SPECIAL FUNCTIONS, CONFORMAL BLOCKS, BETHE ANSATZ, AND $SL(3,\mathbb{Z})$

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Abstract. This is the talk of the second author at the meeting "Topological Methods in Physical Sciences", London, November 2000. We review our work on KZB equations.

KZB equations.
The Knizhnik-Zamolodchikov-Bernard (KZB) equations are a system of differential equations arising in conformal field theory on Riemann surfaces. For each $g, n \in \mathbb{Z}_{\geq 0}$, a simple complex Lie algebra $\mathfrak{g}$, n highest weight $\mathfrak{g}$-modules V_i and a complex parameter κ, we have such a system of equations. In the case of genus $g = 1$, they have the form

$$\kappa \frac{\partial v}{\partial z_j} = -\sum_\nu h_\nu^{(j)} \frac{\partial v}{\partial \lambda_\nu} + \sum_{l,\, l \neq j} r(z_j - z_l, \lambda, \tau)^{(j,l)} v, \qquad j = 1, ..., n,$$

$$4\pi i \kappa \frac{\partial v}{\partial \tau} \;=\; \Delta_\lambda v + \frac{1}{2} \sum_{i,j} s(z, \lambda, \tau)^{(i,j)} v.$$

The unknown function v takes values in the zero weight space $V[0] = \cap_{x \in \mathfrak{h}} \mathrm{Ker}(x)$ of the tensor product $V = V_1 \otimes \cdots \otimes V_n$ with respect to the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. It depends on variables $z_1, ..., z_n \in \mathbb{C}$, modulus τ of the elliptic curve and $\lambda = \sum \lambda_\nu h_\nu \in \mathfrak{h}$, where (h_ν) is an orthonormal basis of $\mathfrak{h}$ with respect to a fixed invariant bilinear form. In the equation, $r, s \in \mathfrak{g} \otimes \mathfrak{g}$ are suitable given tensor valued functions, [FW].

The second equation is called the KZB-heat equation.

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Example. For $\mathfrak{g} = sl_N$, $n = 1$, $V = S^{mN}\mathbb{C}^N$, $\mathfrak{h} = \mathbb{C}^N/\mathbb{C}(1, ..., 1)$, the weight-zero space $V[0]$ is one dimensional, the KZB equations are scalar equations $\partial v/\partial z_1 = 0$ and

$$4\pi i\kappa \frac{\partial v}{\partial \tau} = \sum_{i=1}^{N} \frac{\partial^2 v}{\partial \lambda_i^2} + 2m(m+1) \sum_{1\leq i<j\leq N} \rho'(\lambda_i - \lambda_j, \tau)v\,. \tag{1}$$

Here $'$ denotes the derivative with respect to the first argument and ρ is defined in terms of the first Jacobi theta function,

$$\theta(t,\tau) = -\sum_{j\in\mathbb{Z}} e^{\pi i (j+\frac{1}{2})^2\tau + 2\pi i (j+\frac{1}{2})(t+\frac{1}{2})}, \qquad \rho(t,\tau) = \frac{\theta'(t,\tau)}{\theta(t,\tau)}.$$

Notice that $\rho' = -\wp + c$ where $\wp(t,\tau)$ is the Weierstrass function, $c = c(\tau)$ a function of τ, and we recover in the right hand side of the KZB-heat equation the Hamilton operator $H_{N,m}$ of the elliptic Calogero-Moser quantum N-body system,

$$-H_{N,m} = \sum_{i=1}^{N} \frac{\partial^2}{\partial \lambda_i^2} + 2m(m+1)\sum_{i<j} \rho'(\lambda_i - \lambda_j, \tau)$$

with coupling constant $m(m+1)$.

In particular, if $\mathfrak{g} = sl_2$, then the Cartan subalgebra $\mathfrak{h}$ can be identified with $\mathbb{C}$, and $\lambda \in \mathbb{C}$. For the irreducible $2m+1$ dimensional module V, the KZB-heat equation takes the form

$$2\pi i\kappa \frac{\partial v}{\partial \tau}(\lambda,\tau) \;=\; \frac{\partial^2 v}{\partial \lambda^2}(\lambda,\tau) \;+\; m(m+1)\rho'(\lambda,\tau)\, v(\lambda,\tau)\,. \tag{2}$$

A remarkable fact about all forms of the KZ equations is that they can be realized geometrically. They have solutions defined as hypergeometric integrals depending on parameters.

