$ minimizes $G_{N-1,N}(w, y)$ and the definition of $H_{N-1}(w, y)$, we get

$$\{y : G_{N-1,N}^2(\delta(w + \zeta), y - \beta w - \zeta) > 0\} = \{y : H_{N-1}(\delta(w + \zeta), y - \beta w - \zeta) > 0\}. \quad (23)$$

From the assumption in Equation (19), we get $S_{N-2}(w) \leq S_{N-1}(w) + \beta\delta w$. This proves the quantity in Equation (23) is nonnegative when $y \geq S_{N-2}(w)$.

Claim 2: If (i),(ii), and (iii) of Theorem 1 hold for $n \leq N - 1$, then they hold for $n - 1$.

Proof of Claim 2:

The induction hypothesis implies that we have

$$G_{n,N}(w, y) = c(1 - \alpha)y + \alpha \bar{L}_n(w, y) + (\alpha \Delta_{n+1} + \dots + \alpha^{n-1} \Delta_N)$$

$$\begin{aligned}
& + \alpha \left(\frac{c\beta}{1 - \alpha E[\delta]} E[\delta] + c\beta \right) w + \alpha \left(\frac{c\beta}{1 - \alpha E[\delta]} E[\delta] + c \right) \mu_n \\
& + \alpha \int_0^\infty H_{n+1}(\delta(w + \min(y - \beta w, \zeta)), y - \beta w - \zeta) f_n(\zeta) d\zeta,
\end{aligned}$$

$y = S_n(w)$ minimizes $G_{n,N}(w, y)$, and $G_{n,N}(w, y)$ increases with respect to y , when $y \geq S_n(w)$. $\int_0^\infty H_n(\delta(w + \min(y - \beta w, \zeta)), y - \beta w - \zeta) f_{n-1}(\zeta) d\zeta$ is an increasing function of y when $y \geq S_{n-1}(w)$, Then

$$\begin{aligned}
& G_{n,N}(w, S_n(w)) \\
& = (\alpha \Delta_{n+1} + \dots + \alpha^{n-1} \Delta_N) + \alpha \left(\frac{c\beta}{1 - \alpha E[\delta]} E[\delta] + c\beta \right) w + \alpha \left(\frac{c\beta}{1 - \alpha E[\delta]} E[\delta] + c \right) \mu_n \\
& + c(1 - \alpha) S_n(w) + \alpha \bar{L}_n(w, S_n(w)) \\
& = \frac{c\beta}{1 - \alpha E[\delta]} w + (\Delta_n + \alpha \Delta_{n+1} + \dots + \alpha^{n-1} \Delta_N)
\end{aligned}$$

and correspondingly

$$\begin{aligned}
V_{n,N}(w, x) & = \frac{c\beta}{1 - \alpha E[\delta]} w - cx + (\Delta_n + \alpha \Delta_{n+1} + \dots + \alpha^{n-1} \Delta_N) \\
& + \begin{cases} 0 & \text{if } x \leq S_n(w) \\ G_{n,N}(w, x) - G_{n,N}(w, S_n(w)) & \text{o.w.} \end{cases}
\end{aligned}$$

Therefore from the DP recursion in Equation (13) we get

$$\begin{aligned}
& G_{n-1,N}(w, y) \\
& = cy + \alpha L_{n-1}(w, y) + \alpha \int_0^1 \int_0^{y-\beta w} V_{n,N}(\delta(w + \zeta), y - \beta w - \zeta) f_{n-1}(\zeta) d\zeta g(\delta) d\delta \\
& + \alpha \int_0^1 \int_{y-\beta w}^\infty V_{n,N}(\delta(w + y - \beta w), y - \beta w - \zeta) f_{n-1}(\zeta) d\zeta g(\delta) d\delta \\
& = c(1 - \alpha)y + \alpha \bar{L}_{n-1}(w, y) + (\alpha \Delta_n + \dots + \alpha^n \Delta_N) + \alpha \left(\frac{c\beta}{1 - \alpha E[\delta]} E[\delta] + c\beta \right) w \\
& + \alpha \left(\frac{c\beta}{1 - \alpha E[\delta]} E[\delta] + c \right) \mu_{n-1} \\
& + \alpha \int_0^1 \int_0^\infty H_n(\delta(w + \min(y - \beta w, \zeta)), y - \beta w - \zeta) f_{n-1}(\zeta) d\zeta g(\delta) d\delta,
\end{aligned} \tag{24}$$

which implies that (i) holds for $n - 1$. It is easy to see that $y = S_{n-1}(w)$ minimizes $c(1 - \alpha)y + \alpha \bar{L}_{n-1}(w, y)$. The assumption (19) implies that

$$F_{n-1}^{-1}\left(\frac{\bar{p} - c(1 - \alpha)}{\bar{p} + h}\right) \leq F_n^{-1}\left(\frac{\bar{p} - c(1 - \alpha)}{\bar{p} + h}\right) \quad (25)$$

which shows that $\tau_n(w, S_{n-1}(w)) < 0$. This proves that

$$\int_0^{\tau_n(w, S_{n-1}(w))} H_n(\delta(w + \zeta), S_{n-1}(w) - \beta w - \zeta) f_{n-1}(\zeta) d\zeta = 0. \quad (26)$$

Then $y = S_{n-1}(w)$ minimizes $G_{n-1, N}(w, y)$. Hence, $G_{n-1, N}(w, y)$ increases with respect to y when $y \geq S_{n-1}(w)$. Furthermore, the first order derivative of $\int_0^\infty H_{n-1}(\delta(w + \min(y - \beta w, \zeta)), y - \beta w - \zeta) f_{n-2}(\zeta) d\zeta$ with respect to y is given by

$$\begin{aligned} & \frac{\partial \int_0^{\tau_{n-1}(w, y)} H_{n-1}(\delta(w + \zeta), y - \beta w - \zeta) f_{n-2}(\zeta) d\zeta}{\partial y} \\ = & \frac{\partial \int_0^{\tau_{n-1}(w, y)} [G_{n-1, N}(\delta(w + \zeta), y - \beta w - \zeta) - G_{n-1, N}(\delta(w + \zeta), S_{n-1}(\delta(w + \zeta)))] f_{n-2}(\zeta) d\zeta}{\partial y} \\ = & \int_0^{\tau_{n-1}(w, y)} G_{n-1, N}^2(\delta(w + \zeta), y - \beta w - \zeta) f_{n-2}(\zeta) d\zeta \end{aligned}$$

