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REALITY PROPERTY OF DISCRETE WRONSKI MAP WITH IMAGINARY STEP

E. MUKHIN*, V. TARASOV⋆, AND A. VARCHENKO◇

*,⋆*Department of Mathematical Sciences, Indiana University – Purdue University Indianapolis*
402 North Blackford St, Indianapolis, IN 46202-3216, USA

⋆*St. Petersburg Branch of Steklov Mathematical Institute*
Fontanka 27, St. Petersburg, 191023, Russia

◇*Department of Mathematics, University of North Carolina at Chapel Hill*
Chapel Hill, NC 27599-3250, USA

Abstract. For a set of quasi-exponentials with real exponents, we consider the discrete Wronskian (also known as Casorati determinant) with pure imaginary step $2h$. We prove that if the coefficients of the discrete Wronskian are real and for every its roots the imaginary part is at most $|h|$, then the complex span of this set of quasi-exponentials has a basis consisting of quasi-exponentials with real coefficients. This result is a generalization of the statement of the B. and M. Shapiro conjecture on spaces of polynomials. The proof is based on the Bethe ansatz for the XXX model.

Introduction

It follows from the Fundamental Theorem of Algebra that a polynomial $p(x) \in \mathbb{C}[x]$ has real coefficients (up to an overall constant) if all its roots are real. The B. and M. Shapiro conjecture for the case of Grassmannian, proved in [MTV1], is a generalization of this fact. It claims that if the Wronskian of N polynomials with complex coefficients has real roots only, then the space spanned by these polynomials has a basis consisting of polynomials with real coefficients. For $N = 2$ (the case proved in [EG]), the statement can be formulated in the following attractive form: if all critical points of a rational function are real, then the function is real modulo a fractional linear transformation.

Several generalizations of these results are known, see [EGSV], [MTV2]. In [MTV2], it is shown that if the Wronskian of N quasi-exponentials $p_i(x)\,e^{\lambda_i x}$, where $\lambda_1, \dots, \lambda_N$ are real numbers and $p_1, \dots, p_N$ are polynomials with complex coefficients, has real roots only, then the space spanned by these quasi-exponentials has a basis such that all polynomials have real coefficients. It is also shown that the result holds true if the Wronskian is replaced with the discrete Wronskian with real step and some additional restrictions on the roots of the

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Wronskian are imposed. A similar statement is proved in [MTV2] about spaces of quasi-polynomials of the form $x^{z_i} p_i(x, \log x)$, where z_i are real numbers and $p_i(x, y)$ are polynomials with complex coefficients, and their Wronskians.

In this paper we present yet another instance of this phenomenon. We consider the discrete Wronskian with purely imaginary step $2h$ of a space of N quasi-exponentials $e^{\lambda_i x} p_i(x)$ with real exponents λ_i. We show that if this discrete Wronskian is real and for every its roots the imaginary part is at most $|h|$, then the the space of quasi-exponentials has a basis such that all polynomials have real coefficients.

Our method is similar to that of [MTV1], [MTV2]. Namely, we consider the quantum integrable model of XXX-type associated with a tensor product of vector representations of $\mathfrak{gl}_N$. This model can be solved by the method of the algebraic Bethe ansatz. For each space of quasi-exponentials V of dimension N in a generic position, we associate an eigenvector of all transfer matrices of the model (Bethe vector), such that the eigenvalues of the transfer matrices are the coefficients of the difference operator of order N annihilating V. Then we use properties of the transfer matrices to show an appropriate symmetry of the coefficients, which implies our result.

The paper is organized as follows. In Section 1 we describe our result, see Theorem 1.1, and give some examples. Section 2 provides the required results from the representation theory of the Yangian $Y(\mathfrak{gl}_N)$. Section 3 contains the proof of the main theorem. We discuss curious reformulations of Theorem 1.1 in Section 4. The Appendix contains a proof of the symmetry of transfer matrices, see Proposition 2.2, used in this paper.

1. Formulation of the results

1.1. **The main theorem.** A function of the form $p(x)\, Q^x$, where Q is a nonzero complex number with the argument fixed, and $p(x) \in \mathbb{C}[x]$ is a polynomial, is called *a quasi-exponential function with base* Q. A quasi-exponential function $p(x)\, Q^x$ is called *real* if Q is real and $p(x)$ has real coefficients, $p(x) \in \mathbb{R}[x]$.

