Combining Equational Tree Automata Over AC and ACI Theories*

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Abstract. In this paper, we study combining equational tree automata in two different senses: (1) whether decidability results about equational tree automata over disjoint theories $\mathcal{E}_1$ and $\mathcal{E}_2$ imply similar decidability results in the *combined theory* $\mathcal{E}_1 \cup \mathcal{E}_2$; (2) checking emptiness of a language obtained from the *Boolean combination* of regular equational tree languages. We present a negative result for the first problem. Specifically, we show that the intersection-emptiness problem for tree automata over a theory containing at least one AC symbol, one ACI symbol, and 4 constants is undecidable despite being decidable if either the AC or ACI symbol is removed. Our result shows that decidability of intersection-emptiness is a *non-modular* property even for the union of disjoint theories. Our second contribution is to show a decidability result which implies the decidability of two open problems: (1) If idempotence is treated as a rule $f(x,x) \to x$ rather than an equation $f(x,x) = x$, is it decidable whether an AC tree automata accepts an idempotent normal form? (2) If $\mathcal{E}$ contains a single ACI symbol and arbitrary free symbols, is emptiness decidable for a Boolean combination of regular $\mathcal{E}$-tree languages?

1 Introduction

Tree automata are a theoretical tool with applications in many areas, including sufficient completeness of algebraic specifications [2, 8], protocol verification [4, 5], type inference [3], and theorem proving [13]. Many different frameworks have been proposed for addressing these applications as each framework must balance the often competing goals of expressive power and tractability of different operations. In our own applications [7, 8], the most important properties are a decidable emptiness problem, and closure under Boolean operations and equational congruences. Regular tree automata satisfy two of these properties, however they are not closed under arbitrary equational congruences. For example, the set of terms equivalent modulo associativity to a term in a regular tree language may not be a regular tree language [16].

Many extensions to tree automata have been proposed to remedy this problem, including multitree automata [14], equational tree automata [16], and two-way alternating equational tree automata [25]. These extensions allow one to

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recognize terms equivalent modulo an equational theory, however multitree automata are only defined for AC theories and the other frameworks lack closure under Boolean operations. Due to this problem, *propositional tree automata* were proposed in [9]. They are closed under both an equational theory and Boolean operations — but have an undecidable emptiness problem.

A separate issue in equational tree automata is that few properties are decidable for arbitrary theories. Consequently, most work on equational tree automata focuses on particular equational theories where one or more symbols satisfies combinations of specific equations such as associativity (A), commutativity (C), and idempotence (I). This restriction is unavoidable due to decidability issues, but leaves open the question as to whether these results can be *combined*. For example, tree automata over a theory $\mathcal{E}_{\mathsf{AC}}$ with an AC symbol and free symbols are effectively closed under intersection [18], and tree automata over a theory $\mathcal{E}_{\mathsf{ACI}}$ with an ACI symbol and free symbols are also effectively closed under intersection [24]. Does this imply that tree automata over the combined theory $\mathcal{E}_{\mathsf{AC}} \cup \mathcal{E}_{\mathsf{ACI}}$ are effectively closed under intersection as well?

Our first contribution is to show that tree automata over $\mathcal{E}_{\mathsf{AC}} \cup \mathcal{E}_{\mathsf{ACI}}$ are *not* effectively closed under intersection. Moreover, the intersection-emptiness problem, which is decidable for tree automata over $\mathcal{E}_{\mathsf{AC}}$ and $\mathcal{E}_{\mathsf{ACI}}$ separately, is undecidable for tree automata over the combined theory $\mathcal{E}_{\mathsf{AC}} \cup \mathcal{E}_{\mathsf{ACI}}$. We obtain this result by showing that every *alternating tree language* [25] over a theory $\mathcal{E}$ can be effectively expressed as the intersection of two regular tree languages over a theory $\mathcal{E}'$ containing $\mathcal{E}$ and an additional ACI symbol. Since the emptiness problem for alternating AC-tree automata is undecidable [25], it follows that so is intersection-emptiness for regular tree automata over $\mathcal{E}_{\mathsf{AC}} \cup \mathcal{E}_{\mathsf{ACI}}$. Since emptiness is always decidable for regular equational tree automata, it follows that regular tree automata over $\mathcal{E}_{\mathsf{AC}} \cup \mathcal{E}_{\mathsf{ACI}}$ are not effectively closed under intersection.

Our result implies that both the decidability of intersection-emptiness and effective closure under intersection are *non-modular* properties, even for disjoint theories. Modularity is an important property to have, because it aids in the process of decomposing complex problems into simpler parts which can be reasoned about separately. For example, the Shostak [21] and Nelson-Oppen [15] combination methods have been fundamental to the development of automated theorem provers that combine the capabilities of many different decision procedures. Given the importance of modularity, we decided to further analyze how the interaction between the AC symbol and ACI symbol led to undecidability.

Our second contribution is to define a restricted class of tree automata over a theory $\mathcal{E}$ with AC and ACI symbols which are closed under equational congruences. We further show that the emptiness problem is decidable for the Boolean closure of tree languages in that class — a problem which we call the *propositional emptiness problem* as it closely relates to the emptiness problem for propositional tree automata. The tree automata in the restricted class we consider are called *AC-intersection free* and subjects each ACI symbol $+$ in $\mathcal{E}$ to one of two constraints: (1) either the clauses in the automaton where $+$ appears must satisfy certain syntactic restrictions to avoid simulating the intersection clauses of alternating tree automata; or (2) the idempotence equation $x + x = x$ in $\mathcal{E}$ must be treated as a rewrite rule $x+x \to x$ as in the *tree automata with normalization* framework of [17]. In that framework, some of the equations in $\mathcal{E}$ may be treated as rewrite rules in a confluent and terminating rewrite theory $\mathcal{R}$. Rather than computing the congruence closure of the tree language modulo $\mathcal{E}$, terms are first normalized by rewriting with $\mathcal{R}$ modulo the remaining equations $\mathcal{E}' \subseteq \mathcal{E}$, and then checked for membership in the underlying equational tree languages $\mathcal{L}(\mathcal{A}/\mathcal{E}')$. Their framework has different semantics than standard equational tree automata, but is often able to obtain better closure and decidability properties.

An important consequence of our second contribution is that it simultaneously solves two open problems: (1) We show that the emptiness problem is decidable for tree automata with normalization over idempotence rules and AC equations. This problem was mentioned in [17] and left unsolved. (2) We show that the propositional emptiness problem is decidable for equation tree automata over the theory $\mathcal{E}_{\mathsf{ACI}}$ containing a single ACI symbol and arbitrary free symbols. This problem is interesting, because equational tree automata over $\mathcal{E}_{\mathsf{ACI}}$ are not closed under complementation [23]. Its decidability also has a further implication — propositional emptiness is a non-modular property. Our earlier undecidability result implies that propositional emptiness is undecidable for equational tree automata over $\mathcal{E}_{\mathsf{AC}} \cup \mathcal{E}_{\mathsf{ACI}}$, while propositional emptiness is decidable for $\mathcal{E}_{\mathsf{AC}}$ [18].

One underlying goal in this work is to develop better tree automata techniques for *non-linear theories*. This is important in applications such as sufficient completeness checking where existing techniques either do not support rewriting modulo axioms [2] or are restricted to left-linear rewrite rules [8]. Although sufficient completeness checking is undecidable in general for specifications with non-linear rules and rewriting modulo AC [12], our decidability results show that sufficient completeness is decidable modulo AC when the every non-linear rule in the specification has the form $f(x, x) \to r$. It would be interesting to see if the techniques presented here can be extended to other forms of non-linear rules.

This paper is organized as follows. In Section 2, we review basic concepts from rewriting and tree automata. In Section 3, we show how alternating tree languages can be expressed as the intersection of two regular tree languages. In Section 4, we define a subclass of equational tree automata, which we call *AC-intersection free*, and state a decidability result which solves the two open problems discussed previously, and in Section 5, we present our algorithm for showing the previous decidability result. Finally, we discuss related work and suggest avenues for future research in Section 6.

2 Preliminary Definitions

We assume the reader is familiar with equational logic and rewriting as well as tree automata [1].

2.1 Equational and Rewrite Theories

An *equational theory* $\mathcal{E} = (F, E)$ consists of a signature F together with a set of equations $l = r$ with $l, r \in T_F(X)$. For each term $t \in T_F(X)$, we let $[t]_{\mathcal{E}}$ denote the equivalence class of terms equal to t with respect to the equivalence relation $=_{\mathcal{E}}$ induced by $\mathcal{E}$. We just write $[t]$ for $[t]_{\mathcal{E}}$ when the theory can be inferred from the context, and we let $T_{\mathcal{E}}$ denote the F-algebra whose universe $T_{\mathcal{E}}$ consists of the equivalence classes of T_F formed by $=_{\mathcal{E}}$.

A *rewrite theory* $\mathcal{R}$ is a set of *rewrite rules* of the form $l \to r$ with $l, r \in T_F(X)$. A term $t \in T_F(X)$ rewrites to $u \in T_F(X)$ module $\mathcal{E}$, denoted $t \to_{\mathcal{R}/\mathcal{E}} u$ if there is rule $l \to r \in R$, context C, and substitution θ such that $t =_{\mathcal{E}} C[l\theta]$ and $u =_{\mathcal{E}} C[r\theta]$. A term t is *$\mathcal{R}/\mathcal{E}$-irreducible* if it cannot be further rewritten. We write $t{\downarrow}_{\mathcal{R}/\mathcal{E}} u$ if there is a term $v \in T_F(X)$ such that $t \to^*_{\mathcal{R}/\mathcal{E}} v$ and $u \to^*_{\mathcal{R}/\mathcal{E}} v$. A rewrite theory $\mathcal{R}$ is *terminating* modulo $\mathcal{E}$ if $\to^+_{\mathcal{R}/\mathcal{E}}$ is well-founded. $\mathcal{R}$ is *confluent* if $t \to^*_{\mathcal{R}/\mathcal{E}} u$ and $t \to^*_{\mathcal{R}/\mathcal{E}} v$ implies $u{\downarrow}_{\mathcal{R}/\mathcal{E}} v$. If $\mathcal{R}$ is terminating and confluent modulo $\mathcal{E}$, then for all $t \in T_F$, there effectively exists an $\mathcal{R}/\mathcal{E}$-irreducible term $t{\downarrow}_{\mathcal{R}/\mathcal{E}} \in T_F$ that is unique up to $=_{\mathcal{E}}$. We let $\mathsf{Can}_{\mathcal{R}/\mathcal{E}} \subseteq T_{\mathcal{E}}$ denote the canonical term algebra whose universe is the set of $\mathcal{E}$-equivalence classes of $\mathcal{R}/\mathcal{E}$-irreducible terms.

In this paper, we restrict our attention to equational theories $\mathcal{E}$ only containing axioms with the following forms:

$(x + y) + z = x + (y + z)$	$x + y = y + x$	$x + x = x$
associativity	commutativity	idempotence

Relative to an equational theory $\mathcal{E}$, if a symbol $f \in F$ does not appear in any of the equations, we say it is a *free symbol.* If $f \in F$ appears in associativity and commutativity equations but no other equations, we say that it is an *AC symbol.* Finally, if $f \in F$ appears in associativity, commutativity, and idempotence equations, we say that it is an *ACI symbol.* We shall restrict our attention to equational theories where each symbol is a free, AC, or ACI symbol.

2.2 Tree Automata

We treat tree automata as collections of Horn clauses of particular forms as in [25]. A *regular $\mathcal{E}$-tree automaton* $\mathcal{A}$ is a finite set of Horn clauses each with the form:

$$p(f(x_1, \ldots, x_n)) \Leftarrow p_1(x_1), \ldots, p_n(x_n) \qquad \text{regular clause}$$

where $f \in F$ has arity n and $p, p_1, \ldots, p_n$ are elements of a finite set of unary predicate symbols called the *states* of the automaton. In some definitions, tree automata may also contain *ϵ-clauses* of the form $p(x) \Leftarrow q(x)$, but these can be eliminated without loss of expressive power. We write $\mathcal{A}/\mathcal{E} \vdash p(t)$ if $p(t)$ is entailed by the axioms in $\mathcal{A} \cup \mathcal{E}$. There are a variety of different inference systems for entailment with equivalent semantics, and when it is necessary to refer to a equivalence $\dfrac{t =_{\mathcal{E}} u \qquad \mathcal{A}/\mathcal{E} \vdash p(u)}{\mathcal{A}/\mathcal{E} \vdash p(t)}$

membership $\dfrac{\mathcal{A}/\mathcal{E} \vdash A_1\theta \qquad \ldots \qquad \mathcal{A}/\mathcal{E} \vdash A_n\theta}{\mathcal{A}/\mathcal{E} \vdash A\theta}$ if $A \Leftarrow A_1 \ldots A_n \in \mathcal{A}$

Fig. 1. Inference System for $\mathcal{A}/\mathcal{E}$

specific inference steps, we use the inference rules in Fig. 1, For an equational theory $\mathcal{E} = (F, \varnothing)$ with no equations, we write $\mathcal{A} \vdash p(t)$ for $\mathcal{A}/\mathcal{E} \vdash p(t)$.

We keep the acceptance condition separate from the automaton itself, and since the automaton only recognizes languages that are closed modulo $\mathcal{E}$, we define languages as subsets of $T_{\mathcal{E}}$ rather than T_F. For each state p belonging to $\mathcal{A}$, the *language recognized by p in* $\mathcal{A}$, denoted $\mathcal{L}_p(\mathcal{A}/\mathcal{E}) \subseteq T_{\mathcal{E}}$, is defined by

$$\mathcal{L}_p(\mathcal{A}/\mathcal{E}) = \{\, [t] \in T_{\mathcal{E}} \mid \mathcal{A}/\mathcal{E} \vdash p(t) \,\}. \tag{1}$$

One fundamental result from [25] about regular $\mathcal{E}$-tree automata is:

Theorem 1. *For each theory $\mathcal{E}$ and regular $\mathcal{E}$-tree automaton $\mathcal{A}$,*

$$\mathcal{A}/\mathcal{E} \vdash p(t) \iff (\exists u \in [t]_{\mathcal{E}})\ \mathcal{A} \vdash p(u).$$

For an arbitrary theory $\mathcal{E}$, the class of languages recognized by regular $\mathcal{E}$-tree automata is closed under union, but not under intersection or complementation. Motivated by this fact, an equational tree automata framework called *propositional tree automata* is introduced in [9] that is effectively closed under Boolean operations in all theories. The key idea is to use a propositional formula rather than a set of final states as the acceptance condition for defining the language recognized by the automaton. In this paper, we present an alternative formalization that preserves the basic idea. Given a tree automaton $\mathcal{A}$ with states Q, we extend (1) from languages $\mathcal{L}_p(\mathcal{A}/\mathcal{E})$ recognized by a state p to languages $\mathcal{L}_\phi(\mathcal{A}/\mathcal{E})$ recognized by a propositional formula ϕ constructed from atomic predicates Q and Boolean connectives $\wedge$ and $\neg$:

$$\mathcal{L}_{\phi_1 \wedge \phi_2}(\mathcal{A}/\mathcal{E}) = \mathcal{L}_{\phi_1}(\mathcal{A}/\mathcal{E}) \cap \mathcal{L}_{\phi_2}(\mathcal{A}/\mathcal{E}) \qquad \mathcal{L}_{\neg\phi_1}(\mathcal{A}/\mathcal{E}) = T_{\mathcal{E}} \setminus \mathcal{L}_{\phi_1}(\mathcal{A}/\mathcal{E}).$$

There are many decision problems that have been studied in the context of tree automata. The *membership problem* for $\mathcal{E}$ is the problem of deciding for an equivalence class $[t] \in T_{\mathcal{E}}$, $\mathcal{E}$-tree automaton $\mathcal{A}$ and state p in $\mathcal{A}$ whether $[t] \in \mathcal{L}_p(\mathcal{A}/\mathcal{E})$. The *emptiness problem* for $\mathcal{E}$ is the problem of deciding for an $\mathcal{E}$-tree automaton $\mathcal{A}$ and state p whether $\mathcal{L}_p(\mathcal{A}/\mathcal{E}) = \varnothing$. This problem is decidable in linear time for an arbitrary theory $\mathcal{E}$ using Theorem 1 and standard tree automata techniques [1]. The *intersection-emptiness problem* for $\mathcal{E}$ is the problem of deciding for an $\mathcal{E}$-tree automaton $\mathcal{A}$ and states $p_1, \ldots, p_n$ of $\mathcal{A}$ whether

$\mathcal{L}_{p_1}(\mathcal{A}/\mathcal{E}) \cap \cdots \cap \mathcal{L}_{p_n}(\mathcal{A}/\mathcal{E}) = \varnothing$. Finally, the *propositional emptiness problem* for $\mathcal{E}$ is the problem of deciding for an $\mathcal{E}$-tree automaton $\mathcal{A}$ with states Q and propositional formula ϕ over atomic predicates Q whether $\mathcal{L}_\phi(\mathcal{A}/\mathcal{E}) = \varnothing$.

