A STUDY OF SATURATED TENSOR CONE FOR SYMMETRIZABLE KAC-MOODY ALGEBRAS

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1. Introduction

Let $\mathfrak{g}$ be a symmetrizable Kac-Moody Lie algebra with the standard Cartan subalgebra $\mathfrak{h}$ and the Weyl group W. Let P_+ be the set of dominant integral weights. For $\lambda \in P_+$, let $L(\lambda)$ be the irreducible, integrable, highest weight representation of $\mathfrak{g}$ with highest weight λ. For a positive integer s, define the *saturated tensor semigroup* as

$$\Gamma_s := \{(\lambda_1, \dots, \lambda_s, \mu) \in P_+^{s+1} : \exists\, N > 1 \text{ with } L(N\mu) \subset L(N\lambda_1) \otimes \cdots \otimes L(N\lambda_s)\}.$$

The aim of this paper is to begin a systematic study of Γ_s in the infinite dimensional symmetrizable Kac-Moody case. In this paper, we produce a set of necessary inequalities satisfied by Γ_s, which we describe now. Let $X = G^{\min}/B$ be the standard full KM-flag variety associated to $\mathfrak{g}$, where $G^{\min}$ is the 'minimal' Kac-Moody group with Lie algebra $\mathfrak{g}$ and B is the standard Borel subgroup of $G^{\min}$. For $w \in W$, let $X_w = \overline{BwB/B} \subset X$ be the corresponding Schubert variety. Let $\{\varepsilon^w\}_{w\in W} \subset H^*(X, \mathbb{Z})$ be the (Schubert) basis dual (with respect to the standard pairing) to the basis of the singular homology of X given by the fundamental classes of X_w. The following result is our first main theorem valid for any symmetrizable $\mathfrak{g}$ (cf. Theorem 3.3).

Theorem 1.1. *Let $(\lambda_1, \dots, \lambda_s, \mu) \in \Gamma_s$. Then, for any $u_1, \dots, u_s, v \in W$ such that $n^v_{u_1,\dots,u_s} \neq 0$, where*

$$\varepsilon^{u_1} \dots \varepsilon^{u_s} = \sum_w n^w_{u_1,\dots,u_s}\, \varepsilon^w,$$

we have

$$\left(\sum_{j=1}^{s} \lambda_j(u_j x_i) \right) - \mu(v x_i) \geq 0, \text{ for any } x_i,$$

where $x_i \in \mathfrak{h}$ is dual to the simple roots of $\mathfrak{g}$.

The proof of the theorem relies on the Kac-Moody analogue of the Borel-Weil theorem and the Geometric Invariant Theory (specifically the Hilbert-Mumford index). We conjecture that the above inequalities are sufficient as well to describe Γ_s. In fact, we conjecture a much sharper result, where much fewer inequalities suffice to describe the semigroup Γ_s. To explain our conjecture, we need some more notation.

Let $P \supset B$ be a (standard) parabolic subgroup and let $X_P := G^{\min}/P$ be the corresponding partial flag variety. Let W_P be the Weyl group of P (which is, by definition, the Weyl group of the Levi L of P) and let W^P be the set of minimal length coset representatives of cosets in W/W_P. The projection map $X \to X_P$ induces an injective homomorphism $H^*(X_P,\mathbb{Z}) \to H^*(X,\mathbb{Z})$ and $H^*(X_P,\mathbb{Z})$ has the Schubert basis $\{\varepsilon_P^w\}_{w\in W^P}$ such that ε_P^w goes to ε^w for any $w \in W^P$. As defined by Belkale-Kumar [BK, §6] in the finite dimensional case (and extended here in Section 7 for any symmetrizable Kac-Moody case), there is a new deformed product $\odot_0$ in $H^*(X_P,\mathbb{Z})$, which is commutative and associative. Now, we are ready to state our conjecture (see Conjecture 7.3).

1.2. **Conjecture.** *Let $\mathfrak{g}$ be any indecomposable symmetrizable Kac-Moody Lie algebra and let $(\lambda_1,\dots,\lambda_s,\mu) \in P_+^{s+1}$. Assume further that none of λ_j is W-invariant and $\mu - \sum_{j=1}^s \lambda_j \in Q$, where Q is the root lattice of G. Then, the following are equivalent:*

(a) $(\lambda_1,\dots,\lambda_s,\mu) \in \Gamma_s$.

(b) For every standard maximal parabolic subgroup P in $G^{\min}$ and every choice of $s+1$-tuples $(w_1,\dots,w_s,v) \in (W^P)^{s+1}$ such that ϵ_P^v occurs with coefficient 1 in the deformed product

$$\epsilon_P^{w_1} \odot_0 \cdots \odot_0 \epsilon_P^{w_s} \in \big(H^*(X_P,\mathbb{Z}),\odot_0\big),$$

the following inequality holds:

$$\Big(\sum_{j=1}^{s} \lambda_j(w_j x_P)\Big) - \mu(v x_P) \geq 0, \qquad (I^P_{(w_1,\dots,w_s,v)})$$

where α_{i_P} is the (unique) simple root not in the Levi of P and $x_P := x_{i_P}$.

This conjecture is motivated from its validity in the finite case due to Belkale-Kumar [BK, Theorem 22]. (For a survey of these results in the finite case, see [K_5].) So far, the only evidence of its validity in the infinite dimensional case is shown for $s = 2$ and $\mathfrak{g}$ of types $A_1^{(1)}$ and $A_2^{(2)}$ (cf. Theorems 7.5 and 8.6). In these cases, we explicitly determine Γ_2 and thereby show the validity of the conjecture.

A positive integer d_o is called a *saturation factor* for $\mathfrak{g}$ if for any Λ, Λ', $\Lambda'' \in P_+$ such that $\Lambda - \Lambda' - \Lambda'' \in Q$ and $L(N\Lambda)$ is a submodule of $L(N\Lambda')\otimes L(N\Lambda'')$, for some $N \in \mathbb{Z}_{>0}$, then $L(d_o\Lambda)$ is a submodule of $L(d_o\Lambda') \otimes L(d_o\Lambda'')$.

We prove the following result on saturation factors (cf. Corollaries 6.4 and 8.7).

Theorem 1.3. *For $A_1^{(1)}$, any integer $d_o > 1$ is a saturation factor. For $A_2^{(2)}$, 4 is a saturation factor.*

The proof in these affine rank-2 cases makes use of basic representation theory of the Virasoro algebra (in particular, Lemma 4.1). Let δ be the smallest positive imaginary root of $\mathfrak{g}$. To determine the saturated tensor semigroup, we show that it is enough to know the components of $L(\lambda_1)\otimes L(\lambda_2)$ which are δ-maximal, i.e., the components $L(\mu)\subset L(\lambda_1)\otimes L(\lambda_2)$ such that $L(\mu+n\delta)\not\subseteq L(\lambda_1)\otimes L(\lambda_2)$ for any $n>0$. Let $m^{\mu}_{\lambda_1,\lambda_2}$ be the multiplicity of $L(\mu)$ in $L(\lambda_1)\otimes L(\lambda_2)$. If $L(\mu)$ is a δ-maximal component of $L(\lambda_1)\otimes L(\lambda_2)$, then $\sum_{n\in\mathbb{Z}_{\leq 0}} L(\mu+n\delta)^{\oplus m^{\mu+n\delta}_{\lambda_1,\lambda_2}}$ is a unitarizable coset module for the Virasoro algebra arising from the Sugawara construction for the diagonal embedding $\mathfrak{g}\hookrightarrow\mathfrak{g}\oplus\mathfrak{g}$. Proposition 5.5 for $A_1^{(1)}$ (and the analogous Proposition 8.2 for $A_2^{(2)}$) determining the maximal δ-components plays a crucial role in the proofs.

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2. Notation

We take the base field to be the field of complex numbers $\mathbb{C}$. By a variety, we mean an algebraic variety over $\mathbb{C}$, which is reduced but not necessarily irreducible.

Let G be any symmetrizable Kac-Moody group over $\mathbb{C}$ completed along the negative roots (as opposed to completed along the positive roots as in [K$_3$, Chapter 6]) and $G^{\min}\subset G$ be the 'minimal' Kac-Moody group as in [K$_3$, §7.4]. Let B be the standard (positive) Borel subgroup, B^- the standard negative Borel subgroup, $H=B\cap B^-$ the standard maximal torus and W the Weyl group (cf. [K$_3$, Chapter 6]). Let U (resp. U^-) be the unipotent radical $[B,B]$ (resp. $[B^-,B^-]$) of B (resp. B^-). Let

$$\bar{X}=G/B$$

be the 'thick' flag variety which contains the standard KM-flag variety

$$X=G^{\min}/B.$$

If G is not of finite type, $\bar{X}$ is an infinite dimensional non quasi-compact scheme (cf. [Ka, §4]) and X is an ind-projective variety (cf. [K$_3$, §7.1]). The group $G^{\min}$ acts on $\bar{X}$ and X.

More generally, for any standard parabolic subgroup $P\supset B$, define the partial flag variety

$$X_P=G^{\min}/P,$$

and

$$\bar{X}_P=G/P.$$

Recall that if W_P is the Weyl group of P (which is, by definition, the Weyl Group W_L of its Levi subgroup L), then in each coset of W/W_P we have a unique member w of minimal length. Let W^P be the set of the minimal length representatives in the cosets of W/W_P.

For any $w\in W^P$, define the Schubert cell:

$$C^P_w:=BwP/P\subset G/P$$

endowed with the reduced subscheme structure. Then, it is a locally closed subvariety of the ind-variety G/P isomorphic with the affine space $\mathbb{A}^{\ell(w)}$, $\ell(w)$ being the length of w (cf. [K$_3$, §7.1]). Its closure is denoted by X_w^P, which is an irreducible (projective) subvariety of G/P of dimension $\ell(w)$. We denote the point $wP \in C_w^P$ by $\dot{w}$. We abbreviate C_w^B, X_w^B by C_w, X_w respectively.

Similarly, define the opposite Schubert cell

$$C_P^w := B^- wP/P \subset \bar{X}_P,$$

and the opposite Schubert variety

$$X_P^w := \overline{C^w} \subset \bar{X}_P,$$

both endowed with the reduced subscheme structures. Then, X_P^w is a finite codimensional irreducible subscheme of $\bar{X}_P$ (cf. [K$_3$, Section 7.1] and [Ka, §4]). As above, we abbreviate C_B^w, X_B^w by C^w, X^w respectively.

For any integral weight λ (i.e., any character e^λ of H), we have a $G^{\min}$-equivariant line bundle $\mathcal{L}_B(\lambda)$ on X associated to the character $e^{-\lambda}$ of H. Similarly, we have a G-equivariant line bundle $\mathcal{L}_{B^-}(\lambda)$ on $X^- := G/B^-$ associated to the character e^λ of H.

By the Bruhat decomposition

$$X_P = \sqcup_{w\in W^P} C_w^P,$$

the singular homology $H_*(X_P, \mathbb{Z})$ of X_P with integral coefficients has a basis $\{\mu(X_w^P)\}_{w\in W^P}$, where $\mu(X_w^P) \in H_{2\ell(w)}(X_P, \mathbb{Z})$ denotes the fundamental class of X_w^P. Let $\{\epsilon_P^w\}_{w\in W^P}$ be the dual basis of the singular cohomology $H^*(X_P, \mathbb{Z})$ under the standard pairing of cohomology with homology, i.e.,

$$\epsilon_P^u(\mu(X_v^P)) = \delta_{u,v}, \text{ for any } u, v \in W^P.$$

Thus, $\epsilon_P^w \in H^{2\ell(w)}(X_P, \mathbb{Z})$. If $P = B$, we abbreviate ϵ_P^u by ϵ^u.

Let $\Delta = \{\alpha_1, \dots, \alpha_r\} \subset \mathfrak{h}^*$ be the set of simple roots, $\{\alpha_1^\vee, \dots, \alpha_r^\vee\} \subset \mathfrak{h}$ the set of simple coroots and $\{s_1, \dots, s_r\} \subset W$ the corresponding simple reflections, where $\mathfrak{h} := \operatorname{Lie} H$. Let $\rho \in X(H)$ be any weight satisfying

$$\rho(\alpha_i^\vee) = 1, \quad \text{for all} \quad 1 \le i \le r,$$

where $X(H)$ is the character group of H (identified as a subgroup of $\mathfrak{h}^*$ via the derivative). When G is a finite dimensional semisimple group, ρ is unique, but for a general Kac-Moody group G, it may not be unique.

Choose elements $x_i \in \mathfrak{h}$ such that

$$\alpha_j(x_i) = \delta_{i,j}, \text{ for any } 1 \le i, j \le r. \tag{1}$$

Observe that x_i may not be unique.

Define the set of *dominant integral weights*

$$P_+ := \{\lambda \in X(H) : \lambda(\alpha_i^\vee) \in \mathbb{Z}_+ \, \forall\, 1 \le i \le r\},$$

and the set of *dominant integral regular weights*

$$P_{++} := \{\lambda \in X(H) : \lambda(\alpha_i^\vee) \in \mathbb{Z}_{\ge 1} \, \forall\, 1 \le i \le r\},$$

where $\mathbb{Z}_+$ is the set of non-negative integers. The integrable highest weight (irreducible) modules of $G^{\min}$ are parameterized by P_+. For $\lambda \in P_+$, let $L(\lambda)$ be the corresponding integrable highest weight (irreducible) G-module with highest weight λ.

3. Necessary Inequalities for the Saturated Tensor Semigroup

Fix a positive integer s and define the *saturated tensor semigroup* $\Gamma_s = \Gamma_s(G)$:

$$\Gamma_s := \{(\lambda_1, \dots, \lambda_s, \mu) \in P_+^{s+1} : \exists\, N > 1 \text{ with } L(N\mu) \subset L(N\lambda_1) \otimes \cdots \otimes L(N\lambda_s)\}. \tag{2}$$

It is indeed a semigroup by the anlogue of the Borel-Weil theorem for the Kac-Moody case (see the identity (3) in the proof of Theorem 3.3). We give a certain set of inequalities satisfied by Γ_s. But, we first recall some basic results about the Hilbert-Mumford index.

