Termination of Just/Fair Computations in Term Rewriting ⋆

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Abstract

The main goal of this paper is to apply rewriting termination technology —enjoying a quite mature set of termination results and tools— to the problem of proving automatically the termination of concurrent systems under fairness assumptions. We adopt the thesis that a concurrent system can be naturally modeled as a rewrite system, and develop a theoretical approach to systematically transform, under reasonable assumptions, fair-termination problems into ordinary termination problems of associated relations, to which standard rewriting termination techniques and tools can be applied. Our theoretical results are combined into a practical *proof method* for proving fair-termination that can be automated and can be supported by current termination tools. We illustrate this proof method with some concrete examples and briefly comment on future extensions.

Key words: Concurrent programming, fairness, term rewriting, program analysis, termination.

1 Introduction

Our goal in this paper is the development of new automated methods for proving termination of concurrent systems under fairness assumptions. Specifically,

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we want to contribute new methods that take advantage of the rich set of termination results and tools developed in recent years for term rewriting systems to prove termination of concurrent systems under fairness assumptions. This requires both adopting a certain theoretical stance about the modeling of concurrent systems, and developing new results and techniques to make the rewriting-based termination techniques applicable to fair concurrent systems.

The theoretical stance in question is the thesis that *a concurrent system can be naturally modeled as a rewrite system*. This has by now been amply demonstrated to hold by theoretical approaches such as reduction semantics [2] and rewriting logic [23], and by quite exhaustive studies showing that almost any imaginable concurrent system can be naturally modeled as a rewrite theory (see [25]). Once this theoretical stance is adopted, since fairness is a pervasive property of concurrent systems, needed to establish many properties of interest, the first thing required is to correctly express the fairness notion within the rewriting framework. In this regard, the early work of Porat and Francez [28,29], and the work of Tison for the ground fair termination case [32], complemented by the more recent "localized fairness" notion in [24] offer a good basis. In this setting, a subset $\mathcal{R}_F$ of the rules of a Term Rewriting System (TRS) $\mathcal{R}$, whose rules are conveniently *labelled* can be used to *localize* the desired fairness requirements over particular rules of $\mathcal{R}$.

As we explain in Section 9, other notions of fairness have also been proposed for rewrite systems, with other, quite different, motivations that make such notions inadequate for our purposes, namely, *modeling the fairness of concurrent systems*. For concurrent systems, rewrite rules describe system transitions, and the notion of *fair computation* (also called *strong fairness*) should require that if a rule is infinitely often enabled, then it is infinitely often taken. Similarly, the notion of *just computation* (also called *weak fairness*) should require that if a rule is eventually always enabled, then it is infinitely often taken.

Example 1 *The following TRS models a* scheduler *which is responsible for the distribution of processing in a concurrent operating system, where a number of processes* `proc` *(where* `proc` *is a constant symbol) run independently.*

```
[end]     exec(P) -> stop

[execute] schedule(cons(proc,PS)) -> schedule(shift(exec(proc),PS))

[remove]  schedule(cons(stop,PS)) -> schedule(PS)

[round]   schedule(cons(exec(P),PS)) -> schedule(shift(exec(P),PS))

[shift1]  shift(P,nil) -> cons(P,nil)

[shift2]  shift(P,cons(Q,PS)) -> cons(Q,shift(P,PS))
```

Processes are in one of three states: ready *(*`proc`*),* running *(*`exec(proc)`*), and* finished *(*`stop`*). A "round robin" fair scheduling strategy gives each process a fixed amount of processing time and then shifts the activity to the next one in a list of processes. If a process is ready, then it is executed (rule* `execute`*). If it is running, then the next one is taken (*`round`*). If the process stops, then it is removed from the system (*`remove`*). A running process* `exec(proc)` *finishes when the rule* `end` *is applied. Although the system is clearly nonterminating, computations following a* fair *strategy regarding rules* `end`*,* `execute` *and* `remove` *will terminate. Example 15 below shows how to give a formal proof of this claim by using the results in this paper. Furthermore, the proof can be obtained by using existing tools for proving termination (properties) of rewrite systems.*

The question that this paper then addresses, and presents partial answers to, is: how can rewriting termination techniques and tools be used to *automatically* prove the fair or just-termination of a concurrent system? To the best of our knowledge, except for the quite restricted case of fair-termination of ground term rewriting systems for which Tison's tree automata techniques provide a decision procedure [32], this precise question has not been previously posed or answered in the literature. Yet, we believe that, given the maturity of methods and tools for termination of rewrite systems, this is an important problem to attack, both theoretically and because of its many practical applications.

The related question of finding general methods for proving fair termination of term rewriting systems has indeed been studied before, particularly by Porat and Francez [28,29]. However, their efforts followed Floyd's classical approach, which uses predicates on states (in our setting, ground terms) to achieve termination (see [10, Chapter 2] for a general description of this approach, and also [19]). In particular, their characterization of fair-termination of a rewrite system in terms of the compatibility of a well-founded ordering with all possible *full derivations* [29, Definition 9] does not lend itself to mechanization. Some parallelism can be found with the use of Manna and Ness's versus Lankford's termination criteria in proofs of termination of rewriting (see, e.g., [6]). Manna and Ness' theorem [22] establishes that termination of a TRS $\mathcal{R}$ is equivalent to the existence of a *well-founded ordering* $>$ which is compatible with *all rewriting steps* $s \to t$ (i.e., $s > t$ whenever $s \to t$ for all terms s and t). Since in general there is an infinite number of rewriting steps $s \to t$, Manna and Ness' theorem is not amenable to automation. In contrast, Lankford's theorem establishes that termination of a TRS $\mathcal{R}$ is equivalent to the existence of a *reduction ordering* $>$ (i.e., a stable, monotonic and well-founded ordering on terms) which is compatible with *the rules* (typically finitely many) of the TRS (i.e., $l > r$ for all rules $l \to r$ of $\mathcal{R}$). Provided that a suitable reduction ordering on terms $>$ is available (typically a simplification ordering [4,5,30]), automatic testing of compatibility with the set of rules is then feasible.

The need to check *all* (infinitely many) full derivations (as in Porat and

Francez's approach) makes automatic proofs of fair-termination quite unfeasible. Instead, our approach seeks reasonable conditions under which *just/fair-termination can be reduced to ordinary termination of associated relations*, for which standard rewriting termination techniques and tools can be applied to automate the proof process.

In Section 3, we introduce the notions of justice and fairness we work with. In particular, we introduce the notions of 1-label $\mathcal{R}_F$-justice and 1-label $\mathcal{R}_F$-fairness. Basically they correspond to labelled justice and fairness (in the sense of [24]) where all rules in $\mathcal{R}_F$ (describing the desired just/fair behavior) are labelled in the *same* way. In our setting, these notions are specially relevant because (as shown in the subsequent sections) we are able to *characterize* the corresponding termination notions as termination of (combinations of) more standard reduction relations. In the literature on fairness, though, labels are usually identified with rules, thus leading to the notion of *rule* justice/fairness, where each rule in $\mathcal{R}_F$ is assumed to 'own' a different label (e.g., [29]). Furthermore, when $\mathcal{R}_F = \mathcal{R}$ this becomes the notion of fairness in term rewriting proposed in [28]. Although the notion of 1-label $\mathcal{R}_F$-fairness is new, similar notions of fairness can also be found in the literature [18]. In this sense, our framework (which is based on [24]) can be seen as unifying previous notions of fairness in term rewriting. Moreover, our definitions regarding justice in term rewriting and the associated proof methods are novel in the literature.

In Section 4, we define just/fair termination in this more general setting and show that the problem of proving rule just/fair-termination of a TRS $\mathcal{R}$ (w.r.t. the whole system, as done in [28]) can be treated as the problem of proving just/fair-termination of $\mathcal{R}$ w.r.t. a sub-TRS $\mathcal{R}_F$ of $\mathcal{R}$. We also show how these more general notions of just/fair-termination are related to 1-label just/fair-termination. Actually, the '1-label' properties are often *sufficient conditions* for the 'rule' properties of $\mathcal{R}_F$-just/fair-termination.

In Section 5 we show that, if we take $\mathcal{S} = \mathcal{R} - \mathcal{R}_F$, then the 1-label $\mathcal{R}_F$-just/fair-termination of $\mathcal{R}$ can be proved by proving termination of combinations of (restrictions of) the relations $\to_{\mathcal{S}}$, $\to_{\mathcal{R}_F}$ and $\to_{\mathcal{R}}$. Actually, we prove that 1-label just/fair-termination can be *fully characterized* as termination of combinations of such reduction relations. Furthermore, we prove that termination of some of such reduction relations is actually *necessary* for more general notions like rule $\mathcal{R}_F$-just/fair-termination.

Section 6 shows how to translate such requirements into more standard termination problems, namely: proving or disproving *termination* and *relative termination* of TRSs. Fortunately, such termination problems can be managed by existing termination tools like AProVE [12], C*i*ME [3], Jambox [9], Matchbox [33], MU-TERM [1,20], TTT [13], and TPA [17], among others. Therefore, we get a quite practical approach for proving fair-termination of TRSs which clearly differs from more ad-hoc or restrictive approaches like the ones in [28,29,32]. In the case of just-termination of a TRS, we are not aware of any previous approaches in the literature, except for [24], to either characterize the notion or provide any proof techniques.

In Section 7 we explain how our results can be combined into a unified method, which offers different proof strategies to tackle a fair-termination problem. We show this method in action in proofs of concrete examples in Section 8. We consider the results obtained so far as encouraging, since they can allow proving just/fair-termination automatically.

Comparisons with related work are drawn in Section 9, where we also discuss how this paper extends and improves a previous version published in [21]. Section 10 concludes the paper and discusses future work.

2 Preliminaries on Term Rewriting

Let $R \subseteq A \times A$ be a binary relation on a set A. We denote by R^+ the transitive closure of R and by R^* its reflexive and transitive closure. Given binary relations $R, S \subseteq A \times A$, by $R \circ S$ we mean the relation $\{(x, z) \mid \exists y \in A, x\ R\ y \wedge y\ S\ z\}$. Given $R \subseteq A \times A$ and $B \subseteq A$, we let $R|_B = \{(a, b) \in R \mid a \in B\}$ and $R \cap B^2 = \{(a, b) \in R \mid a, b \in B\}$. An R-sequence is a finite or countably infinite sequence (i.e., either $a_1, a_2, \ldots, a_n$ for some $n \in \mathbb{N}$, or $a_1, a_2, \ldots$) such that for a_i, a_{i+1} two consecutive elements in the sequence, we have $a_i\ R\ a_{i+1}$; we say that such a sequence begins with a_1 (if it is finite, we also say that it *ends* with a_n). An element $a \in A$ is said to be an *R-normal form* if there exists no b such that $a\ R\ b$. The set of all R-normal forms is denoted by NF_R. The complement $\mathsf{RED}_R = A - \mathsf{NF}_R$ is called the set of *R-reducible* elements. We say that b is an R-normal form of a (written $a\ R^!\ b$) if $b \in \mathsf{NF}_R$ and $a\ R^* b$. We say that R is *terminating* iff there is no infinite sequence $a_1\ R\ a_2\ R\ a_3 \cdots$. Given binary relations R and S (on the same set A), we say that S *preserves R-normal forms* if for each $a \in \mathsf{NF}_R$ and $b \in A$, $a\ S\ b$ implies that $b \in \mathsf{NF}_R$.

Throughout this paper, $\mathcal{X}$ denotes a countable set of variables, and $\mathcal{F}$ denotes a signature, i.e., a set of function symbols $\{f, g, \ldots\}$, each having a fixed arity given by a mapping $ar : \mathcal{F} \to \mathbb{N}$. The set of terms built from $\mathcal{F}$ and $\mathcal{X}$ is $\mathcal{T}(\mathcal{F}, \mathcal{X})$. Terms are viewed as labelled trees in the usual way. Positions $p, q, \ldots$ are represented by chains of positive natural numbers used to address subterm positions of t. The set of positions of a term t is denoted $\mathcal{P}os(t)$. The subterm at position p of t is $t|_p$, and $t[s]_p$ is the term t with the subterm at position p replaced by s.

A *rewrite rule* is a sequent of the form $\alpha : l \to r$, with $l, r \in \mathcal{T}(\mathcal{F}, \mathcal{X})$, $l \notin \mathcal{X}$, $\mathcal{V}ar(r) \subseteq \mathcal{V}ar(l)$, and $\alpha \in \mathcal{L}$, a *label* marking the rule and belonging to a set $\mathcal{L}$ of labels. The left-hand side (*lhs*) of the rule is l and r is the right-hand side (*rhs*). A TRS is a pair $\mathcal{R} = (\mathcal{F}, R)$ with R a (possibly infinite) set of rewrite rules. We denote by $\mathcal{L}(\mathcal{R})$ the set of labels used in the rewrite rules of $\mathcal{R}$. A term $t \in \mathcal{T}(\mathcal{F}, \mathcal{X})$ rewrites to s (at position $p \in \mathcal{P}os(t)$ and using the rule $\alpha : l \to r \in \mathcal{R}$), written $t \xrightarrow{p}_{l \to r} s$ (or just $t \to_{\mathcal{R}} s$ or even $t \to s$ if no confusion arises), if $t|_p = \sigma(l)$ and $s = t[\sigma(r)]_p$, for some substitution σ. A rewrite sequence (or $\mathcal{R}$-sequence) A is a finite or infinite sequence $t_1, t_2, \ldots$ of terms t_i such that $t_i \to t_{i+1}$ for all $i \geq 1$ (and $i < n+1$ if t_{n+1} is the last term of a finite sequence of $n+1$ terms). A TRS $\mathcal{R}$ is *terminating* if $\to_{\mathcal{R}}$ is terminating. The set of normal forms of $\mathcal{R}$ ($\mathcal{R}$-normal forms) is denoted by $\mathsf{NF}_{\mathcal{R}}$.

Given TRSs $\mathcal{R} = (\mathcal{F}, R)$ and $\mathcal{S} = (\mathcal{F}, S)$, we denote by $\mathcal{R} \cup \mathcal{S}$ the TRS $(\mathcal{F}, R \cup S)$; also, we write $\mathcal{R} \subseteq \mathcal{S}$ to indicate that $R \subseteq S$.

3 Justice and Fairness in Term Rewriting

Our definitions of justice and fairness in term rewriting are based on the formulation of localized justice/fairness properties given in [24]. Let us first introduce some basic terminology:

(1) We say that a rule $\alpha : l \to r$ is *enabled* on a term t if t contains a redex of this rule, i.e., $t = C[\sigma(l)]$ for some context $C[\,]$ and substitution σ.
(2) An α-rule is a rule with label α. In our setting, a label can mark not just one but possibly several different rules. Thus, given a term t, we can think of a label α as representing the one-step *rewriting computations* which can be performed by using an α-rule to rewrite t to a term t'; we then write $t \to_\alpha t'$ if there is a rule $\alpha : l \to r$ with label α such that $t = C[\sigma(l)]$ and $t' = C[\sigma(r)]$ for some substitution σ. Note that the one-step rewrite relation $\to$ is the union of those relations: $\to = \bigcup_{\alpha \in \mathcal{L}(\mathcal{R})} \to_\alpha$.
(3) We say that a (one-step computation with) label α is *enabled* on a term t if some α-rule is enabled on t.
(4) We say that a (one-step computation with) label α is *taken* in a reduction step $t \to s$ if $t \to_\alpha s$ for some α-rule.