It is known that both the intersection-emptiness and propositional emptiness problem is decidable for regular equational tree automata over a theory $\mathcal{E}_{\mathsf{AC}}$ with AC and free symbols [16]. In contrast, both intersection-emptiness and propositional emptiness are undecidable for regular equational tree automata over a theory $\mathcal{E}_{\mathsf{A}}$ with associative and free symbols [18]. As an example of a tree automata framework where intersection-emptiness is decidable and propositional emptiness is undecidable, we refer the reader to the *monotone AC tree automata* framework of [19].

3 Alternating Tree Automata

One extension to tree automata is the alternating tree automata framework of [22] which was extended to the equational case in [25]. In a Horn-clause representation, an *alternating tree automaton* is a tree automaton which in addition to regular clauses, may also contain *intersection clauses* of the form:

$$p(x) \Leftarrow p_1(x), p_2(x) \qquad\qquad \text{intersection clause.}$$

Alternating $\mathcal{E}$-tree automata are closed under both intersection and union, but are not always closed under complementation. If $\mathcal{E}$ is the free theory, i.e., $\mathcal{E} = (F, \varnothing)$, then the class of languages recognized by alternating and regular automata coincide. However, this is often not the case for other theories. For example, alternating AC-tree automata are strictly more powerful than regular AC-automata. In particular, the emptiness problem is undecidable for alternating AC-tree automata [25].

Our first new result in this paper is to show that every alternating $\mathcal{E}$-tree language is isomorphic to the intersection of two regular $\mathcal{E}'$-tree languages where $\mathcal{E}'$ is the theory obtained by adding a fresh ACI symbol $\circ$ to $\mathcal{E}$.

Theorem 2. *Let $\mathcal{E} = (F, E)$ and $\mathcal{E}' = (F', E')$ be equational theories such that $\mathcal{E}'$ contains the symbols and equations in $\mathcal{E}$ and adds a fresh ACI operator $\circ$.*

Given an alternating $\mathcal{E}$-tree automaton $\mathcal{A}$ with states Q, one can effectively construct a regular $\mathcal{E}'$-tree automaton $\mathcal{B}$ containing the states Q and an additional fresh state k such that

- *For all $p \in Q$ and $t \in T_F$, $\mathcal{A}/\mathcal{E} \vdash p(t) \iff \mathcal{B}/\mathcal{E}' \vdash p(t)$.*
- *For all $t \in T_{F'}$, $\mathcal{B}/\mathcal{E}' \vdash k(t) \iff T_F \cap [t]_{\mathcal{E}'} \neq \varnothing$.*

Proof. Let $\mathcal{B}$ be the automaton containing the following clauses:

- $\mathcal{B}$ contains all of the clauses in $\mathcal{A}$ that are not intersection clauses;
- for each intersection clause $p(x) \Leftarrow p_1(x), p_2(x)$ in $\mathcal{A}$, $\mathcal{B}$ contains the clause $p(x_1 \circ x_2) \Leftarrow p_1(x_1), p_2(x_2)$; and

– for each symbol $f \in F$ with arity n, $\mathcal{B}$ contains the clause $k(f(x_1, \ldots, x_n)) \Leftarrow k(x_1), \ldots, k(x_n)$.

We first show that $\mathcal{A}/\mathcal{E} \vdash p(t)$ implies $\mathcal{B}/\mathcal{E}' \vdash p(t)$ for all $p \in Q$. Since $\mathcal{B}$ contains all the clauses in $\mathcal{A}$ other than the intersection clauses, all we need to show is that $\mathcal{B} \cup \mathcal{E}'$ entails each intersection clause $q(x) \Leftarrow q_1(x), q_2(x)$ in $\mathcal{A}$. This is immediate, because $\mathcal{B}$ must contain the clause $q(x_1 \circ x_2) \Leftarrow q_1(x_1), q_2(x_2)$, and so $\mathcal{B}$ entails $q(x \circ x) \Leftarrow q_1(x), q_2(x)$. The theory $\mathcal{E}'$ contain the axiom $x \circ x = x$, and thus $\mathcal{B} \cup \mathcal{E}'$ entails $q(x) \Leftarrow q_1(x), q_2(x)$.

We now show that $\mathcal{B}/\mathcal{E}' \vdash p(t)$ implies $\mathcal{A}/\mathcal{E} \vdash p(t)$ for all $p \in Q$. If $\mathcal{B}/\mathcal{E}' \vdash p(t)$ then by Theorem 1 there is a term $u \in T_{F'}$ such that $t =_{\mathcal{E}'} u$ such that $\mathcal{B} \vdash p(u)$. We construct a term $v \in T_F$ such that $u =_{\mathcal{E}'} v$ and $\mathcal{A}/\mathcal{E} \vdash p(v)$. Since $t =_{\mathcal{E}'} u =_{\mathcal{E}'} v$ and neither t nor v contain the added symbol $\circ$, it is not difficult to show that $t =_{\mathcal{E}} v$, and thus $\mathcal{A}/\mathcal{E} \vdash p(t)$.

We construct the term $v \in T_F$ from the proof that $\mathcal{B} \vdash p(u)$ by analyzing the proof bottom-up starting from the leaves. Each inference step that does not use a clause containing the idempotence symbol $\circ$ has a direct corresponding inference step using the clauses in $\mathcal{A}$ and can be handled easily. On the other hand, given an inference step of the form

$$\frac{\mathcal{B} \vdash q_1(u_1) \qquad \mathcal{B} \vdash q_2(u_2)}{\mathcal{B} \vdash q(u_1 \circ u_2)}$$

with $q(x_1 \circ x_2) \Leftarrow q_1(x_1), q_2(x_2)$ in $\mathcal{B}$, we first observe that $u_1 =_{\mathcal{E}'} u_2 =_{\mathcal{E}'} u_1 \circ u_2$, because $u_1 \circ u_2$ is a subterm of u, and u is equivalent to $t \in T_F$ which does not contain the symbol $\circ$. By induction, we know that for $i \in [1, 2]$, there is a term $v_i \in T_F$ such that $u_i =_{\mathcal{E}'} v_i$ and $\mathcal{A}/\mathcal{E} \vdash q_i(v_i)$. As $v_1 =_{\mathcal{E}'} u_1 =_{\mathcal{E}'} u_2 =_{\mathcal{E}'} v_2$ and both v_1 and v_2 are in T_F, it follows that $v_1 =_{\mathcal{E}} v_2$, and thus $\mathcal{A}/\mathcal{E} \vdash p_2(v_1)$. By using the intersection clause $p(x) \Leftarrow p_1(x), p_2(x)$ in $\mathcal{A}$, it follows that $\mathcal{A}/\mathcal{E} \vdash p(v_1)$ and thus we are done as $v_1 =_{\mathcal{E}} u_1 =_{\mathcal{E}} u_1 \circ u_2$.

Finally, we show that $\mathcal{B}/\mathcal{E}' \vdash p(t)$ if and only if $T_F \cap [t]_{\mathcal{E}'} \neq \varnothing$ for all $t \in T_{F'}$, by observing that $\mathcal{B} \vdash k(u)$ iff u is in T_F, and so by Theorem 1,

$$\mathcal{B}/\mathcal{E}' \vdash k(t) \iff (\exists u \in [t]_{\mathcal{E}'})\, \mathcal{B} \vdash k(u) \iff T_F \cap [t]_{\mathcal{E}'} \neq \varnothing.$$

□

From this theorem, it follows that for each $p \in Q$, the languages $\mathcal{L}_p(\mathcal{A}/\mathcal{E})$ and $\mathcal{L}_p(\mathcal{B}/\mathcal{E}') \cap \mathcal{L}_k(\mathcal{B}/\mathcal{E}')$ are isomorphic with the bijective mapping

$$h_p : [t]_{\mathcal{E}} \in \mathcal{L}_p(\mathcal{A}/\mathcal{E}) \mapsto [t]_{\mathcal{E}'} \in \mathcal{L}_p(\mathcal{B}/\mathcal{E}') \cap \mathcal{L}_k(\mathcal{B}/\mathcal{E}').$$

Although this connection between alternating and regular languages seems worth further study, our main interest in this result is that allows us to use the result in [25] about the undecidability of emptiness for alternating AC-tree automata to show that intersection-emptiness is undecidable for regular tree automata over a theory $\mathcal{E}$ with both an AC and ACI symbol.

Corollary 1. *If $\mathcal{E}$ is an equational theory with* 4 *constants, an AC symbol, and an ACI symbol, then the intersection-emptiness problem for regular tree automata over $\mathcal{E}$ is undecidable.*

Proof. Let $\mathcal{E}_{\mathsf{AC}}$ denote the equational theory obtained by removing the ACI symbol from $\mathcal{E}$. The theory $\mathcal{E}_{\mathsf{AC}}$ is *torsion-free* according to the definition in [25] with regard to the 4 constants, and consequently the emptiness problem is undecidable for alternating $\mathcal{E}_{\mathsf{AC}}$-tree automata by Prop. 11 in [25]. By Theorem 2, for each alternating automaton $\mathcal{A}$, we can construct a regular $\mathcal{E}$-tree automaton $\mathcal{B}$ such that $\mathcal{L}_p(\mathcal{A}/\mathcal{E}_{\mathsf{AC}}) = \varnothing$ iff $\mathcal{L}_p(\mathcal{B}/\mathcal{E}) \cap \mathcal{L}_k(\mathcal{B}/\mathcal{E}) = \varnothing$. □

The theory $\mathcal{E}$ in the previous statement can be partitioned into disjoint theories $\mathcal{E}_{\mathsf{AC}}$ and $\mathcal{E}_{\mathsf{ACI}}$ where $\mathcal{E}_{\mathsf{AC}}$ contains the AC symbol and $\mathcal{E}_{\mathsf{ACI}}$ contains the ACI symbol and the constants are split freely between them. Intersection-emptiness is decidable for both $\mathcal{E}_{\mathsf{AC}}$ [18] and $\mathcal{E}_{\mathsf{ACI}}$ [24], but as the previous statement shows it is undecidable for $\mathcal{E} = \mathcal{E}_{\mathsf{AC}} \cup \mathcal{E}_{\mathsf{ACI}}$. It follows that intersection-emptiness is a non-modular property for equational tree automata even for combinations of disjoint theories.

4 AC-Intersection Free Tree Automata

Having shown that intersection-emptiness is undecidable in general for equational tree automata over a theory $\mathcal{E}$ with AC and ACI symbols, we have decided to search for a restricted subclass of equational tree automata over $\mathcal{E}$ for which not only is intersection-emptiness decidable, but so is the propositional emptiness problem. Our search for this class began by trying to eliminate the main culprit that led to the undecidability result in Cor. 1 — the ability of clauses with ACI symbols to simulate the intersection clauses of an alternating AC-tree automata.

The solution we have found is to subject each ACI symbol $\circ$ in $\mathcal{E}$ to one of two constraints: (1) either the clauses in the automaton where $\circ$ appears must satisfy certain syntactic restrictions explained below; or (2) the idempotence equation $x \circ x = x$ in $\mathcal{E}$ must be treated as a rewrite rule $x \circ x \to x$ as in the *tree automata with normalization* framework of [17]. We first define the syntactic restrictions:

Definition 1. *Let $\mathcal{E}$ be an equational theory $\mathcal{E}$ in which each symbol is AC, ACI, or free. A regular $\mathcal{E}$-tree automaton $\mathcal{A}$ is* AC-intersection free *iff for each clause in $\mathcal{A}$ with the form $p(x_1 \circ x_2) \Leftarrow p_1(x_1), p_2(x_2)$ where $\circ \in F$ is an ACI symbol, it is the case that for all $q_1, q_2 \in Q$, and AC or ACI symbols $+ \neq \circ$,*

$$p_1(x_1 + x_2) \Leftarrow q_1(x_1), q_2(x_2) \in \mathcal{A} \implies p(x_1 + x_2) \Leftarrow q_1(x_1), q_2(x_2) \in \mathcal{A}.$$

One important observation is that AC-intersection free automata are closed under disjoint unions — that is given two AC-intersection free $\mathcal{E}$-tree automata $\mathcal{A}$ and $\mathcal{B}$ such that the states have been renamed so that the states in $\mathcal{A}$ and $\mathcal{B}$ are disjoint, the union $\mathcal{E}$-tree automaton $\mathcal{C} = \mathcal{A} \cup \mathcal{B}$ is also AC-intersection free. Moreover, $\mathcal{L}_p(\mathcal{A}/\mathcal{E}) = \mathcal{L}_p(\mathcal{C}/\mathcal{E})$ for each state p in $\mathcal{A}$, and $\mathcal{L}_q(\mathcal{B}/\mathcal{E}) = \mathcal{L}_q(\mathcal{C}/\mathcal{E})$ for each state q in $\mathcal{A}$. Since we will soon show that the propositional emptiness problem is decidable for AC-intersection free automata, it follows that the emptiness of an arbitrary Boolean combination of AC-intersection free tree languages is decidable even if the languages are defined in different automata.

This syntactic restriction may be too strong in some applications, and so we also study a different approach to handling idempotence equations that is suggested by the tree automata with normalization framework of [17]. A *tree automaton with normalization* (TAN) $\mathcal{A}$ is equipped with a rewrite system $\mathcal{R}$ that is confluent and terminating modulo an equational theory $\mathcal{E}$. A term t is accepted by TAN $\mathcal{A}$ if its normal form $[t{\downarrow}_{\mathcal{R}/\mathcal{E}}]$ is in the underlying equational tree language $\mathcal{L}(\mathcal{A}/\mathcal{E})$. This framework borrows the fundamental idea in term rewriting, namely that some of the equations in a theory $\mathcal{E}'$ are best handled by orienting them as rewrite rules in a rewrite system $\mathcal{R}$ in a way so that $\mathcal{R}$ is confluent and terminating modulo the remaining equations $\mathcal{E} \subseteq \mathcal{E}'$. As $\mathcal{R}$ is terminating and confluent modulo $\mathcal{E}$, the language is closed with respect to both the equations in $\mathcal{E}$ and the equations obtained from the rules in $\mathcal{R}$.

Our interest in the TAN framework stems from the fact that if $\mathcal{R}_\mathsf{I}$ is a rewrite system containing idempotence rules $f(x, x) \to x$ for some of the AC symbols in a theory $\mathcal{E}$ with free, AC, and ACI symbols, then $\mathcal{R}_\mathsf{I}$ is confluent and terminating modulo $\mathcal{E}$. This suggests that as an alternative to the restrictions in Def. 1, we can treat some of the idempotence equations as rules, and still have a class of tree automata closed modulo both the equations in $\mathcal{E}$ and the underlying equations in $\mathcal{R}_\mathsf{I}$. By handling the idempotence equations as rules, we avoid the problem of simulating intersection clauses, because that simulation relies on applying idempotence in the direction $x \to x + x$.

By requiring that each ACI symbol either satisfies the syntactic constraints in the definition of AC-intersection free automata, or treats the idempotence equation as a rule as in the tree automata with normalization approach, we describe an algorithm in the next section whose correctness implies the following:

Theorem 3. *Let $\mathcal{E} = (F, E)$ be a theory with free, AC, and ACI symbols, and let $\mathcal{R}_\mathsf{I}$ be a rewrite theory where the only axioms are idempotence rules of the form $x + x \to x$ for an AC symbol $+ \in F$.*

If $\mathcal{A}$ is an AC-intersection free $\mathcal{E}$-tree automaton with states Q and ϕ is a propositional formula with atomic predicates Q, it is decidable whether

$$\mathsf{Can}_{\mathcal{R}_\mathsf{I}/\mathcal{E}} \cap \mathcal{L}_\phi(\mathcal{A}/\mathcal{E}) = \varnothing.$$

One important observation about this theorem is that it implies the decidability of two open questions both of which can be viewed as special cases:

The first open question settled by Theorem 3 is the problem of deciding the emptiness of the language accepted by a tree automata with normalization over an equational theory $\mathcal{E}_\mathsf{AC}$ with AC and free symbols and a rewrite system $\mathcal{R}_\mathsf{I}$ containing idempotence equations for some of the AC symbols in $\mathcal{E}_\mathsf{AC}$. Specifically, we want to decide whether $\mathsf{Can}_{\mathcal{R}_\mathsf{I}/\mathcal{E}_\mathsf{AC}} \cap L_p(\mathcal{A}/\mathcal{E}_\mathsf{AC}) = \varnothing$ for each $\mathcal{E}_\mathsf{AC}$-tree automaton $\mathcal{A}$ and state p in $\mathcal{A}$. The problem was mentioned in [17], but left unsolved. Theorem 3 solves this problem, because $\mathcal{E}_{\mathsf{AC}}$ contains no ACI symbols and thus every $\mathcal{E}_{\mathsf{AC}}$-tree automaton is AC-intersection free. One observation made in [17] is that for tree automaton with normalization, the decidability of the emptiness problem only depends on the left hand sides of the rules in $\mathcal{R}$. It follows that if the emptiness problem is decidable when $\mathcal{R}$ contains idempotence rules $x+x \to x$, it is also decidable when $\mathcal{R}$ contains nilpotence rules $x+x \to 0$.