Example. Consider the function

$$v(\lambda,\tau) \;=\; \int_0^1 \left(\frac{\theta(t,\tau)}{\theta'(0,\tau)}\right)^{-\frac{2}{\kappa}} \frac{\theta(\lambda - t,\tau)\theta'(0,\tau)}{\theta(\lambda,\tau)\theta(t,\tau)} g(\lambda - \frac{2}{\kappa}t,\tau)\,dt$$

where $g(\lambda,\tau)$ is any solution of the heat equation

$$2\pi i\kappa \frac{\partial g}{\partial \tau}(\lambda,\tau) \;=\; \frac{\partial^2 g}{\partial \lambda^2}(\lambda,\tau).$$

For instance, $g(\lambda,\tau) = e^{\lambda\mu + \frac{\mu^2}{2\pi i\kappa}\tau}$ for $\mu \in \mathbb{C}$. Then v is a solution of the KZB-heat equation (2) with $m = 1$ [FV1].

Remark. Simplest KZ equations have the classical Gauss hypergeometric function as their solution. It is natural to consider solutions of all KZ type equations as generalized hypergeometric functions. Thus there are hypergeometric functions associated with curves of any genus. The function $v(\lambda,\tau)$ is the simplest elliptic hypergeometric function.

The Gauss hypergeometric function is

$$\frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)} F(a,b,c;z) \;=\; \int_1^\infty t^{a-c}(t-1)^{c-b-1}(t-z)^{-a}\,dt\,.$$

The functions

$$\left(\frac{\theta(t,\tau)}{\theta'(0,\tau)}\right)^a \qquad \text{and} \qquad \frac{\theta(\lambda-t,\tau)\theta'(0,\tau)}{\theta(\lambda,\tau)\theta(t,\tau)}$$

are elliptic analogs of the functions t^a and $1/t$, respectively. The hypergeometric solutions to the KZB equations were discovered through this analogy.

Remark. If κ tends to 0, we are dealing with the eigenfunction problem. Find eigenfunctions of the elliptic Calogero-Moser Hamiltonian,

$$\sum_{i=1}^N \frac{\partial^2 v}{\partial\lambda_i^2} + 2m(m+1)\sum_{1\le i<j\le N}\rho'(\lambda_i-\lambda_j,\tau)\,v \;=\; E\,v\,.$$

The stationary phase method applied to hypergeometric solutions of the KZB equations gives eigenfunctions. For instance, application of the method to the function $v(\lambda,\tau)$ implies that for every $\mu\in\mathbb{C}$, the function $v(\lambda)=e^{\lambda\mu}\theta(\lambda-t_0,\tau)/\theta(\lambda,\tau)$ is an eigenfunction of the operator

$$\frac{\partial^2}{\partial\lambda^2} \;+\; 2\rho'(\lambda,\tau)$$

if t_0 is a critical point of the function $e^{-\mu t}\theta(t,\tau)$. This is a Bethe ansatz type formula, see [FV1, FV2]. For $\mathfrak{g}\;=sl_2$, the Bethe ansatz formulas reduce to Hermite's 1872 solution of the Lamé equation, see [WW].

The fact that solutions have the form of explicitly written integrals is a tool to study solutions as well as equations, for instance , the monodromy properties of solutions or their modular properties with respect to changes of τ.

Modular symmetries of KZB.

For any $\begin{pmatrix} a & b \\ c & d\end{pmatrix}\in SL(2,\mathbb{Z})$, the elliptic curves with moduli τ and $(a\tau+b)/(c\tau+d)$ are isomorphic, the corresponding KZB equations are related. Having a family of solutions one can ask about monodromy properties of solutions with respect to transformations of the lattice $\mathbb{Z}+\tau\mathbb{Z}$.

