BARGAINING THEORY

by

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DECLARATION

The thesis is the result of my own work and includes nothing which is the outcome of work done in collaboration.

An earlier version of chapter 3, under my authorship and as a result of solely my own work, has been awarded the 1987 University of Cambridge Stevenson prize.

ABSTRACT

This thesis is devoted to the study of bargaining using the methods of non-cooperative game theory. In Part I (chapters 1 and 2) we examine the role of commitment in bilateral bargaining. Two different notions of commitment in bargaining are explored in two different non-cooperative infinite-time horizon sequential games with complete and perfect information. In Part II (chapters 3 and 4) we examine the role of outside options in bilateral bargaining. Two models are presented, each model is a non-cooperative infinite-time horizon sequential game with complete information. The two models differ in their approach to modelling the interlacing of the search and bargaining processes. In the final part of this thesis, Part III (chapter 5), we present a theory of a decentralised market, based on the idea that the agents of the market search for partners with whom to trade and when a buyer and a seller meet they initiate a sequential bargaining process over the terms of trade.

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While working on this thesis several individuals have helped me in various ways. I would like to take the opportunity to express my thanks to all of them.

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I am grateful to King's College, Cambridge, for financial support for the past three years.

My parents have been a source of constant inspiration to me.

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I wish to thank Partha Dasgupta for his helpful comments on the first drafts of chapters 3 and 4 and for discussing with me the economic implications of the models discussed in chapter 4.

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[illegible] in the development of the Keynesian theory [illegible] of Cambridge for their helpful comments on [illegible] chapters 4, 5 and 6, in particular on this matter.

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CONTENTS

INTRODUCTION

Bargaining is a pervasive phenomena in modern societies, ranging from trade and wage negotiations to arms control talks to haggling in a bazaar. For this reason alone the study of bargaining is useful and interesting. Moreover, bargaining occupies an important place in economic theory, since the "pure bargaining problem" is at the opposite pole of economic phenomena from "perfect competition". This, therefore, is another important reason to study bargaining.

In this thesis we shall study bargaining using the methods of non-cooperative game theory. Parts I and II of this thesis are concerned with the roles of commitment and outside options in bilateral bargaining. Part III deals with embedding a bilateral bargaining model in a large market context.

The starting point for this thesis is the classic paper by Rubinstein (1982). In that paper, Rubinstein presents a non-cooperative infinite-time horizon sequential game with complete information, which represents a bargaining process. In that game, the two bargainers make offers alternately until agreement on the partition of the surplus is reached. Rubinstein proved the existence of a unique sub-game perfect equilibrium partition. One may interpret the Rubinstein game as one that exemplifies the role of time in bilateral bargaining.

All the bargaining games to be presented in this thesis either are entirely based on this alternating-offers sequential game due to Rubinstein or else incorporate the essence of this alternating-offers notion in an infinite-time horizon sequential game. Furthermore, throughout the thesis our "method of proof" will depend on that presented by Rubinstein in his 1982 paper.

In this thesis we shall restrict attention to bargaining games with complete information, of which Rubinstein's (1982) game is an example. We shall take on board this assumption so as to cast away the issues that arise with incomplete information; and thus be able to focus on the issues that concern us. A future research programme would be to

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In this thesis we shall restrict attention to bargaining games with complete information, of which Rubinstein's (1982) game is an example. We shall take on board this assumption so as to cast away the issues that arise with incomplete information; and thus be able to focus on the issues that concern us. A future research programme would be to add certain kinds of differential information to the models to be presented in this thesis and then investigate the resulting change in the equilibrium outcomes. However, it may be noted that such a research programme will have to wait until the confusion that currently resides in the literature on "the correct analysis of games with incomplete information" is resolved; in this context Binmore (1987b) provides a thought-provoking discussion of the possible causes of this confusion and the possible routes to remedy.

In Part I (chapters 1 and 2) of the thesis we examine the role of commitment in bilateral bargaining. Two different notions of commitment in bargaining are explored in two different non-cooperative infinite-time horizon sequential games with complete and perfect information.

The first notion of commitment that we explore is the following. Bargainers can take actions during the negotiation process to increase the future cost of backing down from a committed bargaining position (for example, not to budge from an offer, or not to accept an opponent's offer). In this notion, making a commitment is costless, but revoking a commitment is costly.

In the second notion of commitment that we explore, the reverse is true: making a commitment is costly, but revoking a commitment is costless. In this notion commitment has a time dimension, in that, a bargainer chooses, strategically, the length of time for which he is committed. Commitment in this second notion is irrevocable during the length of time that a bargainer has chosen to commit himself. Thus, commitment is irrevocable *à la* "Nash demand game" (i.e., a genuine unconditional take-it-or-leave-it offer) only if a bargainer chooses to commit himself for an infinite length of time.

In Part II (chapters 3 and 4) of this thesis we examine the role of outside options in bilateral bargaining. Two models are presented, each model is a non-cooperative infinite-time horizon sequential game with complete information. Both models explicitly take into account the search dimension of the situation: essentially the issue is that the bargainers have to engage in some sort of search in order to find an outside option. The two models differ in their approach to

modelling the interlacing of the search and bargaining processes. Once one takes the search dimension of the situation into account, one has to form a "view" on how the search and bargaining processes ought to be interlaced. Two such "views" are explored in chapters 3 and 4.

In the final part of the thesis, Part III (chapter 5), we develop a theory of a market in which the institution of price formation is decentralised. The agents of the market (i.e., the buyers and the sellers) do not, ex-ante, know each others' location. The theory, therefore, includes a matching technology within which the agents search for partners with whom to trade. When a buyer and a seller meet, they initiate a sequential bargaining process over the terms of trade. In other words, chapter 5 is concerned with embedding a bilateral bargaining model in a large market context.

Throughout the thesis, the solution concept that we shall employ in order to analyse the bargaining games is the sub-game perfect equilibrium concept due to Selten (1965, 1975).

Finally, a word of caution. The chapters are self-contained, in that each chapter can be read independently. This has meant (a) that some statements will appear in each of the chapters, for example, the definitions of a bargaining situation and of the sub-game perfect equilibrium concept, and (b) that the numbering of equations, footnotes and figures in any one chapter is independent from that of any other chapter.

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PART ONE

COMMITMENT

Chapter 1

The role of commitment in bargaining I

1. INTRODUCTION

In this and the next chapter we shall examine the role of commitment in bilateral bargaining. In each of the two chapters, a model will be presented that will capture some of the features of the notion of commitment in bargaining. Each of the models is a non-cooperative infinite-time horizon sequential game with complete and perfect information.

It is now fairly well-established that one cannot usually offer a sensible estimate of what is likely to happen in bilateral bargaining situations without having a view on the roles to be ascribed to (a) commitment, (b) time and (c) information within the bargaining process (see Binmore and Dasgupta (1987, Chap. 1) for an introductory discussion of this point). The commitment aspects of bargaining have hardly been explored in the recent literature on the non-cooperative game-theoretic strategic approach to bargaining. On the other hand, time and information have received a great deal of attention in this literature. There is no doubt that commitment must have an important role in shaping the outcome in bilateral bargaining situations, as is argued forcefully by Schelling in his classic paper (Schelling (1956)), where he focused entirely on the commitment aspects of bargaining. In fact, Schelling views the bargaining process as a struggle between bargainers to commit themselves to - that is, to convince their opponent that they will not retreat from - advantageous bargaining positions.

Schelling defines commitment impressionistically and by way of examples, but the essential idea seems to involve making a demand and 'burning one's bridges', or taking actions during the negotiation process that increase the future cost of backing down from one's demand. In this chapter, a generalised version of this very idea will be formalised in a game-theoretic model with complete information (i.e., no informational asymmetries).

The Nash (1953) demand game can be viewed as a model of commitment in which commitments are <u>irrevocable</u>. In the Nash demand game, there is only one stage, in which bargainers <u>simultaneously</u> make demands. A

demand represents an unconditional take-it-or-leave-it offer. Recently Crawford (1982) has studied a simple two-stage game of incomplete information in which there is uncertainty about the extent to which commitments are genuinely irrevocable. In his game the bargainers, in the first stage, make demands simultaneously. It must be noted, firstly, that it is rarely the case that commitments are irrevocable. Secondly, that the notion of simultaneous demands is not particularly realistic in most bilateral bargaining situations. And thirdly, that bargaining processes ought to be modelled as infinite-time horizon games.

In this chapter we shall formalise Schelling's view of commitment in bargaining, in a non-cooperative infinite-time horizon sequential game with perfect and complete information. We assume complete information to avoid the usual problems associated with incomplete information games, and besides the interest of this chapter is solely in examining the role of commitment in bargaining.

The game (to be presented in secton 2) has the players making offers alternately. In the bargaining process a player can commit himself to, not to budge from his offer, or not to accept his opponent's offer. Commitment to a bargaining position means the following. The player takes actions that make it costly for him to later back down from this bargaining position. Commitment is therefore revocable, but at a cost. Commitment to a bargaining position can lead the bargaining process into a "concession game", a game in which one of the bargainers has to concede in order for the bargaining process to either yield an agreement or proceed to a game of fresh offers and counteroffers.

Let c_i $(i=A,B)$ denote the cost to player i of backing down from his commitment to a bargaining position and δ the (common) discount factor. In section 3 we prove: (a) for any $c_A, c_B \geq 0$ such that (i) $(c_A, c_B) \neq (0,0)$, (ii) $(c_A, c_B) \neq (0, 1-\delta+\varepsilon)$ for any $\varepsilon \geq 0$, and (iii) $(c_A, c_B) \neq (1-\delta+\varepsilon, 0)$ for any $\varepsilon \geq 0$, and for any $\delta < 1$ the bargaining game has a unique subgame perfect equilibrium partition in which the player who makes the first offer (i.e., starts the bargaining) receives payoff equal to one and the other player receives payoff equal to zero. (b) for $c_A = c_B = 0$ and for any $\delta < 1$ the bargaining game has two subgame

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perfect equilibrium partitions, (i) in which the player who makes the first offer receives payoff equal to one and the other player receives payoff equal to zero, and (ii) in which the player who makes the first offer receives payoff equal to $1/(1+\delta)$ and the other player receives payoff equal to $\delta/(1+\delta)$ (i.e., the Rubinstein (1982) solution). (c) for $i \neq j$, $i,j=A,B$: for $c_i=0$ and for any $c_j \geq 1-\delta$ and for any $\delta<1$, if player i makes the first offer, then the bargaining game has two subgame perfect equilibrium partitions, (i) in which player i receives payoff equal to one and player j receives payoff equal to zero, and (ii) in which player i receives payoff equal to $1-\delta$ and player j receives payoff equal to δ, and on the other hand, if player j makes the first offer, then the bargaining game has a unique subgame perfect equilibrium partition in which player j receives payoff equal to one and player i receives payoff equal to zero.

We wish to share our view, that at first sight, result (a) seems counterintuitive. To explain why it seems counterintuitive we take the case where the player who makes the first offer has a negligible cost of backing down from his commitment to a bargaining position, while the other player has a large cost of backing down from his commitment to a bargaining position. In the equilibrium, the player with the "negligible cost" demands the whole surplus and commits himself to not to budge from this demand, and the other player accepts. Now suppose this player (i.e., the responder) deviates, and rejects his opponent's demand; furthermore, he commits himself to not to accept his opponent's demand. His cost of backing down from his commitment is large. Thus we reach a stage in the bargaining process where the players have made mutually incompatible commitments. Commonsense would now suggest that the player with the "negligible cost" (i.e., who made the demand) should "concede", that is back down from his commitment to not to budge from his demand, since he knows that his opponent will not "concede" because his opponent will incur a large cost. Working backwards we see that this equilibrium, in which the player who makes the first offer demands the whole surplus and commits not to budge from this demand, ought not to be sub-game perfect. Note that the fact that the player who makes the first offer (i.e., the "negligible cost" player) can commit himself before the responder (i.e., the "large cost" player) gives the "negligible cost" player absolutely no "clout"

whatsoever. Sub-game perfection and the fact that commitment is revocable at a negligible cost prevents the "negligible cost" player to commit himself to not to budge from demanding the whole surplus.

The commonsense argument is in fact reflected in Schelling's work. Schelling (1956) argues, quite convincingly, that in bargaining "weakness is often strength"; and thus, the player with the large cost of backing down from his commitment should indeed have greater "bargaining strength" than the player with the negligible cost of backing down from his commitment. But we have here, in this chapter, a result that appears to contradict this very commonsensical notion.

The question then is, "where does the commonsense argument break down, and thus, what is a reasonable explanation for result (a)?"

The "negligible cost" player is correct to realise that the "large cost" player will not "concede". But, given the equilibrium, it is not optimal for the "negligible cost" player to "concede" either. This is because if he "concedes", then the bargaining process proceeds into a subgame which begins with the "large cost" player demanding the whole surplus and committing himself to not to budge from this demand. Thus the "negligible cost" player will receive no surplus if he "concedes", but he will incur the (negligible) cost of backing down from his commitment - hence a negative payoff. Therefore, once the players make mutually incompatible commitments, stalemate is the guaranteed outcome. Realising this, the player who has to respond to his opponent's demand will accept any offer (if the offer is coupled with a commitment to not to budge from the offer) no matter how large is his cost of backing down from his commitment to not to accept the offer.

2. THE MODEL

Two players, A and B, are bargaining on the partition of a pie of size one. The pie will be partitioned only after the players reach an agreement. The players make offers alternately. In the bargaining process a player can commit himself to, not to budge from his offer, or not to accept his opponent's offer. Commitment to a bargaining position means the following. The player takes actions that make it costly for him to later back down from this bargaining position. Commitment is therefore revocable, but at a cost. Commitment to a bargaining position can lead the bargaining process into a "concession game", a game in which one of the players has to concede in order for the bargaining process to either yield an agreement or proceed to a game of fresh offers and counteroffers.

The bargaining process is modelled as a non-cooperative infinite-time horizon sequential game with complete and perfect information. The time dimension of bargaining process is discrete, $t \in \{0,1,2,...\}$.

The structure of the subgame at any time t, $t \geq 1$, depends on the "immediate" history.

(1) Suppose at time t-1 player j (j=A,B) made an offer to player i ($i \neq j$, i=A,B) but player j did not commit himself to not to budge from his offer, and player i rejects the offer.

Then, at time t, the subgame (denote it by G_i) has the following structure. Player i makes an offer and decides whether, to commit, or not to commit, himself to not to budge from his offer, i.e., player i chooses $(x,a) \in [0,1] X \{C,NC\}$, where C denotes "to commit" and NC denotes "not to commit"; refer to Figure 1.[1]

If a=NC, then player j moves: he either accepts ("Ac") or rejects ("R") the offer. If player j accepts the offer, then the game ends. If player j rejects the offer, then the game proceeds, at time t+1, to game G_j.

[1] Figure 1 is placed at the end of section 4.

2. [illegible]

Two players, A and B, are bargaining on the partition of a pie of size one. The pie will be partitioned only after the players reach an agreement. The players make offers alternately. [illegible] proposes a player can [illegible] to accept his opponent's [illegible] position means the [illegible] ideally [illegible] threatened as [illegible] bargaining position [illegible] game", a game in which one of the players [illegible] the bargaining process [illegible] case of [illegible].

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If <u>a=c</u>, then player j moves: he either accepts the offer ("Ac") or rejects the offer and commits himself to not to accept the offer ("RC") or rejects the offer but does not commit himself ("RNC"). If player j accepts, then the game ends. If player j chooses "RC", then we are in a situation where the players have made mutually incompatible commitments. Thus, the game proceeds, at time t+1, to a "concession game", to be described below in (2) (denoted by $K_i(x)$, where x is the offer to which player i has committed himself). If player j chooses "RNC", then the game proceeds, at time t+1, to another "concession game", to be described below in (3) (denoted by $L_i(x)$, where x is the offer to which player i has committed himself).

(2) Suppose at time t-1 player i (i=A,B) has committed himself to not to budge from an offer, x ($x \in [0,1]$), and player j ($j \neq i$, j=A,B) has committed himself to not to accept the offer x; they have committed themselves to incompatible bargaining positions.

Then, at <u>time t</u>, the subgame (denote it by $K_i(x)$) has the following structure. Player i moves: he either concedes ("Con") or does not concede ("Ncon"). If player i concedes, then at time t the game proceeds to the subgame G_j, described in (1) above. If player i does not concede, then at time t+1 the game proceeds where player j decides whether to concede ("Con") or not to concede ("Ncon"). If player j concedes, then the game ends. If player j does not concede, then at time t+2 the game proceeds to the subgame $K_i(x)$.

(3) Suppose at time t-1 player i (i=A,B) has committed himself to not to budge from an offer, x ($x \in [0,1]$), and player j ($j \neq i$, j=A,B) has rejected the offer but has not committed himself to not to accept the offer.

Then, at <u>time t</u>, the subgame (denote it by $L_i(x)$) has the following structure. Player i moves: he either concedes ("Con") or does not concede ("Ncon"). If player i concedes, then at time t the game proceeds to the subgame G_j, described in (1) above. If player i does not concede, then at time t+1 the game proceeds where player j decides whether to concede ("Con") or not to concede ("Ncon"). If player j concedes, then the game ends. If player j does not concede, then at time t+2 the game proceeds to the subgame $L_i(x)$.

If $a = c$, then player j elects to either accept the offer ("Ac") or rejects the offer and commits himself to not to accept the offer ("RC") or rejects the offer but does not commit himself ("RNC"). If player j accepts, then the game ends. If player j chooses "RC", then we are in a situation where the players have made mutually incompatible commitments; thus, the game proceeds, at time $t+1$, to a "concession game", to be described below in (2) (denoted by $G_i(a)$, where a is the offer to which player i has committed himself). If player j chooses "RNC", then the game proceeds, at time $t+1$, to another "concession game", to be described below in (3) (denoted by $G_j(a)$, where a is the offer to which player i has committed himself).

(2) Suppose at time $t-1$ player i ($i=A,B$) has committed himself to not to budge from an offer, $a \in [0,1]$, and player j ($j \neq i$, $j=A,B$) has committed himself to not to accept the offer a: they have committed themselves to incompatible bargaining positions.

Then, at time t, the subgame (denote it by $G_i(a)$) has the following structure. Player i moves: he either concedes ("Con") or does not concede ("NCon"). If player i concedes, then at time t the game proceeds to the subgame G_0 described in (1) above. If player i does [illegible], then at time $t+1$ the game proceeds where player j decides [illegible] or not to concede ("NCon"). If player j [illegible]. If player j does not concede, then at [illegible] to the subgame $G_i(a)$.

(3) [illegible] player i ($i=A,B$) has committed himself to not [illegible] an offer, $a \in [0,1]$, and player j ($j \neq i$, $j=A,B$) has not [illegible] but has not committed himself to not to accept the offer.

Then, at time t, the subgame (denote it by $G_j(a)$) has the following structure. Player i moves: he either concedes ("Con") or does not concede ("NCon"). If player i concedes, then at time t the game proceeds to the subgame G_0 described in (1) above. If player i does not concede, then at time $t+1$ the game proceeds where player j decides whether to concede ("Con") or not to concede ("NCon"). If player j concedes, then the game ends. If player j does not concede, then at time $t+2$ the game proceeds to the subgame $G_j(a)$.

The game (i.e., the bargaining process) begins at $t=0$. The game at $t=0$ is G_A or G_B according to whether it is player A or player B who makes the first offer (i.e., starts the bargaining); G_A and G_B are described above (see also Figure 1).

We shall assume that the players maximise expected utility. Let δ ($\delta<1$) be the (common) discount factor. Let c_i, $i=A,B$, ($c_i\geq 0$) be the cost to player i of backing down from a commitment to a bargaining position. Suppose the players agree at time t ($t\geq 0$) to an offer x ($x\in[0,1]$). Let $\{t_k^i\}_{k=1}^{N_i}$ be the times, up until time t, at which player i backs down from a "commitment". Then, the discounted payoff to player i is:

$$x_i\delta^t - c_i \sum_{k=1}^{N_i} \delta^{t_k^i},$$

where $x_A=x$ and $x_B=1-x$.

We shall assume that the game G_i ($i=A,B$) is a game of complete information, i.e., all information is assumed to be common knowledge amongst the players. Furthermore, note that G_i is a game of perfect information.

A strategy for each agent in G_i will tell the agent the choice to make at each and every decision node that he may be at. Each of the two players will have a set of strategies from which to choose a strategy. The solution concept we will use is the subgame perfect equilibrium (SGPE) (Selten (1965, 1975)). A strategy tuple is in SGPE if its restriction to any subgame is in Nash equilibrium.

3. PERFECT EQUILIBRIUM

PROPOSITION: (a) For any $c_A, c_B \geq 0$ such that (i) $(c_A, c_B) \neq (0,0)$, (ii) $(c_A, c_B) \neq (0, 1-\delta+\varepsilon)$ for any $\varepsilon \geq 0$, and (iii) $(c_A, c_B) \neq (1-\delta+\varepsilon, 0)$ for any $\varepsilon \geq 0$, and for any $\delta < 1$ the game G_i $(i=A,B)$ has a unique SGPE partition in which player i receives payoff equal to one and player j $(j \neq i,\ j=A,B)$ receives payoff equal to zero. (b) For $c_A = c_B = 0$ and for any $\delta < 1$ the game G_i $(i=A,B)$ has two SGPE partitions, (i) in which player i receives payoff equal to one and player $j (j \neq i,\ j=A,B)$ receives payoff equal to zero, and (ii) in which player i receives payoff equal to $1/(1+\delta)$ and player j $(j \neq i,\ j=A,B)$ receives payoff equal to $\delta/(1+\delta)$ (i.e., the Rubinstein (1982) solution). (c) For $i \neq j$, $i,j=A,B$ we have: for $c_i = 0$ and for any $c_j \geq 1-\delta$ and for any $\delta < 1$, the game G_i has two SGPE partitions, (i) in which player i receives payoff equal to one and player j receives payoff equal to zero, and (ii) in which player i receives payoff equal to $1-\delta$ and player j receives payoff equal to δ, and the game G_j has a unique SGPE partition in which player j receives payoff equal to one and player i receives payoff equal to zero.

PROOF: Let $i,j=A,B$, $i \neq j$; and $x_j = x$ if $j=A$, $x_j = 1-x$ if $j=B$.

Let $k_i^j(x)$ $(K_i^j(x))$ denote the infimum (supremum) of the payoffs to player j in any SGPE of the subgame $K_i(x)$. Then the infimum (supremum) of the payoffs to player j in any SGPE as of point (2) (in Figure 1) is $\max\{x_j - c_j,\ \delta k_i^j(x)\}$ $(\max\{x_j - c_j,\ \delta K_i^j(x)\})$.

If at point (1) player i chooses "Ncon", then $K_i^j(x) = \delta \max\{x_j - c_j, \delta K_i^j(x)\}$ and $k_i^j(x) = \delta \max\{x_j - c_j,\ \delta k_i^j(x)\}$. Thus,

$$K_i^j(x) = k_i^j(x) = \begin{cases} 0 & \text{if } x_j - c_j \leq 0 \\ \delta(x_j - c_j) & \text{if } x_j - c_j \geq 0 \end{cases} \tag{1a}$$

If, on the other hand, at point (1) player i chooses "Con", then

$$\begin{aligned} k_i^j(x) &= m_j^j \quad \text{and} \\ K_i^j(x) &= M_j^j \quad , \end{aligned} \tag{1b}$$

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where m_j^j (M_j^j) is the infimum (supremum) of the payoffs to player j in any SGPE of the game G_j.

By repeating the above argument one can establish the infimum and supremum of the payoffs to player j in any SGPE of the subgame $L_i(x)$; denote this infimum (supremum) by $l_i^j(x)$ ($L_i^j(x)$).

<u>If</u> at point (4) player i chooses "Ncon", then

$$L_i^j(x)=l_i^j(x)=\delta x_j \tag{2a}$$

<u>If</u>, on the other hand, at point (4) player i chooses "Con", then

$$l_i^j(x)=m_j^j \quad \text{and}$$
$$L_i^j(x)=M_j^j \quad . \tag{2b}$$

Denote the infimum (supremum) of the payoffs to player j in any SGPE as of point (5) (in Figure 1) by I_x^j(S_x^j).

$$I_x^j=\delta\max\{k_i^j(x),l_i^j(x)\} \tag{3a}$$

$$S_x^j=\delta\max\{K_i^j(x),L_i^j(x)\} \tag{3b}$$

The infimum (supremum) of the payoffs to player j in any SGPE as of point (6) is $1-\delta m_j^j$($1-\delta M_j^j$); note that both the infimum and the supremum are independent of x.

Hence,

$$M_i^i=\max\{1-\delta m_j^j,\ 1-\min\{I_x^j:x\in[0,1]\}\},$$

i.e.,
$$M_i^i=1-\min\{\delta m_j^j,\ \min\{I_x^j:x\in[0,1]\}\} \tag{4a}$$

and

$$m_i^i=\max\{1-\delta M_j^j,\ 1-\min\{S_x^j:x\in[0,1]\}\},$$

i.e.,
$$m_i^i=1-\min\{\delta M_j^j,\ \min\{S_x^j:x\in[0,1]\}\}, \tag{4b}$$

where m_i^i(M_i^i) is the infimum (supremum) of the payoffs to player i in any SGPE of the game G_i.

Let us now compute I_x^j and S_x^j for all $x\in[0,1]$; I_x^j and S_x^j are defined by equations (3a) and (3b).

(I) Suppose for an $x \in [0,1]$ at point (1) player i chooses "Ncon" and at point (4) player i chooses "Ncon". Then, using equations (1a), (2a), (3a) and (3b), we obtain:

$$S_x^j = I_x^j = \delta^2 x_j \qquad (5)$$

(II) Suppose for an $x \in [0,1]$ at point (1) player i chooses "Con" and at point (4) player i chooses "Con". Then, using equations (1b), (2b), (3a) and (3b), we obtain:

$$I_x^j = \delta m_j^j \text{ and} \qquad (6a)$$

$$S_x^j = \delta M_j^j \qquad (6b)$$

(III) Suppose for an $x \in [0,1]$ at point (1) player i chooses "Con" and at point (4) player i chooses "Ncon". Then, using equations (1b), (2a), (3a) and (3b), we obtain:

$$I_x^j = \delta \max\{m_j^j,\ \delta x_j\} \text{ and} \qquad (7a)$$

$$S_x^j = \delta \max\{M_j^j,\ \delta x_j\} \qquad (7b)$$

(IV) Suppose for an $x \in [0,1]$ at point (1) player i chooses "Ncon" and at point (4) player i chooses "Con". Then, using equations (1a), (2b), (3a) and (3b), we obtain:

$$I_x^j = \delta \max\{m_j^j,\ \delta(x_j - c_j)\} \text{ and} \qquad (8a)$$

$$S_x^j = \delta \max\{M_j^j,\ \delta(x_j - c_j)\} \qquad (8b)$$

Let us now proceed to compute m_i^i and M_i^i (i=A,B), where m_i^i and M_i^i are defined by equations (4a) and (4b).

CASE A: Suppose when i=A and j=B, for x=1 equation (5) applies, and for any x<1 either equation (5) or (6) or (7) or (8) applies; and suppose when i=B and j=A, for x=0 equation (5) applies, and for any x>0 either equation (5) or (6) or (7) or (8) applies.

Then

$$\min\{I_x^B : x \in [0,1]\} = 0 = \min\{S_x^B : x \in [0,1]\} \text{ and}$$

$$\min\{I_x^A : x \in [0,1]\} = 0 = \min\{S_x^A : x \in [0,1]\} \text{ . Thus,}$$

$$M_A^A = m_A^A = 1 \quad \text{and} \quad M_B^B = m_B^B = 1 \qquad (9)$$

(ii) Suppose for an a=(0,1) at point (i) player 1 chooses "Short" and at point (ii) player 1 chooses "Long". Then, using equations (14), (16) and (18), we obtain:

[illegible] (20)

(iii) Suppose for [illegible] at point (i) player 1 chooses "Low" and at point (ii) player [illegible]. Then, using equations (15), (16), and (18), we

[illegible] (21)

[illegible] (22)

(iv) Suppose [illegible] at point (i) player 1 chooses "Low" [illegible] point (ii) [illegible]. Then, using equations (15), [illegible] (18), we

[illegible] (23)

[illegible] (24)

(v) Suppose [illegible] at point (i) player 1 chooses "High" [illegible] point (ii) [illegible] chooses "High". Then, using equations (14), (17) and (19), we obtain:

[illegible] (25)

[illegible] (26)

[illegible] is computed to [illegible], where [illegible] determined by equations (25) and (26).

[illegible] when t=1 and j=k, for t=1 equation (3) applies, and [illegible] either equation (3) or (4) or (5) or (6) applies; and [illegible] when t=0 and j=k, then [illegible] equation (3) applies) and for any [illegible] equation (5) or (6) or (7) [illegible] applies.

[illegible] and

[illegible]. Then,

[illegible] and

[illegible] (27)

Let us now check whether the equilibrium payoffs, as defined by equation (9), are "consistent" with the <u>supposition</u> made above (Case A). The supposition states that player i (i=A,B) chooses "Ncon" at points (1) and (4), for x=1 if i=A, and for x=0 if i=B. If player i instead chooses "Con", then in the equilibrium player i receives a surplus equal to zero and incurs a cost c_i ($c_i>0$) - hence a <u>strictly</u> negative payoff. Thus, indeed, "Ncon" is the optimal choice. Furthermore, if $c_i=0$, then "Ncon" is also an optimal choice.

Hence, the supposition (Case A) is "consistent" with the equilibrium payoffs, defined by equation (9), for any $c_A, c_B \geq 0$. (In fact, player i chooses "Ncon" for all $x \in [0,1]$ at points (1) and (4)).

Hence, for any $c_A, c_B \geq 0$, $M_A^A = m_A^A = 1$ and $M_B^B = m_B^B = 1$.[2]

A pair of strategies that support this solution is as follows. Player i (i=A,B) always offers x_i^*, where

$$x_i^* = \begin{cases} 1 \ \underline{\text{if}} \ i=A \\ \\ 0 \ \underline{\text{if}} \ i=B \end{cases},$$

and commits himself to not to budge from this offer. Player j ($j \neq i$, j=A,B) accepts any offer <u>if</u> it is coupled with a commitment, and accepts offers smaller than $1-\delta$ <u>if</u> the offer is not coupled with a commitment. Player i never concedes in the subgames $K_i(x)$ and $L_i(x)$ for any $x \in [0,1]$. Player j always concedes in a subgame $K_i(x)$ if $x_j > c_j$, but never concedes if $x_j \leq c_j$; furthermore, player j always concedes in the subgame $L_i(x)$ for any $x \in [0,1]$.

<u>CASE B</u>: <u>Suppose</u> when i=A and j=B, the largest x, $x \in [0,1]$, for which equation (5) applies is $x=1-\varepsilon_A$, for some ε_A such that $1 \geq \varepsilon_A > 0$, and for any $1 \geq x > 1-\varepsilon_A$ either equation (6) or (7) or (8) applies, and for any $0 \leq x < 1-\varepsilon_A$ either equation (5) or (6) or (7) or (8) applies; and <u>suppose</u> when i=B and j=A, the smallest x, $x \in [0,1]$, for which equation (5) applies is $x=\varepsilon_B$, for some ε_B such that $1 \geq \varepsilon_B > 0$, and for any $0 \leq x < \varepsilon_B$ either equation (6) or (7) or (8) applies, and for any $1 \geq x > \varepsilon_B$ either

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[2] Here we can allow $\delta \leq 1$. But the condition $\delta < 1$ is required to prove the "inconsistency" of the other possible suppositions (to be discussed below) with the equilibrium payoffs that they generate; $\delta = 1$ leads to multiple equilibria in these other suppositions, which would then allow us to choose an equilibrium that would be consistent with a given supposition. Thus <u>$\delta < 1$</u> is a necessary requirement for the Proposition above.

Let us now check whether the equilibrium payoffs, as defined by equation (9), are "consistent" with the expectations upon which these [illegible]

[illegible]

[illegible]

A [illegible]

[illegible]

[illegible] and [illegible] [illegible] is not to accept less than [illegible]. Player 2 [illegible] [illegible] [illegible] [illegible] is [illegible] with a commitment, and [illegible] [illegible] smaller than [illegible] [illegible] [illegible] is not [illegible] with [illegible]. Player 1 never concedes in the subgame [illegible] and [illegible] for any [illegible]. Player 2 always concedes in a subgame [illegible] [illegible], but never concedes if [illegible] [illegible]; player 2 always concedes in the subgame [illegible] for any [illegible].

[illegible] [illegible] when [illegible] and [illegible], the largest [illegible], for which equation (5) applies is [illegible], for some [illegible] such that [illegible], and for any [illegible], either equation (6) or (7) or (8) applies, and for any [illegible], either equation (5) or (6) or (7) or (8) applies and [illegible] when [illegible] and [illegible], the smallest [illegible], for which equation (5) applies is [illegible], for some [illegible] such that [illegible], and for any [illegible] either equation (6) or (7) or (8) applies, and for any [illegible] either

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[illegible]

equation (5) or (6) or (7) or (8) applies.

Then $S^{B}_{1-\varepsilon_A} = I^{B}_{1-\varepsilon_A} = \varepsilon_A\delta^2$ and $S^{A}_{\varepsilon_B} = I^{A}_{\varepsilon_B} = \varepsilon_B\delta^2$. Therefore, we obtain that, for $i,j=A,B, i\neq j$,

$$M^{i}_{i} = 1-\min\{\delta m^{j}_{j},\ \varepsilon_i\delta^2\} \text{ and}$$

$$m^{i}_{i} = 1-\min\{\delta M^{j}_{j},\ \varepsilon_i\delta^2\} \tag{10}$$

<u>Assume</u> for $i,j=A,B,i\neq j$, $\varepsilon_i\delta>m^{j}_{j},M^{j}_{j}$. Then, using equation (10), provided $\delta<1$, we obtain:

$$M^{A}_{A}= m^{A}_{A}= 1/(1+\delta) \text{ and}$$
$$M^{B}_{B}= m^{B}_{B}= 1/(1+\delta) \tag{11}$$

Therefore, equation (11) holds if

$$\text{for } i=A,B,\ \varepsilon_i>1/[\delta(1+\delta)] \tag{12}$$

Let us now check whether this equilibrium is "consistent" with the <u>supposition</u> made above (Case B).

When $i=A$ and $j=B$ for any $x>1-\varepsilon_A$ equation (5) does not apply. Thus for any $x>1-\varepsilon_A$ player A chooses "Con" either at point (1) or at point (4) or at both points (1) and (4). Hence, for any $\delta<1$,

$$\delta/(1+\delta) - c_A \geq 0 \tag{13}$$

Since $1/[\delta(1+\delta)]>\delta/(1+\delta)$ equations (12) and (13) $\Rightarrow$

$$\text{for } i=A,B,\ \varepsilon_i > c_A \tag{14}$$

Symmetrically, when $i=B$ and $j=A$, we have

$$\delta/(1+\delta) - c_B \geq 0 \tag{15}$$

and $$\text{for } i=A,B,\ \varepsilon_i > c_B. \tag{16}$$

When $i=A$ and $j=B$ for $x=1-\varepsilon_A$ equation (5) applies, i.e., player A chooses "Ncon" at points (1) and (4). Therefore, using equations (1a) and (2a), we have that $K^{B}_{A}(1-\varepsilon_A)=k^{B}_{A}(1-\varepsilon_A)=\delta[1-(1-\varepsilon_A)-c_B]=\delta[\varepsilon_A-c_B]$ (since $\varepsilon_A>c_B$, cf. equation (16)), and $L^{B}_{A}(1-\varepsilon_A)=l^{B}_{A}(1-\varepsilon_A)=\delta\varepsilon_A$. Hence at points (2) and (3) player B chooses "Con". Thus, for player A to choose "Ncon" at points (1) and (4) it must be the case, that for any $\delta<1$,

$$[1-\varepsilon_A]\delta\geq[\delta/(1+\delta)-c_A]. \tag{17}$$

Symmetrically, when $i=B$ and $j=A$, we have

$$[1-\varepsilon_B]\delta\geq[\delta/(1+\delta)-c_B] \tag{18}$$

[illegible] (7) and (8) applies.

[illegible] Therefore, we obtain [illegible]

[illegible] and

[illegible] (4.3)

[illegible] equation [illegible]

[illegible] (4.4)

[illegible] (4.5)

[illegible] "constrained" with its [illegible]

[illegible] equations (2) [illegible] (i) or (ii) which [illegible] and (ii). Hence, for any [illegible],

[illegible] (4.6)

[illegible] equations [illegible] we have

[illegible] (4.7)

Consequently, when [illegible] we have

[illegible] (4.8)

[illegible] (4.9)

[illegible] equation (8) applies [illegible] (i) and (ii). Therefore, [illegible] we have that [illegible] equation [illegible] Thus, for player A to choose [illegible] (i) and (ii) it must be the case, that for any [illegible],

[illegible] (4.10)

[illegible] we have

[illegible] (4.11)

We now show that (17) is false for some $\delta<1$. This implies a contradiction. Hence the equilibrium, as described in equations (11) and (12), is not "consistent" with the supposition of Case B, above. Thus the equilibrium cannot hold.

