Conformal Boundaries, SPTs, and the Monopole-Fermion Problem

Philip Boyle Smith

Supervisor: Prof. D. Tong

DAMTP
University of Cambridge

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Declaration

I hereby declare that except where specific reference is made to the work of others, the contents of this thesis are original.

To the best of my knowledge, this thesis is not substantially the same as any other work that has been submitted before for any degree or other qualification in this, or any University.

It includes nothing which is the outcome of work done in collaboration, except as specified in the Acknowledgements.

Philip Boyle Smith
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Acknowledgements

This thesis is based on the papers [1–4], as well as prior unpublished work. The following is an attempt to list all the sources of input during its completion.

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Chapter 1

Introduction

Magnetic line defects have played an important role in gauge theories since their original introduction by 't Hooft nearly 40 years ago [5]. They were originally introduced as probes for differentiating between the different phases of non-abelian gauge theories, but have since seen a vast resurgence in interest in recent years due to the programme of generalised symmetries, where they have found their natural setting as the operators transforming under a magnetic 1-form symmetry [6, 7]. The magnetic, and more generally dyonic line operators in supersymmetric gauge theories have also found success in refining our understanding of the action of various dualities such as $\mathcal{S}$-duality in $\mathcal{N} = 4$ SYM, which in turn has illuminated subtle differences between these theories that had previously gone unnoticed [8].

This thesis is devoted to understanding such operators in chiral gauge theories, which can be defined as gauge theories in which the fermionic matter sits in a complex representation of the gauge group. As we shall see shortly, simple topological reasons conspire to make this a highly nontrivial problem, which to the best of my knowledge remains unsolved. This situation is made all the more unfortunate given that chiral gauge theories constitute one of the most challenging classes of gauge theories one can study. Due to the inapplicability of most familiar tools, such as supersymmetry and numerical Monte-Carlo methods, many of the simplest chiral gauge theories remain without a satisfactory dynamical understanding since their introduction in the 1980s (for a review, see [9]). Perhaps, if the available defect operators in these theories were better understood, including their magnetic line operators, then one could finally open a (small) window into their dynamics.

Two Problems with Monopoles

The problems mentioned above are well illustrated by a d=3+1 dimensional Weyl fermion moving in the background of Dirac monopole of strength m, for m a nonzero integer. Without loss of generality, we can consider a left-handed Weyl fermion. When written in spherical coordinates (t, r, θ, ϕ), with $r = 0$ corresponding to the location of the monopole, the fermion field decomposes into an infinite collection of fields that are either in-movers $\psi(t + r)$ or out-movers $\psi(t - r)$. These partial waves, along with their in-mover or out-mover status, are completely determined by the spectrum of the Dirac operator $\not{D}_{S^2,m}$ on a copy of S^2 with a uniform magnetic field carrying m units of magnetic flux. As is well-known [10, 11], the spectrum of this Dirac operator takes the following form:

- For an infinite set of values $\lambda > 0$, there are pairs of eigenvalues $(\lambda, -\lambda)$. Each pair gives rise to a pair of partial waves, an in-mover $\psi_\lambda(t + r)$ and an out-mover $\psi_{-\lambda}(t - r)$. The two are coupled together with a position-dependent mass term

$$\mathcal{L} \supset \frac{\lambda}{r}(\psi_\lambda^* \psi_{-\lambda} + \text{c.c.})$$

  which grows strong near the core of the monopole. This causes an in-moving particle to scatter into an out-moving particle, at a distance of roughly λ/E from the monopole, where E is a characteristic energy scale.

- There is also a finite collection of zero modes, whose existence can be traced to the Atiyah-Singer index theorem,

$$\text{index}(\not{D}_{S^2,m}) = \int_{S^2} \frac{F}{2\pi} = m$$

  When $m > 0$, it turns out there are exactly m zero modes of positive chirality. Each of these gives rise to a partial wave $\psi_{0,i}(t + r)$ that is strictly in-moving, for $i = 1, \ldots, m$. (A similar statement holds for m negative, except with the chirality and in/out status flipped.) In contrast to the situation for non-zero modes, however, there is no accompanying outgoing mode, and so we refer to them as chiral lowest partial waves. Such incoming particles simply travel into the core of the monopole and vanish, violating both charge and energy conservation.

The full partial wave decomposition of the Weyl fermion therefore takes the form

$$\begin{aligned}\psi(x) &= \frac{1}{r}\sum_{i=1}^{m} f_{0,i}(\theta,\phi)\psi_{0,i}(t+r) \\ &+ \frac{1}{r}\sum_{\lambda>0}\Big(f_\lambda(\theta,\phi)\psi_\lambda(t+r) + f_{-\lambda}(\theta,\phi)\psi_{-\lambda}(t-r)\Big)\end{aligned} \tag{1.1}$$

where the $f_{0,i}$ and $f_{\pm\lambda}$ are profile functions known as spinor monopole harmonics, and the $\psi_{0,i}$ and $\psi_{\pm\lambda}$ are dynamical quantum fields.

When the $U(1)$ gauge field is made dynamical, the existence of the chiral lowest partial wave $\psi_{0,i}$ causes problems. As we have seen, it necessarily leads to gauge charge leaking into the monopole. In a gauge theory, the violation of gauge charge is unacceptable, and leads to the same unphysical symptoms (for example, one perturbative manifestation would be a failure of the decoupling of negative-norm states) as would happen if one tried to gauge an anomalous symmetry. But this is of course exactly what we are doing: a single Weyl fermion indeed has an 't Hooft anomaly for $[U(1)^3]$. The bad behaviour of the chiral lowest partial wave is simply a manifestation of this anomaly.

Although the above illustration was doomed from the outset by the anomaly, it did serve to highlight two ingredients that will be crucial in what follows. First is the partial wave reduction (1.1), and second is the existence of a chiral lowest partial wave $\psi_{0,i}$, consisting of m in-moving fields lacking a suitable boundary condition at $r = 0$. (For negative m there is a similar statement, but with $-m$ out-moving fields instead.)

The First Problem

We now turn to a more realistic 4d chiral gauge theory in which all the relevant anomalies cancel. (We will explain in more detail later which anomalies these are.) Such a gauge theory is given by a collection of Weyl fermions, which we take to all have left-handed chirality, with $U(1)$ charge assignments

$$1\,, \quad 5\,, \quad -7\,, \quad -8\,, \quad 9 \tag{1.2}$$

As before, we place at the origin a Dirac monopole, which for simplicity we take to have charge 1. Each fermion now experiences a magnetic charge proportional to the charge assignment of that fermion. We perform the partial wave decomposition (1.1), and keep only the lowest chiral partial waves, as these are the only fields lacking an automatically generated boundary condition at $r = 0$. The result is a d=1+1 dimensional effective field theory on the half-infinite line $r \geq 0$, consisting of the following fields:

chirality	$U(1)_Q$	$SU(2)_{\text{rot}}$
left-mover	1	**1**
left-mover	5	**5**
right-mover	-7	**7**
right-mover	-8	**8**
left-mover	9	**9**

Each positively charged 4d fermion of charge q has given rise to q left-moving 2d fermions of charge q, while for negative charges q the result is instead $|q|$ right-movers of charge q. Each chiral lowest partial wave also transforms as a single multiplet under the group $SU(2)_{\text{rot}}$ of spatial rotations that preserve the monopole; this is shown in the last column. Our goal is to define a suitable boundary condition for these fields that conserves energy, charge, and angular momentum.

There is a general expectation that one can impose boundary conditions preserving any continuous symmetry that does not suffer an 't Hooft anomaly. (See for example [12, 13], where various versions of this statement were shown. Later, in Section 1.2, we will also review evidence for this statement in any number of dimensions.) We should therefore check whether all relevant 2d anomalies vanish before searching for boundary conditions. They are

$$[\text{Grav}^2]_{2d} \qquad [U(1)_Q^2]_{2d} \qquad [SU(2)^2_{\text{rot}}]_{2d}$$

These 2d anomalies descend from the 4d anomalies in a straightforward way. If m is the magnetic charge of the monopole, then

$$\begin{aligned}
[\text{Grav}^2]_{2d} &= \sum_q (mq) &&= m\,[U(1)_Q \cdot \text{Grav}^2]_{4d} \\
[U(1)_Q^2]_{2d} &= \sum_q (mq)(q^2) &&= m\,[U(1)_Q^3]_{4d} \\
[SU(2)^2_{\text{rot}}]_{2d} &= \sum_q \frac{1}{6}(mq)(m^2q^2-1) &&= \frac{m^3}{6}\,[U(1)_Q^3]_{4d} - \frac{m}{6}\,[U(1)_Q \cdot \text{Grav}^2]_{4d}
\end{aligned}$$

All the above anomalies therefore vanish if the two 4d anomalies

$$[U(1)_Q^3]_{4d} \qquad [U(1)_Q \cdot \text{Grav}^2]_{4d}$$

do too. The vanishing of the first is guaranteed in any chiral $U(1)$ gauge theory, hence automatically satisfied. The second is only guaranteed if the gauge theory can be put on a curved background. If not, then the theory still makes sense in flat space, but there are now limitations on what kind of magnetic line operators can be inserted. For a general gauge group G, magnetic lines can only be defined by inserting Dirac monopoles into $U(1)$ subgroups for which the mixed $U(1)$-gravitational anomaly vanishes. (During the completion of this project, similar observations were also made in Maldacena's work on magnetically charged black holes [14].) From now on will assume that both anomalies vanish.

The chiral gauge theory (1.2) is an example of a theory where this condition is met, by virtue of the equations $1+5+9=7+8$ and $1^3+5^3+9^3=855=7^3+8^3$, and in fact is one of the simplest theories of this kind. We therefore expect it should be possible to write down a symmetry-preserving boundary condition for the effective 2d theory. However, because the anomaly cancels in an interesting way, it is far from obvious how to do this. In particular, linear boundary conditions of the kind $\psi_L = \psi_R$ fail to do the job, since these preserve vector-like rather than chiral symmetries. This difficulty is the first problem alluded to above, and is in fact generic among all chiral gauge theories:

> **Problem 1** Given a chiral gauge theory with gauge group G, and a choice of $U(1)_Q \subseteq G$ with vanishing $[U(1)_Q^3]$ and $[U(1)_Q \cdot \mathrm{Grav}^2]$ anomalies, there is generically no known boundary condition for the effective 2d theory preserving $U(1)_Q$, $SU(2)_{\mathrm{rot}}$ and satisfying energy conservation.

There is a slight refinement of this problem which also takes into account any other global symmetries the theory might have. If there is a subgroup $H \subseteq G$ which commutes with $U(1)_Q$, then H also descends to a global symmetry of the 2d effective theory. If furthermore all relevant 4d anomalies vanish, namely $[H^2 \cdot U(1)_Q]_{4d}$ and $[H \cdot U(1)_Q^2]_{4d}$, then H is also free of 't Hooft anomalies in 2d: $[H^2]_{2d} = [H \cdot U(1)_Q]_{2d} = 0$, and we therefore expect that it should be possible to preserve at the boundary. Again, we generically do not know how to construct such boundary conditions, even for theories which evade Problem 1.

Before going on we address a number of comments arising from this discussion:

Q: Should we demand that the global symmetry H in the refined version of Problem 1 also has a vanishing $[H^3]_{\mathrm{4d}}$ anomaly?

A: There is reason to be cautious. As we have said, any non-anomalous symmetry should be possible to preserve at a boundary, and the converse is also expected to be true, which means that we should be cautious in demanding H be preserved if it has a $[H^3]$ anomaly. But in four dimensions, what we have is not a boundary, but a line defect. There is no reason to expect the same conditions on 't Hooft anomalies to be necessary for a symmetry to be preserved at a boundary versus at a line defect. In the current situation, the most natural condition is that all 2d anomalies involving H vanish, and $[H^3]_{\text{4d}}$ has no effect on these anomalies. We therefore make no restriction on $[H^3]_{\text{4d}}$.

Q: Is it reasonable to demand conservation of charge and angular momentum, when these quantum numbers could be soaked up by degrees of freedom living on the monopole worldline?