which is nonnegative when $y \geq S_{n-1}(w)$, since $G_{n-1, N}(w, y)$ increases with respect to y when $y \geq S_{n-1}(w)$,

$$\{y : G_{n-1, N}^2(\delta(w + \zeta), y - \beta w - \zeta) > 0\} = \{y : H_{n-1}(\delta(w + \zeta), y - \beta w - \zeta) > 0\} \quad (27)$$

and

$$\tau_{n-1}(w, S_{n-1}(w)) \leq 0. \quad (28)$$

Then (ii) holds for $n - 1$. This implies that the base stock level policy, which orders up to $S_{n-1}(w)$, is the optimal for the discounted cost function over $n - 1, n, \dots, N$ periods. Clearly, we get

$$G_{n-1, N}(w, S_{n-1}(w)) = \frac{c\beta}{1 - \alpha E[\delta]} w + (\Delta_{n-1} + \alpha \Delta_n + \dots + \alpha^n \Delta_N) \quad (29)$$

Therefore,

$$\begin{aligned} V_{n-1, N}(w, x) &= \frac{c\beta}{1 - \alpha E[\delta]} w - cx + (\Delta_{n-1} + \alpha \Delta_n + \dots + \alpha^n \Delta_N) \\ &+ \begin{cases} 0 & \text{if } x \leq S_{n-1}(w) \\ G_{n-1, N}(w, x) - G_{n-1, N}(w, S_{n-1}(w)) & \text{o.w.} \end{cases} \end{aligned}$$

which implies that (iii) holds for $n - 1$.

The theorem (1) follows from this.

Remark 1 *From Theorem 1 it follows that the optimal policy in state in period n in state (w, x) is to order up to $S_n(w)$ as given in (16). This is called a w -dependent base-stock inventory replenishment policy.*

Special Case: i.i.d. Demands

Consider the special case where $F_n(x) = F(x)$ for $n = 0, 1, \dots, N - 1$, that is, the demands are i.i.d.. In this case, the base-stock level in period n is given by

$$S(w) = \beta w + F^{-1}\left(\frac{\bar{p} - c(1 - \alpha)}{\bar{p} + h}\right) \quad (30)$$

for all $n = 0, \dots, N - 1$. Thus, the optimal policy is the stationary w -dependent base-stock policy. It is unusual to get a stationary optimal policy for a finite horizon problem. We do so because of the special terminal cost.

5 Infinite Horizon ETDC

In the previous section we have proved that in the i.i.d. demands case the stationary w -dependent base-stock policy minimizes the total expected discounted (or undiscounted) cost over any finite horizon N . We denote this stationary policy by π^* . Thus we have shown that

$$\begin{aligned} V_N(w, x) &= V_N^{\pi^*}(w, x) \\ &= E_{\pi^*}\left(\sum_{n=1}^{N-1} \alpha^n C_n(W_n, X_n) + \alpha^N T(W_N, X_N) | W_0 = w, X_n = 0\right) \end{aligned}$$

In this section we shall show that

$$\lim_{N \rightarrow \infty} E_{\pi^*}(\alpha^N T(W_N, X_N) | W_0 = w, X_n = 0) = 0.$$

This will show that π^* also minimizes the expected total discounted cost over the infinite horizon. Our main result is given in the next theorem.

Theorem 2 *Suppose $F_n(x) = F(x)$ for all $n \geq 0$. Then the stationary w -dependent base-stock policy that orders upto $S(w) = \beta w + F^{-1}\left(\frac{\bar{p} - c(1 - \alpha)}{\bar{p} + h}\right)$ in any period in state (w, x) minimizes the infinite horizon discounted cost.*

Proof: From Equation (1), we get

$$W_{n+1} \leq \delta_n[(1 - \beta)W_n + \beta W_n + \zeta_n]. \quad (31)$$

Hence, taking expected values,

$$E_{\pi^*}(W_{n+1}) \leq E(\delta)E(W_n) + \delta\mu, \quad (32)$$

where $\mu = E(\zeta_n)$. Iterating the above we get

$$\begin{aligned} E_{\pi^*}(W_N) &\leq E(\delta)^N E_{\pi^*}(W_0) + (E(\delta) + \dots + E(\delta)^N)\mu \\ &\leq [E(\delta)^N w + \frac{E(\delta)\mu(1 - E(\delta)^N)}{1 - E(\delta)}] \\ &\leq E(\delta)^N w + \frac{\mu}{1 - E(\delta)} \end{aligned}$$

The effect of the initial inventory $X_0 = x$ is to increase the above right hand side at most by $E(\delta)^N x$. Also $E(X_N) \geq -\mu$. Combining the above arguments we get that

$$\begin{aligned} E_{\pi^*}(T(W_N, X_N)|W_0 = w, X_0 = x) &= E_{\pi^*}\left(\frac{c\beta}{1 - E(\delta)}W_N - cX_N|W_0 = w, X_0 = x\right) \\ &\leq \left(\frac{\mu}{1 - E(\delta)} + (w + x)E(\delta)^N\right)\frac{c\beta}{1 - E(\delta)} + c\mu. \end{aligned}$$

Hence,

$$\lim_{N \rightarrow \infty} \alpha^N E_{\pi^*}(T(W_N, X_N)|W_0 = w, X_0 = x) = 0. \quad (33)$$

Hence, it follows that

$$\lim_{N \rightarrow \infty} V_N^{\pi^*}(w, x) = V^{\pi^*}(w, x) = V(w, x). \quad (34)$$

That is, π^* is optimal for the infinite horizon discounted cost case. ■

6 Infinite Horizon Average Cost

We first consider a finite horizon model with no discounting ($\alpha = 1$) with a special terminal cost given by Equation (15) at time N . We focus on the model with independent and identical demands with distributions $F_n(x) = F(x)$. From the special case studied in section 4, we get the following theorem.

Theorem 3 *Let*

$$S(w) = \beta w + F^{-1}\left(\frac{\tilde{p}}{\tilde{p} + h}\right) \quad (35)$$

where $\tilde{p} = p - \frac{c\beta E(\delta)}{1-E(\delta)}$. The stationary base-stock policy that orders up to $S(w)$ in state (w, x) minimizes the N period total cost.