We first observe that there exists an x, $x>1-\varepsilon_A$, such that $1-x-c_B>\delta/(1+\delta)$, provided $\delta<1/\sqrt{2}$. There will exist such an x if $[1-c_B-\delta/(1+\delta)]>[1-\varepsilon_A]\Leftrightarrow[\varepsilon_A-c_B]>[\delta/(1+\delta)]$. From equation (12), $\varepsilon_A>1/[\delta/(1+\delta)]$, and from equation (15), $\delta/(1+\delta)>c_B\Rightarrow 2\delta/(1+\delta)>[c_B+\delta/(1+\delta)]$. Now $1/[\delta(1+\delta)]>2\delta/(1+\delta)$ if $\delta<1/\sqrt{2}$. Thus $[\varepsilon_A-c_B]>\delta/(1+\delta)$. Hence, there exists an x, $x>1-\varepsilon_A$, such that $[1-x-c_B]>\delta/(1+\delta)$, i.e., such that $x<[1-c_B-\delta/(1+\delta)]$. Denote this x by $\hat{x}$. By continuity there exists a γ, $\gamma>0$, such that $\hat{x}=1-\varepsilon_A+\gamma$.

Now, given our supposition, for any $x>1-\varepsilon_A$ player A chooses "Con" either at point (1) or at point (4) or at both points (1) and (4); in particular for $x=\hat{x}$. Since $[1-\hat{x}-c_B]>\delta/(1+\delta)$, which implies $1-\hat{x}>\delta/(1+\delta)$, player B will choose "Con" either at point (2) (if player A chooses "Con" at point (1)) or at point (3) (if player A chooses "Con" at point (4)) or at both points. In order for it to be optimal for player A to choose "Con" we require

$$[\delta/(1+\delta)-c_A]\geq\delta\hat{x}=\delta[1-\varepsilon_A+\gamma],$$

which implies that $[\delta/(1+\delta)-c_A]>\delta[1-\varepsilon_A]$. This is in contradiction to equation (17).

By assuming that the other possibilities hold of equation (10), such as for $i,j=A,B,i\neq j$, m_j^j, $M_j^j>\varepsilon_i\delta$, one can prove by similar arguments that the equilibrium obtained is not "consistent" with the supposition of Case B.

Hence, the supposition of Case B is false, for any $c_A,c_B\geq 0$ and for any $\delta<1$.

CASE C: Suppose when $i=A$ and $j=B$, for $x=1$ equation (5) applies and for any $0\leq x<1$ either equation (5) or (6) or (7) or (8) applies; and suppose when $i=B$ and $j=A$, the smallest x, $x\in[0,1]$, for which equation (5) applies is $x=\varepsilon_B$, for some ε_B such that $1\geq\varepsilon_B>0$, and for any $0\leq x<\varepsilon_B$ either equation (6) or (7) or (8) applies, and for any $1\geq x>\varepsilon_B$ either equation (5) or (6) or (7) or (8) applies.

Then, we obtain that

$$M_A^A = m_A^A = 1 \quad \text{and} \\ M_B^B = m_B^B = 1-\delta^2\varepsilon_B. \tag{19}$$

One can show that this equilibrium, as described by equation (19), is not "consistent" with the supposition of Case C, above, for any $c_A, c_B \geq 0$ and for any $\delta<1$.

CASE D: Suppose when i=B and j=A, for x=0 equation (5) applies and for any $1\geq x>0$ either equation (5) or (6) or (7) or (8) applies; and suppose when i=A and j=B, the largest x, $x\in[0,1]$, for which equation (5) applies is $x=1-\varepsilon_A$, for some ε_A such that $1\geq\varepsilon_A>0$, and for any $1\geq x>1-\varepsilon_A$ either equation (6) or (7) or (8) applies, and for any $0\leq x<1-\varepsilon_A$ either equation (5) or (6) or (7) or (8) applies.

Then, we obtain that

$$M_B^B = m_B^B = 1 \quad \text{and} \\ M_A^A = m_A^A = 1-\delta^2\varepsilon_A. \tag{20}$$

One can show that this equilibrium, as described by equation (20), is not "consistent" with the supposition of Case D, above, for any $c_A, c_B \geq 0$ and for any $\delta<1$.

CASE E: Suppose when i=A and j=B, $\nexists x\in[0,1]$ such that equation (5) applies (i.e., $\forall x\in[0,1]$ either equation (6) or equation (7) or equation (8) applies); and suppose when i=B and j=A, $\nexists x\in[0,1]$ such that equation (5) applies.

Then, we obtain that, for $i,j=A,B, i\neq j$,

$$M_i^i = 1-\delta m_j^j \quad \text{and} \\ m_i^i = 1-\delta M_j^j. \tag{21}$$

Thus, provided $\delta<1$, we obtain:

$$M_A^A = m_A^A = 1/(1+\delta) \quad \text{and} \\ M_B^B = m_B^B = 1/(1+\delta). \tag{22}$$

Let us now check whether this equilibrium is "consistent" with the supposition made above (Case E).

Then we obtain that

$$[illegible] \quad \text{and} \tag{38}$$

[illegible]

[illegible]

(39)

[illegible]

[illegible]

Then we obtain that, for [illegible],

$$[illegible] \quad \text{and} \tag{40}$$

[illegible]

(41)

[illegible]

Since for any $x\in[0,1]$ player i ($i=A,B$) will choose "Con" either at point (1) or at point (4) or at both point (1) and point (4), we must have:

$$\text{for } i=A,B \qquad \delta/(1+\delta)-c_i>0 \tag{23}$$

When $i=A$ and $j=B$, for any $x\in[0,1]$ player A will choose "Con" either at point (1) or at point (4) or at both points (1) and (4). Take $x=[1/(1+\delta)-c_B]$. Since $1-x-c_B=\delta/(1+\delta)$ and $1-x\geq\delta/(1+\delta)$, it implies that player B will choose "Con" either at point (2) (*if* player A chooses "Con" at point (1)) or at point (3) (*if* player A chooses "Con" at point (4)) or at both points. In order for it to be optimal for A to choose "Con" we require

$$\delta/(1+\delta)-c_A\geq\delta[1/(1+\delta)-c_B]$$

i.e.,

$$\delta c_B\geq c_A \tag{24}$$

Symmetrically, when $i=B$ and $j=A$, we require

$$\delta c_A\geq c_B \tag{25}$$

For any $c_A>0$ and $\delta<1$,

$$c_A>\delta c_A \tag{26}$$

For any $c_B>0$ and $\delta<1$,

$$c_B>\delta c_B \tag{27}$$

Combining (24), (25) and (26) we obtain: $\delta c_B\geq c_A>\delta c_A\geq c_B$, i.e., $\delta c_B>c_B$, for any $\delta<1, c_A>0$ and $c_B\geq 0$. This is a contradiction.

Similarly, combining (24), (25) and (27) we obtain: $\delta c_A\geq c_B>\delta c_B\geq c_A$, i.e., $\delta c_A>c_A$, for any $\delta<1, c_B>0$ and $c_A\geq 0$. This is a contradiction.

Thus, the equilibrium payoffs (as defined by equation (22)) are not "consistent" with the supposition made above (Case E), for any c_A, $c_B\geq 0$ such that $(c_A,c_B)\neq(0,0)$ and for any $\delta<1$.

If, on the other hand, $c_A=c_B=0$ and $\delta<1$, then the above arguments ((26), (27)) do not go through. And if $c_A=c_B=0$ and $\delta<1$, then the equilibrium payoffs (defined by (22)) are indeed "consistent" with the supposition of Case E.

CASE F: Suppose when i=A and j=B, $\exists x\in[0,1]$ such that equation (5) applies (i.e., $\forall x\in[0,1]$ either equation (6) or equation (7) or equation (8) applies); and suppose when i=B and j=A, for x=0 equation (5) applies, and for any x>0 either equation (5) or (6) or (7) or (8) applies.

Then, we obtain that

$$M_B^B=m_B^B=1 \quad \text{and} \quad M_A^A=m_A^A=1-\delta. \tag{28}$$

Let us now check whether this equilibrium is "consistent" with the supposition made above (Case F).

When i=A and j=B, the supposition states, that for any $x\in[0,1]$ player A will choose "Con" either at point (1) or at point (4) or at both points (1) and (4).

If $c_A>0$, then this is definitely not optimal, since by choosing "Con" player A receives zero surplus.

If $c_A=0$, then choosing "Con" can be optimal. Firstly, suppose $c_B<1-\delta$. Then $1-c_B-\delta>0$. Thus, there exists an $1>x>0$, say $\hat{x}$, such that $1-c_B-\delta>\hat{x}>0$. Let $\hat{x}=1-c_B-\delta-\varepsilon$ for some ε small. Then $1-x>\delta$ and $1-x-c_B>\delta$. Thus, player B will choose "Con" either at point (2) (if player A chooses "Con" at point (1)) or at point (3) (if player A chooses "Con" at point (4)) or at both points. In order for it to be optimal for A to choose "Con" we require

$$0\geq\hat{x}\delta=(1-c_B-\delta-\varepsilon)\delta,$$

which is a contradiction since $\hat{x}>0$. Thus if $c_A=0$ and $c_B<1-\delta$, then the equilibrium payoffs (as defined by equation (28)) are not "consistent" with the supposition made above (Case F). Secondly, suppose $c_B\geq1-\delta$. Then $1-c_B\leq\delta$. Thus, for all $x\in[0,1]$, $1-x-c_B\leq\delta$. Thus, there does not exist an $x\in[0,1]$ such that player B would choose "Con" given that player A chooses "Con". Hence, if $c_A=0$ and $c_B\geq1-\delta$, then the equilibrium payoffs (as defined by equation (28)) are indeed "consistent" with the supposition made above (Case F).

In conclusion, for any $\delta<1$ and for any $c_B\geq1-\delta$ and for $c_A=0$, $M_A^A=m_A^A=1-\delta$ and $M_B^B=m_B^B=1$.

<u>CASE G</u>: Symmetric to Case F, but with the roles of players A and B reversed. Therefore, by similar arguments one arrives at the following conclusion: for any $\delta<1$ and for any $c_A\geq 1-\delta$ and for $c_B=0$, $M_B^B=m_B^B=1-\delta$ and $M_A^A=m_A^A=1$.

We have considered all the possible suppositions. For any $c_A,c_B\geq 0$ such that (i) $(c_A,c_B)\neq(0,0)$, (ii) $(c_A,c_B)\neq(0,1-\delta+\varepsilon)$ for any $\varepsilon\geq 0$, and (iii) $(c_A,c_B)\neq(1-\delta+\varepsilon,0)$ for any $\varepsilon\geq 0$, and for any $\delta<1$ the game G_i $(i=A,B)$ has a <u>unique</u> SGPE partition (see <u>Case A</u> above) as stated in the Proposition. For $c_A=c_B=0$ and for any $\delta<1$ the game G_i $(i=A,B)$ has <u>two</u> SGPE partitions (see <u>Case A</u> and <u>Case E</u> above) as stated in the Proposition. For $i\neq j$, $i,j=A,B$: for $c_i=0$ and for any $c_j\geq 1-\delta$ and for any $\delta<1$ the game G_i has <u>two</u> SGPE partitions and the game G_j has a <u>unique</u> SGPE partition (see <u>Cases A, F and G</u>) as stated in the Proposition.

Q.E.D.

4. SUMMARY AND CONCLUDING REMARKS

In this chapter we have presented a model that examines the role of commitment in bilateral bargaining; the model is a non-cooperative infinite-time horizon sequential game with complete and perfect information.

The bargaining process, represented by the sequential game, incorporates commitment possibilities *à la* Schelling (see Schelling (1956)): the bargainers can take actions during the bargaining process that increase the future cost of backing down from one's commitment to, not to budge from one's offer, or not to accept an opponent's offer. Thus commitment is revocable, but at a cost.

Mutually incompatible commitments (for example, if one of the bargainers commits himself to not to budge from an offer and the other bargainer commits himself to not to accept the offer) leads the bargaining process into a "concession game", a game in which one of the bargainers has to concede in order for the bargaining process to either yield an agreement or proceed to a game of fresh offers and counteroffers.

The Proposition (section 3) established, (a) for any $c_A, c_B \geq 0$ such that (i) $(c_A, c_B) \neq (0,0)$, (ii) $(c_A, c_B) \neq (0, 1-\delta+\varepsilon)$ for any $\varepsilon \geq 0$, and (iii) $(c_A, c_B) \neq (1-\delta+\varepsilon, 0)$ for any $\varepsilon \geq 0$, and for any $\delta < 1$ the uniqueness of the SGPE partition, (b) for $c_A = c_B = 0$ and for any $\delta < 1$ the multiplicity of the SGPE partition, and (c) for $i \neq j$, $i, j = A, B$: for $c_i = 0$ and for any $c_j \geq 1-\delta$ and for any $\delta < 1$ the uniqueness of the SGPE partition for the game G_j and the multiplicity of the SGPE partition for the game G_i.

In the Introduction (section 1) we argued that at first sight result (a) seemed counterintuitive, and argued that it appears to contradict Schelling's (commonsensical) notion, that in bargaining "weakness is often strength". In order to re-emphasize the apparently paradoxical nature of result (a) we now illustrate a special case. Suppose the responder's cost of backing down from his commitment to not to accept the "first mover's" offer is infinite relative to the cost to the "first mover" of backing down from his commitment to not

to budge from his offer. Then it is as if the "first mover" has no possibility of commitment (i.e., has no actions available to increase the future cost of backing down from his demand), while the responder can make an irrevocable commitment to not to accept his opponent's demand. In this scenario, one would not expect the "first mover" to obtain the whole surplus. But he does; it is as if he, and not the responder, who has available an irrevocable commitment (see the Introduction, where we give an explanation for this, apparently paradoxical, result).

We shall now venture some thoughts on the implications of the result of this chapter, for bargaining theory in general, and Rubinstein's (1982) classic paper in particular.

Observe, that for $c_A=c_B=0$ and for any $\delta<1$ the bargaining game does not possess the uniqueness property. But, if both costs are strictly positive, then the uniqueness property is obtained.

Our bargaining game incorporates the alternating-offers notion in bargaining, due to Rubinstein (1982). In fact, one may view our game as a "generalisation" of the Rubinstein game. However, as we have seen, the uniqueness property of the Rubinstein game does not survive for some values of c_A and c_B. This is indeed disturbing, especially since the usefulness of non-cooperative bargaining models rests mainly on their specifying a unique equilibrium.

We have seen that if both costs are strictly positive, then the Rubinstein solution disappears and the bargaining game has a unique equilibrium. This equilibrium can be called "bad", in the sense, that in real life, the first mover (i.e., the player who starts the bargaining) would not obtain the whole surplus. This observation may imply that something fundamental is wrong with our bargaining model. Maybe it is an "incorrect" model of bargaining? If so, then one may in effect criticise the alternating-offers model of bargaining, on which our bargaining game is built.

An alternative angle from which to study the implications of our result is the following.

to budge from his offer. Then it is as if the "first-mover" has no credibility of commitment (i.e., has no strategic ability to increase the future cost of backing down from his demand), while the responder has some at [illegible] commitment to not to accept his opponent's demand. In this scenario, one would not expect the "first-mover" to obtain the whole surplus. But he [illegible] it is so, and yet the responder [illegible] [illegible] commitment than the [illegible] [illegible] for this, apparently paradoxical, [illegible]

We shall [illegible] to the implications of the [illegible] [illegible] changes in 'shrinkage' and [illegible] [illegible].

[illegible] and the way that the concession game does not [illegible] property. But, it still makes one strictly [illegible] property is [illegible].

But [illegible] interpretation of the alternating-offers model in bargaining (i.e. [illegible] Rubinstein (1982)). In fact, one may view that game as a "special [illegible]" of the concession game. However, as we have seen, the uniqueness property of the Rubinstein game does not survive for some values of c_1 and c_2. This is another distinction, especially since the usefulness of non-cooperative bargaining models comes mainly in their specifying a unique equilibrium.

We have seen that if both costs are strictly positive, then the [illegible] disagreement and the bargaining game has a unique equilibrium. This equilibrium can be called "[illegible]", in the sense, that in real life, the first mover (i.e., the player who starts the negotiation) would not obtain the whole surplus. This observation may imply that something fundamental is wrong with our bargaining model. [illegible] it is an "incorrect" model of bargaining? If so, then one may in [illegible] criticise the alternating-offers model of bargaining, on which our bargaining game is based.

An alternative angle from which to study the implications of our result is the following.

Take the Rubinstein game; commitment possibilities are not allowed and time plays the key role in determining the bargaining outcome.

It is presumably the case that the Rubinstein "bargaining outcome" should be stable in the presence of small perturbations.

For example, if one of the bargainers has available an outside option we know that it will not affect the Rubinstein "bargaining outcome" provided the value of the outside option is small; this is the famous 'Outside Option Principle' discovered by Binmore (1985). Hence, the Rubinstein "bargaining outcome" is indeed stable in the presence of small perturbations from outside of this type.

Now suppose each of the bargainers, in the Rubinstein game, has been given an action which each can take in order to commit himself to, not to budge from his offer, or not to accept the opponent's offer; the action makes it costly for a player to later back down from a committed bargaining position. And suppose these costs, to both players, are negligible. This is a small perturbation to the Rubinstein game.

The game presented in this chapter is one possible way to modelling this perturbation to the Rubinstein game, with c_A and c_B strictly positive but very small (i.e., negligible). As we have seen, this game has a unique "bargaining outcome", which is very different from the Rubinstein "bargaining outcome". Thus, the Rubinstein "bargaining outcome" is not stable in the presence of small perturbations of this type.

Finally, we shall comment on the notion of "commitment" in bargaining. Since there does not exist a unique unambiguous and well defined notion of "commitment" in bargaining there is a need for exploring the various plausible notions that come to the mind.

In this chapter we have modelled a generalised version of Schelling's notion of commitment, which is, that bargainers take actions during the negotiation process to increase the future cost of backing down from a committed bargaining position. Here, commitment is revocable, but at a cost.

Commitment, as modelled in this chapter, does not have what we shall call a time dimension. Here, a bargainer can make a commitment which could, in principle, last for any length of time. There are no costs associated with the length of time for which one is committed; (for example, no lawyer's fees to be paid, which could be a function of the time for which commitment is sought). In this chapter, making a commitment costs nothing, but revoking a commitment is costly. Time has no role to play in the notion of commitment that is adopted in this chapter.

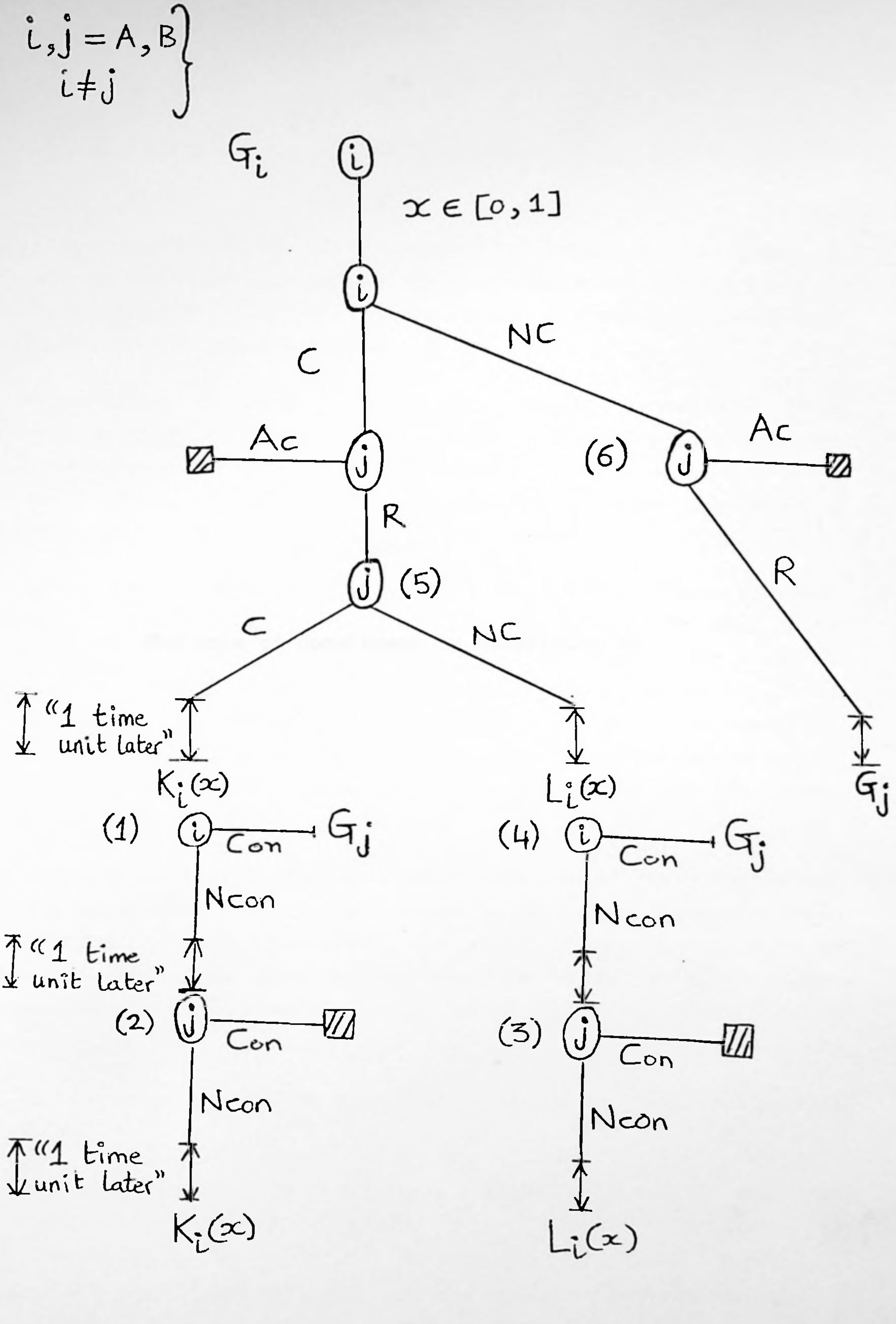
In chapter 2 we shall present a model that explores an alternative notion of commitment in bargaining; the model is a non-cooperative infinite-time horizon sequential game with complete and perfect information.

In that chapter making a commitment to a bargaining position is costly, but there are no costs of revoking a commitment. A bargainer chooses, strategically, the length of time for which he will commit himself to a bargaining position. The costs depend on the length of time chosen; costs being strictly increasing and strictly convex in time. Commitment is irrevocable during the length of time that a bargainer has chosen to commit himself. Thus, commitment is irrevocable *à la* "Nash demand game" (i.e., a genuine unconditional take-it-or-leave-it offer) only if a bargainer chooses to commit himself for an infinite length of time.

commitment, as members of their community. This [illegible] we'll call a time dimension. Hence, a conjecture: [illegible] which would, in principle, last for any length of time [illegible] costs associated with the length of time that action [illegible] (for example, no lawyer's bonus in the world, which [illegible] of the time for which commitment is sought). In [illegible] commitment costs [illegible] but no role to [illegible] this chapter.

In respect [illegible] cation of [illegible] infinite-[illegible] informatio[illegible]

In the [illegible] cally, t[illegible] whereas [illegible] himself to a [illegible] time chosen [illegible] time. Consider [illegible] together has chosen [illegible] irrevocable [illegible] time-to-or-leave-it [illegible] itself for an infinite length of time.



i, j = A, B
i ≠ j
G_i
i
x ∈ [0, 1]
i
C
NC
Ac
j
(6)
j
Ac
R
j
(5)
R
C
NC
"1 time unit later"
K_i(x)
L_i(x)
G_j
(1)
i
Con
G_j
(4)
i
Con
G_j
Ncon
Ncon
"1 time unit later"
(2)
j
Con
(3)
j
Con
Ncon
Ncon
"1 time unit later"
K_i(x)
L_i(x)

Figure 1

Chapter 2

The role of commitment in bargaining II

1. INTRODUCTION

This chapter, together with chapter 1, examines the role of commitment in bilateral bargaining. In this chapter we shall present a model that explores a notion of commitment in bargaining different from the notion explored in chapter 1.

In this chapter the basic idea is, that making a commitment to a bargaining position is costly, but revoking a commitment is costless (for a motivation of this idea see chapter 1, pp.24-25). This idea will be formalised in a non-cooperative infinite-time horizon sequential game with complete and perfect information.

The players make offers alternately. At the time of making an offer a player can choose, strategically, the length of time for which he is committed to his offer. Commitment during that length of time is irrevocable. The cost of making a commitment depends on the length of time for which commitment to an offer is sought. Commitment is irrevocable *à la* "Nash demand game" (i.e., a genuine unconditional take-it-or-leave-it offer) only if a bargainer chooses to commit himself for an infinite length of time.

The game may be interpreted as a generalisation of the alternating-offers model of Rubinstein (1982), in which the proposer has available an extra strategic variable, namely the length of time for which he is committed to his offer. This implies that the "times" at which offers are made are determined endogenously, except the first offer - which is made at time t=0.

In section 3 we prove the existence of a unique sub-game perfect equilibrium.

In the equilibrium, the Rubinstein (1982) solution is obtained (i.e., neither player A nor player B ever commit themselves) if certain conditions hold. These conditions are made relatively transparent if the exogenously fixed minimum time between successive offers - which represents the physical constraints in making an offer - tends to zero. Then, in this limit, the conditions are:

$$\text{for } i \neq j,\ i,j=A,B,\ c_i'(0) \geq r_j[r_i/(r_i+r_j)]$$

where r_i $(i=A,B)$ is the rate of time preference of player $i(r_i>0)$ and $c_i(T)$ is the cost to player i for making a commitment for a length of time T. c_i is a strictly increasing and strictly convex function with $c_i(0)=0$ and $c_i'(0)>0$. The right-hand side of the inequality above is interpreted as the marginal benefit to player i if player i commits himself. Thus, in the equilibrium, the Rubinstein solution is obtained if, at the margin, the cost to both players of committing themselves exceeds their respective benefit of doing so.

2. THE MODEL

Two players, A and B, are bargaining on the partition of a pie of size one. The pie will be partitioned only after the players reach an agreement.

The players make offers alternately. At the time of making an offer a player can choose, strategically, the <u>length of time</u> for which he is committed to his offer. Commitment during that length of time is irrevocable. The cost of making a commitment depends on the length of time for which commitment to an offer is sought. Thus, <u>making</u> a commitment is costly, but <u>revoking</u> a commitment is costless.

The "times" at which offers are made are determined endogenously, except the first offer - which is made at time $t=0$. The player who makes an offer (i.e., the proposer) decides on the length of time for which he is committed to his offer, and his opponent (i.e., the responder) either concedes (i.e., accepts) or rejects and waits to make a counteroffer, which the responder can make only after a certain length of time, namely the length of time for which the proposer has committed himself <u>plus</u> an exogenously fixed minimum time - which represents the physical constraints in making a counteroffer.

The bargaining process is modelled as a non-cooperative infinite-time horizon sequential game with complete and perfect information. The game may be interpreted as a generalisation of the alternating-offers model of Rubinstein (1982), in which the proposer (i.e., the player who makes an offer) has available an extra strategic variable, namely the <u>length of time</u> for which he is committed to his offer.

Let Δ denote the exogenously fixed minimum time between successive offers. Let T denote what we shall call the <u>commitment time</u> (i.e., the length of time above the exogenously fixed minimum time for which a proposer has committed himself to his offer); Δ, $T \in \mathbb{R}_+$.

At time $t=\sum_{i=1}^{2n}(T_i)+(2n)\Delta$, $(n=0,1,2,\ldots)$, player A makes an offer, x_{2n+1}, and chooses a commitment time, T_{2n+1}, where $x_{2n+1} \in [0,1]$ and $T_{2n+1} \in \mathbb{R}_+$. Player B either concedes (i.e., accepts) or rejects. If

player B concedes, then the game ends. On the other hand, if player B rejects, then the game proceeds at time $t=\sum_{i=1}^{2n+1}(T_i)+(2n+1)\Delta$, where player B makes a move. Note that if player B rejects, then he has to wait for a time, of length $T_{2n+1}+\Delta$, before making a move. Nothing happens during that time. Player A has made a commitment and player B does not concede.

At time $t=\sum_{i=1}^{2n+1}(T_i)+(2n+1)\Delta$, $(n=0,1,2,\ldots)$, player B makes an offer, x_{2n+2}, and chooses a commitment time, T_{2n+2}, where $x_{2n+2}\in[0,1]$ and $T_{2n+2}\in\mathbb{R}_+$. Player A either concedes or rejects. Concession terminates the game. Rejection leads the game to a point where player A has to move, at time $t=\sum_{i=1}^{2n+2}(T_i)+(2n+2)\Delta$.

The bargaining game begins at time $t=0$ with player A choosing $(x_1,T_1)\in[0,1]X\mathbb{R}_+$.

We shall assume that the players maximise expected utility. Let $r_i(i=A,B)$ denote the rate of time preference of player i. Let $c_i:\mathbb{R}_+\rightarrow\mathbb{R}_+(i=A,B)$ be a strictly increasing and strictly convex function with $c_i(0)=0$ and $c_i'(0)>0$. $c_i(T)$ denotes the cost to player i for making a commitment for a length of time T.

Suppose the players agree at time $t^*(t^*\geq 0)$ to an offer x^*, $x^*\in[0,1]$; x^* is the share of surplus received by player A and $1-x^*$ is the share of surplus received by player B. Let $(T_1^*,\ldots,T_n^*)$ denote the commitment times chosen by the two players up until time t^*, when they come to an agreement. Thus, $t^*=\sum_{j=1}^{n-1}(T_j^*)+(n-1)\Delta$. If n is odd, then it is player A's offer which is the accepted offer. If, on the other hand, n is even, then it is player B's offer which is the accepted offer. The commitment times of player A are $(T_1^*,T_3^*,\ldots,T_k^*)$, where $k=n$ if n is odd, and $k=n-1$ if n is even. The time at which player A chose the commitment time $T_{2i+1}^*(i=0,1,\ldots,k$, where $k=[(n-1)/2]$ if n is odd, and $k=[(n-2)/2]$ if n is even) is $\sum_{j=1}^{2i}(T_j^*)+(2i)\Delta$. Thus the payoff, discounted to time $t=0$, to player A is:

$$x^*\exp[-r_A(\sum_{j=1}^{n-1}(T_j^*)+(n-1)\Delta)]-\sum_{i=0}^{k}c_A(T_{2i+1}^*)\exp[-r_A(\sum_{j=1}^{2i}(T_j^*)+(2i)\Delta)],$$

where $k=[(n-1)/2]$ if n is odd, and $k=[(n-2)/2]$ if n is even. One can, similarly, define the payoff to player B.

player B succeeds, then the game ends. On the other hand, if player B [illegible] then the game proceeds at time [illegible], where player A makes a move. Note that if player B [illegible] then he has to wait for a time of length [illegible] before making a move. Nothing happens during that time. Player A has made a [illegible] and player B has not [illegible].

[illegible] time [illegible] player B makes an attempt, [illegible] and [illegible] where [illegible] and [illegible] player A [illegible]

[illegible] when player A [illegible]

[illegible] marginal [illegible] positive [illegible] Let [illegible] time [illegible] of player [illegible] Let [illegible] increasing and strictly [illegible] [illegible] denotes the cost to player A of making a [illegible] for a length of time [illegible].

Suppose the [illegible] at time [illegible] the share of surplus received by player A and [illegible] the share of surplus received by player B. Let [illegible] denote the [illegible] times chosen by the two players up until time [illegible] when they [illegible] an agreement. Thus, [illegible] If [illegible] is odd, then it is player A's offer which is the accepted offer [illegible] On the other hand, if [illegible] is even, then it is player B's offer which is the accepted offer. The [illegible] times of player A are [illegible] where [illegible] if [illegible] is odd and [illegible] if [illegible] is even. The time at which player A [illegible] [illegible] if [illegible] is odd, [illegible] if [illegible] is even [illegible] Thus the payoff [illegible] to player A [illegible]

$$[illegible]$$

where [illegible] if [illegible] is odd, and [illegible] if [illegible] is even. The [illegible] [illegible] payoff to player B, We shall assume that the bargaining game described above, which shall be denoted by G, is a game of complete information (i.e., all information is assumed to be common knowledge amongst the players). Furthermore, note that G is a game of perfect information.

A strategy for each agent in G will tell the agent the choice to make at each and every decision node that he may be at. Each of the two players will have a set of strategies from which to choose a strategy. The solution concept we will use is the sub-game perfect equilibrium (SGPE) (Selten (1965, 1975)). A strategy tuple is in SGPE if its restriction to any subgame is in Nash equilibrium.

3. PERFECT EQUILIBRIUM

The game G that we have presented (in section 2) may be interpreted as nothing more than a generalisation of the alternating-offers model of Rubinstein (1982), in which the proposer has available an extra strategic variable, namely the length of time for which he is committed to his offer. The Proposition below deals with the existence and the uniqueness of the SGPE of the game G; it is analogous to Theorem 1 in Rubinstein (1987b) (Thm. 1 of Rubinstein (1987b) is a restatement of Rubinstein's (1982) result).

PROPOSITION.

Let (x^*,t^*,y^*,s^*) be the unique solution for the following pair of programmes:

$$\max_{(y,s)\in S} [1-y-c_B(s)] \quad \text{s.t. } y=[x-c_A(t)]\exp(-r_A(\Delta+s)), \tag{1}$$

$$\max_{(x,t)\in S} [x-c_A(t)] \quad \text{s.t. } 1-x=[1-y-c_B(s)]\exp(-r_B(\Delta+t)), \tag{2}$$

where $(x,t)\in S$, $(y,s)\in S$ and $S=[0,1]\times\mathbb{R}_+$.

Then, the unique SGPE of the game G is the pair of strategies in which player A (player B) always makes the offer $x^*(y^*)$ and commits himself for a length of time $t^*(s^*)$, accepts any proposal $y,s(x,t)$ which leaves him better off than the proposal y^*, $s^*(x^*,t^*)$, and rejects any proposal which is strictly worse for him than the proposal y^*, $s^*(x^*,t^*)$- where (x^*,t^*), (y^*,s^*) are defined above.

REMARKS: (i) The pair of programmes, (1) and (2), above, are analogous to the two fundamental equations in Rubinstein (1987b, Theorem 1). (ii) One can verify that the pair of strategies described in the Proposition are in SGPE. (iii) Lemmas (1) and (2) below prove the existence of a unique solution (x^*,t^*,y^*,s^*) to the two programmes

3. PERFECT EQUILIBRIUM

The game G that we have presented (in
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nd the uniqueness of
Theorem 1 is
restatement

PROPOSITION

Let $(x^{*}, t^{*}$ optimal solution
programmes:

max $(1-y$

$(y,s)\in S$

s.t. $y \geq (x - c_{A}(t^{*}))\exp\{-r_{A}(s+\sigma)\}$,

max $(x - c_{B}(t))$

$(x,t)\in S$

s.t. $1-x \geq (1-y-c_{B}(\sigma))\exp\{-r_{B}(\Delta+t)\}$,

where $(x,t)\in S$, $(y,s)\in S$ and $S=[0,1]\times\mathbb{R}_{+}$

Then, the unique SGPE of the game G is the
which player A (player B) always makes
himself for a length of time $t^{*}(s^{*})$, accepts
which leaves him better off than the
rejects any proposal which is strictly worse
$y^{*}, s^{*}(x^{*}, t^{*})$ -- where (x^{*}, t^{*}), (y^{*}, s^{*}) are

REMARKS. (i) The pair of programmes
analogous to the two fundamental equations
Theorem 1). (ii) One can verify that the
in the Proposition are in some. (iii)
the existence of a unique solution $(x^{*}, t^{*}$

described in the Proposition. (iv) The proof of the uniqueness of the SGPE of our game G is not presented here. That proof is similar to the proof of uniqueness of the SGPE of the Rubinstein (1982) game presented in Rubinstein (1987b) (see his Proof 1 of Theorem 1) - which is based on the method of proof presented by Shaked and Sutton (1984a).

Lemma 1.

Consider the two programmes, (1) and (2), described in the Proposition, above.

Then, there exist functions f_1, f_2, f_3 and f_4, where $f_i: S \rightarrow [0,1]$ $(i=1,3)$, and $f_i: S \rightarrow \mathbb{R}_+$ $(i=2,4)$, such that (1) $\Rightarrow$

$$y = f_1(x,t) \tag{3}$$

$$s = f_2(x,t) \tag{4}$$

and (2) $\Rightarrow$

$$x = f_3(y,s) \tag{5}$$

$$t = f_4(y,s) \tag{6}$$

where f_1, f_2, f_3 and f_4 are defined below.

$$s = f_2(x,t) = \begin{cases} 0 & \text{if } r_A[x - c_A(t)]\exp(-r_A\Delta) \le c'_B(0) \\ \text{defined by equation (7)} & \text{if } r_A[x - c_A(t)]\exp(-r_A\Delta) > c'_B(0) \end{cases}$$

$$r_A[x - c_A(t)]\exp[-r_A(\Delta + s)] = c'_B(s) \tag{7}$$

$$y = f_1(x,t) = \begin{cases} [x - c_A(t)]\exp(-r_A\Delta) & \text{if } r_A[x - c_A(t)]\exp(-r_A\Delta) \le c'_B(0) \\ [x - c_A(t)]\exp(-r_A(\Delta + s)), \text{ where } s \text{ is defined by equation (7)} & \text{if } r_A[x - c_A(t)]\exp(-r_A\Delta) > c'_B(0) \end{cases}$$

$$t = f_4(y,s) = \begin{cases} 0 & \text{if } r_B[1 - y - c_B(s)]\exp(-r_B\Delta) \le c'_A(0) \\ \text{defined by equation (8)} & \text{if } r_B[1 - y - c_B(s)]\exp(-r_B\Delta) > c'_A(0) \end{cases}$$

$$r_B[1-y-c_B(s)]\exp[-r_B(\Delta+t)]=c_B'(t) \tag{8}$$

$$x=f_3(y,s)=\begin{cases} 1-[1-y-c_B(s)]\exp(-r_B\Delta) & \text{if } r_B[1-y-c_B(s)]\exp(-r_B\Delta) \leq c_A'(0) \\ 1-[1-y-c_B(s)]\exp(-r_B(\Delta+t)), \text{ where } t \text{ is defined by equation (8)} & \text{if } r_B[1-y-c_B(s)]\exp(-r_B\Delta) > c_A'(0) \end{cases}$$

Proof.