A: Yes, since such degrees of freedom are automatically incorporated into the definition of a boundary condition by the formalism we review in Section 1.1. This includes the rigid rotor "dyon" degree of freedom that arises on the worldline of a solitonic 't Hooft-Polyakov monopole [15, 16]. One can also ask about Jackiw-Rebbi zero modes [17]. These are not true localised degrees of freedom on the monopole, but are instead localised to an arbitrarily large distance $1/m$ where m is a fermion mass, so technically are not captured by integrating out the monopole core to get an effective boundary condition. Nonetheless, we will see in Section 2.3.2 that these modes are also captured by the boundary states analysis, albeit in a different way.

The Second Problem

Although the first problem above will be our main motivation in what follows, there is a second closely related problem that is worth mentioning alongside. To illustrate it, we start from the non-chiral theory

$$U(1) \;+\; N \text{ Weyls of charge } {+1} \;+\; N \text{ Weyls of charge } {-1} \tag{1.3}$$

where both Weyls are left-handed, and again consider a monopole of unit charge. The 2d effective theory consists of the fields

chirality	$U(1)_Q$	$SU(N)$
left-mover	$+1$	$\mathbf{N}$
right-mover	-1	$\mathbf{N}$

where we have included a column for the non-anomalous $SU(N)$ global symmetry that rotates both sets of N Weyls. (While there is technically a $[SU(N)^3]_{4d}$ anomaly, as we have argued this can be ignored.) The problem then manifests in the scattering of particles off the boundary. When n left-moving excitations scatter from the boundary, they must bounce back as $-n$ right-moving excitations, on $U(1)_Q$ conservation grounds. (Here, negative numbers mean antiparticles.) But this is incompatible with the $SU(N)$ global symmetry, in particular its center $\mathbb{Z}_N$, as in general $n \neq -n \bmod N$. The non-existence of asymptotic states carrying the right quantum numbers to describe the outcomes of scattering events has been noted in [15, 18–20], and is referred to as the "unitarity paradox", or scattering paradox.

Viewed on its own this result is not terribly unsettling. Perhaps it is simply telling us that the magnetic line operator for this $U(1)$ does not exist, with the scattering paradox simply one manifestation of this fact. But our example (1.3) was carefully chosen to show that this is not the case. Our gauge theory can be extended in the UV to

$$SU(2) \ + \ N \text{ Weyls in the fundamental representation} \ + \ \text{adjoint Higgs}$$

while the Dirac monopole is realised by a solitonic 't Hooft Polyakov monopole. (At least for N even, to cancel the Witten anomaly.) Because the latter is a dynamical object of the theory, not a static probe that is inserted by hand, all the information as to boundary conditions and scattering is automatically generated by the theory. The boundary condition is also known, and is referred to as the "dyon" state [21]. The scattering paradox is then the statement that despite all this, the answer to such simple physical questions as the scattering of fermions still remains a mystery:

> **Problem 2** Even in chiral gauge theories that evade Problem 1 (i.e. a satisfactory boundary condition is known), the scattering of fermions off the monopole generically exhibits a "unitary paradox" where there exists no asymptotic state that can carry off the quantum numbers of the incoming state.

The scattering paradox has a long history, but perhaps the first serious investigation was undertaken by Callan in the 1980s. Since then the problem has been attacked either directly or in passing by work of Sen, Polchinski, Maldacena, Witten, and many others.

At first, Callan proposed that the final state was a superposition of states which conserved quantum numbers "on average" [19]. This violates the basic axioms of symmetry in quantum mechanics, and so was quickly replaced with the suggestion that the outgoing state consisted of particles with fractional quantum numbers that were dubbed 'semitons'. (This terminology arose from an abelian bosonisation approach in which the outgoing state consists of fractional kinks, hence the name semi-solitons.) Now the problem had a name, but not a solution. Still, the picture at least clarified that the paradox is related to infra-red divergences in the scattering of massless particles: upon the addition of a fermion mass m, the semitons decay back to regular solitons in a time $\sim 1/m$ [22, 23].

Maldacena claimed to have solved the puzzle in the special case of $N = 4$ flavours, using triality of $SO(8)$ [20]. The claim was that incoming states, which could be thought of as transforming in the $\mathbf{8}_v$ representation, scattered into solitonic states transforming in the $\mathbf{8}_s$ representation. However no four-dimensional explanation of these solitons was given, and neither was the generalisation to other values of N. The author's interpretation is that this is simply a version of the semitons story with more symmetry. Nevertheless, the key insights of the paper, namely the existence of an enhanced $SO(8)$ symmetry and the importance of triality, have undergone a surprising resurgence in more recent years in the context of Symmetry-Protected Topological phases [24, 25]. Because of this, the dyon boundary state at $N = 4$ has a special name: the Maldacena-Ludwig state. We will return to the ML state in Chapter 4, where we will have much more to say about it.

During the completion of this project, two more recent papers appeared on the same subject. The first [26] made an examination of the multi-fermion condensates that surround the 't Hooft-Polyakov monopole [27], in particular how they respond to a fermion scattering off the monopole into its mysterious final state. The authors found ripples in these condensates, generalising to four dimensions the semitons picture found in two dimensions, and furthermore these were found to carry the desired fractional quantum numbers. Their findings suggest that care must be taken in defining asymptotic states in theories in which there are fermion condensates with long (power-law) tails.

The second series of papers [28, 29] introduced a totally different perspective on the problem. Their fundamental insight is that as representations of the Lorentz group, both the asymptotic states and the S-matrix must be modified when one considers scattering of both electrically and magnetically charged particles. Although this approach has not yet resolved the puzzle, it suggests that doing so will require looking more carefully at the kinematics of asymptotic states in the presence of magnetic monopoles.

In this work, we will focus only on the first problem, since it is a concrete question about the existence of certain symmetric fermion boundary conditions, and turns out to have a richer set of connections to contemporary physics. The second problem is less well-posed, and requires dealing with thorny issues of asymptotic states in theories with massless particles, slowly-decaying condensates, and magnetic charges, all at the same time. Nevertheless, aspects of the second problem will resurface at several points throughout the following analysis. For now we turn to the construction of chirally symmetric boundary conditions for Dirac fermions. Since the bulk theory is conformal, the correct framework for doing so is boundary conformal field theory. In the next section we give a brief review of this field.

1.1 Review of Boundary Conformal Field Theory

Boundary conformal field theory is the study of boundary conditions in two-dimensional conformal field theories. The framework was originally introduced by Cardy [30, 31] to describe boundary critical phenomena of statistical systems, but has since found applications in diverse areas such as critical percolation, string theory [32–35], where it forms the natural worldsheet description of D-branes, and later condensed matter [36, 37], where it is closely related to the entanglement structure of topological phases. (We will review the latter connection further in Section 1.2.)

Given an arbitrary 2d CFT, BCFT purports to solve the following problems:

- It provides a systematic description of all possible boundary conditions that can be imposed on a theory.
- It tells us how to compute in the presence of such boundary conditions. This includes partition functions and correlation functions of local operators on Riemann surfaces with arbitrary boundary components.
- It provides constraints on the possible boundary conditions a theory may admit. In some cases, most notably bosonic rational CFTs, these constraints are sufficiently stringent to allow all boundary conditions to be determined.

Our motivation lies especially in the third option; to solve the first fermion-monopole problem, we need to determine all boundary conditions for a given set of 2d Dirac fermions preserving a given chiral symmetry. Reviews of this topic can be found, for example, in [38–40], which we follow below.

The key idea is that thanks to the basic axioms of Euclidean field theory, a boundary condition can be viewed as a state living in the Hilbert space of the theory associated to the boundary. By cutting the theory on a manifold that infinitesimally surrounds the boundary, depicted in Figure 1.1 as a dotted line, and integrating out all degrees of freedom between it and the boundary, one obtains a corresponding boundary state that describes the same physics.

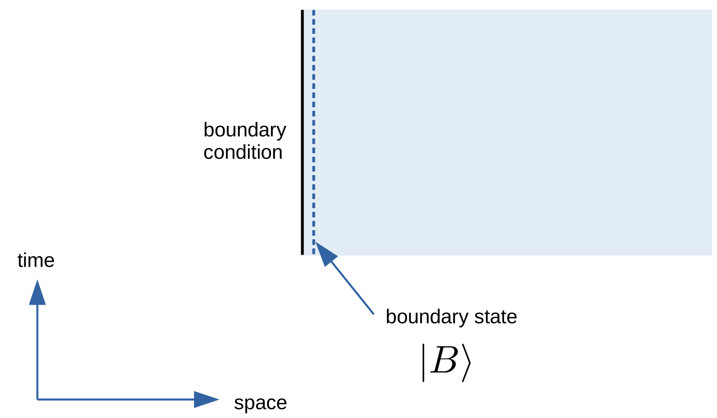


Figure 1.1: Integrating out a neighbourhood of a boundary

When presented with an arbitrary CFT, we generally know neither the allowed boundary conditions nor boundary states. Fortunately, the boundary states must obey a number of consistency conditions that we derive below. There is a general expectation that, like the constraints of crossing symmetry and unitarity in the conformal bootstrap programme, these constraints are sufficient to determine all boundary conditions, at least in principle. In the cases we study, this will also be the case.

1.1.1 Conformal Symmetry

At a timelike boundary, as in Figure 1.1, a key physical requirement is that there is no flux of energy into the boundary. (Momentum conservation, however, is explicitly broken by the presence of the boundary, so we do not impose it.) This condition is expressed by

$$T_{tx}|_{x=0} = 0 \tag{1.4}$$

where $T_{\mu\nu}$ is the stress-energy tensor of the theory, and $x = 0$ is the boundary.

This condition is useful in constraining the boundary state $|B\rangle$. To derive the constraints, it is simplest to perform a conformal transformation to map the boundary to the unit circle, with the bulk living in the region $|z| > 1$, as in Figure 1.2. This is because the boundary state now lives in the Hilbert space of radial quantisation, where the action of the Virasoro symmetry is more transparent. The condition (1.4) becomes

$$\left(z^2\, T(z) - \bar{z}^2\, \bar{T}(z)\right) |B\rangle = 0 \tag{1.5}$$

at every $|z| = 1$. Or, employing the expansions

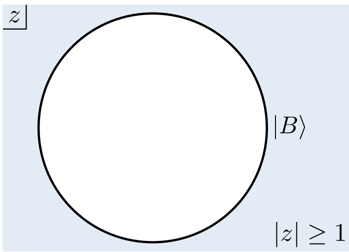


Figure 1.2: The boundary state $|B\rangle$ in radial quantisation.

$$T(z) = \sum_{n=-\infty}^{\infty} z^{-n-2} L_n$$
$$\bar{T}(z) = \sum_{n=-\infty}^{\infty} \bar{z}^{-n-2} \bar{L}_n$$

of the stress-tensor into Virasoro modes in radial quantisation, it becomes

$$(L_n - \bar{L}_{-n})\,|B\rangle = 0 \tag{1.6}$$

for all $n \in \mathbb{Z}$. The solutions of this equation have a very simple form. Recall that in general the structure of the Hilbert space of radial quantisation is

$$\mathcal{H} = \bigoplus_{(h,\bar{h})\in\mathcal{I}} \mathcal{V}_h \otimes \bar{\mathcal{V}}_{\bar{h}}$$

where $\mathcal{I}$ is an indexing multiset, and $\mathcal{V}_h$ and $\mathcal{V}_{\bar{h}}$ label irreducible representations of the left and right Virasoro algebras. The constraint (1.6) is linear, so can be solved separately in each sector $\mathcal{V}_h \otimes \bar{\mathcal{V}}_{\bar{h}}$. As was first shown by Ishibashi [41], the result is that

- If $h \neq \bar{h}$, there are no non-zero solutions.
- If $h = \bar{h}$, all solutions are a multiple of a certain solution $\|h\rangle\rangle$, known as an Ishibashi state.

The most general solution is therefore

$$|B\rangle \;\in\; \mathrm{span}\Big\{ \;\|h\rangle\rangle \;:\; (h,\bar{h}) \in \mathcal{I}\,,\; h = \bar{h} \;\Big\}$$

The above result is essentially the generalisation to the Virasoro algebra of Schur's lemma in representation theory.

