# A geometrical derivation of the Dirac equation 

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#### Abstract

We give a geometrical derivation of the Dirac equation by considering a spin- $\frac{1}{2}$ particle travelling with the speed of light in a cubic spacetime lattice. The mass of the particle acts to flip the multi-component wavefunction at the lattice sites. Starting with a difference equation for the case of one spatial and one time dimensions, we generalize the approach to higher dimensions. Interactions with external electromagnetic and gravitational fields are also considered. One logical interpretation is that only at the lattice sites is the spin- $\frac{1}{2}$ particle aware of its mass and the presence of external fields.


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## I. INTRODUCTION

There are different ways to derive the Dirac equation. But probably there is no derivation more elegant than the one Dirac gave in his book. [1] The derivation based on Wigner's analysis of the irreducible unitary representation of the "Poincare group" (the covering group of the inhomogeneous proper othochronous Lorentz group) certainly is also important. 2] There is another intriguing derivation which Feynman gave (for the $S O(1,1)$ case with one spatial dimension and one time dimension) in his class ${ }^{1}$ and which was given as a problem in his book with Hibbs [3].

In this paper we give another derivation of the Dirac equation. Our approach bears some resemblance to Feynman's and is based on Dirac's observation that the instantaneous velocity operators of the spin- $\frac{1}{2}$ particle (hereafter called by the generic name "the electron") have eigenvalues $\pm c$ and that they anticommute. ${ }^{2}$ (Hereafter, unless clarity demands otherwise, we set the speed of light $c$, as well as Planck's constant $\hbar$, equal to unity.) We assume spacetime to be "filled" with a four-dimensional cubic lattice with lattice length $\Delta x=\Delta y=\Delta z=\Delta t=l$. While it is natural to take $l$ to be the Planck length $\left(\sim 10^{-33}\right.$ cm ), we will simply take it to be a length very small compared to the electron's Compton wavelength, the only intrinsic length available in the problem for a free electron. For the resulting difference equation, the zeroth order term in $\Delta t$ gives a trivial identity while the first order term yields the Dirac equation.

It is interesting that the Dirac equation is invariant under rotations and Lorentz transformations, while the underlying spacetime lattice is not. This situation is one which is not unfamiliar in the spatial dimensions in condensed matter physics. (But a related and more intriguing result was that found by Snyder [4] more than half a century ago, who showed that spacetime being a continuum is not required by Lorentz invariance.) In our approach, an electron's propagation through spacetime can be visualized as consisting of two steps: the

[^1]electron transfers from one spatial lattice site to a neighboring site in one unit of time (thus travelling with the speed of light) and at a lattice site the electron (its multi-component wavefunction, to be more precise) is "flipped" by a mass operator ${ }^{3}$ and interacts with external fields.

In the next section, we consider the case of one spatial dimension (and one time dimension). We treat the case of higher spatial dimensions in Section III. Interactions with external electromagnetic and gravitational fields are considered in Section IV. Further discussions are given in the last Section.

## II. $S O(1,1):(\mathbf{1}+\mathbf{1})$-DIMENSIONAL CASE

We assume that the electron of mass $m$ moves with the speed of light from one lattice site to a neighboring (spatially left or right) site with time $t$ always increasing on the "cubic" spacetime ( $\mathrm{z}, \mathrm{t}$ ) lattice. The wavefunction has two components

$$
\begin{equation*}
\psi(z, t)=\binom{\psi_{+}(z, t)}{\psi_{-}(z, t)} \tag{1}
\end{equation*}
$$

where $\psi_{+}$denotes the component arriving from the event $(z-\Delta t, t-\Delta t)$ while $\psi_{-}$means arriving from $(z+\Delta t, t-\Delta t)$.