The second open question settled by Theorem 3 is the problem of deciding the propositional emptiness of equational tree automata over a theory $\mathcal{E}_{\mathsf{ACI}}$ with a single ACI symbol and free symbols. This problem is interesting, because equational tree automata over $\mathcal{E}_{\mathsf{ACI}}$ are not closed under complementation [23], and so the propositional emptiness problem is not reducible to the emptiness problem in this theory. Theorem 3 solves this problem, because $\mathcal{E}_{\mathsf{ACI}}$ contains only a single ACI symbol, and thus every $\mathcal{E}_{\mathsf{ACI}}$-tree automaton is AC-intersection free. Solving the propositional emptiness problem also shows that both subsumption ($\mathcal{L}_p(\mathcal{A}/\mathcal{E}_{\mathsf{ACI}}) \subseteq \mathcal{L}_q(\mathcal{B}/\mathcal{E}_{\mathsf{ACI}})$) and universality ($\mathcal{L}_p(\mathcal{A}/\mathcal{E}_{\mathsf{ACI}}) = T_{\mathcal{E}_{\mathsf{ACI}}}$) are decidable for equational tree automata over $\mathcal{E}_{\mathsf{ACI}}$, and both problems appear to be open. Additionally, since intersection-emptiness is undecidable for equational tree automata over $\mathcal{E}_{\mathsf{AC}} \cup \mathcal{E}_{\mathsf{ACI}}$ due to Cor. 1, it follows that propositional emptiness over $\mathcal{E}_{\mathsf{AC}} \cup \mathcal{E}_{\mathsf{ACI}}$ is undecidable as well. However, propositional emptiness is decidable for $\mathcal{E}_{\mathsf{AC}}$ [18] and implied to be decidable for $\mathcal{E}_{\mathsf{ACI}}$ by Theorem 3. It follows that propositional emptiness is also a *non-modular property* for the combination of disjoint theories.

5 Decision Procedure

In this section, we define an algorithm that solves the decision problem posed in Theorem 3. We begin with a discussion of our overall approach, and how any solution to check the *emptiness* of a regular equational tree language over a theory containing idempotence axioms appears to also require being able to compute the *size* of a language. We then present results about terms whose root is a free symbol in Section 5.1, and present results abouts terms whose root is an AC or ACI symbol in Section 5.2. In Section 5.3, we present our function for estimating the number of distinct equivalence classes that reach a particular profile. Finally, in Section 5.4, we present the algorithm itself, and verify its correctness.

For this section, $\mathcal{E} = (F, E)$ denotes a theory in which each symbol is AC, ACI, or free, $\mathcal{R}_{\mathsf{I}}$ denotes a rewrite system where the only axioms are idempotence rules of the form $x + x \to x$ for an AC symbol $+ \in F$, and $\mathcal{A}$ denotes a regular AC-intersection free $\mathcal{E}$-tree automaton with states Q. At times it is useful to treat all the idempotence equations as idempotence rules. We let $\mathcal{E}_{\mathsf{AC}} \subseteq \mathcal{E}$ containing only the associativity and commutativity equations in $\mathcal{E}$, and we let $\hat{\mathcal{R}}_{\mathsf{I}}$ denote the rewrite system containing the rules in $\mathcal{R}_{\mathsf{I}}$ as well as a rule $x + x \to x$ for each equation $x + x = x$ in $\mathcal{E}$. It can be observed that $\hat{\mathcal{R}}_{\mathsf{I}}$ is terminating and confluent modulo $\mathcal{E}_{\mathsf{AC}}$, so for all $\mathcal{R}_{\mathsf{I}}/\mathcal{E}$-irreducible terms $t, u \in T_F$, $t =_{\mathcal{E}} u$ iff

$t{\downarrow_{\hat{\mathcal{R}}_\mathsf{I}/\mathcal{E}_\mathsf{AC}}} =_{\mathcal{E}_\mathsf{AC}} u{\downarrow_{\hat{\mathcal{R}}_\mathsf{I}/\mathcal{E}_\mathsf{AC}}}$. For all $[t], [u] \in \mathsf{Can}_{\mathcal{R}_\mathsf{I}/\mathcal{E}}$, we say that $[t]$ is a *flattened subterm* of $[u]$, denoted $[t] \trianglelefteq_\mathsf{flat} [u]$, if either:

- $u{\downarrow_{\hat{\mathcal{R}}_\mathsf{I}/\mathcal{E}_\mathsf{AC}}} =_{\mathcal{E}_\mathsf{AC}} f(u_1, \ldots, u_n)$ with f a free symbol and $t{\downarrow_{\hat{\mathcal{R}}_\mathsf{I}/\mathcal{E}_\mathsf{AC}}} =_{\mathcal{E}_\mathsf{AC}} u_i$ for some $i \in [1, n]$, or
- $u{\downarrow_{\hat{\mathcal{R}}_\mathsf{I}/\mathcal{E}_\mathsf{AC}}} =_{\mathcal{E}_\mathsf{AC}} u_1 + \cdots + u_n$ with $+$ an AC or ACI symbol, $n \geq 2$, $\mathsf{root}(u_i) \neq +$ for all $i \in [1, n]$, and $t{\downarrow_{\hat{\mathcal{R}}_\mathsf{I}/\mathcal{E}_\mathsf{AC}}} =_{\mathcal{E}_\mathsf{AC}} u_i$ for some $i \in [1, n]$.

Our algorithm is similar to the subset construction algorithm in [9] for checking the propositional emptiness of equational tree automata over A and AC symbols. For each $[t] \in T_\mathcal{E}$, the *profile* of $[t]$, denoted $\mathsf{profile}([t])$, is a pair that contains all the information about $[t]$ relevant to the algorithm.

Definition 2. *Let* $\mathsf{profile} : T_\mathcal{E} \to F \times \mathcal{P}(Q)$ *be the function such that:*

$$\mathsf{profile}([t]) = (\mathsf{root}(t{\downarrow_{\hat{\mathcal{R}}_\mathsf{I}/\mathcal{E}_\mathsf{AC}}}), \mathsf{states}_{\mathcal{A}/\mathcal{E}}([t])).$$

where $\mathsf{states}_{\mathcal{A}/\mathcal{E}}([t]) = \{\, p \in Q \mid \mathcal{A}/\mathcal{E} \vdash p(t) \,\}$.

Note that $\mathsf{root}(t{\downarrow_{\hat{\mathcal{R}}_\mathsf{I}/\mathcal{E}_\mathsf{AC}}})$ is uniquely determined as $\mathcal{E}_\mathsf{AC}$ only contains associativity and commutativity axioms which do not change the root symbol of a term.

For an automaton $\mathcal{B}$ with states Q' over a theory $\mathcal{E}' = (F', E')$ with free, A, and AC symbols, we presented a semi-algorithm in [9] for constructing the set

$$\mathsf{det}(\mathcal{B}) = \{\, (f, P) \in F' \times \mathcal{P}(Q') \mid (\exists [t] \in T_{\mathcal{E}'})\ \mathsf{root}([t]) = f \wedge \mathsf{states}_{\mathcal{B}/\mathcal{E}'}([t]) = P \,\}.$$

By computing this set, we can decide if $\mathcal{L}_\phi(\mathcal{B}/\mathcal{E}') \neq \varnothing$ by checking for a profile $(f, P) \in \mathsf{det}(\mathcal{B})$ such that $P \models \phi$ where $P \models \phi$ is defined inductively:

$$P \models \phi_1 \wedge \phi_2 \text{ iff } P \models \phi_1 \text{ and } P \models \phi_2 \qquad P \models \neg\phi \text{ iff } P \not\models \phi \qquad P \models p \text{ iff } p \in P$$

For solving the problem in Theorem 3, this approach is inadequate for two reasons: (1) We want to decide whether $\mathsf{Can}_{\mathcal{R}_\mathsf{I}/\mathcal{E}} \cap \mathcal{L}_\phi(\mathcal{A}/\mathcal{E}) = \varnothing$ rather than deciding whether $\mathcal{L}_\phi(\mathcal{A}/\mathcal{E}) = \varnothing$. (2) Both $\mathcal{E}$ and $\mathcal{R}$ may contain idempotence axioms, and idempotence appears to require constructing a structure that not only allows checking it there *exists* a term with a particular profile, but also *how many* distinct terms have that profile. We illustrate this with an example. Let $\mathcal{E}_\mathsf{ACI}$ be the theory containing an ACI symbol $\circ$ and constants a, b, and c, and let $\mathcal{B}$ be the $\mathcal{E}_\mathsf{ACI}$-tree automaton with the rules:

$$p_1(a) \quad p_1(b) \quad p_2(x_1 \circ x_2) \Leftarrow p_1(x_1), p_1(x_2) \quad p_3(x_1 \circ x_2) \Leftarrow p_1(x_1), p_2(x_2).$$

In this automaton, one can observe that

$$\mathcal{L}_{p_2}(\mathcal{B}/\mathcal{E}_\mathsf{ACI}) = \mathcal{L}_{p_3}(\mathcal{B}/\mathcal{E}_\mathsf{ACI}) = \{\, [a \circ a], [a \circ b], [b \circ b] \,\},$$

and consequently $\mathcal{L}_{p_3 \wedge \neg p_2}(\mathcal{B}/\mathcal{E}_\mathsf{ACI}) = \varnothing$. Now consider the automaton $\mathcal{B}'$ containing the clauses in $\mathcal{B}$ and the additional clause $p_1(c)$. One can observe that $\mathcal{L}_{p_3 \wedge \neg p_2}(\mathcal{B}'/\mathcal{E}_\mathsf{ACI}) = \{\, [a \circ b \circ c] \,\}$. The language $\mathcal{L}_{p_3 \wedge \neg p_2}(\mathcal{B}'/\mathcal{E}_\mathsf{ACI})$ is not empty, because there are 3 distinct elements in $\mathcal{L}_{p_1}(\mathcal{B}'/\mathcal{E}_{\mathsf{ACI}})$, whereas $\mathcal{L}_{p_1}(\mathcal{B}/\mathcal{E}_{\mathsf{ACI}})$ only contains 2 elements. It is relatively straightforward to generalize this idea, so that for any positive integer $n \in \mathbb{N}$ and tree automaton $\mathcal{B}$ over $\mathcal{E}$ with a state p, we can construct an tree automaton $\mathcal{B}'$ over the theory $\mathcal{E}'$ containing $\mathcal{E}$ as well as a fresh ACI symbol $\circ$, and construct a formula ϕ over the states in $\mathcal{B}'$ such that

$$\mathcal{L}_\phi(\mathcal{B}'/\mathcal{E}) \neq \varnothing \iff |\mathcal{L}_p(\mathcal{B}/\mathcal{E})| = n.$$

Since a language may contain a (countably) infinite number of elements, for reasoning about the size of the language, it is helpful to extended basic arithmetic operators to ω. Specifically, we extend addition to ω so that it is still commutative, and satisfies the equations

$$\omega + \omega = \omega, \quad \text{and} \quad n + \omega = \omega,$$

and we extend multiplication to ω so that it is still commutative, and satisfies the equations

$$\omega \times \omega = \omega, \qquad 0 \times \omega = 0, \quad \text{and} \quad n \times \omega = \omega \text{ if } n > 0.$$

Instead of a set, we construct a directed graph $(D_\mathcal{A}, \trianglelefteq_\mathcal{A})$ with the nodes

$$D_\mathcal{A} = \{\, d \in F \times \mathcal{P}(Q) \mid (\exists [t] \in \mathsf{Can}_{\mathcal{R}_\mathsf{I}/\mathcal{E}})\ \mathsf{profile}([t]) = d \,\},$$

and $\trianglelefteq_\mathcal{A}$ contains an edge $d_1 \trianglelefteq_\mathcal{A} d_2$ iff there are $[t], [u] \in \mathsf{Can}_{\hat{\mathcal{R}}_\mathsf{I}/\mathcal{E}_\mathsf{AC}}$ such that $\mathsf{profile}([t]_\mathcal{E}) = d_1$, $\mathsf{profile}([u]_\mathcal{E}) = d_2$, and $[t] \trianglelefteq_\mathsf{flat} [u]$. The edge relation $\trianglelefteq_\mathcal{A}$ will be used later in counting the number of equivalence classes with a given profile.

We incrementally construct $(D_\mathcal{A}, \trianglelefteq_\mathcal{A})$ by starting with the empty graph $(D_0, \trianglelefteq_0) = (\varnothing, \varnothing)$ and applying inference rules to form increasing larger subgraphs $(D_1, \trianglelefteq_1) \subseteq (D_2, \trianglelefteq_2) \subseteq \cdots \subseteq (D_\mathcal{A}, \trianglelefteq_\mathcal{A})$ until saturation. This process terminates as the size of $D_\mathcal{A}$ is at most $|F| \times 2^{|Q|}$. Each profile graph $(D, \trianglelefteq) \subseteq (D_\mathcal{A}, \trianglelefteq_\mathcal{A})$ can be viewed as representing the (possibly infinite) subset of $\mathsf{Can}_{\mathcal{R}_\mathsf{I}/\mathcal{E}}$ that is already explored.

Definition 3. *For each graph $(D, \trianglelefteq) \subseteq (D_\mathcal{A}, \trianglelefteq_\mathcal{A})$, let $\mathsf{Can}_{D,\trianglelefteq}$ denote the smallest set containing each $[t] \in \mathsf{Can}_{\mathcal{R}_\mathsf{I}/\mathcal{E}}$ if $\mathsf{profile}([t]_\mathcal{E}) \in D$ and for all $[u] \in \mathsf{Can}_{\mathcal{R}_\mathsf{I}/\mathcal{E}}$,*

$$[u] \trianglelefteq_\mathsf{flat} [t] \implies [u] \in \mathsf{Can}_{D,\trianglelefteq} \wedge \mathsf{profile}([u]) \trianglelefteq \mathsf{profile}([t]).$$

Furthermore, for each $d \in D$, we let $\mathsf{profile}^{-1}_{D,\trianglelefteq}(d)$ denote the elements in $\mathsf{Can}_{D,\trianglelefteq}$ with profile d, i.e., $\mathsf{profile}^{-1}_{D,\trianglelefteq}(d) = \{\, [t] \in \mathsf{Can}_{D,\trianglelefteq} \mid \mathsf{profile}([t]) = d \,\}$.

The graph $(D_\mathcal{A}, \trianglelefteq_\mathcal{A})$ can be viewed as the graph where every $\mathcal{R}_\mathsf{I}/\mathcal{E}$-irreducible term has been explored.

Lemma 1. *For all $(D, \trianglelefteq) \subseteq (D_\mathcal{A}, \trianglelefteq_\mathcal{A})$,*

$$(D, \trianglelefteq) = (D_\mathcal{A}, \trianglelefteq_\mathcal{A}) \iff \mathsf{Can}_{D,\trianglelefteq} = \mathsf{Can}_{\mathcal{R}_\mathsf{I}/\mathcal{E}}.$$

Proof. We first show that $\mathsf{Can}_{D_{\mathcal{A}},\trianglelefteq_{\mathcal{A}}} = \mathsf{Can}_{\mathcal{R}_{\mathsf{I}}/\mathcal{E}}$. As $\hat{\mathcal{R}}_{\mathsf{I}}$ is confluent and terminating modulo $\mathcal{E}_{\mathsf{AC}}$, it is sufficient to show that for each $\hat{\mathcal{R}}_{\mathsf{I}}/\mathcal{E}_{\mathsf{AC}}$-irreducible term $t \in T_F$, $[t]_{\mathcal{E}} \in \mathsf{Can}_{D_{\mathcal{A}},\trianglelefteq_{\mathcal{A}}}$. We prove this by structural induction on t. By definition, $\mathsf{profile}([t]) \in D_{\mathcal{A}}$, and so we only need to prove that for all $\hat{\mathcal{R}}_{\mathsf{I}}/\mathcal{E}_{\mathsf{AC}}$-irreducible terms $u \in T_F$, if $[u] \trianglelefteq_{\mathsf{flat}} [t]$, then $[u] \in \mathsf{Can}_{D,\trianglelefteq}$ and $\mathsf{profile}([u]) \trianglelefteq_{\mathcal{A}} \mathsf{profile}([t])$. There are two cases to consider:

- If $t = f(t_1, \ldots, t_n)$ where f is a free symbol. In this case, if $u \in T_F$ is a $\hat{\mathcal{R}}_{\mathsf{I}}/\mathcal{E}_{\mathsf{AC}}$-irreducible term such that $[u] \trianglelefteq_{\mathsf{flat}} [t]$ then there is an $i \in [1, n]$ such that $u =_{\mathcal{E}_{\mathsf{AC}}} t_i$. By induction, we know that $[t_i]_{\mathcal{E}} \in \mathsf{Can}_{D_{\mathcal{A}},\trianglelefteq_{\mathcal{A}}}$, and $[t_i] \trianglelefteq [u]$ by definition. Therefore, $[t]_{\mathcal{E}} \in \mathsf{Can}_{D_{\mathcal{A}},\trianglelefteq_{\mathcal{A}}}$.
- Otherwise $u = u_1 + \cdots + u_n$ for some AC or ACI symbol $+$ where $n \geq 2$, $\mathsf{root}(u_i) \neq +$ for all $i \in [1, n]$. In this case, if $u \in T_F$ is a $\hat{\mathcal{R}}_{\mathsf{I}}/\mathcal{E}_{\mathsf{AC}}$-irreducible term such that $[u] \trianglelefteq_{\mathsf{flat}} [t]$, then there is an $i \in [1, n]$ such that $u =_{\mathcal{E}_{\mathsf{AC}}} t_i$. By induction, we know that $[t_i]_{\mathcal{E}} \in \mathsf{Can}_{D_{\mathcal{A}},\trianglelefteq_{\mathcal{A}}}$ and by $[t_i] \trianglelefteq_{\mathsf{flat}} [u]$. Therefore, $[t] \in \mathsf{Can}_{D_{\mathcal{A}},\trianglelefteq_{\mathcal{A}}}$.