Example. Consider equation (2) for $m=0$,

$$2\pi i\kappa\frac{\partial v}{\partial\tau} \;=\; \frac{\partial^2 v}{\partial\lambda^2}\,. \tag{3}$$

If $v(\lambda,\tau)$ is a solution, then $\tilde v(\lambda,\tau)=v(\lambda,\tau+1)$ and

$$\tilde v(\lambda,\tau) \;=\; \frac{1}{\sqrt{\tau}}\,e^{-\frac{\pi i\lambda^2}{2\tau}}\,v(\frac{\lambda}{\tau},\,-\frac{1}{\tau})$$

are solutions too. If $\{v_\mu(\lambda,\tau)\}$ is a family of solutions depending on a parameter μ, then one can ask about "monodromy" relations of the three families of solutions: $\{v_\mu(\lambda,\tau)\}$, $\{v_\mu(\lambda,\tau+1)\}$, and $\{\frac{1}{\sqrt{\tau}}\, e^{-\frac{\pi i\lambda^2}{2\tau}}\, v_\mu(\frac{\lambda}{\tau},\, -\frac{1}{\tau})\}$.

The KZB-heat equation as a flat connection. One can consider equation (2) as the equation for horizontal sections of a connection on a vector bundle over the upper half plane $\mathfrak{H}_+$ whose fiber $F(\tau)$ over a point τ is the space of functions of λ.

If κ is a positive integer not less than 2, then the bundle has a finite dimensional subbundle (of conformal blocks) invariant with respect to the connection and consisting of certain theta functions of level κ, [EFK, FW, FV1].

In this paper we address the following

Problem. *Quantize the KZB heat equation, study modular properties of the quantization.*

It turns out that the KZB-heat equation is quantized to a difference qKZB-heat equation (with step p) in such a way that the $SL(2,\mathbb{Z})$ symmetry of the KZB-heat equation related to the lattice $Z+\tau\mathbb{Z}$ is quantized to an $SL(3,\mathbb{Z})$ symmetry related to the lattice $Z+\tau\mathbb{Z}+p\mathbb{Z}$.

A quantization of the heat equation is a discrete connection over $\mathfrak{H}_+$ with the same fiber, i.e. a linear operator $T(\tau,\tau+p) : F(\tau+p)\to F(\tau)$. This linear operator tends to

$$1 \,+\, \mathrm{const}\, p\, \Big(\frac{\partial^2}{\partial\lambda^2} \,+\, m(m+1)\rho'(\lambda,\tau)\,\Big) \,+\, \dots$$

as $p\to 0$.

For methodological reasons we describe first a quantization of equation (3). For a nonzero $\eta\in\mathbb{C}$, introduce linear operators

$$\begin{aligned} U \quad &: \quad f(\lambda) \,\mapsto\, \frac{i}{\sqrt{4i\eta}}\int_{\eta\mathbb{R}} e^{-\pi i\frac{\lambda\mu}{2\eta}}\, f(-\mu)\, d\mu\,,\\ \alpha \quad &: \quad f(\lambda) \,\mapsto\, e^{-\pi i\frac{\lambda^2}{4\eta}}\, f(\lambda)\,. \end{aligned}$$

Define

$$T(\tau,\tau+p)=\alpha U\alpha\,:\, f(\lambda)\,\mapsto\, \frac{i}{\sqrt{4i\eta}}\int_{\eta\mathbb{R}} e^{-\pi i\frac{(\lambda+\mu)^2}{4\eta}}\, f(-\mu)\, d\mu\,. \tag{4}$$

The translation operator $T(\tau,\tau+p)$ in this case is basically just the Fourier transform.

The qKZB-heat equation is the equation for flat sections,

$$v(\lambda,\tau) \,=\, \frac{i}{\sqrt{4i\eta}}\int_{\eta\mathbb{R}} e^{-\pi i\frac{(\lambda+\mu)^2}{4\eta}}\, v(-\mu,\tau+p)\, d\mu\,. \tag{5}$$

Set $p = -2\kappa\eta$. Let $\eta \to 0$ and λ, τ, κ fixed.