Let us examine programme (1) (cf. the Proposition). Substituting for the constraint, we obtain:

$$\max_{s\in \mathbb{R}_+} [1-[x-c_A(t)]\exp[-r_A(\Delta+s)]-c_B(s)].$$

Let F(s) denote the maximand. Differentiating twice, we obtain:

$F'(s)=r_A[x-c_A(t)]\exp[-r_A(\Delta+s)]-c_B'(s)$ and

$F''(s)=-r_A^2[x-c_A(t)]\exp[-r_A(\Delta+s)]-c_B''(s)$.

Since $x-c_A(t)\geq 0$ and $c_B''(s)>0$, we have that $F''(s)<0$. Thus $F:\mathbb{R}_+\rightarrow\mathbb{R}_+$ is strictly concave, and hence a unique solution exists; denote it by s.

$$\begin{array}{lll} s=0 & \text{if} & F'(0)\leq 0 \\ s>0 & \text{if} & F'(0)>0 \end{array}$$

If $F'(0)>0$, then s is defined by the first-order condition (i.e., $F'(s)=0$). Hence, there exists a function, say $f_2:S\rightarrow\mathbb{R}_+$, where f_2 is as defined in the Lemma.

Substituting $s=f_2(x,t)$ into the constraint of programme (1), we obtain that there exists a function, say $f_1:S\rightarrow[0,1]$, where $y=f_1(x,t)$ and f_1 is as defined in the Lemma.

One can similarly solve programme (2) and show the existence of the functions f_3 and f_4 as defined in the Lemma.

Q.E.D.

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

Lemma 2.

There exists a unique solution (x^*, t^*, y^*, s^*) to equations (3), (4), (5) and (6) of Lemma 1.

Proof.

Case I: $r_A[x-c_A(t)]\exp(-r_A\Delta)\leq c_B'(0)$ and $r_B[1-y-c_B(s)]\exp(-r_B\Delta)\leq c_A'(0)$.

Then, using Lemma 1, we have:

$$y=f_1(x,t)=[x-c_A(t)]\exp(-r_A\Delta),$$
$$s=f_2(x,t)=0,$$
$$x=f_3(y,s)=1-[1-y-c_B(s)]\exp(-r_B\Delta) \text{ and}$$
$$t=f_4(y,s)=0.$$

Solving the above equations, we obtain:

$$x^*=[1-\exp(-r_B\Delta)]/[1-\exp(-\Delta(r_A+r_B))]$$
$$t^*=0$$
$$y^*=[\exp(-r_A\Delta)][[1-\exp(-r_B\Delta)]/[1-\exp(-\Delta(r_A+r_B))]]$$
$$s^*=0.$$

Hence, we have a unique solution under the conditions specified above for Case I, which become:

for $i\neq j, i,j=A,B,$ $c_i'(0)\geq[r_j\exp(-r_j\Delta)[1-\exp(-r_i\Delta)]]/[1-\exp(-\Delta(r_i+r_j))]$.

There are three other cases to be considered. In each of them, unlike Case I above, no explicit solution will be obtained. However, using the Brouwer Fixed Point Theorem and the Gale-Nakaido Univalence Theorem, one can prove the existence and the uniqueness of a solution to equations (3) - (6).

Q.E.D.

The interesting conclusion is brought out by the result of Case I in Lemma 2. The result is the Rubinstein (1982) bargaining solution, which will hold if:

for $i\neq j$, $i,j=A,B,$

$$c_i'(0)\geq[r_j\exp(-r_j\Delta)[1-\exp(-r_i\Delta)]]/[1-\exp(-\Delta(r_i+r_j))] \qquad (9)$$

Note that $s^*=t^*=0$ (i.e., neither player A nor player B ever make a

commitment). Thus, our game G has produced the Rubinstein result <u>if</u> (9) holds.

In order to obtain a more transparent interpretation let $\Delta \to 0$. Then: $x^* = y^* = r_B/(r_A+r_B)$ and $s^* = t^* = 0$ (i.e., the Rubinstein result) <u>if</u>

for $i \neq j$, $i,j=A,B$, $c_i'(0) \geq r_j[r_i/(r_i+r_j)]$ (10)

Note that $r_i/(r_i+r_j)$ is the share of the surplus received by player j. To restate, commitment is never made by either player A or player B if condition (10) holds. Condition (10) establishes a "relationship" between the marginal costs of <u>commitment</u>, $c_i'(0)$ (i=A,B) and the rates of <u>time</u> preference, r_A and r_B.

In fact, $r_j[r_i/(r_i+r_j)]$ (i.e., the right-hand side of the inequality (10)) can be interpreted as the <u>marginal benefit</u> (MB) to player i (when player i and player j have <u>not</u> committed themselves for any length of time, i.e., given $s^* = t^* = 0$). $r_i/(r_i+r_j)$ is the share of the cake (i.e., surplus) received by player j and r_j is player j's rate of time preference. When player i commits himself, at the margin, the amount of cake he has to give to player j, in order to "buy him out", <u>decreases</u> - and thus player i's share increases - and the amount by which it decreases depends on the rate of time preference of player j (i.e., r_j) and, of course, what player j receives if he rejects - which is $r_i/(r_i+r_j)$. Hence, $MB=r_j[r_i/(r_i+r_j)]$.

Let us further illustrate this condition (10) by reference to some special cases. (i) Suppose r_j is very small (say, approximately equal to zero, i.e., $r_j \approx 0$). Then the MB to player i is approximately equal to zero. $r_j \approx 0$ means that player j is very patient, which in turn implies that player j does not care when he receives $r_i/(r_i+r_j)$, which in turn implies that player i <u>cannot</u> gain by committing himself, at the margin (i.e., $MB \approx 0$). (ii) Suppose r_j is very large (say, approximately infinite, i.e., $r_j \approx \infty$). This means that player j is very impatient, which in turn implies that player j cares a lot as to when he receives $r_i/(r_i+r_j)$, which in turn implies that player i <u>can</u> gain a lot by committing himself, at the margin (i.e., $MB \approx \infty$).

[illegible]

PART TWO

OUTSIDE OPTIONS

Chapter 3

Bargaining, search and the 'Outside Option Principle'

1. INTRODUCTION

In his classic paper, Rubinstein (1982) presented a solution to the bilateral bargaining problem using the notion of a perfect equilibrium in a bargaining process. Now suppose one of the two players in the bilateral bargaining situation is free to quit bargaining and instead take up some outside option; the outside option is available with certainty. The outside option and the pie under bargaining are mutually exclusive. "How will the value of the outside option impinge on the bargaining outcome?" Binmore (1985), using an extension of the Rubinstein bargaining model, demonstrated that if the value of the outside option is less than what the player receives in the Rubinstein solution then it will not influence the bargaining outcome, and on the other hand, if the value of the outside option is larger than what the player receives in the Rubinstein solution then the outside option does influence the bargaining outcome - his opponent buys him out by giving him the value of the outside option. This result is known as the 'Outside Option Principle' (see Shaked and Sutton (1984b) for a further discussion).

Now suppose one of the two players in the bilateral bargaining situation is free to quit bargaining and instead take up some outside option. But, now the outside option is not available with certainty; the player has to engage in a process of random search in order to find this outside option. In other words, the player is free to quit bargaining in order to search for his outside option. If the player does not find the outside option, after having searched for some time, then he may resume bargaining. In this situation, how will the

1. INTRODUCTION

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Now suppose one of the two players in the bilateral bargaining situation is free to quit bargaining and instead take up some outside option. But, now the outside option is not available with certainty; the player has to engage in a process of costly search in order to find the outside option. In other words, the player is free to quit bargaining in order to search for the outside option. If the player does not find the outside option, after having searched for some time, then he may resume bargaining. In this situation, [illegible] will the value of the outside option impinge on the bargaining outcome, given that search is costly.

The present chapter will provide an answer to the above question. In fact, in this chapter we will study a more general situation; each of the two players is free to quit bargaining and instead engage in a process of random search in order to find one of his many outside options. The players may resume bargaining, after having searched for some time without success. Once again, we are interested to know how the values of the outside options impinge on the bargaining outcome, given that search is costly.

Thus, in this chapter we study the following situation. Two players are bargaining on the partition of a pie of size one. The pie will be partitioned only after the players reach an agreement. Each of the two players is free to quit bargaining and instead engage in a process of random search in order to find one of his many outside options, which the player may adopt instead of attempting to reach an agreement in the bargaining (i.e., the outside options and the pie under bargaining are mutually exclusive). The players can choose to resume bargaining, after having searched for some time without success.

An example of such a situation is when two insiders, a firm and a worker, are bargaining over the wage. And the worker is free to quit bargaining in order to search for alternative wage offers.

We present a model, in section 2, which is a non-cooperative infinite-time horizon sequential game with complete information. The game incorporates two processes, bargaining and search, both of which depend on time. The bargaining process is the alternating-offers procedure studied by Stahl (1972) and Rubinstein (1982).

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We present a model in Section 2, which is a non-cooperative infinite time horizon sequential game with complete information. The game combines two processes, bargaining and search, both of which depend on time. The bargaining process is the alternating-offers procedure studied by Stahl (1972) and Rubinstein (1982).

Furthermore, the game incorporates a "view" on how the bargaining and search processes ought to be interlaced.

In section 3 we use the subgame perfect equilibrium solution concept (Selten (1965, 1975)) to analyse the game, and we obtain a unique equilibrium partition. We then analyse the limiting case as the time between successive offers tends to zero. There are two important reasons why one is interested in this limit. Firstly, this eliminates the first mover advantage. And secondly, this overcomes the criticism that is often made regarding the rigidity of the timetable for making proposals; these points were first discussed by Binmore (1987a).

Suppose the players did not play the game; then each player would achieve his expected reservation value, which is derived from following a sequentially optimal search rule over his outside options. (See McCall (1965) for a discussion of optimal stopping rules).

Before we state the key result that is obtained, we note that the Binmore (1985) 'Outside Option Principle' extends to the case when each of the two players has <u>one</u> outside option available with <u>certainty</u>; of course, one assumes that the sum of the two outside options is less than one, in order for there to exist mutually beneficial trade amongst the two players.

The key result is, that in the limiting case we obtain the Binmore (1985) 'Outside Option Principle', with the players' <u>expected reservation values</u> treated as the outside options.

An alternative angle from which to view this result is as follows. The game presented in this chapter has produced a result (the limiting case) that would be produced by the Binmore (1985) extension of the Rubinstein game (which produced the 'Outside Option

Furthermore, the game incorporates a 'rule' on how the bargaining and search processes ought to be determined.

In Section 3 we use the extended perfect equilibrium solution concept (Selten (1965, 1975)) to analyse the game, which we show to have a unique equilibrium [illegible]. [illegible] the limiting case as the time between [illegible]. There are two important [illegible]. [illegible] illustrates the [illegible] (Binmore, [illegible]) have appeared in the [illegible] regarding the analysis of the [illegible]; these models were first discussed by [illegible].

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Before we state the key result that is obtained, we note that the Binmore (1985) 'Outside Option Principle' applies to the case when each of the two players has some outside option available, with certainty of course, and assumes that the sum of the two outside options is less than one, in order for there to exist mutually beneficial trade amongst the two players.

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Principle') if in the Binmore game we *a priori* define the expected reservation values of the players to be their outside options available with certainty.

2. THE MODEL

Two players, A and B, are bargaining on the partition of a pie of size one. The pie will be partitioned only after the players reach an agreement. Player k (k = A, B) is free to quit bargaining and instead engage in a process of random search in order to find one of his many outside options, which the player may adopt instead of attempting to reach an agreement in the bargaining (i.e., the outside options and the pie under bargaining are mutually exclusive). Denote the outside options by x_i^k ($i = 1, \ldots, N_k$, $1 \geqslant x_i^k \geqslant 0$), and assume that the options are ordered (i.e., for $i = 1, \ldots, N_k-1$, $x_{i+1}^k > x_i^k$). Player k will find the outside option x_i^k ($i = 1, \ldots, N_k$) with probability p_i^k ($1 \geqslant p_i^k \geqslant 0$ and $\sum_{i=1}^{N_k} p_i^k \leqslant 1$).

The model is a non-cooperative infinite-time horizon sequential game with complete information. The game incorporates two processes, bargaining and search, both of which depend on time. The bargaining process is the alternating-offers procedure studied by Stahl (1972) and Rubinstein (1982). The search process is interlaced with the bargaining process as follows. At any time in the bargaining process when player A makes an offer to player B, B can, either accept the offer, or reject the offer and wait to make a counteroffer (and thus remain on the negotiating table), or reject the offer and leave the negotiating table (i.e., the bargaining process) in order to search for an outside option. At the end of <u>one</u> period of search either an outside option is taken up, in which case the game ends, or the players return to the negotiating table with B making an offer to A. And symmetrically, following B's offer to A, A can choose to interrupt the bargaining process in order to search for an

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We may now proceed to a description of the game.

At time $t = 0$, player A makes an offer to player B (point (1) in Figure 1).[1] Player B can either accept the offer, in which case the game ends, or reject the offer, in which case B has to wait Δ units of time to make a counteroffer, or reject the offer and leave the negotiating table (i.e., the bargaining process) in order to search for an outside option.

If player B chooses, at time $t = 0$, to search for an outside option, then player A has to decide whether to or not to search for an outside option (point (2) in Figure 1). One period of search takes τ units of time.

Suppose player A does not search. Then, at time $t = \tau$, a chance move occurs (point (3)), in which, with probability p_i^B player B finds the outside option x_i^B, and with probability $(1 - \sum_{i=1}^{N_B} p^B)$ player B does not find an outside option, in which case B returns to the negotiating table and makes an offer to A (point (4)). If B finds the outside option x_i^B he either chooses to take it, in which case the game ends, or chooses not to take it, in which case B returns to the negotiating table and makes an offer to A (point (4)).

Suppose player A chooses to search. Then, at time $t = \tau$, a

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1. Figure 1 is placed at the end of section 4.

outside option. Thus, that [illegible] of search, [illegible] game would [illegible] the bargaining [illegible]

[illegible] of the bargaining [illegible]

[illegible] of the game.

[illegible] Figure 1 [illegible]

[illegible] at time $t = 0$, to search for an outside option, then [illegible] to decide whether or not to search the outside option [illegible] period of search [illegible]

Suppose player A does not search. Then, at time $t = 1$, a chance move occurs [illegible], in which, with probability [illegible] player A finds the outside option [illegible], and with probability [illegible] player A does not find an outside option, in which case it returns to the negotiating table and makes an offer [illegible]. If A finds the outside option [illegible] in which case [illegible] the game ends [illegible] to which case B returns to the negotiating table and makes an offer [illegible].

Suppose player A chooses to search. Then, at time $t = 1$, [illegible]

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1. Figure 1 is placed at the end of section [illegible].

chance move occurs (point (5)), in which, (i) with probability $[1 - \sum_{i=1}^{N_B} p_i^B][1 - \sum_{j=1}^{N_A} p_j^A]$ both players, A and B, do not find an outside option, in which case B makes an offer to A (point (6)), (ii) with probability $[1 - \sum_{j=1}^{N_A} p_j^A]p_i^B$ player B finds the outside option x_i^B and player A does not find an outside option, in which case B has to choose either to take it and thus the game ends or not to take it and thus make an offer to A (point (6)), (iii) with probability $[1 - \sum_{i=1}^{N_B} p_i^B]p_j^A$ player A finds the outside option x_j^A and player B does not find an outside option, in which case A has to choose either to take it and thus the game ends or not to take it and thus B makes an offer to A (point (6)), and (iv) with probability $p_i^B p_j^A$ player A finds the outside option x_j^A and player B finds the outside option x_i^B, in which case both players, simultaneously, decide whether to or not to take up their respective outside option. If both players do not take up their respective outside option, then B makes an offer to A (point (6)). If either A or B or both A and B take up their outside option, then the game ends.

To summarise. If at time $t = 0$ player B chose to search, then at time $t = \tau$ (after searching for one round) either the game has ended or the players have returned to the negotiating table with B making an offer to A. On the other hand, if at time $t = 0$ player B chose not to search and simply waited to make a counteroffer, then at time $t = \Delta$ B makes an offer to A.

We now describe the subgame in which player B begins by making an offer to player A. The structure of this subgame is independent of the history of the subgame. Let us denote the game in which B begins with an offer by G_B, and the game in which A begins with an offer by G_A. Therefore, the game at time $t = 0$ is G_A, [illegible] we obtain [illegible] with probabilities [illegible] with [illegible], A and B, we see that the [illegible]

[illegible]

[illegible] and [illegible]

[illegible] B [illegible]

[illegible] the same game.

Interpretation. If at time t [illegible] player A [illegible] to search, then at time t + 1 [illegible]

[illegible] Player A [illegible] Let [illegible] denote the game in which B begins with [illegible] the game in which B begins with [illegible] Therefore, the game [illegible] and the subgame at time $t = \Delta$ and the subgames at time $t = \tau$ (when B makes an offer) is G_B.

We have described, above, the game G_A. Repeat that description, but with A replaced by B and B replaced by A, and one obtains the description of the game G_B. To summarise. If at time $t = \tau$ ($t = \Delta$) A chose to search, then at time $t = 2\tau(t = \Delta + \tau)$ (after searching for one round) either the game has ended or the players have returned to the negotiating table with A making an offer to B. On the other hand, if at time $t = \tau(t = \Delta)$ A chose not to search and simply waited to make a counteroffer, then at time $t = \tau + \Delta$ ($t = 2\Delta$) A makes an offer to B.

Thus, at times $t = 2\Delta$, $t = \Delta + \tau$, $t = \tau + \Delta$ and $t = 2\tau$ (when player A makes an offer to player B) the game has returned to the game G_A. Hence, note the recursive structure of the game G_A that begins at time $t = 0$; the homogeneity of the game permits us to define G_A and G_B independently of the time elapsed since the beginning of the game.

We shall assume that the two players maximise expected utility. Player k (k = A, B) has a von Neumann-Morgenstern utility function U_k $(z, t, m) = z\, \delta^t \beta^m$. z can be either the share of the pie received by player k, _if_ agreement on the partition is achieved, or the value of an outside option belonging to player k, _if_ k takes up an outside option, or the expected reservation value of player k, _if_ player w ($w \neq k$, w = A, B) takes up an outside option leaving player k with his expected reservation value. The expected reservation value is the expected payoff derived from following a sequentially optimal search rule over outside options. t is the time elapsed from time $t = 0$ before z is obtained, and m is the number of periods that player k and the subgame at time $t + 1$ will then correspond to time $t + 1$ [illegible] [illegible] is added to [illegible].

[illegible] the given G_t. [illegible] but with a [illegible] [illegible] If at time $t + 1$ [illegible] [illegible] the [illegible] [illegible] to [illegible] [illegible] with [illegible].

[illegible] $t \le T$ and [illegible] player [illegible] of the game has returned to the game [illegible] the remaining structure of the game [illegible] begins at time [illegible] The [illegible] of the [illegible] by the [illegible] of the time elapsed since the beginning of the game.

We shall assume that the two [illegible]. Player [illegible] with [illegible] [illegible] the value of the [illegible] the [illegible] of [illegible] option, or the expected reservation value of player [illegible] with an expected reservation value. The expected reservation value is the expected reward derived from following a sequentially optimal search rule over [illegible] options. t is the time elapsed from time $t = 0$ [illegible] is obtained, and n is the number of [illegible] that player [illegible] searched before z is obtained. δ is the (common) discount factor (i.e., the cost of time), $0 < \delta < 1$, and β represents a fixed cost per period of search, $0 < \beta < 1$. We shall assume that the total cost of search per period of search includes the cost of time (the τ units of time incurred) and the fixed cost per period of search.

We shall assume that the game G_A is a game of complete information, i.e., all information (including the game tree and the players preferences) is assumed to be common knowledge amongst the players. We note that G_A is a game of imperfect information. The imperfect information arises only in the search process, when both of the players find an outside option and have to decide simultaneously (i.e., without knowing what decision the opponent is taking) whether to or not to take up their respective outside option. We emphasize that nowhere else in the game G_A does there exist imperfect information. We note that this imperfect information is innocuous in that the subgame perfect equilibrium solution concept is sufficient (and necessary) to ensure a unique outcome of the game.

Suppose player k (k = A, B) refused to play the game G_A with player w ($w \neq k$, w = A, B); then player k would achieve his expected reservation value (ERV), R_k, derived from following a sequentially optimal search rule over outside options. Let us compute R_k. There will exist r_k such that it is sequentially optimal for player k to accept outside options x_i^k for $i \geqslant r_k$, and to reject outside options x_i^k for $i \leqslant r_k - 1$; i.e., there exists r_k such that

$$x^k_{r_k} \geqslant \delta^\tau \beta \sum_{i=1}^{N_k} p^k_i x^k_i \geqslant x^k_{r_k-1}$$

[illegible] is attained. [illegible] by the cooperative approach [illegible] that the cost of [illegible] a fixed cost any period of [illegible] that the total cost of [illegible] per period [illegible] value of [illegible] of [illegible]

[illegible]

[illegible] advantage. We note that the [illegible] is [illegible] that the [illegible] equilibrium [illegible] and necessary to ensure a [illegible] of the game.

Because player i ($i = A, B$) [illegible] with player j ($j \neq i$, $j = A, B$) then player i [illegible] the [illegible] value (SEV), [illegible] derived from [illegible] outside options [illegible]. There [illegible] such that it is [illegible] outside options of the [illegible] for [illegible], there [illegible] such that

$$[illegible]$$

(see McCall (1965) for a discussion of optimal stopping rules). r_k will depend on the parameters p_i^k, x_i^k for $i = 1, \ldots, N_k$, δ, τ and β. Given r_k, R_k is defined as follows:

$$R_k = \left[\delta^\tau \beta \sum_{i=r_k}^{N_k} p_i^k x_i^k\right] / \left[1 - \left[1 - \sum_{i=r_k}^{N_k} p_i^k\right] \delta^\tau \beta\right].$$

A strategy for each agent in G_A will tell the agent the choice to make at each and every decision node that he may be at. Each of the two players will have a set of strategies from which to choose a strategy. The solution concept we will use is the subgame perfect equilibrium (SGPE) (Selten (1965, 1975)). A strategy tuple is in SGPE if its restriction to any subgame is in Nash equilibrium.

[illegible] depends on the [illegible] Fig. 5, [illegible]

[illegible]

A similar [illegible]

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3. PERFECT EQUILIBRIUM

We will now analyse the game G_A using the SGPE solution concept. We firstly, in Case I, characterise the unique SGPE partition of the game when player A has no outside option and player B has one outside option. This will elucidate the crux of the main result of this chapter and allow us to draw certain conclusions. We will not present the analysis of the game G_A when player A has many outside options and player B has many outside options. This is because the algebra involved is extremely complicated and lengthy, to say the least. However, we secondly, in Case II, prove the existence and uniqueness of the SGPE partition (and characterise it) when player A has no outside option and player B has many (say N) outside options. Thirdly, in Case III, we prove the existence and uniqueness of the SGPE partition (and characterise it) when player A has one outside option and player B has one outside option.

Case I: Player A has no outside option and player B has one outside option.

Player A's ERV is zero. And player B's ERV, R_B, is as follows:

$$R_B = [\delta^{\tau}\beta px]/[1 - (1-p)\delta^{\tau}\beta] ,$$

(cf. section 2). Note that we have dropped the subscripts and superscripts, since A has no outside option and B has only one outside option; this reduces the notation. p denotes the probability that B will find his outside option x in one period of search. Furthermore, note that $R_B < 1$ for all values $0 < \delta < 1$, $\tau > 0$, $0 < \beta <$

1, $0 \leqslant p \leqslant 1$ and $0 \leqslant x \leqslant 1$. Since $R_A = 0$, $R_A + R_B < 1$ is satisfied for all possible parameter values. $R_A + R_B < 1$ implies that there exists a mutually advantageous trade, i.e., a surplus exists.

The game G_A has a unique SGPE partition, in which agreement is reached at time $t = 0$ and player A receives share M, given by:

$$M = 1/(1 + \delta^{\Delta}) \quad \underline{\text{if}} \quad 1/(1 + \delta^{\Delta}) \geqslant [\delta^{T}\beta px]/[\delta^{\Delta} - (1-p)\delta^{T}\beta],$$

$$= [1 - \delta^{T}\beta[(1-p) + px]]/[1 - (1-p)\delta^{T+\Delta}\beta] \quad \underline{\text{if}}$$

$$1/(1+\delta^{\Delta}) < [\delta^{T}\beta px]/[\delta^{\Delta} - (1-p)\delta^{T}\beta] \ ,$$

and player B receives $1 - M$. (The above result is obtained by putting $N = 1$ in Proposition 1, which will be stated and proved below, in Case II.)

A relatively transparent interpretation is made possible by taking the limit as $\Delta \to 0$. But, there are two further and important reasons why one is interested in this limit. Firstly, this eliminates the first mover advantage. And secondly, this overcomes the criticism that is often made regarding the rigidity of the timetable for making proposals (i.e., after rejecting an offer and choosing not to search a player will typically wish to make his counteroffer at the earliest possible moment, and thus the limiting case can be used as a paradigm for the case in which the players are not formally constrained by an exogenously determined timetable). These points were first discussed by Binmore (1987a). Thus as $\Delta \to 0$, we obtain:

$$M = 1/2 \quad \underline{\text{if}} \quad R_B \leqslant 1/2$$

$$= 1 - R_B \quad \underline{\text{if}} \quad R_B > 1/2 \ ,$$

[illegible]

$$[illegible]$$

[illegible]

$$[illegible]$$

where R_B is the expected reservation value (ERV) of player B.

If one treats R_B as the outside option to player B available with certainty, then in the limit (as $\Delta \to 0$) we have rediscovered the 'Outside Option Principle'. The principle was first discovered by Binmore (1985). The 'Outside Option Principle' ('OOP') can be obtained in our game when player A has no outside option and player B has one outside option available with certainty, and thus B does not have to search for his outside option (or equivalently, $\tau = 0$, $\beta = 1$ and $p = 1$). The 'OOP' (limiting case, as $\Delta \to 0$):

$$\begin{aligned} M &= 1/2 && \underline{\text{if}} \quad x \leqslant 1/2 \\ &= 1-x && \underline{\text{if}} \quad x > 1/2 \ , \end{aligned}$$

where x is the outside option of player B available with certainty.

The 'OOP' refers to the situation when player B has an outside option x, available with certainty. Now suppose B has to search for his outside option x and that search is costly. In section 2, we presented a game which incorporates the bargaining and the search processes; in particular, the game represents a "view" as to how the bargaining process ought to be interlaced with the search process. We are led to the conclusion that it is not the value of x that influences the bargaining outcome, as it does in the case when x is available with certainty, but it is the value of R_B that matters. Furthermore, R_B influences the bargaining outcome as if R_B were the outside option of player B available with certainty.

In Case II and Case III, below, we rediscover the above result in more general environments.

where E_B is the expected reservation value (ERV) of player B.

If one treats E_B as the outside option of player B (i.e., as if player B can obtain E_B with certainty), then [illegible] ... "Outside Option [illegible]" ... Binmore (1985) ... [illegible] ... and player B has ... [illegible] ... and $\delta_A = \delta_B$ ... [illegible]

$$[illegible]$$

where x is the outside option of player B available with certainty.

The [illegible] to the situation when player B has an outside option x, available with certainty. Now suppose B has to search for his outside option x and that search is costly. In section 2, we presented a game which incorporates the bargaining and the search processes; in particular, the game represents a 'view' as to how the bargaining process ought to be interlaced with the search process. We are led to the conclusion that it is not the value of x that influences the bargaining outcome, as it does in the case when x is available with certainty, but it is the value of E_B that matters. Furthermore, E_B influences the bargaining outcome as if E_B were the outside option of player B available with certainty.

In Case II and case III, below, we rediscover the above result in more general environments.

Case II: Player A has no outside option and player B has many (say N) outside options.

Player A's ERV is zero. And, player B's ERV, R_B, is as follows:

$$R_B = \left[\delta^T\beta \sum_{i=r}^{N} p_i x_i\right] / \left[1 - \left[1 - \sum_{i=r}^{N} p_i\right]\delta^T\beta\right] ,$$

(cf. section 2). Note that we have dropped some subscripts and superscripts, since A does not have any outside option; this reduces the notation. p_i ($i = 1, ..., N$) denotes the probability that B will find his outside option x_i in one period of search. May we recall (from section 2) that there exists r such that

$$x_r \geqslant \delta^T\beta \sum_{i=1}^{N} p_i x_i \geqslant x_{r-1} .$$

Furthermore, note that $R_B < 1$ for all parameter values. Thus, $R_A + R_B < 1$ since $R_A = 0$.

The SGPE of the game G_A is analysed using the elegant method proposed by Shaked and Sutton (1984a).

We begin by establishing the following:

Lemma 1: Let $P_A(G_A)$ denote the set of SGPE payoffs to player A in the game G_A. Let M denote the supremum (infimum) of $P_A(G_A)$. Then:

$$M = 1 - \max\left\{\delta^\Delta(1 - \delta^\Delta M),\ F\right\} , \tag{1}$$

where $F = \delta^T\beta\left[\left[1 - \sum_{i=1}^{N} p_i\right](1 - \delta^\Delta M) + \sum_{i=1}^{N} p_i \max(x_i, 1 - \delta^\Delta M)\right]$

Case II. Player A [illegible] and player B [illegible] mixed strategies.

Player A's [illegible] v_B as follows

$$v_B = \left| \ldots \right| \ldots$$

[illegible]

$v_B = \ldots$ [illegible]

Furthermore, note that $v_B \le 1$ for all parameter values. [illegible] $v_B \le 1$ since $R_0 \ge 0$.

The value of the game G_k is [illegible] proposed by [illegible] (1965b).

We begin by establishing the following

Lemma 1. Let $F_k(G_k)$ denote the set of [illegible] of the game G_k. Let M denote the [illegible] of $F_k(G_k)$. Then

$$d = 1 - \max \{ \ldots \} \qquad (4)$$

where [illegible]

Proof: Let $P_A(G_A)$ denote the set of SGPE payoffs to player A in the game G_A and let M denote the supremum of $P_A(G_A)$.

In Figure 1, consider the subgame beginning from point (7), at which player A can choose either to search (i.e., move to point (8)) or not to search and wait to make a counteroffer (i.e., move to point (9)). Player A does not have an outside option, and thus the supremum of the payoff to A in any perfect equilibrium of the subgame beginning at point (7) is $\delta^{\Delta}M$.

Now consider the subgame beginning at point (10), where B makes an offer to A. Any offer by B which gives A more than $\delta^{\Delta}M$ will be accepted by A; and so there is no perfect equilibrium in which B offers more than $\delta^{\Delta}M$. It follows that B will get at least $1 - \delta^{\Delta}M$: in fact, this is the infimum of the payoff received by B in the subgame beginning from point (10).

By repeating the above argument, the infimum of the payoff received by B in the subgame beginning from point (4) and the subgame beginning from point (6) is $1 - \delta^{\Delta}M$.

Now consider the subgame beginning from point (3), at which B finds the outside option x_i ($i = 1, \ldots, N$) with probability p_i and with probability $\left[1 - \sum_{i=1}^{N} p_i\right]$ B does not find an outside option. Suppose B finds the outside option x_i ($i = 1, \ldots, N$). If he takes it up, then B receives at least x_i and if he does not take it up, then B moves to point (4) and receives at least $1 - \delta^{\Delta}M$. Thus, the infimum of the payoff to B in any perfect equilibrium of the subgame beginning from point (3) if he finds x_i is $\max(x_i, 1 - \delta^{\Delta}M)$. Thus the infimum of the payoff to B in any perfect equilibrium of the subgame beginning from

Recall that $V_1(G_1)$ exceeds the sum of random entries in [illegible] of the game G_1 must be at least the expectation of $V_1(G_1)$.

In Figure [illegible] beginning from point (1), at which player [illegible] [illegible] [illegible] [illegible] [illegible] of the subgame (1).

Now [illegible] beginning at point (1), player 2 [illegible] which gives a zero-sum game and [illegible] which a [illegible] that 2 will get at least 1 - [illegible] in fact, [illegible] of the payoff received by 2 in the subgame beginning from point (2).

By repeating the above argument, the variance of the amount received by 2 in the subgame beginning from point (2) into the subgame beginning from point (3) is $1 - 2^{-N}$.

Now consider the subgame beginning from point (3), at which 2 finds the outcome option a_i ($i = 1, \ldots, N$) with probability p_i and with probability $\left(1 - \sum_{i=1}^{N} p_i\right)$ 2 does not find an outcome option. Whenever 2 finds the outcome option a_i ($i = 1, \ldots, N$), if he takes it, then 2 receives at least a_i and if he does not take it, then 2 moves to point (4) and receives at least $1 - 2^{-N}$. [illegible] of the payoff to 2 in any perfect equilibrium of the subgame beginning from point (3) if he finds a_i is merely $1 - 2^{-N}$. Hence the variance of the payoff to 2 in any perfect equilibrium of the subgame beginning from point (3) is

$$\left[1 - \sum_{i=1}^{N} p_i\right](1 - \delta^{\Delta}M) + \sum_{i=1}^{N} p_i \max (x_i, 1 - \delta^{\Delta}M) .$$

In Figure 1, consider the subgame beginning from point (11), at which B has to choose either to search or not to search. The infimum of the payoff to B in any perfect equilibrium of the subgame beginning from point (11) is H, where

$$H = \max \left\{\delta^{\Delta}(1 - \delta^{\Delta}M), F\right\} , \qquad (2)$$

where $F = \delta^{T}\beta\left[\left[1 - \sum_{i=1}^{N} p_i\right](1 - \delta^{\Delta}M) + \sum_{i=1}^{N} p_i \max(x_i, 1 - \delta^{\Delta}M)\right]$

Now consider the game beginning from point (1), at which A makes an offer to B. Any offer by A which gives B less than H will be rejected by B; and so there is no perfect equilibrium in which A offers less than H. It follows that A will get at most 1 - H; this is the supremum of the payoff to A in the game beginning from point (1). Hence, M = 1 - H, where H is defined by equation (2) above. Hence the equation shown in Lemma 2 (i.e., equation (1)) defines the supremum of $P_A(G_A)$.

We defined M as the supremum of $P_A(G_A)$. The above argument may be repeated exactly, but with M defined instead as the infimum of $P_A(G_A)$; and with the words more/less, most/least, supremum/infimum and accept/reject interchanged throughout. Hence equation (1) also defines the infimum of $P_A(G_A)$.

Q.E.D.

Before we characterise the solution of the game G_A we shall make the following assumption: $\tau \geqslant \Delta$ (i.e., the time of one period of search is greater than or equal to the time of one period of bargaining).

Proposition 1: The game G_A has a unique SGPE partition, in which agreement is reached at $t = 0$, and player A receives share M, given by:

$M = 1/(1 + \delta^\Delta)$ if either (i) $x_N \leqslant 1/(1 + \delta^\Delta)$ or (ii) there exists an $m \in \{2, 3, \ldots, N\}$ such that $x_{m-1} \leqslant 1/(1 + \delta^\Delta) < x_m$ and $1/(1 + \delta^\Delta) \geqslant$

$$\left[\delta^\tau \beta \sum_{i=m}^{N} p_i x_i\right] \Big/ \left[\delta^\Delta - (1 - \sum_{i=m}^{N} p_i)\delta^\tau \beta\right]$$

or (iii) $x_1 \geqslant 1/(1 + \delta^\Delta)$ and $1/(1 + \delta^\Delta) \geqslant$

$$\left[\delta^\tau \beta \sum_{i=1}^{N} p_i x_i\right] \Big/ \left[\delta^\Delta - (1 - \sum_{i=1}^{N} p_i)\delta^\tau \beta\right]$$

$$= \left[1 - \delta^\tau \beta \left[\sum_{i=m}^{N} p_i x_i + \left[1 - \sum_{i=m}^{N} p_i\right]\right]\right] \Big/ \left[1 - \left[1 - \sum_{i=m}^{N} p_i\right]\delta^{\tau+\Delta}\beta\right]$$

if there exists an $m \in \{1, 2, 3, \ldots, N\}$ such that

$$x_m \geqslant \left[(1 - \delta^\Delta) + \delta^{\tau+\Delta}\beta \sum_{i=m}^{N} p_i x_i\right] \Big/ \left[1 - \left[1 - \sum_{i=m}^{N} p_i\right]\delta^{\tau+\Delta}\beta\right]$$

$\geqslant x_{m-1}$ and $1/(1 + \delta^\Delta) <$

Before we characterize the solution of the game G_1 we point out the following reasonableness [illegible] the sum of each product [illegible] is greater than [illegible] any period of bargaining.