An $\mathcal{R}$-sequence is *fair* (w.r.t. the labeled rules contained in a sub-TRS $\mathcal{R}_F$ of $\mathcal{R}$) if for each different label α corresponding to rules in $\mathcal{R}_F$, each label α which is infinitely often enabled during the sequence is infinitely often taken. Similarly, an $\mathcal{R}$-sequence is *just* if for each label α for rules in $\mathcal{R}_F$, if from some point on α is *continuously* enabled, then α is infinitely often taken. We make this more precise in the following definition.

Definition 1 (Labelled justice and fairness) *Given a TRS $\mathcal{R}$, a finite or infinite $\mathcal{R}$-sequence $A : t_1 \to_{\mathcal{R}} t_2 \to_{\mathcal{R}} \cdots$, and a label α, we let*

$$I_\alpha^A = \{i \in \mathbb{N} \mid \exists \alpha : l \to r, C_i, \sigma_i, p_i, \text{ such that } t_i = C_i[\sigma_i(l)]_{p_i}\}$$

Let $\mathcal{R}_F \subseteq \mathcal{R}$ be such that it shares no label with $\mathcal{R} - \mathcal{R}_F$, i.e., $\mathcal{L}(\mathcal{R}_F) \cap \mathcal{L}(\mathcal{R} - \mathcal{R}_F) = \varnothing$. We say that the sequence A is:

(1) just (also called *weakly fair*) w.r.t. the rules in $\mathcal{R}_F$ *(abbreviated* $\mathcal{R}_F$-just*) if for all $\alpha \in \mathcal{L}(\mathcal{R}_F)$ if there is $k \in \mathbb{N}$ such that $I_\alpha^A \supseteq \{n \mid n \geq k\}$, then there is an infinite set $J_\alpha^A \subseteq I_\alpha^A$ such that, for all $j \in J_\alpha^A$, $t_j \to_\alpha t_{j+1}$.*

(2) fair (also called *strongly fair*) w.r.t. the rules in $\mathcal{R}_F$ *(abbreviated* $\mathcal{R}_F$-fair*) if for all $\alpha \in \mathcal{L}(\mathcal{R}_F)$ whenever I_α^A is infinite, then there is an infinite set $J_\alpha^A \subseteq I_\alpha^A$ such that, for all $j \in J_\alpha^A$, $t_j \to_\alpha t_{j+1}$.*

The following example illustrates these notions.

Example 2 *Consider the following TRS $\mathcal{R}$:*

α_1 : `a -> b`　　　　α_3 : `b -> c`

α_2 : `b -> a`　　　　α_4 : `a -> c`

where $\mathcal{R}_F$ consists of the rules α_3 and α_4. The $\mathcal{R}$-sequence:

$$\mathtt{a} \to_{\alpha_1} \mathtt{b} \to_{\alpha_2} \mathtt{a} \to_{\alpha_1} \mathtt{b} \to_{\alpha_2} \mathtt{a} \to \cdots$$

is $\mathcal{R}_F$-just (there is no label from $\mathcal{L}(\mathcal{R}_F)$ which is continuously enabled) but it is not *$\mathcal{R}_F$-fair, because the labels of the rules in $\mathcal{R}_F$ are infinitely often enabled but never taken. Assume now that we slightly modify $\mathcal{R}_F$ to $\mathcal{R}'_F$ by making $\alpha_3 = \alpha_4 = \alpha$. Then, the previous sequence is* not *$\mathcal{R}'_F$-just anymore, because one of the two rules in $\mathcal{R}'_F$ is always enabled (hence reductions labelled with α are continuously enabled) but never taken.*

As a simple consequence of Definition 1, a finite $\mathcal{R}$-sequence is always fair and just w.r.t. any $\mathcal{R}_F \subseteq \mathcal{R}$. Also, all $\mathcal{R}$-sequences are fair and just w.r.t. $\mathcal{R}_F = \varnothing$. Calling fairness “strong fairness”, and justice “weak fairness” is justified because of the following.

Proposition 1 *Let $\mathcal{R}$ be a TRS. Any $\mathcal{R}_F$-fair $\mathcal{R}$-sequence A is always $\mathcal{R}_F$-just.*

Proof. Consider an $\mathcal{R}_F$-fair $\mathcal{R}$-sequence $A : t_1 \to_{\mathcal{R}} t_2 \to_{\mathcal{R}} \cdots$. If there are $\alpha \in \mathcal{L}(\mathcal{R}_F)$ and $k \in \mathbb{N}$ such that $I_\alpha^A \supseteq \{n \mid n \geq k\}$, then, by $\mathcal{R}_F$-fairness, α is infinitely often taken along A, i.e., there is an infinite set $J_\alpha^A \subseteq I_\alpha^A$ such that, for all $j \in J_\alpha^A$, $t_j \to_\alpha t_{j+1}$. Thus, A is $\mathcal{R}_F$-just too. □

Definition 1, because of the fact that the same label can be shared by more than one rule, allows us to consider more or less *localized* notions of fairness and justice. In the rest of the paper, we will focus our attention on the following two special cases of fairness and justice.

Definition 2 *Let $\mathcal{R}$ be a TRS and $\mathcal{R}_F \subseteq \mathcal{R}$. An $\mathcal{R}$-sequence A is said to be*

(1) Rule $\mathcal{R}_F$-fair *(resp.* rule $\mathcal{R}_F$-just*) if A is $\mathcal{R}_F$-fair (resp. $\mathcal{R}_F$-just) and every rule in $\mathcal{R}_F$ has a* different label*: $|\mathcal{L}(\mathcal{R}_F)| = |\mathcal{R}_F|$.*
(2) 1-label $\mathcal{R}_F$-fair *(resp.* 1-label $\mathcal{R}_F$-just*) if A is $\mathcal{R}_F$-fair (resp. $\mathcal{R}_F$-just) and all rules in $\mathcal{R}_F$ have the* same label*: $|\mathcal{L}(\mathcal{R}_F)| = 1$.*

Regarding related notions of fairness and justice, we have the following:

(1) Porat and Francez's notion of *rule fairness* for a TRS $\mathcal{R}$ [28, page 289], is captured by rule fairness w.r.t. $\mathcal{R}$ itself (i.e., $\mathcal{R}_F = \mathcal{R}$ in Definition 2.1).
(2) Rule $\mathcal{R}_F$-fairness in Definition 2.1 corresponds to Porat and Francez's *relativized* fairness (denoted $\mathcal{R}_F$-fairness in [28, page 291]).
(3) Justice has not been discussed in the realm of term rewriting systems (except in [24]), although our definition of rule justice w.r.t. $\mathcal{R}$ in Definition 2 can be thought of as a natural translation of well-known notions like Lehmann, Pnueli and Stavi's [19] for concurrent systems.
(4) As far as we are aware of, except for [24], the notions of 1-label $\mathcal{R}_F$-fairness and 1-label $\mathcal{R}_F$-justice have not been discussed before in the literature.

Remark 1 *Concerning Definition 2, in the following we only consider two 'extreme' labellings for the rules in $\mathcal{R}_F$: either (1) each rule in $\mathcal{R}_F$ has a different label ($|\mathcal{L}(\mathcal{R}_F)| = |\mathcal{R}_F|$), or (2) all rules in $\mathcal{R}_F$ have the same label ($|\mathcal{L}(\mathcal{R}_F)| = 1$). When discussing justice and fairness, we will not explicitly distinguish TRSs containing the same rules but having different labellings. Instead, in each case we will indicate the 'rule' or '1-label' uses we are interested in.*

The different notions introduced by Definition 2 are related as follows:

Proposition 2 *Let $\mathcal{R}$ be a TRS and $\mathcal{R}_F$ be a finite sub-TRS of $\mathcal{R}$. Then, all rule $\mathcal{R}_F$-fair $\mathcal{R}$-sequences A are 1-label $\mathcal{R}_F$-fair.*

PROOF. Let us assume that α is the only label in $\mathcal{R}_F$, and suppose that I^A_α is infinite. Then, since the number of rules in $\mathcal{R}_F$ is finite, there must be some rule that is enabled an infinite number of times. By rule fairness that rule is taken infinitely often. But this means that the set J^A_α must be infinite, so we have 1-label fairness. □

The following example shows that, in general, 1-label $\mathcal{R}_F$-fair sequences need not be rule $\mathcal{R}_F$-fair.

Example 3 *Consider the TRS $\mathcal{R}$ and the $\mathcal{R}$-sequence A in Example 2. Assume that $\mathcal{R}_F$ consists of rules α_2 and α_3. Note that A is 1-label $\mathcal{R}_F$-fair but it is not rule $\mathcal{R}_F$-fair.*

The following example shows that there are no similar general results connecting 1-label $\mathcal{R}_F$-justice and rule $\mathcal{R}_F$-justice.

Example 4 *Consider the TRS $\mathcal{R}$ in Example 2.*

(1) Assume that $\mathcal{R}_F$ consists of rules α_3 and α_4. Then, the $\mathcal{R}$-sequence A in Example 2 is rule $\mathcal{R}_F$-just but it is not 1-label $\mathcal{R}_F$-just.
(2) Assume now that $\mathcal{R}_F$ consists of rules α_1 and α_4. Then, the sequence:

$$\mathtt{d(a,a)} \to_{\alpha_1} \mathtt{d(b,a)} \to_{\alpha_2} \mathtt{d(a,a)} \to_{\alpha_1} \mathtt{d(b,a)} \to_{\alpha_2} \mathtt{d(a,a)} \to \cdots$$

(where we assume the existence of a binary symbol `d`*) is 1-label $\mathcal{R}_F$-just but it is not rule $\mathcal{R}_F$-just (rule α_4 is continuously enabled but never taken).*

According to Definition 2, for single rule TRSs $\mathcal{R}_F \subseteq \mathcal{R}$, rule $\mathcal{R}_F$-just (resp. $\mathcal{R}_F$-fair) sequences and 1-label $\mathcal{R}_F$-just (resp. $\mathcal{R}_F$-fair) sequences coincide. Also, if $\mathcal{R}$ itself is a single rule TRS, then the sets of: (*i*) 1-label $\mathcal{R}$-fair sequences, (*ii*) 1-label $\mathcal{R}$-just sequences, and (*iii*) arbitrary $\mathcal{R}$-sequences, coincide.

4 Termination of just/fair sequences

The following definition introduces the termination notions related to just and fair sequences considered above.

Definition 3 (just/fair-termination) *Let $\mathcal{R}$ be a TRS and let $\mathcal{R}_F \subseteq \mathcal{R}$. We say that $\mathcal{R}$ is rule $\mathcal{R}_F$-fairly-terminating (respectively 1-label $\mathcal{R}_F$-fairly-terminating, 1-label $\mathcal{R}_F$-justly-terminating, rule $\mathcal{R}_F$-justly-terminating) if all rule $\mathcal{R}_F$-fair $\mathcal{R}$-sequences (respectively all 1-label $\mathcal{R}_F$-fair $\mathcal{R}$-sequences, all 1-label $\mathcal{R}_F$-just $\mathcal{R}$-sequences, all rule $\mathcal{R}_F$-just $\mathcal{R}$-sequences) are finite.*

Remark 2 *The notion of rule $\mathcal{R}$-fair-termination in Definition 3 coincides with Porat and Francez's notion of fair-termination [28], and that of rule $\mathcal{R}_F$-fair-termination is equivalent to [29, Definition 17].*

As far as we are aware of, the notions of 1-label $\mathcal{R}_F$-fair-termination, 1-label $\mathcal{R}_F$-just-termination and rule $\mathcal{R}_F$-just-termination have not been discussed before in the literature.

Remark 3 *Note that ordinary termination of TRSs is subsumed by Definition 3: if $\mathcal{R}_F = \varnothing$, then all $\mathcal{R}$-sequences are trivially $\mathcal{R}_F$-fair and $\mathcal{R}$ is $\varnothing$-fairly-terminating if and only if $\mathcal{R}$ is terminating. And, clearly, termination of $\mathcal{R}$ implies all just/fair-termination properties considered here. However, the opposite is not true: for instance, the system* {`a -> b`, `a -> a`} *is rule fairly-terminating but not terminating.*

4.1 Extensional vs. intensional just/fair-termination

A perceptive reader might have noticed that our definitions of just/fair rewrite sequence and of just/fair termination are *extensional*, in the precise sense of being based on rewrite sequences. This agrees well with the usual treatment of the rewriting relation as a *binary* relation between terms, and is technically convenient, because it will allow us to consider various *binary relations* on terms as means to prove just/fair termination.

However, in some sense a more faithful modeling of the justice/fairness phenomenon would be obtained by viewing the rewriting relation as a labelled transition system, that is, as a *ternary* relation made up of triples (t, α, t'), displayed as $t \stackrel{\alpha}{\to} t'$. This would lead to what we might call an *intensional* notion of (finite or infinite) labelled rewrite sequence of the form

$$t_1 \stackrel{\alpha_1}{\to} t_2 \stackrel{\alpha_2}{\to} t_3 \stackrel{\alpha_3}{\to} \cdots$$

which we shall call a *computation*. We can view a computation as a pair (A, Γ), where A is the rewrite sequence $t_1 \to t_2 \to t_3 \ldots$, and Γ is the sequence of labels $\alpha_1\, \alpha_2 \ldots$. Then, given $\mathcal{R}_F \subseteq \mathcal{R}$, we would call the computation (A, Γ) *fair* w.r.t. the rules in $\mathcal{R}_F$ if for all $\alpha \in \mathcal{L}(\mathcal{R}_F)$, whenever I^A_α is infinite, then there is an infinite set $J^A_\alpha \subseteq I^A_\alpha$ such that for al $j \in J^A_\alpha$ we have $t_j \stackrel{\alpha}{\to} t_{j+1}$ in (A, Γ), that is, α is the j-th element of the sequence of labels Γ. And we would likewise define the notion of (A, Γ) being *just* w.r.t. the rules in $\mathcal{R}_F$ in the obvious similar way. Note that the intensional notions are *different* form the extensional ones, because in general the projection function $\pi : (A, \Gamma) \mapsto A$ mapping a computation to its underlying rewrite sequence is *not* injective: there can in general be infinitely many labelings Γ for the same rewrite sequence A.

The intensional definitions of $\mathcal{R}$ being $\mathcal{R}_F$-just/fair terminating would then be the obvious ones: all just (resp. fair) computations w.r.t. the rules in $\mathcal{R}_F$ are *finite*.

Since the intensional and extensional notions are different, this raises the reasonable concern of whether the extensional Definition 3 and the above, intensional definition would describe different notions of just/fair termination. But there is nothing to worry about since we have:

Proposition 3 *$\mathcal{R}$ is $\mathcal{R}_F$-just (resp. $\mathcal{R}_F$-fair) terminating in the extensional sense of Definition 3 if and only if it is $\mathcal{R}_F$-just (resp. $\mathcal{R}_F$-fair) terminating in the intensional sense defined above.*

PROOF. Obviously, if (A, Γ) is a nonterminating just (resp. fair) computation w.r.t. the rules in $\mathcal{R}_F$, then A is a nonterminating just (resp. fair) rewrite sequence w.r.t. the rules in $\mathcal{R}_F$. To see the converse implication, the key observation is that the projection function $\pi : (A, \Gamma) \mapsto A$, which is well-defined for all computations, restricts to a *surjective* function from the set of all just *computations* w.r.t. the rules in $\mathcal{R}_F$ to the set of all just *rewrite sequences* w.r.t. the rules in $\mathcal{R}_F$. Likewise, π restricts to a *surjective* function from the set of all fair *computations* w.r.t. the rules in $\mathcal{R}_F$ to the set of all fair *rewrite sequences* w.r.t. the rules in $\mathcal{R}_F$. Therefore, if A is a nonterminating just (resp. fair) rewrite sequence w.r.t. the rules in $\mathcal{R}_F$, then we can always find a (not necessarily unique) nonterminating just (resp. fair) computation (A, Γ) w.r.t. the rules in $\mathcal{R}_F$. □

4.2 A hierarchy of just/fair-termination properties

As remarked in the introduction, we are going to show that 1-label just/fair-termination can be *fully characterized* as termination of combinations of some reduction relations. We will handle these termination problems by using existing termination tools. In this section we show how 1-label just/fair-termination can be used to prove rule $\mathcal{R}_F$-just/fair-termination.