On the other hand, if $\mathsf{Can}_{D,\trianglelefteq} = \mathsf{Can}_{\mathcal{R}_{\mathsf{I}}/\mathcal{E}}$, then for each $[t] \in \mathsf{Can}_{\mathcal{R}_{\mathsf{I}}/\mathcal{E}}$, we know that $\mathsf{profile}([t]) \in D$ and consequently $D = D_{\mathcal{A}}$. Additionally, for all $[t], [u] \in \mathsf{Can}_{\mathcal{R}_{\mathsf{I}}/\mathcal{E}}$, if $[t] \trianglelefteq_{\mathsf{flat}} [u]$, then we know $\mathsf{profile}([t]) \trianglelefteq \mathsf{profile}([u])$. Consequently, $\trianglelefteq = \trianglelefteq_{\mathcal{A}}$. □

5.1 Free Symbols

For each free symbol $f \in F$, we define a function states_f which computes the states of a term $f(t_1, \ldots, t_n)$ when the states for each term t_i are already known:

Definition 4. *Given a free symbol $f \in F$ with arity n, we define the function $\mathsf{states}_f : \mathcal{P}(Q)^n \to \mathcal{P}(Q)$ such that for $P_1, \ldots, P_n \subseteq Q$, $\mathsf{states}_f(P_1, \ldots, P_n) \subseteq Q$ is the smallest set containing a state $p \in Q$ if either:*

- *$\mathcal{A}$ contains $p(f(x_1, \ldots, x_n)) \Leftarrow p_1(x_1), \ldots, p_n(x_n)$ where $p_i \in P_i$ for $i \in [1, n]$,*
- *or $\mathcal{A}$ contains $p(x_1 \circ x_2) \Leftarrow p_1(x_1), p_2(x_2) \in \mathcal{A}$ with $\circ$ an ACI-symbol in $\mathcal{E}$ and $p_1, p_2 \in \mathsf{states}_f(P_1, \ldots, P_n)$.*

The following lemma relates $\mathsf{states}_{\mathcal{A}/\mathcal{E}}$ and states_f:

Lemma 2. *For each term $t = f(t_1, \ldots, t_n) \in T_F$ with f free in $\mathcal{E}$,*

$$\mathsf{states}_{\mathcal{A}/\mathcal{E}}([t]) = \mathsf{states}_f(\mathsf{states}_{\mathcal{A}/\mathcal{E}}([t_1]), \ldots, \mathsf{states}_{\mathcal{A}/\mathcal{E}}([t_n])).$$

Proof. This lemma is straightforward to show if we first make a few observations. For all $p \in Q$, we know by Theorem 1 that $p \in \mathsf{states}_{\mathcal{A}/\mathcal{E}}([t])$ iff there is a term $u \in [t]$ such that $\mathcal{A} \vdash p(u)$. Since u is equivalent modulo $\mathcal{E}$ to a term whose root symbol is the free symbol f, we know that u can only have two possible forms:

1. $u = f(u_1, \ldots, u_n)$ with $u_i =_{\mathcal{E}} t_i$ for $i \in [1, n]$. In this case, as $\mathcal{A} \vdash p(u)$, $\mathcal{A}$ must contain a clause with the form $p(f(x_1, \ldots, x_n)) \Leftarrow p_1(x_1), \ldots, p_n(x_n)$ where $\mathcal{A} \vdash p_i(u_i)$ for $i \in [1, n]$. Furthermore, for $i \in [1, n]$, $u_i =_{\mathcal{E}} t_i$ and $\mathcal{A} \vdash p_i(u_i)$ implies that $p_i \in \mathsf{states}_{\mathcal{A}/\mathcal{E}}([t_i])$.
2. $u = u_1 \circ u_2$ with $\circ$ is an ACI symbol $u_1 =_{\mathcal{E}} u_2$. In this case, as $\mathcal{A} \vdash p(u)$, $\mathcal{A}$ must contain a clause with the form $p(x_1 \circ x_2) \Leftarrow p_1(x_1), p_2(x_2)$ where $\mathcal{A} \vdash p_1(u_1)$ and $\mathcal{A} \vdash p_2(u_2)$. Furthermore, both u_1 and u_2 are smaller terms equivalent modulo $\mathcal{E}$ to u and t, so $p_1, p_2 \in \mathsf{states}_{\mathcal{A}/\mathcal{E}}([t])$.

For $i \in [1, n]$, let $P_i = \mathsf{states}_{\mathcal{A}/\mathcal{E}}([t_i])$. These two cases mirror the two rules used in the definition of states_f, and so it is straightforward to show that for all $u =_{\mathcal{E}} t$, $\mathcal{A} \vdash p(u) \implies p \in \mathsf{states}_f(P_1, \ldots, P_n)$ by induction on the proof used to show that $\mathcal{A} \vdash p(u)$. It is also straightforward to show that $p \in \mathsf{states}_f(P_1, \ldots, P_n) \implies \mathcal{A}/\mathcal{E} \vdash p(t)$ by induction on the inference steps used to construct $\mathsf{states}_f(P_1, \ldots, P_n)$. □

Given a profile graph $(D, \trianglelefteq) \subseteq (D_{\mathcal{A}}, \trianglelefteq_{\mathcal{A}})$, the following lemma is useful for determining the number of distinct equivalence classes in $\mathsf{Can}_{D,\trianglelefteq}$ with profile $(f, P) \in D_{\mathcal{A}}$ where f is a free symbol.

Lemma 3. *For each profile graph $(D, \trianglelefteq) \subseteq (D_{\mathcal{A}}, \trianglelefteq_{\mathcal{A}})$, and profile $(f, P) \in D$ where f is a free symbol,*

$$\left|\mathsf{profile}^{-1}_{D,\trianglelefteq}(f, P)\right| = \sum_{\substack{(f_1,P_1),\ldots,(f_n,P_n) \in D \\ (\forall i \in [1,n])\ (f_i,P_i) \trianglelefteq (f,P) \\ \mathsf{states}_f(P_1,\ldots,P_n) = P}} \prod_{i=1}^{n} \left|\mathsf{profile}^{-1}_{D,\trianglelefteq}(f_i, P_i)\right|. \tag{2}$$

Note that if $S = \varnothing$, the sum $\sum_{x \in S} f(x) = 0$ while the product $\prod_{x \in S} f(x) = 1$.

Proof. For each $[t] \in \mathsf{profile}^{-1}_{D,\trianglelefteq}(f, P)$, we may assume without loss of generality that t is $\hat{\mathcal{R}}_{\mathsf{I}}/\mathcal{E}_{\mathsf{AC}}$-irreducible. It follows that t has the form $f(t_1, \ldots, t_n)$ with $[t_i] \in \mathsf{Can}_{D,\trianglelefteq}$ for each $i \in [1, n]$. Let $P_i = \mathsf{states}_{\mathcal{A}/\mathcal{E}}([t_i])$ for $i \in [1, n]$. As $[t]$ is in $\mathsf{Can}_{D,\trianglelefteq}$, we know that $P_i \trianglelefteq (f, P)$. By Lemma 2, we know that $\mathsf{states}_{\mathcal{A}/\mathcal{E}}([f(t_1, \ldots, t_n)]) = P$ if and only if $\mathsf{states}_f(P_1, \ldots, P_n) = P$. For each tuple $(d_1, \ldots, d_n) \in D^n$, let $\mathsf{Can}_f(d_1, \ldots, d_n) \subseteq \mathsf{Can}_{D,\trianglelefteq}$ denote the set:

$$\mathsf{Can}_f(d_1, \ldots, d_n) = \{\, [f(t_1, \ldots, t_n)] \in \mathsf{Can}_{D,\trianglelefteq} \mid (\forall i \in [1, n])\, \mathsf{profile}([t_i]) = d_i \,\}.$$

For distinct tuples $\vec{d_1}, \vec{d_2} \in D^n$, it is not difficult to observe that the sets $\mathsf{Can}_f(\vec{d_1})$ and $\mathsf{Can}_f(\vec{d_2})$ are disjoint. From these observations, we can conclude that:

$$|\mathsf{Can}_{D,\trianglelefteq}(f, P)| = \sum_{\substack{(f_1,P_1),\ldots,(f_n,P_n) \in D \\ (\forall i \in [1,n])\ (f_i,P_i) \trianglelefteq (f,P) \\ \mathsf{states}_f(P_1,\ldots,P_n) = P}} |\mathsf{Can}_f((f_1, P_1), \ldots, (f_n, P_n))|. \tag{3}$$

Moreover, for each tuple $(d_1, \ldots, d_n) \in D^n$, it is not difficult to show that

$$|\mathsf{Can}_f(d_1, \ldots, d_n)| = \left|\mathsf{profile}^{-1}_{D,\trianglelefteq}(d_1)\right| \times \cdots \times \left|\mathsf{profile}^{-1}_{D,\trianglelefteq}(d_n)\right|. \tag{4}$$

Equation (2) follows immediately from (3) and (4). □

5.2 AC and ACI Symbols

Similar to [9], we define a context free grammar $G(+)$ for each AC or ACI symbol $+ \in F$. Intuitively, the grammar captures inferences in the automaton $\mathcal{A}$ over *flattened* terms with the form $t_1 + \cdots + t_n$ where $\mathsf{root}(t_i) \neq +$ for $i \in [1, n]$.

Definition 5. *Given an AC or ACI symbol $+ \in F$, we define the context free grammar $G(+)$ with terminals $\Sigma(+) = (F \backslash \{+\}) \times \mathcal{P}(Q)$, non-terminals Q, and production rules*

$$\begin{aligned} G(+) = \{\, p := p_1 p_2 \mid p(x_1 + x_2) \Leftarrow p_1(x_1) \wedge p_2(x_2) \in \mathcal{A} \,\} \\ \cup \{\, p := (f, P) \mid (f, P) \in \Sigma(+) \wedge p \in P \,\}. \end{aligned}$$

For each state $p \in Q$, we let $\mathcal{L}_p(G(+))$ denote the language generated by p using the rules in $G(+)$. For each non-terminal $p \in Q$, a Presburger formula $\psi_{G(+),p}(\vec{x})$ can be constructed with free variables $\vec{x} = \{\, x_d \,\}_{d \in \Sigma(+)}$ whose models $M(\psi_{G(+),p}) \subseteq \mathbb{N}^{\Sigma(+)}$ equal the commutative image of $\mathcal{L}_p(G(+))$ [20], i.e.,

$$M(\psi_{G(+),p}) = \{\, \#(w) \mid w \in \mathcal{L}_p(G(+)) \,\}.$$

where $\# : \Sigma(+)^* \to \mathbb{N}^{\Sigma(+)}$ maps each string to the vector counting the number of occurrences of each letter in the string.

We first show that each parse tree for $G(+)$ corresponds to a proof in $\mathcal{A}/\mathcal{E}$:

Lemma 4. *For each term $t = t_1 + \cdots + t_n \in T_F$ where $n \geq 1$, $+$ an AC or ACI symbol, and $\mathsf{root}(t_i{\downarrow_{\hat{\mathcal{R}}_\mathsf{I}/\mathcal{E}_\mathsf{AC}}}) \neq +$ for $i \in [1, n]$,*

$$\#(\mathsf{profile}([t_1]), \ldots, \mathsf{profile}([t_n])) \in M(\psi_{G(+),p}) \implies \mathcal{A}/\mathcal{E} \vdash p(t).$$

Proof. We show this statement by induction on $n \geq 1$. There are two cases to consider:

- If $n = 1$, $G(+)$ must contain the rule $p := \mathsf{profile}([t_1])$. This implies $p \in \mathsf{states}_{\mathcal{A}/\mathcal{E}}([t_1])$, and thus $\mathcal{A}/\mathcal{E} \vdash p(t_1)$.
- Otherwise $n \geq 2$, and $G(+)$ contains a rule $p := p_1 p_2$ which can be viewed as partitioning $t_1, \ldots, t_n$ into two sequences: (1) a sequence $u_1, \ldots, u_m$ with $1 < m < n$ and $\#(\mathsf{profile}([u_1]), \ldots, \mathsf{profile}([u_m])) \in M(\psi_{G(+),p_1})$ and (2) a sequence $v_1, \ldots, v_{n-m}$ with $\#(\mathsf{profile}([v_1]), \ldots, \mathsf{profile}([v_{n-m}])) \in M(\psi_{G(+),p_2})$. Let $u = u_1 + \cdots + u_m$, and let $v = v_1 + \cdots + v_{n_m}$. As $+$ is associative in $\mathcal{E}$, we know that $t =_{\mathcal{E}} u + v$. As both m and $n - m$ are less than n, by induction we know that $\mathcal{A}/\mathcal{E} \vdash p_1(u)$ and $\mathcal{A}/\mathcal{E} \vdash p_2(v)$. By the definition of $G(+)$, we know that $\mathcal{A}$ contains the clause $p(x_1 + x_2) \Leftarrow p_1(x_1), p_2(x_2)$, and consequently $\mathcal{A}/\mathcal{E} \vdash p(t)$. ☐

Due to the possibility of idempotence equations in $\mathcal{E}$, it is more complex to show how a proof that $\mathcal{A}/\mathcal{E} \vdash p(t)$ corresponds to a parse tree using the production rules in $G(+)$. We first show the following lemma, which relies on our assumption that $\mathcal{A}$ is AC-intersection free.

Lemma 5. *For each AC or ACI symbol* $+ \in F$*, if* $\mathcal{A}/\mathcal{E} \vdash p(t)$*, then there is a term* $t_1 + \cdots + t_n \in [t]_\mathcal{E}$ *such that* $\#(\mathsf{profile}([t_1]), \ldots, \mathsf{profile}([t_n])) \in M(\psi_{G(+),p})$ *and* $\mathsf{root}(t_i) \neq +$ *for* $i \in [1, n]$.

