Ergodicity of the adic transformation on the Euler graph

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Abstract

The Euler graph has vertices labelled (n,k) for $n = 0,1,2,...$ and $k = 0,1,...,n$, with $k+1$ edges from (n,k) to $(n+1,k)$ and $n-k+1$ edges from (n,k) to $(n+1,k+1)$. The number of paths from (0,0) to (n,k) is the Eulerian number $A(n,k)$, the number of permutations of 1,2,...,$n+1$ with exactly $n-k$ falls and k rises. We prove that the adic (Bratteli-Vershik) transformation on the space of infinite paths in this graph is ergodic with respect to the symmetric measure.

1. *The Euler Graph*

The Euler graph is an infinite directed graph such that at level n there are $n+1$ vertices labelled $(n,0)$ through (n,n). The vertex (n,k) has $n+2$ total edges leaving it, with $k+1$ edges connecting it to vertex $(n+1,k)$ and $n-k+1$ edges connecting it to vertex $(n+1,k+1)$.

Define X to be the space of infinite edge paths on the Euler graph. X is a compact metric space in a natural way: if two paths $x = x_0x_1x_2...$ and $y = y_0y_1y_2...$ agree for all n less than j and $x_j \neq y_j$, then define $d(x,y) = 2^{-j}$. The number of paths from the root vertex (0,0) to the vertex (n,k) is the Eulerian number, $A(n,k)$, which is the number of

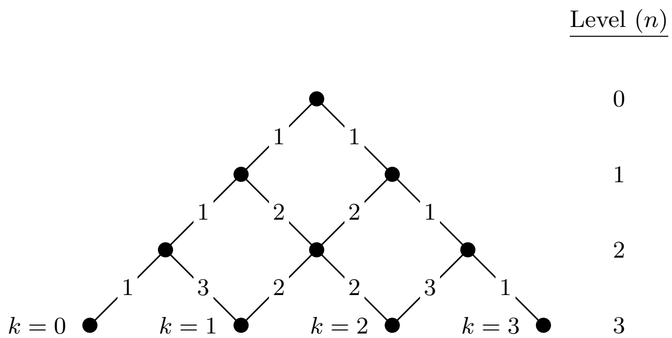

Fig. 1. The first three levels of the Euler graph. The numbers on the diagonals give the number of edges coming out of each vertex, and k represents the label on each vertex.

permutations of $0, 1, ..., n$ with exactly k rises and $n-k$ falls. These numbers satisfy the recursion

$$A(n+1,k) = (n-k+2)A(n,k-1) + (k+1)A(n,k). \tag{1·1}$$

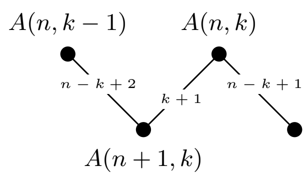

Fig. 2. The Euler Graph gives rise to Equation 1·1.

We put a partial order on the set of paths in X. The edges e_0 through e_{n+1} into the fixed vertex (n,k) with $0 < k < n$ are completely ordered; we illustrate it so that the ordering increases from left to right. If x, y are paths in X, we say that x is less than y if there exists an N such that both x and y pass through vertex $(N+1,k)$, $x_n = y_n$ for all $n > N$, and $x_N < y_N$ with respect to the edge ordering.

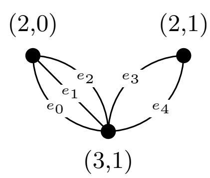

Fig. 3. The order on the edges coming into vertex (3,1): $e_0 < e_1 < e_2 < e_3 < e_4$.

Define $k_n : X \to \{0, 1, \ldots, n-1\}$ by agreeing that if a path x passes through vertex (n,k), then $k_n(x) = k$. We then say that x has a *left turn* at level n if $k_{n+1}(x) = k_n(x)$ and a *right turn* if $k_{n+1}(x) = k_n(x) + 1$. Then define $X_{\max}$ to be the set of paths in X such that there are no greater paths with respect to the above ordering: $X_{\max}$= { the path with no left turns, the path with no right turns}$\cup\{x \in X|$there is a j such that x has a unique left turn at x_j and for all $n \geq j$, x_n is the maximal edge into $(n, j+1)\}$. $X_{\min}$ is the set of paths in X such that there are no smaller paths with respect to the above ordering: $X_{\min}$= { the path with no left turns, the path with no right turns}$\cup\{x \in X|$ there is a j such that x has a unique right turn at x_j and for all $n \geq j$, x_n is the minimal edge into $(n, n-j)\}$ Both $X_{\max}$ and $X_{\min}$ are countable.