Using Propositions 1 and 2, we have the following obvious facts:

Proposition 4 *Let $\mathcal{R}$ be a TRS and let $\mathcal{R}_F \subseteq \mathcal{R}$ be a finite sub-TRS. Then, we have:*

(1) $\mathcal{R}$ is rule $\mathcal{R}_F$-fairly-terminating if $\mathcal{R}$ is 1-label $\mathcal{R}_F$-fairly-terminating.
(2) $\mathcal{R}$ is 1-label $\mathcal{R}_F$-fairly-terminating if $\mathcal{R}$ is 1-label $\mathcal{R}_F$-justly-terminating.
(3) $\mathcal{R}$ is rule $\mathcal{R}_F$-fairly-terminating if $\mathcal{R}$ is rule $\mathcal{R}_F$-justly-terminating.

And, according to Definition 2 again, if $\mathcal{R}_F \subseteq \mathcal{R}$ is a single rule TRS, then $\mathcal{R}$ is rule $\mathcal{R}_F$-fairly-terminating iff $\mathcal{R}$ is 1-label $\mathcal{R}_F$-fairly-terminating. Furthermore, $\mathcal{R}$ is rule $\mathcal{R}_F$-justly-terminating iff $\mathcal{R}$ is 1-label $\mathcal{R}_F$-justly-terminating. We also have the following.

Proposition 5 *Let $\mathcal{R}$ be a TRS and let $\mathcal{R}_F \subseteq \mathcal{R}$ be a finite sub-TRS. If $\mathcal{R}$ is 1-label $\mathcal{R}_F$-fairly-terminating and rule $\mathcal{R}_F$-justly-terminating, then $\mathcal{R}$ is 1-label $\mathcal{R}_F$-justly-terminating.*

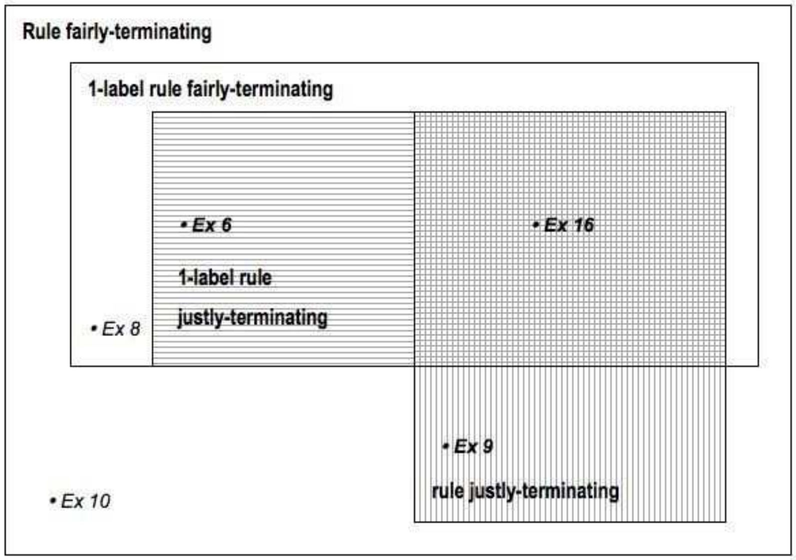

Fig. 1. Comparing Just/fair-termination properties

Proof. By contradiction. If $\mathcal{R}$ is not 1-label $\mathcal{R}_F$-justly-terminating, then there is an infinite 1-label $\mathcal{R}_F$-just sequence $A : t_1 \rightarrow_{\mathcal{R}} t_2 \rightarrow_{\mathcal{R}} \cdots$. Consider I^A_α for the unique label α which is assumed to be used in $\mathcal{R}_F$. Then either:

(1) I^A_α is finite, which means that A is an $\mathcal{R}_F$-fair infinite sequence, thus contradicting 1-label $\mathcal{R}_F$-fair-termination of $\mathcal{R}$, or
(2) I^A_α is infinite but α is *not* continuously enabled on A. So, in particular, no individual rule in $\mathcal{R}_F$ is continuously enabled on A. Therefore, A contradicts rule $\mathcal{R}_F$-just-termination of $\mathcal{R}$, or
(3) There is $k \geq 1$ such that $I^A_\alpha \subseteq \{n \mid n \geq k\}$ and (by the assumption that A is 1-label $\mathcal{R}_F$-just) there is an infinite set $J^A_\alpha \subseteq I^A_\alpha$ such that, for all $j \in J^A_\alpha$, $t_j \rightarrow_\alpha t_{j+1}$. Therefore, A contradicts 1-label $\mathcal{R}_F$-fair-termination of $\mathcal{R}$.

□

Figure 1 summarizes these results; examples mentioned there are introduced and discussed below.

4.3 Simplifying just/fair-termination problems

In contrast to ordinary termination, just-termination and fair-termination are *not* preserved if some of the rules of the TRS are dropped: there are TRSs $\mathcal{R}$ which are $\mathcal{R}_F$-fairly-terminating for some $\mathcal{R}_F \subseteq \mathcal{R}$, but are not $\mathcal{R}'_F$-fairly-terminating for a subset $\mathcal{R}'_F \subset \mathcal{R}_F$.

Example 5 *Consider the following TRS $\mathcal{R}$ [28,32]:*

```
a -> f(a)

a -> b
```

As noticed by Tison, $\mathcal{R}$ is rule fairly-terminating (i.e., fairly-terminating w.r.t. $\mathcal{R}$ itself). Let $\mathcal{R}_F$ be the sub-TRS of $\mathcal{R}$ consisting of the first rule (then take $\mathcal{S} = \mathcal{R} - \mathcal{R}_F$). The following infinite $\mathcal{R}$-sequence (as usual, we underline the contracted redex):

$$\underline{\mathtt{a}} \to_{\mathcal{R}_F} \mathtt{f}(\underline{\mathtt{a}}) \to_{\mathcal{R}_F} \mathtt{f}(\mathtt{f}(\underline{\mathtt{a}})) \to_{\mathcal{R}_F} \cdots$$

is $\mathcal{R}_F$-fair. This shows that $\mathcal{R}$ is not $\mathcal{R}_F$-fairly-terminating.

The key observation is that, given $\mathcal{R}_F, \mathcal{R}'_F \subseteq \mathcal{R}$, where $\mathcal{R}_F$ and $\mathcal{R}'_F$ are label-closed inside $\mathcal{R}$, i.e., $\mathcal{R}_F$ does not share any labels with $\mathcal{R} - \mathcal{R}_F$, and the same holds for $\mathcal{R}'_F$, the set of $\mathcal{R}_F \cup \mathcal{R}'_F$-fair (resp. just) sequences is the *intersection* of the sets of $\mathcal{R}_F$-fair and $\mathcal{R}'_F$-fair (resp. $\mathcal{R}_F$-just and $\mathcal{R}'_F$-just) sequences. Therefore, we have the following obvious sufficient condition in the other direction.

Proposition 6 *A TRS $\mathcal{R}$ is rule $\mathcal{R}_F$-fairly-terminating (resp. 1-label $\mathcal{R}_F$-fairly-terminating, 1-label $\mathcal{R}_F$-justly-terminating, rule $\mathcal{R}_F$-justly-terminating) for some $\mathcal{R}_F \subseteq \mathcal{R}$ if there is a subset $\mathcal{R}'_F \subset \mathcal{R}_F$ not sharing any labels with $\mathcal{R} - \mathcal{R}'_F$, such that $\mathcal{R}$ is $\mathcal{R}'_F$-fairly-terminating (resp. 1-label $\mathcal{R}'_F$-fairly-terminating, 1-label $\mathcal{R}'_F$-justly-terminating, rule $\mathcal{R}'_F$-justly-terminating).*

The subset $\mathcal{R}'_F$ in Proposition 6 can be a *single* rule. For instance, Tison observes that $\mathcal{R}$ in Example 5 is rule fairly-terminating thanks to the rule `a -> b`. As we have seen above, this is a specially interesting case. The system in Example 1, however, is $\mathcal{R}_F$-fairly-terminating provided that $\mathcal{R}_F$ contains all three rules `end`, `execute`, and `remove`. It is easy to see that the absence of one of them destroys fair-termination. Proposition 6 will be used later and has the following obvious consequence.

Corollary 1 *A TRS $\mathcal{R}$ is rule fairly-terminating if there is a subset $\mathcal{R}_F \subseteq \mathcal{R}$ not sharing any labels with $\mathcal{R} - \mathcal{R}_F$, such that $\mathcal{R}$ is rule $\mathcal{R}_F$-fairly-terminating.*

5 Reducing Just/fair-termination to Termination

Termination analysis has recently experienced a remarkable development in the term rewriting community, leading to a new generation of promising methods, tools, and applications. An important goal of this paper is giving an appropriate theoretical basis for just/fair-termination which allows us to take advantage of term rewriting methods and tools in order to develop automatic proof techniques.

In this section, we investigate how to reduce a proof of just/fair-termination to the problem of proving termination of particular (combinations of) reduction relations. Our approach is as follows: for $\mathcal{S} = \mathcal{R} - \mathcal{R}_F$ we *characterize* 1-label just-termination and 1-label fair-termination in terms of termination of combinations of (restrictions of) the relations $\to_{\mathcal{S}}$, $\to_{\mathcal{R}_F}$ and $\to_{\mathcal{R}}$. We show that such characterizations can be used for proving 1-label just-termination and 1-label fair-termination in practice. Regarding rule just-termination and rule fair-termination, we use Proposition 4, but we also provide some necessary conditions for them which are formulated as termination properties as well.

5.1 From 1-label just-termination to termination

In order to ensure that no infinite 1-label $\mathcal{R}_F$-just sequences are possible for a given TRS $\mathcal{R}$ (w.r.t. an intended $\mathcal{R}_F \subseteq \mathcal{R}$), we have to ensure that the following two kinds of $\mathcal{R}_F$-just sequences are not possible:

(1) Infinite sequences A containing an infinite number of terms which are not enabled for any rule in $\mathcal{R}_F$ (i.e., an infinite number of terms in the sequence are in $\mathcal{R}_F$-normal form). This means that A is as follows:

$$t_1 \to^*_{\mathcal{R}} t'_1 \to_{\mathcal{S}} t_2 \to^*_{\mathcal{R}} t'_2 \to_{\mathcal{S}} t_3 \to^*_{\mathcal{R}} \cdots$$

where $t'_i \in \mathsf{NF}_{\mathcal{R}_F}$ for $i \geq 1$.

(2) Infinite sequences A where, from some point on, some rules in $\mathcal{R}_F$ are continuously enabled and $\mathcal{R}_F$ rules are infinitely often taken. This means that A must have the form

$$t_0 \to^*_{\mathcal{R}} t_1 \to^*_{\mathcal{S}} t'_1 \to_{\mathcal{R}_F} t_2 \to^*_{\mathcal{S}} t'_2 \to_{\mathcal{R}_F} t_3 \to^*_{\mathcal{S}} \cdots$$

where $t_i, t'_i \notin \mathsf{NF}_{\mathcal{R}_F}$ for $i \geq 1$ and for each subsequence $t_i = u_{i1} \to_{\mathcal{S}} u_{i2} \to_{\mathcal{S}} \cdots \to_{\mathcal{S}} u_{in_i} = t'_i$, where $n_i \geq 0$, we also have $u_{ij} \notin \mathsf{NF}_{\mathcal{R}_F}$ for $1 \leq j \leq n_i$.

Therefore, since infinite 1-label $\mathcal{R}_F$-just sequences can only be of one of these two forms, we have the following.

Theorem 1 *A TRS $\mathcal{R} = \mathcal{R}_F \cup \mathcal{S}$ with $\mathcal{R}_F$ finite is* 1*-label $\mathcal{R}_F$-justly-terminating if and only if $\to^*_{\mathcal{R}} \circ (\to_{\mathcal{S}} |_{\mathsf{NF}_{\mathcal{R}_F}})$ and $(\to_{\mathcal{S}} \cap \mathsf{RED}^2_{\mathcal{R}_F})^* \circ (\to_{\mathcal{R}_F} \cap \mathsf{RED}^2_{\mathcal{R}_F})$ are terminating.*

PROOF. The *if* part is clear from the previous considerations. For the only if part, assume that $\mathcal{R}$ is 1-label $\mathcal{R}_F$-just-terminating but either $\to^*_{\mathcal{R}} \circ (\to_{\mathcal{S}} |_{\mathsf{NF}_{\mathcal{R}_F}})$ or $(\to_{\mathcal{S}} \cap \mathsf{RED}^2_{\mathcal{R}_F})^* \circ (\to_{\mathcal{R}_F} \cap \mathsf{RED}^2_{\mathcal{R}_F})$ are not terminating. In the first case, we would have an infinite sequence

$$t_1 \to^*_{\mathcal{R}} t'_1 \to_{\mathcal{S}} t_2 \to^*_{\mathcal{R}} t'_2 \to_{\mathcal{S}} t_3 \to_{\mathcal{R}} \cdots$$

where $t'_i \in \mathsf{NF}_{\mathcal{R}_F}$ for $i \geq 1$, which is obviously 1-label $\mathcal{R}_F$-just (because there is an infinite number of terms $t'_i \in \mathsf{NF}_{\mathcal{R}_F}$ where no rule of $\mathcal{R}_F$ is enabled). In the second case, we would have an infinite sequence

$$t_1 \to^*_{\mathcal{S}} t'_1 \to_{\mathcal{R}_F} t_2 \to^*_{\mathcal{S}} t'_2 \to_{\mathcal{R}_F} t_3 \to_{\mathcal{R}} \cdots$$

where $t_i, t'_i \notin \mathsf{NF}_{\mathcal{R}_F}$ for $i \geq 1$ and for each subsequence $t_i = u_{i1} \to_{\mathcal{S}} u_{i2} \to_{\mathcal{S}} \cdots \to_{\mathcal{S}} u_{in_i} = t'_i$, where $n_i \geq 0$, we also have $u_{ij} \notin \mathsf{NF}_{\mathcal{R}_F}$ for $1 \leq j \leq n_i$, which is also 1-label $\mathcal{R}_F$-just. Thus, in both cases we get a contradiction. □

We can use Theorem 1 for proving 1-label $\mathcal{R}_F$-just-termination.