Fix a natural number $N \geqslant 2$. Let $\boldsymbol{Q} = (Q_1, \dots, Q_N)$ be a sequence of nonzero complex numbers with their arguments fixed. We call a complex vector space of dimension N spanned by quasi-exponential functions $p_i(x)\, Q_i^x$, with $i = 1, \dots, N$, a *space of quasi-exponentials with bases* $\boldsymbol{Q}$. The space of quasi-exponentials is called *real* if it has a basis consisting of real quasi-exponential functions.

Let h be a non-zero complex number. *The discrete Wronskian with step* $2h$, also known as Casorati determinant, of functions $f_1(x), \dots, f_N(x)$ is the $N \times N$ determinant

$$\mathrm{Wr}_h^d(f_1, \dots, f_N) \,=\, \det\Big(f_i\big(x + h(2j - N - 1)\big)\Big)_{i,j=1,\dots,N}.$$

Note that the notation here is slightly different from that of [MTV2].

The discrete Wronskians of two bases for a space of functions differ by multiplication by a nonzero number.

Let V be a space of quasi-exponentials with bases $\boldsymbol{Q}$. The discrete Wronskian of any basis for V is a quasi-exponential function of the form $w(x) \prod_{j=1}^N Q_j^x$, where $w(x) \in \mathbb{C}[x]$. The unique representative with a monic polynomial $w(x)$ is called *the discrete Wronskian of* V and is denoted by $\mathrm{Wr}_h^d(V)$.

The main result of this paper is the following theorem.

Theorem 1.1. *Let V be a space of quasi-exponentials with real bases $\boldsymbol{Q} \in (\mathbb{R}^\times)^N$, and let $\mathrm{Wr}^d_h(V) = w(x)\prod_{j=1}^N Q_j^x$, where $w(x) = \prod_{i=1}^n (x - z_i)$. Assume that $\mathrm{Re}\, h = 0$, $w(x) \in \mathbb{R}[x]$, and $|\mathrm{Im}\, z_i| \leqslant |h|$ for all $i = 1, \dots, n$. Then the space V is real.*

Theorem 1.1 is proved in Section 3.

The B. and M. Shapiro conjecture proved in [MTV1] asserts that if all roots of the differential Wronski determinant of a space of polynomials V are real then V is real. The B. and M. Shapiro conjecture follows from Theorem 1.1 by taking the limit $h \to 0$.

1.2. Examples.

Example 1. Let $N = 2$, $\boldsymbol{Q} = (1, Q)$, $p_1(x) = x + a$, $p_2(x) = x + b$. Then the discrete Wronskian has two roots. Let the discrete Wronskian be a real function. Then without loss of generality we can assume its zeros to be at $\pm A$, where A is either real or pure imaginary. We have the following equation on a, b,

$$\mathrm{Wr}^d_h(x + a, Q^x(x + b)) = Q^x(Q^h - Q^{-h})(x + A)(x - A)\,,$$

which has two solutions:

$$a \;=\; -b \;=\; \frac{(Q^h + Q^{-h})h \pm \sqrt{(Q^h - Q^{-h})^2 A^2 + 4h^2}}{(Q^h - Q^{-h})}\,.$$

If $\mathrm{Re}\, h = 0$ and $Q \in \mathbb{R}$, then $|Q^h| = 1$, $Q^h + Q^{-h}$ is real, and $Q^h - Q^{-h}$ is purely imaginary. Hence, a, b are real for real A, and for purely imaginary A such that $|A| \leqslant |h|$.

However, if A is purely imaginary and $|A| > |h|$, then for $Q = e^{\pi/2h}$ the numbers a, b are not real. In this example, Theorem 1.1 gives the largest possible set of values of A such that the numbers a and b are real for all real $Q \neq 0$.

Example 2. Let $N = 2$, $\boldsymbol{Q} = (1, 1)$, $p_1(x) = x + a$, $p_2(x) = x^3 + bx^2 + c$. Then the discrete Wronskian has three roots, which we assume to be at 0, A and B. We also assume that both A and B are real, or A and B are complex conjugate.