Proof. The term $t_1 + \cdots + t_n \in [t]_\mathcal{E}$ can be inductively constructed from the proof used to show $\mathcal{A}/\mathcal{E} \vdash p(t)$. Equivalence steps in that proof are trivial as the inductive hypothesis immediately implies a suitable term can be constructed. For membership steps, there are three cases to consider:

- If $\mathsf{root}(t{\downarrow}_{\hat{\mathcal{R}}_\mathsf{I}/\mathcal{E}_\mathsf{AC}}) \neq +$, then we use Theorem 1 to find a $u =_\mathcal{E} t$ such that $\mathcal{A} \vdash p(u)$. We know that $u{\downarrow}_{\hat{\mathcal{R}}_\mathsf{I}/\mathcal{E}_\mathsf{AC}} =_{\mathcal{E}_\mathsf{AC}} t{\downarrow}_{\hat{\mathcal{R}}_\mathsf{I}/\mathcal{E}_\mathsf{AC}}$, and thus $\mathsf{root}(u{\downarrow}_{\hat{\mathcal{R}}_\mathsf{I}/\mathcal{E}_\mathsf{AC}}) \neq +$. By definition $p \in \mathsf{states}_{\mathcal{A}/\mathcal{E}}([u])$, and so $G(+)$ must contain the rule $p := \mathsf{profile}([u])$. Consequently, $\#(\mathsf{profile}([u])) \in M(\psi_{G(+),p})$, and thus $u{\downarrow}_{\hat{\mathcal{R}}_\mathsf{I}/\mathcal{E}_\mathsf{AC}} \in [t]_\mathcal{E}$ is exactly the term we are looking for.
- If $\mathsf{root}(t) = +$, then we let $t = u + v$. The membership must have the form:
$$\frac{\mathcal{A}/\mathcal{E} \vdash p_1(u) \qquad \mathcal{A}/\mathcal{E} \vdash p_2(v)}{\mathcal{A}/\mathcal{E} \vdash p(u + v)}.$$
We construct $u_1 + \cdots + u_m \in [u]_\mathcal{E}$ and $v_1 + \cdots + v_n \in [v]_\mathcal{E}$ by induction. Let $\vec{u} = \#(\mathsf{profile}([u_1]), \ldots, \mathsf{profile}([u_m]))$ and $\vec{v} = \#(\mathsf{profile}([v_1]), \ldots, \mathsf{profile}([v_n]))$. We know $\mathcal{A}$ contains the clause $p(x_1 + x_2) \Leftarrow p_1(x_1), p_2(x_2)$, and so $G(+)$ contains the rule $p := p_1 p_2$. By induction, we know that $\vec{u} \in M(\psi_{G(+),p_1})$ and $\vec{v} \in M(\psi_{G(+),p_2})$, and so $\vec{u} + \vec{v} \in M(\psi_{G(+),p})$. It follows that the term $(u_1 + \cdots + u_m) + (v_1 + \cdots + v_n) \in [1]_\mathcal{E}$ satisfies the required conditions.
- Otherwise, we know that $\mathsf{root}(t{\downarrow}_{\hat{\mathcal{R}}_\mathsf{I}/\mathcal{E}_\mathsf{AC}}) = +$ while $\mathsf{root}(t) \neq +$. It follows that t must have the form $t = u \circ v$ for some ACI symbol $\circ$, and the membership must have the form:
$$\frac{\mathcal{A}/\mathcal{E} \vdash p_1(u) \qquad \mathcal{A}/\mathcal{E} \vdash p_2(v)}{\mathcal{A}/\mathcal{E} \vdash p(u \circ v)}$$
where $u =_\mathcal{E} v =_\mathcal{E} t$. We construct $u_1 + \cdots + u_n \in [u]_\mathcal{E}$ by induction. We know that $n \geq 2$, as $u_1 + \cdots + u_n =_\mathcal{E} t$ implies $\mathsf{root}(u_1 + \cdots + u_n{\downarrow}_{\hat{\mathcal{R}}_\mathsf{I}/\mathcal{E}_\mathsf{AC}}) = +$. Let $\vec{u} = \#(\mathsf{profile}([u_1]), \ldots, \mathsf{profile}([u_n]))$. By induction we know that $\vec{u} \in M(\psi_{G(+),p_1})$, and thus $G(+)$ contains a rule of the form $p_1 := q_1 q_2$ as $n \geq 2$. It follows that $\mathcal{A}$ contains the rule $p_1(x_1 + x_2) \Leftarrow q_1(x_1), q_2(x_2)$. As $\mathcal{A}$ is AC-intersection free, it must also contain must contain the rule $p(x_1 + x_2) \Leftarrow q_1(x_1), q_2(x_2)$. Thus $G(+)$ contains the $p := q_1 q_2$, and if we swap this rule in for the rule $p_1 := q_1 q_2$ used to show $\vec{u} \in M(\psi_{G(+),p_1})$, it follows that $\vec{u} \in M(\psi_{G(+),p})$. It follows that the term $u_1 + \cdots + u_1 \in [t]_\mathcal{E}$ satisfies the required conditions. □

For each AC symbol $+$, we can show:

Lemma 6. *For each* $\hat{\mathcal{R}}_\mathsf{I}/\mathcal{E}_\mathsf{AC}$*-irreducible term* $t = t_1 + \cdots + t_n \in T_F$ *where* $+$ *is an AC symbol,* $n \geq 1$*, and* $\mathsf{root}(t_i) \neq +$ *for* $i \in [1, n]$,
$$\mathcal{A}/\mathcal{E} \vdash p(t) \iff \#(\mathsf{profile}([t_1]), \ldots, \mathsf{profile}([t_n])) \in M(\psi_{G(+),p}).$$

Proof. By lemma 4, if $\#(\mathsf{profile}([t_1]), \ldots, \mathsf{profile}([t_n])) \in M(\psi_{G(+),p})$, then $\mathcal{A}/\mathcal{E} \vdash p(t)$. On the other hand, if $\mathcal{A}/\mathcal{E} \vdash p(t)$ then by Lemma 5, there is a term $u_1 + \cdots + u_m =_{\mathcal{E}} t$ such that $\#(\mathsf{profile}([u_1]), \ldots, \mathsf{profile}([u_m])) \in M(\psi_{G(+),p})$ and $\mathsf{root}(u_i) \neq +$ for $i \in [1, m]$. As $t_1 + \cdots + t_n =_{\mathcal{E}} u_1 + \cdots + u_m$, and $+$ only appears in associativity and commutativity equations, it follows that $m = n$ and $\#(\mathsf{profile}([u_1]), \ldots, \mathsf{profile}([u_m])) = \#(\mathsf{profile}([t_1]), \ldots, \mathsf{profile}([t_n]))$. Consequently,

$$\#(\mathsf{profile}([t_1]), \ldots, \mathsf{profile}([t_n])) \in M(\psi_{G(+),p}).$$

□

For each ACI symbol $\circ$, we can show:

Lemma 7. *For each $\hat{\mathcal{R}}_{\mathsf{I}}/\mathcal{E}_{\mathsf{AC}}$-irreducible term $t = t_1 \circ \cdots \circ t_n \in T_F$ where $\circ$ is an ACI symbol, $n \geq 1$, and $\mathsf{root}(t_i) \neq \circ$ for $i \in [1, n]$,*

$$\mathcal{A}/\mathcal{E} \vdash p(t) \iff \#(\mathsf{profile}([t_1]), \ldots, \mathsf{profile}([t_n])) \in M((\exists \vec{y})\, \vec{x} \sqsubseteq \vec{y} \wedge \psi_{G(\circ),p}(\vec{y}))$$

where $\vec{x} \sqsubseteq \vec{y}$ is the formula $\bigwedge_{d \in \Sigma(\circ)} x_d \leq y_d \wedge ((y_d > 0) \Rightarrow (x_d > 0))$.

Proof. We let $\vec{x} = \#(\mathsf{profile}([t_1]), \ldots, \mathsf{profile}([t_n]))$ We first show that if there is a $\vec{y} \in \mathbb{N}^{F \times \mathcal{P}(Q)}$ such that $\vec{x} \sqsubseteq \vec{y}$ and $\psi_{G(\circ),p}(\vec{y})$ holds, then $\mathcal{A}/\mathcal{E} \vdash p(t)$. Since $\vec{x} \sqsubseteq \vec{y}$, we know that for each $d \in \Sigma(\circ)$, if $y_d > x_d$, then $x_d > 0$ and consequently there is a term $t_d \in T_F$ such that $t_d = t_i$ for some $i \in [1, n]$. We let $u \in T_F$, denote the term $u = t \circ u_1 \circ \cdots \circ u_m$ where for each $d \in D$ such that $y_d > x_d$, there are exactly $y_d - x_d$ distinct indices $d(1), \ldots, d(y_d - x_d) \in [1, m]$ such that $u_{d(i)} = t_d$. It is not difficult to show that $u =_{\mathcal{E}} t$ as $\circ$ is ACI, and moreover $\vec{x} + \#(\mathsf{profile}(u_1), \ldots, \mathsf{profile}(u_m)) = \vec{y}$. It follows by Lemma 4 that $\mathcal{A}/\mathcal{E} \vdash p(u)$, and thus $\mathcal{A}/\mathcal{E} \vdash p(t)$.

We now show that if $\mathcal{A}/\mathcal{E} \vdash p(t)$, then there is a vector $\vec{y} \in \mathbb{N}^{\Sigma(\circ)}$ such that $\#(\mathsf{profile}([t_1]), \ldots, \mathsf{profile}([t_n])) \sqsubseteq \vec{y}$ and $\vec{y} \in M(\psi_{G(\circ),p})$. In this case, by Lemma 5, there is a term $u_1 \circ \cdots \circ u_m =_{\mathcal{E}} t$ such that

$$\#(\mathsf{profile}([u_1]), \ldots, \mathsf{profile}([u_m])) \in M(\psi_{G(\circ),p}).$$

and $\mathsf{root}(u_i{\downarrow_{\hat{\mathcal{R}}_{\mathsf{I}}/\mathcal{E}_{\mathsf{AC}}}}) \neq \circ$ for $i \in [1, m]$. As t is $\hat{\mathcal{R}}_{\mathsf{I}}/\mathcal{E}_{\mathsf{AC}}$-irreducible, there must be a surjective function $h : [1, m] \to [1, n]$ such that $u_i =_{\mathcal{E}} t_{h(i)}$ for $i \in [1, m]$. Let $\vec{y} = \#(\mathsf{profile}([u_1]), \ldots, \mathsf{profile}([u_m]))$. For each $d \in D$, the existence of h implies $x_d \leq y_d$ and the surjectivity of h implies that $y_d > 0 \implies x_d > 0$. □

For each profile $(+, P) \in F \times \mathcal{P}(Q)$ with $+$ an AC or ACI symbol, we use $G(+)$ in the definition of the Presburger formula $\psi_{+,P}(\vec{x})$ which identifies terms whose profile is $(+, P)$. For AC symbols $+ \in F$, we let

$$\psi_{+,P}(\vec{x}) = |\vec{x}| \geq 2 \wedge \bigwedge_{p \in P} \psi_{G(+),p}(\vec{x}) \wedge \bigwedge_{p \in Q \setminus P} \neg \psi_{G(+),p}(\vec{x})$$

where $|\vec{x}|$ is the sum of the variables $x_d \in \vec{x}$. For ACI symbols $\circ \in F$, we let

$$\psi_{\circ,P}(\vec{x}) = |\vec{x}| \geq 2 \wedge \bigwedge_{p \in P} (\exists \vec{y})\, \vec{x} \sqsubseteq \vec{y} \wedge \psi_{G(\circ),p}(\vec{y}) \wedge \bigwedge_{p \in Q \setminus P} \neg(\exists \vec{y})\, \vec{x} \sqsubseteq \vec{y} \wedge \psi_{G(\circ),p}(\vec{y})$$

The following lemma describes precisely how the models in $M(\psi_{+,P})$ correspond to $\hat{\mathcal{R}}_{\mathsf{I}}/\mathcal{E}_{\mathsf{AC}}$-irreducible terms with a particular profile.

Lemma 8. *For each $\hat{\mathcal{R}}_{\mathsf{I}}/\mathcal{E}_{\mathsf{AC}}$-irreducible term $t = t_1 + \cdots + t_n \in T_F$ where $+ \in F$ is an AC or ACI symbol, $n \geq 1$, and $\mathsf{root}(t_i) \neq +$ for $i \in [1, n]$,*

$$\mathsf{profile}([t]) = (+, P) \iff \#(\mathsf{profile}([t_1]), \ldots, \mathsf{profile}([t_n])) \in M(\psi_{+,P}).$$

Proof. Since t is $\hat{\mathcal{R}}_{\mathsf{I}}/\mathcal{E}_{\mathsf{AC}}$ irreducible, we know that $\mathsf{profile}([t]) = (+, P)$ iff $n > 2$ and $\mathcal{A}/\mathcal{E} \vdash p(t) \iff p \in P$ for $p \in Q$. Let $\vec{x} = \#(\mathsf{profile}(t_1), \ldots, \mathsf{profile}(t_n))$.

- If $+$ is an AC symbol, then by Lemma 6, $\mathcal{A}/\mathcal{E} \vdash p(t) \iff \vec{x} \in M(\psi_{G(+),p})$. It follows that $p \in P \iff \in M(\psi_{G(+),p})$, and consequently

$$\mathsf{profile}([t]) = (+, P) \iff \vec{x} \in M(\psi_{+,P}).$$

- Otherwise $+$ is an ACI symbol, and by Lemma 7, $\mathcal{A}/\mathcal{E} \vdash p(t)$ iff there is a $\vec{y} \in \mathbb{N}^{\Sigma(+)}$ such that $\vec{x} \sqsubseteq \vec{y}$ and $\vec{y} \in M(\psi_{G(+),p}(\vec{y}))$. It follows that $p \in P \iff \vec{x} \in M((\exists \vec{y})\, \vec{x} \sqsubseteq \vec{y} \wedge \psi_{G(+),p}(\vec{y}))$, and consequently

$$\mathsf{profile}([t]) = (+, P) \iff \vec{x} \in M(\psi_{+,P}).$$

□

We now turn our attention to the problem of counting the number of distinct elements in $\mathsf{Can}_{D,\trianglelefteq}$ with profile $(+, P) \in D_{\mathcal{A}}$ where $+$ is an AC or ACI symbol. For doing this, the classical *choose function* $\mathsf{C} : (\mathbb{N} \cup \{\, \omega \,\}) \times \mathbb{N} \to \mathbb{N} \cup \{\, \omega \,\}$ which has been partially extended to ω becomes quite useful.

$$\mathsf{C}(n, k) = n!/(k!(n-k)!), \qquad \mathsf{C}(\omega, 0) = 1, \qquad \text{and} \qquad \mathsf{C}(\omega, k) = \omega \text{ if } k > 0.$$

For a symbol $\circ \in F$ that is ACI in $\mathcal{E}$ or AC in $\mathcal{E}$ and $\mathcal{R}_{\mathsf{I}}$ contains the rule $x \circ x \to x$ appears in $\mathcal{R}$, each equivalence class $[t_1 \circ \cdots \circ t_n] \in \mathsf{Can}_{\mathcal{R}_{\mathsf{I}}/\mathcal{E}}$ can be viewed as a set $\{\, [t_1], \ldots, [t_n] \,\} \subseteq \mathsf{Can}_{\mathcal{R}_{\mathsf{I}}/\mathcal{E}}$. For these symbols, the following classical result about C becomes quite useful:

Proposition 1. *Given a finite or countably infinite set A and natural number $k \leq |A|$, the total number of distinct subsets of A with size k equals $\mathsf{C}(|A|, k)$.* □

For an AC symbols $+ \in F$ where $\mathcal{R}$ does not contain an idempotence rule, each equivalence class $[t_1 + \cdots + t_n] \in \mathsf{Can}_{\mathcal{R}_{\mathsf{I}}/\mathcal{E}}$ can be viewed as a multiset $\{\!\{\, [t_1], \ldots, [t_n] \,\}\!\} \in \mathbb{N}^{\mathsf{Can}_{\mathcal{R}_{\mathsf{I}}/\mathcal{E}}}$. For these symbols, the following classical result about C becomes quite useful:

Proposition 2. *Given a non-empty finite or countably infinite set A and natural number $k \in \mathbb{N}$, the total number of distinct multisets of A is given by the formula* $\mathsf{C}(|A| + k - 1, k)$. $\square$

For a symbol $\circ$ is idempotent in $\mathcal{E}$ or $\mathcal{R}$, we need the following result about the size of $\mathsf{profile}^{-1}_{D,\trianglelefteq}(\circ, P)$:

Lemma 9. *For each profile graph $(D, \trianglelefteq) \subseteq (D_{\mathcal{A}}, \trianglelefteq_{\mathcal{A}})$, and profile $(\circ, P) \in D$ where $\circ$ is a symbol that is idempotent in $\mathcal{E}$ or $\mathcal{R}_{\mathsf{I}}$,*

$$\left|\mathsf{profile}^{-1}_{D,\trianglelefteq}(\circ, P)\right| = \sum_{\vec{u} \in M(\psi_{\circ,P,\mathsf{Can}_{D,\trianglelefteq}})} \prod_{d \trianglelefteq (\circ,P)} \mathsf{C}(\left|\mathsf{profile}^{-1}_{D,\trianglelefteq}(d)\right|, u_d) \tag{5}$$

where $\psi_{\circ,P,\mathsf{Can}_{D,\trianglelefteq}}(\vec{x}) = \psi_{\circ,P}(\vec{x}) \wedge \bigwedge_{d \,\trianglelefteq\, (\circ,P)} x_d \leq \left|\mathsf{profile}^{-1}_{D,\trianglelefteq}(d)\right| \wedge \bigwedge_{d \,\ntrianglelefteq\, (\circ,P)} x_d = 0.$

Proof. For each $\vec{v} \in \mathbb{N}^{\Sigma(\circ)}$, let $\mathsf{profile}^{-1}_{D,\trianglelefteq,\circ}(\vec{v}) \subseteq \mathsf{Can}^*_{D,\trianglelefteq}$ denote the set

$$\begin{aligned}\mathsf{profile}^{-1}_{D,\trianglelefteq,\circ}(\vec{v}) = \{\, \{[t_1], \ldots, [t_n]\} \subseteq \mathsf{Can}_{D,\trianglelefteq} \mid (\forall i, j \in [1, n])\, i \neq j \implies t_i \neq_{\mathcal{E}} t_j \\ \wedge\ \#(\mathsf{profile}([t_1]), \ldots, \mathsf{profile}([t_n])) = \vec{v} \,\}.\end{aligned}$$