Example 6 *Consider again the TRS $\mathcal{R}$ in Example 2 and assume that $\mathcal{R}_F$ consists of the rules α_3 and α_4. Let $\mathcal{S} = \mathcal{R} - \mathcal{R}_F$. Then, we have that $\mathsf{NF}_{\mathcal{R}_F} = \{\mathsf{c}\} \cup \mathcal{X}$ and $\mathsf{RED}_{\mathcal{R}_F} = \{\mathsf{a}, \mathsf{b}\}$. Thus, $\to_{\mathcal{S}} |_{\mathsf{NF}_{\mathcal{R}_F}} = \varnothing$ and hence $\to^*_{\mathcal{R}} \circ (\to_{\mathcal{S}} |_{\mathsf{NF}_{\mathcal{R}_F}}) = \varnothing$, i.e., $\to^*_{\mathcal{R}} \circ (\to_{\mathcal{S}} |_{\mathsf{NF}_{\mathcal{R}_F}})$ is terminating. Also, since no right-hand side in $\mathcal{R}_F$ is in $\mathsf{RED}_{\mathcal{R}_F}$, we have that the relation $(\to^*_{\mathcal{S}} \circ \to_{\mathcal{R}_F}) \cap \mathsf{RED}^2_{\mathcal{R}_F} = \varnothing$ is also terminating. Therefore, $\mathcal{R}$ is 1-label $\mathcal{R}_F$-justly-terminating. As noticed in Example 2, the infinite sequence A in the example is $\mathcal{R}_F$-just; therefore, $\mathcal{R}$ is* not *rule $\mathcal{R}_F$-justly-terminating.*

We give the following necessary condition for 1-label $\mathcal{R}_F$-just-termination.

Proposition 7 *If a TRS $\mathcal{R} = \mathcal{R}_F \cup \mathcal{S}$ with $\mathcal{R}_F$ finite is* 1*-label $\mathcal{R}_F$-justly-terminating, then $\mathcal{R}_F$ is terminating.*

PROOF. By contradiction. If $\mathcal{R}_F$ is not terminating, then there is an infinite $\mathcal{R}_F$-sequence $t_1 \to_{\mathcal{R}_F} t_2 \to_{\mathcal{R}_F} \cdots$, where (obviously) $t_i \in \mathsf{RED}_{\mathcal{R}_F}$ for all $i \geq 1$. Thus, $\to_{\mathcal{R}_F} \cap \mathsf{RED}^2_{\mathcal{R}_F}$ is not terminating and $(\to_{\mathcal{S}} \cap \mathsf{RED}^2_{\mathcal{R}_F})^* \circ (\to_{\mathcal{R}_F} \cap \mathsf{RED}^2_{\mathcal{R}_F})$ is not terminating either. Thus, by Theorem 1 it is not 1-label $\mathcal{R}_F$-justly-terminating. □

We have the following result which is useful to *disprove* rule $\mathcal{R}_F$-just-termination.

Proposition 8 *If a TRS $\mathcal{R} = \mathcal{R}_F \cup \mathcal{S}$ with $\mathcal{R}_F$ finite is rule $\mathcal{R}_F$-just-terminating, then $\to_{\mathcal{R}}^{*} \circ (\to_{\mathcal{S}}|_{\mathsf{NF}_{\mathcal{R}_F}})$ is terminating.*

PROOF. By contradiction, assume that there is an infinite $\to_{\mathcal{R}}^{*} \circ (\to_{\mathcal{S}}|_{\mathsf{NF}_{\mathcal{R}_F}})$-sequence. Such a sequence is $\mathcal{R}_F$-just because there is no rule in $\mathcal{R}_F$ which is continuously enabled. Thus, $\mathcal{R}$ is not rule $\mathcal{R}_F$-just-terminating. □

5.2 From 1-label fair-termination to termination

Now we consider 1-label $\mathcal{R}_F$-fair-termination. First of all, notice that, according to Proposition 4, we can use Theorem 1 for proving 1-label $\mathcal{R}_F$-fair-termination by proving 1-label $\mathcal{R}_F$-just-termination. There are, however, 1-label $\mathcal{R}_F$-fairly-terminating TRSs which are *not* 1-label $\mathcal{R}_F$-justly-terminating.

Example 7 *The TRS $\mathcal{R}$ in Example 2 where $\mathcal{R}_F$ consists of the rule α_3 is* not *1-label $\mathcal{R}_F$-justly-terminating (see Example 2). Furthermore, since $\mathcal{R}_F$ consists of a single rule, $\mathcal{R}$ is* not *rule $\mathcal{R}_F$-justly-terminating. However, $\mathcal{R}$ is 1-label $\mathcal{R}_F$-fairly-terminating (see Example 8 below).*

To prove 1-label $\mathcal{R}_F$-fair-termination directly, rather than by reduction to a proof of 1-label $\mathcal{R}_F$-just-termination, we have to ensure that the following two kinds of infinite sequences are not possible:

(1) Infinite sequences A where only a finite number of positions are $\mathcal{R}_F$-enabled (i.e., after a finite number of steps *all* terms in the sequence are $\mathcal{R}_F$-normal forms). This means that A is as follows:

$$t_0 \to_{\mathcal{R}}^{*} t_1 \to_{\mathcal{S}} t_2 \to_{\mathcal{S}} t_3 \to_{\mathcal{S}} \cdots$$

where $t_i \in \mathsf{NF}_{\mathcal{R}_F}$ for $i \geq 1$.

(2) Infinite sequences A where some rules in $\mathcal{R}_F$ are infinitely often taken. This means that A is as follows:

$$t_1 \to_{\mathcal{S}}^{*} t_1' \to_{\mathcal{R}_F} t_2 \to_{\mathcal{S}}^{*} t_2' \to_{\mathcal{R}_F} t_3 \to_{\mathcal{S}} \cdots$$

Since infinite 1-label $\mathcal{R}_F$-fair sequences must exactly be of one of these two forms, we have the following.

Theorem 2 *A TRS $\mathcal{R} = \mathcal{R}_F \cup \mathcal{S}$ with $\mathcal{R}_F$ finite is 1-label $\mathcal{R}_F$-fair-terminating if and only if $\to_{\mathcal{S}} \cap \mathsf{NF}_{\mathcal{R}_F}^2$ and $\to_{\mathcal{S}}^{*} \circ \to_{\mathcal{R}_F}$ are terminating.*

Example 8 *Consider the TRS $\mathcal{R}$ in Example 2 and assume that $\mathcal{R}_F$ consists of the rule α_3. Let $\mathcal{S} = \mathcal{R} - \mathcal{R}_F$. Then, we have that $\mathsf{NF}_{\mathcal{R}_F} = \{\mathsf{a}, \mathsf{c}\} \cup \mathcal{X}$. Thus, $\to_{\mathcal{S}} \cap \mathsf{NF}_{\mathcal{R}_F}^2 = \{$`a -> c`$\}$ is terminating. Also, since every $\mathcal{R}_F$-step yields* `c` *which is an $\mathcal{R}$-normal form (hence also an $\mathcal{S}$-normal form and $\mathcal{R}_F$-normal* *form), we have that* $\to_{\mathcal{S}}^{*} \circ \to_{\mathcal{R}_F}$ *is also terminating. Therefore,* $\mathcal{R}$ *is 1-label* $\mathcal{R}_F$*-fairly-terminating. In Example 7, we have shown that* $\mathcal{R}$ *is neither rule* $\mathcal{R}_F$*-justly-terminating nor 1-label* $\mathcal{R}_F$*-justly-terminating.*

Example 9 *Consider the following TRS* $\mathcal{R}$*:*

α_1 : `a -> c(a,b)` α_3 : `b -> d`

α_2 : `a -> d`

and assume that $\mathcal{R}_F$ *consists of the rules* α_1 *and* α_2*. Then,* $\mathcal{S} = \mathcal{R} - \mathcal{R}_F$ *consists of the rule* α_3*. The sequence*

$$\underline{\mathtt{a}} \to_{\alpha_1} \mathtt{c(\underline{a},b)} \to_{\alpha_1} \mathtt{c(c(a,b),b)} \to_{\alpha_1} \cdots$$

shows that $\to_{\mathcal{S}}^{*} \circ \to_{\mathcal{R}_F}$ *is not terminating, i.e.,* $\mathcal{R}$ *is* not *1-label* $\mathcal{R}_F$*-fairly-terminating. Therefore, it is not 1-label* $\mathcal{R}_F$*-justly-terminating. However, it is possible to see that* $\mathcal{R}$ *is rule* $\mathcal{R}_F$*-justly terminating: since the sub-TRS consisting of the rules* α_2 *and* α_3 *is clearly terminating, every infinite* $\mathcal{R}$*-sequence must perform an infinite number of applications of* α_1*. But each application of* α_1 *keeps both* α_1 *and* α_2 *enabled. As soon as* α_2 *is applied, the sequence terminates; on the other hand, if only* α_1 *is applied infinitely often and* α_2 *is not applied at all, then the infinite sequence is not just (because* α_2 *is continuously enabled but is never taken). Thus,* $\mathcal{R}$ *is rule* $\mathcal{R}_F$*-justly-terminating.*

Note that termination of $\to_{\mathcal{S}}^{*} \circ \to_{\mathcal{R}_F}$ implies termination of $\mathcal{R}_F$. Therefore, we can give the following easy necessary condition for 1-label $\mathcal{R}_F$-fair-termination.

Proposition 9 *If a TRS* $\mathcal{R} = \mathcal{R}_F \cup \mathcal{S}$ *with* $\mathcal{R}_F$ *finite is* 1*-label* $\mathcal{R}_F$*-fair-terminating, then* $\mathcal{R}_F$ *is terminating.*

Again, we can give the following necessary condition for rule $\mathcal{R}_F$-fair-termination, which can be used to disprove it.

Proposition 10 *If a TRS* $\mathcal{R} = \mathcal{R}_F \cup \mathcal{S}$ *with* $\mathcal{R}_F$ *finite is rule* $\mathcal{R}_F$*-fair-terminating, then* $\to_{\mathcal{S}} \cap \mathsf{NF}^2_{\mathcal{R}_F}$ *is terminating.*

PROOF. If $\to_{\mathcal{S}} \cap \mathsf{NF}^2_{\mathcal{R}_F}$ is not terminating, then there is an infinite sequence

$$t_1 \to_{\mathcal{S}} t_2 \to_{\mathcal{S}} t_3 \to_{\mathcal{S}} \cdots$$

where $t_i \in \mathsf{NF}_{\mathcal{R}_F}$ for $i \geq 1$ which is clearly rule $\mathcal{R}_F$-fair. This contradicts rule $\mathcal{R}_F$-fair-termination of $\mathcal{R}$. □

We can use Proposition 4 and Theorem 2 to obtain a sufficient condition for proving rule $\mathcal{R}_F$-fair-termination of TRSs. Furthermore, Proposition 10 shows that termination of $\to_{\mathcal{S}} \cap \mathsf{NF}^2_{\mathcal{R}_F}$ is necessary for rule $\mathcal{R}_F$-fair-termination. The following example, however, shows that Theorem 2 does *not* extend (in the *only if* direction) to rule $\mathcal{R}_F$-fair-termination (i.e., termination of $\to^*_{\mathcal{S}} \circ \to_{\mathcal{R}_F}$ is *not necessary* for rule $\mathcal{R}_F$-fair-termination).

Example 10 *Consider the following TRS $\mathcal{R}$ [28]:*

```
a -> f(a)          g(a,b) -> c          a -> g(a,b)
```

which is rule fairly-terminating. It is not difficult to see that $\mathcal{R}$ is $\mathcal{R}_F$-fairly-terminating when $\mathcal{R}_F \subset \mathcal{R}$ is given by the two rightmost rules above. However, since $\mathcal{R}_F$ is not *terminating, $\to^*_{\mathcal{S}} \circ \to_{\mathcal{R}_F}$ is nonterminating. Thus, $\mathcal{R}$ is not 1-label $\mathcal{R}_F$-fairly-terminating. Moreover, $\to_{\mathcal{R}_F} \cap \mathsf{RED}^2_{\mathcal{R}_F}$ is not terminating, since we have the $\mathcal{R}_F$-sequence:*

$$\mathtt{a} \to_{\mathcal{R}_F} \mathtt{g(\underline{a},b)} \to_{\mathcal{R}_F} \mathtt{g(g(\underline{a},b),b)} \to_{\mathcal{R}_F} \cdots$$

where all terms are $\mathcal{R}_F$-reducible. Therefore, $\mathcal{R}$ is not 1-label $\mathcal{R}_F$-justly-terminating. Furthermore, the infinite sequence

$$\underline{\mathtt{a}} \to_{\mathcal{R}_F} \mathtt{g(\underline{a},b)} \to_{\mathcal{S}} \mathtt{g(f(\underline{a}),b)} \to_{\mathcal{R}_F} \mathtt{g(f(g(a,b)),b)} \to \cdots$$

is $\mathcal{R}_F$-just: the second rule of $\mathcal{R}_F$ is continuously enabled and infinitely often taken, whereas the first rule of $\mathcal{R}_F$ is not *continuously enabled. Thus, $\mathcal{R}$ is* not *rule justly-terminating.*

We end this section with the following result, which connects the four termination properties which are used in Theorems 1 and 2.

Proposition 11 *Let $\mathcal{R}$ be a TRS, $\mathcal{R}_F \subseteq \mathcal{R}$ and $\mathcal{S} = \mathcal{R} - \mathcal{R}_F$.*

(1) If $\to^*_{\mathcal{S}} \circ \to_{\mathcal{R}_F}$ is terminating, then $(\to_{\mathcal{S}} \cap \mathsf{RED}^2_{\mathcal{R}_F})^* \circ (\to_{\mathcal{R}_F} \cap \mathsf{RED}^2_{\mathcal{R}_F})$ is terminating.
(2) If $\to^*_{\mathcal{R}} \circ (\to_{\mathcal{S}}|_{\mathsf{NF}_{\mathcal{R}_F}})$ is terminating, then $\to_{\mathcal{S}} \cap \mathsf{NF}^2_{\mathcal{R}_F}$ is terminating.

Proof.

(1) Trivial.
(2) Termination of $\to^*_{\mathcal{R}} \circ (\to_{\mathcal{S}}|_{\mathsf{NF}_{\mathcal{R}_F}})$ clearly implies termination of $\to_{\mathcal{S}}|_{\mathsf{NF}_{\mathcal{R}_F}}$. Since $\to_{\mathcal{S}} \cap \mathsf{NF}^2_{\mathcal{R}_F} \subseteq \to_{\mathcal{S}}|_{\mathsf{NF}_{\mathcal{R}_F}}$, termination of $\to_{\mathcal{S}} \cap \mathsf{NF}^2_{\mathcal{R}_F}$ follows.

□

6 Proving Just/Fair-Termination

According to Corollary 1 and Proposition 4, we can prove rule fair-termination and rule $\mathcal{R}_F$-fair-termination of a TRS $\mathcal{R}$ by proving 1-label $\mathcal{R}_F$-fair-termination of $\mathcal{R}$. By Theorems 1 and 2, given a TRS $\mathcal{R}$, $\mathcal{R}_F \subseteq \mathcal{R}$ and $\mathcal{S} = \mathcal{R} - \mathcal{R}_F$, we can characterize 1-label $\mathcal{R}_F$-fair-termination and 1-label $\mathcal{R}_F$-just-termination, by respectively proving termination of the following reduction relations:

1-label $\mathcal{R}_F$-fair-termination	1-label $\mathcal{R}_F$-just-termination
$\to_{\mathcal{S}}^* \circ \to_{\mathcal{R}_F}$	$(\to_{\mathcal{S}} \cap \mathsf{RED}^2_{\mathcal{R}_F})^* \circ (\to_{\mathcal{R}_F} \cap \mathsf{RED}^2_{\mathcal{R}_F})$
$\to_{\mathcal{S}} \cap \mathsf{NF}^2_{\mathcal{R}_F}$	$\to_{\mathcal{R}}^* \circ (\to_{\mathcal{S}} \vert_{\mathsf{NF}_{\mathcal{R}_F}})$

Thus, in the following, we consider how to address these four termination problems in more detail.