This case corresponds to the following equation on a, b, c,

$$\mathrm{Wr}^d_h(x + a, x^3 + bx^2 + c) = 4hx(x - A)(x - B)\,,$$

which has two solutions:

$$\begin{aligned} a \;&=\; -(A + B)/3 \pm 1/3\sqrt{-AB - 3h^2 + A^2 + B^2}, \\ b \;&=\; -(A + B) \mp \sqrt{-AB - 3h^2 + A^2 + B^2}, \\ c \;&=\; (-4/3 + 2h^2)(A + B) + h^2/3\sqrt{-AB - 3h^2 + A^2 + B^2}\,. \end{aligned}$$

Let $\mathrm{Re}\, h = 0$. Then a, b, c are real for real A, B. If A and B are complex conjugate, then a, b, c are real if and only if $3(\mathrm{Im}\, A)^2 - (\mathrm{Re}\, A)^2 \leqslant 3|h|^2$.

The equation $3(\mathrm{Im}\, A)^2 - (\mathrm{Re}\, A)^2 = 3|h|^2$ defines a hyperbola on the complex plane which is tangent to the lines $\mathrm{Im}\, A = \pm|h|$.

2. Transfer matrices and the Bethe subalgebra

In this section we recall the required results from the representation theory of the Yangian $Y(\mathfrak{gl}_N)$.

Let $W = \mathbb{C}^N$ with a chosen basis $v_1, \dots, v_N$. For an operator $M \in \mathrm{End}(W)$, we denote $M_{(i)} = 1^{\otimes(i-1)} \otimes M \otimes 1^{\otimes(n-i)}$. Similarly, for an operator $M \in \mathrm{End}(W^{\otimes 2})$, we denote by $M_{(ij)} \in \mathrm{End}(W^{\otimes n})$ the operator acting as M on the i-th and j-th factors of $W^{\otimes n}$.

Let $E_{ab} \in \mathrm{End}(W)$ be the linear operator with the matrix $(\delta_{ia}\delta_{jb})_{i,j=1,\dots,N}$. Let $R(x)$ be the *rational R-matrix*,

$$R(x) \,=\, 1 + x^{-1} \sum_{a,b=1}^{N} E_{ab} \otimes E_{ba} \,=\, 1 + x^{-1} P\,.$$

where $P \in \mathrm{End}(W^{\otimes 2})$ is the flip map: $P(v \otimes w) = w \otimes v$ for all $v, w \in W$.

The *Yangian* $Y(\mathfrak{gl}_N)$ is the unital associative algebra over $\mathbb{C}$ with generators $T_{ab}^{\{s\}}$, $a, b = 1, \dots, N$, $s \in \mathbb{Z}_{\geqslant 1}$, and relations

$$R_{(12)}(x-y)T_{(1)}(x)T_{(2)}(y) = T_{(2)}(y)T_{(1)}(x)R_{(12)}(x-y)\,, \tag{2.1}$$

where $T(x) = \sum_{a,b=1}^{N} E_{ab} \otimes T_{ab}(x)$ and $T_{ab}(x) = \delta_{ab} + \sum_{s=1}^{\infty} T_{ab}^{\{s\}} x^{-s}$. The Yangian $Y(\mathfrak{gl}_N)$ is a Hopf algebra, with the coproduct and antipode given by

$$\Delta\bigl(T_{ab}(x)\bigr) \,=\, \sum_{i=1}^{N} T_{ib}(x) \otimes T_{ai}(x)\,,$$

$$\sum_{a,b=1}^{N} E_{ab} \otimes S\bigl(T_{ab}(x)\bigr) \,=\, \bigl(T(x)\bigr)^{-1}. \tag{2.2}$$

The Yangian $Y(\mathfrak{gl}_N)$ is a flat deformation of $U\bigl(\mathfrak{gl}_N[t]\bigr)$, the universal enveloping algebra of the current Lie algebra $\mathfrak{gl}_N[t]$.

Given $z \in \mathbb{C}$, define the $Y(\mathfrak{gl}_N)$-module structure on the space W by letting $T_{ab}(x)$ act as $\delta_{ab} + (x-z)^{-1} E_{ba}$. We denote this module $W(z)$ and call it the *evaluation module.*

Let $\boldsymbol{Q} = (Q_1, \dots, Q_N) \in (\mathbb{C}^\times)^N$, and $Q = \mathrm{diag}(Q_1, \dots, Q_N)$ be the diagonal matrix with diagonal entries Q_a. Let $\partial = \partial/\partial x$. Set $X_{ab} = \delta_{ab} - Q_a T_{ab}(x)\, e^{-\partial}$. Define the universal difference operator by

$$\mathcal{D}_{\boldsymbol{Q}} \,=\, \sum_{\sigma \in S_N} (-1)^\sigma\, X_{1\sigma(1)}\, X_{2\sigma(2)} \dots X_{N\sigma(N)}\,.$$