The second place that condition (1.4) is useful is in demonstrating the properties of correlation functions near the boundary. For this, we use a conformal transformation

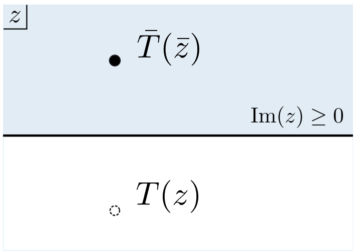


Figure 1.3: The analytic continuation of $T(z)$ is equal to $\bar{T}(\bar{z})$.

to map the theory to the upper half-plane, as in Figure 1.3. The condition (1.4) of no energy flux now takes the form

$$T(z) = \bar{T}(z) \qquad \text{for } z \in \mathbb{R} \tag{1.7}$$

Since $T(z)$ is holomorphic in the upper-half-plane, and $\bar{T}(\bar{z})$ is holomorphic in z for z in the lower-half-plane, and by the above condition the two are equal on the real line, they are therefore analytic continuations of each other inside correlation functions. We can therefore define a single holomorphic stress tensor

$$T(z) := \begin{cases} T(z) & \mathrm{Im}(z) > 0 \\ \bar{T}(\bar{z}) & \mathrm{Im}(z) < 0 \end{cases}$$

in the whole of the complex plane. This observation will come in useful later when we consider the spectrum in the open sector.

Spin-1 Currents

The whole of the above discussion takes on an analogous form when we consider boundary states preserving an enlarged chiral algebra. For example, let us suppose that our CFT has a spin-1 conserved current J^μ. We may additionally ask that this be preserved at a boundary. (For example, J^μ could be the $U(1)_Q$ current in the fermion-monopole problem.) The analogous versions of equations (1.4)-(1.7) are then

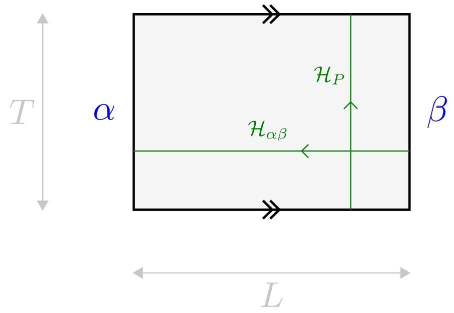
$$\begin{aligned} J_x &= 0 && \text{(at boundary } x = 0\text{)} \\ \left(z\, J(z) + \bar{z}\, \bar{J}(z)\right) |B\rangle &= 0 && \text{(on unit circle)} \\ (J_n + \bar{J}_{-n})|B\rangle &= 0 && \text{(in terms of modes)} \\ J(z) &= \bar{J}(z) && \text{(on real axis)} \end{aligned}$$

The notion of Ishibashi state must also be defined relative to the new chiral algebra, generated by $T(z)$ and $J(z)$. We will make use of these facts in Chapter 2 to construct boundary states.

1.1.2 The Cardy Condition

We have seen that boundary states live in the Hilbert space of radial quantisation, and that symmetries force them to be sums of Ishibashi states. The coefficients of these Ishibashi states however remain undetermined. To derive constraints on them, the key idea is to use modular covariance or, what string theorists refer to as open/closed string duality.

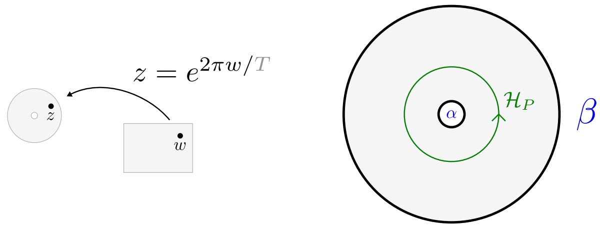
We begin by supposing that $|\alpha\rangle$ and $|\beta\rangle$ are two candidate boundary states, and placing the theory on an interval with boundary conditions α and β at the two ends. We calculate the thermal trace in Euclidean time T, meaning that fermion fields are taken to be antiperiodic in the time direction:



The partition function can be calculated in two different ways, either in the closed sector or in the open sector $\mathcal{H}_{\alpha\beta}$, via

$$\mathcal{Z} = \langle\beta|e^{-LH_P}|\alpha\rangle = \mathrm{Tr}_{\mathcal{H}_{\alpha\beta}}\left(e^{-TH_{\alpha\beta}}\right) \tag{1.8}$$

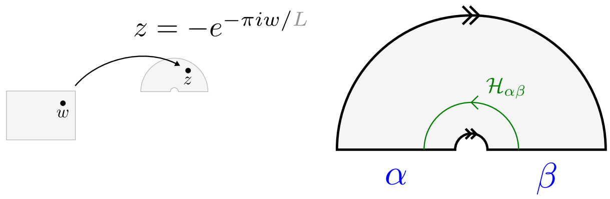
While the closed sector Hilbert space $\mathcal{H}_P$ is known, the open sector Hilbert space $\mathcal{H}_{\alpha\beta}$ depends on the boundary conditions, and is unknown. To make use of the machinery of 2d CFT, we rewrite both sides of this equation by conformally mapping them into a geometry where we can make use of radial quantisation. To deal with the closed-sector term, we map it to an annulus using the following transformation:



An important feature of this transformation is that it takes the originally antiperiodic fermions to be periodic around the origin, which justifies our choice of name $\mathcal{H}_P$ for the Hilbert space. The Hamiltonian H_P maps to the generator of dilations, plus a central charge term arising from the Schwarzian term in the transformation of $T(z)$:

$$H_P = \tfrac{2\pi}{T}(L_0 + \bar{L}_0 - \tfrac{c}{12})$$

For the open-sector term, we instead transform to an antiperiodic half-annulus via



and the corresponding Hamiltonian in this case is

$$H_{\alpha\beta} = \frac{\pi}{L}\left(\frac{1}{2\pi i}\int_{\mathcal{H}_{\alpha\beta}}[dz\, z\, T(z) - d\bar{z}\, \bar{z}\, \bar{T}(z)] - \frac{c}{24}\right)$$

The above integration takes places on a semi-circular contour. But using the analytic continuation of $T(z)$ to the lower half-plane (1.7), the integral can be rewritten as

$$\frac{1}{2\pi i}\oint dz\, z\, T(z)$$

where the contour is now closed, and encircles the origin counter-clockwise. The above formula is in fact the $n = 0$ special case of the more general formula

$$L_n := \frac{1}{2\pi i}\oint dz\, z^{n+1}\, T(z)$$

which defines a set of Virasoro mode operators acting on the open-sector Hilbert space $\mathcal{H}_{\alpha\beta}$. These are distinct from the modes L_n and $\bar{L}_n$ acting on $\mathcal{H}_P$, and turn the space $\mathcal{H}_{\alpha\beta}$ into a representation of the Virasoro algebra in its own right. (If both α and β preserve any other common symmetries, such as a spin-1 current J^μ, then $\mathcal{H}_{\alpha\beta}$ will also furnish a representation of a corresponding current algebra J_n.) In terms of these operators, the Hamiltonian finally becomes

$$H_{\alpha\beta} = \tfrac{\pi}{L}(L_0 - \tfrac{c}{24})$$

The open-closed duality (1.8) now takes the form

$$\langle\beta|e^{-2\pi\frac{L}{T}(L_0+\bar{L}_0-\frac{c}{12})}|\alpha\rangle = \mathrm{Tr}_{\mathcal{H}_{\alpha\beta}}\left(e^{-\pi\frac{T}{L}(L_0-\frac{c}{24})}\right) \tag{1.9}$$

All we shall need from (1.9) can be expressed in a succinct, easy-to-use form as follows. First we define the partition functions

$$\text{open-sector:} \qquad \mathcal{Z}_{\alpha\beta}(q) = \mathrm{Tr}_{\mathcal{H}_{\alpha\beta}}(q^{L_0-\frac{c}{24}}) \tag{1.10}$$

$$\text{closed-sector:} \qquad \mathcal{Z}_P(q) = \langle\beta|q^{\frac{1}{2}(L_0+\bar{L}_0-\frac{c}{12})}|\alpha\rangle \tag{1.11}$$

The arguments of the partition functions in the equality (1.9) are not the same; rather, they are related by a modular transformation. In general we define the standard modular parameters $q \equiv e^{2\pi i\tau}$ and $\tilde{q} \equiv e^{2\pi i\tilde{\tau}}$, where the modular $\mathcal{S}$-transformation relates $\tilde{\tau} = -1/\tau$. We will denote this transformation by $\mathcal{S}(q) = \tilde{q}$. The equality (1.9) can then be written as

$$\mathcal{Z}_{AB}(q) = \mathcal{Z}_P(\mathcal{S}(q)) \tag{1.12}$$

This is *Cardy's condition*. It allows the content of the 'mystery' Hilbert space $\mathcal{H}_{AB}$ to be read off from the right-hand side, which involves a matrix element between two states in the known Hilbert space $\mathcal{H}_P$, namely that of the periodic or Neveu-Schwarz sector in the plane. Equally, the requirement that $\mathcal{Z}_{AB}(q)$ defines a sensible partition function places constraints on the allowed boundary states one can impose.

1.1.3 The Cardy-Lewellen Sewing Conditions

The Cardy conditions give only weak information on a single boundary state, since demanding consistency of the B–B open-sector spectrum falls short of determining $|B\rangle$. Stronger information can be gained by imposing consistency between all possible pairs of boundary states, but this may be infeasible in practice, and may also not provide enough information. In this situation, more information can be gained by considering the Cardy-Lewellen sewing conditions [42]. These conditions encode the requirement of cluster decomposition in the presence of the boundary. In the upper-half-plane geometry, this is the statement that correlators obey

$$\langle \mathcal{O}_1(z)\, \mathcal{O}_2(w) \rangle_B \to \langle \mathcal{O}_1(z) \rangle_B \, \langle \mathcal{O}_2(w) \rangle_B \tag{1.13}$$

as $|\mathrm{Re}(z-w)| \to \infty$ with $\mathrm{Im}(z)$ and $\mathrm{Im}(w)$ fixed, for all local operators $\mathcal{O}_1$, $\mathcal{O}_2$. (Strictly, we mean that the ratio of the two sides tends to one, assuming both sides are nonzero.)

Not all boundary states in fact obey (1.13). It is easy to see that if $|A\rangle$ and $|B\rangle$ are both valid boundary states (obeying the conditions of the previous two sections), then so is $|A\rangle + |B\rangle$. However even if $|A\rangle$ and $|B\rangle$ obey clustering, this need not be true of $|A\rangle + |B\rangle$. The condition (1.13) therefore singles out a privileged subclass of boundary states, the so-called fundamental boundary states, which obey clustering. If $|B_i\rangle$ is a set of fundamental boundary states, then there is an expectation that the full set of boundary states is given by

$$\Big\{ \sum_i n_i |B_i\rangle \;:\; n_i \in \mathbb{Z},\, n_i \geq 0 \Big\}$$

Typically it is also found that the open-sector spectrum of B_i–B_j contains the identity operator with multiplicity δ_{ij}. Whether this follows from the other consistency conditions in general is unknown, and we shall simply impose it by hand. Often, the Cardy-Lewellen sewing conditions also determine information that could have been determined from the Cardy condition; again, the precise overlap between these conditions is unknown.

In bosonic RCFT, the Cardy-Lewellen sewing conditions are particularly powerful. They lead to the construction of a 'classifying algebra' whose representations are in one-to-one correspondence with fundamental boundary states [43]. This has allowed a full classification of the fundamental boundary states to be carried out, for example in the bosonic rational minimal models [44].

Unfortunately these techniques prove much weaker in the irrational theories that are our main interest. For the monopole-fermion problem, we are interested in the CFT of N Dirac fermions. This is irrational with respect to the Virasoro algebra, meaning $\mathcal{H}_P$ decomposes into an infinite number of irreducible representations of $\mathrm{Vir} \times \overline{\mathrm{Vir}}$. Even after we impose a chiral symmetry G, leading to an enlarged chiral algebra, the theory still remains irrational with respect to the new chiral algebra. The classifying algebra approach also only applies to bosonic theories. (The fermionic generalisation only appeared recently, during completion of the project [45].) For these reasons, we will not make use of classifying algebras in this work, and instead impose clustering by hand.

1.2 Review of SPT Phases

A relatively recent development within the condensed matter community is the notion of Symmetry-Protected Topological, or Trivial phase (SPT). As we shall review, there are close connections between boundaries, SPT phases and anomalies [36, 37, 46, 47]. This means that any study of boundary states would be incomplete without commenting on their relationship with SPTs, and we will see that many of our results indeed have clear interpretations on the SPT side. In this section we give a brief review of SPT phases, strongly influenced by Witten's exposition of the subject [48], and give examples of the important such phases that will play a role in this project.