3.1. **Definition.** Let S be any (not necessarily reductive) algebraic group acting on a (not necessarily projective) variety $\mathbb{X}$ and let $\mathbb{L}$ be an S-equivariant line bundle on $\mathbb{X}$. Let $O(S)$ be the set of all one parameter subgroups (for short OPS) in S. Take any $x \in \mathbb{X}$ and $\delta \in O(S)$ such that the limit $\lim_{t \to 0} \delta(t)x$ exists in $\mathbb{X}$ (i.e., the morphism $\delta_x : \mathbb{G}_m \to \mathbb{X}$ given by $t \mapsto \delta(t)x$ extends to a morphism $\widetilde{\delta}_x : \mathbb{A}^1 \to \mathbb{X}$). Then, following Mumford, define a number $\mu^{\mathbb{L}}(x, \delta)$ as follows: Let $x_o \in \mathbb{X}$ be the point $\widetilde{\delta}_x(0)$. Since x_o is $\mathbb{G}_m$-invariant via δ, the fiber of $\mathbb{L}$ over x_o is a $\mathbb{G}_m$-module; in particular, it is given by a character of $\mathbb{G}_m$. This integer is defined as $\mu^{\mathbb{L}}(x, \delta)$.

We record the following standard properties of $\mu^{\mathbb{L}}(x, \delta)$ (cf. [MFK, Chap. 2, §1]):

3.2. **Proposition.** *For any $x \in \mathbb{X}$ and $\delta \in O(S)$ such that $\lim_{t \to 0} \delta(t)x$ exists in $\mathbb{X}$, we have the following (for any S-equivariant line bundles $\mathbb{L}, \mathbb{L}_1, \mathbb{L}_2$):*

(a) $\mu^{\mathbb{L}_1 \otimes \mathbb{L}_2}(x, \delta) = \mu^{\mathbb{L}_1}(x, \delta) + \mu^{\mathbb{L}_2}(x, \delta)$.

(b) *If there exists $\sigma \in H^0(\mathbb{X}, \mathbb{L})^S$ such that $\sigma(x) \neq 0$, then $\mu^{\mathbb{L}}(x, \delta) \geq 0$.*

(c) *If $\mu^{\mathbb{L}}(x, \delta) = 0$, then any element of $H^0(\mathbb{X}, \mathbb{L})^S$ which does not vanish at x does not vanish at $\lim_{t \to 0} \delta(t)x$ as well.*

(d) *For any S-variety $\mathbb{X}'$ together with an S-equivariant morphism $f : \mathbb{X}' \to \mathbb{X}$ and any $x' \in \mathbb{X}'$ such that $\lim_{t \to 0} \delta(t)x'$ exists in $\mathbb{X}'$, we have $\mu^{f^*\mathbb{L}}(x', \delta) = \mu^{\mathbb{L}}(f(x'), \delta)$.*

(e) *(Hilbert-Mumford criterion) Assume that $\mathbb{X}$ is projective, S is connected and reductive and $\mathbb{L}$ is ample. Then, $x \in \mathbb{X}$ is semistable (with respect to $\mathbb{L}$) if and only if $\mu^{\mathbb{L}}(x, \delta) \geq 0$, for all $\delta \in O(S)$.*

In particular, if $x \in \mathbb{X}$ is semistable and δ-fixed, then $\mu^{\mathbb{L}}(x, \delta) = 0$.

The following theorem is one of our main results giving a collection of necessary inequalities defining the semigroup Γ_s.

3.3. **Theorem.** *Let G be any symmetrizable Kac-Moody group and let $(\lambda_1, \cdots, \lambda_s, \mu) \in \Gamma_s$. Then, for any $u_1, \dots, u_s, v \in W$ such that $n^v_{u_1,\dots,u_s} \neq 0$, where*

$$\varepsilon^{u_1} \cdots \varepsilon^{u_s} = \sum_w n^w_{u_1,\dots,u_s}\, \varepsilon^w \in H^*(X, \mathbb{Z}),$$

we have

$$\Big(\sum_{j=1}^s \lambda_j(u_j x_i)\Big) - \mu(v x_i) \geq 0, \quad \text{for any } x_i,$$

where x_i is defined by the equation (1).

Proof. Let

$$Z := \big\{(\bar{g}_1, \dots, \bar{g}_s) \in (X^-)^s : g_1 X^{u_1} \cap \cdots \cap g_s X^{u_s} \cap X_v \neq \emptyset\big\},$$

where $X^- := G/B^-$ and $\bar{g}_j = g_j B^-$. Then, Z contains a nonempty open set by Proposition 3.7. (In fact, by Proposition 3.7, $Z = (X^-)^s$, but we do not need this stronger result.)

Take a nonzero $\sigma \in H^0\big((X^-)^s \times X, \mathcal{L}^N\big)^{G^{\min}}$, where

$$\mathcal{L} := \mathcal{L}_{B^-}(\lambda_1) \boxtimes \cdots \boxtimes \mathcal{L}_{B^-}(\lambda_s) \boxtimes \mathcal{L}_B(\mu).$$

Such a nonzero σ exists, for some $N > 0$, since by [K$_3$, Corollary 8.3.12(a) and Lemma 8.3.9],

$$\begin{aligned} H^0\big((X^-)^s \times X, \mathcal{L}^N\big)^{G^{\min}} &\simeq \operatorname{Hom}_{G^{\min}}\big(L(N\lambda_1)^\vee \otimes \cdots \otimes L(N\lambda_s)^\vee \otimes L(N\mu), \mathbb{C}\big) \\ &\simeq \operatorname{Hom}_{G^{\min}}\big(L(N\mu), [L(N\lambda_1)^\vee \otimes \cdots \otimes L(N\lambda_s)^\vee]^*\big) \\ &\simeq \operatorname{Hom}_{G^{\min}}\big(L(N\mu), [L(N\lambda_1)^\vee \otimes \cdots \otimes L(N\lambda_s)^\vee]^\vee\big) \\ &\simeq \operatorname{Hom}_{G^{\min}}\big(L(N\mu), L(N\lambda_1) \otimes \cdots \otimes L(N\lambda_s)\big) \\ &\neq 0, \end{aligned} \tag{3}$$

since $(\lambda_1, \dots, \lambda_s, \mu) \in \Gamma_s$, where, for a $G^{\min}$-module M, $M^\vee$ denotes the direct sum of the H-weight spaces of the full dual module M^*.

Pick $(\bar{g}_1, \dots, \bar{g}_s) \in Z$ such that $\sigma(\bar{g}_1, \dots, \bar{g}_s, \bar{1}) \neq 0$, where $\bar{1} = 1 \cdot B$. Since $(\bar{g}_1, \dots, \bar{g}_s) \in Z$, there exists $u_1' \geq u_1, \cdots, u_s' \geq u_s$ and $v' \leq v$ such that $g_1 C^{u_1'} \cap \cdots \cap g_s C^{u_s'} \cap C_{v'}$ is nonempty. Now, pick $g \in G^{\min}$ such that

$$gB \in g_1 C^{u_1'} \cap \cdots \cap g_s C^{u_s'} \cap C_{v'}. \tag{4}$$

By Proposition 3.2, for any $\delta \in O(G^{\min})$, $\mu^{\mathcal{L}}(\bar{x}, \delta(t)) \geq 0$, where $\bar{x} = (\bar{g}_1, \dots, \bar{g}_s, \bar{1})$ (since $\sigma(\bar{x}) \neq 0$). By the following Lemma 3.4, applied to the OPS $\delta(t) = g t^{x_i} g^{-1}$, we get

$$\Big(\sum_{j=1}^s \lambda_j(u_j' x_i)\Big) - \mu(v' x_i) \geq 0. \tag{5}$$

But, by [K$_3$, Lemma 8.3.3],

$$(u_j')^{-1}\lambda_j \leq u_j^{-1}(\lambda_j).$$

Thus,

$$\lambda_j(u'_j x_i) \leq \lambda_j(u_j x_i).$$

Similarly,

$$\mu(v' x_i) \geq \mu(v x_i).$$

Thus, from (5), we get

$$\Big(\sum_{j=1}^{s} \lambda_j(u_j x_i)\Big) - \mu(v x_i) \geq 0.$$

This proves the theorem. □

3.4. **Lemma.** *Let $g \in G^{\min}$ be as in the equation (4). Consider the one parameter subgroup $\delta(t) = g t^{x_i} g^{-1} \in O(G^{\min})$. Then,*

(a) $\mu^{\mathcal{L}_{B^-}(\lambda_j)}(g_j B^-, \delta(t)) = \lambda_j(u'_j x_i)$.

(b) $\mu^{\mathcal{L}_B(\mu)}(1 \cdot B, \delta(t)) = -\mu(v' x_i)$.

Proof. (a) $\mu^{\mathcal{L}_{B^-}(\lambda_j)}(g_j B^-, \delta(t)) = \mu^{\mathcal{L}_{B^-}(\lambda_j)}(g^{-1} g_j B^-, t^{x_i})$.
By assumption, $g_j^{-1} g \in U^- u'_j B$. Write

$$g_j^{-1} g = b_j^- u'_j p_j, \quad \text{for some } b_j^- \in U^-,\ p_j \in B.$$

Thus,

$$1 = g^{-1} g_j b_j^- u'_j p_j.$$

Let

$$b_j(t) = b_j^- u'_j t^{-x_i} (u'_j)^{-1} (b_j^-)^{-1} \in B^-.$$

Then,

$$t^{x_i} g^{-1} g_j b_j(t) = t^{x_i} p_j^{-1} t^{-x_i} (u'_j)^{-1} (b_j^-)^{-1}. \tag{6}$$

Consider the G_m-invariant section (via t^{x_i}) of $\mathcal{L}_{B^-}(\lambda_j)$:

$$\begin{aligned}\hat{\sigma}(t) &= \big(t^{x_i}\, g^{-1} g_j, 1\big) \mod B^- \\ &= \big(t^{x_i}\, g^{-1} g_j b_j(t), \lambda_j(b_j(t)^{-1})\big) \mod B^-.\end{aligned}$$

Clearly, $\lim_{t\to 0} t^{x_i}\, g^{-1} g_j b_j(t)$ exists in G by (6).

Now,

$$\begin{aligned}\lambda_j\big(b_j(t)^{-1}\big) &= \lambda_j\big(b_j^- u'_j t^{x_i} (u'_j)^{-1} (b_j^-)^{-1}\big) \\ &= \lambda_j\big(t^{u'_j x_i}\big).\end{aligned}$$

This gives

$$\mu^{\mathcal{L}_{B^-}(\lambda_j)}(g_j B^-, \delta(t)) = \lambda_j(u'_j(x_i)).$$

This proves the (a) part of the lemma.

(b) $\mu^{\mathcal{L}_B(\mu)}(1 \cdot B, \delta(t)) = \mu^{\mathcal{L}_B(\mu)}(g^{-1} B, t^{x_i})$. By assumption,

$$g \in B v' \cdot B.$$

Write

$$g = b v' p, \quad \text{for } b \in U, p \in B.$$

Thus,

$$1 = g^{-1}bv'p.$$

Let

$$b(t) = bv't^{-x_i}(v')^{-1}b^{-1} \in B.$$

Now,

$$t^{x_i}g^{-1}b(t) = t^{x_i}p^{-1}t^{-x_i}(v')^{-1}b^{-1}.$$

Thus,

$$\lim_{t\to 0} t^{x_i}g^{-1}b(t) \text{ exists in } G^{\min}.$$

Consider the G_m-invariant section (via t^{x_i})

$$\begin{aligned}\hat{\sigma}(t) &= (t^{x_i}g^{-1}, 1) \mod B\\ &= \big(t^{x_i}g^{-1}b(t), \mu(b(t))\big) \mod B.\end{aligned}$$

Now,

$$\begin{aligned}\mu(b(t)) &= \mu(bv't^{-x_i}(v')^{-1}b^{-1})\\ &= \mu(t^{-v'x_i}).\end{aligned}$$

This gives

$$\mu^{\mathcal{L}_B(\mu)}(1\cdot B, \delta(t)) = -\mu(v'(x_i)).$$

This proves the (b)-part and hence the lemma is proved. □

3.5. **Definition.** For a quasi-compact scheme Y, an $\mathcal{O}_Y$-module $\mathcal{S}$ is called *coherent* if it is finitely presented as an $\mathcal{O}_Y$-module and any $\mathcal{O}_Y$-submodule of finite type admits a finite presentation.

An $\mathcal{O}_{\bar{X}}$-module $\mathcal{S}$ is called *coherent* if $\mathcal{S}_{|V^S}$ is a coherent $\mathcal{O}_{V^S}$-module for any finite ideal $S \subset W$ (where a subset $S \subset W$ is called an *ideal* if for $x \in S$ and $y \leq x \Rightarrow y \in S$), where V^S is the quasi-compact open subset of $\bar{X}$ defined by

$$V^S = \bigcup_{w\in S} wU^-B/B.$$

Let $K^0(\bar{X})$ denote the Grothendieck group of coherent $\mathcal{O}_{\bar{X}}$-modules $\mathcal{S}$.

Similarly, define $K_0(X) := \lim_{n\to\infty} K_0(X_n)$, where $\{X_n\}_{n\geq 1}$ is the filtration of X giving the ind-projective variety structure (i.e., $X_n = \bigcup_{\ell(w)\leq n} C_w$) and $K_0(X_n)$ is the Grothendieck group of coherent sheaves on the projective variety X_n.

We also define

$$K^{\text{top}}(X) := \operatorname{Invlt}_{n\to\infty} K^{\text{top}}(X_n),$$

where $K^{\text{top}}(X_n)$ is the topological K-group of the projective variety X_n.

Let $* : K^{\text{top}}(X_n) \to K^{\text{top}}(X_n)$ be the involution induced from the operation which takes a vector bundle to its dual. This, of course, induces the involution $*$ on $K^{\text{top}}(X)$.

For any $w \in W$,

$$[\mathcal{O}_{X_w}] \in K_0(X).$$

3.6. **Lemma.** $\big\{[\mathcal{O}_{X_w}]\big\}_{w\in W}$ *forms a basis of* $K_0(X)$ *as a* $\mathbb{Z}$*-module.*

Proof. By [CG, §5.2.14 and Theorem 5.4.17], the result follows. □

For $u \in W$, by [KS, §2], $\mathcal{O}_{X^u}$ is a coherent $\mathcal{O}_{\bar{X}}$-module. In particular, $\mathcal{O}_{\bar{X}}$ is a coherent $\mathcal{O}_{\bar{X}}$-module.