Write

$$\mathcal{D}_{\boldsymbol{Q}} \,=\, 1 - B_{1,\boldsymbol{Q}}(x) e^{-\partial} + B_{2,\boldsymbol{Q}}(x) e^{-2\partial} - \dots + (-1)^N B_{N,\boldsymbol{Q}}(x) e^{-N\partial}\,,$$

where $B_{i,\boldsymbol{Q}}(x)$ are series in x^{-1} with coefficients in $Y(\mathfrak{gl}_N)$. The series $B_{i,\boldsymbol{Q}}(x)$ coincides with the higher transfer matrices, introduced in [KS], see [CT], [MTV3], and formula (A.2). Set $B_{0,\boldsymbol{Q}}(x) = 1$.

The unital subalgebra of $Y(\mathfrak{gl}_N)$ generated by the coefficients of the series $B_{i,\boldsymbol{Q}}(x)$, $i = 1, \dots, N$, is called the *Bethe algebra* and denoted by $\mathcal{B}_{\boldsymbol{Q}}$. The Bethe algebra is commutative [KS].

Given $\boldsymbol{z} = (z_1, \dots, z_n) \in \mathbb{C}^n$, denote by $\boldsymbol{W}(\boldsymbol{z}) = W(z_1) \otimes \dots \otimes W(z_n)$ the tensor product of evaluation modules. For $i = 0, \dots, N$, let $B_{i,\boldsymbol{Q}}(x;\boldsymbol{z})$ be the image of the series $B_{i,\boldsymbol{Q}}(x)$ in $\operatorname{End}\bigl(\boldsymbol{W}(\boldsymbol{z})\bigr)[[x^{-1}]]$. The series $B_{i,\boldsymbol{Q}}(x;\boldsymbol{z})$ sums up to a rational function in x. We have

$$B_{N,\boldsymbol{Q}}(x;\boldsymbol{z}) \,=\, b_{\boldsymbol{Q}}(x;\boldsymbol{z}) \cdot \mathrm{Id}\,, \qquad b_{\boldsymbol{Q}}(x;\boldsymbol{z}) \,=\, Q_1 \dots Q_N \prod_{i=1}^{n} \frac{x - z_i + 1}{x - z_i}\,.$$

Let $v \in \boldsymbol{W}(\boldsymbol{z})$ be an eigenvector of the Bethe algebra $\mathcal{B}_{\boldsymbol{Q}}$,

$$B_{i,\boldsymbol{Q}}(x;\boldsymbol{z})\, v \,=\, B_{i,\boldsymbol{Q},v}(x;\boldsymbol{z})\, v\,, \qquad i = 1, \dots, N\,,$$

where $B_{i,\boldsymbol{Q},v}(x;\boldsymbol{z})$ are rational functions in x with complex coefficients. Let $\mathcal{D}_{\boldsymbol{Q},v}(x;\boldsymbol{z})$ be the scalar difference operator

$$\mathcal{D}_{\boldsymbol{Q},v}(x;\boldsymbol{z}) \,=\, 1 - B_{1,\boldsymbol{Q},v}(x;\boldsymbol{z})e^{-\partial} + B_{2,\boldsymbol{Q},v}(x;\boldsymbol{z})e^{-2\partial} - \dots + (-1)^N B_{N,\boldsymbol{Q},v}(x;\boldsymbol{z})e^{-N\partial}\,.$$

Let $\mathcal{D}$ be a scalar difference operator. We call the space

$$\{\, f(x) \mid \mathcal{D} f(x) = 0,\ \ f(x) \text{ is a linear combination of quasi-exponential functions}\}$$

the *quasi-exponential kernel of the operator* $\mathcal{D}$.

Let $\mathcal{U}$ be the complex span of 1-periodic quasi-exponentials $e^{2\pi\sqrt{-1}\,kx}$, $k \in \mathbb{Z}$.