1.2.1 What is an SPT Phase?

An SPT phase is a d-dimensional quantum field theory, with a global symmetry G, that is gapped and trivially gapped (meaning the theory has a unique gapped ground state on any manifold), but which cannot be deformed to the trivial phase (the empty quantum field theory with $\mathcal{Z} = 1$) without either breaking G or passing through a gapless point. Meanwhile, if two SPTs can be deformed to each other without breaking G or closing the gap, they are said to be the same SPT.

SPT phases are intimately connected to anomalies. To see this, we consider deforming the SPT to the trivial phase while preserving G. By definition, a phase transition must occur somewhere on this path. We now suppose that the deformation is performed along a direction in space, by setting the couplings in the Lagrangian to position-dependent profile functions. The phase transition now manifests itself as an interface on which there typically live localised gapless modes. The interface theory is a $(d-1)$-dimensional quantum field theory, with a global symmetry G, carrying an 't Hooft anomaly for G. Said another way, when a d-dimensional SPT is placed on a manifold with boundary, the $(d-1)$-dimensional boundary theory exhibits an anomaly for G. In fact, this is a one-to-one correspondence: every anomalous theory can be realised at the boundary of a unique SPT phase.

The above fact is at the heart of the modern classification of anomalies by cobordism [49, 50]. Given an SPT phase, the theory can be deformed by taking the mass gap to infinity (relative to any metric on the manifold). The resulting theory is a unitary TQFT, and has the property that its Hilbert space on any manifold consists of a single state, all other excited states having now decoupled, a property called invertibility. Deformation classes of such theories have been shown to be classified by the cobordism group $\Omega^d(\mathrm{B}G)$

[51]. (As we explain later, this statement must be generalised slightly in fermionic theories, or theories with other nontrivial tangent-space structure.) The above group therefore gives a characterisation of all possible anomalies a symmetry G can exhibit.

1.2.2 SPTs and Boundaries

For us, SPT phases will be related to boundaries in two crucial ways:

- A d-dimensional CFT admits boundary conditions preserving G if and only if its corresponding bulk $(d+1)$-dimensional SPT phase is trivial.
- If a d-dimensional CFT admits boundary conditions preserving G, then each boundary condition is associated to a d-dimensional G-symmetric SPT phase.

Below we explain both of these points in more detail, and illustrate them with examples. These examples will also play an important role in all later chapters.

Existence of Boundaries

As stressed in [37], there is a close correspondence between ways to put a theory on a manifold with boundary, and ways to gap a theory preserving certain symmetries. The intuitive correspondence is that, given any interaction that gaps the system, one can turn it on in the Lagrangian with a spatial, step-function profile. At low energies, then this then looks like a boundary condition for the massless fields. If the gapping interaction preserves symmetry G, then so too do the boundary conditions. Thus

$$\text{gappable preserving } G \quad \Longrightarrow \quad \text{boundaryable preserving } G \tag{1.14}$$

In dimension $d \geq 3$, it is known [52] that *any* theory with vanishing anomaly for G can be gapped by turning on interactions preserving G. (In dimension d=2 there is also an expectation that this is possible, though the method of [52] can no longer be applied directly, due to infra-red issues specific to two dimensions such as the Coleman-Mermin-Wagner theorem.) This subtlety aside, we have

$$\text{vanishing anomaly for } G \quad \Longrightarrow \quad \text{gappable preserving } G$$

Finally, if a theory has non-vanishing 't Hooft anomaly for G, then it necessarily resides on the boundary of a $(d+1)$-dimensional bulk SPT phase for symmetry G. The simple identity $\partial^2 = 0$ then suggests that such a system cannot itself admit a boundary, as argued in [37]. Taking the contrapositive of this statement gives the implication

$$\text{boundaryable preserving } G \quad \Longrightarrow \quad \text{vanishing anomaly for } G$$

This closes the circle of implications, showing that the three properties are equivalent.

Note that for the above claims to be true, we must be careful about exactly what we mean by 'gapping', and exactly what boundary states are allowed. For the former, we additionally require that the resulting gapped ground state be nondegenerate when placed on any manifold. The issue is that certain discrete anomalies can be saturated by TQFTs, as shown in [53], so simply having a gap in the spectrum is not enough to guarantee vanishing anomaly. For the latter, we also require the boundary states to be fundamental boundary states in the sense of Section 1.1.3.

Example: Fidkowski-Kitaev and the $\mathbb{Z}_8$ Anomaly

The paradigmatic example of the above ideas is the symmetric gapping of 8 1+1d Majorana fermions preserving $(-1)^{F_L} \times (-1)^{F_R}$ symmetry. Fidkowski and Kitaev showed that for this theory, with action

$$S = i \int dtdx \sum_{i=1}^{8} \psi_i \partial_+ \psi_i + \bar{\psi}_i \partial_- \bar{\psi}_i$$

it is possible to write down an interaction that symmetrically gaps the system, with a trivial ground state [24]. As a first attempt, one might try the mass term

$$\mathcal{L} \ni i \sum_{i=1}^{8} \psi_i \bar{\psi}_i$$

This gaps the system with a trivial ground state, but explicitly breaks the symmetry. Less naïvely, one could try the Gross-Neveu term

$$\mathcal{L} \ni \left(\sum_{i=1}^{8} \psi_i \bar{\psi}_i \right)^2$$

This now preserves the symmetry, and gaps the system, but not trivially: the symmetry is spontaneously broken by the formation of a fermion condensate. However, making use of the fact that there are eight fermions, there is third option: we can write down

$$\mathcal{L} \ni \sum_{i=1}^{8} \psi_i \psi_j \, \alpha^{[ij]}{}_{[kl]} \, \bar{\psi}_k \bar{\psi}_l \tag{1.15}$$

where α is the triality automorphism of $\mathfrak{so}(8)$ that swaps the vector and spinor representations, leaving the conjugate spinor fixed. Fidkowski and Kitaev showed that this interaction succeeds in symmetrically gapping the system with a unique ground state.

These results illustrate our earlier, more general story in the following way. A single 2d Majorana fermion has an anomaly for $(-1)^{F_L} \times (-1)^{F_R}$. The corresponding 3d SPT phase is a pair of massive 3d Dirac fermions of opposite masses $\pm M$, on which $(-1)^{F_L}$ and $(-1)^{F_R}$ act separately. As we take $M \to \infty$, in a suitable regularisation one obtains the partition function $\mathcal{Z}_{3\text{d}}^{\text{SPT}} = e^{\frac{i\pi}{2}(\eta(\mathcal{D}_L)-\eta(\mathcal{D}_R))}$, where $\eta(\mathcal{D}_{L,R})$ is the eta invariant for the 3d Dirac operator $\mathcal{D}_{L,R}$ coupled to the spin structure for $(-1)^{F_{L,R}}$. This partition is known to take values that are eighth roots of unity. This means that for eight copies of the system, the corresponding 3d SPT phase $(\mathcal{Z}_{3\text{d}}^{\text{SPT}})^8$ is now trivial, hence the theory has no anomaly. And indeed, in this case, we have seen above that the theory is gappable.

To complete our story, we need to exhibit the boundary condition corresponding to the Fidkowski-Kitaev gapped phase. This is none other than the Maldacena-Ludwig state we described in the introduction. This identification was first made in [37], and follows essentially because the symmetry preserved by (1.15) uniquely specifies the boundary state. We will make a detailed investigation of the symmetry preserved by more general boundary states in Chapter 4, where the Maldacena-Ludwig state will return as a prominent example. We will also classify the most general boundary state preserving $(-1)^{F_L} \times (-1)^{F_R}$; in accordance with the above analysis, such states will be found to exist only for the number of Majorana fermions a multiple of 8.

Classification of Boundaries

The second way in which SPT phases will feature in BCFT is more straightforward. Typically we expect that each boundary condition of a d-dimensional CFT should be realisable by a wall to a gapped phase. Such a phase is in particular a d-dimensional SPT phase, and so we have a mapping from boundary states to SPT phases.

As explained in detail in [37], this leads to an important connection between the open-sector spectrum of states in BCFT and properties of SPT phases. Suppose that A and B are two distinct SPT phases, each obtained from the same CFT by turning on some relevant interaction. Then if A and B are placed in contact with each other, their interface realises an 't Hooft anomaly for G. Now we deform the set-up so that A and B are separated by a length L of CFT. Because anomalies are robust, the spectrum of states on L still carries the same anomaly for G. As L is taken larger than all other scales, the spectrum of these modes becomes the open-sector BCFT spectrum between boundary states $|A\rangle$ and $|B\rangle$. Thus we can expect to see an imprint of the anomaly in the structure of the open-sector Hilbert space $\mathcal{H}_{AB}$.

The correspondence between gapped phases and boundaries also features in another result. In [36], it was shown that the entanglement spectrum of a gapped phase, in a certain limit, is exactly given by the CFT spectrum on an interval with suitable boundary conditions, determined from the gapped phase in the way we have described above. This result however will not play any role in our work.

Example: Arf and the $\mathbb{Z}_2$ Anomaly

The most important example for us will be the $\mathbb{Z}_2$ anomaly of $(-1)^F$ fermion parity in two dimensions. The corresponding 2d SPT phase in this case is the theory $\mathcal{Z} = (-1)^{\text{Arf}}$, whose boundary theory consists of a single quantum-mechanical Majorana fermion, as we now review.

A single Majorana fermion in quantum mechanics, with action

$$S[\chi(\tau)] = \frac{i}{2}\int d\tau\, \chi(\tau)\, \partial_\tau \chi(\tau) \tag{1.16}$$

provides what is arguably the simplest system suffering an anomaly. To see this, as explained in [54], we can place the above system on a periodic circle (of length 1). The Dirac operator ∂_τ then has a single zero mode. This means that the partition function vanishes, and to get a nonzero result one must instead consider

$$\langle \chi(\tau) \rangle = 1$$

The fact that a correlator of a fermionic quantity is non-vanishing reflects that $(-1)^F$ has been broken by an anomaly.

The anomaly in $(-1)^F$ also manifests directly in the partition function. We can see this by taking two copies of a Majorana fermion, λ_1 and λ_2. Canonical quantisation gives rise to a 2d Clifford algebra $\{\lambda_i, \lambda_j\} = \delta_{ij}$ which acts irreducibly on a Hilbert space of dimension 2. This means that a single Majorana fermion would act on a Hilbert space of dimension $\sqrt{2}$, which is nonsensical. Indeed, the dimension of the Hilbert space is counted by the path integral for a single Majorana mode, with anti-periodic boundary conditions in the temporal direction. This can be computed from the action (1.16), and is given by

$$Z_{\text{Majorana}} = \text{Tr}_{\mathcal{H}}(\mathbb{1}) = \sqrt{2} \tag{1.17}$$

This reflects the fact that there is no way to consistently quantise a single Majorana mode in d=0+1 dimensions. This simple fact is the essence of the mod 2 anomaly, and the telltale factor of $\sqrt{2}$ will be a recurring motif throughout this project.

To make the connection to 2d SPT phases, we proceed as follows. As shown by Jackiw and Rebbi [17], when a 2d dimensional Majorana fermion is given a spatially-varying mass that varies from negative as $x \to -\infty$ to positive as $x \to +\infty$, the interface localises a single copy of the quantum-mechanical Majorana fermion system. In other words, when the positive- and negative-mass phases are placed in contact with each other, the interface realises an anomalous system under $(-1)^F$. This is the hallmark property for the two phases to be distinct SPT phases.

For reasons described earlier, the existence of two SPT phases for $(-1)^F$ should have a reflection in the boundary states for fermions in two dimensions. Investigating this will form much of the subject of Chapter 2. We will also come back to this point in more detail in Section 5.1.1.