Denote this stationary base-stock policy as π^* .

Let the expected cost per period of following policy π for a finite horizon N starting from state (w, x) be defined as

$$g_N^\pi(w, x) = \frac{1}{N+1} E_\Pi\left(\sum_{n=0}^{N-1} C_n(W_n, X_n) + T(W_N, X_N) | W_0 = w, X_0 = x\right), \quad (36)$$

and let $g_N(w, x)$ be the optimal expected cost per period for a finite horizon N starting in state w, x , that is,

$$g_N(w, x) = \inf_{\pi} g_N^\pi(w, x) = \frac{V_N(w, x)}{N+1}. \quad (37)$$

Theorem 3 implies that

$$V_N(w, x) = V_N^{\pi^*}(w, x). \quad (38)$$

Hence, it follows that

$$g_N(w, x) = g_N^{\pi^*}(w, x). \quad (39)$$

In this section, we shall show that

$$\lim_{N \rightarrow \infty} \frac{1}{N+1} (E_{\pi^*}(T(W_N, X_N) | W_0 = w, X_0 = x)) = 0, \quad (40)$$

and

$$\lim_{N \rightarrow \infty} \frac{1}{N} (E_{\pi^*}\left(\sum_{n=0}^{N-1} C_n(W_n, X_n)\right) | W_0 = w, X_0 = x) \quad (41)$$

exists. This will establish that π^* minimizes the expected cost per period over infinite horizon. To prove the limit in Equation 41 exists, we prove the following properties of (W_n, X_n) .

Theorem 4 *Under the policy π^* , $\{(W_n, X_n), n \geq 0\}$ is an irreducible, aperiodic, and positive recurrent DTMC.*

Proof is in the appendix.

Next, we show the existence of the limit of N period average total cost when N goes to infinity.

Theorem 5 $\lim_{N \rightarrow \infty} E(\frac{1}{N} \sum_{n=0}^{N-1} C_n(W_n, X_n) | W_0 = w, X_0 = x)$ exists and is finite for all (w, x) .

Proof: We get the total cost at period n as

$$\begin{aligned}
& E(C_n(W_n, X_n) | W_n = w, X_n = x) \\
&= E(c(y_n - X_n) + h(y_n - \beta W_n - \zeta_n)^+ + p(y_n - \beta W_n - \zeta_n)^- | W_n = w, X_n = x) \\
&\leq E(c(y_n - X_n) + h y_n + p(\zeta_n + \beta W_n) | W_n = w, X_n = x) \\
&= E((c + h) \max(X_n, S(W_n)) + p(\zeta_n + \beta W_n) - c X_n | W_n = w, X_n = x) \\
&= (c + h)E(\max(X_n, S(W_n))) - cE(X_n) + p\mu + p\beta E(W_n).
\end{aligned}$$

When $X_n \geq 0$ then we could rewrite the above quantity as

$$\begin{aligned}
& E(C_n(W_n, X_n) | W_n = w, X_n = x) \\
&\leq (c + h)E(S(W_n)) - cE(X_n) + p\mu + p\beta E(W_n) \\
&\leq (c + h)(\beta E(W_n) + F^{-1}(\frac{\tilde{p}}{\tilde{p} + h})) + p\mu + p\beta E(W_n) \\
&\leq (c + h)F^{-1}(\frac{\tilde{p}}{\tilde{p} + h}) + p\mu + (c + h + p)\beta(\frac{\mu}{1 - E(\delta)} + (w + x)E(\delta)^n).
\end{aligned}$$

When $X_n \leq 0$ we get

$$\begin{aligned}
& E(C_n(W_n, X_n) | W_n = w, X_n = x) \\
&\leq (c + h)E(X_n) - cE(X_n) + p\mu + p\beta E(W_n) \\
&\leq p\mu + p\beta E(W_n) \\
&\leq p\mu + p\beta(\frac{\mu}{1 - E(\delta)} + (w + x)E(\delta)^n).
\end{aligned}$$

Hence, we get

$$\begin{aligned}
& E(C_n(W_n, X_n) | W_0 = w, X_0 = x) \\
&\leq (c + h)F^{-1}(\frac{\tilde{p}}{\tilde{p} + h}) + p\mu + (c + h + p)\beta(\frac{\mu}{1 - E(\delta)} + (w + x)E(\delta)^n) \\
&< \infty.
\end{aligned}$$

The above bound along with the fact that $\{(W_n, X_n), n = 0, \dots, \infty\}$ is a positive recurrent and irreducible implies that

$$\lim_{N \rightarrow \infty} E\left(\frac{1}{N} \sum_{n=0}^{N-1} C_n(W_n, X_n) | W_0 = w, X_0 = x\right) \quad (42)$$

exists and is finite. ■

Finally, we study the property of the special terminal cost and derive the optimal inventory policy to minimize the long-run average cost.

Theorem 6 *The stationary w -dependent base stock level policy π^* minimizes the expected cost per period over infinite horizon.*

Proof: Using the terminal cost in Equation (15) and using the argument in Theorem 3, we get

$$\begin{aligned} E_{\pi^*}(T(W_N, X_N) | W_0 = w, X_0 = x) &= E_{\pi^*}\left(\frac{c\beta}{1 - E(\delta)} W_N - cX_N | W_0 = w, X_0 = x\right) \\ &\leq \left(\frac{\mu}{1 - E(\delta)} + (w + x)E(\delta)^N\right) \frac{c\beta}{1 - E(\delta)} + c\mu \end{aligned}$$

Hence,

$$\lim_{N \rightarrow \infty} \frac{1}{N + 1} E(T(W_N, X_N) | W_0 = w, X_0 = x) = 0. \quad (43)$$

Since the w -dependent base stock policy minimizes the expected cost per period over N periods for each N , it is clear that it minimizes the long-run average cost. ■