Define a pairing

$$\langle\, , \,\rangle : K^0(\bar{X}) \otimes K_0(X) \to \mathbb{Z},\ \langle [\mathcal{S}], [\mathcal{F}] \rangle = \sum_i (-1)^i \chi\big(X_n, \mathcal{T}or_i^{\mathcal{O}_{\bar{X}}}(\mathcal{S}, \mathcal{F})\big),$$

if $\mathcal{S}$ is a coherent sheaf on $\bar{X}$ and $\mathcal{F}$ is a coherent sheaf on X supported in X_n (for some n), where χ denotes the Euler-Poincaré characteristic. Then, as in [K$_4$, Lemma 3.4], the above pairing is well defined.

By [KS, Proof of Proposition 3.4], for any $u \in W$,

$$\mathcal{E}xt^k_{\mathcal{O}_{\bar{X}}}(\mathcal{O}_{X^u}, \mathcal{O}_{\bar{X}}) = 0 \quad \forall k \neq \ell(u). \tag{7}$$

Define the sheaf

$$\omega_{X^u} := \mathcal{E}xt^{\ell(u)}_{\mathcal{O}_{\bar{X}}}\big(\mathcal{O}_{X^u}, \mathcal{O}_{\bar{X}}\big) \otimes \mathcal{L}(-2\rho),$$

which, by the analogy with the Cohen-Macaulay (for short CM) schemes of finite type, will be called the *dualizing sheaf* of X^u.

Now, set the sheaf on $\bar{X}$

$$\begin{aligned} \xi^u &:= \mathcal{L}(\rho)\omega_{X^u} \\ &= \mathcal{L}(-\rho)\mathcal{E}xt^{\ell(u)}_{\mathcal{O}_{\bar{X}}}(\mathcal{O}_{X^u}, \mathcal{O}_{\bar{X}}). \end{aligned}$$

Then, as proved in [K$_4$, Proposition 3.5], for any $u, w \in W$,

$$\langle [\xi^u], [\mathcal{O}_{X_w}] \rangle = \delta_{u,w}. \tag{8}$$

With these preliminaries, we are ready to prove the following result.

3.7. **Proposition.** *With the notation as in the proof of Theorem 3.3, $Z = (X^-)^s$, if ε^v occurs in $\varepsilon^{u_1} \cdots \varepsilon^{u_s}$ with nonzero coefficient.*

Proof. We give the proof in the case $s = 2$. The proof for general s is similar.

For $u, v \in W$, express

$$\varepsilon^u \varepsilon^v = \sum_{\substack{w \\ \ell(w) = \ell(u) + \ell(v)}} n^w_{u,v} \varepsilon^w.$$

Express the product in topological K-theory $K^{\mathrm{top}}(X)$ of $X = G^{\min}/B$:

$$\psi^u_o \psi^v_o = \sum_{\ell(w) \geq \ell(u) + \ell(v)} m^w_{u,v} \psi^w_o,$$

where $\psi^w := *\tau^{w^{-1}}$ (τ^w being the Kostant-Kumar 'basis' of $K_H^{\mathrm{top}}(X)$ as in [KK, Remark 3.14]) and $\{\psi^w_o\}_{w \in W}$ is the corresponding 'basis' of $K^{\mathrm{top}}(X) \simeq \mathbb{Z} \otimes_{R(H)} K_H^{\mathrm{top}}(X)$, cf. [KK, Proposition 3.25]).

Then, by [KK, Proposition 2.30],

$$n^w_{u,v} = m^w_{u,v}, \quad \text{if } \ell(w) = \ell(u) + \ell(v). \tag{9}$$

Let $\Delta : X \to X \times X$ be the diagonal map. Then, by [K$_4$, Proposition 4.1] and the identity (8), for any $u, v, w \in W$, $g_1, g_2 \in G^{\min}$,

$$m^w_{u,v} = \langle [\xi^u \boxtimes \xi^v], [\Delta_* \mathcal{O}_{X_w}]\rangle$$
$$= \langle [\xi^u \boxtimes \xi^v], [(g_1^{-1}, g_2^{-1}) \cdot (\Delta_* \mathcal{O}_{X_w})]\rangle,$$

since $[(g_1^{-1}, g_2^{-1}) \cdot \Delta_* \mathcal{O}_{X_w}] = [\Delta_* \mathcal{O}_{X_w}]$ as elements of $K_0(X \times X)$. Thus,

$$m^w_{u,v} = \langle [\xi^u \boxtimes \xi^v], [(g_1^{-1}, g_2^{-1}) \cdot (\Delta_* \mathcal{O}_{X_w})]\rangle \tag{10}$$
$$:= \sum_i (-1)^i \chi(\bar{X} \times \bar{X}, \mathcal{T}or_i^{\mathcal{O}_{\bar{X}\times\bar{X}}} \Big(\xi^u \boxtimes \xi^v, (g_1^{-1}, g_2^{-1}) \cdot (\Delta_* \mathcal{O}_{X_w})\Big).$$

Now, by definition, the support of ξ^u is contained in X^u and hence the support of the sheaf

$$\mathcal{S}_i := \mathcal{T}or_i^{\mathcal{O}_{\bar{X}\times\bar{X}}} \left(\xi^u \boxtimes \xi^v, (g_1^{-1}, g_2^{-1}) \cdot \Delta_* \mathcal{O}_{X_w}\right)$$

is contained in

$$X^u \times X^v \cap \big((g_1^{-1}, g_2^{-1}) \cdot \Delta(X_w)\big), \tag{11}$$

which is empty if

$$(g_1 X^u) \cap (g_2 X^v) \cap X_w = \emptyset. \tag{12}$$

Thus, if the equation (12) is true, then the Tor sheaf $\mathcal{S}_i = 0\ \forall i \geq 0$. Thus, if the equation (12) is satisfied,

$$m^w_{u,v} = 0.$$

Now, assume that $\ell(w) = \ell(u) + \ell(v)$. Then, by the equation (9),

$$n^w_{u,v} = 0, \quad \text{if the equation (12) is satisfied.}$$

But, since by assumption, $n^w_{u,v} \neq 0$, we see that

$$(g_1 X^u) \cap (g_2 X^v) \cap X_w \neq \emptyset, \quad \text{for any } g_1, g_2 \in G^{\min}.$$

But since $G^{\min}/(G^{\min} \cap B^-) \xrightarrow{\sim} X^-$, we get the proposition. □

4. Tensor Product Decomposition for Affine Kac-Moody Lie Algebras

4.1. **The Virasoro Algebra.** We recall the definition of the Virasoro algebra and its basic representation theory, which we need. The *Virasoro algebra* Vir has a basis $\{C, L_n : n \in \mathbb{Z}\}$ over $\mathbb{C}$ and the Lie bracket is given by

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{1}{12}(m^3 - m)\delta_{m,-n} C \text{ and } [\mathrm{Vir}, \mathrm{C}] = 0.$$

Let $\mathrm{Vir}_0 := \mathbb{C}L_0 \oplus \mathbb{C}C$. Then, a Vir module V is said to be a *highest weight representation* if there exists a Vir_0-eigenvector $v_o \in V$ such that $L_n v_o = 0$ for $n \in \mathbb{Z}_{>0}$ and $U(\bigoplus_{n<0} \mathbb{C}L_n)v_o = V$. Such a V is said to have *highest weight* $\lambda \in \mathrm{Vir}_0^*$ if $Xv_o = \lambda(X)v_o$, for all $X \in \mathrm{Vir}_0$. (It is easy to see that such a v_o is unique up to a scalar multiple and hence λ is unique.) The irreducible highest weight representations of Vir are in 1-1 correspondence with elements of Vir_0^* given by the highest weight. Denote the basis of Vir_0^*

dual to the basis $\{L_0, C\}$ of Vir_0 as $\{h, z\}$. For any $\mu \in \mathrm{Vir}_0^*$, denote the μ-th weight space of V by V_μ, i.e.,

$$V_\mu := \{v \in V : X \cdot v = \mu(X)v \ \forall X \in \mathrm{Vir}_0\}.$$

Define a Vir module V to be *unitarizable* if there exists a positive definite Hermitian form $(\cdot\,,\,\cdot)$ on V so that $(L_n v\,,\, w) = (v\,,\, L_{-n}w)$ for all $n \in \mathbb{Z}$ and $(Cv\,,\, w) = (v\,,\, Cw)$. It is easy to see that if M is a Vir-submodule of V, then $M^\perp$ is also a submodule. Hence, any unitarizable representation of Vir is completely reducible. Note that for a unitarizable highest weight Vir-representation V with highest weight λ, if v_o is a highest weight vector, then

$$0 \le (L_{-n}v_o\,,\, L_{-n}v_o) = (L_n L_{-n}v_o\,,\, v_o) = (2n\lambda(L_0) + \frac{1}{12}(n^3 - n)\lambda(C))(v_o\,,\, v_o) \tag{13}$$

for all $n > 0$. Therefore, both $\lambda(L_0)$ and $\lambda(C)$ must be nonnegative real numbers.

Lemma 4.1. *Let V be a unitarizable, highest weight (irreducible) representation of Vir with highest weight λ.*

(a) If $\lambda(L_0) \neq 0$, then $V_{\lambda+nh} \neq 0$, for any $n \in \mathbb{Z}_+$.

(b) If $\lambda(L_0) = 0$ and $\lambda(C) \neq 0$, then $V_{\lambda+nh} \neq 0$, for any $n \in \mathbb{Z}_{>1}$ and $V_{\lambda+h} = 0$.

(c) If $\lambda(L_0) = \lambda(C) = 0$, then V is one dimensional.

Proof. If $\lambda(L_0) \neq 0$, then by the equation (13) (since both of $\lambda(L_0)$ and $\lambda(C) \in \mathbb{R}_+$), $L_{-n}v_o \neq 0$, for any $n \in \mathbb{Z}_+$.

If $\lambda(L_0) = 0$ and $\lambda(C) \neq 0$, then again by the equation (13), $L_{-n}v_o \neq 0$, for any $n \in \mathbb{Z}_{>1}$. Also, $L_{-1}v_o = 0$.

If $\lambda(L_0) = \lambda(C) = 0$, then (by the equation (13) again), $L_{-n}v_o = 0$, for any $n \in \mathbb{Z}_{\geq 1}$. This shows that V is one dimensional. □

4.2. **Tensor product decomposition: A general method.** Let $\mathfrak{g}$ be the untwisted affine Kac-Moody Lie algebra associated to a finite dimensional simple Lie algebra $\overset{\circ}{\mathfrak{g}}$, i.e.,

$$\mathfrak{g} = \left(\overset{\circ}{\mathfrak{g}} \otimes \mathbb{C}[t, t^{-1}]\right) \oplus \mathbb{C}c \oplus \mathbb{C}d.$$

Let $\overset{\circ}{\mathfrak{h}}$ be a Cartan subalgebra of $\overset{\circ}{\mathfrak{g}}$. Then,

$$\mathfrak{h} := \overset{\circ}{\mathfrak{h}} \otimes 1 \oplus \mathbb{C}c \oplus \mathbb{C}d$$

is the standard Cartan subalgebra of $\mathfrak{g}$. Let $\delta \in \mathfrak{h}^*$ be the smallest positive imaginary root of $\mathfrak{g}$ (so that the positive imaginary roots of $\mathfrak{g}$ are precisely $\{n\delta, n \in \mathbb{Z}_{\geq 1}\}$). Then, δ is given by $\delta_{|\overset{\circ}{\mathfrak{h}} \oplus \mathbb{C}c} \equiv 0$ and $\delta(d) = 1$. For any $\Lambda \in P_+$, let $P(\Lambda)$ be the set of weights of $L(\Lambda)$ and let $P^o(\Lambda)$ be the set of δ-maximal weights of $L(\Lambda)$, i.e.,

$$P^o(\Lambda) = \{\lambda \in \mathfrak{h}^* : \lambda \in P(\Lambda) \text{ but } \lambda + n\delta \notin P(\Lambda) \text{ for any } n > 0\}.$$

For any $\lambda \in X(H)$, define the *δ-character of $L(\Lambda)$ through λ* by

$$c_{\Lambda,\lambda} = \sum_{n\in\mathbb{Z}} \dim L(\Lambda)_{\lambda+n\delta}\, e^{n\delta}.$$

Since δ is W-invariant,

$$c_{\Lambda,\lambda} = c_{\Lambda,w\lambda}, \text{ for any } w \in W. \tag{14}$$

Moreover, $P^o(\Lambda)$ is W-stable. It is obvious that

$$ch\, L(\Lambda) = \sum_{\lambda\in P^o(\Lambda)} c_{\Lambda,\lambda} e^{\lambda}. \tag{15}$$

By [K$_3$, Exercise 13.1.E.8], for any $\lambda \in P(\Lambda')$ and $\Lambda'' \in P_+$, $\Lambda''+\lambda+\rho$ belongs to the Tits cone. Hence, there exists $v \in W$ such that $v^{-1}(\Lambda''+\lambda+\rho) \in P_+$. Moreover, if $\Lambda''+\lambda+\rho$ has nontrivial W-isotropy, then its isotropy group must contain a reflection (cf. [K$_3$, Proposition 1.4.2(a)]). Thus, for such a $\lambda \in P(\Lambda')$, i.e., if $\Lambda''+\lambda+\rho$ has nontrivial W-isotropy,

$$\sum_{w\in W} \varepsilon(w) e^{w(\Lambda''+\lambda+\rho)} = 0. \tag{16}$$

Define

$$\bar{P}_+ := \{\Lambda \in P_+ : \Lambda(d) = 0\}.$$

For any $m \in \mathbb{Z}_+$, let

$$P_+^{(m)} := \{\Lambda \in P_+ : \Lambda(c) = m\},$$

and let

$$\bar{P}_+^{(m)} := \bar{P}_+ \cap P_+^{(m)}.$$

Then, $\bar{P}_+^{(m)}$ provides a set of representatives in $P_+^{(m)}$ mod $(P_+ \cap \mathbb{C}\delta)$.

For any $\Lambda, \Lambda', \Lambda'' \in P_+$, define

$$\begin{aligned} T_\Lambda^{\Lambda',\Lambda''} = \{\lambda \in P^o(\Lambda') : \exists v_{\Lambda,\Lambda'',\lambda} \in W \text{ and } S_{\Lambda,\Lambda'',\lambda} \in \mathbb{Z} \text{ with}\\ \lambda + \Lambda'' + \rho = v_{\Lambda,\Lambda'',\lambda}(\Lambda+\rho) + S_{\Lambda,\Lambda'',\lambda}\delta\}. \end{aligned}$$

Observe that since $\Lambda+\rho+n\delta \in P_{++}$ for any $n \in \mathbb{Z}$, such a $v_{\Lambda,\Lambda'',\lambda}$ and $S_{\Lambda,\Lambda'',\lambda}$ are unique by [K$_3$, Proposition 1.4.2 (a), (b)] (if they exist). Also, observe that

$$T_\Lambda^{\Lambda',\Lambda''} = \emptyset, \text{ unless } \Lambda(c) = \Lambda'(c) + \Lambda''(c) \text{ and } \Lambda' + \Lambda'' - \Lambda \in Q, \tag{17}$$

where Q is the root lattice of $\mathfrak{g}$.