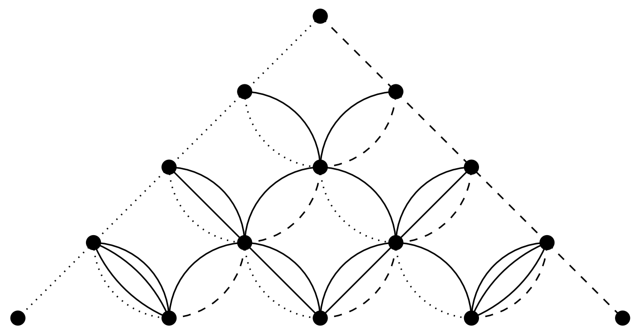

Fig. 4. The dashed paths are maximal, and the dotted paths are minimal. In addition, the paths following the far left edge and the far right edge are both maximal and minimal.

If $x \in X \setminus X_{\max}$, consider the first non-maximal edge, x_j, of x and let y_j be the next greatest edge with respect to the edge ordering. Then define $y_0y_1...y_{j-1}$ to be the minimal path into the source of y_j and let $T(x) = y_0...y_jx_{j+1}x_{j+2}...$(so $T(x)_i = x_i$ for all $i = j+1, j+2, \dots$). Then $T : X \setminus X_{\max} \to X \setminus X_{\min}$ is the *Euler adic.*

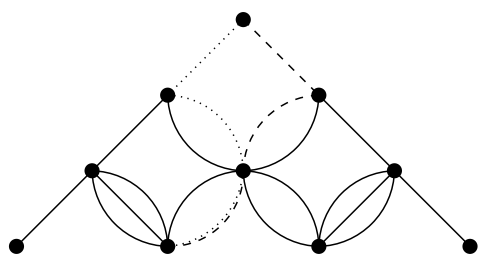

Fig. 5. T maps the dotted path into the dashed path.

Since both $X_{\max}$ and $X_{\min}$ are countable, for any T-invariant, nonatomic measure μ, $\mu(X_{\max}) = \mu(X_{\min}) = 0$.

2. *The Symmetric Invariant Measure*

A *cylinder set* $C = [c_0c_1...c_{n-1}]$ is $\{x \in X | x_i = c_i$ for all $i = 0, 1, ..., n-1\}$. Given any $T-$invariant Borel measure, μ, on X, define the *weight* w_n on an edge c_n connecting level n and $n+1$ to be $\mu([c_0...c_n]|[c_0...c_{n-1}])$ for n greater than 0 and $w_0 = \mu([c_0])$. Then $\mu[c_0...c_n] = w_0...w_n$, where w_i is the weight on the edge c_i. There are two conditions which together are necessary and sufficient to ensure that a measure on X is T-invariant. The first is that if e_0 and e_1 have the same source vertex and the same terminal vertex, then their weights are equal. The second is the *diamond law.* If u_1 is the weight associated with the edges connecting vertex (n, k) to $(n+1, k)$, u_2 is the weight associated to the edges connecting (n, k) to $(n+1, k+1)$, v_1 is the weight on the edges connecting $(n+1, k)$ to $(n+2, k+1)$, and v_2 is the weight on the edges connecting $(n+1, k+1)$ to $(n+2, k+1)$, then $u_1v_1 = u_2v_2$.

Definition. *The* symmetric measure*, η, is determined by assigning weights 1/$(n+2)$ on each edge connecting level n to level $n+1$.*

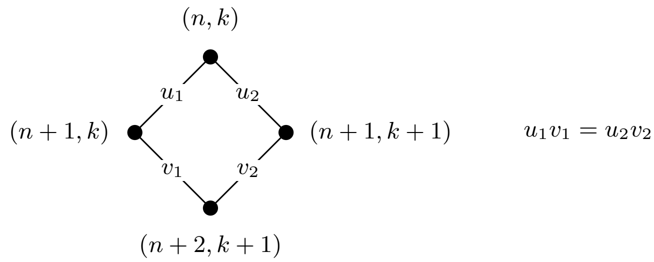

Fig. 6. The diamond law.