For each $\hat{\mathcal{R}}_{\mathsf{I}}/\mathcal{E}_{\mathsf{AC}}$-irreducible term $t \in T_F$ such that $[t]_{\mathcal{E}} \in \mathsf{profile}^{-1}_{D,\trianglelefteq}(\circ, P)$, we know that t must have the form $t = t_1 \circ \cdots \circ t_n$ where $n \geq 2$. Moreover, we can assume that each term t_i is distinct with $\mathsf{root}(t_i) \neq \circ$ and $[t_i]_{\mathcal{E}} \in \mathsf{Can}_{D,\trianglelefteq}$ for $i \in [1, n]$. Let $\vec{x} = \#(\mathsf{profile}([t_1]), \ldots, \mathsf{profile}([t_n]))$. By Lemma 8, we know that $\vec{t} \in M(\psi_{\circ,P})$. As $[t_i] \in \mathsf{Can}_{D,\trianglelefteq}$ for $i \in [1, n]$, we know that $x_d \leq \left|\mathsf{profile}^{-1}_{D,\trianglelefteq}(d)\right|$ for $d \in D$. It follows that $\vec{x} \in M(\psi_{\circ,P})$. For distinct vectors $\vec{u}, \vec{v} \in \mathbb{N}^{\Sigma(\circ)}$, we know that $\mathsf{profile}^{-1}_{D,\trianglelefteq,\circ}(\vec{u})$ and $\mathsf{profile}^{-1}_{D,\trianglelefteq,\circ}(\vec{v})$ are disjoint sets, and consequently,

$$\left|\mathsf{profile}^{-1}_{D,\trianglelefteq}(\circ, P)\right| = \sum_{\vec{u} \in M(\psi_{\circ,P,\mathsf{Can}_{D,\trianglelefteq}})} \left|\mathsf{profile}^{-1}_{D,\trianglelefteq,\circ}(\vec{u})\right|. \tag{6}$$

Moreover, if we partition equivalence classes in each set in $\mathsf{profile}^{-1}_{D,\trianglelefteq,\circ}(\vec{u})$ by their profile, it can be observed that:

$$\left|\mathsf{profile}^{-1}_{D,\trianglelefteq,\circ}(\vec{u})\right| = \prod_{d \trianglelefteq (\circ,P)} \left|\{\, P \subseteq \mathsf{profile}^{-1}_{D,\trianglelefteq}(d) \mid |P| = u_d \,\}\right|. \tag{7}$$

Finally, by Prop. 1, it follows that for each $k \leq \left|\mathsf{profile}^{-1}_{D,\trianglelefteq}(d)\right|$,

$$\left|\{\, P \subseteq \mathsf{profile}^{-1}_{D,\trianglelefteq}(d) \mid |P| = k \,\}\right| = \mathsf{C}(\left|\mathsf{profile}^{-1}_{D,\trianglelefteq}(d)\right|, k). \tag{8}$$

Equation (5) follows immediately from (6), (7), and (8). $\square$

For an AC symbol $+$ is not idempotent in $\mathcal{R}$, we need the following result about the size of $\mathsf{profile}^{-1}_{D,\trianglelefteq}(+, P)$:

Lemma 10. *For each profile graph* $(D, \trianglelefteq) \subseteq (D_{\mathcal{A}}, \trianglelefteq_{\mathcal{A}})$*, and profile* $(+, P) \in D$ *where* $+$ *is an AC symbol in* $\mathcal{E}$ *that is not idempotent in* $\mathcal{R}_{\mathsf{I}}$*,*

$$\left|\mathsf{profile}^{-1}_{D,\trianglelefteq}(+, P)\right| = \sum_{\vec{u} \in M(\psi_{+,P,\mathsf{Can}_{D,\trianglelefteq}})} \prod_{\substack{d \trianglelefteq (+,P) \\ \left|\mathsf{profile}^{-1}_{D,\trianglelefteq}(d)\right| > 0}} \mathsf{C}(\left|\mathsf{profile}^{-1}_{D,\trianglelefteq}(d)\right| + u_d - 1, u_d) \quad (9)$$

where $\psi_{+,P,\mathsf{Can}_{D,\trianglelefteq}}(\vec{x}) = \psi_{+,P}(\vec{x}) \wedge \bigwedge_{\substack{d \trianglelefteq (+,P) \\ \left|\mathsf{profile}^{-1}_{D,\trianglelefteq}(d)\right| = 0}} x_d = 0 \bigwedge_{d \not\trianglelefteq (+,P)} x_d = 0.$

Proof. For each $\vec{v} \in \mathbb{N}^{\Sigma(+)}$, let $\mathsf{profile}^{-1}_{D,\trianglelefteq,+}(\vec{v}) \subseteq \mathsf{Can}^{*}_{D,\trianglelefteq}$ denote the set

$$\{\, \{\!\{[t_1], \ldots, [t_n]\}\!\} \in \mathbb{N}^{\mathsf{Can}_{D,\trianglelefteq}} \mid \#(\mathsf{profile}([t_1]), \ldots, \mathsf{profile}([t_n])) = \vec{v} \,\}.$$

For each $[t]_{\mathcal{E}} \in \mathsf{profile}^{-1}_{D,\trianglelefteq}(+, P)$, we can assume that t is $\hat{\mathcal{R}}_{\mathsf{I}}/\mathcal{E}_{\mathsf{AC}}$-irreducible and $t = t_1 + \cdots + t_n$ where $n \geq 2$, $\mathsf{root}(t_i) \neq +$, and $[t_i]_{\mathcal{E}} \in \mathsf{Can}_{D,\trianglelefteq}$ for $i \in [1, n]$. Let $\vec{t} = \#(\mathsf{profile}([t_1]), \ldots, \mathsf{profile}([t_n]))$. By Lemma 8, we know that $\vec{t} \in M(\psi_{+,P})$. By the definition of $\mathsf{Can}_{D,\trianglelefteq}$ we know that for $i \in [1, n]$, $\mathsf{profile}([t_i]) \trianglelefteq (+, P)$. For $i \in [1, n]$, if t_i has a profile d, then we know that $t_i \in \mathsf{profile}^{-1}_{D,\trianglelefteq,+}(d)$ and consequently $\left|\mathsf{profile}^{-1}_{D,\trianglelefteq,+}(d)\right| > 0$. It follows that $\vec{t} \in M(\psi_{+,P})$. For distinct vectors $\vec{u}, \vec{v} \in \mathbb{N}^{\Sigma(+)}$, we know that $\mathsf{profile}^{-1}_{D,\trianglelefteq}(\vec{u})$ and $\mathsf{profile}^{-1}_{D,\trianglelefteq}(\vec{v})$ are disjoint sets. By putting the last two observations together, we can conclude that:

$$\left|\mathsf{profile}^{-1}_{D,\trianglelefteq}(+, P)\right| = \sum_{\vec{u} \in M(\psi_{+,P,\mathsf{Can}_{D,\trianglelefteq}})} \left|\mathsf{profile}^{-1}_{D,\trianglelefteq}(\vec{u})\right|. \quad (10)$$

Moreover, if we partition the elements of each multiset in $\mathsf{profile}^{-1}_{D,\trianglelefteq}(\vec{u})$ by their profile d, it is not difficult to show that

$$\left|\mathsf{profile}^{-1}_{D,\trianglelefteq}(\vec{u})\right| = \prod_{d \trianglelefteq (+,P)} \left|\{\, \vec{x} \in \mathbb{N}^{\mathsf{profile}^{-1}_{D,\trianglelefteq}(d)} \mid |\vec{x}| = u_d \,\}\right|. \quad (11)$$

Finally, by Prop. 2, it follows that for each $k \in N$ and $d \in D$ where $\mathsf{profile}^{-1}_{D,\trianglelefteq}(d)$ is non-empty,

$$\left|\{\, \vec{x} \in \mathbb{N}^{\mathsf{profile}^{-1}_{D,\trianglelefteq}(d)} \mid |\vec{x}| = k \,\}\right| = \mathsf{C}(\left|\mathsf{profile}^{-1}_{D,\trianglelefteq}(d)\right| + k - 1, k). \quad (12)$$

Equation (9) follows immediately from (10), (11), and (12). $\square$

5.3 Computing the Size of a Language

We next present a function $\mathsf{cnt}_{D,\trianglelefteq} : D \to \mathbb{N} \cup \{\, \omega \,\}$ which for each graph $(D, \trianglelefteq) \subseteq (D_{\mathcal{A}}, \trianglelefteq_{\mathcal{A}})$ and profile $d \in D$, returns an estimate of the number of elements in $\mathsf{Can}_{\mathcal{R}_{\mathsf{I}}/\mathcal{E}}$ with the profile d. We show below that for each $d \in D$,

$$\left|\mathsf{profile}^{-1}_{D,\trianglelefteq}(d)\right| \leq \mathsf{cnt}_{D,\trianglelefteq}(d) \leq \left|\mathsf{profile}^{-1}_{D_{\mathcal{A}},\trianglelefteq_{\mathcal{A}}}(d)\right|. \quad (13)$$

Before showing this, we first must define $\mathsf{cnt}_{D,\trianglelefteq}$.

Definition 6. *For each* $(D, \trianglelefteq) \subseteq (D_{\mathcal{A}}, \trianglelefteq_{\mathcal{A}})$*, let* $\mathsf{cnt}_{D,\trianglelefteq} : D \to \mathbb{N} \cup \{\, \omega \,\}$ *be the function such that* $\mathsf{cnt}_{D,\trianglelefteq}(d) = \omega$ *if* $d \trianglelefteq^{+} d$ *and otherwise*

- *For each constant* $c \in F$*,* $\mathsf{cnt}_{D,\trianglelefteq}(c, P) = 1$*.*
- *For each free symbol* $f \in F$ *with arity* $n > 0$*,*

$$\mathsf{cnt}_{D,\trianglelefteq}(f, P) = \sum_{\substack{(f_1,P_1),\ldots,(f_n,P_n) \,\in\, D \\ (\forall i \in [1,n])\ (f_i,P_i) \,\trianglelefteq\, (f,P) \\ \mathsf{states}_f(P_1,\ldots,P_n) = P}} \ \prod_{i=1}^{n} \mathsf{cnt}_{D,\trianglelefteq}(f_i, P_i).$$

- *For each symbol* $\circ \in F$ *that is idempotent in* $\mathcal{E}$ *or* $\mathcal{R}$*,* $\mathsf{cnt}_{D,\trianglelefteq}(\circ, P) = \omega$ *if* $|M(\psi_{\circ,P,\mathsf{I}})| = \omega$*, and otherwise,*

$$\mathsf{cnt}_{D,\trianglelefteq}(\circ, P) = \sum_{\vec{u} \,\in\, M(\psi_{\circ,P,\mathsf{I}})} \ \prod_{d \trianglelefteq (\circ,P)} \mathsf{C}(\mathsf{cnt}_{D,\trianglelefteq}(d), u_d)$$

where $\psi_{\circ,P,\mathsf{I}}(\vec{x}) = \psi_{\circ,P}(\vec{x}) \wedge \bigwedge_{d \,\trianglelefteq\, (\circ,P)} x_d \leq \mathsf{cnt}_{D,\trianglelefteq}(d) \wedge \bigwedge_{d \,\not\trianglelefteq\, (\circ,P)} x_d = 0.$

- *For each AC symbol* $+ \in F$ *that is not idempotent in* $\mathcal{E}$ *or* $\mathcal{R}$*,* $\mathsf{cnt}_{D,\trianglelefteq}(+, P) = \omega$ *if* $|M(\psi_{+,P,\mathsf{AC}})| = \omega$*, and otherwise,*

$$\mathsf{cnt}_{D,\trianglelefteq}(+, P) = \sum_{\vec{u} \,\in\, M(\psi_{+,P,\mathsf{AC}})} \ \prod_{\substack{d \trianglelefteq (+,P) \\ \mathsf{cnt}_{D,\trianglelefteq}(d) > 0}} \mathsf{C}(\mathsf{cnt}_{D,\trianglelefteq}(d) + u_d - 1, u_d)$$

where $\psi_{+,P,\mathsf{AC}}(\vec{x}) = \psi_{+,P}(\vec{x}) \wedge \bigwedge_{\substack{d \,\trianglelefteq\, (+,P) \\ \mathsf{cnt}_{D,\trianglelefteq}(d) = 0}} x_d = 0 \wedge \bigwedge_{d \,\not\trianglelefteq\, (+,P)} x_d = 0.$

For proving the computability and correctness of $\mathsf{cnt}_{D,\trianglelefteq}$, we define the well-founded ordering $\lhd \subseteq \trianglelefteq$ as follows:

$$d_1 \lhd d_2 \iff d_1 \trianglelefteq d_2 \wedge \neg(d_2 \trianglelefteq^{+} d_2).$$

It can be easily shown that $\lhd^{+}$ is well-founded. As D is finite, if $\lhd^{+}$ were not well-founded, then there must be profiles $d_1, d_2 \in D$ such that $d_1 \lhd^{+} d_2 \lhd^{+} d_1$. This leads to a contradiction, as $\lhd \subseteq \trianglelefteq$ and $d_1 \lhd^{+} d_1$ implies that for all $d \in D$, $d \not\trianglelefteq d_1$.

Lemma 11. *The function* $\mathsf{cnt}_{D,\trianglelefteq}$ *is computable.*

Proof. To show this, observe that if in evaluating $\mathsf{cnt}_{D,\trianglelefteq}(d)$, we recursively call $\mathsf{cnt}_{D,\trianglelefteq}(d')$ for some $d' \in D$, then $d' \lhd d$. Since $\lhd$ is well-founded, it follows that the chain of recursive calls is finite. Most of the other operations are straightforward to implement. For representing elements of $\mathbb{N} \cup \{\, \omega \,\}$, an abstract data type should be used that can represent any natural number as well as the constant ω. Each of the formulas ψ appearing an expression $M(\psi)$ are formulas in Presburger arithmetic, and thus $M(\psi)$ is effectively a semilinear set [6]. It follows that one can easily decide whether $|M(\psi)| = \omega$ and enumerate the vectors if $M(\psi)$ is finite. $\square$

Before we can prove the claim made in equation (13), we need to show how the edge relation $\trianglelefteq_{\mathcal{A}}$ can be used to detect when $\mathsf{Can}_{\mathcal{R}_\mathsf{I}/\mathcal{E}}$ contains an infinite number of equivalence classes with a given profile. To show this, we first define the *size* of a term $t \in T_F$, denoted $\mathsf{size}(t)$ to be the number of symbols in t. Since the associativity and commutativity equations in $\mathcal{E}_{\mathsf{AC}}$ preserve the size of a term, one can observe that if $t =_{\mathcal{E}_{\mathsf{AC}}} u$, then $\mathsf{size}(t) = \mathsf{size}(u)$.

Lemma 12. *If $d_1 \trianglelefteq^+_{\mathcal{A}} d_2$ for $d_1, d_2 \in D_{\mathcal{A}}$, then for all $[t_1] \in \mathsf{Can}_{\mathcal{R}_\mathsf{I}/\mathcal{E}}$ such that $\mathsf{profile}([t_1]_{\mathcal{E}}) = d_1$, there $\exists [t_2] \in \mathsf{Can}_{\mathcal{R}_\mathsf{I}/\mathcal{E}}$ such that $\mathsf{profile}([t_2]_{\mathcal{E}}) = d_2$ and $\mathsf{size}(t_2{\downarrow_{\hat{\mathcal{R}}_\mathsf{I}/\mathcal{E}_{\mathsf{AC}}}}) > \mathsf{size}(t_1{\downarrow_{\hat{\mathcal{R}}_\mathsf{I}/\mathcal{E}_{\mathsf{AC}}}})$.*

Proof. We prove this by induction on the length of the chain of inferences used to show $d_1 \trianglelefteq^+_{\mathcal{A}} d_2$.

The inductive case is easier, and so we prove it first. In this case, we know there is a profile $d' \in D_{\mathcal{A}}$ such that $d_1 \trianglelefteq^+_{\mathcal{A}} d' \trianglelefteq^+_{\mathcal{A}} d_2$. By our first induction hypothesis we know that there is an equivalence class $[t'] \in \mathsf{Can}_{\hat{\mathcal{R}}_\mathsf{I}/\mathcal{E}_{\mathsf{AC}}}$ such that $\mathsf{profile}([t']_{\mathcal{E}}) = d$ and $\mathsf{size}(t_1) < \mathsf{size}(t')$. Our second induction hypothesis then implies the existence of $[t_2] \in \mathsf{Can}_{\hat{\mathcal{R}}_\mathsf{I}/\mathcal{E}_{\mathsf{AC}}}$ such that $\mathsf{profile}([t_2]_{\mathcal{E}}) = d_2$ and $\mathsf{size}(t_1) < \mathsf{size}(t') < \mathsf{size}(t_2)$.