6.1 Termination of $\to_{\mathcal{S}}^* \circ \to_{\mathcal{R}_F}$

Given binary relations $\to_1$ and $\to_2$ on an abstract set A, $\to_1$ is called *relatively noetherian* (or better, *relatively terminating*) with respect to $\to_2$ if every infinite $\to_1 \cup \to_2$-sequence contains only finitely many $\to_1$-steps (see [11, Section 2.1], although the notion goes back to Klop: see also [16, Exercise 2.0.8(11)]). In his PhD thesis [11], Geser has investigated relative termination. In our setting, this notion is interesting due to the following result.

Proposition 12 [11] *Let $\to_1$ and $\to_2$ be binary relations. Then, $\to_2^* \circ \to_1$ is terminating if and only if $\to_1$ is relatively terminating with respect to $\to_2$.*

Thus, according to this result, termination of $\to_{\mathcal{S}}^* \circ \to_{\mathcal{R}_F}$ can be investigated as the relative termination of $\mathcal{R}_F$ w.r.t. $\mathcal{S}$. Fortunately, there are even automatic tools such as Jambox [9], Matchbox [33], and TPA [17], which can be used to prove or disprove relative termination of TRSs.

Example 11 *Consider the TRS $\mathcal{R}$ in Example 5. Let $\mathcal{R}_F$ be the sub-TRS consisting of the rule* `a -> b` *and $\mathcal{S} = \mathcal{R} - \mathcal{R}_F$. The* TPA *tool [17] can be used to prove termination of $\to_{\mathcal{S}}^* \circ \to_{\mathcal{R}_F}$.*

Consider again the system $\mathcal{R}$ in Example 1 with $\mathcal{R}_F$ consisting of the rules `end`, `execute`, *and* `remove` *and $\mathcal{S} = \mathcal{R} - \mathcal{R}_F$ consisting of rules* `round`, `shift1`, *and* `shift2`. *We have used* TPA *to obtain an automatic proof of termination of $\to_{\mathcal{S}}^* \circ \to_{\mathcal{R}_F}$.*

6.2 Termination of $\to_{\mathcal{S}} \cap \mathsf{NF}^2_{\mathcal{R}_F}$

As remarked before, termination of $\to_{\mathcal{S}} \cap \mathsf{NF}^2_{\mathcal{R}_F}$ is guaranteed if $\mathcal{S}$ terminates, but this can lead to a quite restrictive setting. The following result is helpful to overcome this problem.

Proposition 13 *Let $\mathcal{R}$ and $\mathcal{S}$ be two TRSs. Let $\mathcal{S}' = \{l \to r \in \mathcal{S} \mid l, r \in \mathsf{NF}_{\mathcal{R}}\}$. Then, $\to_{\mathcal{S}} \cap \mathsf{NF}^2_{\mathcal{R}}$ is terminating if and only if $\to_{\mathcal{S}'} \cap \mathsf{NF}^2_{\mathcal{R}}$ is terminating.*

PROOF. By definition of $\mathcal{S}'$, we have $\to_{\mathcal{S}'} \cap \mathsf{NF}^2_{\mathcal{R}} = \to_{\mathcal{S}} \cap \mathsf{NF}^2_{\mathcal{R}}$. □

Corollary 2 *Let $\mathcal{R}$ and $\mathcal{S}$ be two TRSs and $\mathcal{S}' = \{l \to r \in \mathcal{S} \mid l, r \in \mathsf{NF}_{\mathcal{R}}\}$. If $\mathcal{S}'$ is terminating, then $\to_{\mathcal{S}} \cap \mathsf{NF}^2_{\mathcal{R}}$ is terminating.*

Example 12 *Consider the TRS $\mathcal{R}$ in Example 5 with $\mathcal{R} = \mathcal{R}_F \cup \mathcal{S}$ as in Example 11. Since $\mathcal{S}'$ defined as in Corollary 2 is empty, $\to_{\mathcal{S}} \cap \mathsf{NF}^2_{\mathcal{R}_F}$ is terminating.*

Consider again the TRS in Example 1 with $\mathcal{R}_F$ and $\mathcal{S}$ as in Example 11. The use of Corollary 2 yields a simpler version $\mathcal{S}'$ of $\mathcal{S}$, which consists of the rules `shift1` *and* `shift2`*. Since $\mathcal{S}'$ can be proved terminating (by using, e.g.,* AProVE*), by Proposition 13, $\to_{\mathcal{S}} \cap \mathsf{NF}^2_{\mathcal{R}_F}$ is also terminating.*

The following example shows the limitations of this approach.

Example 13 *Consider the following TRS $\mathcal{R}$:*

```
f(a,a) -> a

f(a,X) -> f(X,a)
```

Let $\mathcal{R}_F$ be the sub-TRS of $\mathcal{R}$ consisting of the first rule and $\mathcal{S} = \mathcal{R} - \mathcal{R}_F$. Although $\to_{\mathcal{S}} \cap \mathsf{NF}^2_{\mathcal{R}_F}$ is clearly terminating, it is not possible to use Corollary 2 to prove it: both the lhs `f(a,X)` *and rhs* `f(X,a)` *are $\mathcal{R}_F$-normal forms. Thus, $\mathcal{S}' = \mathcal{S}$ is nonterminating.*

6.3 Termination of $(\to_{\mathcal{S}} \cap \mathsf{RED}^2_{\mathcal{R}_F})^* \circ (\to_{\mathcal{R}_F} \cap \mathsf{RED}^2_{\mathcal{R}_F})$

First, note that if we write $\to_2 = \to_{\mathcal{S}} \cap \mathsf{RED}^2_{\mathcal{R}_F}$ and $\to_1 = \to_{\mathcal{R}_F} \cap \mathsf{RED}^2_{\mathcal{R}_F}$, then termination of $(\to_{\mathcal{S}} \cap \mathsf{RED}^2_{\mathcal{R}_F})^* \circ (\to_{\mathcal{R}_F} \cap \mathsf{RED}^2_{\mathcal{R}_F})$ becomes termination of $\to_2^* \circ \to_1$, which is equivalent to the termination of $\to_1$ relative to $\to_2$.

However, by Proposition 11, termination of $\to_{\mathcal{S}}^* \circ \to_{\mathcal{R}_F}$ (which we can try to prove automatically by the methods and tools discussed in Section 6.1) is a sufficient condition for termination of $(\to_{\mathcal{S}} \cap \mathsf{RED}^2_{\mathcal{R}_F})^* \circ (\to_{\mathcal{R}_F} \cap \mathsf{RED}^2_{\mathcal{R}_F})$. Furthermore, if $\to^*_{\mathcal{S}} \circ \to_{\mathcal{R}_F}$ is not terminating, then by Theorem 2 $\mathcal{R}$ is not 1-label $\mathcal{R}_F$-fairly-terminating. Thus, by Proposition 4, $\mathcal{R}$ is *not* 1-label $\mathcal{R}_F$-justly-terminating. Therefore, if termination of $\to^*_{\mathcal{S}} \circ \to_{\mathcal{R}_F}$ can actually be *disproved*, then we are also disproving 1-label $\mathcal{R}_F$-just-termination of $\mathcal{R}$, and proving termination of $(\to_{\mathcal{S}} \cap \mathsf{RED}^2_{\mathcal{R}_F})^* \circ (\to_{\mathcal{R}_F} \cap \mathsf{RED}^2_{\mathcal{R}_F})$ becomes useless. In this respect, it is interesting to note that recent termination tools like Matchbox [33] are able to disprove relative termination.

Therefore, if we can prove or disprove termination of $\to^*_{\mathcal{S}} \circ \to_{\mathcal{R}_F}$, it makes no sense to try to prove termination of $(\to_{\mathcal{S}} \cap \mathsf{RED}^2_{\mathcal{R}_F})^* \circ (\to_{\mathcal{R}_F} \cap \mathsf{RED}^2_{\mathcal{R}_F})$.

Disproving termination of $\to_{\mathcal{R}_F} \cap \mathsf{RED}^2_{\mathcal{R}_F}$ amounts to disproving termination of $(\to_{\mathcal{S}} \cap \mathsf{RED}^2_{\mathcal{R}_F})^* \circ (\to_{\mathcal{R}_F} \cap \mathsf{RED}^2_{\mathcal{R}_F})$. And since termination of $\mathcal{R}_F$ is necessary for termination of $\to_{\mathcal{R}_F} \cap \mathsf{RED}^2_{\mathcal{R}_F}$, we have the following result which can be used to *disprove* termination of $(\to_{\mathcal{S}} \cap \mathsf{RED}^2_{\mathcal{R}_F})^* \circ (\to_{\mathcal{R}_F} \cap \mathsf{RED}^2_{\mathcal{R}_F})$. In fact, it is implicit in the proof of Proposition 7.

Proposition 14 *If* $(\to_{\mathcal{S}} \cap \mathsf{RED}^2_{\mathcal{R}_F})^* \circ (\to_{\mathcal{R}_F} \cap \mathsf{RED}^2_{\mathcal{R}_F})$ *is terminating, then* $\mathcal{R}_F$ *is terminating.*

6.4 Termination of $\to^*_{\mathcal{R}} \circ (\to_{\mathcal{S}} |_{\mathsf{NF}_{\mathcal{R}_F}})$

Clearly, $\to^*_{\mathcal{R}} \circ (\to_{\mathcal{S}} |_{\mathsf{NF}_{\mathcal{R}_F}})$ terminates if $\to^*_{\mathcal{R}} \circ \to_{\mathcal{S}}$ terminates; futhermore, we have the following.

Proposition 15 *Let* $\to, \to_1$ *and* $\to_2$ *be binary relations on a set* A *such that* $\to = \to_1 \cup \to_2$. *Then,* $\to^* \circ \to_2$ *is terminating if and only if* $\to^*_1 \circ \to_2$ *is terminating.*

Proof. The *only if* part is obvious. For the *if* part, just consider that, since $\to = \to_1 \cup \to_2$, any $\to^* \circ \to_2$-sequence can always be written as a sequence of $\to_2$-steps with possible $\to_1$-steps in the middle. This yieds an infinite $\to^*_1 \circ \to_2$-sequence, which contradicts termination of $\to^* \circ \to_2$. □

Therefore, we can prove termination of $\to^*_{\mathcal{R}} \circ (\to_{\mathcal{S}} |_{\mathsf{NF}_{\mathcal{R}_F}})$ by proving relative termination of $\mathcal{S}$ w.r.t. $\mathcal{R}_F$.

Corollary 3 *If* $\to^*_{\mathcal{R}_F} \circ \to_{\mathcal{S}}$ *is terminating, then* $\to^*_{\mathcal{R}} \circ (\to_{\mathcal{S}} |_{\mathsf{NF}_{\mathcal{R}_F}})$ *is terminating.*

Let $\mathcal{R}$ and $\mathcal{S}$ be two TRSs, and let $\mathcal{S}' = \{l \to r \in \mathcal{S} \mid l \in \mathsf{NF}_{\mathcal{R}}\}$. Then, $\to_{\mathcal{S}} |_{\mathsf{NF}_{\mathcal{R}}} = \to_{\mathcal{S}'} |_{\mathsf{NF}_{\mathcal{R}}}$ and $\to^*_{\mathcal{R}} \circ (\to_{\mathcal{S}} |_{\mathsf{NF}_{\mathcal{R}}}) = \to^*_{\mathcal{R}} \circ (\to_{\mathcal{S}'} |_{\mathsf{NF}_{\mathcal{R}}})$. We can use this fact to prove termination of $\to^*_{\mathcal{R}} \circ (\to_{\mathcal{S}} |_{\mathsf{NF}_{\mathcal{R}_F}})$ by proving termination of $\to^*_{\mathcal{R}} \circ \to_{\mathcal{S}'}$, i.e., by proving the relative termination of $\mathcal{S}'$ w.r.t. $\mathcal{R}$.

Corollary 4 *Let $\mathcal{R}$ and $\mathcal{S}$ be two TRSs. Let $\mathcal{S}' = \{l \to r \in \mathcal{S} \mid l \in \mathsf{NF}_{\mathcal{R}}\}$. If $\to^*_{\mathcal{R}} \circ \to_{\mathcal{S}'}$ is terminating, then $\to^*_{\mathcal{R}} \circ (\to_{\mathcal{S}} |_{\mathsf{NF}_{\mathcal{R}_F}})$ is terminating.*

Example 14 *Consider the TRS $\mathcal{R}$ in Example 5 with $\mathcal{R} = \mathcal{R}_F \cup \mathcal{S}$ as in Example 11. Since here $\mathcal{S}'$, defined as in Corollary 4, is empty, $\to^*_{\mathcal{R}} \circ (\to_{\mathcal{S}} |_{\mathsf{NF}_{\mathcal{R}_F}})$ is also empty and hence terminating.*

Furthermore, we have the following result which allows us to prove termination of $\to^*_{\mathcal{R}} \circ (\to_{\mathcal{S}} |_{\mathsf{NF}_{\mathcal{R}_F}})$ by proving *termination* of $\mathcal{S}'$ as above.

Proposition 16 *Let $\mathcal{R}$ and $\mathcal{S}$ be TRSs such that $\mathcal{S}$ preserves the $\mathcal{R}$-normal forms. If $\mathcal{S}' = \{l \to r \in \mathcal{S} \mid l \in \mathsf{NF}_{\mathcal{R}}\}$ is terminating, then $\to^*_{\mathcal{R}\cup\mathcal{S}} \circ (\to_{\mathcal{S}} |_{\mathsf{NF}_{\mathcal{R}}})$ is terminating.*

PROOF. If $\mathcal{S}$ preserves the $\mathcal{R}$-normal forms, then $\to_{\mathcal{S}} |_{\mathsf{NF}_{\mathcal{R}}} = \to_{\mathcal{S}} \cap \mathsf{NF}^2_{\mathcal{R}}$ and

$$(\to^*_{\mathcal{R}\cup\mathcal{S}} \circ (\to_{\mathcal{S}} |_{\mathsf{NF}_{\mathcal{R}}}))^+ = \to^*_{\mathcal{R}\cup\mathcal{S}} \circ (\to_{\mathcal{S}} \cap \mathsf{NF}^2_{\mathcal{R}})^+$$

Clearly, termination of $(\to_{\mathcal{S}} \cap \mathsf{NF}^2_{\mathcal{R}})^+$ implies that of $\to^*_{\mathcal{R}\cup\mathcal{S}} \circ (\to_{\mathcal{S}} \cap \mathsf{NF}^2_{\mathcal{R}})^+$. Termination of $(\to_{\mathcal{S}} \cap \mathsf{NF}^2_{\mathcal{R}})^+$ is equivalent to termination of $\to_{\mathcal{S}} \cap \mathsf{NF}^2_{\mathcal{R}}$. By Proposition 13, termination of $\mathcal{S}'$ implies termination of $\to_{\mathcal{S}} \cap \mathsf{NF}^2_{\mathcal{R}}$ (note that, since $\mathcal{S}' \subseteq \mathcal{S}$ preserves the $\mathcal{R}$-normal forms, we must have $r \in \mathsf{NF}_{\mathcal{R}}$ for all $l \to r \in \mathcal{S}'$; thus Proposition 13 applies to $\mathcal{S}'$ as defined here). □

We can use the previous results as an alternative method for proving termination of $\to^*_{\mathcal{R}} \circ (\to_{\mathcal{S}} |_{\mathsf{NF}_{\mathcal{R}_F}})$ as follows.