This measure clearly satisfies both of the above conditions and hence is T-invariant.

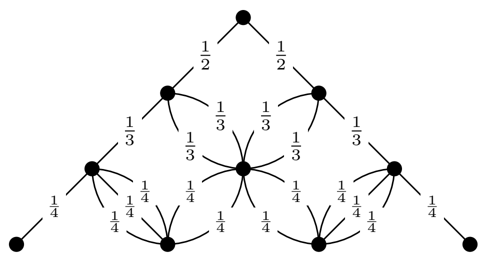

Fig. 7. The Symmetric Measure

3. *The Cutting and Stacking Representation*

We can also view the transformation T as a map on the unit interval defined by "cutting and stacking" which preserves Lebesgue measure, m. Each stage of cutting and stacking corresponds to a level in the Euler graph. At each stage $n = 0, 1, 2, \ldots$ we have $n + 1$ stacks $S_{n,0}, S_{n,1}, \ldots, S_{n,n}$ (corresponding to the vertices (n, k), $0 \le k \le n$, of the Euler graph). Stack $S_{n,k}$ consists of $A(n, k)$ subintervals of $[0, 1]$. Each subinterval corresponds to a cylinder set determined by a path of length n, terminating in vertex (n, k). The transformation $\tilde{T}$ is defined by mapping each level of the stack, except the topmost one, linearly onto the one above it. This corresponds to mapping each non-maximal path of length n to its successor. To proceed to the next stage in the cutting and stacking construction, each stack $S_{n,k}$ is cut into $n + 2$ equal substacks. These are recombined into new stacks in the order prescribed by the way T maps their corresponding cylinder sets. In this way, we obtain a Lebesgue measure-preserving transformation defined almost everywhere on $[0, 1]$.

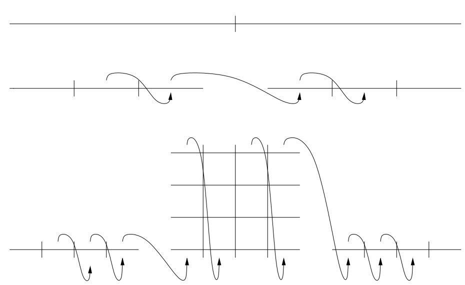

Fig. 8. The Euler adic as a cutting and stacking transformation.

4. *Ergodicity*

In order to prove that the Euler adic T is ergodic with respect to the symmetric measure η, we adapt the proof in [**5**] of ergodicity of the $\mathcal{B}(1/2, 1/2)$ measure for the Pascal adic. For previous proofs of the ergodicity of Bernoulli measures for the Pascal adic, see [**4**],[**10**], [**8**],[**6**],[**7**] and the references that they contain.

PROPOSITION 1. *For each $x \in X$, denote by $I_n(x)$ the cylinder set determined by $x_0x_1...x_{n-1}$. Then for each measurable $A \subseteq X$,*

$$\frac{\eta(A \cap I_n(x))}{\eta(I_n(x))} \to \chi_A(x) \text{ almost everywhere.}$$

Proof. In view of the isomorphism of (X, η) and $([0, 1], m)$, this is just the Lebesgue Density Theorem. □

Denote by ρ the measure $\eta \times \eta$ on $X \times X$.

PROPOSITION 2. *For ρ-almost every $(x, y) \in X \times X$, there are infinitely many n such that $I_n(x)$ and $I_n(y)$ end in the same vertex of the Euler graph, equivalently $(n, k_n(x)) = (n, k_n(y))$.*

This is equivalent to saying that for infinitely many n the number of left turns in $x_1...x_n$ equals the number of left turns in $y_1...y_n$, or that in the cutting and stacking representation the subintervals of [0,1] corresponding to $I_n(x)$ and $I_n(y)$ are in the same stack. This happens because the symmetric measure has a central tendency: if a path is not near the center of the graph at level n, there is a greater probability that at level $n + 1$ it will be closer to the center than before (and the farther from the center, the greater the probability). We defer momentarily the proof of Proposition 2 in order to show how it immediately implies the main result.