In the base case, we know that $d_1 \trianglelefteq_{\mathcal{A}} d_2$. By the definition of $\trianglelefteq_{\mathcal{A}}$, there must be equivalence classes $[u], [v] \in \mathsf{Can}_{\mathcal{R}_\mathsf{I}/\mathcal{E}}$ such that $\mathsf{profile}([u]) = d_1$, $\mathsf{profile}([v]) = d_2$, and $[u] \trianglelefteq_{\mathsf{flat}} [v]$. Assuming $\mathsf{profile}([t_1]) = d_1$, we construct t_2 by analyzing why $[u] \trianglelefteq_{\mathsf{flat}} [v]$. There are two cases to consider:

– If $v{\downarrow_{\hat{\mathcal{R}}_\mathsf{I}/\mathcal{E}_{\mathsf{AC}}}} =_{\mathcal{E}_{\mathsf{AC}}} f(v_1, \ldots, v_n)$ where f is a free symbol and $u =_{\mathcal{E}_{\mathsf{AC}}} v_i$ for some $i \in [1, n]$, then we let

$$t_2 = f(v_1, \ldots, v_{i-1}, t_1, v_{i+1}, \ldots, v_n).$$

Clearly, $[t_1] \trianglelefteq_{\mathsf{flat}} [t_2]$. We know that $\mathsf{states}_{\mathcal{A}/\mathcal{E}}([t_1]) = \mathsf{states}_{\mathcal{A}/\mathcal{E}}([u])$, and consequently $\mathsf{states}_{\mathcal{A}/\mathcal{E}}([t_2]) = \mathsf{states}_{\mathcal{A}/\mathcal{E}}([v])$ by Lemma 2. As $\mathsf{profile}([v]) = d_2 = (f, \mathsf{states}_{\mathcal{A}/\mathcal{E}}([v]))$, it follows that $\mathsf{profile}([t_2]) = d_2$. Finally,

$$\mathsf{size}(t_2{\downarrow_{\hat{\mathcal{R}}_\mathsf{I}/\mathcal{E}_{\mathsf{AC}}}}) = 1 + \sum_{j \in [1,n] \setminus \{i\}} \mathsf{size}(v_j) + \mathsf{size}(t_1{\downarrow_{\hat{\mathcal{R}}_\mathsf{I}/\mathcal{E}_{\mathsf{AC}}}}),$$

and thus $\mathsf{size}(t_2{\downarrow_{\hat{\mathcal{R}}_\mathsf{I}/\mathcal{E}_{\mathsf{AC}}}}) > \mathsf{size}(t_1{\downarrow_{\hat{\mathcal{R}}_\mathsf{I}/\mathcal{E}_{\mathsf{AC}}}})$.

– Otherwise, $v{\downarrow_{\hat{\mathcal{R}}_\mathsf{I}/\mathcal{E}_{\mathsf{AC}}}} =_{\mathcal{E}_{\mathsf{AC}}} v_1 + \cdots + v_n$ with $+$ an AC or ACI symbol, $n \geq 2$, $\mathsf{root}(v_i) \neq +$ for all $i \in [1, n]$, and $u =_{\mathcal{E}_{\mathsf{AC}}} v_i$ for some $i \in [1, n]$. If $t_1{\downarrow_{\hat{\mathcal{R}}_\mathsf{I}/\mathcal{E}_{\mathsf{AC}}}} =_{\mathcal{E}_{\mathsf{AC}}} v_j$ for some $j \in [1, n]$, then we let $t_2 = u$ and it trivially follows that $\mathsf{profile}([t_2]) = d_2$ and $\mathsf{size}(t_2{\downarrow_{\hat{\mathcal{R}}_\mathsf{I}/\mathcal{E}_{\mathsf{AC}}}}) > \mathsf{size}(t_1{\downarrow_{\hat{\mathcal{R}}_\mathsf{I}/\mathcal{E}_{\mathsf{AC}}}})$. Otherwise, we let

$$t_2 = v_1 + \cdots + v_{i-1} + t_1{\downarrow_{\hat{\mathcal{R}}_\mathsf{I}/\mathcal{E}_{\mathsf{AC}}}} + v_{i+1} + \cdots + v_n.$$

We know that $t_1{\downarrow_{\hat{\mathcal{R}}_\mathsf{I}/\mathcal{E}_{\mathsf{AC}}}} \neq_{\mathcal{E}_{\mathsf{AC}}} v_i$ for all $i \in [1, n]$, and thus t_2 is $\hat{\mathcal{R}}_\mathsf{I}/\mathcal{E}_{\mathsf{AC}}$-irreducible. It follows that $\mathsf{size}(t_2{\downarrow_{\hat{\mathcal{R}}_\mathsf{I}/\mathcal{E}_{\mathsf{AC}}}}) > \mathsf{size}(t_1{\downarrow_{\hat{\mathcal{R}}_\mathsf{I}/\mathcal{E}_{\mathsf{AC}}}})$. Since $n > 2$, $\mathsf{root}(u) = +$, and thus $\mathsf{profile}([u]) = d_1 = (+, P)$ for some $P \subseteq Q$. It follows by Lemma 8 that $\#(\mathsf{profile}([u_1]), \ldots, \mathsf{profile}([u_n])) \in M(\psi_{+,P})$. As $\mathsf{profile}([t_1]) = \mathsf{profile}([u_1])$, it follows $\mathsf{profile}([t_2]) = d_2$. $\square$

We can use the previous lemma to make the following observation:

Corollary 2. *For all $d \in D_{\mathcal{A}}$, if $d \trianglelefteq_{\mathcal{A}}^{+} d$, then* $\left|\mathsf{profile}^{-1}_{D_{\mathcal{A}},\trianglelefteq_{\mathcal{A}}}(d)\right| = \omega$.

Proof. For all $d \in D_{\mathcal{A}}$, there is a $[t] \in \mathsf{Can}_{\mathcal{R}_\mathsf{I}/\mathcal{E}}$ such that $\mathsf{profile}([t]) = d$. If $d \trianglelefteq_{\mathcal{A}}^{+} d$, then we can use Lemma 12 to construct an infinite sequence $[t_1], [t_2], \dots \in \mathsf{Can}_{\mathcal{R}_\mathsf{I}/\mathcal{E}}$ of equivalence classes each with profile d and where $\mathsf{size}(t_i{\downarrow}_{\hat{\mathcal{R}}_\mathsf{I}/\mathcal{E}_\mathsf{AC}}) < \mathsf{size}(t_j{\downarrow}_{\hat{\mathcal{R}}_\mathsf{I}/\mathcal{E}_\mathsf{AC}})$ for $i < j$. It follows that for all distinct $i, j \in \mathbb{N}$, $t_i{\downarrow}_{\hat{\mathcal{R}}_\mathsf{I}/\mathcal{E}_\mathsf{AC}} \neq_{\mathcal{E}_\mathsf{AC}} t_j{\downarrow}_{\hat{\mathcal{R}}_\mathsf{I}/\mathcal{E}_\mathsf{AC}}$, and thus are also distinct modulo $\mathcal{E}$. Consequently, $\mathsf{Can}_{\mathcal{R}_\mathsf{I}/\mathcal{E}}$ contains an infinite number of equivalence classes with profile d. □

We are now ready to prove our previous claim in equation (13).

Lemma 13. *For all profile graphs $(D, \trianglelefteq) \subseteq (D_{\mathcal{A}}, \trianglelefteq_{\mathcal{A}})$ and $d \in D$,*

$$\left|\mathsf{profile}^{-1}_{D,\trianglelefteq}(d)\right| \leq \mathsf{cnt}_{D,\trianglelefteq}(d) \leq \left|\mathsf{profile}^{-1}_{D_{\mathcal{A}},\trianglelefteq_{\mathcal{A}}}(d)\right|. \tag{13}$$

Proof. We prove (13) for all $d \in D$ by induction on d with respect to the well-founded relation $\lhd$. In our inductive proof, there are four cases to consider:

- If $d \trianglelefteq^{+} d$, then $\mathsf{cnt}_{D,\trianglelefteq}(d) = \omega$, and thus $\left|\mathsf{profile}^{-1}_{D,\trianglelefteq}(d)\right| \leq \mathsf{cnt}_{D,\trianglelefteq}(d)$. On the other hand, as $\trianglelefteq$ is a subset of $\trianglelefteq_{\mathcal{A}}$, we know that $d \trianglelefteq_{\mathcal{A}}^{+} d$. By Cor. 2, it follows that $\left|\mathsf{profile}^{-1}_{D_{\mathcal{A}},\trianglelefteq_{\mathcal{A}}}(d)\right| = \omega$.
- If $d = (c, P)$ with c a constant, then because $D \subseteq D_{\mathcal{A}}$, we know there is an equivalence class $[t] \in \mathsf{Can}_{\mathcal{R}_\mathsf{I}/\mathcal{E}}$ such that $\mathsf{root}(t{\downarrow}_{\hat{\mathcal{R}}_\mathsf{I}/\mathcal{E}_\mathsf{AC}}) = c$ and $\mathsf{states}_{\mathcal{A}/\mathcal{E}}([t]) = P$. However, $\mathsf{root}(t{\downarrow}_{\hat{\mathcal{R}}_\mathsf{I}/\mathcal{E}_\mathsf{AC}}) = c$ implies that $t =_{\mathcal{E}} c$. As there is only one equivalence class containing c, it follows that $\left|\mathsf{profile}^{-1}_{D,\trianglelefteq}(d)\right| = 1$ and $\left|\mathsf{profile}^{-1}_{D_{\mathcal{A}},\trianglelefteq_{\mathcal{A}}}(d)\right| = 1$.
- If $d = (f, P)$ with f a free symbol with arity $n > 0$, then by using Lemma 3 with both $(D, \trianglelefteq)$ and $(D_{\mathcal{A}}, \trianglelefteq_{\mathcal{A}})$, we can reduce (13) for d to the problem of showing (13) for all $d' \trianglelefteq d$. However, this follows trivially by our induction hypothesis as $d \not\trianglelefteq^{+} d$ and $d' \trianglelefteq d$ implies $d' \lhd d$.
- If $d = (\circ, P)$ with $\circ \in F$ idempotent in $\mathcal{E}$ or $\mathcal{R}$, then we first note that for all $d' \in D$, $d' \trianglelefteq d \implies d' \lhd d$ as $d \not\trianglelefteq^{+} d$. It follows that we may assume that equation (13) holds for each $d' \trianglelefteq d$. This implies that $M(\psi_{\circ,P,\mathsf{Can}_{D,\trianglelefteq}}) \subseteq M(\psi_{\circ,P,\mathsf{I}}) \subseteq M(\psi_{\circ,P,\mathsf{Can}_{D_{\mathcal{A}},\trianglelefteq_{\mathcal{A}}}})$, and consequently by using Lemma 9, we can reduce the problem of showing (13) for all $d \in D$ to two problems: (1) for all $d' \trianglelefteq d$ and $k \leq \left|\mathsf{profile}^{-1}_{D,\trianglelefteq}(d')\right|$,

  $$\mathsf{C}(\left|\mathsf{profile}^{-1}_{D,\trianglelefteq}(d')\right|, k) \leq \mathsf{C}(\mathsf{cnt}_{D,\trianglelefteq}(d'), k),$$

  and (2) for all for all $d' \trianglelefteq d$ and $k \leq \mathsf{cnt}_{D,\trianglelefteq}(d')$,

  $$\mathsf{C}(\mathsf{cnt}_{D,\trianglelefteq}(d'), k) \leq \mathsf{C}(\left|\mathsf{profile}^{-1}_{D_{\mathcal{A}},\trianglelefteq_{\mathcal{A}}}(d')\right|, k).$$

  Both of these problems follow easily from our induction hypothesis and the definition of C.

Starting with the empty graph $(D_0, \trianglelefteq_0) = (\varnothing, \varnothing)$, we freely apply either of the rules below to construct $(D_{i+1}, \trianglelefteq_{i+1})$ from $(D_i, \trianglelefteq_i)$ subject to the condition that a rule may only be applied if $(D_{i+1}, \trianglelefteq_{i+1}) \neq (D_i, \trianglelefteq_i)$. The rules are applied until completion to obtain the graph $(D_*, \trianglelefteq_*)$.

$$\frac{\text{choose free symbol } f \in F \text{ and } (f_1, P_1), \ldots, (f_n, P_n) \in D_i}{\begin{array}{l} D_{i+1} := D_i \cup \{ (f, \mathsf{states}_f(P_1, \ldots, P_n)) \} \\ \trianglelefteq_{i+1} := \trianglelefteq_i \cup \{ ((f_j, P_j), (f, \mathsf{states}_f(P_1, \ldots, P_n))) \mid j \in [1, n] \} \end{array}}$$

$$\frac{\text{choose AC or ACI symbol } + \in F \text{ and } P \subseteq Q \text{ s.t. } (\exists \vec{x})\, \psi_{+,P,D_i,\trianglelefteq_i}(\vec{x})}{\begin{array}{l} D_{i+1} := D_i \cup \{ (f, P) \} \\ \trianglelefteq_{i+1} := \trianglelefteq_i \cup \{ (d, (f, P)) \mid d \in D_i \wedge (\exists \vec{x})\, \psi_{+,P,D_i,\trianglelefteq_i}(\vec{x}) \wedge x_d > 0 \} \end{array}}$$

where if $+$ is idempotent in $\mathcal{E}$ or $\mathcal{R}$, then

$$\psi_{+,P,D_i,\trianglelefteq_i}(\vec{x}) = \psi_{+,P}(\vec{x}) \wedge \bigwedge_{d \in \Sigma(+) \cap D_i} x_d \leq \mathsf{cnt}_{D_i,\trianglelefteq_i}(d) \wedge \bigwedge_{d \in \Sigma(+) \setminus D_i} x_d = 0.$$

and if $+$ is not idempotent in $\mathcal{E}$ or $\mathcal{R}$, then

$$\psi_{+,P,D_i,\trianglelefteq_i}(\vec{x}) = \psi_{+,P}(\vec{x}) \wedge \bigwedge_{\substack{d \in \Sigma(+) \cap D_i \\ \mathsf{cnt}_{D_i,\trianglelefteq_i}(d) = 0}} x_d = 0 \wedge \bigwedge_{d \in \Sigma(+) \setminus D_i} x_d = 0.$$

Fig. 2. Inference System for Constructing $(D_*, \trianglelefteq_*)$

- Otherwise $d = (+, P)$ with $+ \in F$ an AC symbol that is not idempotent in $\mathcal{R}$. We first note that for all $d' \trianglelefteq d$, $d' \triangleleft d$ as $d \not\trianglelefteq^+ d$. It follows that we may assume (13) for each $d' \trianglelefteq d$. This implies that $M(\psi_{+,P,\mathsf{Can}_{D,\trianglelefteq}}) \subseteq M(\psi_{+,P,\mathsf{AC}}) \subseteq M(\psi_{+,P,\mathsf{Can}_{D_\mathcal{A},\trianglelefteq_\mathcal{A}}})$, and consequently by using Lemma 10, we can reduce the problem of showing (13) for all $d \in D$ to two problems: (1) for all $d' \trianglelefteq d$ and $k \in \mathbb{N}$ where $\left|\mathsf{profile}^{-1}_{D,\trianglelefteq}(d')\right| > 0$,

  $$\mathsf{C}(\mathsf{cnt}_{D,\trianglelefteq}(d') + k - 1, k) \leq \mathsf{C}(\left|\mathsf{profile}^{-1}_{D_\mathcal{A},\trianglelefteq_\mathcal{A}}(d')\right| + k - 1, k),$$

  and (2) for all $d' \trianglelefteq d$ and $k \in \mathbb{N}$ where $\mathsf{cnt}_{D,\trianglelefteq}(d') > 0$,

  $$\mathsf{C}(\mathsf{cnt}_{D,\trianglelefteq}(d') + k - 1, k) \leq \mathsf{C}(\left|\mathsf{profile}^{-1}_{D_\mathcal{A},\trianglelefteq_\mathcal{A}}(d')\right| + k - 1, k).$$

  Both of these problems follow easily from our induction hypothesis and the definition of the choose function C. $\square$

5.4 Constructing $(D_\mathcal{A}, \trianglelefteq_\mathcal{A})$

The algorithm for a constructing the profile graph $(D_*, \trianglelefteq_*)$ is given Fig. 2. We show that $(D_*, \trianglelefteq_*) = (D_\mathcal{A}, \trianglelefteq_\mathcal{A})$ in two steps. First, we show that $(D_*, \trianglelefteq_*) \subseteq (D_\mathcal{A}, \trianglelefteq_\mathcal{A})$ by showing that if $(D_i, \trianglelefteq_i) \subseteq (D_\mathcal{A}, \trianglelefteq_\mathcal{A})$, then any graph $(D_{i+1}, \trianglelefteq_{i+1})$ obtained by applying one of the inference rules in Fig. 2 is a subgraph of $(D_\mathcal{A}, \trianglelefteq_\mathcal{A})$. Since the initial graph $(D_0, \trianglelefteq) = (\varnothing, \varnothing) \subseteq (D_\mathcal{A}, \trianglelefteq_\mathcal{A})$, this implies that $(D_*, \trianglelefteq_*) \subseteq (D_\mathcal{A}, \trianglelefteq_\mathcal{A})$. Second, we prove that $\mathsf{Can}_{D_*,\trianglelefteq_*} = \mathsf{Can}_{\mathcal{R}_\mathsf{I}/\mathcal{E}}$, which by Lemma 1 implies that $(D_*, \trianglelefteq_*) = (D_\mathcal{A}, \trianglelefteq_\mathcal{A})$.