Theorem. *Let $v_\eta(\lambda,\tau)$ be a family of solutions of (5) with asymptotics $v_\eta(\lambda,\tau) = v^0(\lambda,\tau) + \eta v^1(\lambda,\tau) + \ldots$ Then $v^0(\lambda,\tau)$ satisfies (3).*

Proof. The stationary phase asymptotic expansion of the right hand side of (5) is

$$v^0(\lambda,\tau) + \eta v^1(\lambda,\tau) + \frac{i\eta}{\pi}(2\pi i\kappa \frac{\partial v^0}{\partial \tau}(\lambda,\tau) - \frac{\partial^2 v^0}{\partial \lambda^2}(\lambda,\tau)) + O(\eta^2) + \ldots$$

□

Introduce a function

$$u(\lambda,\mu,\tau,p,\eta) = e^{-\pi i \frac{\lambda\mu}{2\eta}}\,. \tag{6}$$

For every $\mu \in \mathbb{C}$, the function u is a projective solution of the qKZB-heat equation (5),

$$u(\lambda,\mu,\tau,p,\eta) = e^{-\pi i \frac{\mu^2}{4\eta}}\,(T(\tau,\tau+p)u)(\lambda,\mu,\tau+p,p,\eta),$$

in particular, $v(\lambda,\mu,\tau,p,\eta) = e^{\pi i \frac{\mu^2}{4\eta}\frac{\tau}{p} - \pi i \frac{\lambda\mu}{2\eta}}$ is a true solution.

Remark [FV3]. Let κ be a positive integer. The functions

$$\theta_{j,\kappa}(\lambda,\tau) = \sum_{r\in\mathbb{Z}+j/2\kappa} e^{2\pi i\kappa(r^2\tau + r\lambda)}, \qquad j \in \mathbb{Z}/2\kappa\mathbb{Z},$$

form the space $\Theta_\kappa(\tau)$ of theta functions of level κ. Let $E(\tau) = \{f \in \Theta_\kappa(\tau) \,|\, f(-\lambda) = -f(\lambda)\}$ be the space of odd theta functions. The translation operator $T(\tau,\tau+p)$ maps $E(\tau+p)$ to $E(\tau)$ if $-p/2\eta = \kappa$.

This statement is based on the identity

$$\theta_{j,\kappa}(\lambda,\tau) = \frac{i}{\sqrt{4i\eta}} \int_{2\eta\mathbb{R}} e^{-\frac{i\pi}{4\eta}(\lambda+\mu)^2}\theta_{j,\kappa}(-\mu,\tau-2\eta\kappa)\,d\mu\,.$$

The space $E(\tau)$ is the quantization of the finite dimensional space of conformal blocks.

Modular properties of the q-heat operator (in this case of the Fourier transform).

The group $SL(3,\mathbb{Z})$ is generated by the elementary matrices $e_{i,j}$, $i \neq j$. The elementary matrix $e_{i,j}$ is the element of $SL(3,\mathbb{Z})$ which differ from the identity matrix by having the i,j matrix element equal to 1. The relations can be chosen [M] to be

$$\begin{aligned} e_{i,j}e_{k,l} &= e_{k,l}e_{i,j}, \qquad i \neq l, \quad j \neq k, \\ e_{i,j}e_{j,k} &= e_{i,k}e_{jk}e_{i,j}, \\ (e_{1,3}e_{3,1}^{-1}e_{1,3})^4 &= 1. \end{aligned}$$

Consider $\mathbb{C}^3$ with coordinates x_1, x_2, x_3. The group $SL(3,\mathbb{Z})$ acts on $\mathbb{C}^3$ in the standard way. Consider the trivial bundle over $\mathbb{C}^3$ with the same fiber F over a point $x \in \mathbb{C}^3$. We define a projectively flat connection over the orbit of a point. Namely, for any generator $e_{i,j}$, we define a linear operator

$$\varphi_{i,j}(x) \,:\, F(e_{i,j}^{-1}x) \,\to\, F(x)$$

so that all relations in $SL(3,\mathbb{Z})$ are projectively satisfied. That means that the linear operator corresponding to the left hand side of a relation is equal to the linear operator corresponding to the right hand side of the relation multiplied by a number.