Proposition 4.2. *For any Λ' and $\Lambda'' \in P_+$,*

$$ch\left(L(\Lambda') \otimes L(\Lambda'')\right) = \sum_{\Lambda\in\bar{P}_+^{(m)}} ch\, L(\Lambda) \Big(\sum_{\lambda\in T_\Lambda^{\Lambda',\Lambda''}} \varepsilon(v_{\Lambda,\Lambda'',\lambda}) c_{\Lambda',\lambda} e^{S_{\Lambda,\Lambda'',\lambda}\delta} \Big),$$

where $m := \Lambda'(c) + \Lambda''(c)$.

Moreover, $\sum_{\lambda\in T_\Lambda^{\Lambda',\Lambda''}} \varepsilon(v_{\Lambda,\Lambda'',\lambda})c_{\Lambda',\lambda}e^{S_{\Lambda,\Lambda'',\lambda}\delta}$ *is the character of a unitary representation (though, in general, not irreducible) of the Virasoro algebra* Vir *with central charge*

$$\dim \overset{\circ}{\mathfrak{g}} \cdot \Big(\frac{m'}{m'+g} + \frac{m''}{m''+g} - \frac{m}{m+g}\Big),$$

where $m' := \Lambda'(c), m'' := \Lambda''(c)$ *and* g *is the dual Coxeter number of* $\overset{\circ}{\mathfrak{g}}$*.*

Proof. By the Weyl-Kac character formula (cf. [K$_3$, Theorem 2.2.1]) and the identity (15), for any $\Lambda', \Lambda'' \in P_+$,

$$\begin{aligned}
&\left(\sum_{w\in W}\varepsilon(w)e^{w\rho}\right)\cdot ch\,L(\Lambda')\cdot ch\,L(\Lambda'')\\
&\qquad = \left(\sum_{\lambda\in P^o(\Lambda')} c_{\Lambda',\lambda}e^{\lambda}\right)\cdot\left(\sum_{w\in W}\varepsilon(w)e^{w(\Lambda''+\rho)}\right)\\
&\qquad = \sum_{\lambda\in P^o(\Lambda')} c_{\Lambda',\lambda}\sum_{w\in W}\varepsilon(w)e^{w(\Lambda''+\lambda+\rho)}, \text{ by (14)}\\
&\qquad = \sum_{\Lambda\in\bar{P}_+^{(m)}}\sum_{\lambda\in T_\Lambda^{\Lambda',\Lambda''}} c_{\Lambda',\lambda}\sum_{w\in W}\varepsilon(w)e^{w(v_{\Lambda,\Lambda'',\lambda}(\Lambda+\rho))+S_{\Lambda,\Lambda'',\lambda}\delta}, \text{ by (16)}\\
&\qquad = \sum_{\Lambda\in\bar{P}_+^{(m)}}\sum_{\lambda\in T_\Lambda^{\Lambda',\Lambda''}} c_{\Lambda',\lambda}\sum_{w\in W}\varepsilon(w)\varepsilon(v_{\Lambda,\Lambda'',\lambda})e^{w(\Lambda+\rho)}e^{S_{\Lambda,\Lambda'',\lambda}\delta}\\
&\qquad = \sum_{\Lambda\in\bar{P}_+^{(m)}}\sum_{w\in W}\varepsilon(w)e^{w(\Lambda+\rho)}\sum_{\lambda\in T_\Lambda^{\Lambda',\Lambda''}}\varepsilon(v_{\Lambda,\Lambda'',\lambda})c_{\Lambda',\lambda}e^{S_{\Lambda,\Lambda'',\lambda}\delta}.
\end{aligned}$$

Thus,

$$ch\,\big(L(\Lambda')\otimes L(\Lambda'')\big) = \sum_{\Lambda\in\bar{P}_+^{(m)}} ch\,L(\Lambda)\Big(\sum_{\lambda\in T_\Lambda^{\Lambda',\Lambda''}}\varepsilon(v_{\Lambda,\Lambda'',\lambda})c_{\Lambda',\lambda}e^{S_{\Lambda,\Lambda'',\lambda}\delta}\Big).$$

To prove the second part of the proposition, use [KR, Proposition 10.3]. This proves the proposition. □

4.3. **Remark.** For an affine Kac-Moody Lie algebra $\mathfrak{g}$, if we consider the tensor product decomposition of $L(\Lambda')\otimes L(\Lambda'')$ with respect to the derived subalgebra $\mathfrak{g}'$ (i.e., without the d-action), then the components $L(\Lambda)$ are precisely of the form $\Lambda \in \Lambda' + \Lambda'' + \overset{\circ}{Q}$, where $\overset{\circ}{Q}$ is the root lattice of $\overset{\circ}{\mathfrak{g}}$ (cf. [KW]). Thus, the determination of the eigen semigroup and the saturated eigen semigroup is fairly easy for $\mathfrak{g}'$.

Let $\theta = \sum_{i=1}^{\ell} h_i\alpha_i$ be the highest root of $\overset{\circ}{\mathfrak{g}}$ (with respect to a choice of the positive roots), written as a linear combination of the simple roots

$\{\alpha_1, \ldots, \alpha_\ell\}$ of $\mathring{\mathfrak{g}}$. Let

$$S := \{\sum_{i=0}^{\ell} n_i\alpha_i : n_i \geq 0 \text{ for any } i \text{ and } 0 \leq n_i < h_i \text{ for some } 0 \leq i \leq \ell\},$$

where $h_0 := 1$.

Proposition 4.4. *Let $\mathfrak{g}$ be an untwisted affine Kac-Moody Lie algebra as above. Then, for any $\Lambda \in P_+$ with $\Lambda(c) > 0$,*

$$P^o(\Lambda)_+ = S(\Lambda) \cap P_+,$$

where $P^o(\Lambda)_+ := P^o(\Lambda) \cap P_+$ and $S(\Lambda) = \{\Lambda - \beta : \beta \in S\}$.

Proof. Take $\lambda \in S(\Lambda)$. Then, for any $n \geq 1$,

$$\Lambda - (\lambda + n\delta) = \big(\sum_{i=0}^{\ell} n_i\alpha_i\big) - n\delta = (n_0 - n)\alpha_0 + \sum_{i=1}^{\ell}(n_i - nh_i)\alpha_i,$$

since $\alpha_0 := \delta - \theta$. Now, the coefficient of some α_i in the above sum is negative, for any positive n, since $\lambda \in S(\Lambda)$. Thus, $\lambda + n\delta$ could not be a weight of $L(\Lambda)$ for any positive n. Therefore, if $\lambda \in P(\Lambda) \cap S(\Lambda)$, then it is δ-maximal.

By [Kac, Proposition 12.5(a)], if $\Lambda(c) \neq 0$, then $S(\Lambda) \cap P_+ \subset P(\Lambda)$. Therefore, $S(\Lambda) \cap P_+ \subset P^o(\Lambda)_+$.

Conversely, take $\lambda \in P^o(\Lambda)_+$. Then, $\lambda \in P(\Lambda) \cap P_+$ and $\lambda + \delta \notin P(\Lambda)$. Express $\lambda = \Lambda - n_0\alpha_0 - \sum_{i=1}^{\ell} n_i\alpha_i$, for some $n_i \in \mathbb{Z}_+$. Then,

$$\lambda + \delta = \Lambda - (n_0 - 1)\alpha_0 - \sum_{i=1}^{\ell}(n_i - h_i)\alpha_i.$$

Again applying [Kac, Proposition 12.5(a)], $\lambda + \delta \notin P(\Lambda)$ if and only if $\lambda + \delta \not\leq \Lambda$, i.e., for some $0 \leq i \leq \ell$, $n_i < h_i$. Thus, $\lambda \in S(\Lambda)$. This proves the proposition. □

5. $A_1^{(1)}$ Case

In this section, we consider $\mathfrak{g} = \widehat{\mathfrak{sl}_2} = \big(\bigoplus_{n\in\mathbb{Z}} \mathbb{C}t^n \otimes \mathfrak{sl}_2\big) \oplus \mathbb{C}c \oplus \mathbb{C}d$. In this case $\mathfrak{h}^* = \mathbb{C}\alpha \oplus \mathbb{C}\delta \oplus \mathbb{C}\Lambda_0$, where α is the simple root of $\mathfrak{sl}_2$ and $\Lambda_{0|\mathring{\mathfrak{h}}\oplus\mathbb{C}d} \equiv 0$ and $\Lambda_0(c) = 1$. Then, Λ_0 is a zeroeth fundamental weight. The simple roots of $\widehat{\mathfrak{sl}_2}$ are $\alpha_0 := \delta - \alpha$ and $\alpha_1 := \alpha$. The simple coroots are $\alpha_0^\vee := c - \alpha^\vee$ and $\alpha_1^\vee := \alpha^\vee$. It is easy to see that an element of $\mathfrak{h}^*$ of the form $m\Lambda_0 + \frac{j}{2}\alpha$ belongs to P_+ if and only if $m, j \in \mathbb{Z}_+$ and $m \geq j$.

Specializing Proposition 4.4 to the case of $\mathfrak{g} = \widehat{\mathfrak{sl}_2}$, we get the following.

5.1. **Corollary.** *For $\mathfrak{g} = \widehat{\mathfrak{sl}_2}$ and $\Lambda = m\Lambda_0 + \frac{j}{2}\alpha \in P_+$,*

$$P^o(\Lambda)_+ = \left\{\Lambda - k\alpha,\ \Lambda - l(\delta - \alpha)\ :\ k, l \in \mathbb{Z}_+ \text{ and } k \leq \frac{j}{2},\ l \leq \frac{m-j}{2}\right\}. \tag{18}$$

Proof. The corollary follows from Proposition 4.4 since $m_1\Lambda_0 + \frac{m_2}{2}\alpha + m_3\delta$ belongs to P_+ if and only if $m_1, m_2 \in \mathbb{Z}_+$ and $m_1 \geq m_2$. □

Let π be the projection $\mathfrak{h}^* = \mathbb{C}\Lambda_0 \oplus \mathbb{C}\alpha \oplus \mathbb{C}\delta \to \mathbb{C}\Lambda_0 \oplus \mathbb{C}\alpha$.

5.2. **Lemma.** *Let* $\mathfrak{g} = \widehat{\mathfrak{sl}_2}$. *For* $\Lambda = m\Lambda_0 + \frac{j}{2}\alpha \in P_+$ *(i.e.,* $m, j \in \mathbb{Z}_+$ *and* $m \geq j$*) such that* $m > 0$,

$$\pi(P^o(\Lambda)) = \{\Lambda + k\alpha : k \in \mathbb{Z}\}. \tag{19}$$

Moreover, for any $k \in \mathbb{Z}$, *let* n_k *be the unique integer such that* $\Lambda + k\alpha + n_k\delta \in P^o(\Lambda)$. *Then, writing* $k = qm + r, 0 \leq r < m$, *we have:*

$$n_k = n_r - q(k + r + j). \tag{20}$$

Proof. The assertion (19) follows from the identity (18) together with the action of the affine Weyl group $W \simeq \mathring{W} \times (\mathbb{Z}\alpha^\vee)$ on $\mathfrak{h}^*$, where $\mathring{W}$ is the Weyl group of $\mathfrak{sl}_2$ and $\mathbb{Z}\alpha^\vee$ acts on $\mathfrak{h}^*$ via:

$$T_{n\alpha^\vee}(\mu) = \mu + n\mu(c)\alpha - [n\mu(\alpha^\vee) + n^2\mu(c)]\delta, \quad \text{for } n \in \mathbb{Z}, \mu \in \mathfrak{h}^*. \tag{21}$$

Since $P^o(\Lambda)$ is W-stable, the identity (20) can be established from the action of the affine Weyl group element $T_{-q\alpha^\vee}$ on $\Lambda + k\alpha + n_k\delta$. □

The value of n_r for $0 \leq r < m$ can be determined from the identity (18) by applying $T_{\alpha^\vee}, T_{\alpha^\vee} \cdot s_1$ to $\Lambda - k\alpha$ and applying $1, T_{\alpha^\vee} \cdot s_1$ to $\Lambda - l(\delta - \alpha)$, where s_1 is the nontrivial element of $\mathring{W}$. We record the result in the following lemma.

5.3. **Lemma.** *With the notation as in the above lemma, the value of* n_r *for any integer* $0 \leq r < m$ *is given by*

$$n_r = \begin{cases} -r, & \text{for } 0 \leq r \leq m - j \\ m - j - 2r & \text{for } m - j \leq r < m. \end{cases}$$

5.4. **Lemma.** *Take the following elements in* P_+*:*

$$\Lambda = m\Lambda_0 + \frac{j}{2}\alpha,\ \Lambda' = m'\Lambda_0 + \frac{j'}{2}\alpha,\ \Lambda'' = m''\Lambda_0 + \frac{j''}{2}\alpha,$$

where $m := m' + m''$ *and we assume that* $m' > 0$. *Then,*

$$\pi\left(T_\Lambda^{\Lambda', \Lambda''}\right) = \{\Lambda' + k\alpha \ : \ k \in \mathbb{Z}, k \equiv \frac{1}{2}\left(j - j' - j''\right) \text{ or } k \equiv -\frac{1}{2}\left(j + j' + j''\right) - 1 \bmod M\},$$

where $M := m + 2$. *In particular, by the equation (17),* $T_\Lambda^{\Lambda', \Lambda''}$ *is nonempty if and only if* $\frac{j - j' - j''}{2} \in \mathbb{Z}$.

Moreover, for $\lambda = \Lambda' + k\alpha + n_k\delta \in T_\Lambda^{\Lambda', \Lambda''}$,

$$v_{\Lambda, \Lambda'', \lambda} = \begin{cases} T_{\frac{k - \frac{1}{2}(j - j' - j'')}{M}\alpha^\vee}, & \text{if } k \equiv \frac{1}{2}\left(j - j' - j''\right) \mod M \\ s_1 T_{-\frac{k + \frac{1}{2}(j + j' + j'') + 1}{M}\alpha^\vee}, & \text{if } k \equiv -\frac{1}{2}\left(j + j' + j''\right) - 1 \bmod M, \end{cases}$$

where $T_{n\alpha^\vee}$ is defined by the equation (21). *Further,*

$$S_{\Lambda,\Lambda'',\lambda} = n_k + \frac{\left(k - \frac{1}{2}(j - j' - j'')\right)\left(k + \frac{1}{2}(j + j' + j'') + 1\right)}{M}.$$

Proof. Follows from the fact that $W = \overset{\circ}{W} \rtimes \mathbb{Z}\alpha^\vee$ and that $\rho = 2\Lambda_0 + \frac{1}{2}\alpha$. $\square$

We have the following very crucial result.