Proposition. [illegible] in which agreement is [illegible]

[illegible]

[illegible] where

[illegible]

[illegible]

[illegible]

[illegible] if there exists [illegible] such that

[illegible]

[illegible] and [illegible]

$$\left[\delta^{\tau}\beta \sum_{i=m}^{N} p_i x_i\right] \Big/ \left[\delta^{\Delta} - \left[1 - \sum_{i=m}^{N} p_i\right]\delta^{\tau}\beta\right],$$

(where we define $x_0 = 0$),

and player B receives $1 - M$.

Proof: We begin by showing that the equation stated in Lemma 1 (i.e., equation (1)) has a unique solution. Let us firstly solve this equation for M. The equation is solved by considering three mutually exclusive cases, (i) $x_N \leqslant 1 - \delta^{\Delta}M$, (ii) $1 - \delta^{\Delta}M \leqslant x_1$, and (iii) there exists an $m \in \{2, 3, \ldots, N\}$ such that $x_{m-1} \leqslant 1 - \delta^{\Delta}M \leqslant x_m$.

(i) $\underline{x_N \leqslant 1 - \delta^{\Delta}M}$. Thus, for $i = 1, 2, \ldots, N$, $\max(x_i, 1 - \delta^{\Delta}M) = 1 - \delta^{\Delta}M$. Therefore, equation (1) becomes, $M = 1 - \max\{\delta^{\Delta}(1 - \delta^{\Delta}M), \delta^{\tau}\beta(1 - \delta^{\Delta}M)\}$. We have assumed that $\tau \geqslant \Delta$. Thus, $\delta^{\Delta} \geqslant \delta^{\tau}$. Hence, $M = 1 - \delta^{\Delta}(1 - \delta^{\Delta}M)$; and therefore,

$$M = 1/(1 + \delta^{\Delta}) \quad \underline{\text{if}} \quad x_N \leqslant 1/(1 + \delta^{\Delta}) . \tag{3}$$

(ii) $\underline{x_1 \geqslant 1 - \delta^{\Delta}M}$. Thus, for $i = 1, 2, \ldots, N$, $\max(x_i, 1 - \delta^{\Delta}M) = x_i$. Therefore equation (1) becomes, $M = 1 - \max\left[\delta^{\Delta}(1 - \delta^{\Delta}M),\right.$

$$\left.\delta^{\tau}\beta\left[\left[1 - \sum_{i=1}^{N} p_i\right](1 - \delta^{\Delta}M) + \sum_{i=1}^{N} p_i x_i\right]\right].$$

Thus,

$$M = 1/(1 + \delta^{\Delta}) \ \underline{\text{if}} \ x_1 \geqslant 1/(1 + \delta^{\Delta}) \ \underline{\text{and}} \ 1/(1 + \delta^{\Delta}) \geqslant$$

$$\left[\delta^{\tau}\beta \sum_{i=1}^{N} p_i x_i\right] \Big/ \left[\delta^{\Delta} - \left[1 - \sum_{i=1}^{N} p_i\right]\delta^{\tau}\beta\right]$$

$$= \left[1 - \delta^{T}\beta\left[\left[1 - \sum_{i=1}^{N} p_i\right] + \sum_{i=1}^{N} p_i x_i\right]\right] \Big/ \left[1 - \left[1 - \sum_{i=1}^{N} p_i\right]\delta^{T+\Delta}\beta\right]$$

$$\underline{\text{if}} \quad x_1 \geqslant \left[(1 - \delta^{\Delta}) + \delta^{T+\Delta}\beta \sum_{i=1}^{N} p_i x_i\right] \Big/ \left[1 - \left[1 - \sum_{i=1}^{N} p_i\right]\delta^{T+\Delta}\beta\right]$$

$$\underline{\text{and}} \qquad 1/(1 + \delta^{\Delta}) < \left[\delta^{T}\beta \sum_{i=1}^{N} p_i x_i\right] \Big/ \left[\delta^{\Delta} - \left[1 - \sum_{i=1}^{N} p_i\right]\delta^{T}\beta\right] \qquad (4)$$

(iii) there exists an $m \in \{2, 3, \ldots, N\}$ such that $\underline{x_{m-1} \leqslant 1 - \delta^{\Delta}M \leqslant x_m}$. Equation (1) becomes, $M = 1 - \max\Big[\delta^{\Delta}(1 - \delta^{\Delta}M),$

$$\delta^{T}\beta\left[\left[1 - \sum_{i=m}^{N} p_i\right](1 - \delta^{\Delta}M) + \sum_{i=m}^{N} p_i x_i\right]\Big].$$

Thus,

$$M = 1/(1 + \delta^{\Delta}) \ \underline{\text{if}} \text{ there exists an } m \in \{2, 3, \ldots, N\} \text{ such that}$$

$$x_{m-1} \leqslant 1/(1 + \delta^{\Delta}) \leqslant x_m \quad \underline{\text{and}}$$

$$1/(1 + \delta^{\Delta}) \geqslant \left[\delta^{T}\beta \sum_{i=m}^{N} p_i x_i\right] \Big/ \left[\delta^{\Delta} - \left[1 - \sum_{i=m}^{N} p_i\right]\delta^{T}\beta\right]$$

$$= \left[1 - \delta^{T}\beta\left[\sum_{i=m}^{N} p_i x_i + \left[1 - \sum_{i=m}^{N} p_i\right]\right]\right] \Big/ \left[1 - \left[1 - \sum_{i=m}^{N} p_i\right]\delta^{T+\Delta}\beta\right]$$

$\underline{\text{if}}$ there exists an $m \in \{2, 3, \ldots, N\}$ such that

$$x_m \geqslant \left[(1 - \delta^{\Delta}) + \delta^{T+\Delta}\beta \sum_{i=m}^{N} p_i x_i\right] \Big/ \left[1 - \left[1 - \sum_{i=m}^{N} p_i\right]\delta^{T+\Delta}\beta\right] \geqslant x_{m-1}$$

$$\underline{\text{and}} \quad 1/(1 + \delta^{\Delta}) < \left[\delta^{T}\beta \sum_{i=m}^{N} p_i x_i\right] \Big/ \left[\delta^{\Delta} - \left[1 - \sum_{i=m}^{N} p_i\right]\delta^{T}\beta\right] \qquad (5)$$

Combining equations (3), (4) and (5) gives us the solution of equation (1) for M, which is as stated in the Proposition.

Thus, the equation stated in Lemma 1 does indeed have a unique solution - whence it follows that the supremum and infimum of the set $P_A(G_A)$ coincide.

It is straightforward to show that this solution is in fact supported by a pair of strategies which involve immediate agreement at time t = 0. This follows from the fact that $M \geqslant R_A = 0$ and $1 - M \geqslant R_B$. Player A receives M as defined in the Proposition and player B receives 1 - M.

Hence, the game G_A has a unique SGPE partition.

Q.E.D.

Let us now examine the limiting case, as $\Delta \to 0$. We discussed the reasons for doing so under Case I. As $\Delta \to 0$, we obtain, from Proposition 1:

$M = 1/2$ <u>if</u> either (i) $x_N \leqslant 1/2$ or (ii) there exists an

$m \in \{2, 3, \ldots, N\}$ such that $x_{m-1} \leqslant 1/2 < x_m$ <u>and</u>

$1/2 \geqslant E_m$

or (iii) $x_1 \geqslant 1/2$ <u>and</u> $1/2 \geqslant E_1$ (6)

$= 1 - E_m$ <u>if</u> there exists an $m \in \{1, 2, 3, \ldots, N\}$ such that

$x_m \geqslant E_m \geqslant x_{m-1}$ <u>and</u> $1/2 < E_m$,

(where we define $x_0 = 0$) , (7)

Combining equations (4), (5) and (6) gives us the solution of equation (1) for N, which is as stated in the proposition.

Then, the equation [illegible] and indeed have a unique solution x whence [illegible] uniqueness and independence of the set $P_k(Q_k)$ conditions.

It is [illegible] equation for N. They supported by [illegible] naturally appealing if then $1 \geq 0$ [illegible] that $N = p_1 + N$ and $1 - N \geq p_2$. Player [illegible] the intersection and player 2 receives [illegible]

(Hence, [illegible] law holds.

Q.E.D.

Let us now consider the [illegible] cases with $n = 3$. The [illegible] the scheme for [illegible] more than 3. As $n = 3$, we obtain from Proposition 1:

$N = 1/2$ if either (i) $x_1 < 1/2$ or (ii) there exists an

$n \in \{2, 3, \ldots, N\}$ such that $x_{n-1} < 1/2 < x_n$ and

$1/2 \geq \xi_n$

or (iii) $x_1 \geq 1/2$ and $1/2 \geq \xi_1$ (6)

$N = 1 - \xi_n$ if there exists an $n \in \{1, 2, 3, \ldots, N\}$ such that

$x_n \geq \xi_n > x_{n-1}$ and $1/2 < \xi_n$

(where we define $x_0 = 0$). (7)

where $$E_m = \left[\delta^{\tau}\beta \sum_{i=m}^{N} p_i x_i\right] \Big/ \left[1 - \left[1 - \sum_{i=m}^{N} p_i\right]\delta^{\tau}\beta\right] .$$

Before we can interpret the above limiting case we need to go through a few points.

Firstly, note that if player B refused to play the game with player A, then he would achieve his expected reservation value, denoted by R_B, which is derived from following a sequentially optimal search rule over outside options. Thus, there would exist r such that

$$x_r \geqslant \delta^{\tau}\beta \sum_{i=1}^{N} p_i x_i \geqslant x_{r-1} \tag{8}$$

i.e., player B would accept outside options $x_i \geqslant x_r$ and reject outside options $x_i \leqslant x_{r-1}$. r would depend on the values of the parameters p_i, x_i for i = 1, ..., N, δ, τ and β. R_B is defined as follows:

$$R_B = \left[\delta^{\tau}\beta \sum_{i=r}^{N} p_i x_i\right] \Big/ \left[1 - \left[1 - \sum_{i=r}^{N} p_i\right]\delta^{\tau}\beta\right] .$$

Secondly, note that E_m, which has emerged in the limiting case, is the expected payoff to player B <u>if</u> B followed a search rule such that he accepted outside options $x_i \geqslant x_m$ and rejected outside options $x_i \leqslant x_{m-1}$. Thus $E_r = R_B$.

Thirdly, we have that

$$x_m \geqslant E_m \Rightarrow x_m \geqslant \delta^{\tau}\beta \sum_{i=1}^{N} p_i x_i \ , \quad \forall m \tag{9}$$

where $\quad \bar{s}_m = \left[c'x \sum_{i=m}^{N} v_i \right] \Big/ \left[1 - \sum_{i=m}^{N} u_i c'x \right]$

Before we can [illegible] we need to go through a few steps.

Firstly, [illegible] to play the game with payoffs [illegible] expected [illegible] remaining makes denoted by [illegible] immediately against search [illegible] would cause a game has

$$\bar{s}_j \geq 0 \qquad [illegible] \qquad (8)$$

the payer [illegible] accept outside options $x_j \geq s_j$ and reject outside options $x_j < s_{j-1}$ would depend on the values of the parameters q_i, c_i for $i = 1, \ldots, N$, N, r and M. R_N is defined as follows.

$$R_N = \left[c'x \sum_{i=1}^{N} q_i c_i \right] \Big/ \left[1 - \sum_{i=1}^{N} q_i c'x \right]$$

Secondly, note that $\bar{s}_N$, which has appeared in the desired case, is the expected payoff to player 1 if X followed a search rule which that he accepted outside options $x_j \geq s_j$ and rejected outside options $x_j < s_{j-1}$. Thus $R_N \leq \bar{s}_N$.

Thirdly, we have that

$$[illegible] \qquad (9)$$

and $$x_{m-1} \leq E_m \Leftarrow x_{m-1} \leq \delta^T \beta \sum_{i=1}^{N} p_i x_i \ , \quad \forall m \qquad (10)$$

Thus, from equations (8) and (9), we have,

$$\forall m \leq r - 1 \qquad x_m \not\geq E_m \ ,$$

i.e., $$\forall m \leq r - 1 \qquad x_m \leq E_m \qquad (11)$$

And from equations (8) and (10), we have,

$$\forall m \geq r + 1 \qquad x_{m-1} \not\leq E_m ,$$

i.e., $$\forall m \geq r + 1 \qquad x_{m-1} \geq E_m \qquad (12)$$

and $$\forall m \leq r \qquad x_{m-1} \leq E_m \qquad (13)$$

Fourthly, we have that

$$x_r \geq E_r = R_B \qquad (14)$$

Now we are well equipped to interpret the limiting case. The second equation of the limiting case, i.e., equation (7), is:

$M = 1 - E_m$ <u>if</u> there exists an $m \in \{1, 2, 3, \ldots, N\}$ such that

$$x_m \geq E_m \geq x_{m-1} \ \underline{\text{and}} \ 1/2 > E_m \ \text{(where we define } x_0 = 0\text{)}$$

Using equation (11) we rule out $m \leq r - 1$. Using equation (12) we rule out $m \geq r + 1$. And, using equations (13) and (14), we have $x_r \geq E_r \geq x_{r-1}$. Note that $E_r = R_B$. Thus, equation (7), the second

[illegible]

and $\quad$ [illegible] $\quad$ (7.[illegible])

Thus these equations [illegible]

[illegible] $\quad$ ([illegible])

[illegible]

[illegible] $\quad$ ([illegible])

[illegible] $\quad$ ([illegible])

[illegible]

[illegible] $\quad$ ([illegible])

Now we are now equipped to interpret the [illegible] cases. The general solution of the [illegible] cases, [illegible] equation [illegible] is

[illegible]

[illegible]

[illegible]

equation of the limiting case boils down to:

$$M = 1 - R_B \quad \underline{\text{if}} \quad 1/2 < R_B .$$

(Note that we have not included $x_r \geqslant R_B \geqslant x_{r-1}$ as a condition for $M = 1 - R_B$, since it is always true given the properties of r, cf. equations (13) and (14) above.)

The first equation of the limiting case, i.e., equation (6), is:

$$M = 1/2 \quad \underline{\text{if}} \quad \text{either (i) } x_N \leqslant 1/2 \text{ or (ii) there exists an}$$

$$m \in \{1, 2, 3, \ldots, N\} \text{ such that } x_{m-1} \leqslant 1/2 < x_m \ \underline{\text{and}}$$

$$1/2 \geqslant E_m ,$$

(where we define $x_0 = 0$).

Using equation (11) we rule out $m \leqslant r-1$. We now demonstrate that

$$\forall m \geqslant r + 1, \quad x_{m-1} \leqslant 1/2 < x_m \ \underline{\text{and}} \ 1/2 \geqslant E_m \iff R_B \leqslant 1/2$$

($\Rightarrow$). $\quad E_m = A'/B'$, where $A' = \delta^T \beta \sum_{i=m}^{N} p_i x_i$ and

$$B' = 1 - \left[1 - \sum_{i=m}^{N} p_i\right] \delta^T \beta . \quad R_B = [A' + C]/[B' + D],$$

$$\text{where } C = \delta^T \beta \sum_{i=r}^{m-1} p_i x_i \quad \text{and} \quad D = \delta^T \beta \sum_{i=r}^{m-1} p_i .$$

$E_m \leqslant 1/2 \Rightarrow A' \leqslant [1/2]B'$. Since for $i = r, \ldots, m-1$,

$x_i \leqslant 1/2$, we have, $C \leqslant [1/2]D$. Thus, $A' + C$

solution of the limiting cases with, for

[illegible]

Note that we have [illegible] a constant [illegible]

[illegible] the [illegible]

[illegible] (1) [illegible]

The first [illegible]

[illegible]

[illegible]

[illegible]

[illegible]

$\leqslant [1/2](B' + D) \Rightarrow R_B \leqslant 1/2$.

($\Leftarrow$). We have that, $E_{m-1} \geqslant E_m \Leftrightarrow x_{m-1} \geqslant E_m$, $\forall m \geqslant r + 1$. From equation (12) we are given, $\forall m \geqslant r+1$ $x_{m-1} \geqslant E_m$. Thus, $\forall m \geqslant r + 1$ $E_{m-1} \geqslant E_m$, i.e., $E_r = R_B = \max_{m \geqslant r} \{E_m\}$. Thus, $R_B \leqslant 1/2 \Rightarrow \forall m \geqslant r + 1$ $E_m \leqslant 1/2$, $\Rightarrow \forall m \geqslant r + 1$, $E_m \leqslant 1/2$ <u>and</u> $x_{m-1} \leqslant 1/2 < x_m$.

We now show that $x_N \leqslant 1/2 \Rightarrow R_B \leqslant 1/2$. $x_N \leqslant 1/2 \Rightarrow \forall m \geqslant r$ $x_m \leqslant 1/2$, $\Rightarrow \delta^T\beta \sum_{i=r}^{N} p_i x_i \leqslant \delta^T\beta \left[\sum_{i=r}^{N} p_i\right][1/2]$. Now, $[1/2]\left[\delta^T\beta \sum_{i=r}^{N} p_i\right] \leqslant [1/2]\left[1 - \delta^T\beta + \delta^T\beta \sum_{i=r}^{N} p_i\right]$. Thus, $R_B \leqslant 1/2$.

Finally, note that, $x_{r-1} \leqslant 1/2 < x_r$ <u>and</u> $1/2 \geqslant E_r = R_B \Leftrightarrow R_B \leqslant 1/2$.

Hence the second equation of the limiting case, equation (6), boils down to:

$M = 1/2$ <u>if</u> $R_B \leqslant 1/2$.

IN SUMMARY: as $\Delta \to 0$, we obtain,

$M = 1/2$ <u>if</u> $R_B \leqslant 1/2$

$\quad = 1 - R_B$ <u>if</u> $R_B > 1/2$.

<u>Case III</u>: Player A has one outside option and player B has one outside option.

Player k (k = A, B) has a ERV R_k defined as follows:

$$R_k = [p\, x_k\, \delta^T\beta]/[1 - (1 - p)\delta^T\beta] ,$$

[illegible]

[illegible]

[illegible]

[illegible] Player A has [illegible] strategies [illegible] Player B has [illegible]

[illegible] as follows:

[illegible]

(cf. section 2). Note that we have dropped some subscripts and superscripts; this reduces the notation. p denotes the probability that player k will find his outside option x_k in one period of search; we have assumed that $p_A = p_B = p$. This simplifies the analysis. We shall assume that the parameters p, x_A, x_B, δ, τ and β take values such that $R_A + R_B < 1$, so that mutually beneficial trade exists.

The SGPE of the game G_A is again analysed using the method proposed by Shaked and Sutton (1984a). We have defined the game G_B as the subgame of G_A that begins with B making an offer to A (cf. section 2 and Figure 1). Let $P_k(G_j)$ denote the set of SGPE payoffs to player k (k = A, B) in the game G_j (j = A, B).

We begin by establishing the following:

Lemma 2: Let $M_j^k(m_j^k)$ denote the supremum (infimum) of the set $P_k(G_j)$, then equations (15)-(18) below are satisfied by, (a) $x = M_A^A$, $y = m_B^B$, $z_1 = m_A^B$, $z_2 = M_B^A$, and (b) $x = m_A^A$, $y = M_B^B$, $z_1 = M_A^B$, $z_2 = m_B^A$.

Equation (15):

$$y = 1 - \max\left\{\delta^{\Delta}x,\ \delta^{\tau}\beta[(1-p)x + p\max(x_A, x)]\right\}$$

if either (i) $x_A \leqslant x$ and $x_B \leqslant z_1$ or (ii) $x_A \leqslant x$, $x_B \geqslant z_1$ and $z_1 \geqslant \beta[px_B + (1-p)z_1]$ or (iii) $x_A \geqslant x$, $x_B \geqslant z_1$ and $(1-p)z_1 + pR_B \geqslant \beta[px_B + p(1-p)R_B + (1-p)^2 z_1]$ or (iv) $x_A \geqslant x$ and $x_B \leqslant z_1$.

$$= 1 - \max\left\{\delta^{\Delta}x,\ \delta^{\tau}\beta[p^2 x_A + p(1-p)\max(x_A, x) + p(1-p)R_A +\right.$$

$(1 - p)^2 x]\}$

<u>if</u> either (i) $x_B \geqslant z_1$, $x_A \leqslant x$ <u>and</u> $z_1 \leqslant \beta[px_B + (1 - p)z_1]$ or (ii)

$x_B \geqslant z_1$, $x_A \geqslant x$ <u>and</u> $(1 - p)z_1 + pR_B \leqslant \beta[px_B + p(1 - p)R_B + (1 - p)^2 z_1]$.

<u>Equation (16)</u>: $z_2 = 1 - y$.

<u>Equation (17)</u>:

$$x = 1 - \max\left\{\delta^{\Delta} y,\ \delta^{T}\beta[(1 - p)y + p\max(x_B, y)]\right\}$$

<u>if</u> either (i) $x_B \leqslant y$ <u>and</u> $x_A \leqslant z_2$ or (ii) $x_B \leqslant y$, $x_A \geqslant z_2$ <u>and</u>

$z_2 \geqslant \beta[px_A + (1 - p)z_2]$ or (iii) $x_B \geqslant y$, $x_A \geqslant z_2$ <u>and</u>

$(1 - p)z_2 + pR_A \geqslant \beta[px_A + p(1 - p)R_A + (1 - p)^2 z_2]$

or (iv) $x_B \geqslant y$ <u>and</u> $x_A \leqslant z_2$,

$$= 1 - \max\left\{\delta^{\Delta} y,\ \delta^{T}\beta[p^2 x_B + p(1 - p)\max(x_B, y) + p(1 - p)R_B + (1 - p)^2 y]\right\}$$

<u>if</u> either (i) $x_A \geqslant z_2$, $x_B \leqslant y$ <u>and</u> $z_2 \leqslant \beta[px_A + (1 - p)z_2]$

or (ii) $x_A \geqslant z_2$, $x_B \geqslant y$ <u>and</u> $(1 - p)z_2 + pR_A \leqslant \beta[px_A + p(1 - p)R_A + (1 - p)^2 z_2]$.

Equation (18): $z_1 = 1 - x$.

For purposes of convenience we will use the following notation in the proof below: let S_x^k and I_x^k denote the supremum and infimum, respectively, of the payoff to player k in any SGPE of the subgame beginning from point (x), where point (x) denotes the points labelled in Figure 1.

Proof: Let $M_j^k(m_j^k)$ denote the supremum (infimum) of the set $P_k(G_j)$.

In Figure 1, consider the subgame beginning from point (12), at which (i) with probability p player A finds his outside option x_A and (ii) with probability (1 - p) player A does not find x_A, in which case A makes an offer to B at point (13). Thus, with probability (1 - p), $S_{12}^A = M_A^A$ and $I_{12}^B = m_A^B$. Now suppose player A finds x_A. Then, the most A gets by taking his outside option is, of course, x_A, and the most A gets by not taking his outside option is M_A^A. Thus, with probability p, $S_{12}^A = \max(x_A, M_A^A)$. Now if $\max(x_A, M_A^A) = x_A$, then with probability p, $S_{12}^A = x_A$, and with probability p, $I_{12}^B = R_B$. On the other hand, if $\max(x_A, M_A^A) = M_A^A$, then with probability p, $S_{12}^A = M_A^A$, and thus with probability p, $I_{12}^B = m_A^B$. IN SUMMARY,

$$S_{12}^A = (1 - p)M_A^A + p \max(x_A, M_A^A) , \tag{19}$$

and

$$I_{12}^B = \begin{cases} (1 - p)m_A^B + pm_A^B & \underline{\text{if}} \quad x_A \leqslant M_A^A \\ \\ (1 - p)m_A^B + pR_B & \underline{\text{if}} \quad x_A > M_A^A . \end{cases} \tag{20}$$

Now consider, in Figure 1, the subgame beginning from point (14), at which, (i) with probability p^2 both players, A and B, find their respective outside option, x_A and x_B, (ii) with probability p(1 - p) player A finds x_A but player B does not find x_B, (iii) with probability (1 - p)p player B finds x_B but player A does not find x_A, and (iv) with probability $(1 - p)^2$ neither A nor B finds his outside option, in which case A makes an offer to B at point (15). Thus, with probability $(1 - p)^2$ $S^A_{14} = M^A_A$ and $I^B_{14} = m^B_A$. Now suppose B finds x_B and A does not find x_A. Then, with probability (1 - p)p, $I^B_{14} = \max(x_B, m^B_A)$,

$$\text{and} \quad S^A_{14} = \begin{cases} R_A & \underline{\text{if}} \quad x_B \geqslant m^B_A \\ \\ M^A_A & \underline{\text{if}} \quad x_B < m^B_A \end{cases} .$$

Similarly, if A finds x_A and B does not find x_B, then with probability p(1 - p), $S^A_{14} = \max(x_A, M^A_A)$, and

$$I^B_{14} = \begin{cases} R_B & \underline{\text{if}} \quad x_A \geqslant M^A_A \\ \\ m^B_A & \underline{\text{if}} \quad x_A < M^A_A \end{cases} .$$

Now suppose both players, A and B, find their respective outside option, x_A and x_B. Then with probability p^2,

$$S^A_{14} = \begin{cases} x_A & \underline{\text{if}} \quad x_B \geqslant m^B_A \\ \\ \max(x_A, M^A_A) & \underline{\text{if}} \quad x_B < m^B_A \end{cases}$$

[illegible]

[illegible]

[illegible]

[illegible]

$$\text{and } I^A_{14} = \begin{cases} x_B & \underline{\text{if}} \quad x_A \geqslant M^A_A \\ \max(x_B, m^B_A) & \underline{\text{if}} \quad x_A < M^A_A \end{cases}$$

IN SUMMARY:

$$S^A_{14} = \begin{cases} p^2 x_A + p(1-p)\max(x_A, M^A_A) + p(1-p)R_A + (1-p)^2 M^A_A \\ \qquad \underline{\text{if}} \quad x_B \geqslant m^B_A \\ p\max(x_A, M^A_A) + (1-p)M^A_A \qquad \underline{\text{if}} \quad x_B < m^B_A \end{cases} \tag{21}$$

$$I^B_{14} = \begin{cases} p^2 x_B + p(1-p)\max(x_B, m^B_A) + p(1-p)R_B + (1-p)^2 m^B_A \\ \qquad \underline{\text{if}} \quad x_A \geqslant M^A_A \\ p\max(x_B, m^B_A) + (1-p)m^B_A \qquad \underline{\text{if}} \quad x_A < M^A_A \end{cases} \tag{22}$$

In Figure 1, consider the subgame beginning from point (8), at which B has to choose either to search or not to search. If player B chooses not to search, then he gets at least $\delta^\tau I^B_{12}$, where I^B_{12} is defined in equation (20) above and δ^τ represents the cost of waiting (τ being the time per period of search). If player B chooses to search, then he gets at least $\delta^\tau \beta I^B_{14}$, where I^B_{14} is defined in equation (22) above and β represents the fixed cost per period of search. Thus, $I^B_8 = \max\{\delta^\tau I^B_{12}, \delta^\tau \beta I^B_{14}\}$. If $\delta^\tau I^B_{12} > \delta^\tau \beta I^B_{14}$, then $I^B_8 = \delta^\tau I^B_{12}$, and thus, $S^A_8 = \delta^\tau \beta S^A_{12}$, where S^A_{12} is defined by equation

(19). If, on the other hand, $\delta^{\tau} I^B_{12} \leqslant \delta^{\tau}\beta I^B_{14}$, then $I^B_8 = \delta^{\tau}\beta I^B_{14}$, and thus $S^A_8 = \delta^{\tau}\beta S^A_{14}$, where S^A_{14} is defined by equation (21).

TO SUMMARISE:

$$S^A_8 = \begin{cases} \delta^{\tau}\beta\, S^A_{12} & \underline{\text{if}} \quad \delta^{\tau} I^B_{12} \geqslant \delta^{\tau}\beta I^B_{14} , \\ \delta^{\tau}\beta\, S^A_{14} & \underline{\text{if}} \quad \delta^{\tau} I^B_{12} < \delta^{\tau}\beta I^B_{14} . \end{cases} \tag{23}$$

Let us now consider the subgame beginning from point (7), where player A has to choose either to search or not to search. Thus,

$$S^A_7 = \max(\delta^{\Delta} M^A_A , \; S^A_8) , \tag{24}$$

where S^A_8 is defined in equation (23).

At point (10), in Figure 1, player B makes an offer to player A. Any offer by player B which gives player A more than S^A_7 (as defined above in equation (24)) will be accepted by player A; and so there is no perfect equilibrium in which B offers more than S^A_7 It follows that B will get at least $1 - S^A_7$; and thus, $I^B_{10} = 1 - S^A_7$. Now since the subgame beginning from point (10) is the game G_B, we have $I^B_{10} = m^B_B$. Thus,

$$m^B_B = 1 - S^A_7 \tag{25}$$

And, of course, $M^A_B = S^A_7$,

$$\text{i.e.,} \qquad M^A_B = 1 - m^B_B \tag{26}$$

By substituting for S_7^A, using equation (24), and then substituting for S_8^A, using equation (23), and then substituting for S_{14}^A, I_{14}^B, S_{12}^A and I_{12}^B, using equations (21), (22), (19) and (20), and finally rearranging, we obtain that equation (25) is as equation (15) (which is stated in Lemma 2) with $x = M_A^A$, $y = m_B^B$ and $z_1 = m_A^B$. And note that equation (26) is equation (16) (which is stated in Lemma 2) with $z_2 = M_B^A$ and $y = m_B^B$.

Thus, we have shown that equations (15) and (16), which are stated in Lemma 2, are satisfied by $x = M_A^A$, $y = m_B^B$, $z_1 = m_A^B$ and $z_2 = M_B^A$.

By a symmetric argument one can show that equations (17) and (18) are satisfied by $x = M_A^A$, $y = m_B^B$, $z_1 = m_A^B$ and $z_2 = M_B^A$. In fact, the symmetry of the argument is revealed by the symmetry between equation (15) and equation (17), and between equation (16) and equation (18); take equation (15) and interchange y/x, x_A/x_B, z_1/z_2. R_B/R_A, and one obtains equation (17).

Thus equations (15)-(18) are indeed satisfied by, $x = M_A^A$, $y = m_B^B$, $z_1 = m_A^B$ and $z_2 = M_B^A$.

Now, the argument that led us to discover that equations (15) and (16) are satisfied by $x = M_A^A$, $y = m_B^B$, $z_1 = m_A^B$ and $z_2 = M_B^A$ may be repeated exactly, but with m_B^B replaced by M_B^B, M_A^A replaced by m_A^A, M_B^A replaced by m_B^A and m_A^B replaced by M_A^B and the words supremum/infimum accept/reject, most/least, more/less interchanged throughout. And one discovers that equations (15) and (16) are satisfied by $x = m_A^A$, $y = M_B^B$, $z_1 = M_A^B$ and $z_2 = m_B^A$.

And symmetrically, $x = m_A^A$, $y = M_B^B$, $z_1 = M_A^B$ and $z_2 = m_B^A$ satisfy equations (17) and (18).

TO SUMMARISE: We have shown that equations (15)-(18) are indeed satisfied by (a) $x = M_A^A$, $y = m_B^B$, $z_1 = m_A^B$, $z_2 = M_B^A$, and (b) $x = m_A^A$, $y = M_B^B$, $z_1 = M_A^B$, $z_2 = m_B^A$.

Q.E.D.

We now characterise the solution of the game G_A.

Proposition 2: The game G_A has a unique SGPE partition, in which agreement is reached at $t = 0$, and player A receives the share x, given by:

$x = 1/(1 + \delta^{\Delta})$ if either (i) $x_A \leqslant \alpha_1/(1 + \delta^{\Delta})$ and $x_B \leqslant \alpha_1/(1 + \delta^{\Delta})$ or (ii) $\alpha_1/(1 + \delta^{\Delta}) \leqslant x_A \leqslant \alpha_3/(1 + \delta^{\Delta})$ and $1/(1 + \delta^{\Delta}) \leqslant x_B \leqslant \alpha_1/(1 + \delta^{\Delta})$ or (iii) $1/(1 + \delta^{\Delta}) \leqslant x_A \leqslant \alpha_3/(1 + \delta^{\Delta})$ and $\alpha_1/(1 + \delta^{\Delta}) \leqslant x_B \leqslant \alpha_3/(1 + \delta^{\Delta})$,

$= [1 - \delta^{T}\beta[(1 - p) + px_B]]/[1 - \delta^{T+\Delta}\beta(1 - p)]$ if either (i) $x_B > \alpha_1/(1 + \delta^{\Delta})$ and $x_A \leqslant [\delta^{\Delta}[1 - \delta^{T}\beta[(1 - p) + px_B]]]/[1 - \delta^{T+\Delta}\beta(1 - p)]$ or (ii) $x_B > \alpha_3/[1 + \delta^{\Delta}]$ and $\delta^{\Delta}[1 - \delta^{T}\beta[(1 - p) + px_B]]/[1 - \delta^{T+\Delta}\beta(1 - p)] \leqslant x_A \leqslant \alpha_3\delta^{\Delta}[1 - \delta^{T}\beta[(1 - p) + px_B]]/[1 - \delta^{T+\Delta}\beta(1 - p)]$

$= [1 - \delta^{\Delta}[1 - px_A\delta^{T}\beta]]/[1 - (1 - p)\delta^{T+\Delta}\beta]$ if either (i) $x_A > \alpha_1/(1 + \delta^{\Delta})$ and $x_B \leqslant \delta^{\Delta}[1 - \delta^{\Delta}[1 - px_A\delta^{T}\beta]]/$

$[1 - (1 - p)\delta^{T+\Delta}\beta]$, or (ii) $x_A > \alpha_3/(1 + \delta^{\Delta})$ <u>and</u>

$$\delta^{\Delta}[1 - \delta^{\Delta}[1 - px_A\delta^{T}\beta]]/[1 - (1 - p)\delta^{T+\Delta}\beta] \leqslant x_B \leqslant$$

$$\alpha_3\delta^{\Delta}[1 - \delta^{\Delta}[1 - px_A\delta^{T}\beta]]/[1 - (1 - p)\delta^{T+\Delta}\beta]$$

where $\alpha_1 = [\delta^{\Delta} - (1 - p)\delta^{T}\beta]/p\delta^{T}\beta$ and

$$\alpha_3 = [[\delta^{\Delta} - (1 - p)^2\delta^{T}\beta][1 - (1 - p)\delta^{T}\beta]]/[[p\delta^{T}\beta][1 - (1 - p)^2\delta^{T}\beta]],$$

(and we note that $\alpha_3 \geqslant \alpha_1 > 1$), and player B receives $1 - x$.

<u>Proof</u>: We show in the appendix that the four equations (15)-(18), which are stated in Lemma 2, have a unique solution. Let (x, y, z_1, z_2) denote this unique solution. Thus, $x = M_A^A = m_A^A$, $y = M_B^B = m_B^B$, $1 - y = M_B^A = m_B^A$, $1 - x = M_A^B = m_A^B$.

It is straightforward to show that this solution is in fact supported by a pair of strategies which involve immediate agreement at $t = 0$. This follows from the fact that $x \geqslant R_A$ and $1 - x \geqslant R_B$, where x is as stated in Proposition 2. Player A receives x and player B receives $1 - x$.

Hence the game G_A has a unique SGPE partition.

Q.E.D.

We now turn to an interpretation of this result. Once again, we look at the limiting case, as $\Delta \to 0$. (See Case I where we discuss the interest and importance of this limiting case). As $\Delta \to 0$, we obtain:

$x = 1/2$ <u>if</u> $R_A \leqslant 1/2$ <u>and</u> $R_B \leqslant 1/2$

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$$= 1 - R_B \quad \underline{\text{if}} \quad R_B > 1/2 \quad \underline{\text{and}} \quad R_A \leqslant 1 - R_B$$

$$= R_A \quad \underline{\text{if}} \quad R_A > 1/2 \quad \underline{\text{and}} \quad R_B \leqslant 1 - R_A ,$$

where R_A and R_B denote the expected reservation values of player A and player B, respectively. (Note, $\lim_{\Delta \to 0} \alpha_3 = \lim_{\Delta \to 0} \alpha_1$, and $[x_A]/[\lim_{\Delta \to 0} \alpha_3] = R_A$, $[x_B]/[\lim_{\Delta \to 0} \alpha_3] = R_B$).

Thus, once again, in the limiting case, we obtain the 'Outside Option Principle', provided we reinterpret the notion of the outside option: i.e., we treat R_A and R_B as the outside options of player A and player B, respectively.

4. SUMMARY AND CONCLUDING REMARKS

In section 2 we presented a model of the following situation. Two players are bargaining on the partition of a pie of size one. The pie will be partitioned only after the players reach an agreement. Each of the two players is free to quit bargaining and instead engage in a process of random search in order to find one of his many outside options. The players can choose to resume bargaining, after having searched for some time without success.

An example of such a situation is when two insiders, a firm and a worker, are bargaining over the wage. And the worker is free to quit bargaining in order to search for alternative wage offers.

The central question of interest is, "how will the values of the outside options impinge on the bargaining outcome, given that search is costly?"

From the analysis of the model, conducted in section 3, we arrived at the following answer to the above question. In the limit, as the time between successive offers tends to zero, the bargaining outcome is characterised by the Binmore (1985) 'Outside Option Principle', with the players' <u>expected reservation values</u> treated as the outside options.

The game presented in this chapter, in which we derived the above result, explicitly takes into account the <u>search dimension</u> of the situation that we have modelled. This meant that we have had to formulate a "view" on how the search and bargaining processes ought to be interlaced. In order to model the bargaining situation (in which the players have many outside options, where each option is available with some probability) one has to have a model of how

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[illegible]

In section 3 we [illegible] interesting attributes [illegible]

[illegible]

The general [illegible] with the results of the [illegible]

[illegible]

From the analysis of the model developed in section 4, we [illegible]

[illegible]

The case presented in this chapter, in which we [illegible]

[illegible]

the search and bargaining processes are interlaced; in most such situations, in real life, bargainers have to engage in some sort of search in order to find an outside option.