To complete the illustration, we need to describe the corresponding invertible TQFT for this anomaly, i.e. a system whose boundary realises the quantum-mechanical Majorana fermion. This is almost what is accomplished by the previous paragraph. The only change needed to make the leap from theories on $\mathbb{R}^2$ to arbitrary Riemann surfaces Σ is the addition of a Pauli-Villars regulator. We take this to be a Majorana fermion of positive mass $+M$. Once a regulator is chosen, the positive mass phase $+M$ then becomes trivial, while the negative mass phase $-M$ becomes the nontrivial SPT phase. Its partition function is

$$\frac{\mathcal{Z}_{\text{maj}}[\Sigma; -M]}{\mathcal{Z}_{\text{maj}}[\Sigma; +M]}$$

As we take the limit $M \to +\infty$ relative to the metric on Σ, we finally obtain the topological theory

$$\mathcal{Z}_{\text{2d}}^{\text{SPT}}[\Sigma] = (-1)^{\text{Arf}[\Sigma]}$$

where the Arf invariant is defined to be 0 or 1 according to whether the spin structure on Σ is even or odd. (An even spin structure is one for which $\not{D}$ has an even number of zero modes, a property which remains invariant as the metric is changed thanks to the mod 2 index theorem.) The Arf theory is better known in condensed matter physics as the low-energy limit of the Kitaev chain [55]. Recent applications of this topological field theory can be found in [12, 56–66].

Chapter 2

Chiral Boundary States

Motivated by the fermion-monopole problem, our goal in this chapter is to construct interesting boundary states for Dirac fermions in d=1+1 dimensions. Specifically, a set of N Dirac fermions is equivalent to a set of $2N$ Majorana fermions, the latter having a manifest $SO(2N)_L \times SO(2N)_R$ symmetry, and we wish to construct boundary states preserving an interesting (i.e. chiral), but non-anomalous subgroup G. Although these symmetries do not suffer from 't Hooft anomalies, the anomaly cancels in an nontrivial way which means that it's not obvious how to impose boundary conditions that are consistent with the symmetry. Even though the theory is free in the bulk, this does not help in the construction of boundary states, as these may include strongly interacting degrees of freedom on the boundary. In fact the construction of all such boundary conditions is an unsolved problem, except in the case $N = 1$ [67, 68].

There are two approaches one can take to this problem, a bottom-up and a top-down approach. In the top-down approach, one imposes the symmetry G from the start. One then writes down the most general Ansatz for a family of boundary states obeying symmetry G, and attempts to constrain the unknown data using the machinery of boundary conformal field theory. While this approach is ideal, it has a serious drawback: it is intractable. Namely, the above programme fails when the rank of rank(G) falls below $c = N$ due to an explosion in the amount of unknown data to be determined. To the best of my knowledge, analytic progress in constructing boundary states has only ever been made in the case rank(G) $\geq c$. We will term such theories as 'quasi-rational', since although they still remain irrational with respect to the chiral algebra formed from $T(z)$ and the generators of G, the problem of constructing boundary states remains manageable.

Our best hope therefore lies in the second, bottom-up approach. Here one simply constructs as many boundary states for N Dirac fermions as possible. One can then check which ones preserve the desired chiral symmetry G. In view of the limitations discussed above, as a necessary crutch in constructing the boundary states, we need to impose a symmetry of rank N. The simplest possibility, and in fact the most general possibility we need to consider, is an abelian symmetry $U(1)^N$.

Our goal in this chapter is therefore to construct boundary states for abelian chiral symmetries. We will derive simple expressions, in terms of the fermion charge assignments, for the boundary central charge and for the ground state degeneracy of the system when two different boundary conditions are imposed at either end of an interval. Following the discussion in Section 1.2.2, we will show that all such boundary states fall into one of two classes, related to SPT phases supported by $(-1)^F$, which are characterised by the existence of an unpaired Majorana zero mode. We will explain our main results later in this introduction, but first it will prove useful to give a simple example to set the scene.

A Simple Example: A Single Fermion

We can illustrate some of the issues involved in constructing boundary states by looking at a single Dirac fermion in d=1+1 dimensions. A single Dirac fermion exhibits a $U(1)_V \times U(1)_A$ symmetry. Neither the vector nor axial symmetry has a 't Hooft anomaly, but there is a mixed anomaly between them. This suggests that we should be able to impose boundary conditions that preserve either $U(1)_V$ or $U(1)_A$, but not both.

Indeed, it is not difficult to write down classes of boundary conditions that relate the left-moving fermion ψ_L to the right-moving fermion ψ_R and do the job. We could, for example, consider the boundary condition

$$V[\theta]: \quad \psi_L = e^{i\theta}\psi_R \tag{2.1}$$

This preserves the vector symmetry $U(1)_V$ at the expense of the axial symmetry $U(1)_A$. The boundary condition depends on a phase $e^{i\theta}$, whose existence can be traced to the broken $U(1)_A$.

Alternatively, we could impose the boundary condition

$$A[\theta]: \quad \psi_L = e^{i\theta}\psi_R^\dagger \tag{2.2}$$

This now preserves the axial symmetry but breaks the vector. In the context of condensed matter physics, this axial boundary condition is referred to as *Andreev reflection*. Physically, an electron bounces off the boundary and returns as a hole, a phenomenon that is seen when a wire is attached to a superconductor. Again, the boundary condition is parametrised by a phase.

Compatibility of Boundary Conditions

Our primary interest is in theories that live on an interval. If we attempt to impose different boundary conditions on each end, there are a number of questions that arise. Most importantly, we can ask: is the resulting theory consistent? If it is, we can also ask: how many ground states does the theory have?

The essential physics can already be seen in the single Dirac fermion. At each end, we get a choice of vector (2.1) or axial (2.2) boundary condition, each specified by a phase, θ_1 at one end and θ_2 at the other. There are two possibilities for the resulting physics:

- $V[\theta_1] - V[\theta_2]$ or $A[\theta_1] - A[\theta_2]$: With VV or AA boundary conditions, the system generically has a single ground state. However, in the special case that $\theta_1 = \theta_2$, the Dirac fermion has a single complex zero mode. This increases the ground state degeneracy to 2.

- $A[\theta_1] - V[\theta_2]$ or $V[\theta_1] - A[\theta_2]$: With mixed AV or VA boundary conditions, there is a single Majorana zero mode[1] for all θ_1 and θ_2.

  A single, quantum mechanical Majorana mode is a particularly simple example of an anomalous quantum system. Perhaps the quickest way to see this is to note that a single Majorana zero mode contributes $\sqrt{2}$ to the counting of states in the partition function [94]. We learn that while both V and A boundary conditions are possible, they are not mutually compatible.

[1]To see this, it is simplest to split each Weyl fermion into its Majorana-Weyl components: $\psi_L = \chi_L^1 + i\chi_L^2$ and $\psi_R = \chi_R^1 + i\chi_R^2$. A constant spinor is compatible with the two boundary conditions (2.1) and (2.2) only if

$$\begin{pmatrix} \chi_L^1 \\ \chi_L^2 \end{pmatrix} = R[-\theta_1] \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} R[\theta_2] \begin{pmatrix} \chi_L^1 \\ \chi_L^2 \end{pmatrix}$$

where $R[\theta]$ is the 2×2 matrix that implements a rotation by θ. But the combination of these three matrices is a reflection about some axis and so always has a real eigenvector with eigenvalue $+1$. The same argument applied to the VV and AA case gives a rotation matrix $R[\theta_2 - \theta_1]$ which has eigenvalue $+1$ only when $\theta_1 = \theta_2$.

Summary of Results

The story described above becomes more complicated when we have two or more fermions. This is because there are now non-anomalous chiral symmetries where it is less obvious how to implement the boundary condition.

For example, consider two free Dirac fermions. We may wish to place the theory on a manifold with boundary, now preserving the $U(1)$ global symmetry under which the two left-moving fermions have charges +3 and +4, and the two right-moving fermions have charges +5 and 0. This symmetry does not suffer a 't Hooft anomaly, by virtue of the fact that

$$3^2 + 4^2 = 5^2 + 0^2 \tag{2.3}$$

Yet any linear boundary condition, like (2.1) or (2.2), relating left- and right-moving fermions will not respect this symmetry.

In such situations, there are a number of ways to proceed. One could incorporate additional degrees of freedom on the boundary such that it is possible to write down boundary conditions that are linear in the fermions but continue to respect the symmetry. The fermion-rotor model of [18] provides an example of this kind.

Alternatively, one could attempt to quantise the theory by imposing the non-linear boundary condition $n^\mu J_\mu = 0$ where J_μ is the current and n^μ is normal to the boundary. Although it is not known how to do this in higher dimensions, in d=1+1, the formalism of boundary conformal field theory allows one to proceed exactly in this manner. The purpose of this chapter is to understand some of the properties of boundaries that preserve chiral symmetries like (2.3).

Specifically, as explained in the introduction, we will consider N Dirac fermions and, on a given boundary, insist that a $U(1)^N$ subgroup of the chiral symmetry is preserved. Below we will give a preview of the two main results that follow. For this, we first need to introduce a little notation.

We assign the left-moving fermions charges $Q_{\alpha,i}$ and the right-moving fermions charges $\bar{Q}_{\alpha i}$, where $\alpha = 1, \ldots, N$ labels the $U(1)$ symmetry, and $i = 1, \ldots, N$ labels the fermion. Typically, these charges differ so that we are dealing with a chiral symmetry. We insist that these symmetries do not suffer from mixed 't Hooft anomalies, which means that our charge matrices must obey the constraints

$$Q_{\alpha i}Q_{\beta i} = \bar{Q}_{\alpha i}\bar{Q}_{\beta i} \tag{2.4}$$

From these charge matrices, we can build a rational orthogonal matrix

$$\mathcal{R}_{ij} = (\bar{Q}^{-1})_{i\alpha}Q_{\alpha j}$$

The choice of such a matrix specifies the $U(1)^N$ symmetry that is preserved by the boundary. A general boundary state is then characterised by a choice of $\mathcal{R}$, together with a bunch of phases that are analogous to the $e^{i\theta}$ factors that we met in (2.1) and (2.2).

One final piece of notation: to each charge matrix $\mathcal{R}$ we can associate a lattice $\Lambda[\mathcal{R}] \subseteq \mathbb{Z}^N$. This lattice consists of all integer-valued vectors, $\lambda_i \in \mathbb{Z}$ which satisfy

$$\Lambda[\mathcal{R}] = \left\{ \lambda \in \mathbb{Z}^N \ : \ \mathcal{R}\lambda \in \mathbb{Z}^N \right\}$$

Now we are in a position to describe our results. The first is a simple expression for the Affleck-Ludwig boundary central charge [69]; this is given by

$$g_{\mathcal{R}} = \sqrt{\mathrm{Vol}(\Lambda[\mathcal{R}])} \tag{2.5}$$

where $\mathrm{Vol}(\Lambda[\mathcal{R}])$ is the volume of the primitive unit cell of the lattice Λ. The same result, in a rather different context, can be found in [33].

If each fermion is given a simple boundary condition (2.1) or (2.2), it is simple to check that $g_{\mathcal{R}} = 1$. More complicated, chiral boundary conditions have $g_{\mathcal{R}} > 1$. Typically $g_{\mathcal{R}}$ is not an integer, although its square always is.

Our second result is concerned with the situation in which we place the fermions on an interval, with different symmetries $\mathcal{R}$ and $\mathcal{R}'$ preserved at the two ends. In this case, we derive an elegant formula for the number of ground states $G[\mathcal{R},\mathcal{R}']$ of the system. For generic values of the phases, we find

$$G[\mathcal{R},\mathcal{R}'] = \frac{\sqrt{\mathrm{Vol}(\Lambda[\mathcal{R}])\,\mathrm{Vol}(\Lambda[\mathcal{R}'])}}{\mathrm{Vol}(\Lambda[\mathcal{R},\mathcal{R}'])}\sqrt{\det{}'(\mathbb{1}-\mathcal{R}^T\mathcal{R}')} \tag{2.6}$$

where the intersection lattice $\Lambda[\mathcal{R},\mathcal{R}']$ is defined to be those integer vectors λ which obey $\mathcal{R}\lambda = \mathcal{R}'\lambda \in \mathbb{Z}^N$. For special values of the phases, the ground state degeneracy can be enhanced in way that we detail later in the text.

