REMARKS ON THE STRUCTURE CONSTANTS OF THE VERLINDE ALGEBRA ASSOCIATED TO sl_3

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The Verlinde fusion algebra is an associative commutative algebra associated to a Wess-Zumino-Witten model of conformal field theory [V,F,GW,K,S]. Such a model is labelled by a simple Lie algebra $\mathfrak{g}$ and a natural number k called level. The Verlinde algebra $A(\mathfrak{g},k)$ is a finitely generated algebra with generators V_λ enumerated by irreducible $\mathfrak{g}$-modules admissible for the model. The structure constants $N^\nu_{\lambda,\mu}$ of the multiplication $V_\lambda \cdot V_\mu = \sum_\nu N^\nu_{\lambda,\mu} V_\nu$ are non-negative integers important for applications. (We use the formula in [K, Sec.13.35] as a definition of the structure constants.)

Example 1. The algebra $A(sl_2,k)$ has $k+1$ generators $V_0,...,V_k$. For fixed λ,ν and varying μ, the structure constants $N^{\mu+\nu}_{\lambda,\mu}$ are either zero or form the characteristic function of an interval with respect to μ. Namely, $N^{\mu+\nu}_{\lambda,\mu}=0$, if $\lambda-\nu$ is odd or if $|\nu|>\lambda$. If $\lambda-\nu$ is even and $|\nu|<\lambda$, then $N^{\mu+\nu}_{\lambda,\mu}=1$ for $\mu\in[(\lambda-\nu)/2,k-(\lambda+\nu)/2]$ and $N^{\mu+\nu}_{\lambda,\mu}=0$ otherwise.

It is interesting that after an affine change of the variable the function $N^{\mu+\nu}_{\lambda,\mu}$ of μ becomes the weight function of the irreducible sl_2-module with highest weight $k-\lambda$.

In this paper we give a similar formula for the structure constants of the Verlinde algebra associated to sl_3.

1. Weight Functions.

Let $\mathcal{P}=\mathbb{Z}^3/\mathbb{Z}\cdot(1,1,1)$ be the two dimensional weight lattice of sl_3. Let $L_1=(1,0,0), L_2=(0,1,0), L_3=(0,0,1), \alpha_1=(1,-1,0), \alpha_2=(0,1,-1), \alpha_3=(-1,0,1)$, considered as elements of $\mathcal{P}$.

For a natural number k introduce coordinates on $\mathcal{P}$:

$$y_1(\mu):=(\alpha_1,\mu), \qquad y_2(\mu):=(\alpha_2,\mu), \qquad y_3(\mu):=k+(\alpha_3,\mu)=k-y_1-y_2,$$

where $(x,y)=x_1y_1+x_2y_2+x_3y_3$.

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Definition 1. *A triangle* in $\mathcal{P}$ is a set Δ of the form

$$\Delta = \{\mu \in \mathcal{P} \,|\, y_i(\mu) \geqslant A_i, i = 1, 2, 3\}$$

for some integers A_i.

The number $k - A_1 - A_2 - A_3$ is called *the size* of the triangle. It is the integral length of its edges.

Definition 2. A pair consisting of a natural number m and a triangle Δ of size ℓ is called *appropriate* if $\ell \geqslant 2m - 2$.

Definition 3. *The weight function* $w_{m,\Delta}$ associated to an appropriate pair m, Δ is the following function

$$w_{m,\Delta} : \mathcal{P} \to \mathbb{Z}_{\geqslant 0},$$

which is zero outside Δ, and its level sets inside Δ are shown in the picture. The level sets of $w_{m,\Delta}$ are defined inductively. The function $w_{m,\Delta}$ is equal to zero at the corner triangles of Δ of size $m-2$. $w_{m,\Delta}$ is equal to 1 at the boundary integral points of the remaining part of Δ. Denote by n the number $min\{m-1, \ell-2m+2\}$. Assume that the points of Δ where $w_{m,\Delta} < j$ for $j < n$ are already defined, define the set $w_{m,\Delta} = j$ as the set of boundary integral points of the convex hull of the remaining integral points of Δ. If the set $w_{m,\Delta} = n$ is already defined, put $w_{m,\Delta} = n + 1$ at the remaining part of Δ.

2. Main Result.

Fix a natural number k. A weight $\lambda \in \mathcal{P}$ is called *admissible* at level k if $y_i(\lambda) \geq 0, i = 1, 2, 3$. Denote by V_λ the irreducible sl_3-module with highest weight λ. For $\lambda, \nu \in \mathcal{P}$, denote by $d_\lambda(\nu)$ the dimension of the weight subspace of V_λ of weight ν.

The generators of the Verlinde algebra $A(sl_3, k)$ are labelled by irreducible sl_3-modules V_μ with admissible highest weights.

Let $N^\nu_{\lambda,\mu}$ be the structure constants of $A(sl_3, k)$. Here $\lambda, \mu, \nu \in \mathcal{P}$, and we set $N^\nu_{\lambda,\mu} = 0$, if at least one of the indices is not admissible.