Proposition 2.1 ([MTV4])**.** *Let* $\boldsymbol{z}$ *and* $\boldsymbol{Q}$ *be generic. For every* N*-dimensional complex space* V *of quasi-exponentials with bases* $\boldsymbol{Q}$ *such that*

$$\mathrm{Wr}^{d}_{1/2}(V) \,=\, \prod_{i=1}^{n} \bigl(x - z_i + (N+1)/2\bigr) \prod_{j=1}^{N} Q_j^x\,,$$

there exists an eigenvector $v \in \boldsymbol{W}(z)$ *of the Bethe algebra* $\mathcal{B}_{\boldsymbol{Q}}$ *such that the quasi-exponential kernel of the operator* $\mathcal{D}_{\boldsymbol{Q},v}(x;\boldsymbol{z})$ *has the form* $V \otimes \mathcal{U}$. □

Recall that the module $\boldsymbol{W}(\boldsymbol{z})$ as a vector space is $W^{\otimes n}$. Let $\langle\cdot\,,\cdot\rangle$ be the standard sesquilinear form on $W^{\otimes n}$:

$$\langle v_{a_1} \otimes \dots \otimes v_{a_n}, v_{b_1} \otimes \dots \otimes v_{b_n} \rangle = \prod_{i=1}^{n} \delta_{a_i b_i}\,.$$

We assume that the form is linear with respect to the first argument and semilinear with respect to the second one. The form is clearly a positive definite form.

The following proposition is the key technical fact about the Bethe algebra we need for the proof of Theorem 1.1.

Proposition 2.2. *We have*

(1) *For every* $i = 1, \dots, n-1$, *and* $j = 1, \dots, N$,

$$\check{R}_{(i,i+1)}(z_i - z_{i+1})\, B_{j,\boldsymbol{Q}}(x;\boldsymbol{z}) \,=\, B_{j,\boldsymbol{Q}}(x;\sigma_{i,i+1}\boldsymbol{z})\, \check{R}_{(i,i+1)}(z_i - z_{i+1})\,, \tag{2.3}$$

where $\check{R}_{(i,i+1)}(x) = xP_{(i,i+1)} + 1$ *and* $\sigma_{i,i+1}\boldsymbol{z} = (z_1, \dots, z_{i-1}, z_{i+1}, z_i, z_{i+2}, \dots, z_n)\,.$

(2) *For every* $j=0,\dots,N,$ *and* $v,w\in \boldsymbol{W}(\boldsymbol{z})$,

$$\big\langle B_{j,\boldsymbol{Q}}(x;\boldsymbol{z})\,v,w\big\rangle \,=\, b_{\boldsymbol{Q}}(x;\boldsymbol{z})\,\big\langle v,B_{N-j,\bar{\boldsymbol{Q}}^{-1}}(-\bar{x}-1;-\bar{\boldsymbol{z}})\,w\big\rangle\,, \tag{2.4}$$

where $-\bar{\boldsymbol{z}}=(-\bar{z}_1,\dots,-\bar{z}_n)$, $\bar{\boldsymbol{Q}}^{-1}=(\bar{Q}_1^{-1},\dots,\bar{Q}_N^{-1})$, *and the bar denotes the complex conjugation*

Proposition 2.2 is proved in Appendix.

3. Proof of Theorem 1.1

Recall that $Q_1,\dots,Q_N$ are nonzero real numbers, $h\neq 0$ is a purely imaginary number and V is a space of quasi-exponentials V with bases $\boldsymbol{Q}$. We also assume that

$$\mathrm{Wr}^d_h(V)\prod_{j=1}^N Q_j^{-x} \,=\, \prod_{i=1}^n (x-z_i)$$

is a polynomial with real coefficients, and $|\operatorname{Im} z_i|\leqslant |h|$ for all $i=1,\dots,n$.

Lemma 3.1. *Assume that the conclusion of Theorem 1.1 holds for generic* $\boldsymbol{Q},\boldsymbol{z}=(z_1,\dots,z_n)$ *satisfying the assumption of Theorem 1.1. Then the conclusion of Theorem 1.1 holds for arbitrary* $\boldsymbol{Q},\boldsymbol{z}$ *satisfying the assumption of Theorem 1.1.*