Corollary 5 *Let $\mathcal{R}$ be a TRS, $\mathcal{R}_F \subseteq \mathcal{R}$, and $\mathcal{S} = \mathcal{R} - \mathcal{R}_F$ be such that $\mathcal{S}$ preserves the $\mathcal{R}_F$-normal forms. Let $\mathcal{S}' = \{l \to r \in \mathcal{S} \mid l \in \mathsf{NF}_{\mathcal{R}_F}\}$. If $\mathcal{S}'$ is terminating, then $\to^*_{\mathcal{R}} \circ (\to_{\mathcal{S}} |_{\mathsf{NF}_{\mathcal{R}_F}})$ is terminating.*

In Section 8 we use these results for proving just/fair termination.

7 A Method for Proving Just/Fair-Termination as Termination

The results presented in this paper provide a characterization of 1-*label* rule just/fair-termination as termination of a number of combinations of relations (Theorems 1 and 2). According to Theorem 1, proving 1-label $\mathcal{R}_F$-just-termination is equivalent to proving:

T_{J1} : Termination of $(\to_{\mathcal{S}} \cap \mathsf{RED}^2_{\mathcal{R}_F})^* \circ (\to_{\mathcal{R}_F} \cap \mathsf{RED}^2_{\mathcal{R}_F})$, and

T_{J2} : Termination of $\to^*_{\mathcal{R}} \circ (\to_{\mathcal{S}} |_{\mathsf{NF}_{\mathcal{R}_F}})$.

By Theorem 2, proving 1-label $\mathcal{R}_F$-fair-termination is equivalent to proving:

T_{F1} : Termination of $\to^*_{\mathcal{S}} \circ \to_{\mathcal{R}_F}$, and

T_{F2} : Termination of $\to_{\mathcal{S}} \cap \mathsf{NF}^2_{\mathcal{R}_F}$.

More general problems such as rule $\mathcal{R}_F$-just/fair-termination are *reduced* to the former ones (see Figure 1). In general, however, in doing this we only get sufficient conditions for such problems (see Proposition 4).

7.1 Proving rule just/fair-termination

Proposition 4 provides the basis for proving rule $\mathcal{R}_F$-just/fair-termination by proving 1-label $\mathcal{R}_F$-just/fair-termination. Regarding *refutations* of just/fair $\mathcal{R}_F$-termination properties, we can use Propositions 8 and 10 (among others) to translate them into refutations of termination of concrete (combinations of) reduction relations.

7.1.1 Rule $\mathcal{R}_F$-fair-termination

PROBLEM F0: Given a TRS $\mathcal{R}$, is $\mathcal{R}$ fairly-terminating?

Since Porat and Francez's notion of fair-termination of a TRS $\mathcal{R}$ is rule $\mathcal{R}$-fair-termination (Remark 2), we can use Proposition 6 to prove fair-termination of $\mathcal{R}$ by proving rule $\mathcal{R}_F$-fair-termination of $\mathcal{R}$ for some $\mathcal{R}_F \subseteq \mathcal{R}$. Hence, we *look* for a finite sub-TRS $\mathcal{R}_F \subseteq \mathcal{R}$ and go to Problem F1 below to try to prove the new reformulation of the problem. Regarding the choice of $\mathcal{R}_F$, see Remark 6 below.

PROBLEM F1: Given a TRS $\mathcal{R}$ and a finite sub-TRS $\mathcal{R}_F \subseteq \mathcal{R}$, is $\mathcal{R}$ rule $\mathcal{R}_F$-fairly-terminating?

In order to *disprove* $\mathcal{R}_F$-fair-termination of $\mathcal{R}$, we can use Proposition 10 for *disproving* rule $\mathcal{R}_F$-fair-termination of $\mathcal{R}$ by disproving T_{F2}. Additionally, if $\mathcal{R}_F$ is a single rule TRS, then rule $\mathcal{R}_F$-fair-termination and 1-label $\mathcal{R}_F$-fair-termination are equivalent; by Proposition 9, we can disprove $\mathcal{R}_F$-fair-termination of $\mathcal{R}$ by disproving termination of $\mathcal{R}_F$.

According to Proposition 6, we can *prove* $\mathcal{R}_F$-fair-termination of $\mathcal{R}$ by *looking for* a subset $\mathcal{R}'_F \subseteq \mathcal{R}_F$ and, by Proposition 4, proving 1-label $\mathcal{R}'_F$-fair-termination of $\mathcal{R}$ (see Section 7.2 below). If $\mathcal{R}$ is proved 1-label $\mathcal{R}'_F$-fairly-terminating, then it is also rule $\mathcal{R}_F$-fairly-terminating; otherwise, nothing can be said with the present methods.

Remark 4 *In particular, choosing $\mathcal{R}_F = \varnothing$ amounts to proving $\varnothing$-fair-termination of $\mathcal{R}$ which, as discussed in Section 4, is equivalent to proving termination of $\mathcal{R}$ (Remark 3), which of course implies all kinds of just/fair-termination properties discussed here.*

7.1.2 Rule $\mathcal{R}_F$-just-termination

PROBLEM J0: Given a TRS $\mathcal{R}$, is $\mathcal{R}$ justly-terminating?

Following Porat and Francez's notion of fair-termination of a TRS $\mathcal{R}$, we can speak of just-termination of $\mathcal{R}$ as rule $\mathcal{R}$-just-termination. According to Proposition 6 we can *look for* a *single rule* sub-TRS $\mathcal{R}_F \subseteq \mathcal{R}$ and go to Problem J1 below to try to prove the new formulation of the problem.

Remark 5 *In contrast to fair computations, there is no general connection between rule and* 1*-label $\mathcal{R}_F$-just-termination. For this reason, we cannot deal with arbitrary sub-TRSs $\mathcal{R}_F \subseteq \mathcal{R}$ and we restrict the attention to single-rule TRSs $\mathcal{R}_F$ for which we can prove rule $\mathcal{R}_F$-just-termination by proving* 1*-label $\mathcal{R}_F$-just-termination.*

PROBLEM J1: Given a TRS $\mathcal{R}$ and a single rule sub-TRS $\mathcal{R}_F \subseteq \mathcal{R}$, is $\mathcal{R}$ rule $\mathcal{R}_F$-justly-terminating?

Since our techniques actually prove 1-label $\mathcal{R}_F$-just-termination, we try to prove 1-label $\mathcal{R}_F$-just-termination of $\mathcal{R}$ (see Section 7.2 below).

Regarding *refutations* of rule $\mathcal{R}_F$-just-termination, we can use Proposition 7 to disprove 1-label $\mathcal{R}_F$-fair-termination of $\mathcal{R}$ by disproving termination of $\mathcal{R}_F$. Furthermore, we can use Proposition 8 to disprove rule $\mathcal{R}_F$-just-termination, even when $\mathcal{R}_F$ consists of *more* than one rule.

Remark 6 *According to our method, which proves rule $\mathcal{R}_F$-just/fair-termi-*

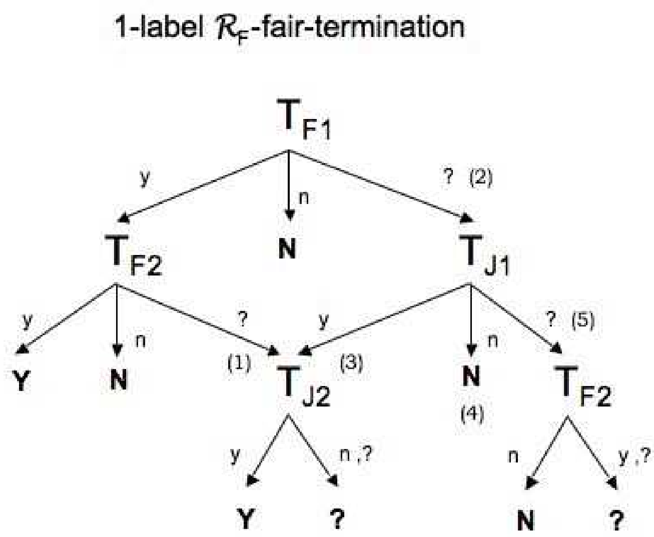

Fig. 2. Proving 1-label $\mathcal{R}_F$-fair-termination

nation by proving 1*-label* $\mathcal{R}_F$*-just/fair-termination,* $\mathcal{R}_F$ *(or* $\mathcal{R}'_F \subseteq \mathcal{R}_F$*) should be chosen to be* terminating*. Otherwise, by Propositions 7 and 9,* $\mathcal{R}$ *is* not 1*-label* $\mathcal{R}_F$*-justly/fairly-terminating.*

7.2 Proving 1-label just/fair-termination

The results in Section 5 are used here, sometimes in combination with those in Section 3, to establish two decision graphs which can be used to prove 1-label $\mathcal{R}_F$-just/fair-termination. In particular, we use the following two facts:

Fact 1: 1-label $\mathcal{R}_F$-just-termination implies 1-label $\mathcal{R}_F$-fair-termination (Proposition 4).
Fact 2: T_{F1} implies T_{J1} (Proposition 11).

The decision graphs are shown in Figures 2 and 3. In order to use them, the nodes should be visited from top to bottom, performing the corresponding termination tests which are supposed to either give a positive answer (y), a negative answer (n), or fail/abort (*?*). The answer indicates which is the next node to be visited: the arc whose label corresponds to the answer should be followed. We briefly justify both decision graphs as follows:

1-label $\mathcal{R}_F$*-fair-termination*

- The two arcs with label 'n' which leave the nodes T_{F1} and T_{F2} clearly lead to *disproving* 1-label $\mathcal{R}_F$-fair-termination, because both T_{F1} and T_{F2}

are necessary conditions for the property. When both proofs succeed, we conclude the 1-label $\mathcal{R}_F$-fair-termination of $\mathcal{R}$.

- If, after proving T_{F1}, the proof of T_{F2} fails, then we can use Fact 1 and try to prove 1-label $\mathcal{R}_F$-just-termination; however, since at this stage we know that T_{F1} holds, by Fact 2 we only need to check T_{J2} (node with label (1) in the diagram). If T_{J2} holds, then we conclude the 1-label $\mathcal{R}_F$-fair-termination of $\mathcal{R}$. Otherwise, we cannot say anything.
- If nothing can be said about T_{F1}, then we can use Fact 1 and try to prove 1-label $\mathcal{R}_F$-just-termination (2). Since T_{F1} and T_{J1} are connected by Fact 2, we use this knowledge during the proof of T_{J1}: see the next bullets.
- If the proof of T_{J1} fails, then we cannot conclude anything about the 1-label $\mathcal{R}_F$-fair-termination of $\mathcal{R}$.
- If we can prove T_{J1}, then we still need to check T_{J2} as above (3).
- If we can *disprove* T_{J1}, then by Fact 2 this also disproves T_{F1} and then we can conclude that $\mathcal{R}$ is *not* 1-label $\mathcal{R}_F$-fairly-terminating (4).
- Finally, if the proof of T_{J1} fails, we can still try to *disprove* 1-label $\mathcal{R}_F$-fair-termination by disproving T_{F2} (node with label (5)). However, at this node of the decision graph, where we know that T_{F1} failed, if T_{F2} can be proved (or fails) we cannot conclude anything regarding 1-label $\mathcal{R}_F$-fair-termination of $\mathcal{R}$.

1-label $\mathcal{R}_F$-just-termination

- The two arcs with label 'n' which leave the nodes T_{J1} and T_{J2} clearly lead to *disproving* 1-label $\mathcal{R}_F$-just-termination, because both T_{J1} and T_{J2} are necessary conditions for the property. When both proofs succeed, we conclude the 1-label $\mathcal{R}_F$-just-termination of $\mathcal{R}$.
- If, after proving T_{J1}, the proof of T_{J2} fails, then we can still *disprove* 1-label $\mathcal{R}_F$-just-termination by disproving T_{F2}: indeed, if T_{F2} does not hold, then $\mathcal{R}$ is *not* 1-label $\mathcal{R}_F$-fairly-terminating and, by Fact 1, it is not 1-label $\mathcal{R}_F$-justly-terminating. Otherwise (i.e., T_{F2} holds or fails to be proved), we cannot conclude anything regarding 1-label $\mathcal{R}_F$-just-termination of $\mathcal{R}$.
- If nothing can be said about T_{J1}, then we can use Fact 2 and try to prove T_{F1} (node with label (1)), see the next bullets.
- If we can prove T_{F1}, by Fact 2 T_{J1} holds but we still need to check T_{J2}.
- If we can *disprove* T_{F1}, then $\mathcal{R}$ is *not* 1-label $\mathcal{R}_F$-fairly-terminating and, by Fact 1, we can also conclude that $\mathcal{R}$ is *not* 1-label $\mathcal{R}_F$-justly-terminating (2).
- If the proof of T_{F1} fails (node with label (3)), then we can still *disprove* 1-label $\mathcal{R}_F$-just-termination by disproving T_{F2} as explained above.

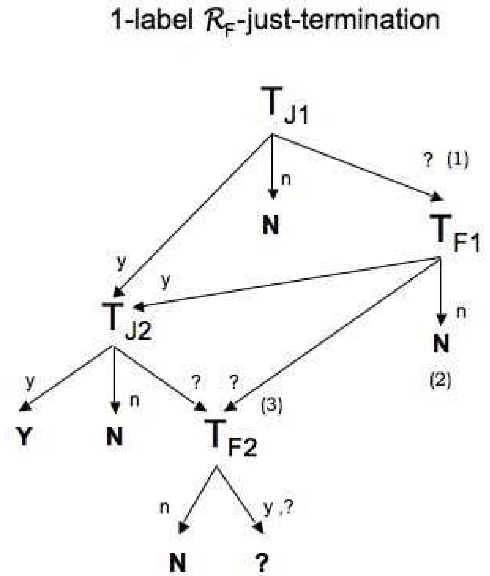

Fig. 3. Proving 1-label $\mathcal{R}_F$-just-termination

7.3 Proving T_{F1}, T_{F2}, T_{J1}, and T_{J2}

As discussed in Section 6, it is possible to use standard termination techniques (and the corresponding termination tools) for proving the required termination properties T_{F1}, T_{F2}, T_{J1}, and T_{J2}.

(1) As discussed in Section 6.1, T_{F1}, i.e., termination of $\to_{\mathcal{S}}^* \circ \to_{\mathcal{R}_F}$, is equivalent to relative termination of $\mathcal{R}_F$ w.r.t. $\mathcal{S}$. Relative termination problems in term rewriting can be specified by means of the TPDB format[1] and are accepted by several termination tools like Jambox, Matchbox, and TPA. Proofs and refutations of relative termination can be obtained in this way. We can also disprove T_{F1} by disproving termination of $\mathcal{R}_F$. Fortunately, termination tools like AProVE and TTT, among others, are able to disprove termination of TRSs (although they do not currently handle relative termination problems).
(2) The second termination problem, T_{F2}, i.e., termination of $\to_{\mathcal{S}} \cap \mathsf{NF}^2_{\mathcal{R}_F}$ can be proved by proving termination of $\mathcal{S}' = \{l \to r \in \mathcal{S} \mid l, r \in \mathsf{NF}_{\mathcal{R}_F}\}$ (Corollary 2).
(3) Regarding T_{J1}, i.e., termination of $(\to_{\mathcal{S}} \cap \mathsf{RED}^2_{\mathcal{R}_F})^* \circ (\to_{\mathcal{R}_F} \cap \mathsf{RED}^2_{\mathcal{R}_F})$, we do not provide any particular technique to directly address it. Rather, we propose (and motivate, see Section 6.3) proving termination of $\to_{\mathcal{S}}^* \circ \to_{\mathcal{R}_F}$, i.e., the relative termination of $\mathcal{R}_F$ w.r.t. $\mathcal{S}$. Again, it is possible

[1] See the *Termination Problems Data Base* (TPDB): `http://www.lri.fr/~marche/tpdb`

to disprove T_{J1} by disproving termination of $\mathcal{R}_F$ (Proposition 14).