Thus, any alternative game that purports to model a bilateral bargaining situation with outside options, where each option is available with some probability, must take account of the search dimension; which will then require the need to form a "view' on how the search and bargaining processes ought to be interlaced.

An interesting observation concerning the result (the limiting case) produced by our game is the following. The result can be obtained in a simpler game; in fact, in a game in which the search dimension of the situation is completely ignored. This game is the Binmore (1985) extension of the Rubinstein game (which produced the 'Outside Option Principle'). Simply define, *a priori*, the expected reservation values of the players to be their outside options available with certainty. Then apply the Binmore game and one obtains our result (limiting case).

We have made the above observation in order to point out the "robustness" of the Binmore (1985) 'Outside Option Principle', provided one redefines the notion of an outside option. We suggest that the concept of the outside option ought to be redefined as the expected reservation value (and an outside option in the "traditional" meaning of the term can be called something else, such as an outside offer). Therefore, a player's Outside Option is his expected payoff achieved from following a sequentially optimal search rule over his outside offers. Thus, in the case when a player has one outside offer available with certainty, the value of his Outside Option equals the value of his outside offer.

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[illegible]

We have seen that [illegible]

We must emphasize that the above observation does not imply that one should ignore the search dimension of the situation when attempting to model the bargaining situation. It happens to be a coincidence that our result (limiting case) can be brought out by the Binmore game. The two games are very different. The "right" way to model the bilateral bargaining situation with outside options, where each option is available with some probability, is to incorporate the search dimension. However, what we may say is, that for purposes of "applications" one could use the Binmore game, with the modified definition of an outside option (suggested above) rather than our "complex" game.

Finally, we shall now make some comments on an important issue. The issue is, what is the correct way, if there is a correct way, to model the interlacing of the bargaining and search processes.

One approach to this issue is that adopted in the current chapter. The basic idea being that a player cannot search for an outside option while bargaining; he has to withdraw from the bargaining process in order to engage in search (see the Introduction to chapter 5 for an argument that justifies this approach to modelling the interlacing of the bargaining and search processes).

The central idea of an alternative approach to modelling the interlacing of the bargaining and search processes is the following. The players can search for an outside option during the bargaining process, i.e., between two successive offers. This approach was first explored by Sutton (1986, pp. 713-714). However, he considers the case in which each bargainer has only one outside option, available with some probability. The result that he obtains is very different

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from the result of the current chapter (and thus his result does not support any modified form of the 'Outside Option Principle', as does the result of the current chapter). In chapter 4 we shall generalise Sutton's (1986) game; we shall examine the case in which the bargainers have <u>many</u> outside options. We shall do this in order to check the "robustness" of the general conclusions obtained by Sutton.

Wolinsky (1987) also presents a bilateral bargaining game (which he later embeds in a large market) in which the Sutton approach is adopted. Wolinsky, however, generalises the search aspect by letting the bargainers choose "how hard they wish to search".

Although in the Introduction to chapter 5 we present an argument to justify the approach adopted by us in the current chapter - the "approach" then being embedded in a large market context in chapter 5 - it may be the case that there may not be a conclusive argument to justify either of the two approaches. The two "approaches" may be viewed as representing two different institutional frameworks. Thus, at this stage of our knowledge, there is no correct way of modelling the interlacing of the search and bargaining processes.

Finally, we now mention a recent paper by Shaked (1987).[2] His paper subscribes to the approach taken in the current chapter.

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2. It may be noted that the first draft of the current chapter was submitted for the University of Cambridge Stevenson prize on 21 April 1987 (and in fact it was later awarded the Stevenson prize). Furthermore, the first draft was written before the papers by Shaked (1987) and Wolinsky (1987) appeared.

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He does not say this explicitly, though, because his concern is of a different nature. Shaked has raised a rather different issue, which however fits within the issues under discussion. The question he addresses is, "at which points during the bargaining process should a player be allowed to withdraw, in order to search for an outside option?"

In the current chapter, may we recall, it is the responder who decides whether to withdraw or to continue bargaining. Shaked argues that for "Hi-tech" markets this may not be a realistic model, although for Bazaars it may. He presents a model in which it is the proposer who decides whether to withdraw or to continue bargaining.

Last but not least, we note that this issue is important for the modelling of large markets with sequential bargaining, since at the heart of those models must lie a "view" on how to interlace the bargaining and search processes.

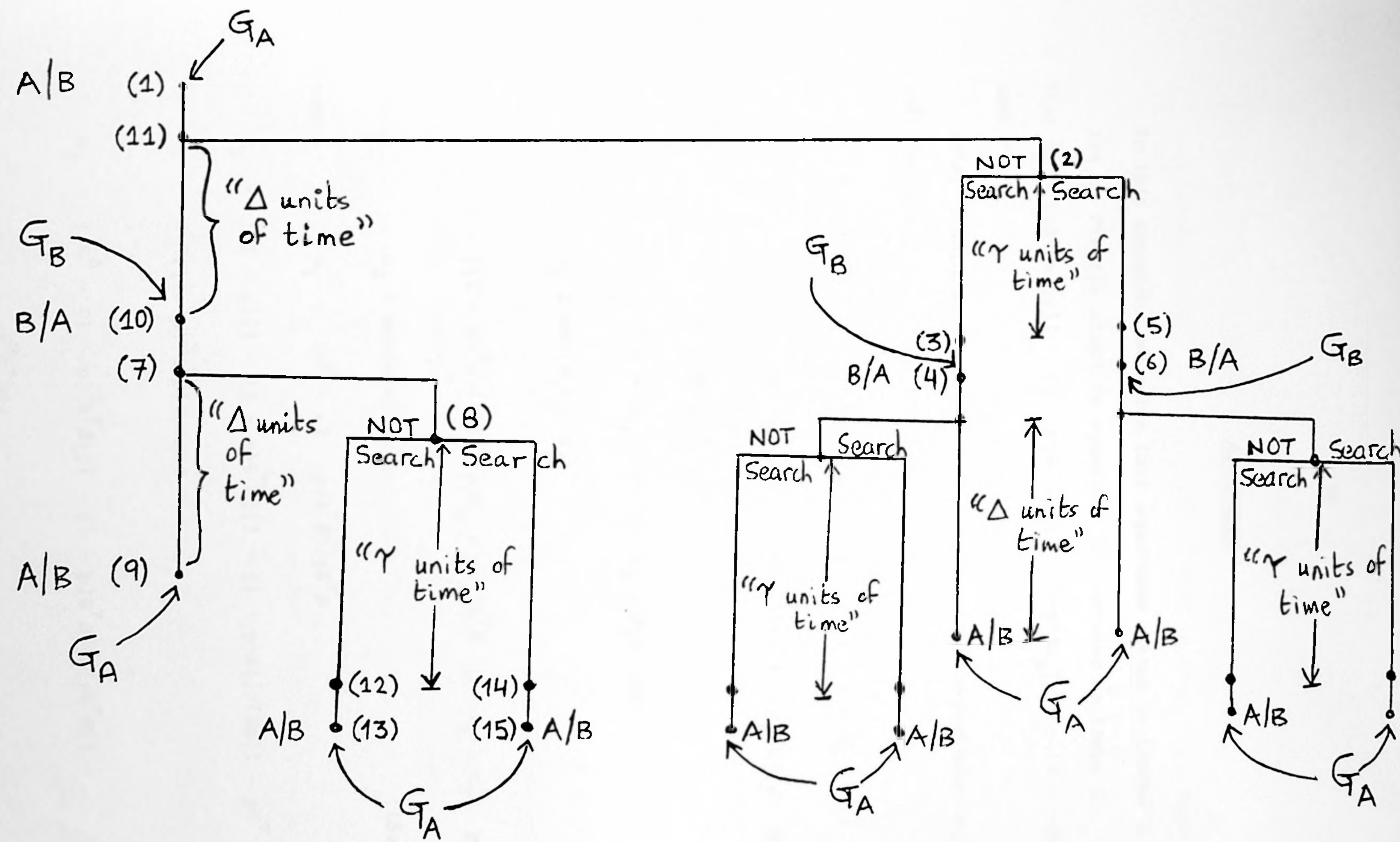

FIGURE 1

The notation a/b reads, a makes an offer to b.

APPENDIX

In this appendix we solve four equations stated in Lemma 2.

Let us firstly simplify equation (15) (stated in Lemma 2). Note that $R_B = [p\delta^T\beta x_B]/[1 - (1-p)\delta^T\beta]$, $R_A = [p\delta^T\beta x_A]/[1 - (1-p)\delta^T\beta]$ and $T \geqslant \Delta$.

After manipulating the max $(\cdot, \cdot)$ and the inequalities, equation (15) becomes:

$$
\begin{aligned}
y &= 1 - \delta^\Delta x \quad \underline{\text{if}} \quad \text{either (i) } x_A \leqslant x \quad \text{or (ii) } x \leqslant x_A \leqslant \alpha_1 x \\
&\qquad \underline{\text{and}}\ x_B \leqslant \max\{\alpha_2 z_1, z_1\} \quad \text{or (iii) } x \leqslant x_A \leqslant \alpha_3 x \quad \underline{\text{and}} \\
&\qquad x_B \geqslant \max\{\alpha_2 z_1, z_1\}. \\
&= 1 - [(1-p)x + px_A]\delta^T\beta \quad \underline{\text{if}} \quad x_A \geqslant \alpha_1 x \quad \underline{\text{and}} \\
&\qquad x_B \leqslant \max\{\alpha_2 z_1, z_1\} \\
&= 1 - [(1-p)^2 x + (1-p)R_A + px_A]\delta^T\beta \quad \underline{\text{if}} \quad x_A \geqslant \alpha_3 x \quad \underline{\text{and}} \\
&\qquad x_B > \max\{\alpha_2 z_1, z_1\} \qquad\qquad (15a)
\end{aligned}
$$

where $\alpha_1 = [\delta^\Delta - (1-p)\delta^T\beta]/p\delta^T\beta$,

$$\alpha_2 = [(1-p)[1 - (1-p)\delta^T\beta][1 - (1-p)\beta]]/[\beta p[1 - p\delta^T - (1-p)^2\delta^T\beta]], \text{ and}$$

$$\alpha_3 = [[\delta^\Delta - (1-p)^2\delta^T\beta][1 - (1-p)\delta^T\beta]]/[p\delta^T\beta[1 - (1-p)^2\delta^T\beta]] \, .$$

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[illegible]

[illegible]

[illegible]

[illegible]

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[Note that $\alpha_3 \geqslant \alpha_1 \geqslant 1$ and $\alpha_1 \geqslant \alpha_2$]

And symmetrically, equation (17) (stated in Lemma 2) becomes:

$$x = 1 - \delta^{\Delta}y \quad \underline{\text{if}} \quad \text{either (i) } x_B \leqslant y \text{ or (ii) } y \leqslant x_B \leqslant \alpha_1 y$$

$$\underline{\text{and}} \quad x_A \leqslant \max\{\alpha_2 z_2, z_2\} \text{ or (iii) } y \leqslant x_B \leqslant \alpha_3 y \quad \underline{\text{and}}$$

$$x_A \geqslant \max\{\alpha_2 z_2, z_2\}$$

$$= 1 - [(1 - p)y + px_B]\delta^{\tau}\beta \quad \underline{\text{if}} \quad x_B \geqslant \alpha_1 y \quad \underline{\text{and}}$$

$$x_A \leqslant \max\{\alpha_2 z_2, z_2\}$$

$$= 1 - [(1 - p)^2 y + (1 - p)pR_B + px_B]\delta^{\tau}\beta \quad \underline{\text{if}} \quad x_B \geqslant \alpha_3 y \quad \underline{\text{and}}$$

$$x_A > \max\{\alpha_2 z_2, z_2\}. \tag{17a}$$

Now secondly, we substitute for z_2 in equation (17a), above, using equation (16) (stated in Lemma 2) and for z_1 in equation (15a) above, using equation (18) (stated in Lemma 2).

And finally, one can now solve for x and y: one does this carefully, taking note of the parameter restrictions embodied in the "inequalities". z_1 and z_2 are obtained using equations (16) and (18). Hence, one can show that the four equations (15)-(18) have a <u>unique</u> solution.

Chapter 4

A note on bargaining with many outside options

1. INTRODUCTION

In chapter 3 we presented a non-cooperative sequential game that modelled bilateral bargaining situations in which the players have many outside options, where each outside option is available with some probability. We argued there that one needs to form a "view" on how the search and bargaining processes ought to be interlaced if one models such a bargaining situation. In fact, the game presented in chapter 3 represents one approach to this important issue. We mentioned there that Sutton (1986, pp. 713-714) has presented a game in which an alternative approach to this issue is taken. However, he considers the case in which each bargainer has only one outside option, available with some probability. In this chapter we will generalise Sutton's game; we shall allow each of the two players to have many outside options, where each outside option is available with some probability. We do this in order to check the "robustness" of the general conclusions obtained by Sutton.

Sutton obtains the following result (the limiting case - when the time between successive offers tends to zero). The outside option belonging to a player, which is available with some probability, does not influence the bargaining outcome if its value is less than the Rubinstein (1982) solution (i.e., 1/2); if its value is greater than the Rubinstein solution, then it influences the bargaining outcome in a particular way.

In this chapter we show that even if the value of an outside option (available with some probability) is greater than the Rubinstein solution it need not influence the bargaining outcome.

Take, for example, the case when a bargainer has two outside

options, where each outside option is available with some probability. Suppose the larger of the two outside options influences the bargaining outcome. Then, we shall show, that even if the smaller of the two outside options has a value greater than the Rubinstein solution it need not influence the bargaining outcome. For it to influence the bargaining outcome its value has to be much greater than the Rubinstein solution: the precise value required will depend on the value of the larger of the two outside options.

2. THE MODEL

Two players, A and B, are bargaining on the partition of a pie of size one. The pie will be partitioned only after the players reach an agreement. Player k (k = A, B) has N_k outside options; denote the outside options by x_i^k (i = 1, 2, ..., N_k) and assume that the options are ordered, i.e., for i = 1, ..., $N_k - 1$, $x_{i+1}^k > x_i^k$. The outside option x_i^k is available with probability p_i^k. The bargaining game is a Rubinstein-type game (Rubinstein (1982)); the players make offers alternately.

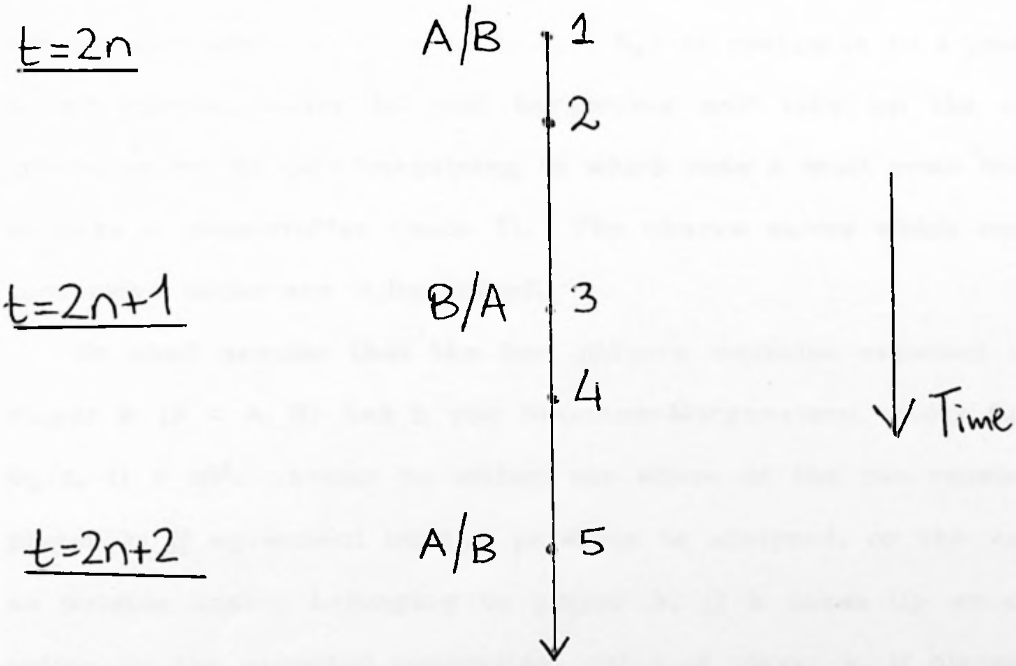

Figure 1

At each time t = 2n (n = 0, 1, 2, ...,) player A makes an offer to player B (node 1 in Figure 1). Player B either accepts or rejects. If he accepts, then the game ends. Otherwise, a chance move occurs, in which, with probability p_i^B (i = 1, 2, ..., N_B) the outside option

t=2n
t=2n+1
B/A
A/B
t=2n+2
Time

x_i^B $(i = 1, 2, \ldots, N_B)$ is available to player B (node 2). If an outside option is available to B, then he can choose, either to quit bargaining and take up the outside option, or not to quit bargaining in which case B must await his turn to make a counteroffer (node 3). Note that with probability $(1 - \sum_{i=1}^{N_B} p_i^B)$ no outside option is available to B (node 2), and thus, B must await his turn to make a counteroffer (node 3).

Following B's offer to A, at time $t = 2n + 1$ (node 3), A either accepts or rejects. If he accepts, then the game ends. Otherwise, a chance move occurs, in which, with probability p_i^A $(i = 1, 2, \ldots, N_A)$ the outside option x_i^A $(i = 1, 2, \ldots, N_A)$ is available to A (node 4); A can choose, either to quit bargaining and take up the outside option, or not to quit bargaining in which case A must await his turn to make a counteroffer (node 5). The chance moves which occur at successive nodes are independent.

We shall assume that the two players maximise expected utility. Player k $(k = A, B)$ has a von Neumann-Morgenstern utility function $U_k(z, t) = z\delta^t$. z can be either the share of the pie received by player k, _if_ agreement on the partition is achieved, or the value of an outside option belonging to player k, _if_ k takes up an outside option, or the expected reservation value of player k, _if_ player w $(w \neq k, w = A, B)$ takes up an outside option leaving player k with his expected reservation value. t is the time elapsed from time $t = 0$ before z is obtained. δ is the (common) discount factor, $0 < \delta < 1$.

Denote the game described above by G. G is a game of complete information (i.e., the game tree and the agents preferences are assumed to be common knowledge amongst the agents). Furthermore,

[illegible]

note that G is a game of perfect information.

Suppose player k (k = A, B) refused to play the game G with player w (w $\neq$ k, w = A, B); then player k would achieve his expected reservation value (ERV), R_k, derived from following a sequentially optimal search rule over outside options. Let us compute R_k. There will exist r_k such that it is sequentially optimal for player k to accept outside options x_i^k for i $\geq r_k$, and to reject outside options x_i^k for i $\leq r_k - 1$; i.e., there exists r_k such that

$$x_{r_k}^k \geq \delta \sum_{i=1}^{N_k} p_i^k x_i^k > x_{r_k-1}^k,$$

(see McCall (1965) for a discussion of optimal stopping rules). r_k will depend on the parameters p_i^k, x_i^k for i = 1, ..., N_k, and δ. We have that, given r_k, R_k is defined as follows:

$$R_k = \left[\sum_{i=r_k}^{N_k} p_i^k x_i^k\right] \Big/ \left[1 - (1 - \sum_{i=r_k}^{N_k} p_i^k)\delta\right].$$

We shall assume that $R_A + R_B < 1$, so that mutually beneficial trade is possible.

A strategy for each agent in G will tell the agent the choice to make at each and every decision node that he may be at. Each of the two players will have a set of strategies from which to choose a strategy. The solution concept we will use is the subgame perfect equilibrium (SGPE) (Selten (1965, 1975)). A strategy tuple is in SGPE if its restriction to any subgame is in Nash equilibrium.

3. PERFECT EQUILIBRIUM

The SGPE of the game G is found using the method proposed by Shaked and Sutton (1984a).

We begin by establishing the following:

Lemma: Let M be the supremum (infimum) of the payoff to A in any SGPE of G, then:

$$M = 1 - \sum_{i=s}^{N_B} p_i^B x_i^B - \delta\left[1 - \sum_{i=s}^{N_B} p_i^B\right]\left[1 - \sum_{i=r}^{N_A} p_i^A x_i^A - \left[1 - \sum_{i=r}^{N_A} p_i^A\right]\delta M\right],$$

if $x_r^A > \delta M \geqslant x_{r-1}^A$ and $x_s^B > \delta(1 - K) \geqslant x_{s-1}^B$, $(r = 1, \ldots, N_A + 1,$

$s = 1, \ldots, N_B + 1)$, where $1 - K = 1 - \sum_{i=r}^{N_A} p_i^A x_i^A -$

$\left[1 - \sum_{i=r}^{N_A} p_i^A\right]\delta M$.

Proof: Let M be the supremum of the payoff to player A in any SGPE of G.

Consider the subgame beginning with an offer made by player A at time t = 2 (node 5 in Figure 1). This subgame has the same structure as the original game G apart from a rescaling of payoffs, and so the supremum of the payoff to A in any SGPE of this game is again M.

Now consider the subgame beginning at node 4. At node 4,

player A has available his outside option x_i^A with probability p_i^A, and with probability $\left[1 - \sum_{i=1}^{N_A} p_i^A\right]$ none of his outside options are available. If A chooses not to take up his outside option x_i^A (if it becomes available), then he must await his turn to make a counteroffer (at node 5). From the paragraph above we know that the supremum of the payoff to A in any SGPE of the subgame beginning at node 5 is M. Discounted to node 4, this equals δM. Thus the supermum of the payoff to A in any SGPE of the subgame beginning at node 4 is K, where

$$K = \left[1 - \sum_{i=1}^{N_A} p_i^A\right]\delta M + \sum_{i=1}^{N_A} \left[p_i^A \max(\delta M,\ x_i^A)\right], \quad \text{that is,}$$

$$K = \left[1 - \sum_{i=r}^{N_A} p_i^A\right]\delta M + \sum_{i=r}^{N_A} p_i^A x_i^A, \quad \underline{\text{if}} \quad x_r^A > \delta M \geqslant x_{r-1}^A,$$

$$(r = 1, \ldots, N_A + 1) \tag{1}$$

Now consider the subgame beginning at node 3, where B makes an offer to A. Any offer by B which gives A more than K (where K is defined in equation (1)) will be accepted by A; and so there is no SGPE in which B offers more than K. It follows that B will get at least 1 − K; in fact, this is the infimum of the payoff received by B in the subgame beginning from node 3.

Now consider the subgame beginning at node 2. At node 2 player B has available his outside option x_i^B with probability p_i^B, and with probability $\left[1 - \sum_{i=1}^{N_B} p_i^B\right]$ none of his outside options are available. If B chooses not to take up his outside option x_i^B (if it becomes available), then he must await his turn to make a

counteroffer (at node 3). The infimum of the payoff to B in any SGPE of the subgame beginning from node 3 is 1 − K, where K is defined in equation (1). Discounted to node 2, this equals $\delta(1 - K)$. Thus the infimum of the payoff to B in any SGPE of the subgame beginning from node 2 is L, where

$$L = \left[1 - \sum_{i=1}^{N_B} p_i^B\right]\delta(1 - K) + \sum_{i=1}^{N_B}\left[p_i^B \max(\delta(1 - K),\ x_i^B)\right],\ \text{that is,}$$

$$L = \left[1 - \sum_{i=s}^{N_B} p_i^B\right]\delta(1 - K) + \sum_{i=s}^{N_B} p_i^B x_i^B \ \underline{\text{if}}\ \ x_s^B > \delta(1 - K) \geqslant x_{s-1}^B,$$

$$(s = 1, \ldots, N_B + 1) \qquad (2)$$

Now consider node 1 in Figure 1 - the node at which the game G begins. Any offer by A which gives B less than L (where L is defined in equation (2)) will be rejected by B; and so there is no SGPE in which A offers less than L. It follows that A will get at most 1 − L; this is the supremum of the payoff to A in the game beginning from node 1. Hence,

$$M = 1 - L \qquad (3)$$

Now substitute for K in equation (2), where K is defined in equation (1), and then substitute for L in equation (3), where L is defined in equation (2), and we obtain that M is as defined in the Lemma; i.e., the equation shown in the Lemma defines the supremum of the payoff to player A in any SGPE of the game G.

We defined M as the supremum of the payoff to player A in any SGPE of G. The above argument may be repeated exactly, but with M

defined instead as the infimum of the payoff to player A in any SGPE of G; and, with the words more/less, most/least, supremum/infimum and accept/reject interchanged throughout. Hence the equation shown in the Lemma also defines the infimum of the payoff to player A.

Q.E.D.

We now characterise the solution of the game G.

<u>Proposition</u>: The game G has a unique SGPE partition, in which player A receives a share of M, where M is given by:

$$M = \left[1 - \sum_{i=s}^{N_B} p_i^B x_i^B - \left[1 - \sum_{i=s}^{N_B} p_i^B\right]\left[1 - \sum_{i=r}^{N_A} p_i^A x_i^A\right]\delta\right] \Big/ \left[1 - \left[1 - \sum_{i=r}^{N_A} p_i^A\right]\left[1 - \sum_{i=s}^{N_B} p_i^B\right]\delta^2\right], \quad \underline{\text{if}}\ \ x_r^A > \delta M \geqslant x_{r-1}^A\ \ \underline{\text{and}}$$

$$x_s^B > \delta(1 - K) \geqslant x_{s-1}^B, \quad (r = 1, \ldots, N_A + 1,$$

$$s = 1, \ldots, N_B + 1),$$

where

$$(1 - K) = \left[1 - \sum_{i=r}^{N_A} p_i^A x_i^A - \left[1 - \sum_{i=r}^{N_A} p_i^A\right]\left[1 - \sum_{i=s}^{N_B} p_i^B x_i^B\right]\delta\right] \Big/ \left[1 - \left[1 - \sum_{i=s}^{N_B} p_i^B\right]\left[1 - \sum_{i=r}^{N_A} p_i^A\right]\delta^2\right],$$

and player B receives 1 - M.

<u>Remark</u>: The game in which player B moves first has a unique SGPE partition, in which player B receives 1 - K (as defined above in the Proposition) if

$$x_r^A > \delta M \geqslant x_{r-1}^A \quad \underline{\text{and}} \quad x_s^B > \delta(1 - K) \geqslant x_{s-1}^B, \quad (r = 1, \ldots, N_A + 1,$$
$$s = 1, \ldots, N_B + 1) ,$$

where M is as defined above in the Proposition, and player A receives K.

<u>Proof</u>: We begin by showing that the equation stated in the preceding Lemma has a unique solution.

Solving that equation for M, and then substituting M into 1 - K, we obtain M as defined in the Proposition above. Thus the equation stated in the preceding Lemma does indeed have a unique solution - whence it follows that the supremum and infimum of the payoff to player A in any SGPE of G coincide.

It is straightforward to show that this solution is in fact supported by a pair of strategies which involve immediate agreement at time $t = 0$. This follows from the assumption that $R_A + R_B < 1$, where R_k $(k = A, B)$ is the reservation value of player k. Player A receives the share M as defined in the Proposition and player B receives 1 - M.

Hence, the game G has a unique SGPE partition.

Q.E.D.

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[illegible] The [illegible] of [illegible] [illegible]
[illegible] [illegible]
[illegible]

$$[illegible]$$

where M [illegible]
[illegible]

Proof. [illegible]
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[illegible] (4.7) and (4.8) [illegible]
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If [illegible]
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Hence, the [illegible] has a [illegible] property.

Remark: An implication of the (characterisation) Proposition is the following. Suppose the r biggest outside options of player A and the s biggest outside options of player B influence the bargaining outcome. We now ask whether the next biggest outside option of, say, player A (i.e., x^A_{r-1}) will or will not influence the bargaining outcome. It will not influence the bargaining outcome if the threat of having recourse to it (if it becomes available) is empty. The threat will be empty if x^A_{r-1} is less than or equal to the payoff that player A will receive if A does not take up the option; this payoff will depend on both the r biggest outside options of player A and the s biggest outside options of player B since these options influence the the bargaining outcome. Thus, no matter how big x^A_{r-1} is, provided it is less than or equal to this payoff it will not influence the bargaining outcome.

The result contained in the Proposition, above, is best interpreted by reference to some special cases.

Case I: Player k (k = A, B) has one outside option available with probability p (i.e., the Sutton case); then,

$$M = 1/(1 + \delta) \quad \text{if} \quad x^A_1,\ x^B_1 \leqslant \delta/(1 + \delta) \tag{4.1}$$

$$= [1 - \delta(1 - px^A_1)]/[1 - (1 - p)\delta^2] \quad \text{if} \quad x^A_1 > \delta/(1 + \delta),$$

$$x^B_1 \leqslant F(x^A_1) \tag{4.2}$$

$$= [1 - px^B_1 - (1 - p)\delta]/[1 - (1 - p)\delta^2] \quad \text{if} \quad x^B_1 > \delta/(1 + \delta),$$

$$x^A_1 \leqslant F(x^B_1) \tag{4.3}$$

Remark. The construction of [illegible] is

the following. [illegible]

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The [illegible] introduced in the [illegible] of [illegible]

[illegible] by reference to [illegible].

Lemma [illegible]. [illegible]

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$$[illegible] \qquad (4.1)$$

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$$[illegible] \qquad [illegible]$$

$$= [1 - px_1^B - (1 - p)(1 - px_1^A)\delta]/[1 - (1 - p^2)\delta^2] \quad \underline{if}$$

$$x_1^A > F(x_1^B),\ x_1^B > F(x_1^A) \tag{4.4}$$

where $F(x_1^k) \;=\; [1 - px_1^k - (1 - p)\delta]/[1 - (1 - p)\delta^2]$, $k = A, B$.

Proof: (cf. the Proposition above). Put $N_A = N_B = 1$ and $p_1^A = p_1^B = p$. Then, $s = 2$ and $r = 2$ gives (4.1), $s = 2$ and $r = 1$ gives (4.2), $s = 1$ and $r = 2$ gives (4.3), and, $s = 1$ and $r = 1$ gives (4.4).

A relatively transparent interpretation is made possible by taking a limiting case: change the time interval between successive offers from 1 to Δ, replace the probability p by $p\Delta$, and the discount factor δ by δ^Δ. Then in the limit $\Delta \to 0$ we obtain (see Sutton (1986, pp. 713-714)) $F(x_1^k) \to 1/2$ and:

$$M \;=\; 1/2 \quad \underline{if} \quad x_1^A,\ x_1^B \leqslant 1/2 \tag{5.1}$$

$$= w[1/2] + (1 - w)x_1^A \quad \underline{if} \quad x_1^A > 1/2,\ x_1^B \leqslant 1/2 \tag{5.2}$$

$$= w[1/2] + (1 - w)(1 - x_1^B) \quad \underline{if} \quad x_1^B > 1/2,\ x_1^A \leqslant 1/2 \tag{5.3}$$

$$= w[1/2] + (1 - w)[x_1^A + (1/2)(1 - x_1^A - x_1^B)] \quad \underline{if} \quad x_1^A,\ x_1^B > 1/2 \tag{5.4}$$

where $w = 1/[1 - (p/\ln \delta^2)]$; $(1 - w)$ can be interpreted as a measure of the likelihood that the outside option will be available.

Thus, from case I (i.e., the Sutton case) above, we see that the

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outside option belonging to a player, which is available with probability p, does <u>not</u> influence the bargaining outcome if its value is less than the Rubinstein (1982) solution (i.e., 1/2); if its value is greater than 1/2, then it influences the bargaining outcome in a particular way (cf. equations (5.1)-(5.4)).

In case II, below, we show that this conclusion does not hold when a player has <u>two</u> outside options, where each option is available with some probability.

<u>Case II</u>: Player A has no outside options and player B has two outside options available with probabilities p_1^B and p_2^B; then,

$$M = 1/(1 + \delta) \quad \underline{\text{if}} \quad x_2^B \leqslant \delta/(1 + \delta) \tag{6.1}$$

$$= [1 - p_2^B x_2^B - (1 - p_2^B)\delta]/[1 - (1 - p_2^B)\delta^2] \quad \underline{\text{if}} \quad x_2^B > \delta/(1 + \delta)$$

$$\underline{\text{and}} \qquad x_1^B \leqslant \delta/(1 + \delta) + [\delta^2 p_2^B/(1 - (1 - p_2^B)\delta^2)][x_2^B - (\delta/(1 + \delta))] \tag{6.2}$$

$$= [1 - p_2^B x_2^B - p_1^B x_1^B - (1 - p_2^B - p_1^B)\delta]/[i - (1 - p_2^B - p_1^B)\delta^2] \quad \underline{\text{if}}$$

$$x_2^B > \delta/(1 + \delta)$$

$$\underline{\text{and}} \quad x_1^B > \delta/(1 + \delta) + [\delta^2 p_2^B/(1 - (1 - p_2^B)\delta^2)][x_2^B - (\delta/(1 + \delta))] \tag{6.3}$$

<u>Proof</u>: (cf. the Proposition above). Put $N_A = 0$ and $N_B = 2$. Then, s = 3 gives (6.1), s = 2 gives (6.2), and s = 1 gives (6.3).

Change the time interval between successive offers from 1 to Δ,

[illegible]

[illegible]

[illegible]

[illegible]

$$[illegible]$$

$$[illegible]$$

$$[illegible]$$

[illegible]

[illegible]

replace p_i^B (i = 1, 2) by $p_i^B\Delta$, and δ by δ^Δ. Then in the limit $\Delta \to 0$ we obtain:

$$M = 1/2 \quad \underline{\text{if}} \quad x_2^B \leqslant 1/2 \tag{7.1}$$

$$= w_2[1/2] + (1 - w_2)(1 - x_2^B) \quad \underline{\text{if}} \quad x_2^B > 1/2 \quad \underline{\text{and}}$$

$$x_1^B \leqslant 1/2 + (1 - w_2)(x_2^B - (1/2)) \tag{7.2}$$

$$= [(w_2\ w_1)(1/2) + w_2(1 - w_1)(1 - x_1^B) + w_1(1 - w_2)x_2^B]/$$

$$[w_2 + w_1(1 - w_2)] \quad \underline{\text{if}} \quad x_2^B > 1/2 \quad \underline{\text{and}}$$

$$x_1^B > 1/2 + (1 - w_2)(x_2^B - (1/2)) \tag{7.3}$$

where $w_k = 1/[1 - (p_k^B/\ln \delta^2)]$, k = 1, 2; $(1 - w_k)$ can be interpreted as a measure of the likelihood that the outside option x_k^B (k = 1, 2) will be available.

As can be seen, from equation (7.2), that even if the smaller of the two outside options, namely x_1^B, has a value greater than the Rubinstein solution (i.e., 1/2) it need <u>not</u> influence the bargaining outcome.

surface of [illegible] by [illegible] and [illegible] to [illegible] from [illegible] — [illegible]
we obtain

$$[illegible] \qquad (7.4)$$

$$[illegible]$$

$$[illegible] \qquad (7.4a)$$

$$[illegible]$$

$$[illegible]$$

$$[illegible] \qquad (7.5)$$

whereas [illegible] and [illegible] may be interpreted as a measure of the likelihood [illegible] the [illegible] of [illegible] will be available.

As can be seen, from equation (7.4), that even if the number of [illegible] the two [illegible] equations, namely [illegible], has a value greater than the [illegible] [illegible], [illegible] it need not influence the integrating process.

PART THREE

GENERAL EQUILIBRIUM

Chapter 5

Price formation in a decentralised market

1. INTRODUCTION

In this chapter we present a theory of a market in which the institution of price formation is decentralised. The agents of the market (i.e., the buyers and the sellers) do not, ex-ante, know each others' location. The theory, therefore, includes a matching technology within which the agents search for partners with whom to trade. When a buyer and a seller meet, they initiate a sequential bargaining process over the terms of trade. If they reach an agreement they leave the market.

Models of this type have been explored recently by various authors. These models can be distinguished by two main important factors: the approach adopted with respect to (i) the basic bargaining problem, and (ii) the matching and search problem. In the models presented in Diamond (1982), Diamond and Maskin (1979) and Mortensen (1981, 1982), the matching and search process is modelled explicitly as a non-cooperative game, but the events that follow a match are not modelled. Rather, these authors assume that the parties reach an agreement on the terms of the contract suggested by the Nash Bargaining Solution. On the other hand, in the models presented in Rubinstein and Wolinsky (1985, 1986), Gale (1986, 1987) and Binmore and Herrero (1984), the matching and search process is not modelled explicitly *à la* Diamond-Maskin-Mortensen, but they do explicitly model the bargaining process as a non-cooperative game, in which the bargaining procedure is described in detail. Wolinsky (1987) presents the only paper that combines the model of matching and search of Mortensen (1982) with a sequential bargaining model.

The *raison d'être* of this chapter is now discussed.

(A) Firstly, the model that will be presented seems more general and less restrictive than any of the previous models mentioned above.

The matching and search model *à la* Diamond-Maskin-Mortensen has two major assumptions: (i) all agents of the same type are assumed, *a priori*, to search with the same strategy and (ii) the matching rate is a linear or a quadratic function of the search choices made by each

type of agent. In our matching and search model (which will have a framework similar to that of Diamond-Maskin-Mortensen) we will not make these two assumptions. We shall, in fact, assume (i) that agents of the same type can choose different search strategies and (ii) that the matching rate is some function of the search choices of all the agents. Thus, for example, in our model the rate at which a particular seller and a particular buyer meet will be affected by the search choices of the other agents.