Proposition 5.5. *Fix Λ, Λ' and Λ'' as in Lemma 5.4 and asume that $\frac{j-j'-j''}{2} \in \mathbb{Z}$ and both of $m', m'' > 0$. Then, the maximum of $\left\{S_{\Lambda,\Lambda'',\lambda} : \lambda \in T_\Lambda^{\Lambda',\Lambda''} \text{ and } \varepsilon(v_{\Lambda,\Lambda'',\lambda}) = 1\right\}$ is achieved precisely when $\pi(\lambda) = \Lambda' + \frac{1}{2}(j - j' - j'')\alpha$.*

Proof. By Lemma 5.4 and the explicit description of the length function of $T_{n\alpha^\vee}$ (cf. [K$_3$, Exercise 13.1.E.3]),

$$\pi\{\lambda \in T_\Lambda^{\Lambda',\Lambda''} : \varepsilon(v_{\Lambda,\Lambda'',\lambda}) = 1\} = \{\Lambda' + k_l\alpha : l \in \mathbb{Z}\},$$

where $M := m + 2$ and $k_l := \frac{j-j'-j''}{2} + lM$. Take $\lambda = \Lambda' + k_l\alpha \in \pi(T_\Lambda^{\Lambda',\Lambda''})$ for $l \in \mathbb{Z}$. Write $k_l = q_l m' + r_l$ for $q_l \in \mathbb{Z}$ and $0 \le r_l < m'$. Then, by Lemmas 5.2, 5.3 and 5.4, for $\lambda = \Lambda' + k_l\alpha$ (setting $J := \frac{j-j'-j''}{2}$),

$$\begin{aligned}
S_{\Lambda,\Lambda'',\lambda} &= n_{r_l} - \frac{(J + j' + lM + r_l)(J + lM - r_l)}{m'} + l(lM + 1 + j) \\
&= l^2 M(1 - \frac{M}{m'}) + l(1 + j - \frac{M(j - j'')}{m'}) - \frac{(j - j'')^2 - j'^2}{4m'} + \frac{r_l^2}{m'} + \frac{r_l j'}{m'} + n_{r_l} \\
&= l^2 M(1 - \frac{M}{m'}) + l(1 + j - \frac{M}{m'}(j - j'')) - \frac{(j - j'')^2 - j'^2}{4m'} + p(k_l),
\end{aligned}$$

where

$$p(k_l) := \frac{r_l^2}{m'} + \frac{r_l}{m'}j' + n_{k_l}.$$

Let $P = P_{m',j'} : \mathbb{R} \to \mathbb{R}$ be the following function:

$$P(s) \quad := \quad \begin{cases} \frac{(s - \frac{m'}{2}k)^2}{m'} - \frac{(j')^2}{4m'}, & \text{if } \left|s - \frac{m'}{2}k\right| \le \frac{j'}{2} \text{ for some } k \in 2\mathbb{Z} \\ \frac{(s - \frac{m'}{2}k)^2}{m'} - \frac{(m' - j')^2}{4m'}, & \text{if } \left|s - \frac{m'}{2}k\right| \le \frac{m' - j'}{2} \text{ for some } k \in 2\mathbb{Z} + 1. \end{cases}$$

Let $k_s \in \mathbb{Z}$ be such a k. (Of course, k_s depends upon m' and j'.)

Claim 5.6. $P(s) = p(s - \frac{j'}{2})$ for $s \in \frac{j'}{2} + \mathbb{Z}$.

Proof. Clearly, both of P and p are periodic with period m'. So, it is enough to show that $P(s) = p(s - \frac{j'}{2})$, for $s - \frac{j'}{2}$ equal to any of the integral points of the interval $[-j', m' - j']$. By Lemma 5.3 and the identity (20), for any integer $-j' \le r \le 0$,

$$p(r) = \frac{1}{m'}r(r + j'),$$

and for any integer $0 \leq r \leq m' - j'$,

$$p(r) = \frac{r(r+j')}{m'} - r.$$

From this, the claim follows immediately. □

Fix $m' > 0$. Let

$$I := \{(t, j', m'', j'', j) \in \mathbb{R}^5 : 0 \leq j' \leq m',\, 1 \leq m'',\, 0 \leq j'' \leq m'',\, 0 \leq j \leq m' + m''\}.$$

Define $F : I \to \mathbb{R}$ by

$$F : (t, j', m'', j'', j) \mapsto t^2 M(1 - \frac{M}{m'}) + t\big(j(1 - \frac{M}{m'}) + 1 + \frac{M}{m'} j''\big) + \frac{(j')^2 - (j - j'')^2}{4m'} + P(\frac{1}{2}\left(j - j''\right) + tM).$$

Thus, F is a continuous, piecewise smooth function with failure of differentiability along the set

$$\{(t, j', m'', j'', j) \in I \,:\, \frac{1}{2}(j \pm j' - j'') + tM \in m'\mathbb{Z}\}.$$

Claim 5.7. Let $\Delta(t) = \Delta(t, j', m'', j'', j) := F(t+1, j', m'', j'', j) - F(t, j', m'', j'', j)$. Then, on I,

(1) Δ is a nonincreasing function of t
(2) Δ is increasing with respect to j''
(3) Δ is nonincreasing in j
(4) $\Delta(0)$ is decreasing in m''
(5) $\Delta(-1)$ is nondecreasing in m''.

Proof. We compute and give bounds for the partial derivatives of Δ, where they exist.

$$\begin{aligned}\Delta(t) &= 2tM(1 - \frac{M}{m'}) + \big((j+M)(1 - \frac{M}{m'}) + 1 + \frac{M}{m'} j''\big) \\ &\quad + P(tM + M + \frac{1}{2}(j - j'')) - P(tM + \frac{1}{2}(j - j'')).\end{aligned}$$

Hence,

$$\begin{aligned}\partial_t \Delta(t) &= 2M(1 - \frac{M}{m'}) + M\big(P'(tM + M + \frac{1}{2}(j - j'')) - P'(tM + \frac{1}{2}(j - j''))\big) \\ &= 2M(1 - \frac{M}{m'}) + 2\frac{M}{m'}(M - \frac{m'}{2}k_1 + \frac{m'}{2}k_0) \\ &= 2M(1 - \frac{k_1 - k_0}{2}),\end{aligned}$$

where $k_1 := k_{(t+1)M + \frac{1}{2}(j - j'')}$ and $k_0 := k_{tM + \frac{1}{2}(j - j'')}$. Since $2 \leq k_1 - k_0$, we see that $\partial_t \Delta \leq 0$, wherever $\partial_t \Delta$ exists. Since Δ is continuous everywhere and differentiable on all but a discrete set, Δ is nonincreasing in t.

$$\partial_{j''}\Delta(t) = \frac{M}{m'} - \frac{1}{2}\left(P'(tM + M + \frac{1}{2}(j-j'')) - P'(tM + \frac{1}{2}(j-j''))\right).$$

Now, $|P'| \leq 1$, so $\frac{M}{m'} + 1 \geq \partial_{j''}\Delta \geq \frac{M}{m'} - 1 = \frac{m''+2}{m'} > 0$.

For (3):

$$\begin{aligned}\partial_j\Delta(t) &= 1 - \frac{M}{m'} + \frac{1}{2}\big(P'(tM + M + \frac{1}{2}(j-j'')) - P'(tM + \frac{1}{2}(j-j''))\big)\\ &= 1 - \frac{M}{m'} + \frac{1}{m'}\left(M - \frac{m'}{2}k_1 + \frac{m'}{2}k_0\right)\\ &= 1 - \frac{k_1 - k_0}{2} \leq 0.\end{aligned}$$

(4) and (5) follow from the following calculation:

$$\begin{aligned}\partial_{m''}\Delta =& 2t(1 - 2\frac{M}{m'}) + (1 - 2\frac{M}{m'} + \frac{1}{m'}(j''-j))\\ &+ (t+1)P'(tM + M + \frac{1}{2}(j-j'')) - tP'(tM + \frac{1}{2}(j-j'')).\end{aligned}$$

Hence,

$$\begin{aligned}\partial_{m''}\Delta(0) &= 1 - 2\frac{M}{m'} + \frac{1}{m'}(j''-j) + P'(M + \frac{1}{2}(j-j''))\\ &\leq 1 - 2\frac{M}{m'} + \frac{m''}{m'} + 1\\ &= \frac{-m''-4}{m'} < 0,\end{aligned}$$

and

$$\begin{aligned}\partial_{m''}\Delta(-1) &= -2(1 - 2\frac{M}{m'}) + (1 - 2\frac{M}{m'} + \frac{1}{m'}(j''-j)) + P'(-M + \frac{1}{2}(j-j''))\\ &= -1 + 2\frac{M}{m'} + \frac{1}{m'}(j''-j) + P'(-M + \frac{1}{2}(j-j''))\\ &= -1 + 2\frac{M}{m'} + \frac{1}{m'}(j''-j) - 2\frac{M}{m'} + \frac{1}{m'}(j-j'') - k_0\\ &= -1 - k_0.\end{aligned}$$

Note that $k_0 \leq -1$ since $-\frac{(j-j'')}{2} - M < -\frac{m'}{2}$. Thus, $\partial_{m''}\Delta(-1) \geq 0$. $\square$

Claim 5.8. *The maximum of* $F = F(-, j', m'', j'', j) : \mathbb{Z} \to \mathbb{R}$ *occurs at* 0.

Proof. We show that $\Delta(-1) > 0 > \Delta(0)$. Since Δ is nonincreasing in t, it would follow that $F(0) > F(t)$ for all $t \in \mathbb{Z}_{\neq 0}$.

Let us begin with $\Delta(-1)$. By the previous claim 5.7, $\Delta(-1)$ is as small as possible when $m'' = 1$, $j'' = 0$, and $j = m' + 1$. So, let us compute with these values:

$$\begin{aligned}
\Delta(-1) &\geq \frac{6}{m'} + 1 + P(\frac{1}{2}m' + \frac{1}{2}) - P(-2 - \frac{1}{2}m' - \frac{1}{2}) \\
&= \frac{6}{m'} + 1 + \frac{(\frac{1}{2}m' + \frac{1}{2} - \frac{1}{2}m'k_1)^2}{m'} - \frac{(2 + \frac{1}{2}m' + \frac{1}{2} + \frac{1}{2}m'k_0)^2}{m'} \\
&+ \begin{cases} \frac{m'}{4} - \frac{j'}{2} & \text{if } k_0 \text{ odd, } k_1 \text{ even} \\ 0 & \text{if } k_1 - k_0 \text{ even} \\ \frac{j'}{2} - \frac{m'}{4} & \text{if } k_1 \text{ odd, } k_0 \text{ even.} \end{cases}
\end{aligned}$$

Note that for $m' \geq 5$, the possible values of (k_1, k_0) are $(1, -1)$; $(1, -2)$; or $(2, -2)$. So, the result, that $\Delta(-1) > 0$, is established by considering such pairs directly and by cases for smaller m'.

For $\Delta(0)$, we take $m'' = 1$, $j'' = 1$, and $j = 0$.

$$\begin{aligned}
\Delta(0) &= (\frac{-3(3 + m')}{m'} + 1 + \frac{3 + m'}{m'}) + P(\frac{1}{2} + 2 + m') - P(-\frac{1}{2}) \\
&= 1 - \frac{2(3 + m')}{m'} + P(\frac{1}{2} + 2 + m') - P(-\frac{1}{2}) \\
&= 1 - \frac{2(3 + m')}{m'} + \frac{(\frac{1}{2} + 2 + m' - \frac{1}{2}m'k_1)^2}{m'} - \frac{(\frac{1}{2} + \frac{1}{2}m'k_0)^2}{m'} \\
&+ \begin{cases} \frac{m'}{4} - \frac{j'}{2} & \text{if } k_0 \text{ odd, } k_1 \text{ even} \\ 0 & \text{if } k_1 - k_0 \text{ even} \\ \frac{j'}{2} - \frac{m'}{4} & \text{if } k_1 \text{ odd, } k_0 \text{ even.} \end{cases}
\end{aligned}$$

For $m' \geq 5$, the possible values of (k_1, k_0) are $(3, -1)$; $(3, 0)$; or $(2, 0)$. So, again the result, that $\Delta(0) < 0$, is established by considering such pairs directly and by cases for smaller m'. □

This completes the proof of the proposition. □

Remark 5.9. We have shown that $F(l, j', m'', j'', j) = S_{\Lambda,\Lambda'',\lambda}$ for integral values of l. If l is not an integer, then $\lambda_l := \Lambda' + (lM + J)\alpha$ may not be in $\pi(T_\Lambda^{\Lambda',\Lambda''})$, in which case $S_{\Lambda,\Lambda'',\lambda_l}$ is not defined. On the other hand, if $\lambda_l \in \pi(T_\Lambda^{\Lambda',\Lambda''})$, we note that the equality $F(l, j', m'', j'', j) = S_{\Lambda,\Lambda'',\lambda_l}$ holds, as can be seen by letting $k_l = lM - \frac{1}{2}(j + j' + j'') - 1$ in the above proof.

Now, let us apply the same analysis to the case that $\varepsilon(v_{\Lambda,\Lambda'',\lambda}) = -1$. By Lemma 5.4, this corresponds to $k_l = -\frac{1}{2}(j + j' + j'') - 1 + lM$. For $\lambda = \Lambda' + k_l\alpha$, let us denote the function $S_{\Lambda,\Lambda'',\lambda}$ by $G_{\mathbb{Z}}(l) = G_{\mathbb{Z}}(l, j', m'', j'', j)$. Thus, $G_{\mathbb{Z}} : \mathbb{Z} \to \mathbb{Z}$.

5.10. **Lemma.** *Define the function $G = G(-, j', m'', j'', j) : \mathbb{R} \to \mathbb{R}$ by*

$$G(t, j', m'', j'', j) = F(t - \frac{j+1}{M}, j', m'', j'', j).$$

Then, $G_{|\mathbb{Z}} = G_{\mathbb{Z}}$.