It is not at all obvious that the expression for ground state degeneracy $G[\mathcal{R},\mathcal{R}']$ is an integer. In fact, we claim that $G[\mathcal{R},\mathcal{R}']$ is either an integer, or is $\sqrt{2}$ times an integer,

$$G[\mathcal{R},\mathcal{R}'] \in \mathbb{Z} \cup \sqrt{2}\,\mathbb{Z} \tag{2.7}$$

The case of $\sqrt{2}\,\mathbb{Z}$ is telling us that the system has an unpaired Majorana zero mode, and hence the two boundary conditions are mutually anomalous.

Furthermore, we show that all symmetries $\mathcal{R}$ fall into one of two classes which, following the discussion of a single fermion above, we denote as class $\mathcal{V}$ and class $\mathcal{A}$. Any choice of boundary conditions $\mathcal{R}$ and $\mathcal{R}'$ from within the same class results in an integer ground state degeneracy. In contrast, if $\mathcal{R}$ and $\mathcal{R}'$ are chosen from different classes, then there is an unpaired Majorana zero mode.

The Relationship to Gapped Systems

For the d=1+1 chiral symmetries considered in this chapter, there is a long literature devoted to the question of when these systems can be gapped, starting with the influential work of Haldane [70]. (See, for example, [71–73] for further developments.) It was shown in [37] that the possible boundary states that one can build are entirely equivalent to Haldane's so-called "null vector condition"[2].

When the boundary condition is viewed as a gapped phase, the two classes $\mathcal{V}$ and $\mathcal{A}$ that we described above translate into the $\mathbb{Z}_2$ classification of SPT phases protected by $(-1)^F$ from in Section 1.2.2. The question of whether there is an SPT interpretation of the full ground state degeneracy (2.6) remains open.

Our calculation also provides a good example of how a difficult interacting problem can be solved by mapping it to a simpler proxy problem in boundary conformal field theory. The Haldane gapped phases underlying our boundary states are in general interacting, non-integrable systems. It would have been hard to directly determine whether the low-lying spectrum between two such gapped phases supports an anomaly. But as we have seen, this information is not lost upon mapping to BCFT, where things become simpler.

[2]Since we are dealing with Dirac fermions, viewed as edge modes they have a trivial K-matrix, $K = \text{diag}(\mathbb{1}_N, -\mathbb{1}_N)$. Applied to this case, Haldane's criterion simply states that it's possible to find a gapping potential provided that the charge vectors $l_{\alpha i} = (Q_{\alpha i}, -\bar{Q}_{\alpha i})$ obey $l_{\alpha i} K_{ij} l_{\beta j} = 0$, which is simply the anomaly condition (2.4). In the continuum, the gapping process can be described by first performing an exactly marginal deformation that shifts the Narain moduli, then performing a relevant deformation by a sum of N independent multi-fermion operators.

Plan

In Section 2.1 we construct the boundary states preserving a given $U(1)^N$ symmetry. We give a partial proof that the boundary central charge is given by (2.5). This proof is completed in Section 2.2 where we consider theories on an interval, with different boundary conditions imposed at each end.

We also derive the formula (2.6) in Section 2.2. Most of the effort is taken up with the showing that, for large classes of examples, the ground state degeneracy obeys (2.7), with all states falling into one of two classes. (This is far from trivial and there remain a number of special cases where we have been unable to prove the result, but have compelling numerical evidence.)

Finally, in Section 2.3, we give a number of examples of boundary conditions. We also include several appendices which detail technical results that are omitted from the main text.

2.1 Construction of Boundary States

In this section we construct all possible boundary conditions that one can impose on N Dirac fermions in d=1+1 dimensions, subject to the requirement that there is vanishing flux of both energy and of a chosen $U(1)^N$ current flowing into the boundary.

2.1.1 Ishibashi States for Free Fermions

We start with the theory of N Dirac fermions in d=2 dimensions. Our convention for the action and currents can be found in Appendix 2.A. In the absence of a boundary, these fermions enjoy a $SO(2N)_L \times SO(2N)_R$ chiral symmetry. Our aim is to study boundaries that preserve some choice of subgroup

$$U(1)^N \subset U(1)^N_L \times U(1)^N_R \subseteq SO(2N)_L \times SO(2N)_R \tag{2.8}$$

Each $U(1)_\alpha$, with $\alpha = 1, \ldots, N$, is specified by the charges $Q_{\alpha i}$ for each of the $i = 1, \ldots, N$ left-moving fermions and, independently, charges $\bar{Q}_{\alpha i}$ for each of the $i = 1, \ldots, N$ right-moving fermions.

We begin by working in the closed sector, with Hilbert space $\mathcal{H}_P$. We will apply the strategy described in Section 1.1. The $\mathfrak{u}(1)^N$ current algebra consists of holomorphic and anti-holomorphic currents J_i and $\bar{J}_i$, with $i = 1, \ldots, N$, whose mode expansion is

$$[J_{i,n}, J_{j,m}] = [\bar{J}_{i,n}, \bar{J}_{j,m}] = n\delta_{ij}\delta_{n+m,0}$$

The preserved $U(1)_\alpha$ symmetries have currents

$$\mathcal{J}_{\alpha,n} = Q_{\alpha i}J_{i,n} \quad \text{and} \quad \bar{\mathcal{J}}_{\alpha,n} = \bar{Q}_{\alpha i}\bar{J}_{i,n} \tag{2.9}$$

The requirement that no $U(1)_\alpha$ current flows into the boundary amounts to saying that

$$(\mathcal{J}_{\alpha,n} + \bar{\mathcal{J}}_{\alpha,-n})\,|A\rangle = 0 \tag{2.10}$$

For solutions to exist, we must have the vanishing commutator

$$[\mathcal{J}_{\alpha,n} + \bar{\mathcal{J}}_{\alpha,-n}, \mathcal{J}_{\beta,m} + \bar{\mathcal{J}}_{\beta,-m}] = n\delta_{n+m,0}(Q_{\alpha i}Q_{\beta i} - \bar{Q}_{\alpha i}\bar{Q}_{\beta i})$$

This tells us that the charges of the left- and right-movers must satisfy the N^2 constraints

$$Q_{\alpha i}Q_{\beta i} = \bar{Q}_{\alpha i}\bar{Q}_{\beta i} \tag{2.11}$$

This is precisely the requirement that there is no mixed 't Hooft anomaly between the $U(1)_\alpha$ and $U(1)_\beta$ symmetries. From now on, we assume that all such anomalies vanish.

Our description of a $U(1)^N$ subgroup in terms of charges may be intuitive, but suffers from an inherent redundancy: any redefinition of the charges by

$$Q_{\alpha i} \to U_{\alpha\beta}Q_{\beta i} \qquad \bar{Q}_{\alpha i} \to U_{\alpha\beta}\bar{Q}_{\beta i}$$

with $U_{\alpha\beta}$ unimodular does not change the $U(1)^N$ subgroup they describe. One way of eliminating this redundancy is to introduce the matrix

$$\mathcal{R}_{ij} = (\bar{Q}^{-1})_{i\alpha}Q_{\alpha j} \tag{2.12}$$

which is rational and orthogonal. The possible anomaly-free $U(1)^N$ subgroups of $U(1)^N_L \times U(1)^N_R \subset SO(2N)_L \times SO(2N)_R$ are then in one-to-one correspondence with such matrices. For these reasons, we will use both $(Q, \bar{Q})$ and $\mathcal{R}$ in what follows when specifying the $U(1)^N$ symmetry.

The construction of the boundary states requires further knowledge about the structure of $\mathcal{H}_P$. Under the current algebra generated by $J_{i,n}$ and $\bar{J}_{i,n}$, the Hilbert space decomposes into charge sectors. In each sector, there is a ground state $|\lambda, \bar{\lambda}\rangle$ with charges

$$J_{i,0}|\lambda,\bar{\lambda}\rangle = \lambda_i|\lambda,\bar{\lambda}\rangle \quad , \quad \bar{J}_{i,0}|\lambda,\bar{\lambda}\rangle = \bar{\lambda}_i|\lambda,\bar{\lambda}\rangle \tag{2.13}$$

where $\lambda_i, \bar{\lambda}_i \in \mathbb{Z}$.[3] These ground states obey $J_{i,n}|\lambda,\bar{\lambda}\rangle = \bar{J}_{i,n}|\lambda,\bar{\lambda}\rangle = 0$ for $n \geq 1$. Excitations above the ground state are then constructed by acting with $J_{i,-n}$ and $\bar{J}_{i,-n}$ for $n \geq 1$. The condition (2.10) that $U(1)_\alpha$ is preserved can be imposed as separate condition on each charge sector $(\lambda, \bar{\lambda})$, and reads

$$\left(\mathcal{R}_{ij} J_{j,n} + \bar{J}_{i,-n}\right)|A\rangle = 0 \tag{2.14}$$

Importantly, not all charge sectors $(\lambda, \bar{\lambda})$ admit solutions to (2.14). The $n = 0$ equation tells us that we must restrict to those charge sectors that obey

$$\bar{\lambda}_i = -\mathcal{R}_{ij}\lambda_j \tag{2.15}$$

Not all λ will give rise to integer-valued solutions of this equation. Instead, λ must lie in a certain sub-lattice of $\mathbb{Z}^N$, defined by

$$\Lambda[\mathcal{R}] = \left\{ \lambda \in \mathbb{Z}^N \ : \ \mathcal{R}\lambda \in \mathbb{Z}^N \right\} \tag{2.16}$$

The allowed charge sectors are then $(\lambda, \bar{\lambda}) = (\lambda, -\mathcal{R}\lambda)$ for $\lambda \in \Lambda[\mathcal{R}]$. In each such sector, the condition (2.14) is solved by Ishibashi states which take the form [41]

$$\|\lambda, \bar{\lambda}; \mathcal{R}\rangle\rangle = \exp\left(-\sum_{n=1}^{\infty} \frac{1}{n}\mathcal{R}_{ij}\, \bar{J}_{i,-n} J_{j,-n}\right)|\lambda, \bar{\lambda}\rangle \tag{2.17}$$

We can now write down the most general boundary state preserving the symmetry. It takes the form of a linear sum of Ishibashi states, over the allowed charge sectors:

$$|a; \mathcal{R}\rangle = \sum_{\lambda \in \Lambda[\mathcal{R}]} a_\lambda\, \|\lambda, -\mathcal{R}\lambda; \mathcal{R}\rangle\rangle \tag{2.18}$$

The Sugawara construction then ensures that since the state preserves each $U(1)_\alpha$, it also has no net energy inflow. Ishibashi states of the form (2.17) were also considered in [36, 37, 46, 47]. It remains only to determine the complex coefficients a_λ.

[3] Our phase convention for the $|\lambda, \bar{\lambda}\rangle$ is detailed in Appendix 2.A. However, in almost all of what follows, this choice will play no role.

2.1.2 Clustering and the Cardy Condition

The coefficients a_λ in (2.18) are constrained by two sets of consistency conditions. The first of these conditions is the requirement that correlation functions obey clustering. In this context, these are known as the Cardy-Lewellen sewing conditions [31, 42]. A nice review can be found in [39], with applications in [68, 74]. As imposing these sewing conditions is somewhat intricate, we relegate the details to Appendix 2.B where we show that the ratios of the coefficients a_λ must obey

$$\frac{a_\lambda}{a_0} = e^{i\gamma_{\mathcal{R}}(\lambda)}\, e^{i\theta\cdot\lambda} \tag{2.19}$$

This ratio is a phase, but with various parameters that we are free to choose. In particular, there are N phases θ_i. These are the generalisation of the phases that we met in (2.1) and (2.2).

The ratio (2.19) also includes the factor $e^{i\gamma_{\mathcal{R}}(\lambda)}$. The definition of this phase is explained in Appendix 2.B. It does not play a role in many of the physical results that we derive below. For this reason, we do not elaborate on it any further in the main text.