A central feature of the non-cooperative strategic approach to the basic bargaining problem adopted by Rubinstein-Wolinsky-Gale-Binmore-Herrero (with the exception of Wolinsky (1987)) is the following. In these models the decision to abandon the bargaining between a matched seller and buyer is not a strategic decision; the matched seller and buyer are "forced" to leave each other. This, in our view, is not genuine sequential bargaining *à la* Rubinstein (1982). In our model, as in the model of Wolinsky (1987), when a buyer and a seller meet they can bargain as long as they like; the choice to leave each other is modelled explicitly as a strategic decision.

At this point we shall mention the difference between the bargaining procedure adopted by us and that adopted by Wolinsky (1987). Both our and Wolinsky's bargaining procedures are modified versions of the classic alternating-offers model of Rubinstein (1982). The key ideas that lie at the heart of the difference between the two bargaining procedures can be put as follows. In our model a matched pair of agents who are in the process of bargaining (over the terms of trade) cannot search for alternative partners while bargaining, whereas in Wolinsky (1987) the matched pair can search while bargaining. Let us elucidate. When a matched pair of agents go through a round of bargaining, firstly one of the two agents is chosen at random to make the offer. This step is assumed in both our and Wolinsky's models. Now, in Wolinsky's model the agent (who has to react to the offer) has two choices, either to accept the offer, or to reject the offer and wait for a fixed time interval before proceeding to another round of bargaining, whereas in our model the agent has, in addition to these two choices, another choice: he can withdraw from the bargaining process (i.e., "the negotiating table") and search for type of agent. In her semantic and lexical model, which will have a framework similar to that of [illegible] case grammar, we will see [illegible]

A special feature [illegible]

At this point, we shall consider the differences between the [illegible] an alternative partner. However, in Wolinsky's model, during the time interval between successive offers the two agents can search for alternative partners, and can therefore switch to alternative partners during this time interval and thus they may not proceed to bargain with each other in the next round of bargaining.

We have the following argument in favour of the particular assumptions that we have chosen. The argument is based on answering the question, "what is the *raison d'être* of the interval of time between successive offers in Rubinstein-type alternating-offers bargaining models?" The starting point is to observe that one can criticise Rubinstein-type models with regard to the rigidity of the timetable for making proposals; what constrains the players to the use of this timetable (i.e., why have the length of time between successive offers fixed exogenously). Binmore (1987a) has shown that it is reasonable to use a Rubinstein-type model in which the intervals between successive proposals are vanishingly small as a paradigm for the case in which the players are not formally constrained by an exogenously determined timetable. One could argue further that we should, at the outset, put the time between successive offers <u>equal</u> to zero. But, as is well known, we then face the problem of indeterminacy of the bargaining outcome. However, one could counter-argue that time between successive offers must be strictly positive if only for physical reasons. So let there remain an interval of time between successive offers, but interest will center on the interval tending to zero. Thus, the *raison d'être* of introducing a time interval between successive offers is to capture the notion of "frictions" in the bargaining process, which happens to eliminate the indeterminacy of the bargaining outcome. One should view the time gap between successive offers as an integral part of the successive bargaining rounds, and thus an integral part of the bargaining process. The time interval between successive offers is part and parcel of the bargaining process, the process of the sequence of offers and counteroffers. One cannot use this time gap for purposes of search. Search takes time, and thus if an agent wishes to search for an alternative partner, then he must withdraw from the bargaining process.

[illegible]

(B) Secondly, most of the models mentioned above focus attention on symmetric equilibria (i.e., equilibria in which agents of the same type choose the same strategy). In our theory we allow for non-symmetric equilibria (i.e., agents of the same type are allowed to choose different strategies). However, we prove that the unique sub-game perfect equilibrium outcome is symmetric. A property of the equilibrium is that all sellers (who are identical) choose the same strategy, and all buyers (who are identical) choose the same strategy.

(C) The third, and perhaps the most important, justification for the current chapter is the new results and new insights it provides in answering the question, "to what extent is the conventional wisdom, that all 'frictionless' markets are Walrasian, correct?"

The main theorem will establish that all transactions take place at different prices (i.e., non-uniform prices emerge in equilibrium). This is because the demand and supply conditions change as traders leave the market (and there are no new traders who enter).[1] The price at which a matched pair trade depends on the state of the market, and three types of frictions. Firstly, the bargaining friction, which is captured by the exogenously fixed time interval between successive offers. We shall call this friction the Internal friction. Secondly, the market friction, which is captured by the search costs, and represents the extent to which the market environment (in particular, the fact that there are more buyers than there are sellers)[2] impinges on the equilibrium price. We shall call this friction the External friction. Thirdly, there is the (common) rate of time preference.

As the External friction tends to zero, keeping the Internal friction and the rate of time preference strictly positive, all the equilibrium prices tend to the competitive equilibrium price. An explanation and interpretation for this is as follows. As the External friction tends to zero, the sellers (since they are on the short side

[1] We will assume that all traders enter the market at one single time and that the market continues to operate until all possible transactions are completed. This assumption is also adopted by Rubinstein and Wolinsky (1986) and Binmore and Herrero (1984, section 8).

[2] We analyse the model with the assumption that there are more buyers than there are sellers. (The case where there are more sellers than there are buyers can be analysed in a similar manner - and the results would be 'symmetric'. The case where the number of sellers equals the number of buyers is somewhat different).

of the market) are able to play off the buyers, one against another, in order to obtain the whole surplus. An alternative angle from which to view this is as follows. As the External friction tends to zero, we will show that the expected time taken for an agent to "getting matched" tends to zero. Since the sellers are on the short side of the market it is they (and not the buyers) who can exploit this to their advantage. Suppose a seller is matched with a buyer. Since the Internal friction is strictly positive and since the expected time taken for the seller to find an alternative buyer tends to zero the seller will always prefer to change partners than to continue bargaining with his current buyer. Thus, in essence, the matched seller and buyer play a one-shot game in which the seller announces a take-it-or-leave-it offer to the buyer.

On the other hand, as the Internal friction tends to zero, keeping the External friction and the rate of time preference strictly positive, all the equilibrium prices tend to the bilateral bargaining equilibrium price. An explanation and interpretation for this is as follows. The External friction becomes infinitely large relative to the Internal friction, and thus the market environment (in particular, the fact that there are more buyers than there are sellers) does not impinge on the equilibrium prices, and thus a matched seller and buyer become locked in a bilateral bargaining game - hence the equilibrium prices are the bilateral bargaining equilibrium price.

As both the External and the Internal frictions approach zero, keeping the rate of time preference strictly positive, (i) all the equilibrium prices approach the competitive equilibrium price, _if_ the External friction approaches zero at a higher speed than the Internal friction, and (ii) all the equilibrium prices approach the bilateral bargaining equilibrium price, _if_ the Internal friction approaches zero at a higher speed than the External friction.

It will become clear that the External friction and the Internal friction are working, on the equilibrium prices, in the opposite directions. As both of these frictions tend to zero, the effect of the External friction favours competition, while the effect of the Internal friction favours bilateral bargaining. Thus, competitiveness is characterised not by the removal of all frictions but rather by the

removal of certain frictions.

Section 2 of the chapter describes the model. In section 3, we prove the existence and uniqueness of the sub-game perfect equilibrium outcome. Then, in section 4, we discuss the nature of the equilibrium and derive some key implications; these implications provide new insight in our understanding of the range of validity of the Walrasian outcome.

2. THE MODEL

The agents in the model are buyers and sellers. All the sellers are identical. And all the buyers are identical. A seller has one unit of an indivisible commodity for sale, and a buyer is seeking to buy one and only one unit of this commodity. A seller's value (i.e., reservation price) of the commodity is normalised at zero, and a buyer's value (i.e., reservation price) is normalised at one; thus, there exists a unit surplus between any buyer and seller.

When a buyer and a seller are matched, they bargain over the partition of the unit surplus associated with the match; i.e., they bargain over the terms of trade (or, if you like, the price of the commodity). After they reach an agreement they leave the market.

The market considered in the model operates over time. The time dimension is continuous, real time. The market opens at time $t=0$, with M buyers and N sellers, where $M,N \in \mathbb{N}$. We will assume that no new agent of either type enters the market after time $t=0$. The market terminates when all possible transactions are executed (namely, $\min\{N,M\}$ transactions).

REMARKS: (1) The central idea common to all the models, cited in the Introduction, is that pairs of agents meet through some matching technology and then there is pairwise bargaining. However, the authors differ in their choice of the following three mutually exclusive assumptions regarding the relationship between the traders' presence in the market and time: (i) the market is in a steady state in terms of the number of sellers and buyers in it - the completion of one transaction is immediately followed by the pairing of a new buyer and seller, (ii) all traders enter the market at one single time and the market continues to operate until all possible transactions are completed, and (iii) the number of sellers and buyers considering to enter the market is constant over time. There is no *a priori* reason to suggest that either one of the assumptions is "superior" to the others. One has to investigate the implications of all the three assumptions. However, we note that models which differ with respect to the particular assumption chosen are not strictly comparable (see

Rubinstein (1987a) for a further discussion on this point). In this chapter we have adopted assumption (ii); this assumption is also adopted by Rubinstein and Wolinsky (1986) and Binmore and Herrero (1984, section 8).

(2) In section 3 we will analyse the model with the assumption, M>N. (The case where M<N can be analysed in a very similar manner. However, the case where M=N is somewhat different).

There are two processes to be modelled, the matching process and the bargaining process. We, firstly, describe the matching process in isolation from the bargaining process. Secondly, we describe the bargaining process in isolation from the matching process. And thirdly, we combine the two processes to describe precisely how the market operates.

The Matching Process

The central idea is that the agents of the model (i.e., the buyers and the sellers) do not, ex-ante, know each others' location. Thus, an agent of one type will "search", within the domain of a particular matching process, in order to find the location of an agent of the opposite type.

Formally, the matching process will be represented by a stochastic process. And the parameters of the stochastic process will be influenced by the "search" choices of the agents. An agent chooses a non-negative real number that represents his "search". One can interpret this non-negative real number as the intensity of search.

We will assume that only unmatched agents can "search". Matched agents in the process of bargaining cannot "search" (cf. the Introduction (section 1) for a justification of this assumption).

The time dimension of the stochastic process is continuous; $t \in \mathbb{R}_+$. The state space of the system is defined by the set W, where $W=\{w:w \subseteq U\}$, where U is the set of all agents, and w denotes a set of unmatched agents. Let X(t) denote the state of the system at time t, $t \in \mathbb{R}_+$. X(0)=U (thus at t=0 all agents are unmatched).

Assumption 1

Suppose at time $t=T$ $X(T)=w$, where $w\in W$. Then, when the stochastic system moves out of state w, it will move to one and only one of the following states: $\{w^{ij}: s_i\in w$ and $b_j\in w\}$, where $w^{ij}=[w$ minus (seller i and buyer j)], denoting seller i by s_i and buyer j by b_j.

Assumption 1 means that at any instant in time only one pair can get matched, or no two pairs can get matched at the same instant of time.

Assumption 2

Let $X(T)=w$, where $w\in W$. Suppose $X(T+t_1)=w$, and suppose at time $t=T+t_1+h$ all agents belonging to w excluding seller i and buyer j are unmatched. Then, the probability that seller i (s_i) and buyer j (b_j) remain unmatched at time $t=T+t_1+h$ is independent of the length of time, before time $t=T+t_1$, that s_i and b_j were unmatched; this length of time is t_1, and thus the probability is independent of t_1.

Assumption 2 embodies the Markovian property of 'complete lack of memory'; with this assumption the stochastic process will be Markovian.

Let $w\in W$ and $s_i\in w$, $b_j\in w$. Then, let T^w_{ij} denote the random time taken to move from state w to state w^{ij}.

Assumption 2 $\Leftrightarrow$ T^w_{ij} is an exponential random variable with parameter λ^w_{ij}, $\lambda^w_{ij}>0$. (See Feller (1968), chap. XVII.6).

Assumption 3

$\forall w\in W$, $\{T^w_{ij}: s_i\in w$ and $b_j\in w\}$ are independent random variables.

Let $w\in W$. Given Assumption 1, and that $\{T^w_{ij}: s_i\in w$ and $b_j\in w\}$ are independent exponential random variables with parameters $\{\lambda^w_{ij}: s_i\in w$ and $b_j\in w\}$, we have (a) $\forall s_i\in w$, $T^w_i=\min_{b_j\in w}\{T^w_{ij}\}$ is an exponential random variable with parameter $\sum_{b_j\in w}\lambda^w_{ij}$, (b) $\forall b_j\in w$, $T^w_j=\min_{s_i\in w}\{T^w_{ij}\}$ is an exponential random variable with parameter $\sum_{s_i\in w}\lambda^w_{ij}$, and (c) $T^w=\min_{s_i\in w}\min_{b_j\in w}\{T^w_{ij}\}$ is an exponential random variable with parameter $\sum_{s_i\in w}\sum_{b_j\in w}\lambda^w_{ij}$.

[illegible]

Assumption 1 [illegible]

[illegible] Then, when the stationary [illegible] it will move to [illegible] and only one of the [illegible]

Assumption [illegible]

Assumption [illegible]

[illegible] Markovian property [illegible] stochastic process will be [illegible]

[illegible]

Assumption 3 [illegible]

[illegible]

T^w is the random time to move out of state w (i.e., the random time for one pair to get matched, given that the system is in state w).

Suppose that X(T)=w. And suppose a transition occurs at time T+t. Given Assumption 1, we have, the probability that this transition is to the state w^{ij} is $[\lambda^w_{ij}/(\sum_{s_m\in w}\sum_{b_n\in w}\lambda^w_{mn})]$, for any $s_i\in w$ and $b_j\in w$.

The description of the motion of the stochastic process is as follows: the process sojourns in a given state w for a random length of time whose distribution function is an exponential distribution with parameter $\sum_{s_i\in w}\sum_{b_j\in w}\lambda^w_{ij}$. When leaving state w the process enters one and only one of the following states $\{w^{ij}:s_i\in w \text{ and } b_j\in w\}$ with probabilities $\{[\lambda^w_{ij}/(\sum_{s_m\in w}\sum_{b_n\in w}\lambda^w_{mn})]: s_i\in w \text{ and } b_j\in w\}$.

The parameters of the stochastic system are $\{\lambda^w_{ij}:w\in W \text{ and } s_i\in w, b_j\in w\}$. One can interpret the parameter λ^w_{ij} as, the rate at which seller i and buyer j get matched, given that the state of the stochastic system is w. We shall now describe how the search intensities chosen by the unmatched agents influence these parameters. Let e^n_s and e^m_b denote the search intensities chosen by seller n and buyer m, respectively.

Assumption 4

Both e^n_s and e^m_b depend on the state of the stochastic system, w; but are independent of time, t.

REMARKS: (1) Note that the search intensities depend on time to the extent that the state of the stochastic system will depend on time.

(2) We have assumed that the search intensities are independent of time in order to avoid needless technicalities that would arise when defining the payoff functions.

We now observe that, for any w∈W, the <u>number</u> of sellers and the <u>number</u> of buyers is related as follows. Let w be any element of W. And let k be the numbers of sellers. Then there are M-N+k buyers. This is because N-k sellers have, either left the market, or are in the process of bargaining, in either case each of the N-k sellers is

[illegible]

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[illegible]

[illegible]

[illegible]

[illegible]

matched with some buyer. Since M is the total number of buyers, there are M-(N-k)=M-N+k unmatched buyers. Now define $|w|$=number of unmatched sellers, given that $w\in W$; and of course, M-N+$|w|$=number of unmatched buyers, given that $w\in W$. Finally, define N functions as follows:

$$F^l:\mathbb{R}_+^{l+(M-N+l)}\rightarrow\mathbb{R}_+,\ l=N,N-1,\ldots,1\ .$$

We shall assume that F^l is continuous and twice differentiable on its domain. Let $w\in W$ and $s_i\in w$, $b_j\in w$. Then, we define λ^w_{ij} as:

$$\lambda^w_{ij}=F^k(e^i_s,e^j_b,\underline{e}^w_{ij}),$$

where $|w|=k$, $k\in\{1,2,3,\ldots,N-1,N\}$, and $\underline{e}^w_{ij}=(\{e^n_s:s_n\in w,n\neq i\},\{e^m_b:b_m\in w,m\neq j\})$.

Assumption 5 : (assumptions on F^k)

(a) F^k is strictly increasing and concave in e^i_s and in e^j_b. (b) F^k is strictly decreasing and convex in e^n_s $(s_n\in w,n\neq i)$ and in $e^m_b(b_m\in w,m\neq j)$. (c) Suppose all sellers belonging to w choose the same search intensity (denote it by e_s) and all buyers belonging to w choose the same search intensity (denote it by e_b). Then,

(i) $$\frac{\partial F^k(\bullet)}{\partial e^p_s}=\frac{\partial F^k(\bullet)}{\partial e^q_s}\quad\text{for all } p\neq q\neq i,\ s_p,s_q\in w\ ,$$

and

(ii) $$\frac{\partial F^k(\bullet)}{\partial e^p_b}=\frac{\partial F^k(\bullet)}{\partial e^q_b}\quad\text{for all } p\neq q\neq j,\ b_p,b_q\in w\ .$$

(d)

(i) $$\frac{\partial^2F^k(\bullet)}{\partial e^p_s\partial e^q_s}=0\text{ for all } p\neq q,\ s_p,\ s_q\in w.$$

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

(ii) $$\frac{\partial^2 F^k(\bullet)}{\partial e_s^p \partial e_b^k} = 0 \text{ for all } s_p \in w \text{ and } b_k \in w.$$

(iii) $$\frac{\partial^2 F^k(\bullet)}{\partial e_b^m \partial e_b^n} = 0 \text{ for all } m \neq n,\ b_m,\ b_n \in w.$$

(i.e., all mixed derivatives are zero).

REMARK: Assumption 5 (i.e., the assumptions on F^k), (a)-(d), is needed to prove the uniqueness of the Nash equilibrium for the search game, given our method of proof.

This completes the model of search and matching.

Before we proceed, we define three functions derived from F^k, that will be used later in the analysis of the model.

Suppose all sellers belonging to w choose the same search intensity (denote it by e_s) and all buyers belonging to w choose the same search intensity (denote it by e_b). Then, $\lambda_{ij}^w = F^k(e_s, e_b, \underline{e}_k)$, where $\underline{e}_k = \underline{e}_{ij}^w$ with $e_s^n = e_s$ ($s_n \in w, n \neq i$) and $e_b^m = e_b$ ($b_m \in w, m \neq j$). Define $f_1^k : \mathbb{R}_+^2 \to \mathbb{R}_+$ as follows:

$$f_1^k(e_s, e_b) = F^k(e_s, e_b, \underline{e}_k) .$$

Thus, $\lambda_{ij}^w = f_1^k(e_s, e_b)$; when sellers choose the same search intensity, and buyers choose the same search intensity, the rate at which any buyer and any seller meet is independent of the faces (i.e., independent of i and j).

Given Assumption 5(c), define $f_2^k : \mathbb{R}_+^2 \to \mathbb{R}_+$ and $f_3^k : \mathbb{R}_+^2 \to \mathbb{R}_+$ as follows:

[illegible]

[illegible]

[illegible] when each player chooses the same search intensity, [illegible] choose the same search intensity, the rate at which [illegible] is independent of [illegible] of i and j.

[illegible] define [illegible] as follows:

$$f_2^k(e_s, e_b) = \frac{\partial F^k(\bullet)}{\partial e_s^p} = \frac{\partial F^k(\bullet)}{\partial e_s^q}, \text{ and}$$

$$f_3^k(e_s, e_b) = \frac{\partial F^k(\bullet)}{\partial e_b^p} = \frac{\partial F^k(\bullet)}{\partial e_b^q}.$$

The Bargaining Process

Suppose at some time t ($t \geq 0$) a buyer and a seller get matched. Then, they begin bargaining as follows.

Firstly, one of the parties is selected at random, with probability 1/2, to propose a partition of the unit surplus to which the other party then reacts with acceptance ("A") <u>or</u> rejection ("R") <u>or</u> rejection and search ("RS"). Acceptance of a proposal ends the bargaining (at time t) and both parties leave the market having executed a transaction. If a proposal is rejected, then the same (above) bargaining procedure is repeated Δ time later, at time $t+\Delta$. Rejection and search implies that the parties abandon each other and return to the matching process (at time t) to search for altenative partners.

REMARKS: (1) Suppose the two parties abandon each other in order to search for alternative partners. And suppose one of them finds an alternative partner. If the party who has found a new partner can remember the "address" (i.e., location) of his previous partner, then he could use this as a threat against his new partner. This means that the above bargaining procedure has to be amended so as to include this possibility. This would lead us to formulate a rather complicated bargaining procedure. In order to avoid this, and in order to retain our initially proposed bargaining procedure, we asssume that when two parties abandon each other (in order to search for alternative partners) they forget each others' "address", and thus cannot return to bargain with each other at a later time in future. (However, we shall assume that the parties have perfect recall and can identify

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

each other by face in future, if they meet again, i.e., if the matching process matches them again).

(2) Unmatched agents cannot meet matched agents who are in the process of bargaining. This is because matched agents who are in the process of bargaining do not search (i.e., "advertise"); search requires time and thus the matched agents who are in the process of bargaining have to abandon the bargaining process if they want to search (cf. the Introduction, where we presented an argument to justify this assumption).

The bargaining procedure adopted here is a modified version of that studied in chapter 3. The model studied in chapter 3 has offers being made alternatively and not determined by a random mechanism; this model is based on the classic works of Rubinstein (1982) and Stahl (1972). The idea of having a random mechanism determine the proposer was first suggested by Binmore (1987a), who applied the idea to the Rubinstein (1982) model. One advantage of having the proposer being determined by a random mechanism is that the bargaining procedure becomes completely symmetric for any $\Delta>0$. This will simplify the analysis. (As a matter of fact, all the recent papers in this literature use the random mechanism to determine the proposer).

The Evolution of the Market/Order of Events.

Let us now proceed to describe precisely how the market operates, i.e., let us combine the search and matching process with the bargaining process, and show how they are interlaced.

The market operates in continuous, real time. Let us consider the market at any time T ($T\geq 0$); assuming that the market is still in operation.

At time T, each and every agent has to be in <u>one and only one</u> of the following classes: (i) <u>Class 1</u> (C_1) := unmatched agents, (ii) <u>Class 2</u> (C_2) := matched agents who will have a round of bargaining at time T (these will include the newly matched pair, if there is one, and the pairs who last had a bargaining round at time $T-\Delta$ and decided to continue bargaining), (iii) <u>Class 3</u> (C_3) := matched pairs who will not have a round of bargaining since they last had a round of

bargaining at some time s, where $T>s>\max(0,T-\Delta)$, (iv) Class 4 (C_4) := pairs who have left the market, having reached an agreement.

Therefore, at time T, there will exist a partition of the set, U, of all agents into four mutually exclusive classes, namely C_1, C_2, C_3 and C_4. Let p_T denote the partition of U at time T. And let $\mathbb{P}$ denote the set of all possible partitions of U, at time T, into the four mutually exclusive classes, C_i i=1,2,3,4.

Define the state of the market at time T by its partition, p_T, of U.

The decisions that are taken instantaneously, but sequentially, at time T are as follows:

First: matched pairs of Class 2 (C_2) go through a bargaining round.

Second: members of C_1 and matched pairs of C_2 who abandoned each other at the First stage (above), during the bargaining round, choose, simultaneously, search intensities.

A history of the market up until time T, but excluding events at time T, will include all the events that have occured from time t=0 to time t=T; this is characterised by the following data:

(I) For all t ($T\geq t\geq 0$) p_t, the state of the market at time t.

(II) For all t ($T>t\geq 0$) and $\forall$ matched pair $\in C_2(t)$, we have, (a) the proposer, (b) the proposal, and (c) the reaction of the responder.

(III) For all t ($T>t\geq 0$) and $\forall$ agent $\in C_1(t)$ and $\forall$ agent $\in \overline{C_2(t)}$, we have, his search intensity choice, where $\overline{C_2(t)}$ ($\subseteq C_2(t)$) are those matched pairs who abandoned each other at the bargaining round.

Let g^T denote such a market history, and G^T denote the set of all possible market histories at time T.

Preferences

We shall assume that all agents maximise expected utility. Suppose an agent reaches an agreement at time t, t>0. Let x denote the share

of the unit surplus that the agent receives. The agent may have searched for some part of the time during which he was in the market. Let e denote the search intensity chosen by the agent. This may depend on the state of the stochastic system, e(w) (cf. Assumption 4). Denote the states during which the agent searched by $\{w_1,\ldots,w_n\}$. Let $[t_i,t_{i+1}]$ denote the time interval during which, (a) the stochastic system was in state w_i, $i=1,\ldots,n$, <u>and</u> (b) the agent searched with intensity $e(w_i)$. Let $c : \mathbb{R}_+ \to \mathbb{R}_+$ be a strictly increasing, strictly convex, continuous and twice differentiable function. And $c(0)=0$. c is the (common) cost of search function.

Therefore the utility to the agent is :

$$xe^{-rt} - \sum_{i=1}^{n} \int_{t_i}^{t_{i+1}} c[e(w_i)]e^{-rs}ds$$

$$= xe^{-rt} - \frac{1}{r} \sum_{i=1}^{n} c[e(w_i)] \, [e^{-rt_i} - e^{-rt_{i+1}}] ,$$

where r is the (common) rate of time preference, $r>0$.

Informational Assumptions

We shall assume that the model and the preferences are common knowledge amongst the agents. Thus the game is one of <u>complete</u> information.

A history of the market at any time T is characterised by three elements (I)-(III), described earlier under the subsection headed, "the Evolution of the Market". We shall assume that elements (I) and (II) are common knowledge amongst the agents but element (III) is not common knowledge. In fact, an agent never knows the search intensities chosen by the other agents. Therefore the game is one of <u>imperfect</u> information.

We shall assume that there is <u>perfect recall</u>; in particular, an agent does not forget the faces of the agents he met in the past.

Strategies, Outcome and Equilibrium

Let us now define a strategy for an agent i; i can be any seller or any buyer. May we recall that G^T denotes the set of all market histories at time T, excluding the events at time T. For any $g^T \in G^T$, g^T minus element (III) is that part of a market history at time T which is common knowledge amongst the agents. Let h^T denote this. Let H^T denote the set of all such h^T, i.e.,

$$H^T = \{ h^T : h^T = [g^T/\text{element (III)}] \text{ , where } g^T \in G^T \} .$$

$\forall h^T \in H^T$, at time T, agent i can be in one and only one of the four classes, C_k, k=1,2,3,4. We are interested in histories after which agent i has to take a decision. Let us partition the set H^T as follows: $H^T = H_i^T(C_1) \cup H_i^T(C_2) \cup H_i^T(C_3) \cup H_i^T(C_4)$, where $H_i^T(C_k)$ is the subset of market histories that lead agent i to be in class k (k=1,2,3,4) at time T.

(1) $\forall h^T \in H_i^T(C_4)$, agent i is not in the market, and thus he has no decision to take.

(2) $\forall h^T \in H_i^T(C_3)$, agent i has no decision to take at time T.

(3) $\forall h^T \in H_i^T(C_2)$, agent i is matched and has a bargaining round. There are three decision points at which the agent might have to make a decision, (i) what proposal to make (if he is selected to propose), (ii) how to react to the proposal (if he is not selected to propose), and (iii) which search intensity to adopt (if agent i or his partner decide to leave each other).

Define four functions, f_1^T, f_2^T, f_3^T and f_4^T, as follows: $f_1^T : H_i^T(C_2) \times E_i^T \rightarrow [0,1]$, $f_2^T : H_i^T(C_2) \times E_i^T \times [0,1] \times \{\text{"RS"}\} \rightarrow \mathbb{R}_+$, $f_3^T : H_i^T(C_2) \times E_i^T \times [0,1] \rightarrow \{\text{"A"}, \text{"R"}, \text{"RS"}\}$, $f_4^T : H_i^T(C_2) \times E_i^T \times [0,1] \times \{\text{"RS"}\} \rightarrow \mathbb{R}_+$, where $E_i^T = \{e_i : e_i$ denotes the search intensities chosen by agent i from time t=0 up until time t=T$\}$. Note that f_2^T represents the decision if the opponent "rejects and searches", while f_4^T represents the decision if the agent "rejects and searches". Let $f^T = (f_1^T, f_2^T, f_3^T, f_4^T)$. f^T defines decisions on any $h^T \in H_i^T(C_2)$.

(4) $\forall h^T \in H_i^T(C_1)$, agent i is not matched at time T. Now $\forall h^T \in H_i^T(C_1)$ there may exist pairs of agents that are in C_2. Given an $h^T \in H_i^T(C_1)$, let $F[C_2; h^T \in H_i^T(C_1)]$ denote the matched pairs of C_2. Let $e[F(.;.)]$ denote a possible set of proposers, offers and replies associated with the matched pairs of C_2. And let $B[F(.;.)]$ denote the set of all possible $e[F(.;.)]$'s. Now define a function f_5^T,

$$f_5^T : H_i^T(C_1) \times E_i^T \times B[F(C_2; h^T \in H_i^T(C_1))] \rightarrow \mathbb{R}_+ \ .$$

Define $f^T = (f_1^T, f_2^T, f_3^T, f_4^T, f_5^T)$. And define $F_i = \{f^T : \infty > T \geq 0\}$. F_i is a strategy for agent i. Let A_i denote the set of all strategies for agent i.

Associated with a (M+N) strategy vector $\{F_i\}_{i=1}^{M+N}$ is an outcome of the game. Let us define what we mean by an outcome.

Let $\{F_i\}_{i=1}^{M+N}$ be a (M+N) strategy vector. Then $\forall T$ and $\forall g^T \in G^T$, define, $(x_{ij}(g^T), y_{ij}(g^T))$ $\forall$seller i, buyer $j \in U$, and $e_i(g^T)$ $\forall$agent $i \in U$ as follows:

$x_{ij}(g^T) \in [0,1]$ IF given $g^T \in G^T$ seller i and buyer j have a bargaining round at time T (with the buyer chosen to propose) and they reach an agreement where $x_{ij}(g^T)$ is the terms of trade (i.e., price) agreed at.

undefined IF given $g^T \in G^T$ seller i and buyer j, either do not have a bargaining round at time T, or if they do (and it is the buyer who is chosen to propose) they do not reach an agreement.

$y_{ij}(g^T)$ is similarly defined, but with the seller being chosen to propose.

$e_i(g^T) \in \mathbb{R}_+$ IF given $g^T \in G^T$ agent i chooses a search intensity.

$= 0$ IF given $g^T \in G^T$ agent i does not "search".

Then, an OUTCOME of the game, given $\{F_i\}_{i=1}^{M+N}$ is defined as :

$\{x_{ij}(g^T):\forall T, \forall g^T \in G^T$ and $\forall$seller i, buyer $j \in U\}$

union $\{y_{ij}(g^T):\forall T, \forall g^T \in G^T$ and $\forall$seller i, buyer $j \in U\}$

union $\{e_i(g^T):\forall T, \forall g^T \in G^T$ and $\forall$agent $i \in U\}$.

We shall adopt the subgame perfect equilibrium (SGPE) solution concept (Selten (1965, 1975)). A SGPE is a (M+N) vector of strategies, one for each agent, such that its restriction to any proper subgame is a Nash equilibrium.

Let b_i^T denote the personal history of agent i up to some decision point at time T; may we recall that at time T there are a sequence of decision points. Let $u_i[\{F_i\}_{i=1}^{M+N}](b_i^T)$ denote the value at time T of the expected utility to agent i who has experienced the history of b_i^T and who employs the strategy F_i while the others, agent j ($j \neq i$), employ $\{F_j\}_{j \neq i}$.

A SGPE is a (M+N) vector $\{F_i^*\}_{i=1}^{M+N}$ where $\forall i$, $F_i^* \in A_i$, and for all possible histories b_i^T,

$$u_i[\{F_i^*\}_{i=1}^{M+N}](b_i^T) \geq u_i[F_i, \{F_j^*\}_{\substack{j \neq i \\ j=1}}^{M+N}](b_i^T) \text{ , for all } F_i \in A_i \text{ .}$$

3. EXISTENCE AND UNIQUENESS OF EQUILIBRIUM

In this section we shall prove the existence and uniqueness of the sub-game perfect equilibrium (SGPE) outcome. In the following section we will discuss the nature of the equilibrium and derive some key implications.

We first state the Theorem that deals with the existence and uniqueness of the equilibrium outcome, and then prove the Theorem.

THEOREM

(a) Given that there are N sellers and M buyers, there exists a unique SGPE outcome of the game, in which a matched pair of agents trade instantaneously; where $N, M \in \mathbb{N}$ and $M>N$.

(b) The price at which a matched pair of agents trade depends only on the number of unmatched sellers and the number of unmatched buyers at the time when the pair is matched; i.e., the price is $x(k)$ or $y(k)$ according to whether it is the buyer or the seller who proposes, where k is the number of unmatched sellers, $k=N-1, N-2, \ldots 1, 0$, and, of course, there are $M-N+k$ unmatched buyers. (Thus, for example, $x(N-1)$ or $y(N-1)$ is the price at which the first matched pair trade). Note that the price does not depend on a particular seller or a particular buyer, and depends on history only to the extent that history determines a state of the stochastic system, w (which in turn defines the value of k). Furthermore, note that non-uniform prices have emerged, which is due to the fact that demand-supply conditions change, as traders leave the market.

(c) In any state of the stochastic system, all the unmatched sellers choose the same search intensity, and all the unmatched buyers choose the same search intensity. The search intensity chosen in any state, $w, w \in W$, does not depend on w, but depends only on the number of unmatched sellers belonging to w, and the number of unmatched buyers belonging to w; i.e., $e_s(k+1)$ and $e_b(k+1)$ denote the search intensities chosen by an unmatched seller and an unmatched buyer, respectively, given that there are k+1 unmatched sellers, where k=N-

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1,N-2,...,1,0, and, of course, there are M-N+(k+1) unmatched buyers. Note that the search intensity chosen depends on history only to the extent that history determines a state of the stochastic system, w (which in turn defines the value of k).

(d) Let $\underline{x}_k=\{x(l)\}_{l=0}^{k}$, $\underline{y}_k=\{y(l)\}_{l=0}^{k}$ and $\underline{X}_k=\{(x(l)+y(l))/2\}_{l=0}^{k}$, (k=N-1,N-2,...,1,0). Then $\{x(k),y(k)\}_{k=0}^{N-1}$ are defined inductively by the following two equations:

$$x(k) = \max \{ h_1(\underline{x}_k,\underline{y}_k) , h_2(\underline{X}_k) \} \tag{1}$$

$$1-y(k) = \max \{ h_3(\underline{x}_k,\underline{y}_k) , h_4(\underline{X}_k) \} \tag{2}$$

where h_i i=1,2,3,4 are defined below.

$$h_1(\underline{x}_k,\underline{y}_k) = e^{-r\Delta} \sum_{l=0}^{k} P_{kl}(\Delta) \, [(x(l)+y(l))/2] \tag{3}$$

$$h_3(\underline{x}_k,\underline{y}_k) = e^{-r\Delta} \sum_{l=0}^{k} P_{kl}(\Delta) \, [(1-x(l)+1-y(l))/2] \tag{4}$$

where $P_{kl}(\Delta)=\mathrm{Prob}[X(t+\Delta)=w_2|X(t)=w_1]$ $\forall t$, and where $w_1,w_2 \in W$, $|w_1|=k$ and $|w_2|=l$, i.e., the probability that k-l matches occur from time t to time $t+\Delta$. It is independent of t, since the search intensities are assumed to be independent of t (cf. Assumption 4) $\Rightarrow$ matching rates are independent of t $\Rightarrow$ parameters of the stochastic process are independent of t. $h_2(\underline{X}_k)=V_s(k+1)$ and $h_4(\underline{X}_k)=V_b(k+1)$, where $V_s(k+1)$ and $V_b(k+1)$ are defined inductively by the following equations:

$$V_s(k+1) = \frac{[X(k)+kV_s(k)][M-N+k+1]f_1^{k+1}(e_s(k+1),e_b(k+1))-c(e_s(k+1))}{r + [k+1][M-N+k+1]f_1^{k+1}(e_s(k+1),e_b(k+1))} \tag{5}$$

$$V_b(k+1)=\frac{[(1-X(k))+(M-N+k)V_b(k)][k+1]f_1^{k+1}(e_s(k+1),e_b(k+1))-c(e_b(k+1))}{r+[k+1][M-N+k+1]f_1^{k+1}(e_s(k+1),e_b(k+1))} \quad (6)$$

where $X(k)=\frac{1}{2}[x(k)+y(k)]$, $V_s(0)=V_b(0)=0$, and where $\{e_s(k+1),e_b(k+1)\}$ is the unique solution to the following equations:

$$[X(k)-V_s(k+1)]\left[\frac{\partial f_1^{k+1}(e_s(k+1),e_b(k+1))}{\partial e_s}-kf_2^{k+1}(e_s(k+1),e_b(k+1))\right]$$

$$[M-N+k+1]+[V_s(k)-V_s(k+1)]kf_2^{k+1}(e_s(k+1),e_b(k+1))=c'(e_s(k+1)) \quad (7)$$

$$[1-X(k)-V_b(k+1)]\left[\frac{\partial f_1^{k+1}(e_s(k+1),e_b(k+1))}{\partial e_b}-[M-N-k]f_3^{k+1}(e_s(k+1),e_b(k+1))\right]$$

$$[k+1]+[V_b(k)-V_b(k+1)][M-N+k]f_3^{k+1}(e_s(k+1),e_b(k+1))=c'(e_b(k+1)) \quad (8)$$

The method of proof is as follows. It is based on the Principle of Induction. Take any element of $\mathbb{N}$, say $\bar{M}$. Given this $\bar{M}\in\mathbb{N}$, we will firstly prove the Theorem for $(N,M)=(1,\bar{M}+1)$. We will, secondly, assume that the Theorem is true for $(N,M)=(L,\bar{M}+L)$, where $L\in\mathbb{N}$, $L\geq1$, and then show that the Theorem is true for $(N,M)=(L+1,\bar{M}+L+1)$. Then, by the Principle of Induction, the Theorem is true for $(N,M)=(n,\bar{M}+n)$, $\forall n\in\mathbb{N}$. Since $\bar{M}$ is any element of $\mathbb{N}$, we will have proved that the Theorem is true $\forall N,M\in\mathbb{N}$ such that $M>N$.