7 Model with Emergency Supply

In this section, we study the N -period emergency supply model where rather than backlogging the manufacturer has to satisfy the unmet demands by purchasing the items from an emergency supplier. We use p as the per unit emergency purchase cost where $p > c$. Note there are no shortages in this model. Let β_n be the failure fraction in period n and δ_n be the warranty expiration rate in period n . We assume that $\{\beta_n, n = 0, \dots, N - 1\}$, and $\{\delta_n, n = 0, \dots, N - 1\}$ are two independent sequences of i.i.d. random variables with density distributions $k(\cdot)$ and $g(\cdot)$ respectively. The dynamic of the system is given by

$$\begin{aligned}
X_{n+1} &= \max(Y_n - \beta_n W_n - \zeta_n, 0) \\
W_{n+1} &= \delta_n(W_n + \zeta_n)
\end{aligned}$$

Let $V_N^\pi(w, x)$ and $V_N(w, x)$ be defined as in Section 3. Following the methodology of Section 4, we can write the optimality recursions as follows

$$\begin{aligned}
&V_{N,N}(w, x) = 0; \\
&G_{n,N}(w, y) \\
&= cy + L_n(w, y) \\
&+ \alpha \int_0^1 \int_0^1 \int_0^{y-\beta w} V_{n+1,N}(\delta(w + \zeta), y - \beta w - \zeta) f_n(\zeta) d\zeta g(\delta) d\delta k(\beta) d\beta \\
&+ \int_0^1 \int_0^1 \int_{y-\beta w}^\infty V_{n+1,N}(\delta(w + \zeta), 0) f_n(\zeta) d\zeta g(\delta) d\delta k(\beta) d\beta \\
&n = 0, \dots, N-1 \\
&V_{n,N}(w, x) = \min_{y \geq x} \{G_{n,N}(w, y)\} - cx, \quad n = 0, \dots, N-1,
\end{aligned} \tag{44}$$

where

$$L_n(w, y) = \int_0^1 \int_0^1 [h \int_0^{y-\beta w} (y-\zeta-\beta w) f_n(\zeta) d\zeta + p \int_{y-\beta w}^\infty (\zeta-y+\beta w) f_n(\zeta) d\zeta] g(\delta) d\delta k(\beta) d\beta.$$

The minimum N -period ETDC is given by $V_N(w, x) = V_{0,N}(w, x)$.

Theorem 7 *There exists a function $S_n(w)$ such that the optimal policy in period n is to order upto $S_n(w)$.*

Proof: We prove it using a series of claims by using induction.

Claim 1: $G_{N-1,N}(w, y)$ is convex with respect to y .

Proof of Claim 1:

From Equation (44), we get

$$G_{N-1,N}(w, y) = cy + L_{N-1}(w, y),$$

and from the definition of $L_{N-1}(w, y)$, the second order derivative of $L_{N-1}(w, y)$ with respect to y can be shown to be

$$L_{N-1}^{22}(w, y) = \int_0^1 \int_0^1 \frac{\partial^2 L_{N-1}(w, y)}{\partial y^2} g(\delta) d\delta k(\beta) d\beta = \int_0^1 (p+h) f_{N-1}(y-\beta w) k(\beta) d\beta \geq 0. \tag{45}$$

This proves our claim 1.

Claim 2: If $G_{n,N}(w, y)$ is convex with respect to y , then $V_{n,N}(w, x)$ is convex with respect to x .

Proof of Claim 2:

Let $y = S_n(w)$ be the solution to

$$\frac{\partial G_{n,N}(w, y)}{\partial y} = 0. \quad (46)$$

The convexity of $G_{n,N}(w, y)$ implies that the optimal policy in period n is order up to $S_n(w)$. Then

$$V_{n,N}(w, x) = \begin{cases} G_{n,N}(w, S_n(w)) - cx & \text{if } x \leq S_n(w) \\ G_{n,N}(w, x) - cx & \text{o.w.} \end{cases} \quad (47)$$

We get the first order derivative of $V_{n,N}(w, x)$ with respect to x as

$$V_{n,N}^2(w, x) = \begin{cases} -c & \text{if } x \leq S_n(w) \\ G_{n,N}^2(w, x) - c & \text{o.w.} \end{cases} \quad (48)$$

Since $G_{n,N}^2(w, S_n(w)) = 0$, $V_{n,N}^2(w, x)$ is continuous at $x = S_n(w)$, i.e., $V_{n,N}^2(w, S_n(w)^+) = V_{n,N}^2(w, S_n(w)^-) = -c$. The second order derivative then can be shown to be

$$V_{n,N}^{22}(w, x) = \begin{cases} 0 & \text{if } x \leq S_n(w) \\ G_{n,N}^{22}(w, x) & \text{o.w.} \end{cases}$$

Thus $V_{n,N}(w, x)$ is convex with respect to x .

Claim 3: If $V_{n,N}(w, x)$ is convex with respect to x , then $G_{n-1,N}(w, y)$ is convex with respect to y .

Proof of Claim 3:

From the definition, we get

$$\begin{aligned} G_{n-1,N}(w, y) &= cy + L_{n-1}(w, y) \\ &+ \alpha \int_0^1 \int_0^1 \int_0^{y-\beta w} V_{n,N}(\delta(w + \zeta), y - \zeta - \beta w) f_{n-1}(\zeta) d\zeta g(\delta) d\delta k(\beta) d\beta \\ &+ \alpha \int_0^1 \int_0^1 \int_{y-\beta w}^\infty V_{n,N}(\delta(w + \zeta), 0) f_{n-1}(\zeta) d\zeta g(\delta) d\delta k(\beta) d\beta. \end{aligned}$$

The second order derivative of $G_{n-1,N}(w, y)$ with respect to y can be shown to be

$$\begin{aligned}
& G_{n-1,N}^{22}(w, y) \\
&= \frac{\partial^2 G_{n-1,N}(w, y)}{\partial y^2} \\
&= \frac{\partial^2 L_{n-1}(w, y)}{\partial y^2} + \alpha \int_0^1 \int_0^1 \frac{\partial^2 \int_0^\infty V_{n,N}(\delta(w + \zeta), (y - \zeta - \beta w)^+) f_{n-1}(\zeta) d\zeta}{\partial y^2} g(\delta) d\delta k(\beta) d\beta \\
&= L_{n-1}^{22}(w, y) + \alpha \int_0^1 \int_0^1 \left[\int_0^{y-\beta w} V_{n,N}^{22}(\delta(w + \zeta), y - \beta w - \zeta) f_{n-1}(\zeta) d\zeta \right. \\
&+ \left. V_{n,N}^2(\delta(w + \zeta), 0) f_{n-1}(y - \beta w) \right] g(\delta) d\delta k(\beta) d\beta
\end{aligned}$$

From the Equation (48) and (45), the above quantity can be simplified as

$$\begin{aligned}
G_{n-1,N}^{22}(w, y) &= \alpha \int_0^1 \int_0^1 \left[\int_0^{y-\beta w} V_{n,N}^{22}(\delta(w + \zeta), y - \beta w - \zeta) f_{n-1}(\zeta) d\zeta \right. \\
&+ \left. (p + h - \alpha c) f_{n-1}(y - \beta w) \right] g(\delta) d\delta k(\beta) d\beta \\
&\geq 0
\end{aligned}$$

The above quantity is greater than zero by the assumptions, which means that $G_{n-1,N}(w, y)$ is convex with respect to y . ■

Theorem 7 then follows by the above induction.