THEOREM. *The Euler adic T is ergodic with respect to the symmetric measure, η.*

Proof. Suppose that $A \subseteq X$ is measurable and T-invariant and that $0 < \eta(A) < 1$. By Proposition 1,

$$\frac{\eta(A \cap I_n(x))}{\eta(I_n(x))} \to 1 \text{ and } \frac{\eta(A^c \cap I_n(y))}{\eta(I_n(y))} \to 1 \text{ for } \rho\text{-almost every } (x, y) \in A \times A^c.$$

Hence for almost every $(x, y) \in A \times A^c$ we can pick an $n_0 = n_0(x, y)$ such that for all $n \geq n_0$,

$$\frac{\eta(A \cap I_n(x))}{\eta(I_n(x))} > \frac{1}{2} \qquad \text{and} \qquad \frac{\eta(A^c \cap I_n(y))}{\eta(I_n(y))} > \frac{1}{2}. \tag{4·1}$$

Then, by Proposition 2, we can choose $n \geq n_0$ such that $I_n(x)$ and $I_n(y)$ end in the same vertex, and hence there is $j \in \mathbb{Z}$ such that $T^j(I_n(x)) = I_n(y)$. Since A is T-invariant, this contradicts (4·1). Then we must have $\eta(A) = 0$ or $\eta(A) = 1$, and so T is ergodic with respect to η. □

It remains to prove Proposition 2.

LEMMA 1. *On $(X \times X, \rho)$, for each $n = 1, 2, \ldots$ let $D_n(x, x') = |k_n(x) - k_n(x')|$, and let $\mathcal{F} = \mathcal{B}((x_1, x'_1), \ldots, (x_n, x'_n))$ denote the σ-algebra generated by $(x_1, x'_1), \ldots, (x_n, x'_n)$.*

Let $\sigma(x,x')$ be a stopping time with respect to $(\mathcal{F}_n)$ such that $D_{\sigma(x,x')}(x,x') > 0$. Fix $M > 0$ and let

$$\tau(x,x') = \inf\{n > \sigma(x,x') : D_n \in \{0, M\}\}.$$

For $n = 0, 1, 2, \ldots,$ let

$$Y_n(x,x') = \begin{cases} D_{\sigma(x,x')}(x,x') & \text{if} \quad 0 \le n \le \sigma(x,x') \\ D_n(x,x') & \text{if} \quad \sigma(x,x') < n \le \tau(x,x') \\ D_{\tau(x,x')}(x,x') & \text{if} \quad n \ge \tau(x,x') \end{cases}$$

Then $(Y_n(x,x') : n = 0, 1, 2, \ldots)$ is a supermartingale with respect to $(\mathcal{F}_n)$.

Proof. We have to check the defining inequality for supermartingales only for the range of n where $Y_n = D_n$, since otherwise $Y_n(x,x')$ is constant in n.

If x turns to the left at stage n, then $k_{n+1}(x) = k_n(x)$, but if x turns to the right $k_{n+1}(x) = k_n(x) + 1$. From Figure 2 we see that

$$\eta\{k_{n+1}(x) = k_n(x) | x_1 \ldots x_n\} = \frac{k_n(x) + 1}{n + 2} \tag{4·2}$$

and

$$\eta\{k_{n+1}(x) = k_n(x) + 1 | x_1 \ldots x_n\} = \frac{n - k_n(x) + 1}{n + 2}. \tag{4·3}$$

Without loss of generality assume that $k_n(x') > k_n(x)$. Note that

$$D_{n+1} = \begin{cases} D_n & \text{on the set} \quad A = \{k_{n+1}(x) = k_n(x), k_{n+1}(x') = k_n(x')\} \cup \\ & \qquad \{k_{n+1}(x) = k_n(x) + 1, k_{n+1}(x') = k_n(x') + 1\} \\ D_n + 1 & \text{on the set} \quad B = \{k_{n+1}(x) = k_n(x), k_{n+1}(x') = k_n(x) + 1\} \\ D_n - 1 & \text{on the set} \quad C = \{k_{n+1}(x) = k_n(x) + 1, k_{n+1}(x') = k_n(x')\}. \end{cases}$$