We now proceed to prove that the Theorem is true for $(N,M)=(1,\bar{M}+1)$, where $\bar{M}\in\mathbb{N}$.

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The method of proof is as follows. [illegible] the theorem [illegible]

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We now proceed to prove that the theorem is [illegible]

Define two sets A_s and A_b as follows.

A_s = { x : x is a SGPE payoff to the seller in a subgame starting with the seller's offer }

A_b = { x : x is a SGPE payoff to the seller in a subgame starting with some buyer's offer }

The sets A_s and A_b are the same for all buyers; this is because the buyers are identical, and have the same strategy set and same payoff function.

Let $m_i = \inf A_i$ and $M_i = \sup A_i$ for $i=b,s$. We firstly show that both $(x,y)=(M_b,M_s)$ and $(x,y)=(m_b,m_s)$ are solutions for the following system of equations:

$$x = \max \{ \tfrac{1}{2}[x+y]e^{-r\Delta}, F_s(X) \} \tag{9}$$

$$1-y = \max \{ \tfrac{1}{2}[1-x+1-y]e^{-r\Delta}, F_b(X) \} \tag{10}$$

where $X = \frac{1}{2}[x+y]$, and

$$F_s(X) = \frac{X[\bar{M}+1]\; f_1^1(e_s^*,e_b^*) - c(e_s^*)}{r + [\bar{M}+1]\; f_1^1(e_s^*,e_b^*)} \tag{11}$$

$$F_b(X) = \frac{[1-X]\; f_1^1(e_s^*,e_b^*) - c(e_b^*)}{r + [\bar{M}+1]\; f_1^1(e_s^*,e_b^*)} \tag{12}$$

where (e_s^*,e_b^*) is the unique solution to the following equations:

$$[X-F_s(X)][\bar{M}+1]\frac{\partial f_1^1(e_s^*,e_b^*)}{\partial e_s} = c'(e_s^*) \tag{13}$$

$$[1-X-F_b(X)][\frac{\partial f_1^1(e_s^*,e_b^*)}{\partial e_b} - \bar{M}f_3^1(e_s^*,e_b^*)] - \bar{M}\,F_b(X)f_3^1(e_s^*,e_b^*) = c'(e_b^*) \tag{14}$$

We secondly show that the system (9)-(10) has a unique solution, say (x^*,y^*). This, therefore, implies that $M_b=m_b$ and $M_s=m_s$. Hence, the sets A_s and A_b are singletons. Thus the equilibrium price is unique and independent of the buyer's identity; trade will occur instantaneously, between the seller, and the buyer who gets matched first with the seller.

We shall, in the process, also show that all buyers search with the same intensity, e_b^*, and that the seller searches with intensity e_s^*, where (e_s^*, e_b^*) is the unique solution to the two equations, (13) and (14), as defined above.

Lemmas 1-5, below, will prove that $(x,y)=(M_b,M_s)$ is a solution of (9) and (10).

LEMMA 1

If $x\in A_s$, $y\in A_b$ and $z=\max\{\frac{1}{2}[x+y]e^{-r\Delta}, F_s(X)\}$, where $F_s(X)$ is as defined by equation (11), then $z\in A_b$.

PROOF

Consider the following strategies in a subgame starting with a buyer's offer. The buyer offers price z, where z is as defined in the Lemma above, and the seller agrees to z and any price above it. If the seller deviates, and thus "rejects" or "rejects and searches", then all players follow equilibrium strategies that support the seller's payoff x or y, according to whether in the next bargaining round (whenever it will take place) it is the seller or some buyer who is

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selected to propose.

$z = \max\{\frac{1}{2}[x+y]e^{-r\Delta}, F_s(X)\}$, and thus the seller will not profit from either "rejecting" or "rejecting and searching"; if he "rejects", then he gets a payoff equal to $\frac{1}{2}[x+y]e^{-r\Delta}$, and if he "rejects and searches", then he gets a payoff equal to $F_s(X)$, where $F_s(X)$ is as defined by equation (11). (We will later on prove that $F_s(X)$ is indeed the payoff).

Suppose $F_s(X) > \frac{1}{2}[x+y]e^{-r\Delta}$. If the buyer offers less than z, which equals $F_s(X)$, then the seller will "reject and search". And thus the buyer will get a payoff equal to $F_b(X)$, where $F_b(X)$ is as defined in equation (12). (We will later on prove that $F_b(X)$ is indeed the payoff). Furthermore, we shall also prove later on that $1 \geq F_s(X) + F_b(X)$. Thus $1-z = 1-F_s(X) \geq F_b(X)$, and hence the buyer cannot profit from offering less than z.

Now suppose $F_s(X) \leq \frac{1}{2}[x+y]e^{-r\Delta}$. If the buyer offers less than z, which is equal to $\frac{1}{2}[x+y]e^{-r\Delta}$, then the seller will "reject". And thus the buyer will get a payoff equal to $[(1-x+1-y)/2]e^{-r\Delta}$. Since $1-z = 1-\frac{1}{2}[x+y]e^{-r\Delta} > [(1-x+1-y)/2]e^{-r\Delta}$, the buyer cannot profit by deviating.

Therefore the strategies are in equilibrium. The equilibrium payoff is z, and hence $z \in A_b$.

Q.E.D.

In the proof of Lemma 1, above, we made three claims (without proof): (i) and (ii) that $F_s(X)$ and $F_b(X)$ are the payoffs to the seller and a buyer, respectively, if the seller "rejects and searches" (i.e., leaves his partner in order to search for an alternative partner), and <u>given</u> that all players follow equilibrium strategies that support the seller's payoff x or y, according to whether in the next bargaining round (whenever it will take place) it is the seller or some buyer who is selected to propose, and (iii) that $F_s(X) + F_b(X) \leq 1$. To prove these claims at this stage would interrupt the flow of the current argument. We will therefore defer the proof of these claims till later.

LEMMA 2

$M_b=\max\{\frac{1}{2}[M_s+M_b]e^{-r\Delta},F_s(\hat{M})\}$, where $\hat{M}=\frac{1}{2}[M_s+M_b]$.

PROOF

By Lemma 1, for any $x\in A_s$ and for any $y\in A_b$, we have, $M_b\geq\max\{\frac{1}{2}[x+y]e^{-r\Delta},F_s(X)\}$, and hence, $M_b\geq\max\{\frac{1}{2}[M_s+M_b]e^{-r\Delta},F_s(\hat{M})\}$. We now show that

$$M_b\leq\max\{\frac{1}{2}[M_s+M_b]e^{-r\Delta},F_s(\hat{M})\} \tag{15}$$

Suppose the inequality defined in (15) is false. Then $\exists z_1\in A_b$ such that $M_b\geq z_1>\max\{\frac{1}{2}[M_s+M_b]e^{-r\Delta},F_s(\hat{M})\}$.

This means that there exists a perfect equilibrium in a subgame that starts with some buyer's offer such that the payoff to the seller in this equilibrium is z_1. However, if the buyer deviated and started this subgame by offering the seller the price equal to $\max\{\frac{1}{2}[M_s+M_b]e^{-r\Delta},F_s(\hat{M})\}$, then the perfectness implies that the seller will accept. Since the buyer strictly prefers this, he will indeed deviate profitably by offering a price equal to $\max\{\frac{1}{2}[M_s+M_b]e^{-r\Delta},F_s(\hat{M})\}$, and thus $\nexists z_1\in A_b$ such that $M_b\geq z_1>\max\{\frac{1}{2}[M_s+M_b]e^{-r\Delta},F_s(\hat{M})\}$. Hence the inequality defined by (15) must hold.

Q.E.D.

LEMMA 3

$1-M_s\leq\max\{\frac{1}{2}[1-M_s+1-M_b]e^{-r\Delta},F_b(\hat{M})\}$, where $\hat{M}=\frac{1}{2}[M_s+M_b]$ and $F_b(X)$ is as defined in equation (12).

PROOF

By arguments similar to those in the proof of Lemma 1, one can prove the following: if $x\in A_s$, $y\in A_b$ and $1-z=\max\{\frac{1}{2}[1-x+1-y]e^{-r\Delta},F_b(X)\}$, then $z\in A_s$.

Thus, for any $x\in A_s$ and for any $y\in A_b$, we have, $M_s\geq1-\max\{\frac{1}{2}[1-x+1-y]e^{-r\Delta},F_b(X)\}$. And hence, $M_s\geq1-\max\{\frac{1}{2}[1-M_s+1-M_b]e^{-r\Delta},F_b(\hat{M})\}$.

Q.E.D.

LEMMA 4

In all perfect equilibria in a subgame that starts with the seller's offer to a certain buyer, this buyer's payoff is at least $1-M_s$. In the perfect equilibria in a subgame that starts with a buyer's offer, the payoff to this buyer is at least $1-M_b$.

PROOF

Consider a subgame that starts with the seller offering to buyer i. Suppose that there exists a perfect equilibrium in this subgame in which buyer i gets $1-M_s-\varepsilon$. This perfect equilibrium must be such that there is no immediate agreement, for otherwise the seller's payoff would be above M_s in contradiction to the definition of M_s.

Let p be a price between M_s and $M_s+\varepsilon$, and consider the following candidate for an equilibrium in this subgame. The seller offers p and buyer i accepts it. If the seller demands more than p or if buyer i "rejects" or "rejects and searches", then all players continue as in the original perfect equilibrium. Thus, given our initial hypothesis, this is indeed a perfect equilibrium and the seller's payoff is $p>M_s$, in contradiction to the definition of M_s. Thus, the initial hypothesis is false and so in all perfect equilibria of this subgame buyer i's payoff is at least $1-M_s$.

The second statement of the lemma follows immediately from the definition of M_b.

Q.E.D.

LEMMA 5

$1-M_s=\max\{\frac{1}{2}[1-M_s+1-M_b]e^{-r\Delta},F_b(\hat{M})\}$ where $\hat{M}=\frac{1}{2}[M_s+M_b]$.

PROOF

Given Lemma 3 it is sufficient to show that

$$1-M_s\geq\max\{\tfrac{1}{2}[1-M_s+1-M_b]e^{-r\Delta},F_b(\hat{M})\} \tag{16}$$

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Assume (16) does not hold. Then $\exists z_2 \in A_s$ such that $M_s \geq z_2 > 1-\max\{\frac{1}{2}[1-M_s+1-M_b]e^{-r\Delta}, F_b(\hat{M})\}$, i.e., $1-M_s \leq 1-z_2 < \max\{\frac{1}{2}[1-M_s+1-M_b]e^{-r\Delta}, F_b(\hat{M})\}$.

The $\max\{\frac{1}{2}[1-M_s+1-M_b]e^{-r\Delta}, F_b(\hat{M})\}$ is the minimum payoff guaranteed to any buyer, if he "rejects" or "rejects and searches" (to be shown below). Thus, in equilibrium, no buyer will offer or accept price p such that 1-p is smaller than the RHS of the inequality, defined in (16), and so $\nexists z_2 \in A_s$.

If a buyer "rejects" he gets at least $1-M_s$ or $1-M_b$ according to the selection of the proposer (cf. Lemma 4); thus his (discounted) payoff is $\frac{1}{2}[1-M_s+1-M_b]e^{-r\Delta}$. If he "rejects and searches", then he gets a payoff equal to $F_b(\hat{M})$.

Q.E.D.

Lemmas 1-5 prove that $(x,y)=(M_b,M_s)$ is a solution for the system of equations defined by (9) and (10).

By similar arguments one can show that $(x,y)=(m_b,m_s)$ is a solution of (9) and (10).

Before we prove that (9) and (10) have a unique solution, we shall prove the claims made in the Lemmas on $F_s(X)$ and $F_b(X)$, where $X=\frac{1}{2}[x+y]$: (i) and (ii) that $F_s(X)$ and $F_b(X)$ are the payoffs to the seller and a buyer, respectively, if no one is matched, and given that all players follow equilibrium strategies that support the seller's payoff x or y, according to whether in the next bargaining round (whenever it will take place) it is the seller or some buyer who is selected to propose ($x \in A_s$ and $y \in A_b$), and (iii) that $F_s(X)+F_b(X) \leq 1$.

Suppose at time t no one is matched. And suppose that when the seller is matched with some buyer, the players follow equilibrium strategies that support the seller's payoff x or y according to whether it is the seller or some buyer who is selected to propose ($x \in A_s$ and $y \in A_b$). What is the equilibrium payoff to the seller and buyer k (for $k=1,\ldots,\bar{M}+1$), as of time t?

Let $V_s(t)$, $\{V_b^k(t)\}_{k=1}^{\bar{M}+1}$ denote the equilibrium payoffs. These equilibrium payoffs are determined by the equilibrium search strategy choices.

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$$V_s(t) = \max_{e_s \geq 0} \left[\int_t^\infty [Xe^{-r(u-t)} - (\int_t^u c(e_s) e^{-r(q-t)} dq)] \lambda e^{-\lambda(u-t)} du \right] \tag{17}$$

For $k=1,2,\ldots,\bar{M}+1$,

$$V_b^k(t) = \max_{e_b^k \geq 0} \left[\int_t^\infty [(\lambda_k/\lambda) (1-X) e^{-r(u-t)} + (1 - (\lambda_k/\lambda)) \cdot 0 \right.$$

$$\left. - (\int_t^u c(e_b^k) e^{-r(q-t)} dq)] \lambda e^{-\lambda(u-t)} du \right] \tag{18}$$

where $X=\frac{1}{2}[x+y]$, $\lambda_k=\lambda_{sk}^w=F^1(e_s, e_b^k, \underline{e}_{sk}^w)$, the rate at which the seller (denoted by s) and buyer k are matched, where w denotes the set of unmatched agents (i.e., the seller and the $\bar{M}+1$ buyers), $|w|=1$ and $\underline{e}_{sk}^w = \{e_b^l : b_l \in w, l \neq k\}$;

$$\lambda = \sum_{k=1}^{\bar{M}+1} \lambda_k,$$

the rate at which the seller is matched with some buyer (i.e., the rate at which a match forms, since there is only one seller).

We note that the search intensities, e_s, $\{e_b^k\}_{k=1}^{\bar{M}+1}$, are independent of time, and depend only on the state w (cf. Assumption 4). This implies that λ_k (for $k=1,\ldots,\bar{M}+1$) is independent of time. With this observation we can simplify equations (17) and (18). We obtain (note that the equilibrium payoffs are therefore independent of t):

$$V_s = \max_{e_s \geq 0} P_s(e_s, \underline{e}_b) \tag{17a}$$

and for $k=1,\ldots,\bar{M}+1$,

$$V_b^k = \max_{e_b^k \geq 0} P_b^k(e_b^k, \underline{e}_b^k, e_s) \tag{18a}$$

where $\underline{e}_b = \{e_b^l\}_{l=1}^{\bar{M}+1}$, $\underline{e}_b^k = \{e_b^l\}_{l \neq k}$, and P_s, P_b^k are defined below:

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$$P_s(e_s,\underline{e}_b) = \frac{\lambda X - c(e_s)}{r+\lambda}, \text{ and } P_b^k(e_b^k,\underline{e}_b^k,e_s) = \frac{[1-X]\ \lambda_k - c(e_b^k)}{r+\lambda}.$$

A joint search strategy, $\hat{\underline{e}} = (\hat{\underline{e}}_s, \{\hat{e}_b^k\}_{k=1}^{\bar{M}+1})$, is a Nash equilibrium of the search game if and only if $\hat{\underline{e}}$ solves equations (17a) and (18a).

We shall make the following assumption (Assumption 6, below), which will ensure that if there exists a solution to equations (17a) and (18a), then it is an interior solution (i.e., $\hat{\underline{e}}>\underline{0}$).

Assumption 6

(a) $$\frac{\partial P_s(0,\underline{e}_b)}{\partial e_s} > 0, \quad \forall \underline{e}_b \in \mathbb{R}_+^{\bar{M}+1}$$

(b) $$\frac{\partial P_b^k(0,\underline{e}_b^k,e_s)}{\partial e_b^k} > 0, \quad \forall \underline{e}_b^k \in \mathbb{R}_+^{\bar{M}}, e_s \in \mathbb{R}_+.$$

PROPOSITION 1

There exists a unique solution to equations (17a) and (18a) (denote it by $\hat{\underline{e}}$); with the property that $\hat{e}_b^k = e_b^*$ for $k=1,\ldots,\bar{M}+1$ (i.e., the Nash equilibrium is symmetric).

The proof consists of three claims. Claim 1 shows that there exists a solution to equations (17a) and (18a). Given Assumption 6 the solution(s) are interior. Using this fact, (i) Claim 2 shows that all Nash equilibria are symmetric, and (ii) Claim 3 shows that there exists a unique symmetric Nash equilibrium.

Claim 1

There exists a solution to equations (17a) and (18a) (i.e., there exists a Nash equilibrium).

Proof

The existence of a Nash equilibrium will be proved by appealing to the classical theorem on existence of (pure-strategy) Nash equilibria, (see Debreu (1952)). Thus all we have to do is to verify that the conditions are met.

(1) The strategy sets $\{e_s : e_s \in \mathbb{R}_+\}$ and $\{e_b^k : e_b^k \in \mathbb{R}_+\}$ $(k=1,\ldots,\bar{M}+1)$ are convex. But, they are are not compact. However, we will show below that there exists real positive scalars, Q_s, $Q_b^k (k=1,\ldots,\bar{M}+1)$, such that the strategy sets can be restricted. The new strategy sets will be $\{e_s : e_s \in [0,Q_s]\}$ and $\{e_b^k : e_b^k \in [0,Q_b^k]\}$ $(k=1,\ldots,\bar{M}+1)$. These are both convex and compact.

(2) The payoff functions, P_s, $P_b^k (k=1,\ldots,\bar{M}+1)$, are well defined, continuous and bounded; this follows from the properties of the functions F^1 and c. We will show that P_s and P_b^k $(k=1,\ldots,\bar{M}+1)$ are strictly concave in e_s and e_b^k $(k=1,\ldots,\bar{M}+1)$, respectively.

Thus the conditions will be met.

$$\frac{\partial P_s}{\partial e_s} = \frac{\left[X \dfrac{\partial \lambda}{\partial e_s} - c'(e_s)\right][\lambda + r] - [\lambda X - c(e_s)]\dfrac{\partial \lambda}{\partial e_s}}{[\lambda + r]^2} \tag{19}$$

$$\frac{\partial^2 P_s}{\partial e_s^2} = \frac{1}{[\lambda+r]^2}\left[\, rX \frac{\partial^2 \lambda}{\partial e_s^2} - [\lambda+r]c''(e_s) + \frac{\partial^2 \lambda}{\partial e_s^2}[c(e_s)]\right]$$

$$- \frac{2}{[\lambda+r]^3}\frac{\partial \lambda}{\partial e_s}\left[\, rX \frac{\partial \lambda}{\partial e_s} - [\lambda+r]c'(e_s) + \frac{\partial \lambda}{\partial e_s}[c(e_s)]\right] \tag{20}$$

Given our assumptions on the functions F^1 and c, it is not clear whether

$$\frac{\partial^2 P_s}{\partial e_s^2} < 0.$$

We shall now prove the following statement:

> If there exists a stationary point (for the function P_s, in the interior of $\mathbb{R}_+$), then it is unique and it is a local maximum. —— (I)

It will then become clear that P_s is indeed strictly concave in e_s.

If there exists any stationary points (i.e., local maxima or local minima or saddle points) in the interior of $\mathbb{R}_+$, then it must be the case:

$$\frac{\partial P_s(\hat{e}_s)}{\partial e_s} = 0, \text{ where } \hat{e}_s \text{ is such a stationary point.}$$

$$\Leftrightarrow [\ rX \frac{\partial \lambda(\hat{e}_s)}{\partial e_s} - [\lambda(\hat{e}_s)+r]c'(\hat{e}_s) + \frac{\partial \lambda(\hat{e}_s)}{\partial e_s}[c(\hat{e}_s)]\ ] = 0 \tag{20a}$$

(using equation (19)).

Using equation (20), we have

$$\frac{\partial^2 P_s(\hat{e}_s)}{\partial e_s^2} = \frac{1}{[\lambda+r]^2}[\ rX \frac{\partial^2 \lambda}{\partial e_s^2} - [\lambda+r]c''(\hat{e}_s) + \frac{\partial^2 \lambda}{\partial e_s^2}[c(\hat{e}_s)]\ ],$$

for any stationary point. Note that the second term of the second derivative becomes zero, using equation (20a). Given our assumptions

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using equation (20), we have

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at any stationary point. Note that the second term of the second derivative becomes zero using equation (20a), given our assumptions on the functions c and F^1(c is strictly convex and F^1 is concave in e_s), we have

$$\frac{\partial^2 P_s(\hat{e}_s)}{\partial e_s^2} < 0.$$

Thus all stationary points are local maxima. But since P_s is continuous in e_s, between any two local maxima there must be a local minimum. Hence, if there exists a stationary point, it is a unique local maximum. This concludes the proof of statement (I), above.

Now note that (a) P_s is continuous in e_s and (b) $P_s \to -\infty$ as $e_s \to +\infty$, $\forall \underline{e}_b \in \mathbb{R}_+^{\bar{M}+1}$. Statement (I) together with (a) and (b) imply that P_s is strictly concave in e_s.

Since $\forall \underline{e}_b$, $P_s \to -\infty$ as $e_s \to +\infty$, there exists $Q_s \in \mathbb{R}_+$ such that $\forall \underline{e}_b$ the unique optimal choice of the seller is less than or equal to Q_s.

One can similarly prove that P^k is strictly concave in e_b^k, and show the existence of $Q_b^k \in \mathbb{R}_+$.

Q.E.D.

Claim 2

Let $\underline{e}$ be an interior solution of equations (17a) and (18a). Then, $e_b^k = e_b$ for $k=1,\ldots,\bar{M}+1$ (i.e., all Nash equilibria are symmetric).

Proof

Since $\underline{e}$ is an interior solution, $\underline{e}$ satisfies the first-order conditions with equality. Thus, using equations (17a) and (18a), we obtain:

$$(X-V_s) \sum_{j=1}^{\bar{M}+1} \frac{\partial F^1(e_s, e_b^j, \underline{e}_{s\,j}^w)}{\partial e_s} = c'(e_s) \qquad (21)$$

and for $k=1,...,\bar{M}+1$,

$$[1-X-V_b^k]\frac{\partial F^1(e_s, e_b^k, \underline{e}_{sk}^w)}{\partial e_b^k} - V_b^k \sum_{\substack{j\neq k\\ j=1}}^{\bar{M}+1} \frac{\partial F^1(e_s, e_b^j, \underline{e}_{sj}^w)}{\partial e_b^k} = c'(e_b^k) \qquad (22)$$

We will show below that, for a given arbitrary vector of equilibrium expected utility payoffs $(V_s, \{V_b^k\}_{k=1}^{\bar{M}+1})$, there exists a unique solution to (21) and (22), say $\underline{e}$.

Thus, $e_s = E_1(V_s, \underline{V}_b)$ and for $k=1,...,\bar{M}+1$

$$e_b^k = E_2(V_s, V_b^k, \underline{V}_b^k) \qquad (23)$$

where E_1 and E_2 are functions.

The equilibrium expected utility payoffs must be the same for all buyers (i.e., $V_b^k = V_b$ for $k=1,...,\bar{M}+1$). Otherwise a buyer with a lower payoff could imitate a buyer with a higher payoff; this is possible since all buyers are identical. This in itself does not imply that the buyers will choose the same strategy; it is possible that they obtain the same payoff by choosing different strategies.

However, using the fact that $V_b^k = V_b$ for all k, with equation (23), we obtain that $e_b^k = e_b$ for $k=1,...,\bar{M}+1$.

We now prove the existence of a unique solution $\underline{e}$ to (21) and (22), given an arbitrary vector of equilibrium expected utility payoffs $(V_s, \{V_b^k\}_{k=1}^{\bar{M}+1})$.

We know that a solution $\underline{e}$ ($\underline{e} > \underline{0}$) exists. In order to prove the uniqueness of $\underline{e}$, we shall appeal to the Gale-Nakaido Univalence Theorem. So we have to show that the Jacobian of the system (21)-(22) is negative quasi-definite for all $\underline{e} \in \mathbb{R}_{++}^{\bar{M}+2}$.

Let the Jacobian J of the system (21)-(22) with respect to $\underline{e}$ at any $\underline{e} \in \mathbb{R}_{++}^{\bar{M}+2}$ be denoted by $J=[a_{ij}]_{i,j}$. Using equations (21) and (22) one computes a_{ij} for all i and j. And then, using the assumptions on the functions F^1 (cf. Assumption 5) and c, one obtains that $a_{ij} < 0$ for all i=j and $a_{ij} = 0$ for all $i \neq j$. Thus J is negative quasi-definite.

Q.E.D.

Claim 3

There exists a unique symmetric Nash equilibrium.

Proof

Let (e_s^*, e_b^*) be a symmetric Nash equilibrium; $e_s^*>0$ and $e_b^*>0$. Existence is guaranteed by Claims 1 and 2.

Imposing symmetry on equations (17a), (18a), (21) and (22), we obtain:

$$rV_s = (X-V_s)(\bar{M}+1)f_1^1(e_s^*,e_b^*) - c(e_s^*) \tag{24}$$

$$rV_b = [1-X-(\bar{M}+1)V_b]f_1^1(e_s^*,e_b^*) - c(e_b^*) \tag{25}$$

$$[X-V_s][\bar{M}+1]\frac{\partial f_1^1(e_s^*,e_b^*)}{\partial e_s} = c'(e_s^*) \tag{26}$$

$$[1-X-V_b]\left[\frac{\partial f_1^1(e_s^*, e_b^*)}{\partial e_b} - \bar{M}f_3^1(e_s^*, e_b^*)\right] - \bar{M}\, V_b\, f_3^1(e_s^*, e_b^*) = c'(e_b^*) \tag{27}$$

Firstly, using the Gale-Nakaido Univalence Theorem one obtains that, for a given arbitrary vector (V_s,V_b) of equilibrium expected utility payoffs, there exists a unique solution to (26) and (27), say (e_s^*,e_b^*). And thus, there exist functions E_1 and E_2, where

$$\begin{aligned} e_s^* &= E_1(V_s,V_b) \\ e_b^* &= E_2(V_s,V_b) \end{aligned} \tag{28}$$

Secondly, using the Implicit Function Theorem one obtains that, (i) equation (24) defines V_s as a function (say E_s) of V_b, and (ii) equation (25) defines V_b as a function (say E_b) of V_s. Thus,

$$
\begin{aligned}
V_s &= E_s(V_b) \\
V_b &= E_b(V_s)
\end{aligned}
\tag{29}
$$

$\Rightarrow$ $V_s=E_s[E_b(V_s)]$. Let $E(\cdot)=E_s(E_b(\cdot))$, and thus $V_s=E(V_s)$, where $E:[0,1]\rightarrow[0,1]$.

Below we shall prove: (i) $E'(V_s)>0$ (i.e., E is strictly increasing), (ii) $E''(V_s)>0$ (i.e., E is strictly convex), (iii) $E(1)<1$, and (iv) E is continuous. Thus, we obtain that E has a unique fixed point, which in turn implies that there exists a unique symmetric Nash equilibrium.

(i) $E'(V_s)=E'_s(V_b).E'_b(V_s)$. Using equations (24)-(29) one computes $E'_s(V_b)$ and $E'_b(V_s)$. And then, using the assumptions on functions F^1 (cf. Assumption 5) and c, one obtains that $E'_s(V_b)<0$ and $E'_b(V_s)<0$. Thus $E'(V_s)>0$. (ii) $E''(V_s)=E'_s(V_b).E''_b(V_s)+E''_s(V_b).E'_b(V_s)$. In (i) we have computed $E'_s(V_b)$ and $E'_b(V_s)$, and therefore, by differentiating, one computes $E''_s(V_b)$ and $E''_b(V_s)$. And then, using the assumptions on the functions F^1 (cf. Assumption 5) and c, one obtains that $E''_b(V_s)<0$ and $E''_s(V_b)<0$. Thus $E''(V_s)>0$. (iii) Suppose $E(1)\not<1$, i.e., suppose $E(1)=1$. Then $V_s=1$ would be an equilibrium expected payoff to the seller. Now $V_s\leq X$ and $X\leq 1$. But if $X=1$, then $V_s<1$. Thus, for all X, $V_s<1$, i.e., a contradiction. (iv) E is continuous. This follows since E_s and E_b are continuous.

Q.E.D.

Hence, we have proved that there exists a unique interior Nash equilibrium for the game of search, with the property that all buyers choose the same strategy. Let the strategy choices be denoted by (e^*_s,e^*_b), and the associated payoffs be denoted by (V_s,V_b). (e^*_s,e^*_b,V_s,V_b) is the unique solution to the equations (24), (25), (26) and (27).

Equations (11), (12), (13) and (14) are equations (24), (25), (26) and (27), respectively, with $F_s(X)=V_s$ and $F_b(X)=V_b$. Hence, we have proved claims (i) and (ii) made in Lemmas 1-5 on $F_s(X)$ and $F_b(X)$.

Claim (iii) (i.e., $F_s(X)+F_b(X)\leq 1$) follows from the fact that $V_s\leq X$ and $V_b\leq 1-X$.

All we have to do now is to prove that equations (9) and (10) have a unique solution. (9) and (10) can be rewritten in the following compact notation:

$$\underline{x} = h(\underline{x}) \tag{30}$$

where $\underline{x}=[x,y]$ and $h(\underline{x})=[h_1(\underline{x}),h_2(\underline{x})]$, with $h_1(\underline{x})=\max(\frac{1}{2}[x+y]e^{-r\Delta}, F_s(X))$ and $h_2(\underline{x})=1-\max(\frac{1}{2}[x+y]e^{-r\Delta}, F_b(X))$, where $F_s(X)$ and $F_b(X)$ are defined by equations (11) and (12), respectively.

We shall first use the Brouwer Fixed Point Theorem to prove that h has at least one fixed point (i.e., existence). We then use the (Banach) Contraction Fixed Point Theorem to prove that h has precisely one fixed point (i.e., uniqueness).

LEMMA 6 (Existence)

h has at least one fixed point.

PROOF

(i) $\forall(x,y)\in[0,1]x[0,1]$, $\frac{1}{2}[x+y]e^{-r\Delta}\in[0,1]$ and $\frac{1}{2}[1-x+1-y]e^{-r\Delta}\in[0,1]$.
(ii) $\forall(x,y)\in[0,1]x[0,1]$, $X=\frac{1}{2}[x+y]e^{-r\Delta}\in[0,1]$. $\forall X\in[0,1]$, $F_s(X)\in[0,1]$ and $F_b(X)\in[0,1]$. This is because $0\leq F_s(X)\leq X$, and $0\leq F_b(X)\leq(1-X)/M$.

From (i) and (ii) we obtain that $h:[0,1]x[0,1]\rightarrow[0,1]x[0,1]$. Since $F_s$and F_b are continuous functions of X, we have that h is continous on [0,1]x[0,1]. Therefore, by the Brouwer Fixed Point Theorem, h has at least one fixed point.

Q.E.D.

LEMMA 7 (Uniqueness)

h has one and only one fixed point.

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PROOF

$h:[0,1]\times[0,1]\rightarrow[0,1]\times[0,1]$. By Lemma 6 h has at least one fixed point. Below we shall show that h is a Contraction. Thus by the (Banach) Contraction Fixed Point Theorem h has one and only one fixed point.

Definition: h is a contraction if $\exists\lambda\in[0,1]$ such that $\forall \underline{x}_1,\underline{x}_2\in[0,1]\times[0,1]$, $d[h(\underline{x}_1),h(\underline{x}_2)]\leq\lambda d[\underline{x}_1,\underline{x}_2]$, where d is the Euclidean metric.

Since h is differentiable the contraction condition in the definition, above, is equivalent to:

(a) $|\ \partial h_1(\underline{x})/\partial x\ |\ +\ |\ \partial h_1(\underline{x})/\partial y\ |\ \leq\ \lambda$ (31)

and

(b) $|\ \partial h_2(\underline{x})/\partial x\ |\ +\ |\ \partial h_2(\underline{x})/\partial y\ |\ \leq\ \lambda$ (32)

where $h_1(\underline{x})=\max\{\frac{1}{2}[x+y]e^{-r\Delta},\ F_s(X)\}$ and $h_2(\underline{x})=1-\max\{\frac{1}{2}[x+y]e^{-r\Delta},\ F_b(X)\}$ and where $X=\frac{1}{2}[x+y]$.

Below we shall prove that each of the following four functions is a contraction: (i) $\frac{1}{2}[x+y]e^{-r\Delta}$, (ii) $F_s(X)$, (iii) $1-\frac{1}{2}[1-x+1-y]e^{-r\Delta}$ and (iv) $1-F_b(X)$. Thus $\exists$ positive scalars λ_1, λ_2, λ_3 and λ_4, all strictly less than 1, such that

$$|\ \partial[\tfrac{1}{2}(x+y)e^{-r\Delta}]/\partial x\ |\ +\ |\ \partial[\tfrac{1}{2}(x+y)e^{-r\Delta}]/\partial y\ |\ \leq\ \lambda_1 \tag{33}$$

$$|\ dF_s(X)/dX\ |\ \leq\ \lambda_2 \tag{34}$$

$$|\ \partial[1-\tfrac{1}{2}(1-x+1-y)e^{-r\Delta}]/\partial x\ |\ +\ |\ \partial[1-\tfrac{1}{2}(1-x+1-y)e^{-r\Delta}]/\partial y\ |\ \leq\ \lambda_3 \tag{35}$$

$$|\ \partial[1-F_b(X)]/\partial X\ |\ \leq\ \lambda_4 \tag{36}$$

Now suppose $h_1(\underline{x})=F_s(X)$, i.e., suppose $\max\{\frac{1}{2}[x+y]e^{-r\Delta},\ F_s(X)\}=F_s(X)$. Then, $|\ \partial h_1(\underline{x})/\partial x\ |\ +\ |\ \partial h_1(\underline{x})/\partial y\ |\ =\ |\ [dF_s(X)/dX](1/2)\ |\ +\ |\ [dF_s(X)/dX](1/2)\ |\ =\ |\ dF_s(X)/dX\ |\ \leq\ \lambda_2$, using (34), and since $X=\frac{1}{2}[x+y]$. Similarly, suppose $h_2(\underline{x})=1-F_b(X)$. Then, $|\ \partial h_2(\underline{x})/\partial x\ |\ +\ |\ \partial h_2(\underline{x})\partial y\ |\ =\ |\ d[1-F_b(X)]/dX\ |\ \leq\ \lambda_4$, using (36).

Let $\hat{\lambda}=\max\{\lambda_1,\lambda_2,\lambda_3,\lambda_4\}$. $0<\hat{\lambda}<1$, since $0<\lambda_i<1$ for all i, where i=1,2,3,4.

It is clear that there exists a λ, namely $\lambda=\hat{\lambda}$, such that (31) and (32) are satisfied. Thus h is a contraction.

All we need to do now is to verify that $\exists\ \lambda_1,\lambda_2,\lambda_3,\lambda_4$, where $0<\lambda_i<1$ for i=1,2,3,4, s.t. (33)-(36) are satisfied.

(a) $|\ \partial[\frac{1}{2}(x+y)e^{-r\Delta}]/\partial x\ |+|\ \partial[\frac{1}{2}(x+y)e^{-r\Delta}]/\partial y\ | =e^{-r\Delta}<1$, since $\Delta>0$ and $r>0$.

Choose $\lambda_1=[1+e^{-r\Delta}]/2$; and thus, $0<\lambda_1<1$.

(b) Similarly choose $\lambda_3=[1+e^{-r\Delta}]/2$; and thus, $0<\lambda_3<1$.

(c) $F_s(X)\leq X$, $\forall X\in[0,1]$, with strict inequality if $X>0$. $F_s:[0,1]\rightarrow[0,1]$ with $F_s(0)=0$.

Let $\alpha_1=\sup\{[F_s(X)/X]:X\in[0,1]\}$. Now $[F_s(X)/X]<1$, $\forall X>0$.

Therefore $\alpha_1<1$. $\forall X\neq 0$, $[F_s(X)/X]<\alpha_1$, i.e., $F_s(X)-\alpha_1 X<0$. For $X=0$, $F_s(X)=0=X$, i.e., $F_s(X)-\alpha_1 X=0$.