Hence, $S_{\Lambda,\Lambda'',\lambda}$ has a maximum when $l = 0$ or $l = 1$.

Proof. By the proof of Proposition 5.5 and Remark 5.9, $S_{\Lambda,\Lambda'',\lambda+(j+1)\alpha} = F(l)$, for $\lambda = \Lambda' + k_l\alpha$. Since $\lambda = \Lambda' + (-\frac{1}{2}(j+j'+j'') - 1 + lM)\alpha$, by Proposition 5.5, $S_{\Lambda,\Lambda'',\lambda} = F(l - \frac{j+1}{M})$. This proves the lemma. □

From Lmma 5.10 and the definition of F, it is easy to see that

$$G(1-t, m'-j', m'', m''-j'', m'+m''-j) + \frac{1}{2}(j'+j''-j) = G(t, j', m'', j'', j), \tag{22}$$

for any $t \in \mathbb{R}$. Hence, if the maximum of $G_{\mathbb{Z}}$ occurs at 1, it is equal to

$$G(0, m'-j', m'', m''-j'', m'+m''-j) + \frac{1}{2}(j'+j''-j). \tag{23}$$

We also record the following identity, which is easy to prove from the definition of F.

$$F(0, m'-j', m'', m''-j'', m'+m''-j) + \frac{1}{2}(j'+j''-j) = F(0, j', m'', j'', j). \tag{24}$$

As a corollary of Proposition 5.5 and Lemma 5.10, we get the following 'Non-Cancellation Lemma'.

5.11. **Corollary.** *Let $\Lambda, \Lambda', \Lambda''$ be as in Proposition 5.5 and let*

$$\mu_\Lambda^{\Lambda',\Lambda''} := \max\left\{S_{\Lambda,\Lambda'',\lambda} : \lambda \in T_\Lambda^{\Lambda',\Lambda''} \text{ and } \varepsilon(v_{\Lambda,\Lambda'',\lambda}) = 1\right\},$$
$$\bar{\mu}_\Lambda^{\Lambda',\Lambda''} := \max\left\{S_{\Lambda,\Lambda'',\lambda} : \lambda \in T_\Lambda^{\Lambda',\Lambda''} \text{ and } \varepsilon(v_{\Lambda,\Lambda'',\lambda}) = -1\right\}.$$

Assume that $\mu_\Lambda^{\Lambda',\Lambda''} = \bar{\mu}_\Lambda^{\Lambda',\Lambda''}$. Then,

$$\mu_\Lambda^{\Lambda'',\Lambda'} \neq \bar{\mu}_\Lambda^{\Lambda'',\Lambda'}.$$

Proof. We proceed in two cases:

Case I. Suppose the maximum $\bar{\mu}_\Lambda^{\Lambda',\Lambda''}$ occurs when $\pi(\lambda) = \Lambda' - (\frac{1}{2}(j+j'+j'') + 1)\alpha$ (cf. Lemma 5.10). This means that the δ-maximal weights of $L(\Lambda')$ through $\Lambda' - (\frac{1}{2}(j+j'+j'')+1)\alpha$ and through $\Lambda' + \frac{1}{2}(j-j'-j'')\alpha$ have the same δ coordinate (cf. Proposition 5.5). By (next) Lemma 5.12, we know that this occurs if and only if one of the following two conditions are satisfied:

(1) $\left|\frac{1}{2}(j-j'')\right| \leq \frac{j'}{2}$ and $\frac{1}{2}(j+j'')+1 \leq \frac{j'}{2}$, or

(2) $\frac{1}{2}(j+j'')+1 = \frac{1}{2}(j-j'')$.

The latter is clearly impossible, while the former condition is fulfilled precisely when $\frac{1}{2}(j+j'')+1 \leq \frac{j'}{2}$.

So, for the equality $\mu_\Lambda^{\Lambda',\Lambda''} = \bar{\mu}_\Lambda^{\Lambda',\Lambda''}$ in this case, the neccesary and sufficient condition is:

$$\frac{1}{2}(j+j'')+1 \leq \frac{j'}{2}. \tag{25}$$

Case II. Suppose the maximum $\bar{\mu}_\Lambda^{\Lambda',\Lambda''}$ occurs when $\pi(\lambda) = \Lambda' - (\frac{1}{2}(j+j'+j'') + 1 - M)\alpha$. Then, by the identities (23) and (24), we get

$$G(0, m'-j', m'', m''-j'', m'+m''-j) = F(0, m'-j', m'', m''-j'', m'+m''-j). \tag{26}$$

So, from the case I, we get in this case II, $\mu_\Lambda^{\Lambda',\Lambda''} = \bar{\mu}_\Lambda^{\Lambda',\Lambda''}$ if and only if

$$\frac{1}{2}\big((m'+m''-j) + (m''-j'')\big) + 1 \leq \frac{1}{2}(m'-j'). \tag{27}$$

So, if either of the inequalities (25) or (27) is satisfied, then none of them can be satisfied for the triple $(\Lambda, \Lambda', \Lambda'')$ replaced by $(\Lambda, \Lambda'', \Lambda')$. This proves the corollary. □

Lemma 5.12. *Suppose* $\Lambda' - (\frac{1}{2}(j+j'+j'')+1)\alpha + n_1\delta$ *and* $\Lambda' + \frac{1}{2}(j-j'-j'')\alpha + n_2\delta$ *are* δ*-maximal weights of* $L(\Lambda')$*. Then* $n_1 = n_2$ *if and only if*

$$\left|\frac{1}{2}(j-j'')\right| \leq \frac{j'}{2} \quad \textit{and} \quad \frac{1}{2}(j+j'') + 1 \leq \frac{j'}{2},$$

or $\frac{1}{2}(j+j'') + 1 = \frac{1}{2}(j-j'')$.

Proof. Fix an integer n and consider the set $P_n = \{\nu \in P(\Lambda') : \Lambda' - \nu = k\alpha + n\delta, k \in \mathbb{Z}\}$. We give a description of $P_n \cap P^o(\Lambda')$. Clearly, $P_n = \{\lambda, \lambda - \alpha, \ldots, \lambda - \langle\lambda, \alpha^\vee\rangle\alpha\}$ for some $\lambda = \lambda_n$ and that this λ is uniquely determined by n (cf. [K$_3$, Exercise 2.3.E.2]). Suppose that some $\mu \in P_n$ is not δ-maximal, then none of $\{\mu, \ldots, \mu - \langle\mu, \alpha^\vee\rangle\alpha\}$ are δ-maximal, since if $\mu + k\delta \in P(\Lambda')$, then the whole string $\{\mu+k\delta, \ldots, \mu+k\delta-\langle\mu,\alpha^\vee\rangle\alpha\} \subset P(\Lambda')$. In particular, if $\lambda - \alpha$ is δ-maximal, then so is λ. Hence, $\mathfrak{g}_{\delta-\alpha}L(\Lambda')_\lambda = 0$ and $\mathfrak{g}_\alpha L(\Lambda')_\lambda = 0$. Therefore, λ is the highest weight Λ'. Thus, $P_n \cap P^o(\Lambda')$ is either empty, or $\lambda = \Lambda'$ (in the case that $n = 0$), or the set $\{\lambda, s_1\lambda\}$. From this and Corollary 5.1 the lemma follows easily. □

6. Saturation factor for the $A_1^{(1)}$ Case

We assume that $\mathfrak{g} = \widehat{\mathfrak{sl}_2}$ in this section.

Definition 6.1. Let $\Lambda' \in P_+^{(m')}, \Lambda'' \in P_+^{(m'')}$ and $\Lambda \in P_+^{(m'+m'')}$. Then, we call $L(\Lambda + n\delta)$ the *δ-maximal component of* $L(\Lambda') \otimes L(\Lambda'')$ *through* Λ if $L(\Lambda + n\delta)$ is a submodule of $L(\Lambda') \otimes L(\Lambda'')$ but $L(\Lambda + m\delta)$ is not a component for any $m > n$.

Theorem 6.2. *Let* $\Lambda', \Lambda'', \Lambda$ *be as in Proposition 5.5 . Then,* $L(\Lambda + n\delta)$ *is a* δ*-maximal component of* $L(\Lambda') \otimes L(\Lambda'')$ *if* $n = \min(n_1, n_2)$*, where* n_1 *is such that* $\Lambda - \Lambda'' + n_1\delta \in P^o(\Lambda')$ *and* n_2 *is such that* $\Lambda - \Lambda' + n_2\delta \in P^o(\Lambda'')$.

Proof. This follows immediately by combining Propositions 4.2, 5.5 and Lemma 5.4. □

Lemma 6.3. *Fix a positive integer* N*. Let* $\Lambda \in \bar{P}_+$ *and let* $\lambda \in \Lambda + Q$*, where* Q *is the root lattice* $\mathbb{Z}\alpha \oplus \mathbb{Z}\delta$ *of* $\widehat{\mathfrak{sl}_2}$*. Then,* $N\lambda \in P^o(N\Lambda)$ *if and only if* $\lambda \in P^o(\Lambda)$.

Proof. The validity of the lemma is clear for $\lambda \in P^o(\Lambda)_+$ from Corollary 5.1. But since $P^o(\Lambda) = W \cdot (P^o(\Lambda)_+)$, and the action of W on $\mathfrak{h}^*$ is linear, the lemma follows for any $\lambda \in P^o(\Lambda)$. □

Corollary 6.4. *Let $d_o \in \mathbb{Z}_{>1}$. Let Λ, Λ', $\Lambda'' \in P_+$ be such that $\Lambda - \Lambda' - \Lambda'' \in Q$ and $L(N\Lambda)$ is a submodule of $L(N\Lambda') \otimes L(N\Lambda'')$, for some $N \in \mathbb{Z}_{>0}$. Then, $L(d_o\Lambda)$ is a submodule of $L(d_o\Lambda') \otimes L(d_o\Lambda'')$.*

Such a d_o is called a saturation factor.

Proof. If $\Lambda'(c) = 0$ or $\Lambda''(c) = 0$, then

$$L(N\Lambda') \otimes L(N\Lambda'') \simeq L(N(\Lambda' + \Lambda'')),$$

for any $N \geq 1$. Thus, the corollary is clearly true in this case. So, let us assume that both of $\Lambda'(c) > 0$ and $\Lambda''(c) > 0$. Let $L(N\Lambda + n\delta)$ be the δ-maximal component of $L(N\Lambda') \otimes L(N\Lambda'')$ through $L(N\Lambda)$, for some $n \geq 0$. For any $\Psi \in P_+$, let $\bar{\Psi} \in \bar{P}_+$ be the projection $\pi(\Psi)$ defined just before Lemma 5.2. Applying Theorem 6.2 to $\bar{\Lambda}', \bar{\Lambda}'', \bar{\Lambda}$, and observing that

$$L(\bar{\Psi} + k\delta) \simeq L(\bar{\Psi}) \otimes L(k\delta) \tag{28}$$

and $L(k\delta)$ is one dimensional, we get that there is a δ-maximal component $L(\Lambda + \widetilde{n}\delta)$ of $L(\Lambda') \otimes L(\Lambda'')$ through $L(\Lambda)$, for some (unique) $\widetilde{n} \in \mathbb{Z}$.

Again applying Theorem 6.2 to $N\bar{\Lambda}', N\bar{\Lambda}'', N\bar{\Lambda}$, and observing (using Corollary 5.1) that

$$P^o(N\bar{\Psi}) \supset NP^o(\bar{\Psi}), \tag{29}$$

we get that $L(N\Lambda + N\widetilde{n}\delta)$ is the δ-maximal component of $L(N\Lambda') \otimes L(N\Lambda'')$ through $L(N\Lambda)$. Thus, $n = N\widetilde{n}$. In particular,

$$\widetilde{n} \geq 0. \tag{30}$$

Let

$$\sum_{\lambda \in T_{\bar{\Lambda}}^{\Lambda',\Lambda''}} \varepsilon(v_{\bar{\Lambda},\Lambda'',\lambda}) c_{\Lambda',\lambda} e^{S_{\bar{\Lambda},\Lambda'',\lambda}\delta} = \sum_{k \in \mathbb{Z}_+} c_k e^{(\Lambda(d)+\widetilde{n}-k)\delta}, \tag{31}$$

for some $c_k \in \mathbb{Z}_+$ with c_0 nonzero. By Proposition 4.2, this is the character of a unitarizable Virasoro representation with each irreducible component having the same nonzero central charge. Thus, by Lemma 4.1, for any $k > 1$, we get $c_k \neq 0$.

By the above argument, $L(d_o\Lambda + d_o\widetilde{n}\delta)$ is the δ-maximal component of $L(d_o\Lambda') \otimes L(d_o\Lambda'')$ through $L(d_o\Lambda)$. If $\widetilde{n} = 0$, we get that

$$L(d_o\Lambda) \subset L(d_o\Lambda') \otimes L(d_o\Lambda'').$$

If $\widetilde{n} > 0$, then $d_o\widetilde{n}$ being > 1, by the analogue of (31) for $d_o\Lambda', d_o\Lambda''$ and $d_o\Lambda$, $L(d_o\Lambda) \subset L(d_o\Lambda') \otimes L(d_o\Lambda'')$. (Here we have used that $L_0 = -d + p$ on any $\mathfrak{g}$-isotypical component of $L(\Lambda') \otimes L(\Lambda'')$ with highest weight in $\Lambda + \mathbb{Z}\delta$, for a number p depending only upon $\bar{\Lambda}, \Lambda'$ and Λ'', cf. [KR, Identity 10.25 on page 116].) This proves the corollary. □

Remark 6.5. We note that $L(2\Lambda_0-\delta)$ is not a component of $L(\Lambda_0)\otimes L(\Lambda_0)$ (cf. [Kac, Exercise 12.16]). But, of course, $L(2\Lambda_0)$ is a δ-maximal component. By the identity (31), we know that $L(2d_o\Lambda_0 - d_o\delta)$ must be a component of $L(d_o\Lambda_0)\otimes L(d_o\Lambda_0)$, for any $d_o > 1$. So d_o can not be taken to be 1 in Corollary 6.4.

7. A Conjecture

In this section, G is any symmetrizable Kac-Moody group. We recall the following definition of the deformed product due to Belkale-Kumar [BK]. (Even though they gave the definition in the finite case, the same definition works in the symmetrizable Kac-Moody case, though with only one parameter.)