Next we assume that, at the lattice site $(z, t)$, the arriving components are partially turned around by a unitary matrix:

$$
\begin{equation*}
\binom{\psi_{+}(z, t)}{\psi_{-}(z, t)}=\mathcal{F}\binom{\psi_{+}(z-\Delta t, t-\Delta t)}{\psi_{-}(z+\Delta t, t-\Delta t)} \tag{2}
\end{equation*}
$$

with the "flip operator" $\mathcal{F}$ defined by

$$
\begin{equation*}
\mathcal{F} \equiv e^{-i F m \Delta t} \tag{3}
\end{equation*}
$$

Here $F$ is a hermitian $2 \times 2$ matrix which we call the "flip matrix" and give the most obvious form

$$
F=\sigma_{x} \equiv \sigma_{1}=\left(\begin{array}{ll}
0 & 1  \tag{4}\\
1 & 0
\end{array}\right)
$$

[^2]with $\sigma_{1}$ being the first Pauli matrix. We will approximate Eq. (2) by a differential equation by first writing
\[

$$
\begin{equation*}
\binom{\psi_{+}(z-\Delta t, t-\Delta t)}{\psi_{-}(z+\Delta t, t-\Delta t)}=\mathcal{T} \psi(z, t) \tag{5}
\end{equation*}
$$

\]

with the "transfer" operator $\mathcal{T}$ given by

$$
\begin{equation*}
\mathcal{T}=e^{-\Delta t\left(\frac{\partial}{\partial t}+\sigma_{3} \frac{\partial}{\partial z}\right)} \tag{6}
\end{equation*}
$$

where $\sigma_{3} \equiv \operatorname{diag}(1,-1)$ is the third Pauli matrix. Then the difference equation Eq. (2) takes the form

$$
\begin{equation*}
\psi(z, t)=\mathcal{F} \mathcal{T} \psi(z, t) \tag{7}
\end{equation*}
$$

The difference equation becomes a differential equation if we limit ourselves to the zeroth order (given by the identity $\psi(z, t)=\psi(z, t))$ and the first order term in $\Delta t$. The first order equation is

$$
\begin{equation*}
i \frac{\partial}{\partial t} \psi(z, t)=m \sigma_{1} \psi(z, t)-i \sigma_{3} \frac{\partial}{\partial z} \psi(z, t) \tag{8}
\end{equation*}
$$

the Dirac equation ${ }^{4}$ in $(1+1)$ dimensions! There is no spin in the $S O(1,1)$ case, the little group of $p^{\mu}$ being trivial. In passing we mention that, for the Dirac equation to hold to all orders in $\Delta t$, due to the fact that $\sigma_{1}$ and $\sigma_{3}$ do not commute with each other, it is necessary to replace $\mathcal{F} \mathcal{T}$ in Eq. (7) by their "symmetrical" product $e^{-\Delta t\left(i m \sigma_{1}+\frac{\partial}{\partial t}+\sigma_{3} \frac{\partial}{\partial z}\right)}$. We will not pursue this issue any further and will assume that $\Delta t$ is sufficiently small that higher order terms are negligible, i.e., the Dirac equation is a good approximation to the original difference equation.

## III. HIGHER SPATIAL DIMENSIONS

We start by reminding ourselves that, for the Dirac equation, the velocity operators $\frac{1}{i \hbar}[\vec{x}, H]=c \vec{\alpha}$ not only have eigenvalues $\pm c$, but they also anticommute with each other. ${ }^{5}$ The latter fact makes the generalization of our approach to more than one spatial dimension

[^3]non-trivial. Before we proceed to the $(3+1)$-dimensional case, let us first discuss the two spatial dimensional $S O(2,1)$ case.

As in the preceeding section (for the $S O(1,1)$ case), we use $\sigma_{3}$ to give the dependence on the $z$ coordinate and $\sigma_{1}$ for the flip operator. Of the three Pauli matrices, we have only $\sigma_{2}$ left; so let us call the second spatial coordinate the $y$ coordinate. Now the problem is to express the dependence of $\psi$ on $y$ in the sense that $\sigma_{3}$ gives the dependence on $z$.