For fixed λ, ν consider $N^{\mu+\nu}_{\lambda,\mu}$ as a function of $\mu \in \mathcal{P}$.

Theorem 1.
1. If $d_\lambda(\nu) = 0$, then $N^{\mu+\nu}_{\lambda,\mu} = 0$.
2. If $d_\lambda(\nu) > 0$, then there is an appropriate pair m, Δ such that $m = d_\lambda(\nu)$ and

$$N^{\mu+\nu}_{\lambda,\mu} = w_{m,\Delta}(\mu)$$

for all μ.

Below we describe the triangle Δ.

Assume that $d_\lambda(\nu) > 0$. Set

$$z_{i,\lambda}(\nu) = min\{|m| - 1 \,|\, m \in \mathbb{Z}, d_\lambda(\nu + m\alpha_i) < d_\lambda(\nu)\}$$

for $i = 1, 2, 3$.

Definition 4. A point $\nu \in \mathcal{P}$ is of *type I* (resp. *II*) if the product $(\alpha_1, \nu) \cdot (\alpha_2, \nu) \cdot (\alpha_3, \nu)$ is non-positive (resp. non-negative).

For ν of type I, let i and j be such that $(\alpha_i, \nu) \geq 0, (\alpha_j, \nu) \geq 0$. For ν of type II, let i and j be such that $(\alpha_i, \nu) \leq 0, (\alpha_j, \nu) \leq 0$. In both cases let ℓ be the remaining index in $\{1, 2, 3\}$.

Theorem 2. *Assume that $d_\lambda(\nu) > 0$. Then the triangle Δ in Theorem 1 has the following form. If ν is of type I, then*

$$\Delta = \{\mu \in \mathcal{P} \,|\, y_i(\mu) \geq z_{i,\lambda}(\nu),\, y_j(\mu) \geq z_{j,\lambda}(\nu),\, y_\ell(\mu) \geq z_{\ell,\lambda}(\nu) - (\alpha_\ell, \nu)\}.$$

If ν is of type II, then

$$\Delta = \{\mu \in \mathcal{P} \,|\, y_i(\mu) \geq z_{i,\lambda}(\nu) - (\alpha_i, \nu),\, y_j(\mu) \geq z_{j,\lambda}(\nu) - (\alpha_j, \nu),\, y_\ell(\mu) \geq z_{\ell,\lambda}(\nu)\}.$$

Example 2. Let $\nu = 0$. Then $d_\lambda(0) = 0$ unless $\lambda = (a+3b)L_1 - aL_3$ or $\lambda = aL_1 - (a+3b)L_3$ for some non-negative integers a and b. If λ has this form, then $d_\lambda(0) = a+1$ and the triangle of Theorem 1 is

$$\Delta = \{\mu \in \mathcal{P} \,|\, y_i(\mu) \geq b, i = 1, 2, 3\}.$$

Remark. For an irreducible sl_3-module V_λ, consider its weight function $d_\lambda : \mathcal{P} \to \mathbb{Z}_{\geq 0}$. It is easy to see that, after an affine change of variables, the function d_λ becomes the weight function of an appropriate pair m, Δ, cf. [FH, Sec. 13]. Namely, the affine change of variables $r_\lambda : \mathcal{P} \to \mathcal{P}, \ell_1 L_1 + \ell_2 L_2 \mapsto (\ell_1 + \ell_2)\alpha_1 + \ell_2\alpha_2 + \lambda$, transforms d_λ into the weight function $w_{m,\Delta}$, where $m = (\alpha_1, \lambda) + 1$ and Δ is a triangle of size $(\alpha_1 + 2\alpha_2, \lambda)$.

Conversely, any weight function $w_{m,\Delta}$ after a suitable affine change of variables becomes the weight function of an irreducible sl_3-module.

Remark. It would be interesting to find an analog of these theorems for the sl_4-Verlinde algebra.

3. Application.

Consider the Wess-Zumino-Witten model associated to sl_3 at level k. Consider the space of conformal blocks associated to a torus with one marked point labelled by an admissible sl_3-module V_λ. Denote by $D(\lambda, k)$ the dimension of this space. From the fusion rules [TUY], it follows that $D(\lambda, k) = \sum_\mu N^\mu_{\lambda,\mu}$.

Corollary. *$D(\lambda, k) = 0$ unless $\lambda = (a+3b)L_1 - aL_3$ or $\lambda = aL_1 - (a+3b)L_3$ for some non-negative integers a and b. If λ has this form, then*

$$D(\lambda, k) = \sum_\mu w_{m,\Delta}(\mu)$$

where $m = a+1$ and Δ is described in Example 2. Moreover, $D(\lambda, k)$ is equal to the dimension of the irreducible sl_3-module with highest weight $(k - 3b - 2a)L_1 - aL_3$.