The following lemma is essential to showing that $(D_*, \trianglelefteq_*) \subseteq (D_\mathcal{A}, \trianglelefteq_\mathcal{A})$:

Lemma 14. *For all $(D_i, \trianglelefteq_i) \subseteq (D_\mathcal{A}, \trianglelefteq_\mathcal{A})$, if $(D_{i+1}, \trianglelefteq_{i+1})$ is obtained from $(D_i, \trianglelefteq_i)$ by an inference step using the rules in Fig. 2, then $(D_{i+1}, \trianglelefteq_{i+1}) \subseteq (D_\mathcal{A}, \trianglelefteq_\mathcal{A})$.*

Proof. We consider three different cases separately:

- In the first case, suppose $(D_{i+1}, \trianglelefteq_{i+1})$ is obtained by applying the first rule after choosing the free symbol $f \in F$ and profiles $(f_1, P_1), \ldots, (f_n, P_n) \in D_i$. Let $P = \mathsf{states}_f(P_1, \ldots, P_n)$. We must show that $(f, P) \in D_\mathcal{A}$, and $(f_j, P_j) \trianglelefteq_\mathcal{A} (f, P)$ for all $j \in [1, n]$. As $D_i \subseteq D_\mathcal{A}$, we know that for each $j \in [1, n]$, there is an equivalence class $[t_j] \in \mathsf{Can}_{\mathcal{R}_\mathsf{I}/\mathcal{E}}$ such that $\mathsf{profile}([t_j]) = (f_j, P_j)$. Let $t = f(t_1, \ldots, t_n)$. As f is free, we know that $[t] \in \mathsf{Can}_{\mathcal{R}_\mathsf{I}/\mathcal{E}}$, and therefore $\mathsf{profile}([t]) \in D_\mathcal{A}$ Observe that $\mathsf{profile}([t]) = (f, P)$ by Lemma 2, and thus $(f, P) \in D_\mathcal{A}$. For $j \in [1, n]$, observe that $[t_j] \trianglelefteq_\mathsf{flat} [t]$, and thus $(f_j, P_j) \trianglelefteq_\mathcal{A} (f, P)$.
- In the second case, suppose $(D_{i+1}, \trianglelefteq_{i+1})$ is obtained by applying the second rule after choosing the symbol $\circ \in F$ that is idempotent in $\mathcal{E}$ or $\mathcal{R}$ and choosing a set $P \subseteq Q$. It is sufficient to show that $(\circ, P) \in D_\mathcal{A}$, and for each $\vec{x} \in M(\psi_{+,P,D_i,\trianglelefteq_i})$, if $x_d > 0$, then $d \trianglelefteq_\mathcal{A} (\circ, P)$. We know that there is at least one $\vec{x} \in M(\psi_{+,P,D_i,\trianglelefteq_i})$. For each $d \in \Sigma(\circ) \cap D_i$, we know that $x_d \leq \mathsf{cnt}_{D_i,\trianglelefteq_i}(d)$. By Lemma 13, it follows that there are at least x_d distinct equivalence classes $[t_{d(1)}], \ldots, [t_{d(x_d)}] \in \mathsf{profile}^{-1}_{D_\mathcal{A},\trianglelefteq_\mathcal{A}}(D)$. Without loss of generality, we may assume that each term $t_{d(i)}$ is $\hat{\mathcal{R}}_\mathsf{I}/\mathcal{E}_\mathsf{AC}$-irreducible. Let $t = t_1 \circ \cdots \circ t_n$ be a term where each term t_j corresponds to a unique term $t_{d(k)}$ for some $d \in D_i$ and $k \in [1, x_d]$. The term t is $\hat{\mathcal{R}}_\mathsf{I}/\mathcal{E}_\mathsf{AC}$-irreducible, and $\#(\mathsf{profile}([t_1]), \ldots, \mathsf{profile}([t_n])) = \vec{x}$. It follows that $\mathsf{profile}([t]) = (\circ, P)$ by Lemma 8, and so $(\circ, P) \in D_\mathcal{A}$. For each $d \in \Sigma(\circ)$, if $x_d > 0$, then $[t_{d(1)}] \trianglelefteq_\mathsf{flat} [t]$, and consequently $d \trianglelefteq_\mathcal{A} (\circ, P)$.
- In the third case, suppose $(D_{i+1}, \trianglelefteq_{i+1})$ is obtained by applying the second rule after choosing the symbol $+ \in F$ that is AC in $\mathcal{E}$ and not idempotent in $\mathcal{R}$ and choosing a set $P \subseteq Q$. It is enough to show that $(+, P) \in D_\mathcal{A}$, and for each $\vec{x} \in M(\psi_{+,P,D_i,\trianglelefteq_i})$, if $x_d > 0$, then $d \trianglelefteq_\mathcal{A} (+, P)$. We know that there is at least one $\vec{x} \in M(\psi_{+,P,D_i,\trianglelefteq_i})$. For each $d \in \Sigma(+)$, if $x_d > 0$, then we know $d \in D_i$ and $\mathsf{cnt}_{D_i,\trianglelefteq_i}(d) > 0$. It follows by Lemma 13 that there is an equivalence class $[t_d] \in \mathsf{Can}_{D_\mathcal{A},\trianglelefteq_\mathcal{A}}$ with profile d. Without loss of generality, we may assume that t_d is $\hat{\mathcal{R}}_\mathsf{I}/\mathcal{E}_\mathsf{AC}$-irreducible. Let $t = t_1 + \cdots + t_n$ be a term in which for each $d \in \Sigma(+)$, there are exactly x_d distinct indices $d(1), \ldots, d(x_d) \in [1, n]$ such that $t_{d(i)} = t_d$. It follows that $\#(\mathsf{profile}([t_1]), \ldots, \mathsf{profile}([t_n])) = \vec{x}$, and consequently $\mathsf{profile}([t]) = (+, P)$ by Lemma 8, and so $(+, P)$. For each $d \in \Sigma(+)$ if $x_d > 0$, then $[t_d] \trianglelefteq_\mathsf{flat} [t]$, and thus $d \trianglelefteq_\mathcal{A} (+, P)$. $\square$

The previous lemma implies that $(D_n, \trianglelefteq_n) \subseteq (D_\mathcal{A}, \trianglelefteq_\mathcal{A})$ for all $n \in N$. Since $(D_*, \trianglelefteq_*) = (D_n, \trianglelefteq_n)$ for some $n \in \mathbb{N}$, it follows that $(D_*, \trianglelefteq *) \subseteq (D_\mathcal{A}, \trianglelefteq_\mathcal{A}$.

Corollary 3.

$$(D_*, \trianglelefteq_*) \subseteq (D_{\mathcal{A}}, \trianglelefteq_{\mathcal{A}})$$

□

We now show that $(D_*, \trianglelefteq_*)$ can be viewed as having explored all the elements in $\mathsf{Can}_{\mathcal{R}_\mathsf{I}/\mathcal{E}}$.

Lemma 15.

$$\mathsf{Can}_{D_*,\trianglelefteq_*} = \mathsf{Can}_{\mathcal{R}_\mathsf{I}/\mathcal{E}}.$$

Proof. As $\hat{\mathcal{R}}_\mathsf{I}$ is confluent and terminating, it is enough to show by structural induction that for each $\hat{\mathcal{R}}_\mathsf{I}/\mathcal{E}_{\mathsf{AC}}$-irreducible term t, the equivalence class $[t]_\mathcal{E} \in \mathsf{Can}_{D_*,\trianglelefteq_*}$ There are three cases to consider:

- In the first case, suppose $t = f(t_1, \ldots, t_n)$ with f a free symbol. By induction $t_i \in \mathsf{Can}_{D_*,\trianglelefteq_*}$ for $i \in [1, n]$, and consequently $\mathsf{profile}(t_i) \in D_*$. Let $\mathsf{profile}(t_i) = (f_i, P_i)$, and let $\mathsf{states}_{\mathcal{A}/\mathcal{E}}([t]) = P$. By Lemma 2, we know that $\mathsf{states}_f(P_1, \ldots, P_n) = P$. Since the first rule in Fig. 2 can not be applied to generate a larger graph, we know that $\mathsf{profile}([t]) \in D_*$ and for all $i \in [1, n]$, $\mathsf{profile}([t_i]) \trianglelefteq_* \mathsf{profile}([t])$. Consequently, $[t] \in \mathsf{Can}_{D_*,\trianglelefteq_*}$.
- In the second case, suppose $t = t_1 \circ \cdots \circ t_n$ where $\circ$ an symbol that is idempotent in $\mathcal{E}$ or $\mathcal{R}$ and $\mathsf{root}(t_i) \neq \circ$ for $i \in [1, n]$. By induction we know that $t_i \in \mathsf{Can}_{D_*,\trianglelefteq_*}$ for $i \in [1, n]$. Let $P = \mathsf{states}_{\mathcal{A}/\mathcal{E}}([t])$ and let $\vec{x} = \#(\mathsf{profile}([t_1]), \ldots, \mathsf{profile}([t_n]))$. By Lemma 8 we know that $\vec{x} \in M(\psi_{\circ,P})$. As all of the subterms $t_1, \ldots, t_n$ are distinct and also in $\mathsf{Can}_{D_*,\trianglelefteq_*}$, we know by Lemma 13 that $x_d \leq \mathsf{cnt}_{D_i,\trianglelefteq_i}(d)$. It follows that $\vec{x} \in M(\psi_{\circ,P,D_i,\trianglelefteq_i})$, and since the second rule in Fig. 2 cannot be applied to generate a larger graph, we know that $\mathsf{profile}([t]) \in D_*$ and for all $i \in [1, n]$, $\mathsf{profile}([t_i]) \trianglelefteq \mathsf{profile}([t])$. Consequently, $[t] \in \mathsf{Can}_{D_*,\trianglelefteq_*}$.
- In the final case, suppose $t = t_1 + \cdots + t_n$ with $+$ an AC symbol in $\mathcal{E}$ that is not idempotent in $\mathcal{R}$ and $\mathsf{root}(t_i) \neq +$ for $i \in [1, n]$. By induction we know that $t_i \in \mathsf{Can}_{D_*,\trianglelefteq_*}$ for $i \in [1, n]$. Let $P = \mathsf{states}_{\mathcal{A}/\mathcal{E}}([t])$ and let $\vec{x} = \#(\mathsf{profile}([t_1]), \ldots, \mathsf{profile}([t_n]))$. By Lemma 8 we know that $\vec{x} \in M(\psi_{+,P})$. As all of the subterms $t_1, \ldots, t_n$ are in $\mathsf{Can}_{D_*,\trianglelefteq_*}$, we know by Lemma 13 that if $x_d > 0$, then $\mathsf{cnt}_{D_i,\trianglelefteq_i}(d) > 0$. It follows that $\vec{x} \in M(\psi_{+,P,D_i,\trianglelefteq_i})$, and since the second rule in Fig. 2 cannot be applied to generate a larger graph, we know that $\mathsf{profile}([t]) \in D_*$ and for all $i \in [1, n]$, $\mathsf{profile}([t_i]) \trianglelefteq \mathsf{profile}([t])$. Consequently, $[t] \in \mathsf{Can}_{D_*,\trianglelefteq_*}$. □

We are now able to prove the main result of this section.

Theorem 4. *The graph $(D_{\mathcal{A}}, \trianglelefteq_{\mathcal{A}})$ is effectively constructible.*

Proof. We know by Lemma 15 that $\mathsf{Can}_{D_*,\trianglelefteq_*} = \mathsf{Can}_{\mathcal{R}_\mathsf{I}/\mathcal{E}}$. As $(D_*, \trianglelefteq_*)$ is a subgraph of $(D_{\mathcal{A}}, \trianglelefteq_{\mathcal{A}})$ by Cor. 3, it follows by Lemma 1 that $(D_*, \trianglelefteq_*) = (D_{\mathcal{A}}, \trianglelefteq_{\mathcal{A}})$. However, $(D_*, \trianglelefteq_*)$ can be constructed by applying each inference rule in Fig. 2 a finite number of times. It is decidable whether an inference rule can be applied, because each choice ranges over a finite set, the function $\mathsf{cnt}_{D,\trianglelefteq}$ is computable by Lemma 11 and each formula $\psi_{\circ,P,D_i,\trianglelefteq_i}$ is expressible in Presburger arithmetic after the value for $\mathsf{cnt}_{D_i,\trianglelefteq_i}(d)$ has been replaced with its computed numerical value. $\square$

Theorem 3 can be as a corollary of Theorem 4.

Theorem 3. *Let $\mathcal{E} = (F, E)$ be a theory with free, AC, and ACI symbols, and let $\mathcal{R}_\mathsf{I}$ be a rewrite theory where the only axioms are idempotence rules of the form $x + x \to x$ for an AC symbol $+ \in F$.*

If $\mathcal{A}$ is an AC-intersection free $\mathcal{E}$-tree automaton with states Q and ϕ is a propositional formula with atomic predicates Q, it is decidable whether

$$\mathsf{Can}_{\mathcal{R}_\mathsf{I}/\mathcal{E}} \cap \mathcal{L}_\phi(\mathcal{A}/\mathcal{E}) = \varnothing.$$

Proof. By structural induction on ϕ, it is easy to show that

$$\mathcal{L}_\phi(\mathcal{A}/\mathcal{E}) \neq \varnothing \iff (\exists P \subseteq Q)\mathsf{states}_{\mathcal{A}/\mathcal{E}}(P) \models \phi.$$

It follows that

$$\begin{aligned} \mathsf{Can}_{\mathcal{R}_\mathsf{I}/\mathcal{E}} \cap \mathcal{L}_\phi(\mathcal{A}/\mathcal{E}) \neq \varnothing &\iff (\exists [t] \in \mathsf{Can}_{\mathcal{R}_\mathsf{I}/\mathcal{E}})\, \mathsf{states}_{\mathcal{A}/\mathcal{E}}([t]) \models \phi \\ &\iff (\exists (f, P) \in D_\mathcal{A})\, P \models \phi. \end{aligned}$$

Since $D_\mathcal{A}$ is finite and effectively constructible by Theorem 4, it follows that the question of whether $\mathsf{Can}_{\mathcal{R}_\mathsf{I}/\mathcal{E}} \cap \mathcal{L}_\phi(\mathcal{A}/\mathcal{E}) = \varnothing$ is decidable. $\square$

6 Related Work and Conclusions

Our main contributions in this paper are: (1) We showed that every alternating equational tree language can be expressed as the intersection of two regular equational tree languages by adding a fresh ACI symbol to the theory. This implies that intersection-emptiness is undecidable for regular equational tree automata over a theory with an AC and ACI symbol. (2) We studied the issue of modularity in equational tree automata and showed that both intersection-emptiness and propositional emptiness are non-modular properties even for disjoint theories. (3) We presented a subclass of regular equational tree automata over theories with AC and ACI symbols and showed the decidability of propositional emptiness for that subclass. This result further implied that propositional emptiness is decidable for equational tree automata with one ACI symbol and tree automata with normalization over a rewrite theory with idempotence rules and AC symbols.

One of our goals was to obtain decidability results over non-linear theories. In this direction there are numerous papers on extending tree automata techniques to better handle non-linearity in adding constraints to the automata rules [1, Chapter 4] as well as extending that idea to handle some equational theories [11]. The problem of deciding whether a non-equational tree language accepts an irreducible term for any set of linear or non-linear rules was shown in [2], however the approach used here is quite different. The technique of counting the number of distinct terms was influenced by similar issues in deciding the emptiness of multitree automata [14], and our realization that Presburger arithmetic is useful in the ACI case was inspired by the generalization of Parikh's theorem to arbitrary Kleene algebras in [10].

Although we have solved two open problems, our work suggests additional questions that are worth exploring, including: (1) If we impose stronger conditions on the theories such as linearity or collapse-freeness, can we combine disjoint equational theories in a modular way? (2) Can the semi-decision procedure for the associative case in [9] be extended to handle AC-intersection free automata over theories with any combination of associativity, commutativity, and idempotence? (3) Although ground reducibility modulo AC is undecidable in general for non-linear rules [12], what other non-linear rules exist where emptiness is decidable for tree automata with normalization modulo AC?

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