In the next theorem, we state a useful property of $V_{n,N}(w, x)$ function that can be used to derive structural results for the base stock function $S_n(w)$.

Theorem 8 $V_{n,N}^{12}(w, x) = \frac{\partial V_{n,N}(w, x)}{\partial w \partial x} \leq 0$.

Claim 1: $G_{N-1,N}^{12}(w, y) \leq 0$.

Proof of Claim 1:

From the definition of $G_{N-1,N}(w, y)$, we get $G_{N-1,N}^{12}(w, y)$ as follows.

$$G_{N-1,N}^{12}(w, y) = \int_0^1 L_{N-1}^{12}(w, y) k(\beta) d\beta = \int_0^1 [-\beta(h + p) f_{N-1}(y - \beta w)] k(\beta) d\beta \leq 0 \quad (49)$$

Claim 2: If $G_{n,N}^{12}(w, y) \leq 0$, then $V_{n,N}^{12}(w, y) \leq 0$.

Proof of Claim 2:

From Equation (48), we get

$$V_{n,N}^{12}(w, x) = \begin{cases} 0 & \text{if } x \leq S_n(w) \\ G_{n,N}^{12}(w, x) & \text{o.w.} \end{cases}$$

which implies that $V_{n,N}^{12}(w, x) \leq 0$.

Claim 3: If $V_{n,N}^{12}(w, x) \leq 0$, then $G_{n-1,N}^{12}(w, x) \leq 0$.

Proof of Claim 3:

From the definition, we get

$$\begin{aligned} G_{n-1,N}^{12}(w, y) &= L_{n-1}^{12}(w, y) + \alpha \int_0^1 \int_0^1 \left[\int_0^{y-\beta w} V_{n,N}^{12}(w, x) \delta + (-\beta) V_{n,N}^{22}(w, x) f_{n-1}(\zeta) d\zeta \right. \\ &\quad \left. + \alpha V_{n,N}^2(\delta(w + \zeta), 0) (-\beta) f_{n-1}(y - \beta w) \right] g(\delta) d\delta k(\beta) d\beta \\ &= \int_0^1 \int_0^1 [(-\beta)(p + h - \alpha c) f_{n-1}(y - \beta w) \\ &\quad + \alpha \int_0^{y-\beta w} V_{n,N}^{12}(w, x) \delta + (-\beta) V_{n,N}^{22}(w, x) f_{n-1}(\zeta) d\zeta] g(\delta) d\delta k(\beta) d\beta \end{aligned}$$

The above quantity is less and equal to zero due to our assumptions.

By proving the above three claims, the theorem follows via the induction.

Given that $V_{n,N}(w, x)$ is submodular, we get the following property of the base stock function $S_n(w)$.

Theorem 9 *The optimal base stock level $S_n(w)$ is an increasing function with respect to w , and $\frac{dS_n(w)}{dw} \geq \beta$, when β and δ are fixed at each period, and the identical demand distribution at each period, i.e., $F_n(x) = F(x)$.*

Proof: First, from the definition, we get

$$G_{N-1,N}(w, y) = c(y - x) + L(w, y)$$

and let $y = S_{N-1}(w, y)$ denote the solution to the first order condition as follows.

$$G_{N-1,N}^2(w, y) = c + L^2(w, y) = 0.$$

Then the optimal base stock level $S_{N-1}(w)$ can be derived as

$$F(y - \beta w) = \frac{p - c}{p + h}$$

$$S_{N-1}(w) = y = F^{-1}\left(\frac{p - c}{p + h}\right) + \beta w$$

Therefore $\frac{dS_{N-1}(w)}{dw} = \beta$. Theorem 7, i.e., the convexity of $G_{n,N}(w, y)$ with respect to y implies that we could derive the optimal base stock function $S_n(w)$ by solving the first order condition as a function to $G_{n,N}^2(w, S_n(w)) = 0$. From the Equation (44), we get

$$c + L^2(w, y) + \alpha \int_0^{y - \beta w} V_{n+1,N}^2(\delta(w + \zeta), y - \beta w - \zeta) f(\zeta) d\zeta = 0. \quad (50)$$

Taking derivative with respect to w on both sides, we get

$$\begin{aligned} L^{22}(w, y) \frac{dS_n(w)}{dw} + L^{12}(w, S_n(w)) + \alpha \int_0^{S_n(w) - \beta w} (V_{n+1,N}^{12}(\delta(w + \zeta), S_n(w) - \beta w - \zeta) \delta \\ + V_{n+1,N}^{22}(\delta(w + \zeta), S_n(w) - \beta w - \zeta) \left(\frac{dS_n(w)}{w} - \beta\right)) f(\zeta) d\zeta + \alpha V_{n+1,N}^2(\delta(w + \zeta), 0) \left(\frac{dS_n(w)}{w} - \beta\right) f(S_n(w) - \beta w) \\ = 0 \end{aligned}$$

which reduces to

$$\begin{aligned} - (L^{12}(w, S_n(w)) + \alpha \int_0^{S_n(w) - \beta w} V_{n+1,N}^{12}(\delta(w + \zeta), y - \beta w - \zeta) \delta f(\zeta) d\zeta) \\ = L^{22}(w, S_n(w)) \frac{dS_n(w)}{dw} + \alpha \int_0^{S_n(w) - \beta w} V_{n+1,N}^{22}(\delta(w + \zeta), S_n(w) - \beta w - \zeta) \left(\frac{dS_n(w)}{w} - \beta\right) f(\zeta) d\zeta. \end{aligned}$$