To solve this problem we appeal to rotational invariance and make explicit use of the spin- $\frac{1}{2}$ property of $\psi$. Let $\mathcal{R}$ be the rotation over $\pi / 2$ from the $y$ axis to the $z$ axis. Then $\mathcal{U}(\mathcal{R})$, which represents this rotation, is given by $\mathcal{U}(\mathcal{R})=\frac{1}{\sqrt{2}}\left(1-i \sigma_{1}\right)$. Thus, to the $\sigma_{3} \frac{\partial}{\partial z}$ term in the "transfer" operator $\mathcal{T}$ we add

$$
\begin{equation*}
\mathcal{U}^{-1}(\mathcal{R}) \sigma_{3} \frac{\partial}{\partial y} \mathcal{U}(\mathcal{R})=\sigma_{2} \frac{\partial}{\partial y}, \tag{9}
\end{equation*}
$$

and obtain

$$
\begin{equation*}
i \frac{\partial}{\partial t} \psi(y, z, t)=\left(m \sigma_{1}-i \sigma_{2} \frac{\partial}{\partial y}-i \sigma_{3} \frac{\partial}{\partial z}\right) \psi(y, z, t) . \tag{10}
\end{equation*}
$$

Note that the flip operator, which is used in the preceeding section to invert the $z$-motion, also inverts the $y$-motion. In Dirac's notation [1], we identify $\sigma_{1}=\alpha_{m}, \sigma_{2}=\alpha_{2}, \sigma_{3}=\alpha_{3}$.

Now we recall the general rule [5] that spinors in $2 n$ dimensions and in $2 n+1$ dimensions have $2^{n}$ components. Thus for the case of three spatial dimensions and one temporal dimension $(S O(3,1))$, we need 4 -spinors. And we need, besides $\alpha_{m}, \alpha_{2}, \alpha_{3}$, an extra $\alpha$, all four of which anticommute with one another and have eigenvalues $\pm 1$. Following Dirac [1], we introduce two independent sets of Pauli matrices $\vec{\rho}$ and $\vec{\sigma}$. The $\rho$ 's and the $\sigma$ 's anticommute among each set, whereas the $\rho$ 's and the $\sigma$ 's commute. As we want to make $\alpha_{m}$ look like a flip matrix, we pick

$$
\alpha_{m}=\rho_{1} \sigma_{1}=\left(\begin{array}{llll}
0 & 0 & 0 & 1  \tag{11}\\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0
\end{array}\right)
$$

We complete the set (by the same argument we have used above for the $S O(2,1)$ case) with

$$
\begin{equation*}
\alpha_{1}=\rho_{2} \sigma_{1}, \quad \alpha_{2}=\rho_{3} \sigma_{1}, \quad \alpha_{3}=\sigma_{3} \mathbf{1} \tag{12}
\end{equation*}
$$

bigger in some directions), one cannot specify all three components of the instantaneous velocity simultaneously without running into inconsistencies of predicting a tachyonic speed of $\sqrt{3}$ times the speed of light.

Here $\mathbf{1}$ is the $2 \times 2$ unit matrix. In passing, we mention that it is easy to treat the case of $(4+1)$-dimensional spacetime, as we can now identify $\alpha_{4}$ with $\sigma_{2} \mathbf{1}$.

For the case of $(3+1)$ dimensions, the equation we obtain is

$$
\begin{equation*}
i \frac{\partial}{\partial t} \psi=\left(\alpha_{m} m-\alpha_{1} i \frac{\partial}{\partial x}-\alpha_{2} i \frac{\partial}{\partial y}-\alpha_{3} i \frac{\partial}{\partial z}\right) \psi \tag{13}
\end{equation*}
$$

It is of relevance to remark that the last three terms on the right hand side of Eq. (13) approximate the small finite steps of motion with the speed of light before the event $(x, y, z, t)$ is reached. The term $\alpha_{m} m$ represents the unitary transformation $e^{-i F m d t}$ which takes place at that event (see Eqs. (22) and (3)). Thus the electron is not aware of the fact that it has a mass until it hits a lattice site. If it has no mass, then it is not flipped and it moves at the constant speed of light. If it is massive and is at "rest", then it must be that the electron zigzags around with the speed of light and returns to its original spatial lattice site and wanders around again and returns again etc.