Remark. The operator $\varphi_{2,1}(p,\tau,\eta) \,:\, F(p,\tau-p,\eta) \,\to\, F(p,\tau,\eta)$ will correspond to the qKZB-heat operator.

Introduce linear operators

$$\begin{aligned} U(x_1,x_2,x_3) \quad &: \quad f(\lambda) \,\mapsto\, \int_{x_3\mathbb{R}} e^{-\pi i \frac{\lambda\mu}{2x_3}}\, f(-\mu)\, d\mu\,, \\ \alpha(x_3) \quad &: \quad f(\lambda) \,\mapsto\, e^{-\pi i \frac{\lambda^2}{4x_3}}\, f(\lambda)\,, \\ \beta(x_1,x_2,x_3) \quad &: \quad f(\lambda) \,\mapsto\, e^{-\pi i \frac{\lambda^2}{4}\frac{x_1}{x_2x_3}}\, f(\lambda)\,. \end{aligned}$$

Set

$$\begin{aligned} \varphi_{1,3}(x_1,x_2,x_3) &= 1, \\ \varphi_{2,3}(x_1,x_2,x_3) &= 1, \\ \varphi_{1,2}(x_1,x_2,x_3) &= \alpha(x_3), \\ \varphi_{3,2}(x_1,x_2,x_3) &= \beta(x_1,x_2-x_3,x_3), \\ \varphi_{2,1}(x_1,x_2,x_3) &= \alpha(x_3)\,U(x_2,x_2-x_1,x_3)\,\alpha(x_3), \\ \varphi_{3,1}(x_1,x_2,x_3) &= \beta(x_1-x_3,-x_3,x_2)^{-1}\,U(x_1-x_3,-x_3,x_2)^{-1}\,\beta(x_3,x_2,x_3-x_1)^{-1}. \end{aligned}$$

Theorem. *The operators $\varphi_{i,j}$ define a projectively flat connection over orbits of the $SL(3,\mathbb{Z})$ action.*

Remark. Consider $\mathbb{C}^3$ and the projectivization of the dual space, $P(\mathbb{C}^3)^*$. Consider $X \subset (\mathbb{C}-0)^3 \times P(\mathbb{C}^3)^*$ where

$$X = \{((x_1,x_2,x_3),(y_1:y_2:y_3))\,|\,x_1y_1+x_2y_2+x_3y_3=0\,\}\,.$$

The natural projection $X \to \mathbb{C}^3 - 0$ is a projective line bundle. The group $SL(3,\mathbb{Z})$ acts on X, $g:(x,y)\mapsto (gx,(g^t)^{-1}y)$ for any $g \in SL(3,\mathbb{Z})$. Fix an affine coordinate on fibers, $t:(y_1:y_2:y_3)\mapsto y_2/y_1$. Then for any i,j, $i\neq j$, we have

$$e_{i,j}:(e_{i,j}^{-1}x,t)\mapsto (x,f_{i,j}(x,t))$$

where $f_{1,3}(x,t) = f_{2,3}(x,t) = t$, $f_{1,2}(x,t) = t-1$, $f_{3,2}(x,t) = (tx_3+x_1)/(x_3-x_2)$, $f_{2,1}(x,t) = t/(1-t)$, $f_{3,1}(x,t) = t(x_3-x_1)/(tx_2+x_3)$.

The $SL(3,\mathbb{Z})$-action on X is closely related to the projectively flat connection described in the Theorem. Namely, set

$$G(\lambda;x_1,x_2,x_3;t) \,=\, e^{\frac{\pi i}{4}\lambda^2\frac{t}{x_3}}\,.$$

Call a Gaussian in the fiber $F(x_1,x_2,x_3)$ a function of the form const$\cdot G(\lambda;x_1,x_2,x_3;t)$ for some number t. The linear operators $\varphi_{i,j}(x)\,:\,F(e_{i,j}^{-1}x)\to F(x)$ preserve the Gaussians. Moreover, for all i,j, we have

$$\varphi_{i,j}(x)\,:\,G(\lambda;x_1,x_2,x_3;t)\mapsto \mathrm{const}\cdot G(\lambda;x_1,x_2,x_3;f_{i,j}(t,x))\,.$$