7.1. **Definition.** Let P be any standard parabolic subgroup of G. Recall from Section 2 that $\{\epsilon_P^w\}_{w\in W^P}$ is a basis of the singular cohomology $H^*(X_P,\mathbb{Z})$. Write the standard cup product in $H^*(X_P,\mathbb{Z})$ in this basis as follows:

$$\epsilon_P^u\cdot\epsilon_P^v = \sum_{w\in W^P} n_{u,v}^w\epsilon_P^w, \text{ for some (unique) } n_{u,v}^w\in\mathbb{Z}. \tag{32}$$

Introduce the indeterminate τ and define a deformed cup product $\odot$ as follows:

$$\epsilon_P^u\odot\epsilon_P^v = \sum_{w\in W^P}\tau^{(u^{-1}\rho+v^{-1}\rho-w^{-1}\rho-\rho)(x_P)}n_{u,v}^w\epsilon_P^w, \tag{33}$$

where $x_P := \sum_{\alpha_i\in\Delta\setminus\Delta(P)} x_i$, $\Delta(P)$ is the set of simple roots of the Levi L of P and, as in Section 2, Δ is the set of simple roots of G.

The following lemma is a generalization of the corresponding result in the finite case (cf. [BK, Proposition 17]).

7.2. **Proposition.** *(a) The product $\odot$ is associative and clearly commutative.*

(b) Whenever $n_{u,v}^w$ is nonzero, the exponent of τ in the above is a nonnegative integer.

Proof. The proof of the associativity of $\odot$ is identical to the proof given in [BK, Proof of Proposition 17 (b)].

(b) The proof of this part follows the proof of [BK, Theorem 43]. Consider the decreasing filtration $\mathcal{A}=\{\mathcal{A}_m\}_{m\geq 0}$ of $H^*(X_P,\mathbb{C})$ defined as follows:

$$\mathcal{A}_m := \bigoplus_{w\in W^P:(\rho-w^{-1}\rho)(x_P)\geq m}\mathbb{C}\epsilon_P^w.$$

A priori $\{\mathcal{A}_m\}_{m\geq 0}$ may not be a multiplicative filtration.

We next introduce another filtration $\{\bar{\mathcal{F}}_m\}_{m\geq 0}$ of $H^*(X_P,\mathbb{C})$ in terms of the Lie algebra cohomology. Recall that $H^*(X_P,\mathbb{C})$ can be identified canonically with the Lie algebra cohomology $H^*(\mathfrak{g},\mathfrak{l})$, where $\mathfrak{l}$ is the Lie algebra of the Levi subgroup L of P (cf. [K_2, Theorem 1.6]). The underlying cochain complex $C^\bullet = C^\bullet(\mathfrak{g},\mathfrak{l})$ for $H^*(\mathfrak{g},\mathfrak{l})$ can be written as

$$C^\bullet := [\wedge^\bullet(\mathfrak{g}/\mathfrak{l})^*]^{\mathfrak{l}} = \mathrm{Hom}_{\mathfrak{l}}\big(\wedge^\bullet(\mathfrak{u}_P)\otimes\wedge^\bullet(\mathfrak{u}_P^-),\mathbb{C}\big),$$

where $\mathfrak{u}_P$ (resp. $\mathfrak{u}_P^-$) is the nil-radical of the Lie algebra of P (resp. the opposite parabolic subgroup P^-). Define a decreasing multiplicative filtration $\mathcal{F} = \{\mathcal{F}_m\}_{m\geq 0}$ of the cochain complex $C^\bullet$ by subcomplexes:

$$\mathcal{F}_m := \mathrm{Hom}_{\mathfrak{l}}\left(\frac{\wedge^\bullet(\mathfrak{u}_P)\otimes\wedge^\bullet(\mathfrak{u}_P^-)}{\bigoplus_{s+t\leq m-1}\wedge^\bullet_{(s)}(\mathfrak{u}_P)\otimes\wedge^\bullet_{(t)}(\mathfrak{u}_P^-)}, \mathbb{C}\right),$$

where $\wedge^\bullet_{(s)}(\mathfrak{u}_P)$ (resp. $\wedge^\bullet_{(s)}(\mathfrak{u}_P^-)$) denotes the subspace of $\wedge^\bullet(\mathfrak{u}_P)$ (resp. $\wedge^\bullet(\mathfrak{u}_P^-)$) spanned by the $\mathfrak{h}$-weight vectors of weight β with *P-relative height*

$$\mathrm{ht}_P(\beta) := \mid \beta(x_P) \mid = s.$$

Now, define the filtration $\bar{\mathcal{F}} = \{\bar{\mathcal{F}}_m\}_{m\geq 0}$ of $H^*(\mathfrak{g},\mathfrak{l}) \simeq H^*(X_P,\mathbb{C})$ by

$$\bar{\mathcal{F}}_m := \text{Image of } H^*(\mathcal{F}_m) \to H^*(C^\bullet).$$

The filtration $\mathcal{F}$ of $C^\bullet$ gives rise to the cohomology spectral sequence $\{E_r\}_{r\geq 1}$ converging to $H^*(C^\bullet) = H^*(X_P,\mathbb{C})$. By [K$_3$, Proof of Proposition 3.2.11], for any $m \geq 0$,

$$E_1^m = \bigoplus_{s+t=m}[H^\bullet_{(s)}(\mathfrak{u}_P)\otimes H^\bullet_{(t)}(\mathfrak{u}_P^-)]^{\mathfrak{l}},$$

where $H^\bullet_{(s)}(\mathfrak{u}_P)$ denotes the cohomology of the subcomplex $(\wedge^\bullet_{(s)}(\mathfrak{u}_P))^*$ of the standard cochain complex $\wedge^\bullet(\mathfrak{u}_P)^*$ associated to the Lie algebra $\mathfrak{u}_P$ and similarly for $H^\bullet_{(t)}(\mathfrak{u}_P^-)$. Moreover, by loc. cit., the spectral sequence degenerates at the E_1 term, i.e.,

$$E_1^m = E_\infty^m. \tag{34}$$

Further, by the definition of P-relative height,

$$[H^\bullet_{(s)}(\mathfrak{u}_P)\otimes H^\bullet_{(t)}(\mathfrak{u}_P^-)]^{\mathfrak{l}} \neq 0 \Rightarrow s = t.$$

Thus,

$$E_1^m = 0, \qquad \text{unless } m \text{ is even and}$$
$$E_1^{2m} = [H^\bullet_{(m)}(\mathfrak{u}_P)\otimes H^\bullet_{(m)}(\mathfrak{u}_P^-)]^{\mathfrak{l}}.$$

In particular, from (34) and the general properties of spectral sequences (cf. [K$_3$, Theorem E.9]), we have a canonical algebra isomorphism:

$$\mathrm{gr}(\bar{\mathcal{F}}) \simeq \bigoplus_{m\geq 0}\left[H^\bullet_{(m)}(\mathfrak{u}_P)\otimes H^\bullet_{(m)}(\mathfrak{u}_P^-)\right]^{\mathfrak{l}}, \tag{35}$$

where $\left[H^\bullet_{(m)}(\mathfrak{u}_P)\otimes H^\bullet_{(m)}(\mathfrak{u}_P^-)\right]^{\mathfrak{l}}$ sits inside $\mathrm{gr}(\bar{\mathcal{F}})$ precisely as the homogeneous part of degree $2m$; homogeneous parts of $\mathrm{gr}(\bar{\mathcal{F}})$ of odd degree being zero.

Finally, we claim that, for any $m \geq 0$,

$$\mathcal{A}_m = \bar{\mathcal{F}}_{2m} : \tag{36}$$

Following Kumar [K$_1$], take the d-∂ harmonic representative $\hat{s}^w$ in $C^\bullet$ for the cohomology class ϵ_P^w. An explicit expression is given in [K$_1$, Proposition 3.17]. From this explicit expression, we easily see that

$$\mathcal{A}_m \subset \bar{\mathcal{F}}_{2m}, \text{ for all } m \geq 0. \tag{37}$$

Moreover, from the definition of $\mathcal{A}$, for any $m \geq 0$,

$$\dim \frac{\mathcal{A}_m}{\mathcal{A}_{m+1}} = \#\big\{w \in W^P : (\rho - w^{-1}\rho)(x_P) = m\big\}.$$

Also, by the isomorphism (35) and [K$_3$, Theorem 3.2.7],

$$\dim \frac{\bar{\mathcal{F}}_{2m}}{\bar{\mathcal{F}}_{2m+1}} = \#\big\{w \in W^P : (\rho - w^{-1}\rho)(x_P) = m\big\}.$$

Thus,

$$\dim \frac{\mathcal{A}_m}{\mathcal{A}_{m+1}} = \dim \frac{\bar{\mathcal{F}}_{2m}}{\bar{\mathcal{F}}_{2m+1}}. \tag{38}$$

Of course,

$$\mathcal{A}_0 = \bar{\mathcal{F}}_0. \tag{39}$$

Thus, combining the equations (37), (38) and (39), we get (36). It is easy to see that the filtration $\{\bar{\mathcal{F}}_{2m}\}_{m\geq 0}$ is multiplicative and hence so is $\{\mathcal{A}_m\}_{m\geq 0}$. This proves the (b) part of the proposition. □

The cohomology of X_P obtained by setting $\tau = 0$ in $(H^*(X_P,\mathbb{Z})\otimes\mathbb{Z}[\tau], \odot)$ is denoted by $(H^*(X_P,\mathbb{Z}), \odot_0)$. Thus, as a $\mathbb{Z}$-module, it is the same as the singular cohomology $H^*(X_P,\mathbb{Z})$ and under the product $\odot_0$ it is associative (and commutative).

The following conjecture is a generalization of the corresponding result in the finite case due to Belkale-Kumar [BK, Theorem 22].

7.3. **Conjecture.** *Let G be any indecomposable symmetrizable Kac-Moody group (i.e., its generalized Cartan matrix is indecomposable, cf. [K$_3$, § 1.1]) and let $(\lambda_1, \dots, \lambda_s, \mu) \in P_+^{s+1}$. Assume further that none of λ_j is W-invariant and $\mu - \sum_{j=1}^s \lambda_j \in Q$, where Q is the root lattice of G. Then, the following are equivalent:*

(a) $(\lambda_1, \dots, \lambda_s, \mu) \in \Gamma_s$.

(b) For every standard maximal parabolic subgroup P in G and every choice of $s+1$-tuples $(w_1, \dots, w_s, v) \in (W^P)^{s+1}$ such that ϵ_P^v occurs with coefficient 1 in the deformed product

$$\epsilon_P^{w_1} \odot_0 \cdots \odot_0 \epsilon_P^{w_s} \in \big(H^*(X_P,\mathbb{Z}), \odot_0\big),$$

the following inequality holds:

$$\Big(\sum_{j=1}^s \lambda_j(w_j x_P)\Big) - \mu(v x_P) \geq 0, \qquad (I^P_{(w_1,\dots,w_s,v)})$$

where α_{i_P} is the (unique) simple root in $\Delta \setminus \Delta(P)$ and $x_P := x_{i_P}$.

7.4. **Remark.** (a) By Theorem 3.3, the above inequalities $I^P_{(w_1,\dots,w_s,v)}$ are indeed satisfied for any $(\lambda_1, \dots, \lambda_s, \mu) \in \Gamma_s$.

(b) If G is an affine Kac-Moody group, then the condition that $\lambda \in P_+$ is W-invariant is equivalent to the condition that $\lambda(c) = 0$.

7.5. **Theorem.** *Let $\mathfrak{g} = \widehat{\mathfrak{sl}_2}$. Let $\lambda, \mu, \nu \in P_+$ be such that $\lambda + \mu - \nu \in Q$ and both of $\lambda(c)$ and $\mu(c)$ are nonzero. Then, the following are equivalent:*
(a) $(\lambda, \mu, \nu) \in \Gamma_2$.
(b) The following set of inequalities is satisfied for all $w \in W$ and $i = 0, 1$.

$$\lambda(x_i) + \mu(wx_i) - \nu(wx_i) \geq 0, \quad \textit{and}$$
$$\lambda(wx_i) + \mu(x_i) - \nu(wx_i) \geq 0.$$

In particular, Conjecture 7.3 is true for $\mathfrak{g} = \widehat{\mathfrak{sl}_2}$ and $s = 2$.

Proof. By Lemma 5.2, there exist (unique) $n_1, n_2 \in \mathbb{Z}$ such that

$$\nu - \mu + n_1\delta \in P^o(\lambda), \quad \text{and} \quad \nu - \lambda + n_2\delta \in P^o(\mu).$$

Let $n := \min\,(n_1, n_2)$. By our description of the δ-maximal components as in Theorem 6.2 applied to $\bar{\lambda}, \bar{\mu}, \bar{\nu}$ and using the identity (28), we see that $L(\nu + n\delta)$ is a δ-maximal component of $L(\lambda) \otimes L(\mu)$. Thus, by the equation (29), for any $N \geq 1$, $L(N\nu + Nn\delta)$ is a δ-maximal component of $L(N\lambda) \otimes L(N\mu)$. In particular, by Proposition 4.2 and Lemma 4.1,

$$L(N\nu) \subset L(N\lambda) \otimes L(N\mu) \quad \text{for some } N > 1 \quad \text{if and only if } n \geq 0. \tag{40}$$

By [Kac, Proposition 12.5 (a)], if a weight $\gamma + k\delta \in P(\lambda)$ (for some $k \in \mathbb{Z}_+$), then $\gamma \in P(\lambda)$. Thus,

$$n \geq 0 \quad \text{if and only if } \nu \in \big(P(\lambda) + \mu\big) \cap \big(P(\mu) + \lambda\big). \tag{41}$$

We next show that

$$P(\lambda) = (\lambda + Q) \cap C_\lambda, \tag{42}$$

where $C_\lambda := \{\gamma \in \mathfrak{h}^* : \lambda(x_i) - \gamma(wx_i) \geq 0 \text{ for all } w \in W \text{ and all } x_i\}$. Clearly,

$$P(\lambda) \subset (\lambda + Q) \cap C_\lambda.$$

Since $\lambda + Q$ and C_λ are W-stable, and $\lambda + Q$ is contained in the Tits cone (by [K_3, Exercise 13.1.E.8(a)]), $(\lambda + Q) \cap C_\lambda = W \cdot \big((\lambda + Q) \cap C_\lambda \cap P_+\big)$.