Proof. The proof follows the reasoning in Sections 3.1, 3.2 of [MTV2]. □

To use the results from Section 2, we make a change of variables. Set

$$\widetilde{Q}_i=Q_i^{2h}\,,\qquad \tilde{z}_i=\frac{z_i}{2h}\,,\qquad i=1,\dots,N\,,$$

and let $\widetilde{V}$ be the space of quasi-exponentials with bases $\widetilde{\boldsymbol{Q}}\in\mathbb{C}^N$, given by

$$\widetilde{V} \,=\, \big\{\,p\big(2hx+h\,(N+1)\big)\,\widetilde{Q}^x \;\,|\;\, p(x)\,Q^x\in V\,\big\}\,. \tag{3.1}$$

Then

$$\mathrm{Wr}^d_{1/2}(\widetilde{V}) \,=\, \prod_{i=1}^n \big(x-\tilde{z}_i+(N+1)/2\big)\prod_{j=1}^N \widetilde{Q}_j^x\,.$$

Since the polynomial $\prod_{i=1}^n(x-z_i)$ has real coefficients, there is k, where $0\leqslant 2k\leqslant n$, and an enumeration of $z_1,\dots,z_n$ such that that $\bar{z}_{2i-1}=z_{2i}$, for $i=1,\dots,k$, and $z_{2k+1},\dots,z_n$ are real. Define the form $\langle\cdot\,,\cdot\rangle_k$ on $W^{\otimes n}$ by the formula

$$\langle v,w\rangle_k \,=\, \Big\langle v\,,\prod_{i=1}^k \check{R}_{(2i-1,2i)}(\tilde{z}_{2i-1}-\tilde{z}_{2i})\,w\Big\rangle\,.$$

Lemma 3.2. *Assume* $|\operatorname{Im} z_i|<|h|$ *for all* $i=1,\dots,2k$. *Then the form* $\langle\cdot\,,\cdot\rangle_k$ *is positive definite.*

Proof. The linear operators $\check{R}_{(2i-1,2i)}(\tilde{z}_{2i-1}-\tilde{z}_{2i})$ with $i=1,\dots,k$ are pairwise commuting. Moreover, under our assumptions each $\check{R}_{(2i-1,2i)}(\tilde{z}_{2i-1}-\tilde{z}_{2i})$ is a positive definite selfadjoint operator with respect to the standard form $\langle\cdot\,,\cdot\rangle$. The lemma follows. □

By Lemma 3.1 we can assume that $\boldsymbol{Q}$ and $\boldsymbol{z}$ are generic. Then by Proposition 2.1, there exists an eigenvector $v \in \boldsymbol{W}(\tilde{\boldsymbol{z}})$ of the Bethe algebra $\mathcal{B}_{\widetilde{\boldsymbol{Q}}}$ such that $\widetilde{V}$ is in the kernel of the scalar difference operator $\mathcal{D}_{\widetilde{\boldsymbol{Q}},v}(x;\tilde{\boldsymbol{z}})$.

By Proposition 2.2, we have

$$(B_{j,\widetilde{\boldsymbol{Q}}}(x;\tilde{\boldsymbol{z}}))^* = \overline{b_{\widetilde{\boldsymbol{Q}}}(x;\tilde{\boldsymbol{z}})} B_{N-j,\widetilde{\boldsymbol{Q}}}(-\bar{x}-1;\tilde{\boldsymbol{z}}),$$

where the asterisk denotes the Hermitian conjugation with respect to the form $\langle \cdot\,,\cdot\rangle_k$. Since the form $\langle \cdot\,,\cdot\rangle_k$ is positive definite, it yields

$$\overline{B_{j,\widetilde{\boldsymbol{Q}},v}(x;\tilde{\boldsymbol{z}})} = \overline{b_{\widetilde{\boldsymbol{Q}}}(x;\tilde{\boldsymbol{z}})}\, B_{N-j,\widetilde{\boldsymbol{Q}},v}(-\bar{x}-1;\tilde{\boldsymbol{z}}). \tag{3.2}$$

Let $p(x)\,Q_i^x \in V$, where $p(x)$ is monic. Set $\tilde{p}(x) = p\bigl(2hx + h\,(N+1)\bigr)$, see (3.1). Then $\mathcal{D}_{\widetilde{\boldsymbol{Q}},v}(x;\tilde{\boldsymbol{z}})\bigl(\tilde{p}(x)\widetilde{Q}_i^x\bigr) = 0$, and by formula (3.2),

$$\mathcal{D}_{\widetilde{\boldsymbol{Q}},v}(x;\tilde{\boldsymbol{z}})\bigl(\,\overline{\tilde{p}(-\bar{x}-N-1)}\,\widetilde{Q}_i^x\bigr) = 0\,.$$