Using the Equation (45) for $L^{22}(w, y)$, Equation (50) for $L^{12}(w, y)$, and Equation (48), we can rewrite the above equation as

$$\begin{aligned} - (-\beta(p + h)f(y - \beta w) + \alpha \int_0^{S_n(w) - \beta w} V_{n+1,N}^{12}(\delta(w + \zeta), y - \beta w - \zeta) \delta f(\zeta) d\zeta) \\ = (p + h - \alpha c) \frac{dS_n(w)}{dw} f(S_n(w) - \beta w) \\ + \alpha \int_0^{S_n(w) - \beta w} [V_{n+1,N}^{22}(\delta(w + \zeta), y - \beta w - \zeta) \left(\frac{dS_n(w)}{dw} - \beta\right)] f(\zeta) d\zeta + c\beta\alpha f(S_n(w) - \beta w) \end{aligned}$$

which is rearranged as

$$\begin{aligned} - (\alpha \int_0^{S_n(w) - \beta w} V_{n+1,N}^{12}(\delta(w + \zeta), S_n(w) - \beta w - \zeta) \delta f(\zeta) d\zeta) \\ = ((p + h - \alpha c)f(S_n(w) - \beta w) + \alpha \int_0^{S_n(w) - \beta w} V_{n+1,N}^{22}(\delta(w + \zeta), S_n(w) - \beta w - \zeta) f(\zeta) d\zeta) \left(\frac{dS_n(w)}{dw} - \beta\right) \end{aligned}$$

From the previous Theorem 7 and Theorem 8, we get $\frac{dS_n(w)}{dw} - \beta \geq 0$. ■

Note that βw is the expected number of new items that are needed for warranty. However, the optimal policy is to stock greater than that taking into account the penalty costs associated with the repairs. We end this section with two conjectures for the infinite horizon i.i.d. demands case.

Conjecture 1:

There exist a function $S(w)$ such that the policy that orders up to $S(w)$ in any period when there are w items under warranty minimizes the infinite horizon ETDC.

Conjecture 2:

There exist a function $S(w)$ such that the policy that orders up to $S(w)$ in any period when there are w items under warranty minimizes the infinite horizon expected cost per period.

8 Computational Results

In this section, we numerically investigate the benefits of using an integrated inventory policy over the current ad hoc policy used by the firm where they took only new demand into consideration. We compared the performance over 343 problem instances with the following parameter values- $c = 2, \delta = 0.96, \alpha = 0.95$. Then we varied $h = 0.05, 0.1, 0.15, 0.2, 0.25, 0.30, p = 8, 10, 12, 15, 20, 25, 30$ and $\beta = 0.01, 0.05, 0.1, 0.15, 0.2, 0.25, 0.3$. In these instances we computed the total N periods ($N = 100$) discounted cost of the two policies and averaged it across 1000 simulations. In all our experiments we considered a stationary and uniform demand distribution every period that could be anywhere from 0 to 100 units. In the following passages we highlight our key insights.

Across all these experiments, the average cost improvement due to the integrated policy was 30.7% with a maximum improvement of 61.8% (see Tables 1,2,3). From Figure 3 and 4, we observe the the cost improvement monotonically increases with the stock-out penalty cost p , and monotonically decreases with the holding cost h . This is intuitive since the integrated policy stocks more than the original policy in most of the cases.

Although one would expect that performance improvement of the integrated policy is greater when failure rate is higher, interestingly, we find that as the failure rate increases, the cost improvements first increase and then decrease (see Figure 5). In order

	p = 8	10	12	15	20	25	30
AVG %	0.127	0.19	0.24	0.30	0.38	0.44	0.48
MAX %	0.21	0.28	0.34	0.41	0.496	0.57	0.62
MIN %	0.01	0.012	0.016	0.02	0.026	0.03	0.04

Table 1: Average Cost Improvement Over p

	h=0.01	0.05	0.1	0.15	0.2	0.25	0.3
AVG %	0.315	0.313	0.31	0.308	0.305	0.303	0.3
MAX %	0.618	0.616	0.614	0.611	0.609	0.607	0.605
MIN %	0.01	0.01	0.01	0.01	0.01	0.01	0.01

Table 2: Average Cost Improvement Over h

	$\beta=0.01$	0.05	0.1	0.15	0.2	0.25	0.3
AVG %	0.02	0.25	0.38	0.41	0.40	0.37	0.34
MAX %	0.04	0.395	0.57	0.61	0.62	0.61	0.59
MIN %	0.01	0.12	0.193	0.192	0.158	0.11	0.05

Table 3: Average Cost Improvement Over β

to understand this phenomenon better, we plotted the procurement, holding and stock-out costs separately in Figure 6. We find that the procurement cost of the integrated inventory control policy increases significantly with the failure rate. The procurement increases in the standard policy are just a reflection of the fact that more items need to be ordered to get to the base stock level since most of the demand is being backlogged at higher failure rates. That is also the reason why the stock-out costs of the standard policy are increasing in the failure rate while the stock-out costs of the integrated policy hold steady. There is not a significant difference in the holding costs across the two cases. Thus, as the failure rate increases, the integrated policy benefits from lower stock out costs up to a certain point. Beyond that, the additional improvements in stock-out costs are offset by higher procurement costs that the integrated policy has to incur. Therefore, the integrated policy's improvement first increases and then decreases with the failure rate.

9 Summary and Future Plans

Motivated by the inventory planning issues faced by a seller of items under warranty in this paper we analyzed a setting where seller faces demand from two sources: new demand, and demand to replace failed items under warranty. We considered backlogging and emergency supply cases and studied both the discounted cost and the average cost cases. We proved the optimality of the w -dependent base stock ordering policy where the base stock level is a function of w , the number of items currently under warranty. For the special case, where an i.i.d. fraction of sold items go out of warranty every period and the demand for new products is stationary, we prove the optimality of a stationary w -dependent base stock policy for finite and infinite horizon cases. Through our computational study we provide interesting insights on the benefits of using an integrated policy and show that on average this leads to a 31% improvement over an inventory policy that only considers new demands.