In particular, if $k = 2a + 3b$, the smallest level admissible for V_λ, then $D(\lambda, 2a + 3b) = \dim V_{-aL_3} = (a+1)(a+2)/2$, see in [FH, 15.17] a formula for the dimension.

Remark. Computation of $D(\lambda, k)$ was the starting point of this work.

In the next sections we sketch a proof of Theorems 1 and 2.

4. Formula for the Structure Constants.

Let $W^\wedge$ be the group of affine transformations of the plane $\mathcal{P}$ generated by reflections s_1, s_2, s_3, where s_i is the reflection at the line $y_i = 0$ for $i = 1, 2$, and s_3 is the reflection at the line $y_3 = -3$.

Define another action of $W^\wedge$ on $\mathcal{P}$ by $w{*}\lambda = w(\lambda-\alpha_3)+\alpha_3$. Let $\epsilon : W^\wedge \to \{1,-1\}$ be the homomorphism taking reflections to -1.

The structure constants of the Verlinde algebra $A(sl_3,k)$ are given by the formula

$$N_{\lambda,\mu}^{\mu+\nu} = \sum_{w\in W^\wedge} \epsilon(w)\cdot d_\lambda(\nu+\mu-w*\mu), \tag{1}$$

if $\lambda,\mu,\mu+\nu$ are admissible at level k. This formula is an easy combination of the definition of the structure constants in [K,13.35] and formula 12.31 in [FH].

5. Proof of the Theorems.

Formula (1) holds if $\lambda,\mu,\mu+\nu$ are admissible at level k. For fixed λ and ν, this means that μ belongs to the triangle

$$\Delta_I = \{\mu\in\mathcal{P}\,|\,y_i(\mu)\geq 0, y_j(\mu)\geq 0, y_\ell(\mu)\geq -(\alpha_\ell,\nu)\},$$

if ν is of type I, and to the triangle

$$\Delta_{II} = \{\mu\in\mathcal{P}\,|\,y_i(\mu)\geq -(\alpha_i,\nu), y_j(\mu)\geq -(\alpha_j,\nu), y_\ell(\mu)\geq 0\},$$

if ν is of type II.

We consider all terms of formula (1) as functions of $\mu\in\Delta_I$, resp. of $\mu\in\Delta_{II}$.

Consider the following 13 elements of $W^\wedge$:

$$S = \{\, id,\ s_a,\ s_bs_c, s_cs_b, s_bs_cs_b\,|\,a=1,2,3,(b,c)=(1,2),(1,3),(2,3)\,\}.$$

Rewrite (1) as

$$N_{\lambda,\mu}^{\mu+\nu} = \sum_{w\in S}\epsilon(w)\cdot d_\lambda(\nu+\mu-w*\mu) + \sum_{w\in W^\wedge - S}\epsilon(w)\cdot d_\lambda(\nu+\mu-w*\mu), \tag{2}$$

Lemma 1. *If $\lambda,\mu,\mu+\nu$ are admissible, then all terms of the second sum in (2) are equal to zero.*

The lemma is an easy corollary of admissibility.

To prove Theorems 1 and 2 we compute explicitly 13 functions of μ of the first sum in (2).

The function corresponding to $w=id$ is the constant function $d_\lambda(\nu)$.

From now on we assume that ν is of type I and describe the remaining 12 functions. Type II is considered similarly.

Lemma 2. *The function $d_\lambda(\nu+\mu-s_\ell*\mu)$ as a function of $\mu\in\Delta_I$*
is equal to $d_\lambda(\nu)$, if $y_\ell(\mu)<z_{\ell,\lambda}-(\alpha_\ell,\nu)$,
is equal to 0, if $y_\ell(\mu)\geq z_{\ell,\lambda}+d_\lambda(\nu)-(\alpha_\ell,\nu)-1$,
is equal to $z_{\ell,\lambda}+d_\lambda(\nu)-(\alpha_\ell,\nu)-y_\ell(\mu)-1$ otherwise.
For $a=i,j$, the function $d_\lambda(\nu+\mu-s_a*\mu)$ as a function of $\mu\in\Delta_I$
is equal to $d_\lambda(\nu)$, if $y_a<z_{a,\lambda}$,
is equal to 0, if $y_a\geq z_{a,\lambda}+d_\lambda(\nu)-1$,
is equal to $z_{a,\lambda}+d_\lambda(\nu)-y_a(\mu)-1$ otherwise.

Lemma 3. *The function $d_\lambda(\nu+\mu-(s_is_js_i)*\mu)$ as a function of $\mu\in\Delta_I$ is equal to $d_\lambda(\nu-(y_i(\mu)+y_j(\mu)+2)\alpha_\ell)$.*

The function $d_\lambda(\nu+\mu-(s_is_j)*\mu)$ as a function of $\mu\in\Delta_I$ is equal to $d_\lambda(\nu-(y_i(\mu)+1)\alpha_\ell+(y_j(\mu)+1)\alpha_j)$.

Similar descriptions hold for the remaining functions of the set S. Combining these descriptions we get Theorems 1 and 2.

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