Therefore,

$$F_s(X)-\alpha_1 X\leq 0,\ \forall X\in[0,1] \tag{37}$$

where α_1 is a positive scalar strictly less than one.

Let X_1,X_2 be any elements of $[0,1]$. Then $[F_s(X_i)-\alpha_1 X_i][F_s(X_i)+\alpha_1 X_i]\leq 0$ for i=1,2, using (37); i.e., $[F_s(X_i)]^2-\alpha_1^2 X_i^2\leq 0$ for i=1,2. Thus $[F_s(X_1)]^2+[F_s(X_2)]^2\leq\alpha_1^2[X_1^2+X_2^2]$, $\forall X_1,X_2\in[0,1]$, i.e., $\sqrt{[F_s(X_1)]^2+[F_s(X_2)]^2}\leq\alpha_1\sqrt{X_1^2+X_2^2}$.

Hence $\forall X_1,X_2\in[0,1]$ $\exists$ a positive real scalar strictly less than 1, namely α_1, where $\alpha_1=\sup\{F_s(X)/X:X\in(0,1]\}$, such that $d[F_s(X_1),F_b(X_2)]\leq\alpha_1 d[X_1,X_2]$. Thus $\lambda_2=\alpha_1$.

(d) the existence of λ_4 can be proved using an argument similar to the one used above ((c)) to prove the existence of λ_2.

Q.E.D.

It is easy to check that we have indeed proved that the Theorem is true for $(N,M)=(1,\bar{M}+1)$. We now proceed to the second stage of the proof: assume that the Theorem is true for $(N,M)=(L,\bar{M}+L)$, where $L\in\mathbb{N}$ and $L\geq 1$, and then show that the Theorem is true for $(N,M)=(L+1,\bar{M}+L+1)$.

Define two sets B_s and B_b as follows.

B_s = { x : x is a SGPE payoff to a seller in a subgame starting with the seller's offer, when the seller is matched with some buyer, and all the other L sellers and $\bar{M}+L$ buyers are unmatched }

B_b = { x : is a SGPE payoff to a seller in a subgame starting with some buyer's offer, when the seller is matched with some buyer, and all the other L sellers and $\bar{M}+L$ buyers are unmatched }

REMARK: Since (i) the sellers are identical, having identical strategy sets and payoff functions, and (ii) the buyers are identical, having identical strategy sets and payoff functions, the set B_s and B_b are independent of the particular seller and the particular buyer who are matched.

Let $n_i=\inf B_i$ and $N_i=\sup B_i$ for $i=b,s$. We firstly show that both $(x,y)=(N_b,N_s)$ and $(x,y)=(n_b,n_s)$ are solutions for the following system of equations:

$$x = \max \{ J_1(x,y),\ J_2(X) \} \tag{38}$$

$$1-y = \max \{ J_3(x,y),\ J_4(X) \} \tag{39}$$

where $X=\frac{1}{2}[x+y]$ and J_i for $i=1,2,3,4$ are defined below.

$$J_1(x,y)=(1/2)[x+y]e^{-r\Delta}P_{LL}(\Delta)+e^{-r\Delta}\sum_{l=0}^{L-1} P_{Ll}(\Delta)\ (1/2)[x(l)+y(l)] \tag{40}$$

$$J_3(x,y)=(1/2)[1-x+1-y]e^{-r\Delta}P_{LL}(\Delta)+e^{-r\Delta}\sum_{l=0}^{L-1} P_{Ll}(\Delta)\ (1/2)[1-x(l)+1-y(l)] \tag{41}$$

where $\{x(l),y(l)\}_{l=0}^{L-1}$, and $\{P_{Ll}(\Delta)\}_{l=0}^{L}$ are defined in the Theorem (part(d)).

$J_2(X)$ and $J_4(X)$ are defined below:

$$J_2(X) = \frac{[X+LD_s][\bar{M}+L+1]f_1^{L+1}(e_s^*,e_b^*) - c(e_s^*)}{r + [L+1]\ [\bar{M}+L+1]f_1^{L+1}(e_s^*,e_b^*)} \tag{42}$$

$$J_4(X) = \frac{[1-X+D_b[\bar{M}+L]][L+1]f_1^{L+1}(e_s^*,e_b^*) - c(e_b^*)}{r + [L+1]\ [\bar{M}+L+1]f_1^{L+1}(e_s^*,e_b^*)} \tag{43}$$

where $D_s=V_s(L)$, $D_b=V_b(L)$; $V_s(L)$ and $V_b(L)$ are defined in the Theorem (part (d)), $X=\frac{1}{2}(x+y)$, and where (e_s^*,e_b^*) is the unique solution to the following two equations:

$$[X-J_2(X)]\left[\frac{\partial f_1^{L+1}(e_s^*,e_b^*)}{\partial e_s} - Lf_2^{L+1}(e_s^*,e_b^*)\right][\bar{M}+L+1]$$

$$+\ [D_s-J_2(X)]Lf_2^{L+1}(e_s^*,e_b^*) = c'(e_s^*) \tag{44}$$

$$[1-X-J_4(X)]\left[\frac{\partial f_1^{L+1}(e_s^*,e_b^*)}{\partial e_b} - [\bar{M}+L]f_3^{L+1}(e_s^*,e_b^*)\right][L+1]$$

$$+\ [D_b-J_4(X)][\bar{M}+L]f_3^{L+1}(e_s^*,e_b^*) = c'(e_b^*) \tag{45}$$

We secondly show that the system (38)-(39) has a unique solution, say (x^*,y^*). This, therefore, implies that $N_b=n_b$ and $N_s=n_s$. Hence, the sets B_s and B_b are singletons. Thus the equilibrium price for the first matched pair is unique and independent of the particular seller and the particular buyer who get matched first; trade will occur instantaneously between the seller and the buyer who get matched first.

We shall, in the process, also show that up until the first pair get matched, all buyers search with the same intensity, e_b^*, all sellers search with the same intensity, e_s^*, where (e_s^*, e_b^*) is the unique solution to the equations (44) and (45).

Once the first matched pair leave the market there are then $(N,M)=(L,\bar{M}+L)$ agents. The Theorem is assumed true for this. Thus, we will have proved that the Theorem is true for $(N,M)=(L+1,\bar{M}+L+1)$.

By arguments similar to those presented in Lemmas 1-5 one can show that both $(x,y)=(N_b,N_s)$ and $(x,y)=(n_b,n_s)$ are solutions for the system of equations defined by (38)-(39). In these arguments one would have to prove that $J_2(X)$ and $J_4(X)$ are the payoffs to a seller and a buyer, respectively, if none of the L+1 sellers are matched (which implies that none of the $\bar{M}+L+1$ buyers are matched), and given that all players follow equilibrium strategies that support a seller's payoff x or y according to whether in the next bargaining round (whenever it takes place, and whomsoever are the matched pair) it is the seller or the buyer who is selected to propose; $x \in B_s$ and $y \in B_b$. We will prove this. Then we shall prove that (38) and (39) have a unique solution.

Suppose at time t all L+1 sellers are unmatched (and therefore, all $\bar{M}+L+1$ buyers are unmatched). And suppose that when the first match occurs, the players follow equilibrium strategies that support the seller's payoff x or y according to whether it is the seller or the buyer who is selected to propose ($x \in B_s$ and $y \in B_b$). What is the equilibrium payoff to a seller l (for $l=1,\ldots,L+1$) and to a buyer k (for $k=1,\ldots,\bar{M}+L+1$), as of time t?

Let $\{V_s^l(t)\}_{l=1}^{L+1}$, $\{V_b^k(t)\}_{k=1}^{\bar{M}+L+1}$ denote the equilibrium payoffs. These payoffs are determined by the equilibrium search strategy choices.

For $l=1,\ldots,L+1$

$$V_s^l(t) = \max_{e_s^l \geq 0} [\int_t^\infty [(\lambda_l^s/\lambda)\, xe^{-r(u-t)} + \sum_{\substack{i=1 \\ i \neq l}}^{L+1} \sum_{j=1}^{\bar{M}+L+1} (\lambda_{ij}^w/\lambda)\, V_s^l(u;i,j)e^{-r(u-t)}$$

$$- (\int_t^u c(e_s^l)e^{-r(q-t)}dq)]\lambda e^{-\lambda(u-t)}du] \tag{46}$$

where $X=\frac{1}{2}[x+y]$, $\lambda_{ij}^{w}=F^{L+1}(e_s^i, e_b^j, \underline{e}_{ij}^{w})$, the rate at which seller i and buyer j get matched, where w denotes the set of unmatched agents (i.e., the L+1 sellers and $\bar{M}$+L+1 buyers), $|w|$ = L+1; $\lambda = [\sum_{i=1}^{L+1}\sum_{j=1}^{\bar{M}+L+1}\lambda_{ij}^{w}]$ the rate at which a match occurs, $\lambda_l^s=\sum_{j=1}^{\bar{M}+L+1}\lambda_{lj}^{w}$, the rate at which seller l gets matched. $V_s^l(u;i,j)$ is the equilibrium expected utility payoff to seller l, if he is not part of the first matched pair but the first matched pair is seller i and buyer j, and the match occurs at time u. Since the Theorem is true for (N,M)=(L,$\bar{M}$+L), we have $V_s^l(u;i,j)=V_s(L)$, i.e., independent of time and independent of i and j, dependent only on L, and independent of l.

For k=1,...,$\bar{M}$+L+1

$$V_b^k(t)=\max_{e_b^k\geq 0}[\int_t^{\infty}[\;(\lambda_k^b/\lambda)\quad(1-X)e^{-r(u-t)}+\;(1-\;(\lambda_k^b/\lambda)\;)V_b(L)e^{-r(u-t)}$$
$$-(\int_t^u c(e_b^k)e^{-r(q-t)}dq)]\lambda e^{-\lambda(u-t)}du] \qquad (47)$$

where $\lambda_k^b=\sum_{i=1}^{L+1}\lambda_{ik}^{w}$, the rate at which buyer k gets matched.

We note that the search intensities, $\{e_s^l\}_{l=1}^{L+1}$, $\{e_b^k\}_{k=1}^{\bar{M}+L+1}$, are independent of time, and depend only on the state w (cf. Assumption 4). This implies that λ_{ij}^{w} (for i=1,...,L+1 and j=1,...,$\bar{M}$+L+1) is independent of time. With this observation we can simplify equations (46) and (47). We obtain (note that the equilibrium payoffs are therefore independent of t):

For l=1,...,L+1

$$V_s^l = \max_{e_s^l\geq 0} P_s^l(e_s^l, \underline{e}_s^l, \underline{e}_b) \qquad (46a)$$

and for k=1,...,$\bar{M}$+L+1

$$V_b^k = \max_{e_b^k\geq 0} P_b^k(e_b^k, \underline{e}_b^k, \underline{e}_s) \qquad (47a)$$

where $\underline{e}_s=\{e_s^l\}_{l=1}^{L+1}$, $\underline{e}_s^l=\{e_s^n\}_{n\neq l}$, $\underline{e}_b=\{e_b^k\}_{k=1}^{\bar{M}+L+1}$, $\underline{e}_b^k=\{e_b^n\}_{n\neq k}$. P_s^l and P_b^k are defined below:

$$P_s^l = \frac{\lambda_l^s X + [\lambda-\lambda_l^s]D_s - c(e_s^l)}{r + \lambda}, \text{ where } D_s=V_s(L) \text{ and}$$

$$P_b^k = \frac{\lambda_k^b[1-X] + [\lambda-\lambda_k^b]D_b - c(e_b^k)}{r + \lambda}, \text{ where } D_b=V_b(L).$$

A joint search strategy, $\hat{\underline{e}}=(\hat{\underline{e}}_s,\hat{\underline{e}}_b)$, is a Nash equilibrium of the search game if and only if $\hat{\underline{e}}$ solves equations (46a) and (47a).

We shall make the following assumption (Assumption 7, below), which will ensure that if there exists a solution to equations (46a) and (47a), then it is an interior solution (i.e., $\hat{\underline{e}}>\underline{0}$).

Assumption 7

(a) $$\frac{\partial P_s^l(0,\underline{e}_s^l,\underline{e}_b)}{\partial e_s^l} > 0, \quad \forall \underline{e}_s^l \in \mathbb{R}_+^L, \ \underline{e}_b \in \mathbb{R}_+^{\bar{M}+L+1}$$

(b) $$\frac{\partial P_b^k(0,\underline{e}_b^k,\underline{e}_s)}{\partial e_b^k} > 0, \quad \forall \underline{e}_b^k \in \mathbb{R}_+^{\bar{M}+L}, \ \underline{e}_s \in \mathbb{R}_+^{L+1}$$

PROPOSITION 2

There exists a unique solution to equations (46a) and (47a) (denote it by $\hat{\underline{e}}$); with the property that $\hat{e}_s^l=e_s^*$ for $l=1,\ldots,L+1$, and $\hat{e}_b^k=e_b^*$ for $k=1,\ldots,\bar{M}+L+1$ (i.e., the Nash equilibrium is symmetric).

By using the method of proof that was used earlier to prove Proposition 1, and arguments similar to those presented there, one can prove Proposition 2.

Let V_s^* and V_b^* denote the equilibrium payoffs to a seller and a buyer, respectively. Then $(e_s^*,e_b^*,V_s^*,V_b^*)$ will be the unique solution to

equations (42), (43), (44) and (45) with $J_2(X)=V_s^*$ and $J_4(X)=V_b^*$. Note that $e_s^*=e_s(L+1)$, $e_b^*=e_b(L+1)$, $V_s^*=V_s(L+1)$ and $V_b^*=V_b(L+1)$, where $e_s(L+1)$, $e_b(L+1)$, $V_s(L+1)$ and $V_b(L+1)$ are defined in the Theorem (equations (5)-(8)). Thus $J_2(X)=V_s(L+1)$ and $J_4(X)=V_b(L+1)$.

All we have to do now is to prove that equations (38) and (39) have a unique solution. Let (38) and (39) be represented as follows: $\underline{x}=h(\underline{x})$, where $\underline{x}=[x,y]$ and $h(\underline{x})=[h_1(\underline{x}),h_2(\underline{x})]$, and $h_1(\underline{x})=\max(J_1(\underline{x}),J_2(X))$, $h_2(\underline{x})=1-\max(J_3(\underline{x}),J_4(X))$ (cf. equations (38) and (39)).

A proof similar to that of Lemma 6 will prove that h has at least one fixed point. $\forall X\in[0,1]$, $J_2(X)\in[0,1]$ and $J_4(X)\in[0,1]$. Firstly, note that $J_i(X)\geq 0$ for $i=2,4$ since $J_2(X)=V_s(L+1)$ and $J_4(X)=V_b(L+1)$. Secondly, by induction one proves that $V_s(L)\leq(1/L)[\sum_{l=0}^{L-1} X(l)]$, and then one shows that $J_2(X)\leq[1/(L+1)][X+A]$, $\forall X\in[0,1]$, <u>with</u> strict inequality if $X\neq 0$, where $A=\sum_{l=0}^{L-1} X(l)$ and $X(l)=\frac{1}{2}[x(l)+y(l)]$, which in turn implies that $J_2(X)\leq 1$. Thirdly, by induction one proves that $V_b(L)\leq[1/(\bar{M}+L)][\sum_{l=0}^{L-1}[1-X(l)]]$, and then one shows that $J_4(X)\leq[1/(\bar{M}+L+1)][(1-X)+B]$, $\forall X\in[0,1]$, <u>with</u> strict inequality if $X\neq 1$, where $B=\sum_{l=0}^{L-1}[1-X(l)]$, which in turn implies that $J_4(X)\leq 1$. $\forall(x,y)\in[0,1]\mathrm{X}[0,1]$, $J_1(\underline{x})\in[0,1]$ and $J_3(\underline{x})\in[0,1]$. By the continuity of J_1, J_2, J_3 and J_4 on their respective domains, one obtains the continuity of h on $[0,1]\mathrm{X}[0,1]$.

A proof similar to that of Lemma 7 will prove that h has one and only one fixed point. J_1 and J_3 are contractions. One shows that J_2 and J_4 are contractions by using the facts that (a) $J_2(X)\leq[1/(L+1)][X+A]$ $\forall X\in[0,1]$, <u>with</u> strict inequality if $X\neq 0$, where $A=\sum_{l=0}^{L-1}[X(l)]$ and $X(l)=\frac{1}{2}[x(l)+y(l)]$ and (b) $J_4(X)\leq[1/(\bar{M}+L+1)][(1-X)+B]$ $\forall X\in[0,1]$, <u>with</u> strict inequality if $X\neq 1$, where $B=\sum_{l=0}^{L-1}[1-X(l)]$.

Thus, we have completed the proof of the Theorem.

[illegible]

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4. AN ANALYSIS OF THE SOLUTION

In this section we shall examine how the equilibrium outcome, in particular the equilibrium terms of trade (i.e., prices), depend on the parameters of the model. But, before we do this, we shall first make two observations:

(1) As mentioned in the Introduction (section 1), we have not focused on symmetric equilibria, as various authors (who are mentioned in the Introduction) have done. The Theorem has established the existence of a unique SGPE outcome, allowing for non-symmetric strategies. Furthermore, the Theorem has established that the unique SGPE outcome is symmetric (i.e., all buyers choose the same strategy, and all sellers choose the same strategy).

(2) The Theorem has established that all the transactions take place at different prices (i.e., non-uniform prices emerge in equilibrium). A reason for this is that the demand and supply conditions change as traders leave the market (and there are no new traders who enter). Note that two matched pairs never trade at the same instant in time, since the matching process matches agents one at a time. (Of course, it is possible, for example, that two matched pairs trade within one-millionth of a second, or that all trades occur within one-billionth of a second).

The price at which a matched pair trade depends on the state of the market, and three types of frictions. Firstly, the bargaining friction, which is captured by the parameter Δ, the time between successive offers ($\Delta>0$). We shall call this friction parameter the Internal friction parameter. Secondly, the market friction, which is captured by the cost of search function, $c(\cdot)$, and represents the extent to which the market environment (in particular, the fact that there are more buyers than there are sellers) impinges on the equilibrium price. We shall call this friction parameter the External friction parameter. Thirdly, there is the rate of time preference r ($r>0$).

We first of all examine what happens to the equilibrium prices as the External friction parameter becomes negligible, keeping the Internal friction parameter and the rate of time preference strictly positive.

The External friction parameter is represented by the cost function, $c(\cdot)$. Since it does not make sense to say that $c(\cdot)$ becomes negligible we shall assume that $c(e) = \alpha e^2$ where $\alpha>0$ (which satisfies our proposed properties of c, i.e., strictly increasing and strictly convex). And thus, we let α be the External friction parameter, and hence we will examine what happens to the equilibrium prices as $\alpha \to 0$, keeping $\Delta>0$ and $r>0$.

Let $\bar{x}(k)=\lim_{\alpha\to 0} x(k;\alpha,\Delta)$ and $\bar{y}(k)=\lim_{\alpha\to 0} y(k;\alpha,\Delta)$, for $k=0,1,\ldots,N-1$, where $\{x(k;\alpha,\Delta),y(k;\alpha,\Delta)\}_{k=0}^{N-1}$ are the equilibrium prices, defined in the Theorem. Let $\bar{V}_s(k+1)=\lim_{\alpha\to 0} V_s(k+1;\alpha,\Delta)$ and $\bar{V}_b(k+1)=\lim_{\alpha\to 0} V_s(k+1;\alpha,\Delta)$, for $k=0,1,\ldots,N-1$, where $\{V_s(k+1),V_b(k+1)\}_{k=0}^{N-1}$ are defined in the Theorem.

Firstly, we note that for any $e\in\mathbb{R}_+$, $\alpha e^2 \to 0$ as $\alpha\to 0$ (i.e., the search costs approach zero as $\alpha\to 0$). Secondly, if, either $1>\bar{x}(k)>0$ and $1\geq\bar{y}(k)\geq 0$ $(k=0,\ldots,N-1)$, or $1\geq\bar{x}(k)\geq 0$ and $1>\bar{y}(k)>0$ $(k=0,\ldots,N-1)$, then (since search costs approach zero as $\alpha\to 0$) we obtain that $e_s(k+1)\to+\infty$ and $e_b(k+1)\to+\infty$ as $\alpha\to 0$. Thirdly, if $\bar{x}(k)=\bar{y}(k)=1$ $(k=0,\ldots,N-1)$, then $e_s(k+1)\to+\infty$ and $e_b(k+1)\to 0$ as $\alpha\to 0$. Fourthly, if $\bar{x}(k)=\bar{y}(k)=0$ $(k=0,\ldots,N-1)$, then $e_s(k+1)\to 0$ and $e_b(k+1)\to+\infty$ as $\alpha\to 0$. Since for all k $(k=0,1,\ldots,N-1)$ either $e_s(k+1)\to+\infty$ or $e_b(k+1)\to+\infty$ or both $e_s(k+1)\to+\infty$ and $e_b(k+1)\to+\infty$ as $\alpha\to 0$, we obtain that for all k $(k=0,1,\ldots,N-1)$ $f_1^{k+1}(e_s(k+1),e_b(k+1))\to+\infty$ as $\alpha\to 0$, where $f_1^{k+1}(e_s(k+1),e_b(k+1))$ is the equilibrium rate at which any buyer and any seller get matched, given that there are k+1 unmatched sellers, and M-N+k+1 unmatched buyers. Thus, $[1/(f_1^{k+1}(e_s(k+1),e_b(k+1)))]\to 0$ as $\alpha\to 0$ for all k $(k=0,\ldots,N-1)$, i.e., the expected time taken for an agent to "getting matched" tends to zero as the External friction parameter tends to zero, whatever the state of the market, and keeping $\Delta>0$ and $r>0$.

Using equations (5) and (6), we therefore have, $\bar{V}_s(k+1)=[1/(k+1)][\bar{X}(k)+k\bar{V}_s(k)]$ for all k, $k=0,1,\ldots,N-1$, where $\bar{X}(k)=\frac{1}{2}$

$[\bar{x}(k)+\bar{y}(k)]$, and $\bar{V}_b(k+1)=[1/(M-N+k+1)][1-\bar{X}(k)+[M-N+k]\bar{V}_b(k)]$ for all k, $k=0,\ldots,N-1$.

Claim 4

For $k=1,2,\ldots,N$, (a) $\bar{V}_s(k)=[1/k]\sum_{l=0}^{k-1}[\bar{X}(l)]$, and (b) $\bar{V}_b(k)=[1/(M-N+k)]\sum_{l=0}^{k-1}[1-\bar{X}(l)]$.

The proof of Claim 4 can be shown using the Principle of Induction, given that $V_s(0)=V_b(0)=0$ ($\Rightarrow \bar{V}_s(0)=\bar{V}_b(0)=0$) - thus we shall omit the proof.

Since $f_1^{k-1}(e_s(k+1),e_b(k+1))\rightarrow+\infty$ as $\alpha\rightarrow0$ for all k ($k=0,\ldots,N-1$), we obtain that for all k ($k=1,\ldots N$) $P_{kl}(\Delta)\rightarrow0$ as $\alpha\rightarrow0$, for any $l\neq0$ and for any $\Delta>0$, and $P_{k0}(\Delta)\rightarrow1$ as $\alpha\rightarrow0$, for any $\Delta>0$.

Now we are ready to compute $\{\bar{x}(k),\bar{y}(k)\}_{k=0}^{N-1}$. Using equations (1) and (2), and the results obtained above, we have, for $k=0,1\ldots,N-1$,

$$\bar{x}(k)=\max\{e^{-r\Delta}(1/2)[\bar{x}(0)+\bar{y}(0)],\ (1/(k+1))[\sum_{l=0}^{k}\bar{X}(l)]\} \tag{48}$$

and

$$1-\bar{y}(k)=\max\{e^{-r\Delta}(1/2)[1-\bar{x}(0)+1-\bar{y}(0)],\ (1/(M-N+k+1))[\sum_{l=0}^{k}[1-\bar{X}(l)]]\} \tag{49}$$

where $\bar{X}(l)=\frac{1}{2}[\bar{x}(l)+\bar{y}(l)]$.

We know that $M>N$. Let $(M/N)=\beta$ where $\beta>1$. Thus $M=\beta N$. Substitute in equation (49) for M using $M=\beta N$. Then let $N\rightarrow+\infty$. Thus $[1/(M-N+k+1)] = [1/(N(\beta-1)+k+1)] = [(1/N)/((\beta-1)+((k+1)/N))]\rightarrow0$ as $N\rightarrow+\infty$.

Thus, in addition to $\alpha\rightarrow0$, we will assume that the number of agents becomes infinitely large. This will simplify the analysis below.

Equation (49) becomes, for $k=0,1,\ldots,N-1$,

$$1-\bar{y}(k)=e^{-r\Delta}[[1-\bar{x}(0)+1-\bar{y}(0)]/2] \tag{49a}$$

Put $k=0$ into equation (48). And we obtain that $\bar{x}(0)=[[\bar{x}(0)+\bar{y}(0)]/2]$, i.e., $\bar{x}(0)=\bar{y}(0)$. Now substitute this (i.e., $\bar{x}(0)=\bar{y}(0)$) into equation (49a) with $k=0$, and we obtain that $\bar{x}(0)=\bar{y}(0)=1$. Now substitute $\bar{x}(0)=\bar{y}(0)=1$ into equation (49a), and we obtain that,

$$\text{for } k=0,1,\ldots,N-1,\ \bar{y}(k)=1. \tag{50}$$

Now substitute equation (50), and the result that $\bar{x}(0)=1$, into equation (48), and we obtain that $\bar{x}(k)=\max\{e^{-r\Delta}, [1/(k+1)] \sum_{l=0}^{k} [[\bar{x}(l)+1]/2]\}$.

Claim 5

For $k=0,1,\ldots,N-1$, $\bar{x}(k)=1$.

Proof (by induction)

$\bar{x}(0)=1$ is given. Now we assume $\bar{x}(k-1)=1$, and deduce $\bar{x}(k)=1$. Thus $\bar{x}(k)=\max\{e^{-r\Delta}, [1/(k+1)][[[\bar{x}(k)+1]/2]+k]\}$.

(i) assume $[1/(k+1)][[[\bar{x}(k)+1]/2]+k]>e^{-r\Delta}$ (51)

Then $\bar{x}(k)=[1/(k+1)][[[\bar{x}(k)+1]/2]+k]$, and thus $\bar{x}(k)=1$. Substitute this into (51), and hence $(1/(k+1))[1+k]>e^{-r\Delta}$, i.e., $1>e^{-r\Delta}$, which is true since $\Delta>0$ and $r>0$.

(ii) assume $(1/(k+1))[((\bar{x}(k)+1)/2)+k]<e^{-r\Delta}$ (52)

Then $\bar{x}(k)=e^{-r\Delta}$. Substitute this into (52), and hence $(1/(k+1))[((e^{-r\Delta}+1)/2)+k]<e^{-r\Delta} \Leftrightarrow e^{-r\Delta}+1+2k<2(k+1)e^{-r\Delta} \Leftrightarrow 1<e^{-r\Delta}$, a contradiction. Thus (52) cannot hold.

Q.E.D.

Thus equation (50) and Claim 5 imply, for $k=0,1,\ldots,N-1$, $\bar{x}(k)=\bar{y}(k)=1$. (Hence, for $k=0,1,\ldots,N-1$, $e_s(k+1)\rightarrow+\infty$ and $e_b(k+1)\rightarrow 0$ as $\alpha\rightarrow 0$).

Therefore, as the External friction parameter (α) tends to zero, and as the number of agents become infinitely large (keeping [M/N] constant, a <u>constant strictly greater than</u> 1), $x(k;\alpha)\rightarrow 1$ and $y(k;\alpha)\rightarrow 1$, for $k=0,1,\ldots,N-1$, i.e., all the equilibrium prices tend to the competitive equilibrium price; the sellers take all the surplus. Note that the Internal friction parameter (Δ) and the rate of time preference (r) are kept strictly positive.

Before we give an explanation and interpretation of this result, let us point out that the assumption, "that the number of agents

becomes infinitely large (keeping [M/N]>1)", may not be necessary to obtain this result. It is, of course, a sufficient assumption. We used it since it simplifies equation (49), and therefore the subsequent analysis.

An explanation and interpretation for the above result is as follows. As the External friction parameter, α, tends to zero, the sellers (since they are on the short side of the market) are able to play off the buyers, one against another, in order to obtain the whole surplus. The fact that the Internal friction parameter Δ is strictly positive is important. (We shall show later on that if the Internal friction parameter, Δ, also tends to zero then the above result may not be obtained).

As the External friction parameter (α) tends to zero, we have shown that the expected time taken for an agent to "getting matched" tends to zero. Since the sellers are on the short side of the market it is they (and not the buyers) who can exploit this to their advantage. Suppose a seller is matched with a buyer. Since $\Delta>0$ and since the expected time taken for the seller to find an alternative buyer tends to zero (as $\alpha\to 0$), the seller will always prefer to change partners than to continue bargaining with his current buyer. Thus, in essence, the matched seller and buyer play a one-shot game in which the seller announces a take-it-or-leave-it offer to the buyer.

For $k=0,1,\ldots,N-1$, $\lim_{\Delta\to 0}[\lim_{\alpha\to 0} x(k;\alpha,\Delta)]=1$ and $\lim_{\Delta\to 0}[\lim_{\alpha\to 0} y(k;\alpha,\Delta)]=1$.

The above repeated limits are equivalent to the following double limits: for $k=0,1,\ldots,N-1$,

$$\lim_{(\alpha,\Delta)\to(0,0)} x(k;\alpha,\Delta)=1$$
$$\lim_{(\alpha,\Delta)\to(0,0)} y(k;\alpha,\Delta)=1 \quad , \tag{53}$$

allowing (α,Δ) to approach $(0,0)$ along the path indicated in Figure 1, below; (Call this path P_c).

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

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[illegible]

[illegible]

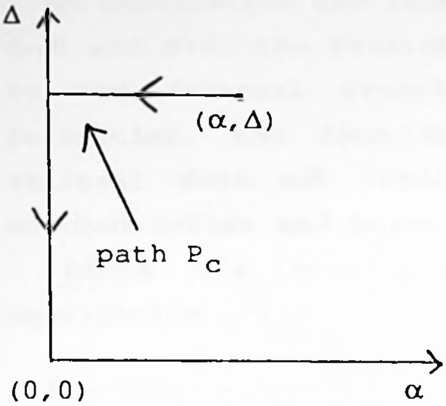

FIGURE 1

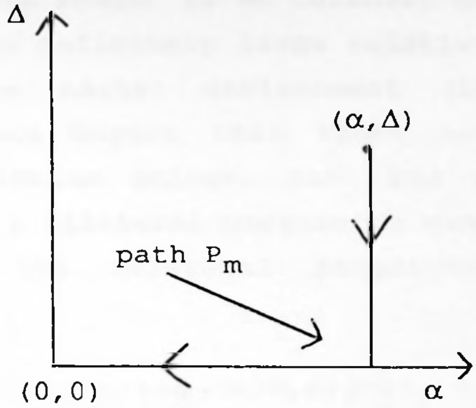

FIGURE 2

Thus, as both the Internal and the External friction parameters tend to zero, along the path P_c, all the equilibrium prices tend to the competitive equilibrium price.

We now proceed to examine what happens to the equilibrium prices as the Internal friction parameter becomes negligible (i.e., $\Delta \to 0$), keeping the External friction parameter and the rate of time preference strictly positive.

Let $\hat{x}(k)=\lim_{\Delta\to 0} x(k;\alpha,\Delta)$ and $\hat{y}(k)=\lim_{\Delta\to 0} y(k;\alpha,\Delta)$, for $k=0,1,\ldots,N-1$, where $\{x(k;\alpha,\Delta),\ y(k;\alpha,\Delta)\}_{K=0}^{N-1}$ are the equilibrium prices.

We first observe that as $\Delta\to 0$, $P_{kl}(\Delta)\to 0$ for $l\neq k$ and $P_{kk}(\Delta)\to 1$. Secondly, we note that, for $k=0,1,\ldots,N-1$, $\hat{V}_s(k+1)\leq \hat{X}(k)=\frac{1}{2}[\hat{x}(k)+\hat{y}(k)]$ and $\hat{V}_b(k+1)\leq 1-\hat{X}(k)=\frac{1}{2}[1-\hat{x}(k)+1-\hat{y}(k)]$.

Thus, using equations (1) and (2), and the two observations made above, we have, for $k=0,1,\ldots,N-1$, $\hat{x}(k)=e^{-r\Delta}[(\hat{x}(k)+\hat{y}(k))/2]$ and $1-\hat{y}(k)=e^{-r\Delta}[(1-\hat{x}(k)+1-\hat{y}(k))/2]$, which gives, for $k=0,1,\ldots,N-1$, $\hat{x}(k)=\frac{1}{2}$ and $\hat{y}(k)=\frac{1}{2}$.

Therefore, as the Internal friction parameter (Δ) tends to zero, for $k=0,1,\ldots,N-1$, $x(k;\alpha,\Delta)\to\frac{1}{2}$ and $y(k;\alpha,\Delta)\to\frac{1}{2}$, i.e., all the equilibrium prices tend to the bilateral bargaining equilibrium price. Note that the External friction parameter and the rate of time

preference are kept strictly positive.

An explanation and interpretation for this result is as follows. As $\Delta \to 0$ and $\alpha > 0$, the External friction becomes infinitely large relative to the Internal friction, and thus the market environment (in particular, the fact that there are more buyers than there are sellers) does not impinge on the equilibrium prices, and thus a matched seller and buyer become locked in a bilateral bargaining game - hence the equilibrium prices are the bilateral bargaining equilibrium price.

For $k=0,1,\ldots,N-1$, $\lim_{\alpha \to 0} [\lim_{\Delta \to 0} x(k;\alpha,\Delta)] = \frac{1}{2}$ and $\lim_{\alpha \to 0} [\lim_{\Delta \to 0} y(k;\alpha,\Delta)] = \frac{1}{2}$.

The above repeated limits are equivalent to the following double limits: for $k=0,1,\ldots,N-1$,

$$\lim_{(\alpha,\Delta) \to (0,0)} x(k;\alpha,\Delta) = \frac{1}{2}$$

$$\lim_{(\alpha,\Delta) \to (0,0)} y(k;\alpha,\Delta) = \frac{1}{2}, \tag{54}$$

allowing (α,Δ) to approach $(0,0)$ along the path indicated in Figure 2, above; (Call this path P_m).

Thus, as both the Internal and the External friction parameters tend to zero, along the path P_m, all the equilibrium prices tend to the bilateral bargaining equilibrium price.

This result is in sharp contrast to the result stated in equation (53) and depicted in Figure 1. An explanation and interpretation for this is as follows. As $(\alpha,\Delta) \to (0,0)$ along the path P_c (shown in Figure 1), it is the External friction parameter that "dominates" the Internal friction parameter, and thus the market environment impinges on the equilibrium prices - hence the result is the competitive equilibrium price. On the other hand, if $(\alpha,\Delta) \to (0,0)$ along the path P_m (shown in Figure 2), it is the Internal friction parameter that "dominates" the External friction parameter, and thus the market environment does not impinge on the equilibrium prices at all - hence preference are kept perfectly positive

[illegible]

equilibrium point.

[illegible]

This result is in sharp contrast to the result stated in equation [illegible] and depicted in Figure 1. [illegible]

[illegible] internal friction parameters [illegible] and that the market [illegible] the equilibrium prices - hence the result is the competitive equilibrium price. On the other hand, if [illegible] along the path [illegible] external friction parameter [illegible] does not impinge on the equilibrium prices [illegible] the result is the bilateral bargaining equilibrium price.

Thus the External friction parameter and the Internal friction parameter are working, on the equilibrium prices, in opposite directions. As both of these friction parameters tend to zero, the effect of the External friction favours competition, while the effect of the Internal friction favours bilateral bargaining. Furthermore, it is not the removal of all frictions which makes the market competitive; the competitive outcome is obtained if the External friction approaches zero at a higher speed than the Internal friction.

In the light of our results, we make some comments on the results obtained by Rubinstein and Wolinsky (1986). We choose their paper for two reasons: (i) it contains some of the recent thoughts on the literature, and (ii) their model contains the same assumption as our model regarding the relationship between the traders' presence in the market and time, namely, that all traders enter the market at one single time and the market continues to operate until all possible transactions are completed, and thus their model is directly comparable to our model.

We first note that anonymity (i.e., impersonal interaction) *à la* Rubinstein and Wolinsky (1986) (see also Rubinstein (1987a), pp.21-23) is <u>not</u> necessary to obtain the competitive equilibrium outcome. In our model we allow the players' strategies to depend on the entire history, and then there exist conditions under which the unique sub-game perfect equilibrium outcome coincides with the competitive equilibrium outcome. What matters are the two friction parameters, namely the Internal and the External friction parameters. Furthermore, from our results, we conclude that the multiplicity of equilibria obtained in Rubinstein and Wolinsky (1986) (see their Proposition 1) is due to the fact that the discount factor is equal to 1. They suggest this not to be the case, and say that, "... here the number of buyers is strictly greater than the number of sellers and it is rather natural to expect that this fact alone would exert sufficient pressure to guarantee the competitive price and that the absence of frictions would just reinforce it", (p.8). In their model there is <u>no</u> friction parameter that captures the market environment (in particular, the fact that there are more buyers than there are sellers). And thus, the

market environment cannot impinge on the equilibrium outcome. Furthermore, the absence of frictions would not reinforce the competitive equilibrium as has been shown in our results.

market-determined (pure) leaves the equilibrium outcome. Furthermore, the absence of frictions would not challenge the competitive equilibrium as has been shown in our results.

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