The simplified assumptions in our model helped us to formulate and theoretically characterize the optimal policy for this problem motivated by the projector firm, thereby advancing our knowledge in the area of stylized inventory models. However, our models

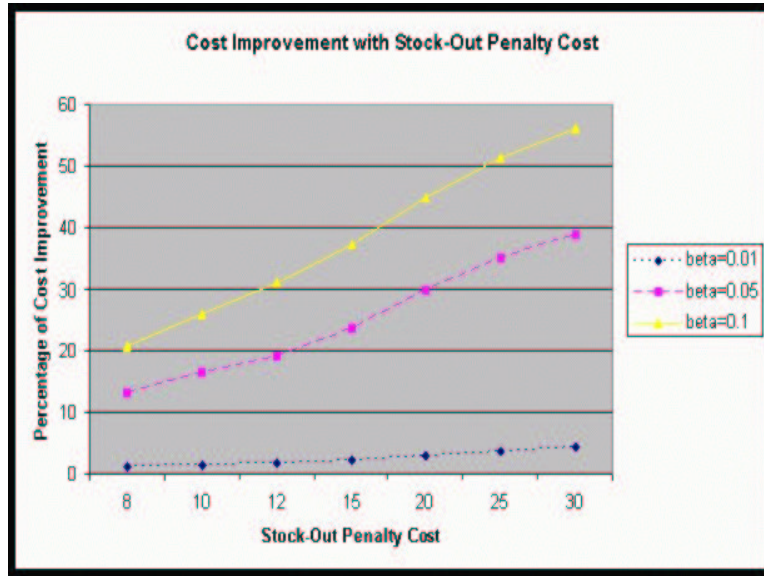


Figure 3: The Cost Improvement with Increasing Stock-out Penalty Cost Rate p

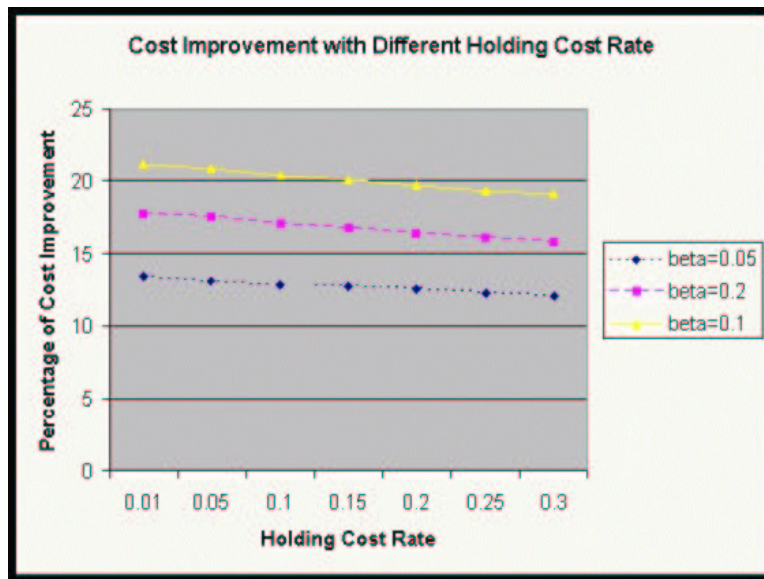


Figure 4: The Cost Improvement with Increasing Holding Cost Rate h

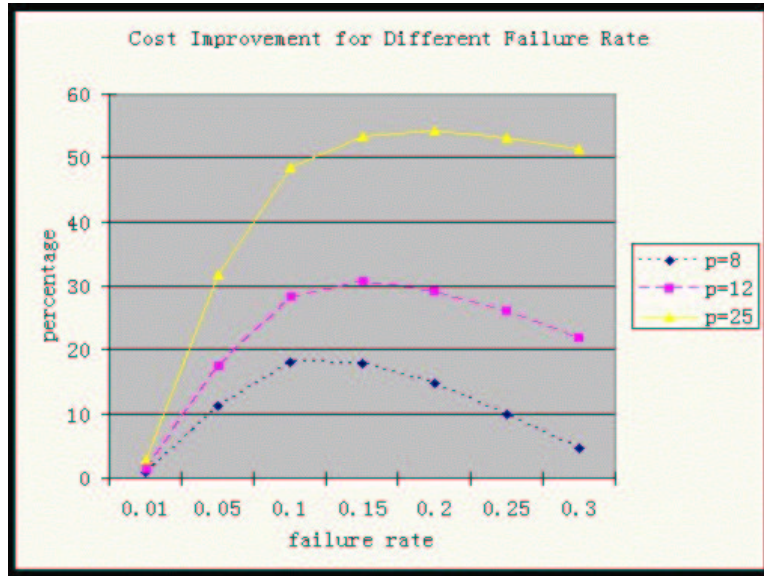


Figure 5: The Cost Improvement with Increasing Failure Rate β

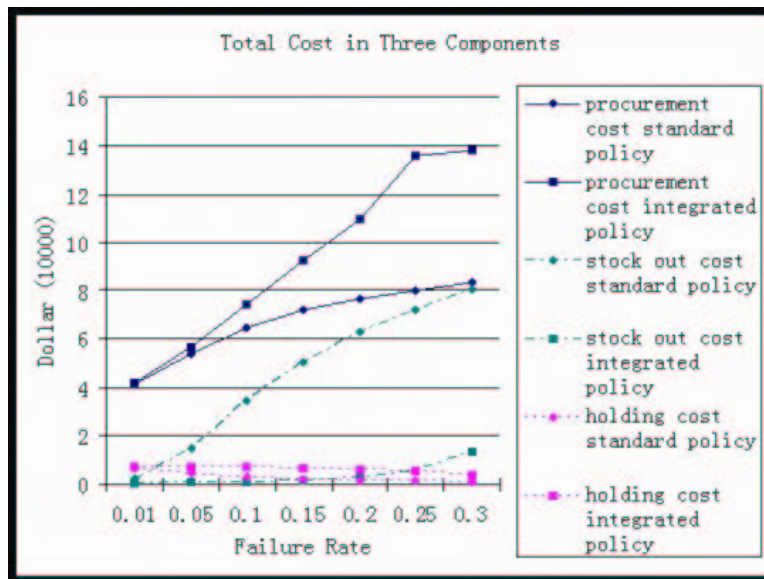


Figure 6: Three Cost Components

have a few shortcomings. Firstly, we assumed that a proportion of items fail independently under warranty following the bernoulli trials. However, in many real situations failures may be due to a defect in assembly or a supplied part that may lead a batch of items to fail around the same time. In order to analyze such a situation, a general warranty returns model needs to be studied. Secondly, in our model we assume that a random proportion of items go out of warranty every period. Further, our model assumes a renewable warranty in that all replaced items are sold as new. However, with the recent advances in information technology, it is possible for firms to keep track of exact age of items in the field at any given time. The state space associated with such a model is going to be larger and as a result more difficult to analyze. Finally, in our model we neglect the possibility of remanufacturing/repair for the items that fail. A model that combines both repair and demand changes is likely to be interesting in terms of analysis and insights. We plan to study these models in the future.