For generic $\widetilde{\boldsymbol{Q}}$, the equation $\mathcal{D}_{\widetilde{\boldsymbol{Q}},v}(x;\tilde{\boldsymbol{z}})\bigl(q(x)\,\widetilde{Q}_i^x\bigr) = 0$ determines a polynomial $q(x)$ uniquely up to multiplication by a constant. Hence $\overline{\tilde{p}(-\bar{x}-N-1)} = \tilde{p}(x)$, because the top coefficients of both polynomials are the same. Taking $y = 2hx + h\,(N+1)$ gives $\overline{p(\bar{y})} = p(y)$. Theorem 1.1 is proved.

4. Matrix reformulation

In this section we give a matrix formulation of Theorem 1.1.

Let $\mathcal{Z}$ be an $N\times N$ matrix with entries $\mathcal{Z}_{ii} = a_i$ for $i = 1,\dots,N$, and

$$\mathcal{Z}_{ij} = \frac{1}{\sin(\lambda_i - \lambda_j)}\,, \qquad 1 \leqslant i < j \leqslant N\,.$$

Here $a_1,\dots,a_N$ and $\lambda_1,\dots,\lambda_N$ are complex numbers. For $i = 1,\dots,n$, set

$$p_i(x) = x - a_i - \sum_{j\neq i}\cot(\lambda_i - \lambda_j)\,.$$

Let V be the space of quasi-exponentials spanned by $p_i(x)\,e^{\lambda_i x}$, $i = 1,\dots,N$.

Lemma 4.1. *We have* $\det(x-\mathcal{Z}) = \mathrm{Wr}^d_{\sqrt{-1}}(V)\,e^{-\sum_{i=1}^N \lambda_i x}$.

Proof. The statement can be obtained by the same calculation as in Section 6.1 of [MTV2]. □

Theorem 4.2. *Let $\lambda_1,\dots,\lambda_N$ be real numbers such that $\lambda_i - \lambda_j \notin \pi\mathbb{Z}$ for all $i,j = 1,\dots,$ N. Assume that the characteristic polynomial $\det(x-\mathcal{Z}) = \prod_{i=1}^N (x-z_i)$ has real coefficients and $|\operatorname{Im} z_i| \leqslant 1$ for all $i = 1,\dots,N$. Then the numbers $a_1,\dots,a_N$ are real.*

Proof. The statement follows from Theorem 1.1 and Lemma 4.1. □

If $a_i = b_i/\varepsilon$, $\lambda_i = \varepsilon\mu_i$, $z_i = \varepsilon x_i$ for all $i = 1,\dots,N$, and $\varepsilon \to 0$, then Theorem 4.2 turns into Theorem 6.4 of [MTV2].

Appendix

Given $\boldsymbol{i} = (i_1 < \dots < i_k)$ and $\boldsymbol{j} = (j_1 < \dots < j_k)$, set

$$T^{\wedge k}_{\boldsymbol{ij}}(x) \,=\, \sum_{\sigma\in S_k} (-1)^\sigma\, T_{i_1 j_{\sigma(1)}}(x)\, T_{i_2 j_{\sigma(2)}}(x-1)\dots T_{i_k j_{\sigma(k)}}(x-k+1)\,. \tag{A.1}$$

Denote $\operatorname{qdet}(x) = T^{\wedge N}_{\boldsymbol{ii}}(x)$, where $\boldsymbol{i} = (1,\dots,N)$. We have

$$B_{k,\boldsymbol{Q}}(x) \,=\, \sum_{\boldsymbol{i}=(i_1<\dots<i_k)} Q_{i_1}\dots Q_{i_k}\, T^{\wedge k}_{\boldsymbol{ii}}(x)\,. \tag{A.2}$$

For every $\boldsymbol{i} = (i_1 < \dots < i_k)$, set $\boldsymbol{i}' = (i'_1 < \dots < i'_{N-k})$, where

$$\{i_1,\dots,i_k,i'_1,\dots,i'_{N-k}\} = \{1,2,\dots,N\}\,.$$

The following result is known, see (1.11) in [NT].

Theorem A.1. [NT] *We have*

$$S\bigl(T^{\wedge k}_{\boldsymbol{ij}}(x)\bigr)\ \operatorname{qdet}(x) \,=\, T^{\wedge(N-k)}_{\boldsymbol{j}'\boldsymbol{i}'}(x-k)\,, \tag{A.3}$$

where S is the antipode.