While clustering imposes constraints on the ratios of the coefficients a_λ, it does not determine the overall normalisation. The upshot is that we are left with a family of boundary states, depending on the phases θ_i, which preserve the symmetry $\mathcal{R}$ and take the form

$$|\theta;\mathcal{R}\rangle = g_{\mathcal{R}} \sum_{\lambda\in\Lambda[\mathcal{R}]} e^{i\gamma_{\mathcal{R}}(\lambda)}\, e^{i\theta\cdot\lambda}\, \|\lambda, -\mathcal{R}\lambda;\mathcal{R}\rangle\!\rangle \tag{2.20}$$

We have taken the opportunity to rebrand the overall normalisation as $g_{\mathcal{R}} \equiv a_0$. This is appropriate, for $g_{\mathcal{R}}$ can be identified as the Affleck-Ludwig central charge of our boundary states [69],

$$g_{\mathcal{R}} = \langle 0, 0|\theta;\mathcal{R}\rangle$$

This boundary central charge has a number of avatars; it can be thought of as the boundary contribution to the free energy $\mathcal{Z}_{AB}(q)$ or, relatedly, to the boundary entropy. For the boundary states (2.20), we claim that the correct normalisation is

$$g_{\mathcal{R}} = \sqrt{\mathrm{Vol}(\Lambda[\mathcal{R}])} \tag{2.21}$$

where Vol($\Lambda[\mathcal{R}]$) is the volume of the primitive unit cell of the lattice Λ. The boundary central charge has the property that $g_{\mathcal{R}} \geq 1$ but, as is to be expected, $g_{\mathcal{R}}$ need not be an integer. The same result for the central charge, albeit in a rather different setting, was previously derived in [33] where it appeared as the tension of a D-brane.

The normalisation $g_{\mathcal{R}}$ is fixed by the Cardy condition (1.12). This states that the matrix element $\mathcal{Z}_P(q)$ computed between any two boundary states must have the interpretation of a partition function on an interval. For a general conformal field theory, this is the requirement that the partition function $\mathcal{Z}_{AB}(q)$ can be written as the sum of Virasoro characters in the open-string picture, weighted by positive integers.

For us, there are two parts to the story. In this section, we will consider the Cardy condition with the same symmetry $\mathcal{R}$ imposed at the two ends of the interval. In this case the whole system has an unbroken $U(1)^N$ symmetry and the Virasoro characters should be replaced by those of the appropriate chiral algebra. We will show that the normalisation (2.21) is the minimal choice that satisfies the Cardy condition. Applications of this condition can be found, for example, in [21, 32].

Ultimately, however, the Cardy condition is a statement about different boundary conditions A and B on each end of the interval, so we should study the system with two different symmetries $\mathcal{R}$ and $\mathcal{R}'$. We will turn to this in Section 2.2 and show that the result (2.21) continues to hold.

To proceed, we construct the Virasoro generators through the usual Sugawara construction,

$$L_n = \frac{1}{2}\sum_{i=1}^{N}\sum_{m=\infty}^{\infty} :J_{i,m}J_{i,n-m}: \qquad \bar{L}_n = \frac{1}{2}\sum_{i=1}^{N}\sum_{m=\infty}^{\infty} :\bar{J}_{i,m}\bar{J}_{i,n-m}: \tag{2.22}$$

The matrix element between two states, $|\theta;\mathcal{R}\rangle$ and $|\theta';\mathcal{R}\rangle$, each of which preserves the same symmetry, is

$$\mathcal{Z}_P(q) = \langle\theta';\mathcal{R}|(-1)^F q^{\frac{1}{2}(L_0+\bar{L}_0-c/12)}|\theta;\mathcal{R}\rangle$$

where, for us, the bulk central charge is $c = N$. The factor of $(-1)^F$ is present because, if $|A\rangle$ describes some boundary condition, then the *same* boundary condition at the other end is described by $\langle A|\,(-1)^F$ rather than $\langle A|$[4]. Here F is the holomorphic fermion number and should not be confused with the total fermion number $F+\bar{F}$.

It might seem odd that we had to single out F over $\bar{F}$. But there is actually no arbitrariness, as $F=\bar{F}$ holds for any valid boundary state. To see that this holds for our states $|\theta;\mathcal{R}\rangle$, note that acting on the ground state (2.13) in each charge sector $(\lambda,\bar{\lambda})$, the holomorphic fermion parity is given by

$$(-1)^F|\lambda,\bar{\lambda}\rangle=(-1)^{\sum_i\lambda_i}|\lambda,\bar{\lambda}\rangle=(-1)^{\lambda^2}|\lambda,\bar{\lambda}\rangle$$

where $\lambda^2=\sum_i\lambda_i^2$. Similarly, the antiholomorphic fermion number is $(-1)^{\bar{F}}=(-1)^{\bar{\lambda}^2}$. But since we restrict to charge sectors obeying $\bar{\lambda}=-\mathcal{R}\lambda$, we necessarily have $\lambda^2=\bar{\lambda}^2$ and so $F=\bar{F}$, as is necessary for a fermion in the presence of a boundary.

With the same matrices $\mathcal{R}$ specifying both boundary states, the $\mathcal{R}$-dependence in the exponent of (2.17) cancels when taking the inner product. (This uses the fact that $\mathcal{R}^T\mathcal{R}=1$.) Instead, the $\mathcal{R}$-dependence manifests itself only in the choice of lattice $\Lambda[\mathcal{R}]$ that we sum over, with the matrix element given by

$$\begin{aligned}\mathcal{Z}_P(q)&=g_{\mathcal{R}}^2\sum_{\lambda\in\Lambda[\mathcal{R}]}e^{i(\theta-\theta')\cdot\lambda}(-1)^{\lambda^2}q^{\frac{1}{4}(\lambda^2+\bar{\lambda}^2)}\prod_{n=1}^{\infty}\frac{q^{-N/24}}{(1-q^n)^N}\\&=g_{\mathcal{R}}^2\sum_{\lambda\in\Lambda[\mathcal{R}]}e^{i(\theta-\theta')\cdot\lambda}(-1)^{\lambda^2}\frac{q^{\frac{1}{2}\lambda^2}}{\eta(\tau)^N}\end{aligned}$$

where, in the Dedekind eta function, we've reverted to the argument τ, related to q via $q=e^{2\pi i\tau}$. The modular $\mathcal{S}$-transform of this partition function is

$$\mathcal{Z}_{AB}(q)=\int d^Nx\left(g_{\mathcal{R}}^2\sum_{\lambda\in\Lambda[\mathcal{R}]}e^{i(\theta-\theta')\cdot\lambda}(-1)^{\lambda^2}e^{2\pi ix\cdot\lambda}\right)\frac{q^{\frac{1}{2}x^2}}{\eta(\tau)^N}\tag{2.23}$$

In order that (2.23) can be interpreted as an interval partition function of the form $\mathrm{Tr}_{\mathcal{H}_{AB}}(q^{L_0-\frac{c}{24}})$, it must be a sum of Virasoro characters weighted by positive-integer coefficients. Actually, since both boundary conditions preserve the same $U(1)^N$ symmetry, these characters must fit together into representations of the corresponding chiral algebra,

$$[\mathcal{J}_{\alpha,n},\mathcal{J}_{\beta,m}]=n\delta_{n+m,0}\mathcal{M}_{\alpha\beta}$$

[4]This can be seen, for example, by computing the partition function of a single Dirac fermion. If $|A\rangle$ corresponds to the vector-like boundary condition (2.1) given by $\psi=e^{i\theta}\bar{\psi}$, then $\langle A|$ corresponds to $\psi=-e^{i\theta}\bar{\psi}$. The need for this minus sign was also discussed in [37] (see footnote 69).

where we've introduced $\mathcal{M}_{\alpha\beta} = Q_{\alpha i} Q_{\beta i} = \bar{Q}_{\alpha i} \bar{Q}_{\beta i}$. Irreducible representations of this algebra are labelled by common eigenvalues of $\mathcal{J}_{\alpha,0}$. We denote these eigenvalues as $\boldsymbol{\lambda}_\alpha$, by analogy with (2.13). The Sugawara construction (2.22) tells us that the Virasoro character associated to such an irrep is

$$q^{\frac{1}{2}\boldsymbol{\lambda}^T \mathcal{M}^{-1} \boldsymbol{\lambda}} \frac{1}{\eta(\tau)^N}$$

Since $\mathcal{M}$ is positive-definite, the power of q is ≥ 0. This means that the partition function (2.23) must be the sum of terms $q^h \eta(\tau)^{-N}$ with $h \geq 0$, weighted by positive integers. Note that any real numbers h are acceptable, because in general, the $\boldsymbol{\lambda}_\alpha$ need not obey any quantisation condition in the open sector.

The above requirement is easily seen to hold. First write $(-1)^{\lambda^2} = e^{i\pi \sum_{i=1}^N \lambda_i} = e^{i\pi \cdot \lambda}$ in the integrand of (2.23). Then we can apply the standard identity

$$\sum_{\lambda \in \Lambda[\mathcal{R}]} e^{2\pi i y \cdot \lambda} = \frac{1}{\mathrm{Vol}(\Lambda[\mathcal{R}])} \sum_{\mu \in \Lambda[\mathcal{R}]^\star} \delta^N(y - \mu)$$

where $\Lambda[\mathcal{R}]^\star$ is the dual lattice, defined by the condition that $\mu \cdot \lambda \in \mathbb{Z}$ for all $\mu \in \Lambda[\mathcal{R}]^\star$ and $\lambda \in \Lambda[\mathcal{R}]$. The choice of $g_\mathcal{R}$ in (2.21) was designed to cancel the $1/\mathrm{Vol}(\Lambda[\mathcal{R}])$ factor that arises in this sum. The upshot is that the partition function (2.23) becomes

$$\mathcal{Z}_{AB}(q) = \sum_{\mu \in \Lambda[\mathcal{R}]^\star} q^{\frac{1}{2}(\mu + \frac{\theta - \theta'}{2\pi} + \frac{1}{2})^2}$$

which is of the form promised.

2.2 Boundaries Preserving Different Symmetries

The consistency conditions of the previous section resulted in a natural guess for a large family of boundary states $|\theta; \mathcal{R}\rangle$,

$$|\theta; \mathcal{R}\rangle = \sqrt{\mathrm{Vol}(\Lambda[\mathcal{R}])} \sum_{\lambda \in \Lambda[\mathcal{R}]} e^{i\gamma_\mathcal{R}(\lambda)} e^{i\theta \cdot \lambda} \, \| \lambda, -\mathcal{R}\lambda; \mathcal{R} \rangle\!\rangle \tag{2.24}$$

However, the argument of the previous section does not fix the normalisation completely. For example, one could pick a positive integer $n_\mathcal{R}$ for each $\mathcal{R}$, and multiply each state by $\sqrt{n_\mathcal{R}}$, and they would continue to satisfy all the conditions we have imposed so far.

One can demonstrate that for simple boundary conditions like those considered in the introduction, no such rescaling is necessary: the boundary states $|\theta;\mathcal{R}\rangle$ already reproduce the correct partition functions, computed via canonical quantisation. However, for more general boundary states which cannot be realised as linear boundary conditions on fermion fields, checking the normalisation this way is not an option.

The first goal of this section is to show that the whole family of boundary states $|\theta;\mathcal{R}\rangle$ are, in fact, correctly normalised. To do this, we will check Cardy's condition between boundary states preserving different symmetries. We find that the partition function $\mathcal{Z}_{AB}$ is indeed always sensible, and that this comes about in a non-trivial way. The simplest interpretation is that all the integers $n_{\mathcal{R}}$ should be chosen to be 1. This normalisation is also consistent with the states (2.24) being fundamental boundary states in the sense of Section 1.1.3; it is reassuring that our calculation independently recovers this property.