Conversely, take $\gamma \in (\lambda + Q) \cap C_\lambda \cap P_+$. Then, $(\lambda - \gamma)(x_i) \geq 0$ and $(\lambda - \gamma)(c) = 0$ and hence $\lambda - \gamma \in \oplus_i \mathbb{Z}_+\alpha_i$, i.e., $\lambda \geq \gamma$. Thus, by [Kac, Proposition 12.5(a)], $\gamma \in P(\lambda)$. This proves (42). Now, combining (40), (41) and (42), we get $L(N\nu) \subset L(N\lambda) \otimes L(N\mu)$ for some $N > 1$ if and only if for all $w \in W$ and $i = 0, 1$,

$$\lambda(x_i) - (\nu - \mu)(wx_i) \geq 0, \quad \text{and} \quad \mu(x_i) - (\nu - \lambda)(wx_i) \geq 0.$$

This proves the equivalence of (a) and (b) in the theorem.

To prove the 'In particular' statement of the theorem, let P_0 (resp. P_1) be the maximal parabolic subgroup of $G = \widehat{\mathrm{SL}_2}$ with $\Delta(P_0) = \{\alpha_1\}$ (resp. $\Delta(P_1) = \{\alpha_0\}$). For any $n \geq 0$, let

$$w_n := \dots s_0 s_1 s_0 \ (n\text{-factors}) \quad \text{and} \quad v_n := \dots s_1 s_0 s_1 \ (n\text{-factors}).$$

Then, by [K3, Exercise 11.3.E.4], $H^*(G/P_0)$ has a $\mathbb{Z}$-basis $\{\epsilon^n_{P_0}\}_{n\geq 0}$, where $\epsilon^n_{P_0} := \epsilon^{w_n}_{P_0}$. Moreover,

$$\epsilon^n_{P_0} \cdot \epsilon^m_{P_0} = \binom{n+m}{n} \epsilon^{n+m}_{P_0}.$$

In particular, $\epsilon^{n+m}_{P_0}$ appears with coefficient one in $\epsilon^n_{P_0} \cdot \epsilon^m_{P_0}$ if and only if at least one of n or m is 0.

By using the diagram automorphism of $\widehat{\mathrm{SL}_2}$, one similarly gets that $H^*(G/P_1)$ has a $\mathbb{Z}$-basis $\{\epsilon^n_{P_1}\}_{n\geq 0}$, where $\epsilon^n_{P_1} := \epsilon^{v_n}_{P_1}$, with the product given by

$$\epsilon^n_{P_1} \cdot \epsilon^m_{P_1} = \binom{n+m}{n} \epsilon^{n+m}_{P_1}.$$

Moreover, from the definition of the deformed product $\odot_0$, clearly

$$\epsilon^0_{P_0} \odot_0 \epsilon^m_{P_0} = \epsilon^0_{P_0} \cdot \epsilon^m_{P_0},$$

and similarly for P_1. From this the 'In particular' statement of the theorem follows. ☐

7.6. **Remark.** (1) It is easy to see that if $\lambda = m\delta$ for some $m \in \mathbb{Z}$, then the equivalence of (a) and (b) in the above theorem breaks down.

(2) Though we have proved Conjecture 7.3 for $\widehat{\mathrm{SL}_2}$ only for $s = 2$, it is quite likely that a similar proof will prove it for any s (for $\widehat{\mathrm{SL}_2}$).

8. The $A_2^{(2)}$ case

By a method similar to that of $A_1^{(1)}$, we handle the $A_2^{(2)}$ case, with minor modifications where necessary. Write $\mathfrak{h} = \mathbb{C}c \oplus \mathbb{C}\alpha^\vee \oplus \mathbb{C}d$ and $\mathfrak{h}^* = \mathbb{C}\omega_0 \oplus \mathbb{C}\alpha \oplus \mathbb{C}\delta$, where $\alpha(\alpha^\vee) = 2$, $\delta(d) = 1$, $\omega_0(c) = 1$, and all other values are 0. Then $(\mathfrak{h}, \{\alpha_0 := \delta - 2\alpha, \alpha_1 := \alpha\}, \{\alpha_0^\vee := c - \frac{1}{2}\alpha^\vee, \alpha_1^\vee := \alpha^\vee\})$ is a realization of the GCM

$$\begin{pmatrix} 2 & -1 \\ -4 & 2 \end{pmatrix}$$

of $A_2^{(2)}$. The fundamental weights are ω_0 and $\omega_1 = \frac{1}{2}\omega_0 + \frac{1}{2}\alpha$. This easily allows one to compute the dominant δ-maximal weights. Analogous to Corollary 5.1, we have the following:

8.1. **Lemma.** *Let λ be a dominant integral weight. Then, the dominant δ-maximal weights of $L(\lambda)$ are the dominant weights of the form*

$$P_+ \cap \{\lambda - j\alpha,\ \lambda + k(2\alpha - \delta),\ \lambda + \alpha - \delta + l(2\alpha - \delta)\ :\ j, k, l \in \mathbb{Z}_{\geq 0}\}.$$

Moreover, $P^o(\lambda)$ is the W-orbit of the above.

Again, to determine the saturated tensor cone, it is enough to describe the δ-maximal components. Thus, to determine the δ-maximal components, by virtue of proposition 4.2, we must find the highest δ-degree term in $\sum_{\lambda \in T_\Lambda^{\Lambda',\Lambda''}} \varepsilon(v_{\Lambda,\Lambda'',\lambda}) c_{\Lambda',\lambda} e^{S_{\Lambda,\Lambda'',\lambda}\delta}$. This computation is done in a somewhat similar manner as in the $A_1^{(1)}$ case, but there are some important modifications. First, we need to use two different piecewise smooth functions to describe the δ-maximal weights of $L(\lambda)$. An upper function A^+ interpolates the δ-maximal weights which are in the W-orbit of the dominant weights of the form

$$\{\lambda - j\alpha,\ \lambda + k(2\alpha - \delta)\ :\ j,k \in \mathbb{Z}_{\geq 0}\},$$

while another function A^- interpolates the δ-maximal weights in the W-orbit of the dominant weights of the form

$$\{\lambda - j\alpha,\ \lambda + \alpha - \delta + l(2\alpha - \delta)\ :\ j,l \in \mathbb{Z}_{\geq 0}\}.$$

Now, all of the arguments made in the $\widehat{\mathfrak{sl}_2}$ case must be made for two extensions of $S_{\Lambda,\Lambda'',\lambda}$ to non-integral values, using A^+ and A^- respectively. Let $\Lambda := m_0\omega_0 + m_1\omega_1$, $\Lambda' := m_0'\omega_0 + m_1'\omega_1$, and $\Lambda'' := m_0''\omega_0 + m_1''\omega_1$. The following result is an analogue of Proposition 5.5 and Lemma 5.10 for the $A_2^{(2)}$ case.

Proposition 8.2. *Let $\Lambda, \Lambda', \Lambda''$ be as above. Assume that both of $\Lambda'(c)$ and $\Lambda''(c) > 0$ and $\Lambda - \Lambda' - \Lambda'' \in Q$, where $Q = \mathbb{Z}\alpha + \mathbb{Z}\delta$ is the root lattice of $A_2^{(2)}$. Then, the maximum $\mu_\Lambda^{\Lambda',\Lambda''}$ of the set*

$$\left\{S_{\Lambda,\Lambda'',\lambda} :\ \lambda \in T_\Lambda^{\Lambda',\Lambda''},\ \varepsilon(v_{\Lambda,\Lambda'',\lambda}) = 1\right\}$$

occurs when $\lambda \equiv \Lambda' + \frac{1}{2}(m_1 - m_1' - m_1'')\alpha \mod \mathbb{C}\delta$. The maximum $\bar{\mu}_\Lambda^{\Lambda',\Lambda''}$ of the set

$$\left\{S_{\Lambda,\Lambda'',\lambda} :\ \lambda \in T_\Lambda^{\Lambda',\Lambda''},\ \varepsilon(v_{\Lambda,\Lambda'',\lambda}) = -1\right\}$$

occurs when $\lambda \equiv \Lambda' - \big(\frac{1}{2}(m_1' + m_1'' + m_1) + 1\big)\alpha \mod \mathbb{C}\delta$ or when $\lambda \equiv \Lambda' - \big(\frac{1}{2}(m_1' + m_1'' + m_1) - 2(\Lambda'(c) + \Lambda''(c) + 1)\big)\alpha \mod \mathbb{C}\delta$.

8.3. **Corollary.** *Let $\Lambda, \Lambda', \Lambda''$ be as in Proposition 8.2. Assume further that $\Lambda'(c) \geq 2$, $\Lambda''(c) \geq 2$, $m_1', m_1'' \neq 1$. Then, if $\mu_\Lambda^{\Lambda',\Lambda''} = \bar{\mu}_\Lambda^{\Lambda',\Lambda''}$, we have*

$$\mu_\Lambda^{\Lambda'',\Lambda'} \neq \bar{\mu}_\Lambda^{\Lambda'',\Lambda'}.$$

The proof of Corollary 8.3 requires a description of the situations in which $\mu_\Lambda^{\Lambda',\Lambda''} = \bar{\mu}_\Lambda^{\Lambda',\Lambda''}$. We reduce these situations to certain cases, and show that in most of these cases, if the roles of Λ' and Λ'' are interchanged, then (as in the $\widehat{\mathfrak{sl}_2}$ case) the equality does not occur. In the remaining cases, we show that $\Lambda'(c) < 2$, $\Lambda''(c) < 2$, $m_1' = 1$, or $m_1'' = 1$.

Theorem 8.4. *Let $\Lambda, \Lambda', \Lambda''$ be as in Proposition 8.2. Then, $L(\Lambda + n\delta)$ is a δ-maximal component of $L(\Lambda') \otimes L(\Lambda'')$ if $n = \min(n_1, n_2)$, where n_1 is such that $\Lambda - \Lambda'' + n_1\delta \in P^o(\Lambda')$ and n_2 is such that $\Lambda - \Lambda' + n_2\delta \in P^o(\Lambda'')$.*

Lemma 8.5. *Fix a positive integer N. Let $\Lambda \in \bar{P}_+$ and let $\lambda \in \Lambda + Q$. Then, $N\lambda \in P^o(N\Lambda)$ if and only if $\lambda \in P^o(\Lambda)$.*

Combining the above results, we get a description of Γ_2, which is identical to that of $\widehat{\mathfrak{sl}_2}$ (cf. Theorem 7.5).

8.6. **Theorem.** *Let $\mathfrak{g} = A_2^{(2)}$. Let $\lambda, \mu, \nu \in P_+$ be such that $\lambda + \mu - \nu \in Q$ and both of $\lambda(c)$ and $\mu(c)$ are nonzero. Then, the following are equivalent:*

(a) $(\lambda, \mu, \nu) \in \Gamma_2$.

(b) The following set of inequalities is satisfied for all $w \in W$ and $i = 0, 1$.

$$\lambda(x_i) + \mu(wx_i) - \nu(wx_i) \geq 0, \quad \textit{and}$$
$$\lambda(wx_i) + \mu(x_i) - \nu(wx_i) \geq 0.$$

In particular, Conjecture 7.3 is true for this case as well for $s = 2$.

The 'In particular' statement of the above theorem follows by using the description of the cup product in the cohomology of the full flag variety of $A_2^{(2)}$ given by Kitchloo [Ki].

It is clear that if the level of $L(\Lambda')$ or $L(\Lambda'')$ is zero, then the tensor product has a single component. Thus, it is already saturated. Assume now that the levels of both of $L(\Lambda')$ and $L(\Lambda'')$ are > 0. Then, since there are representations of level $\frac{1}{2}$, the conditions of Corollary 8.3 are satisfied for any $N\Lambda$, $N\Lambda'$, $N\Lambda''$ with $\Lambda - \Lambda' - \Lambda'' \in Q$, provided $N \geq 4$. Hence:

Corollary 8.7. *For $A_2^{(2)}$, 4 is a saturation factor.*

8.8. **Remark.** When the Kac-Moody Lie algebra $\mathfrak{g}$ is infinite dimensional, then the saturated tensor semigroup Γ_s is *not* finitely generated, for any $s \geq 2$. Thus, it is not clear a priori that there exists a saturation factor for such a $\mathfrak{g}$.

References

[BK] P. Belkale and S. Kumar. Eigenvalue problem and a new product in cohomology of flag varieties, *Inventiones Math.* **166** (2006), 185–228.

[CG] N. Chriss and V. Ginzburg. *Representation Theory and Complex Geometry*, Birkhäuser (1997).

[Kac] V. Kac. *Infinite Dimensional Lie Algebras*, Third Edition, Cambridge University Press (1990).

[KR] V. Kac and A. Raina. *Bombay Lectures on Highest Weight Representations of Infinite Dimensional Lie Algebras*, World Scientific Publishing Company Pvt. Ltd. (1987).

[KW] V. Kac and M. Wakimoto. Modular and conformal invariance constraints in representation theory of affine algebras, *Advances in Math.* **70** (1988), 156–234.

[Ka] M. Kashiwara. The flag manifold of Kac-Moody Lie algebra, in: *Algebraic Analysis, Geometry, and Number Theory* (J-I. Igusa, ed.), The Johns Hopkins University Press, Baltimore, 1989, pp. 161–190.

[KS] M. Kashiwara and M. Shimozono. Equivariant K-theory of affine flag manifolds and affine Grothendieck polynomials, *Duke Math. J.* **148** (2009), 501-538.

[Ki] N. Kitchloo. On the topology of Kac-Moody groups, Preprint (2011).

[KK] B. Kostant and S. Kumar. T-equivariant K-theory of generalized flag varieties, *J. Diff. Geometry* **32** (1990), 549–603.

[K_1] S. Kumar. Geometry of Schubert cells and cohomology of Kac-Moody Lie algebras, *J. Diff. Geometry* **20** (1984), 389–431.

[K_2] S. Kumar. Rational homotopy theory of flag varieties associated to Kac-Moody groups, In: *Infinite Dimensional Groups with Applications*, MSRI Publications vol. **4** (1985), 233–273.

[K_3] S. Kumar. *Kac-Moody Groups, their Flag Varieties and Representation Theory*, Progress in Mathematics, Vol. **204**, Birkhäuser (2002).

[K_4] S. Kumar. Positivity in T-Equivariant K-theory of flag varieties associated to Kac-Moody groups, Preprint (2012).

[K_5] S. Kumar. Additive Eigenvalue Problem (a survey), Preprint (2013).

[MFK] D. Mumford, J. Fogarty and F. Kirwan. *Geometric Invariant Theory, 3rd edn.*, Ergebnisse der Mathematik und Ihrer Grenzgebiete, Vol. **34**, Springer (1994).

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