(4) Finally, regarding T_{J2}, i.e., termination of $\to^*_{\mathcal{R}} \circ (\to_{\mathcal{S}} |_{\mathsf{NF}_{\mathcal{R}_F}})$, in Section 6.3 we provide a number of possibilities for proving this property. We propose to sequentially proceed as follows: for $\mathcal{S}' = \{l \to r \in \mathcal{S} \mid l \in \mathsf{NF}_{\mathcal{R}_F}\}$:
   (a) If $\mathcal{S}$ preserves the $\mathcal{R}_F$-normal forms, then prove termination of $\mathcal{S}'$ (Corollary 5).
   (b) Prove termination of $\to^*_{\mathcal{R}} \circ \to_{\mathcal{S}'}$, i.e., the relative termination of $\mathcal{S}'$ w.r.t. $\mathcal{R}$ (Corollary 4).
   (c) Prove termination of $\to^*_{\mathcal{R}_F} \circ \to_{\mathcal{S}}$, i.e., the relative termination of $\mathcal{S}$ w.r.t. $\mathcal{R}_F$ (Corollary 3).

   Of course, since all these proof methods are just *sufficient conditions* for proving T_{J2}, we can only use the positive answers. Nothing can be concluded from negative ones (refutations).

8 More examples

In this section we describe several examples of nonterminating systems which are justly- or fairly-terminating and, for most of them, show how to formally prove this property using our results. We start with our main example by proving that it is fairly-terminating (Problem F0).

Example 15 *(Continuing Example 1) Let $\mathcal{R}_F$ be composed by the rules* `end`, `execute`*, and* `remove`*. Then, $\mathcal{S}$ contains rules* `round`, `shift1`*, and* `shift2`*. As shown in Examples 11 and 12, T_{F1} and T_{F2} hold. Thus, by Theorem 2,* 1*-label $\mathcal{R}_F$-fair-termination of $\mathcal{R}$ follows. By Proposition 4, rule $\mathcal{R}_F$-fair-termination holds and by Proposition 6, $\mathcal{R}$ is fairly-terminating.*

Examples from Porat and Francez's papers

The following example was proved fairly-terminating by Porat and Francez. We refine their result by showing that it is actually *justly-terminating* according to our definitions in Section 3.

Example 16 *The following TRS $\mathcal{R}$ [28, page 289, second example, 1)]:*

```
a -> b          a -> f(a)
```

is justly-terminating: take $\mathcal{R}_F = \{$`a -> b`*$\}$. Then, $\mathcal{S} = \mathcal{R} - \mathcal{R}_F$ consists of the rule* `a -> f(a)`*. Then, $\to^*_{\mathcal{S}} \circ \to_{\mathcal{R}_F}$ is clearly terminating. By Proposition 11, $(\to_{\mathcal{S}} \cap \mathsf{RED}^2_{\mathcal{R}_F})^* \circ (\to_{\mathcal{R}_F} \cap \mathsf{RED}^2_{\mathcal{R}_F})$ is terminating. Now consider $\mathcal{S}' = \{l \to r \in \mathcal{S} \mid l \in \mathsf{NF}_{\mathcal{R}_F}\} = \varnothing$ which is also terminating. Obviously, $\mathcal{S}$ preserves* *the $\mathcal{R}_F$-normal forms. Thus, by Corollary 4, $\to_{\mathcal{R}}^* \circ (\to_{\mathcal{S}}|_{\mathsf{NF}_{\mathcal{R}_F}})$ is terminating. Therefore, by Theorem 1, $\mathcal{R}$ is* 1*-label $\mathcal{R}_F$-justly-terminating and $\mathcal{R}$ is rule $\mathcal{R}_F$-justly-terminating. Finally, by Proposition 6, we conclude just-termination of $\mathcal{R}$.*

Fair-termination of $\mathcal{R}$ in Example 16 above follows from the just-termination proved in the example by using Proposition 4.

Porat and Francez also showed fair-termination of the following example. We prove it by using our techniques and also show that it is a *proper* case of fair-termination in the sense that it is *not* justly-terminating.

Example 17 *The following TRS $\mathcal{R}$ [28, page 290, second example]:*

```
b -> c          b -> a          f(X,X) -> f(a,b)
```

is fairly-terminating. Take $\mathcal{R}_F = \{$`b -> c`*$\}$ and $\mathcal{S} = \mathcal{R} - \mathcal{R}_F$. Then, $\to_{\mathcal{S}}^* \circ \to_{\mathcal{R}_F}$ is clearly terminating and so is $\mathcal{S}' = \{l \to r \in \mathcal{S} \mid l, r \in \mathsf{NF}_{\mathcal{R}_F}\} = \varnothing$. By Corollary 2, $\to_{\mathcal{S}} \cap \mathsf{NF}^2_{\mathcal{R}_F}$ is terminating. Thus, by Theorem 2, $\mathcal{R}$ is* 1*-label $\mathcal{R}_F$-fairly-terminating and $\mathcal{R}$ is rule $\mathcal{R}_F$-fairly-terminating. Finally, by Proposition 6, we conclude fair-termination of $\mathcal{R}$. Note that $\mathcal{R}$ is* not *justly-terminating: the infinite sequence*

$\underline{\texttt{f(a,a)}} \to_{\mathcal{R}} \texttt{f(a,}\underline{\texttt{b}}\texttt{)} \to_{\mathcal{R}} \underline{\texttt{f(a,a)}} \to_{\mathcal{R}} \cdots$

is just, because no rule is continuously enabled.

The following example is also fairly-terminating but it is not justly-terminating.

Example 18 *Consider the following TRS $\mathcal{R}$ [28, page 294, example 1]:*

```
a -> c          a -> b          b -> a
```

This TRS is fairly-terminating: Take $\mathcal{R}_F = \{$`a -> c`*$\}$. Then, $\to_{\mathcal{S}}^* \circ \to_{\mathcal{R}_F}$ is clearly terminating and $\mathcal{S}' = \{l \to r \in \mathcal{S} \mid l, r \in \mathsf{NF}_{\mathcal{R}_F}\} = \varnothing$ also is. Thus, by Theorem 2, $\mathcal{R}$ is* 1*-label $\mathcal{R}_F$-fairly-terminating and $\mathcal{R}$ is rule $\mathcal{R}_F$-fairly-terminating. Finally, by Proposition 6, we conclude fair-termination of $\mathcal{R}$. The system is not rule justly terminating: the infinite sequence*

$\texttt{a} \to_{\mathcal{R}} \texttt{b} \to_{\mathcal{R}} \texttt{a} \to_{\mathcal{R}} \texttt{b} \to_{\mathcal{R}} \cdots$

is just, because no rule is continuously enabled.

Example 19 *Consider the following TRS $\mathcal{R}$ [28, page 292, remark]:*

```
a -> f(a)          g(a,b) -> c          a -> g(a,b)
```

This TRS is fairly-terminating but, as discussed in Example 10, we cannot prove it by using our current results.

Lottery

Consider the following scenario: a lottery where a finite number of balls are rolling inside a container assumed here to be circular. Eventually, a ball will be removed to pick a number and, of course, the repeated extraction of balls will make the whole process terminating. The following TRS can be used to model this process:

```
[extract]   cons(X,XS) -> XS

[shift]     cons(X,cons(Y,XS)) -> cons(Y,snoc(XS,X))

[circular1] snoc(nil,X) -> cons(X,nil)

[circular2] snoc(cons(X,XS),Y) -> cons(X,snoc(XS,Y))
```

Here, $\mathcal{R}_F$ consists of the rule `extract`, which represents the extraction of a ball. The remaining rules (`shift`, `circular1` and `circular2`) are collected into a nonterminating TRS $\mathcal{S}$ which represents a finite list whose elements are shifted in a circular fashion over and over again.

Let us prove that $\mathcal{R}$ is fairly-terminating w.r.t. $\mathcal{R}_F$. According to Corollary 4 and Theorem 2, we *have* to prove that both $\rightarrow_{\mathcal{S}} \cap \mathsf{NF}^2_{\mathcal{R}_F}$ and $\rightarrow^*_{\mathcal{S}} \circ \rightarrow_{\mathcal{R}_F}$ are terminating. Regarding termination of $\rightarrow^*_{\mathcal{S}} \circ \rightarrow_{\mathcal{R}_F}$, by Proposition 12 this is equivalent to proving that $\mathcal{R}_F$ is relatively terminating with respect to $\mathcal{S}$. We have used TPA to obtain an automatic proof of this. Regarding termination of $\rightarrow_{\mathcal{S}} \cap \mathsf{NF}^2_{\mathcal{R}_F}$, we can use Proposition 13 to obtain a sub-TRS $\mathcal{S}'$ of $\mathcal{S}$ which only contains `circular1`. The TRS $\mathcal{S}'$ is obviously terminating. Thus, by Corollary 2, $\rightarrow_{\mathcal{S}} \cap \mathsf{NF}^2_{\mathcal{R}_F}$ is also terminating and we conclude that $\mathcal{R}$ is $\mathcal{R}_F$-fairly-terminating.

Noisy channel

Consider the following scenario: there are three agents A, B, and C. Agents A and B have to perform tasks a and b (respectively) in a distributed fashion. Agent C receives information about their completion through a two-component channel. Agent A (resp. B), writes “a”, (resp. “b”) on the corresponding channel to communicate to C that his/her task has been finished. Once the tasks performed by A and B have both terminated, C closes the channel. However, the channel is *noisy* in such a way that, when both values are in it, they may get lost. Thus, both A and B may have to repeat their respective signals before the channel is closed. The following TRS can be used to model this process:

```
[A]    [null,Y] -> [a,Y]

[B]    [X,null] -> [X,b]

[C]    [a,b] -> done

[loss] [a,b] -> [null,null]
```

The key point here is that if rule `C` is fair, then the system is terminating. Thus, we consider $\mathcal{R}_F$ consisting of rule `C`.

Let us prove that $\mathcal{R}$ is fairly-terminating w.r.t. $\mathcal{R}_F$. Let $\mathcal{S} = \mathcal{R} - \mathcal{R}_F$, i.e., $\mathcal{S}$ contains the rules `A`, `B` and `loss` (and it is nonterminating). According to Corollary 4 and Theorem 2, we *have* to prove that both $\to_{\mathcal{S}} \cap \mathsf{NF}^2_{\mathcal{R}_F}$ and $\to^*_{\mathcal{S}} \circ \to_{\mathcal{R}_F}$ are terminating. Regarding termination of $\to^*_{\mathcal{S}} \circ \to_{\mathcal{R}_F}$, by Proposition 12 this is equivalent to proving that $\mathcal{R}_F$ is relatively terminating with respect to $\mathcal{S}$. Again, we have used TPA to obtain an automatic proof of this. Regarding termination of $\to_{\mathcal{S}} \cap \mathsf{NF}^2_{\mathcal{R}_F}$, we use Proposition 13 to obtain a simpler version $\mathcal{S}'$ of $\mathcal{S}$, namely, $\mathcal{S}'$ containing rules `A` and `B`. Termination of $\to_{\mathcal{S}} \cap \mathsf{NF}^2_{\mathcal{R}_F}$ is equivalent to termination of $\to_{\mathcal{S}'} \cap \mathsf{NF}^2_{\mathcal{R}_F}$. The TRS $\mathcal{S}'$ is easily proved terminating. Hence, $\mathcal{R}$ is $\mathcal{R}_F$-fairly-terminating.

9 Related work

Porat and Francez's pioneered the research on fair computations and fair-termination in term rewriting. In [28,29], they introduce the notion of fair computation in term rewriting and give a definition of termination of such computations (fair-termination). As remarked in Sections 3 and 4, these notions are subsumed by ours. They also investigated how to prove fair termination, but followed Floyd's classical predicate-based approach, which is completely different from ours. In fact, they do not discuss how to automatically prove fair-termination in their approach. Tison also investigated decidability of fair-termination of rewriting and gave some results for ground TRSs [32].

Various other approaches to fairness within term rewriting have been developed so far. In particular, the notion of fairness as related to the removal of (residuals) of *redexes* rather than concerning the application of rules is well-known after O'Donnell's work [26] on the so-called *outermost-fair reduction* *strategy* and the corresponding normalization results [26,14]. O'Donnell's notion of fairness was intended to provide a basis for computing the normal form of terms. In those works, a (finite or infinite) reduction sequence $t_1 \to t_2 \to \cdots$ is fair if for all $i \geq 1$, and (position of a) redex Δ in t_i, there is $j > i$ such that t_j does not contain any residual of Δ [31, Definition 4.9.10] (see also [16]). It is not difficult to see that this notion of fairness is not comparable to ours. Following these works, fairness plays a very important role in infinitary rewriting as an essential ingredient of strategies which intend to approximate infinitary normal forms [15]. The introduced notions, however, follow the previous style and become, then, incomparable to ours.

Termination techniques have been recently proposed as suitable tools for proving *liveness properties of fair computations* [18]. As in our approach, Koprowski and Zantema define fairness by using two TRSs. According to [18, Section 2.2], an infinite reduction in $\mathcal{R}_F \cup \mathcal{S}$ is called *fair* (w.r.t. $\mathcal{R}_F$) if it contains infinitely many $\mathcal{R}_F$-steps [2]. Nothing is said regarding finite reduction sequences. No distinction between enabled and taken steps is made. Actually, infinite fair sequences in Koprowski and Zantema's sense are of the form

$$t_1 \to^*_{\mathcal{S}} t'_1 \to_{\mathcal{R}_F} t_2 \to^*_{\mathcal{S}} t'_2 \to_{\mathcal{R}_F} \cdots$$

where rules in $\mathcal{R}_F$ are infinitely often enabled (in t'_i) and infinitely often taken (in t'_i again, for $i \geq 1$) without paying attention to the particular rules in $\mathcal{R}_F$ which are enabled or taken. So, infinite fair sequences in Koprowski and Zantema's sense are easily seen to be 1-label $\mathcal{R}_F$-fair. However, there are infinite 1-label $\mathcal{R}_F$-fair sequences which are *not* fair according to [18].

Example 20 *Consider the TRS* $\mathcal{R}$

```
a -> b

b -> b
```

Let $\mathcal{R}_F$ *consists of the first rule (thus,* $\mathcal{S}$ *consists of the second rule). Then, the infinite sequence*

$$\texttt{b} \to_{\mathcal{S}} \texttt{b} \to_{\mathcal{S}} \cdots$$

is not *fair (w.r.t.* $\mathcal{R}_F$*) in Koprowski and Zantema's sense (there is no* $\mathcal{R}_F$*-rewriting step) but it is indeed* 1*-label* $\mathcal{R}_F$*-fair (rules in* $\mathcal{R}_F$ *are never enabled).*

Thus, Koprowski and Zantema's fairness does not really coincide with 1-label

[2] Actually, Koprowski and Zantema's paper use $\mathcal{R}^=$ and $\mathcal{R}$ instead of our notation $\mathcal{S}$ and $\mathcal{R}_F$, respectively.