From (4·2) and (4·3),

$$E_\rho(D_{n+1} - D_n | \mathcal{F}_n) = 0 \cdot \rho(A|\mathcal{F}_n) + 1 \cdot \rho(B|\mathcal{F}_n) - 1 \cdot \rho(C|\mathcal{F}_n)$$

$$= \frac{1}{n+2}[(k_n(x) + 1)(n - k_n(x') + 1) - (k_n(x') + 1)(n - k_n(x) - 1)] \le 0.$$

Hence $E_\rho(D_{n+1}|\mathcal{F}_n) \le D_n$. $\square$

LEMMA 2. $\dfrac{k_n(x)}{n} \to \dfrac{1}{2}$ *in measure.*

Proof. Let $u_n(x) = 2k_n(x) - n$ for all n. We will show that $u_n/n \to 0$ in measure. We begin by computing the variance of u_n. Note that if $k_{n+1}(x) = k_n(x)$ then $u_{n+1} = u_n - 1$, and if $k_{n+1}(x) = k_n(x) + 1$ then $u_{n+1} = u_n + 1$. Following the calculations in [**9**], and using (4·2) and (4·3),

$$E_\eta(u_{n+1}|u_n) = (n+1)/(n+2)u_n;$$

so, since $u_0 = 0$, $E(u_n) = 0$ for all $n = 1, 2, \ldots$. Similarly,

$$E_\eta(u_{n+1}^2 | u_n) = (u_n - 1)^2 \left(\frac{k_n(x) + 1}{n+2}\right) + (u_n + 1)^2 \left(\frac{n - k_n(x) + 1}{n+2}\right)$$

$$= (u_n - 1)^2 \left(\frac{u_n + n + 2}{2(n+2)} \right) + (u_n + 1)^2 \left(\frac{n - u_n + 2}{2(n+2)} \right)$$

$$= \frac{n u_n^2}{n+2} + 1.$$

Then

$$E_\eta(u_{n+1}^2 | u_{n-1}) = \frac{n}{n+2} \left(\frac{(n-1)u_{n-1}^2}{n+1} + 1 \right) + 1,$$

and continuing this recursively we see that

$$V(u_{n+1}) = E_\eta(u_{n+1}^2) = \frac{1}{(n+1)(n+2)} \sum_{i=0}^{n} (i+1)(i+2)$$

$$= \frac{n+3}{3}.$$

Then by Chebyshev's Inequality,

$$\eta \left\{ \left| \frac{u_n}{n} \right| \geq \epsilon \right\} \leq \frac{c}{n\epsilon^2} \to 0 \text{ as } n \to \infty,$$

so that $\dfrac{u_n}{n} \to 0$ in measure, i.e. $\dfrac{k_n(x)}{n} \to \dfrac{1}{2}$ in measure. □

Proof of Proposition 2. From Lemma 1, (D_n) is a supermartingale with respect to $\mathcal{F}_n = (\mathcal{B}((x_1, x_1'), \ldots, (x_n, x_n')))$. Fix $M > 0$ and define stopping times $\sigma(x, x') = \inf\{n | k_n(x) \neq k_n(x')\}$ and $\tau(x, x') = \inf\{n > \sigma(x, x') | D_n \in \{0, M\}\}$. Then $E_\rho(D_\tau) \leq E_\rho(D_\sigma) = 1$. If τ is finite almost everywhere, then

$$E_\rho(D_\tau) = M(\rho\{D_\tau = M\}) + 0(\rho\{D_\tau = 0\}), \text{ so that}$$

$\rho\{D_n \neq 0$ for any $n > \sigma(x, x')\} \leq \rho\{D_\tau = M\} \leq 1/M$ for all M. Letting $M \to \infty$ implies that $\rho\{D_n \neq 0$ for any $n > \sigma(x, x')\} = 0$. Hence with ρ-probability 1 there is an n_0 for which $k_{n_0}(x) = k_{n_0}(x')$. Repeat this process with $\sigma(x, x') = \inf\{n > n_0(x, x') | k_n(x) \neq k_n(x')\}$ to see that with ρ-probability 1, $k_n(x) = k_n(x')$ infinitely many times. It remains to show that τ is finite almost everywhere.