## IV. INTERACTIONS WITH EXTERNAL FIELDS

To put the Dirac equation in a covariant form, we follow the usual procedure of writing $\alpha_{m}=\gamma^{0}\left(\right.$ with $\left.\left(\gamma^{0}\right)^{2}=1\right)$ and multiplying Eq. (13) by $\gamma^{0}$ to yield

$$
\begin{equation*}
\left(i \gamma^{\mu} \partial_{\mu}-m\right) \psi=0 \tag{14}
\end{equation*}
$$

where ( $\mu$ running over $0,1,2,3$ )

$$
\begin{equation*}
\gamma^{\mu} \equiv \gamma^{0} \alpha_{\mu} \tag{15}
\end{equation*}
$$

with $\alpha_{0} \equiv I$, the identity matrix and $\partial_{\mu} \equiv\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$. From the way we have derived the Dirac equation, we can trace the $i \gamma^{\mu} \partial_{\mu}$ term to the "transfer" of the electron at the speed of light between the lattice sites while the $m$ term comes from the "flip" unitary transformation at the lattice sites.

The introduction of an electromagnetic field $A_{\mu}$ is straightforward by using the prescription of minimal substitution in Eq. (14)

$$
\begin{equation*}
i \partial_{\mu} \Rightarrow i \partial_{\mu}+e A_{\mu} \tag{16}
\end{equation*}
$$

Although the $e \alpha_{\mu} A_{\mu}$ term goes together with the $\alpha_{\mu} \partial_{\mu}$ term in the minimal subsitution rule, it is tempting to keep the $\partial_{\mu}$ term identified with the transfer between lattice sites and put
the $e \alpha_{\mu} A_{\mu}$ term together with the $F m$ flip term as taking place at the lattice sites. (But we should keep in mind that, since the Dirac matrices do not commute among themselves, beyond the first order term, there is a difference between associating the interaction term with the "transfer" operator $\mathcal{T}$ or the "flip" operator $\mathcal{F} .{ }^{6}$ )

To incorporate gravitational interactions one needs the tetrad (or vierbein) formalism [6]. One introduces at every event $x$ a set of local inertial coordinates with a tetrad $e_{a}^{\mu}$ of axes labelled by the Minkowski index $a, b, c$ running over $0,1,2,3$. 7] Then the metric in any general noninertial coordinate system is given by $g_{\mu \nu}=e_{\mu}^{a} \eta_{a b} e_{\nu}^{b}$ in terms of the flat Minkowski metric $\eta_{a b}$. Gravitational interactions are introduced via the substitution rule

$$
\begin{equation*}
\gamma^{a} \partial_{a} \Rightarrow \gamma^{a} e_{a}^{\nu}\left(\partial_{\nu}-\frac{1}{4} i \omega_{b c \nu} \sigma^{b c}\right) \tag{17}
\end{equation*}
$$

where $\omega_{a \nu}^{b}=\left(\partial_{\nu} e_{a}^{\mu}+\Gamma_{\nu \lambda}^{\mu} e_{a}^{\lambda}\right) e_{\mu}^{b}$ in terms of the affine connection $\Gamma_{\nu \lambda}^{\mu}$ and $\sigma^{b c}=\frac{1}{2} i\left[\gamma^{b}, \gamma^{c}\right]$. At every spacetime lattice site labelled by $x$, we have a tetrad $e_{a}^{\mu}$. In our interpretation, the electron travels with the speed of light between lattice sites; this is represented by $\gamma^{a} e_{a}^{\nu} \partial_{\nu}$. Then at the lattice site there is a unitary transformation which, in addition to the mass "flip", now contains the interaction term $\gamma^{a} e_{a}^{\nu} \omega_{b c \nu} \sigma^{b c}$.