$\mathcal{R}_F$-fairness. Furthermore, the notions of fairness we have discussed in this paper are more expressive: rule $\mathcal{R}_F$-fairness is not captured by [18] as soon as we use different labels for the rules in $\mathcal{R}_F$.

Koprowski and Zantema's paper does not try to give any notion of fair-termination of TRSs (since this is not their concern, they actually do not mention any of the papers on this topic in the literature [28,29,32]). An attempt to give such a definition in a reasonable way (e.g., "there is no infinite fair sequence") would actually lead to *characterizing* such a notion of fair-termination as relative termination (i.e., termination of $\rightarrow_{\mathcal{S}}^* \circ \rightarrow_{\mathcal{R}_F}$). In particular, the TRS $\mathcal{R}$ in Example 20 would be considered as fairly-terminating (indeed, $\rightarrow_{\mathcal{S}}^* \circ \rightarrow_{\mathcal{R}_F}$ is terminating).

Our analysis, however, has shown that, even for the 1-label $\mathcal{R}_F$-fair case, relative termination is only *one* of the ingredients of 1-label $\mathcal{R}_F$-fair-termination (see Theorem 2). Furthermore, we have shown that termination of $\rightarrow_{\mathcal{S}} \cap \mathsf{NF}^2_{\mathcal{R}_F}$ (the other ingredient) is *necessary* for all other notions of fair-termination investigated here (Proposition 10). Nothing about that is discussed in [18], perhaps in agreement with the fact that they do not consider the problem of termination of fair computations. According to our results, for instance, $\mathcal{R}$ in Example 20 is not 1-label $\mathcal{R}_F$-fairly-terminating because $\rightarrow_{\mathcal{S}} \cap \mathsf{NF}^2_{\mathcal{R}_F} = \rightarrow_{\mathcal{S}}$ is not terminating.

As far as we are aware of, [21] is the first attempt to use standard termination techniques and tools for proving fair-termination of rewriting. This paper is an extended and revised version of [21]. The main differences are:

(1) Justice is not treated in any way in [21].
(2) The termination notions for labelled justice and fairness, and for 1-label $\mathcal{R}_F$-just-computation and 1-label $\mathcal{R}_F$-fair-computation are new.
(3) The results which characterize 1-label $\mathcal{R}_F$-just-termination and 1-label $\mathcal{R}_F$-fair-termination by means of termination of (combinations of) standard reduction relations are new.
(4) The necessary conditions for all just/fair-termination properties discussed in this paper are new.
(5) Except for termination of $\rightarrow_{\mathcal{S}}^* \circ \rightarrow_{\mathcal{R}_F}$, the termination properties which are used and discussed in Sections 5 and 6 are new. The results for proving and disproving them are also new.
(6) The detailed proof/refutation method described in Section 5 is new.
(7) We have given more examples, both from the related literature and as a way to motivate concurrent system applications.

10 Conclusions and future work

On the basis of the notion of *localized* justice/fairness properties in term rewriting [24], we have defined the notions of labelled justice and fairness (Definition 1). Although fairness has been investigated for term rewriting by Porat and Francez [28], the termination notions for *labelled* justice and fairness that we have introduced here seem to be new in the literature. Furthermore, Porat and Francez's notions of fairness are covered as special cases of our definition. We have defined the two specialized notions of rule and 1-label $\mathcal{R}_F$-justice, and rule and 1-label $\mathcal{R}_F$-fairness (Definition 2). Roughly speaking, the difference between rule and 1-label $\mathcal{R}_F$-justice/fairness arises when we distinguish each rule with a different label (in the first case) or we do not distinguish their labels at all (in the second case). We have investigated the connections between these two notions (Propositions 1 and 2).

We have defined rule and 1-label $\mathcal{R}_F$-just-termination and rule and 1-label $\mathcal{R}_F$-fair-termination (Definition 3). Again, Porat and Francez's notions of fair-termination are particular cases of ours (Remark 2). We have investigated the connections between these notions of termination. Our results are summarized in Figure 1. Specifically, 1-label $\mathcal{R}_F$-fairly-terminating TRSs $\mathcal{R}$ are rule $\mathcal{R}_F$-fairly-terminating. In contrast, there are 1-label $\mathcal{R}_F$-justly-terminating TRSs which are *not* rule $\mathcal{R}_F$-justly-terminating (and viceversa!). Nevertheless, rule $\mathcal{R}_F$-justly-terminating TRSs which are 1-label $\mathcal{R}_F$-*fairly*-terminating are 1-label $\mathcal{R}_F$-justly-terminating. Finally, every 1-label $\mathcal{R}_F$-justly-terminating TRS $\mathcal{R}$ is 1-label $\mathcal{R}_F$-fairly-terminating. If $\mathcal{R}_F$ is a single rule TRS, then rule fair-termination and 1-label fair-termination coincide. The same happens regarding just-termination.

We have shown that the problem of proving 1-label $\mathcal{R}_F$-just-termination of a TRS $\mathcal{R}$ w.r.t. a sub-TRS $\mathcal{R}_F$ (where we let $\mathcal{S} = \mathcal{R} - \mathcal{R}_F$) is *equivalent* to the problems of proving termination of $(\to_{\mathcal{S}} \cap \mathsf{RED}^2_{\mathcal{R}_F})^* \circ (\to_{\mathcal{R}_F} \cap \mathsf{RED}^2_{\mathcal{R}_F})$, and $\to^*_{\mathcal{R}} \circ (\to_{\mathcal{S}}|_{\mathsf{NF}_{\mathcal{R}_F}})$ (Theorem 1). Similarly, proving 1-label $\mathcal{R}_F$-fair-termination is equivalent to proving termination of $\to^*_{\mathcal{S}} \circ \to_{\mathcal{R}_F}$ and $\to_{\mathcal{S}} \cap \mathsf{NF}^2_{\mathcal{R}_F}$ (Theorem 2).

We have given necessary conditions for rule $\mathcal{R}_F$-just-termination (Proposition 8) and rule $\mathcal{R}_F$-fair-termination (Proposition 10). We have also given useful necessary conditions for 1-label just-termination (Proposition 7) and 1-label fair-termination (Proposition 9)

We have investigated how to prove termination of such particular relations. Termination of $\to^*_{\mathcal{S}} \circ \to_{\mathcal{R}_F}$ is equivalent to the relative termination of $\mathcal{R}_F$ w.r.t. $\mathcal{S}$. Regarding termination of the other reduction relations, we have given a number of results which allow us to use standard methods and tools for proving them (Corollaries 2, 3, 4 and 5). Also, we provide a number of results for disproving some of these properties (Proposition 14). By using our results, we can reduce the automatic proof of just/fair termination to that of standard termination problems, namely: proving and disproving termination and relative termination of TRSs, which can be addressed by existing termination tools. We have shown how to combine the results in this paper to provide a practical *proof method* for proving fair-termination (Section 7).

A number of interesting issues remain to be investigated. For instance, Example 10 (which we cannot handle at present with our method) shows that a deeper analysis is needed to extend the use of termination techniques (and tools) for proving fair-termination. Regarding future extensions of our techniques, we think that the following ones are interesting to consider:

(1) Investigating more precise methods for proving and disproving the main termination properties enumerated in Section 6.
(2) Just/fair-termination *modulo* a set of equations. Fairness modulo a set of equations (and the corresponding termination notion) was already considered by Porat and Francez [29], but without exploiting standard termination techniques and tools.
(3) Another important aspect of fairness is that, in many applications, only initial expressions satisfying concrete properties are expected to exhibit a fairly-terminating behavior. Indeed, this can be crucial to achieve fair-termination in some cases.
(4) The role of typing information in fair-termination. It is well-known that types play an important role in termination. As shown in [7,8], it is possible to deal with termination of sorted TRS by reducing this problem to the problem of proving termination of a TRS (without sorts). We believe that a similar treatment could be useful for fair-termination.
(5) The implementation of our techniques, and their associated proof method, in a system like MTT [7,8] which is able to use external tools to solve termination problems is also envisaged. This will enable the possibility of more experimentation on practical examples, probably in the context of some of the extensions 1-4 above.

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References

[1] B. Alarcón, R. Gutiérrez, J. Iborra, and S. Lucas. Proving Termination of Context-Sensitive Rewriting with MU-TERM. *Electronic Notes in Theoretical*

Computer Science, 188:105-115, 2007.

[2] G. Berry and G. Boudol. The Chemical Abstract Machine. *Theoretical Computer Science* 96(1):217–248, 1992.

[3] E. Contejean and C. Marché, B. Monate and X. Urbain. Proving termination of rewriting with C*i*ME. In A. Rubio, editor, *Proc. of 6th International Workshop on Termination, WST'03*, pages 71-73, Technical Report DSIC II/15/03, Valencia, Spain, 2003. Available at `http://cime.lri.fr`.

[4] N. Dershowitz. A note on simplification orderings. *Information Processing Letters*, 9(5):212-215, 1979.

[5] N. Dershowitz. Orderings for term rewriting systems. *Theoretical Computer Science*, 17(3):279–301, 1982.

[6] N. Dershowitz. Termination of rewriting. *Journal of Symbolic Computation*, 3:69-115, 1987.

[7] F. Durán, S. Lucas, J. Meseguer, C. Marché, and X. Urbain. Proving Termination of Membership Equational Programs. In P. Sestoft and N. Heintze, editors, *Proc. of ACM SIGPLAN 2004 Symposium on Partial Evaluation and Program Manipulation, PEPM'04*, pages 147-158, ACM Press, New York, 2004.

[8] F. Durán, S. Lucas, J. Meseguer, C. Marché, and X. Urbain. Proving Operational Termination of Membership Equational Programs. *Higher-Order and Symbolic Computation*, to appear, 2007.

[9] J. Endrullis. Jambox, 2006. `http://joerg.endrullis.de`.

[10] N. Francez. Fairness. Springer-Verlag, Berlin, 1986.

[11] A. Geser. Relative Termination. PhD Thesis. Fakultät für Mathematik und Informatik. Universität Passau, 1990.

[12] J. Giesl, P. Schneider-Kamp, and R. Thiemann. AProVE 1.2: Automatic Termination Proofs in the Dependency Pair Framework. In U. Furbach and N. Shankar, editors, *Proc. of Third International Joint Conference on Automated Reasoning, IJCAR'06*, LNAI 4130:281-286, Springer-Verlag, Berlin, 2006. Available at `http://www-i2.informatik.rwth-aachen.de/AProVE`.

[13] N. Hirokawa and A. Middeldorp. Tyrolean Termination Tool. In J. Giesl, editor, *Proc. of 16th International Conference on Rewriting Techniques and Applications, RTA'05*, LNCS 3467:175-184, 2005. Available at `http://cl2-informatik.uibk.ac.at`.

[14] G. Huet and J.J. Lévy. Computations in orthogonal term rewriting systems I, II. In J.L. Lassez and G. Plotkin, editors, *Computational logic: essays in honour of J. Alan Robinson*, pages 395-414 and 415-443. The MIT Press, Cambridge, MA, 1991.

[15] R. Kennaway, J.W. Klop, M.R. Sleep, and F.-J. de Vries. Transfinite Reductions in Orthogonal Term Rewriting Systems. *Information and Computation* 119(1):18-38, 1995.

[16] J.W. Klop. Term Rewriting Systems. In S. Abramsky, D.M. Gabbay, and T.S.E. Maibaum. *Handbook of Logic in Computer Science*, volume 2, pages 1-116. Oxford University Press, 1992.

[17] A. Koprowski. TPA: Termination Proved Automatically. In F. Pfenning, editor, *Proc of the 18th International Conference on Rewriting Techniques and Applications, RTA'06*, LNCS 4098:257-266, Springer Verlag, Berlin, 2006. `http://www.win.tue.nl/tpa`

[18] A. Koprowski and H. Zantema. Proving Liveness with Fairness using Rewriting. In B. Gramlich, editor, *Proc. of the 5th International Workshop on Frontiers of Combining Systems, FroCoS'05*, LNAI 3717:232-247, 2005.

[19] D. Lehmann, A. Pnueli, and J. Stavi. Impartiality, Justice and Fairness: the ethics of concurrent termination. In S. Even and O. Kariv, editors, *Proc. of the 8th International Colloquium on Automata, Languages, and Programming, ICALP'81*, LNCS 115:264-277, Springer-Verlag, Berlin, 1981.

[20] S. Lucas. MU-TERM: A Tool for Proving Termination of Context-Sensitive Rewriting In V. van Oostrom, editor, *Proc. of 15h International Conference on Rewriting Techniques and Applications, RTA'04*, LNCS 3091:200-209, Springer-Verlag, Berlin, 2004. Available at `http://www.dsic.upv.es/~slucas/csr/termination/muterm`.

[21] S. Lucas and J. Meseguer. Termination of Fair Computations in Term Rewriting. In G. Sutcliff and A. Voronkov, editors, *Proc. of 12th Conference on Logic for Programming, Artificial Intelligence and Reasoning, LPAR'05*, LNAI 3835:184-198, Springer-Verlag, Berlin, 2005.

[22] Z. Manna and S. Ness. On the termination of Markov algorithms. In *Proc. of the Third Hawaii International Conference on System Science*, pages 789-792, 1970.

[23] J. Meseguer. Conditional Rewriting Logic as a Unified Model of Concurrency. *Theoretical Computer Science* 96(1):73-55, 1992.

[24] J. Meseguer. Localized Fairness: A Rewriting Semantics. In J. Giesl, editor, *Proc. of the 16th International Conference on Rewriting Techniques and Applications, RTA'05*, LNCS 3467:250-263, Springer-Verlag, Berlin, 2005.

[25] N. Martí-Oliet and J. Meseguer. Rewriting logic: roadmap and bibliography. *Theoretical Computer Science* 285(2):121–154, 2002.

[26] M.J. O'Donnell. Computing in Systems Described by Equations. LNCS 58, Springer-Verlag, Berlin, 1977.

[27] E. Ohlebusch. *Advanced Topics in Term Rewriting.* Springer-Verlag, Berlin, 2002.

[28] S. Porat and N. Francez. Fairness in term rewriting systems. In J.-P. Jouannaud, editor, *Proc. of the 1st International Conference on Rewriting Techniques and Applications, RTA'85*, LNCS 202:287-300, Springer-Verlag, Berlin, 1985.

[29] S. Porat and N. Francez. Full-commutation and fair-termination in equational (and combined) term rewriting systems. In J.H. Siekmann, editor, *Proc. of the 8th International Conference on Automated Deduction, CADE'86*, LNCS 230:21-41, Springer-Verlag, Berlin, 1986.

[30] J. Steinbach. Simplification orderings: History of results. *Fundamenta Informaticae*, 24:47–88, 1995.

[31] TeReSe, editor, *Term Rewriting Systems*, Cambridge University Press, 2003.

[32] S. Tison. Fair termination is decidable for ground systems. In N. Dershowitz, editor, *Proc. of the 3rd International Conference on Rewriting Techniques and Applications, RTA'89*, LNCS 355:462-476, Springer-Verlag, Berlin, 1989.

[33] J. Waldmann. Matchbox: A Tool for Match-Bounded String Rewriting. In V. van Oostrom, editor, *Proc. of the 15h International Conference on Rewriting Techniques and Applications, RTA'04*, LNCS 3091:85-94, Springer-Verlag, Berlin, 2004. Available at `http://141.57.11.163/matchbox`.