For the $Y(\mathfrak{gl}_N)$-module $W(\boldsymbol{z})$, let $\rho_{\boldsymbol{z}} : Y(\mathfrak{gl}_N) \to \operatorname{End}(W^{\otimes N})$ be the corresponding algebra homomorphism. Denote

$$T(x;\boldsymbol{z}) \,=\, \sum_{a,b=1}^N E_{ab}\otimes \rho_{\boldsymbol{z}}\bigl(T_{ab}(x)\bigr)\,, \qquad \widetilde{T}(x,\boldsymbol{z}) \,=\, \sum_{a,b=1}^N E_{ab}\otimes \rho_{\boldsymbol{z}}\bigl(S\bigl(T_{ab}(x)\bigr)\bigr)\,.$$

We have

$$T(x;\boldsymbol{z}) \,=\, R_{(0N)}(x-z_N)\dots R_{(02)}(x-z_2)R_{(01)}(x-z_1)\,, \tag{A.4}$$

and

$$\widetilde{T}(x,\boldsymbol{z}) \,=\, (R_{(01)}(x-z_1))^{-1}(R_{(02)}(x-z_2))^{-1}\dots(R_{(0N)}(x-z_N))^{-1}\,,$$

see (2.2). Here we enumerate the factors of $W^{\otimes(N+1)}$ by $0,1,\dots,N$. Since $\bigl(R(x)\bigr)^t = R(x)$ and $R(x)\,R(-x) = 1-x^{-2}$, we obtain

$$\bigl(\widetilde{T}(x;\boldsymbol{z})\bigr)^t \,=\, T(-x;-\boldsymbol{z})\,\frac{b(x-1;\boldsymbol{z})}{b(x;\boldsymbol{z})}\,, \qquad b(x;\boldsymbol{z}) \,=\, \prod_{i=1}^N \frac{x-z_i+1}{x-z_i}\,. \tag{A.5}$$

Let $T^{\wedge k}_{\boldsymbol{ij}}(x;\boldsymbol{z}) = \rho_{\boldsymbol{z}}\bigl(T^{\wedge k}_{\boldsymbol{ij}}(x)\bigr)$. Then by (A.1) and (A.5),

$$S\bigl(T^{\wedge k}_{\boldsymbol{ij}}(x)\bigr) \,=\, T^{\wedge k}_{\boldsymbol{ji}}(k-1-x;-\boldsymbol{z})\,\frac{b(x-k;\boldsymbol{z})}{b(x;\boldsymbol{z})}\,.$$

In addition, it is known that $\rho_{\boldsymbol{z}}\bigl(\operatorname{qdet}(x)\bigr) \,=\, b(x;\boldsymbol{z})$. Hence taking into account (A.3) and (A.2), after simple transformations we get

$$T^{\wedge k}_{\boldsymbol{ij}}(x) \,=\, b(x;\boldsymbol{z})\, T^{\wedge(N-k)}_{\boldsymbol{i}'\boldsymbol{j}'}(-x-1;-\boldsymbol{z})$$

and

$$B_{k,\boldsymbol{Q}}(x) \,=\, b(x;\boldsymbol{z})\, B_{N-k,\boldsymbol{Q}^{-1}}(-x-1;-\boldsymbol{z})\,.$$

It is also known that

$$\left\langle B_{j,\boldsymbol{Q}}(x;\boldsymbol{z})\,v,w\right\rangle \,=\, \left\langle v,B_{j,\overline{\boldsymbol{Q}}}(\bar{x};\bar{\boldsymbol{z}})\,w\right\rangle,$$

for every $j=1,\dots,N$, and $v,w\in \boldsymbol{W}(\boldsymbol{z})$, see for example, Proposition 4.11 in [MTV3]. This proves formula (2.4) in Proposition 2.2.

Formula (2.3) follows from (A.2), (A.4), and the Yang-Baxter equation

$$\check{R}_{(i,i+1)}(z_i-z_{i+1})\,R_{(0,i+1)}(x-z_{i+1})\,R_{(0i)}(x-z_i) \,=\, \\ =\, R_{(0,i+1)}(x-z_i)\,R_{(0i)}(x-z_{i+1})\,\check{R}_{(i,i+1)}(z_i-z_{i+1})\,.$$

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