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10 Appendix: Proof of Theorem 4

Proof: From the definition of the base stock policy and the equations (1),

$$\begin{aligned} X_{n+1} &= \max(X_n, S(W_n)) - \beta W_n - \zeta_n, \\ W_{n+1} &= \delta_n(W_n + \min(\max(X_n, S(W_n)) - \beta W_n, \zeta_n)), \end{aligned}$$

where $S(W_n)$ is as in the Equation (35). This implies that $\{(W_n, X_n), n \geq 0\}$ is a DTMC. It is easy to show that it is irreducible and aperiodic. We prove the positive recurrence by using the Foster's criterion (Meyn and Tweedie [11]). We choose the following test function $v(W, X) = |X| + W$. Then

$$\begin{aligned} &E(v(W_{n+1}, X_{n+1}) - v(W_n, X_n) | W_n = w, X_n = x) \\ &= E(W_{n+1} + |X_{n+1}| | W_n = w, X_n = x) - w - |x| \end{aligned}$$

When $x \geq 0$, we have

$$\begin{aligned} &E(|X_{n+1}| - X_n | W_n = w, X_n = x) \\ &= E(|\max(X_n, S(W_n)) - \beta W_n - \zeta_n| - X_n | W_n = w, X_n = x) \\ &\leq \begin{cases} -\beta w + \mu & \text{if } x \geq S(w) \\ F^{-1}(\frac{\tilde{p}}{\tilde{p}+h}) + \mu - x, & \text{o.w.} \end{cases} \\ &\leq F^{-1}(\frac{\tilde{p}}{\tilde{p}+h}) + \mu - x, \end{aligned}$$

and

$$\begin{aligned} &E(W_{n+1} - W_n | W_n = x, X_n = x) \\ &= E(E(\delta)(W_n + \min(\max(X_n, S(W_n)) - \beta W_n, \zeta_n)) - W_n | W_n = w, X_n = x) \\ &= \begin{cases} E(E(\delta)(w + \min(x - \beta w, \zeta)) - w | W_n = w, X_n = x) & \text{if } x \geq S(w) \\ E(E(\delta)(w + \min(S(w) - \beta w, \zeta_n)) - w | W_n = w, X_n = x), & \text{o.w.} \end{cases} \\ &\leq \begin{cases} E(\delta)\mu - (1 - E(\delta))w & \text{if } x \geq S(w) \\ E(\delta)\mu - (1 - E(\delta))w, & \text{o.w.} \end{cases} \\ &\leq -(1 - E(\delta))w + E(\delta)\mu. \end{aligned} \tag{51}$$

Therefore

$$\begin{aligned}
& E(v(W_{n+1}, X_{n+1}) - v(W_n, X_n) | W_n = w, X_n = x) \\
&= E(W_{n+1} + |X_{n+1}| - W_n - X_n | W_n = w, X_n = x) \\
&\leq F^{-1}\left(\frac{\tilde{p}}{\tilde{p} + h}\right) + (1 + E(\delta))\mu - (1 - E(\delta))w - x.
\end{aligned}$$

The last expression is < 0 if $(1 - E(\delta))w + x > F^{-1}\left(\frac{\tilde{p}}{\tilde{p} + h}\right) + (1 + E(\delta))\mu$.

When $x \leq 0$, we have

$$\begin{aligned}
& E(|X_{n+1}| - |X_n| | (W_n = w, X_n = x)) \\
&= E(|\max(X_n, S(W_n)) - \beta W_n - \zeta_n| + X_n | (W_n = w, X_n = x)) \\
&\leq F^{-1}\left(\frac{\tilde{p}}{\tilde{p} + h}\right) + \mu + x \\
&\leq 0 \text{ if } x \leq -\mu - F^{-1}\left(\frac{\tilde{p}}{\tilde{p} + h}\right),
\end{aligned}$$

and

$$\begin{aligned}
& E(W_{n+1} - W_n | (W_n = w, X_n = x)) \\
&= E(E(\delta)(W_n + \min(\max(X_n, S(W_n)) - \beta W_n, \zeta_n)) - W_n | (W_n = w, X_n = x)) \\
&= E(\delta)(w + \min(F^{-1}\left(\frac{\tilde{p}}{\tilde{p} + h}\right), \zeta_n) - w) \\
&\leq E(\delta)\mu - (1 - E(\delta))w.
\end{aligned}$$

Therefore,

$$\begin{aligned}
& E(v(W_{n+1}, X_{n+1}) - v(W_n, X_n) | W_n = w, X_n = x) \\
&= E(W_{n+1} + |X_{n+1}| - W_n - X_n | W_n = w, X_n = x) \\
&\leq F^{-1}\left(\frac{\tilde{p}}{\tilde{p} + h}\right) + \mu + x + E(\delta)\mu - (1 - E(\delta))w.
\end{aligned}$$

The last expression is < 0 if $-x + (1 - E(\delta))w > F^{-1}\left(\frac{\tilde{p}}{\tilde{p} + h}\right) + (1 + E(\delta))\mu$.

Now define $A = \{(w, x) : (1 - E(\delta))w + |x| \leq F^{-1}\left(\frac{\tilde{p}}{\tilde{p} + h}\right) + (1 + E(\delta))\mu\}$. Note that A is a finite set. Based on the above properties, we have shown that

$$E(W_{n+1} + |X_{n+1}| - W_n - |X_n| | W_n = w, X_n = x) < 0 \text{ if } (W, X) \notin A. \quad (52)$$

Using the Foster's criterion, this implies that $\{(W_n, X_n), n \geq 0\}$ is an irreducible and positive recurrent DTMC. ■