We have a fixed M; fix also a large L. Fix a small enough ϵ so that if $k_n(x)/n,\ k_n(x')/n$ are in the interval $(1/2 - \epsilon, 1/2 + \epsilon)$, then

$$\frac{k_{n+i}(x)}{n+i}, \frac{n - k_{n+i}(x)}{n+i}, \frac{k_{n+i}(x')}{n+i}, \frac{n - k_{n+i}(x')}{n+i} \geq \frac{1}{4} \text{ for } i = 0, 1, \ldots, ML.$$

In other words, starting from $(n, k_n(x))$ all the probabilities of going left or right for both x and x' are at least 1/4 for ML steps. Let $A_n = \{(x, x') \in X \times X | k_n(x)/n, k_n(x')/n \in (1/2 - \epsilon, 1/2 + \epsilon)\}$, and note that $\rho(A_n) \to 1$ as $n \to \infty$, by the convergence in measure.

Let $B_n = \{(x, x') \in X \times X | k_{n+i}(x) = k_n(x), k_{n+i}(x') = k_n(x') + i$ for all $i = 0, 1 \ldots, M\}$.

For every n, $\{x | \tau(x) = \infty\} \cap A_n \subset A_n \cap B_n^c \cap B_{n+M}^c \cap \cdots \cap B_{n+(L-1)M}^c = G_n$, since (x, x') in B_n implies $D_{n+i}(x, x')$ is either 0 or M for some $i \leq M$. Conditioned on the set A_n, the sets $B_n, B_{n+M}, \ldots, B_{n+(L-1)M}$ are not independent, because at each step the probabilities of going left or right, given by sums of the weights on the edges, are changing. But since the probabilities of going left or right at each step are all near 1/2, so that the probability of each event we are considering is near the probability that it would be assigned by a genuine symmetric random walk, we can estimate the measure of G_n.

For each $j = 0, 1, \ldots, L-1$, abbreviate $E_j = B^c_{n+jM}$. Then for each pair of vertices $v = ((jM-1,k),(jM-1,k'))$,we have $\rho(E_j|v) \leq (1-1/4^{2M})$. Thus

$$\rho(E_j|E_{j-1}\cap\cdots\cap E_0\cap A_n) = \sum_{\substack{\text{vertices } v \text{ at}\\ \text{level } jM-1}} \rho(E_j|v)\rho(v|E_{j-1}\cap\cdots\cap E_0\cap A_n) \leq (1-1/4^{2M}),$$

and iterating gives $\rho(E_{L-1}\cap\cdots\cap E_0|A_n) \leq \left(1-1/4^{2M}\right)^L$.

Therefore $\rho(\tau=\infty|A_n) \leq (1-1/4^{2M})^L$ for all L. Letting $n\to\infty$ and then $L\to\infty$, we conclude that $\rho\{\tau=\infty\}=0$. $\square$

Remark 1. In fact $k_n(x)/n \to 1/2$ almost everywhere. We can see this as follows. Continue to let $u_n(x) = 2k_n(x) - n$ as in Lemma 2. Since $E((n+2)u_{n+1}|(n+1)u_n) = (n+1)u_n,\ S_n = (n+1)u_n$ forms a mean-0 martingale. If $X_n = S_n - S_{n-1}$, then the X_n are a martingale difference sequence in L^2, thus mean 0 and orthogonal. The variance of X_n is

$$E(X_n^2) = E(S_n^2) - E(S_{n-1}^2) = \frac{3n^2+5n}{3}.$$

If we let $b_n = n^2$, then $\sum E(X_n^2)/b_n^2 < \infty$, so by the extension to martingales of Kolmogrov's Criterion for the Strong Law of Large Numbers (see [**3**, p.238]) $S_n/b_n \to 0$ almost everywhere, that is to say, $u_n/n \to 0$ almost everywhere.

Remark 2. It would be interesting to determine further dynamical properties of this system, such as weak mixing, rigidity, singularity of the spectrum, and whether the rank is infinite. So far we can show that the symmetric measure (η) is the only fully supported invariant ergodic measure [**2**], and that (X,T,η) is totally ergodic and loosely Bernoulli, [**1**].

We thank the referee for helpful comments.

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