Remark. The Theorem easily follows from the following two main equations satisfied by the Fourier transform, which we call

the *q-heat equation,*

$$\alpha(x_3)\,U(x_1,x_1+x_2,x_3)\,\alpha(x_3)\,U(x_1+x_2,x_2,x_3)\,\alpha(x_3)\,=\,U(x_1,x_2,x_3)\,, \tag{7}$$

and the *modular equation,*

$$\begin{aligned} U(x_3,x_2,-x_1)\,\beta(-x_3,x_2,x_1)\,U(x_1,-x_3,x_2)\,=& \\ \beta(x_2,x_3,x_1)\,U(x_1,x_2,x_3)\,\beta(x_1,x_2,x_3)\,.& \end{aligned} \tag{8}$$

Proof of the q-heat equation.

$$\begin{aligned} e^{-\pi i\frac{\lambda^2}{4x_3}}\int e^{-\pi i\frac{\lambda\nu}{2x_3}}\left(e^{-\pi i\frac{\nu^2}{4x_3}}\int e^{\pi i\frac{\nu\mu}{2x_3}}\,e^{-\pi i\frac{\mu^2}{4x_3}}\,f(-\mu)\,d\mu\right)d\nu \,=& \\ \int\left(\int e^{-\pi i\frac{(\lambda+\mu-\nu)^2}{4x_3}}d\nu\right)e^{-\pi i\frac{\lambda\mu}{2x_3}}\,f(-\mu)\,d\mu\,=& \\ \mathrm{const}(x_3)\int e^{-\pi i\frac{\lambda\mu}{2x_3}}\,f(-\mu)\,d\mu\,.& \end{aligned}$$

Proof of the modular equation.

$$\begin{aligned} \int e^{\pi i\frac{\lambda\nu}{2x_1}}e^{\pi i\frac{\nu^2}{4}\frac{x_3}{x_1x_2}}\left(\int e^{\pi i\frac{\nu\mu}{2x_2}}\,f(-\mu)\,d\mu\right)d\nu\,=& \\ \int\left(\int e^{\pi i\frac{1}{4x_1x_2x_3}(x_1\mu+x_2\lambda+x_3\nu)^2}\,d\nu\right)e^{-\frac{\pi i}{4}(\lambda^2\frac{x_2}{x_1x_3}+2\frac{\lambda\mu}{x_3}+\mu^2\frac{x_1}{x_2x_3})}f(-\mu)\,d\mu\,=& \\ \mathrm{const}(x_1,x_2,x_3)\,e^{-\pi i\frac{\lambda^2}{4}\frac{x_2}{x_1x_3}}\int e^{-\pi i\frac{\lambda\mu}{2x_3}}\,e^{-\pi i\frac{\mu^2}{4}\frac{x_1}{x_2x_3}}\,f(-\mu)\,d\mu\,.& \end{aligned}$$

Similarly, one can quantize equation (2) for $m = 1$,

$$2\pi i\kappa \frac{\partial v}{\partial \tau}(\lambda,\tau) \,=\, \frac{\partial^2 v}{\partial \lambda^2}(\lambda,\tau) \,+\, 2\rho'(\lambda,\tau)\, v(\lambda,\tau)\,. \tag{9}$$

Introduce a function

$$u(\lambda,\mu,\tau,p,\eta) \,=\, e^{-\pi i\frac{\lambda\mu}{2\eta}} \int_0^1 \Omega_{2\eta}(t,\tau,p)\, \frac{\theta(\lambda+t,\tau)\theta(\mu+t,p)}{\theta(t-2\eta,\tau)\theta(t-2\eta,p)}\, dt \tag{10}$$

where

$$\Omega_a(t,\tau,p) = \prod_{j,k=0}^{\infty} \frac{(1-e^{2\pi i(t-a+j\tau+kp)})(1-e^{2\pi i(-t-a+(j+1)\tau+(k+1)p)})}{(1-e^{2\pi i(t+a+j\tau+kp)})(1-e^{2\pi i(-t+a+(j+1)\tau+(k+1)p)})}\,.$$

This is the analog for $m = 1$ of function (6).