## V. DISCUSSIONS

We have presented a novel derivation of the Dirac equation, hoping to shed new light on the physics of the electron. Motivated by the distinct possibility that the underlying spacetime is discrete at small scales, we have started with a discrete "cubic" lattice. The resulting Dirac equation emerges as the lowest nontrivial order of approximation. Thus the observed Lorentz invariance does not preclude the existence of a discrete spacetime at small scales.

Is our approach useful? We think so. (1) The very fact that the underlying spacetime is discrete means that there is automatically an ultraviolet cutoff which may be used to ameliorate divergence problems in nonrenormalizable theories like (perhaps) quantum gravity. (2) Our starting point is a difference equation rather than a differential equation. While difference equations are more tedious to deal with analytically, they may hold some advantages in numerical calculations.

[^4]We conclude with some speculations and a couple of open questions. In the scenario we have proposed, the electron travels between lattice sites with the speed of light. Only at the lattice sites does the electron "feel" its mass and perhaps also the presence of all external fields. ${ }^{7}$ (Since it is a Yukawa-type interaction which, via the Higgs mechanism, generates mass for the electron, it seems reasonable to assume that at least Yukawa-type interactions take place only at the lattice sites where the mass operator makes its presence felt.) But if gravitational interactions also take place mainly at the lattice sites, does that mean spacetime vertices somehow play an important role in concentrating curvature? And if so, how is this description of geometry and topology related to the Regge calculus [8], for example? These problems deserve further investigations.

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[^1]:    ${ }^{1}$ L. Brown, private communication (2001).
    ${ }^{2}$ The Hamiltonian for a free electron is given by $H=c \alpha_{i} p_{i}+\rho_{3} m c^{2}$ (in Dirac's notation) with anticommuting $\alpha_{i}$ and $\rho_{3}$. The $i$-th component of the velocity is $\frac{d x_{i}}{d t}=(i \hbar)^{-1}\left[x_{i}, H\right]=c \alpha_{i}$. Thus $\frac{d x_{i}}{d t}$ has as eigenvalues $\pm c$, corresponding to the eigenvalues $\pm 1$ of $\alpha_{i}$. This result is actually implied by the uncertainty principle. Dirac 1] also shows that $c \alpha_{i}$ consists of two parts, a constant part $c^{2} p_{i} H^{-1}$, connected with the momentum by the classical relativistic formula, and an oscillatory part whose frequency is high, being at least $m c^{2} / \pi \hbar$.

[^2]:    ${ }^{3}$ This is in consonance with the mass term being equivalent to a helicity flip.

[^3]:    ${ }^{4}$ In our representation of the Dirac matrices, the positive-energy spinor for the plane-wave solution takes the form, aside from a normalization constant, $u(p) \sim\left(-\sigma_{1} \sqrt{p^{2}+m^{2}}-i \sigma_{2} p-m\right) u(0)$, with $u(0)$ being the two-component spinor $(1,-1)$, and the negative-energy spinor $v(p) \sim\left(\sigma_{1} \sqrt{p^{2}+m^{2}}+i \sigma_{2} p-m\right) v(0)$, with $v(0)=(1,1)$.
    ${ }^{5}$ Here an analogy with spin can be made. Just as one cannot specify all three components of the spin simultaneously in quantum mechanics without running into inconsistencies (of predicting a spin $\sqrt{3}$ times

[^4]:    ${ }^{6}$ For the $S O(1,1)$ case, the $A_{\mu}$ term can be incorporated into the flip operator in Eq. (2).

[^5]:    ${ }^{7}$ It is natural to visualize interactions taking place at lattice sites (in conjunction with the mass flip). After all, interactions occur at spacetime points and spacetime points are the lattice sites if the underlying spacetime is discrete.