Define the translation operator for $m = 1$ as

$$T(\tau,\tau+p)\,:\, f(\lambda) \,\mapsto\, -\frac{1}{4\pi\sqrt{i\eta}}\, e^{-\pi i\frac{\lambda^2}{4\eta}} \int_{\eta\mathbb{R}} u(\lambda,\mu,\tau,\tau+p,\eta) \qquad \times$$
$$\frac{\theta(4\eta,\tau+p)\theta'(0,\tau+p)}{\theta(\mu-2\eta,\tau+p)\theta(\mu+2\eta,\tau+p)} \quad e^{-\pi i\frac{\mu^2}{4\eta}}\, f(-\mu)\, d\mu\,.$$

The qKZB-heat equation is the equation

$$v(\lambda,\tau) \,=\, (T(\tau,\tau+p)v)(\lambda,\tau+p)\,. \tag{11}$$

It turns out that the translation operator for $m = 1$ has properties analogous to the properties of the Fourier transform, see [FV3, FV4, FV5]. Namely,

1. *The semiclassical limit of equation (11) is equation (9).*
2. *For every $\mu \in \mathbb{C}$, the function u in (10) is a projective solution of the qKZB-heat equation (11),*

   $$u(\lambda,\mu,\tau,p,\eta) \,=\, e^{-\pi i\frac{\mu^2}{4\eta}}\, (T(\tau,\tau+p)u)(\lambda,\mu,\tau+p,p,\eta),$$

   in particular, $v(\lambda,\mu,\tau,p,\eta) = e^{\pi i\frac{\mu^2}{4\eta}\frac{\tau}{p}} u(\lambda,\mu,\tau,p,\eta)$ is a true solution.
3. *The integral operator T has an $SL(3,\mathbb{Z})$ symmetry similar to the $SL(3,\mathbb{Z})$ symmetry of the Fourier transform.*

Remark. The fact that u is a (projective) solution of the qKZB-heat equation is a relation of the form

$$u(\lambda,\nu,\tau,p) \,=\, u(\lambda,\mu,\tau,\tau+p) * u(\mu,\nu,\tau+p,p) \tag{12}$$

where $*$ is a suitable convolution. If τ, p tend to infinity, then the function u has a trigonometric limit. In this limit, equation (12) becomes a simplest example of the Macdonald-Mekhta identity which has the form $u(\lambda,\nu) \,=\, u(\lambda,\mu) * u(\mu,\nu)$. The theory of the qKZB-heat equation is an elliptic analog of Macdonald's theory, see [EV1].

Remark. We have

$$\Omega_a(t,\tau,p) \;=\; \frac{\Gamma(t+a,\tau,p)}{\Gamma(t-a,\tau,p)}$$

where

$$\Gamma(t,\tau,p) = \prod_{j,k=0}^{\infty} \frac{1-e^{2\pi i(-t+(j+1)\tau+(k+1)p)}}{1-e^{2\pi i(t+j\tau+kp)}}$$

is the elliptic gamma function. The elliptic gamma function has an $SL(3,\mathbb{Z})$ symmetry [FV5] based on the following two main equations, which we call

the *q-heat equation,*

$$\Gamma(t+\tau,\tau,\tau+p)\,\Gamma(t,\tau+p,p) \;=\; \Gamma(t,\tau,p),$$

and the *modular equation,*

$$\Gamma(\frac{t}{p},\frac{\tau}{p},-\frac{1}{p}) \;=\; e^{i\pi Q(t;\tau,p)}\,\Gamma(\frac{t-p}{\tau},-\frac{1}{\tau},-\frac{p}{\tau})\,\Gamma(t,\tau,p),$$

where

$$\begin{aligned} Q(t;\tau,p) \;&=\; \frac{1}{3\tau p}t^3 - \frac{\tau+p-1}{2\tau p}t^2 + \frac{\tau^2+p^2+3\tau p-3\tau-3p+1}{6\tau p}t \\ &\quad + \frac{1}{12}(\tau+p-1)(\tau^{-1}+p^{-1}-1), \end{aligned}$$

cf. (7), (8).

The q-heat and modular equations for the gamma function imply the $SL(3,\mathbb{Z})$ symmetry of the translation operator for $m=1$.

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