To start, we consider an interval in which different $U(1)^N$ symmetries are preserved at each end. The associated symmetries are those described by $\mathcal{R}$ and $\mathcal{R}'$ respectively, and the matrix element is

$$\mathcal{Z}_P(q) = \langle\theta';\mathcal{R}'|(-1)^F q^{\frac{1}{2}(L_0+\bar{L}_0-c/12)}|\theta;\mathcal{R}\rangle \tag{2.25}$$

This time, the $\mathcal{R}$ matrices in the exponent (2.17) of the two states do not cancel. A direct evaluation gives

$$\mathcal{Z}_P(q) = g_{\mathcal{R}}g_{\mathcal{R}'}\left[\sum_{\lambda\in\Lambda[\mathcal{R},\mathcal{R}']} e^{i(\gamma_{\mathcal{R}}(\lambda)-\gamma_{\mathcal{R}'}(\lambda))}\, e^{2\pi i(\frac{\theta-\theta'}{2\pi}+\frac{1}{2})\cdot\lambda}\, q^{\frac{1}{2}\lambda^2}\right]\frac{1}{q^{N/24}}\prod_{n=1}^{\infty}\frac{1}{\det(\mathbb{1}-q^n\mathcal{R}^T\mathcal{R}')}$$

Here we have introduced a new lattice $\Lambda[\mathcal{R},\mathcal{R}']$, which arises from the need to sum over only those charge sectors $(\lambda,\bar{\lambda})$ compatible with both symmetries—that is, satisfying both $\bar{\lambda}=-\mathcal{R}\lambda$ and $\bar{\lambda}=-\mathcal{R}'\lambda$. For these reasons, we shall call it the 'intersection lattice'. It is defined by

$$\Lambda[\mathcal{R},\mathcal{R}'] = \left\{\, \lambda\in\mathbb{Z}^N \;:\; \mathcal{R}\lambda=\mathcal{R}'\lambda\in\mathbb{Z}^N \,\right\} \tag{2.26}$$

We would like to compute the transformation of the partition function $\mathcal{Z}_P(q)$ under the modular $\mathcal{S}$-transformation. We start by dealing with the factor

$$q^{N/24}\prod_{n=1}^{\infty}\det\left(1-q^n\mathcal{R}^T\mathcal{R}'\right) = \prod_r q^{1/24}\prod_{n=1}^{\infty}(1-rq^n)$$

where the product $\prod_r$ is over the N eigenvalues of $\mathcal{R}^T\mathcal{R}'$. Since this is an orthogonal matrix, its eigenvalues are either ± 1 or occur in complex-conjugate pairs of phases. To establish notation for this, we introduce

$$n_{\pm} = \text{Number of } \pm 1 \text{ eigenvalues}$$

We then write the remainder as $e^{\pm 2\pi i t}$, where t ranges over some multiset $T \subset (0, \frac{1}{2})$. The various contributions of these eigenvalues to the product are

$$\begin{aligned} +1 \quad &\Rightarrow \quad q^{1/24}\prod_{n=1}^{\infty}(1-q^n) = \eta(\tau) \\ -1 \quad &\Rightarrow \quad q^{1/24}\prod_{n=1}^{\infty}(1+q^n) = \frac{\eta(2\tau)}{\eta(\tau)} \\ e^{\pm 2\pi i t} \quad &\Rightarrow \quad q^{1/12}\prod_{n=1}^{\infty}\left(1-e^{2\pi i t}q^n\right)\left(1-e^{-2\pi i t}q^n\right) = \frac{1}{2\sin(\pi t)}\frac{\theta_1(t|\tau)}{\eta(\tau)} \end{aligned}$$

where we've adopted the theta-function conventions of [40]. For each of these, the modular $\mathcal{S}$-transformations are given by

$$\begin{aligned} \eta(\tau) \quad &\longrightarrow \quad \sqrt{-i\tau}\,\eta(\tau) \\ \frac{\eta(2\tau)}{\eta(\tau)} \quad &\longrightarrow \quad \frac{1}{\sqrt{2}}\frac{\eta(\tau/2)}{\eta(\tau)} \\ \frac{1}{2\sin(\pi t)}\frac{\theta_1(t|\tau)}{\eta(\tau)} \quad &\longrightarrow \quad -\frac{i\,q^{t^2/2}}{2\sin(\pi t)}\frac{\theta_1(t\tau|\tau)}{\eta(\tau)} \end{aligned}$$

Next, we deal with the factor in $\mathcal{Z}_P(q)$ involving the sum over lattice sites. We need to write the factor of $e^{i(\gamma_{\mathcal{R}}(\lambda)-\gamma_{\mathcal{R}'}(\lambda))}$ as an exponential linear in λ. For this, we appeal to a fact from Appendix 2.B, which states that for all $\lambda \in \Lambda[\mathcal{R},\mathcal{R}']$,

$$e^{i(\gamma_{\mathcal{R}}(\lambda)-\gamma_{\mathcal{R}'}(\lambda))} = (-1)^{s\cdot\lambda}$$

for some vector $s \in \Lambda[\mathcal{R},\mathcal{R}']^{\star}$. The exact expression for s won't concern us here, especially since its value is actually convention-dependent. With the sum now in the form of a theta function, we can proceed as before, this time using the modular $\mathcal{S}$-transformation property

$$\sum_{\lambda\in\Lambda[\mathcal{R},\mathcal{R}']} e^{2\pi i y\cdot\lambda}q^{\frac{1}{2}\lambda^2} \quad \longrightarrow \quad \sqrt{-i\tau}^{\,\dim(\Lambda[\mathcal{R},\mathcal{R}'])}\frac{1}{\mathrm{Vol}(\Lambda[\mathcal{R},\mathcal{R}'])}\sum_{\mu\in\Lambda[\mathcal{R},\mathcal{R}']^{\star}} q^{\frac{1}{2}(\mu+\Pi(y))^2}$$

where $\Pi(y)$ denotes the orthogonal projection of the vector y onto the subspace spanned by $\Lambda[\mathcal{R},\mathcal{R}']$. Combining everything so far, we have

$$\mathcal{Z}_{AB}(q) = g_{\mathcal{R}}g_{\mathcal{R}'}\left[\frac{\sqrt{-i\tau}^{\,\dim(\Lambda[\mathcal{R},\mathcal{R}'])}}{\mathrm{Vol}(\Lambda[\mathcal{R},\mathcal{R}'])}\sum_{\mu\in\Lambda[\mathcal{R},\mathcal{R}']^\star} q^{\frac{1}{2}(\mu+\Pi(\frac{s}{2}+\frac{\theta-\theta'}{2\pi}+\frac{1}{2}))^2}\right]$$
$$\times\left[\frac{1}{\sqrt{-i\tau}\,\eta(\tau)}\right]^{n_+}\left[\frac{\sqrt{2}\,\eta(\tau)}{\eta(\tau/2)}\right]^{n_-}\prod_{t\in T}\left[\frac{2i\sin(\pi t)}{q^{t^2/2}}\frac{\eta(\tau)}{\theta_1(t\tau|\tau)}\right]$$

Importantly, factors of $\sqrt{-i\tau}$ appear in two places: there are $\dim(\Lambda[\mathcal{R},\mathcal{R}'])$ factors from the lattice factor, and $-n_+$ from the $+1$ eigenvalues of $\mathcal{R}^T\mathcal{R}'$. If we are to interpret this as the partition function of a theory on the interval, these must cancel meaning that we must have $\dim(\Lambda[\mathcal{R},\mathcal{R}']) = n_+$. Happily this is the case, as can be seen from the definition (2.26), which says that λ is constrained to obey $\mathcal{R}^T\mathcal{R}'\lambda = \lambda$. Another immediate simplification is to make the replacement

$$(\sqrt{2}\,)^{n_-}\prod_{t\in T}2\sin(\pi t) = \sqrt{\det{}'(\mathbb{1}-\mathcal{R}^T\mathcal{R}')}$$

where $\det'$ denotes the product over non-zero eigenvalues.

The upshot is that the $\mathcal{S}$-transformed partition function is given by

$$\mathcal{Z}_{AB}(q) = \frac{\sqrt{\mathrm{Vol}(\Lambda[\mathcal{R}])\,\mathrm{Vol}(\Lambda[\mathcal{R}'])}}{\mathrm{Vol}(\Lambda[\mathcal{R},\mathcal{R}'])}\sqrt{\det{}'(\mathbb{1}-\mathcal{R}^T\mathcal{R}')}$$
$$\times\sum_{\mu\in\Lambda[\mathcal{R},\mathcal{R}']^\star} q^{\frac{1}{2}(\mu+\Pi(\frac{s}{2}+\frac{\theta-\theta'}{2\pi}+\frac{1}{2}))^2} \tag{2.27}$$
$$\times\left[\frac{\eta(\tau)}{\eta(\tau/2)}\right]^{n_-}\prod_{t\in T}\left[\frac{1}{q^{t^2/2}}\frac{i\,\eta(\tau)}{\theta_1(t\tau|\tau)}\right]\frac{1}{\eta(\tau)^{n_+}}$$

We have separated the terms into three groups, each of which will play its own distinct role in what follows.

2.2.1 Ground State Degeneracy

The partition function (2.27) describes the fermions on the interval, with different boundary conditions on the left and right, corresponding to $|\theta';\mathcal{R}'\rangle$ and $|\theta;\mathcal{R}\rangle$ respectively. We would like to compute the number of ground states of this system.

Consider first the final term $1/\eta(\tau)^{n_+}$. The integer n_+ has yet a third interpretation: the intersection of the two $U(1)^N$ symmetry groups preserved by the two boundaries $\mathcal{R}$

and $\mathcal{R}'$ is $U(1)^{n_+}$. To see this, note that a common $U(1)$ symmetry corresponds to a pair of vectors $s_\alpha \in \mathbb{Z}^N$ and $s'_\alpha \in \mathbb{Z}^N$ such that $(Q_{i\alpha}, \bar{Q}_{i\alpha})s_\alpha = (Q'_{i\alpha}, \bar{Q}'_{i\alpha})s'_\alpha$. In terms of the vector $Q_{i\alpha}s_\alpha$, these conditions again reduce to the requirement that $Q_{i\alpha}s_\alpha$ is an eigenvector of $\mathcal{R}^T\mathcal{R}'$ with eigenvalue $+1$.

We can then run a similar argument to what we saw in Section 2.1.2: because the boundary conditions preserve a common $U(1)^{n_+}$, the Hilbert space must furnish a representation of the $\mathfrak{u}(1)^{n_+}$ current algebra. The structure of such representations forces the partition function to contain a factor of $1/\eta(\tau)^{n_+}$. Thus, the final term of (2.27) is necessarily present in order that the partition function be valid, but as far as the degeneracy is concerned, it can be discarded.

Other terms of (2.27) have no bearing on either the validity of the partition function or the degeneracy. In particular, for these purposes we can completely ignore

$$\frac{\eta(\tau)}{\eta(\tau/2)} = q^{1/48}\prod_{n=1}^{\infty}(1+q^{n/2})$$
$$\frac{i\,\eta(\tau)}{\theta_1(t\tau|\tau)} = q^{-1/12}q^{t/2}\prod_{n=0}^{\infty}\left(1-q^{n+t}\right)^{-1}\left(1-q^{n+1-t}\right)^{-1}$$

as both are power series with positive integer coefficients and leading coefficient unity.[5] Using these expressions, one can also check that all powers of q occurring in (2.27) have exponent $\geq -N/24$. That is, all Virasoro weights in the open sector are ≥ 0, as is consistent for a unitary theory.

The lattice term

$$\sum_{\mu\in\Lambda[\mathcal{R},\mathcal{R}']^\star} q^{\frac{1}{2}\left(\mu+\Pi\left(\frac{s}{2}+\frac{\theta-\theta'}{2\pi}+\frac{1}{2}\right)\right)^2} \tag{2.28}$$

is more interesting. For generic values of the phases, parameterised by θ and θ', this power series has leading coefficient unity. However, at certain symmetrical values of the phases, the coefficient of the leading term may jump from 1 to a higher value. This corresponds to the kind of behaviour we saw in the introduction, where the ground state degeneracy of a single Dirac fermion on an interval is typically 1, but may jump to 2 when the boundary state phases align.

[5]Both of these factors also supply a factor of $\frac{1}{\eta(\tau)}$. For the first, this follows from the identity $\frac{\eta(\tau)}{\eta(\tau/2)} = \frac{1}{\eta(\tau)}\sum_{n=0}^{\infty} q^{(n+1/2)^2/4}$. For the second, such a representation is also possible, although not in simple closed form. So (2.27) actually contains many more than n_+ copies of $\frac{1}{\eta(\tau)}$.

Not all the phases affect the physics. Rather, only the orthogonal projection of $\theta - \theta'$ onto $\Lambda[\mathcal{R}, \mathcal{R}']$, which can naturally be thought of as living in $\mathrm{Hom}(\Lambda[\mathcal{R}, \mathcal{R}'], U(1)) \cong U(1)^{n_+}$, enters into the exponent of (2.28). This implies that the less compatible the boundary conditions, the fewer means we have to affect them. This mirrors what we saw in the introduction for a single Dirac fermion.

In what follows, we first assume generic values of the phases so that (2.28) has no degeneracy. Later, when we discuss specific examples, we will explore how the ground state degeneracy jumps at specific values of the phases.

After stripping off all of the terms discussed so far, what's left of the partition function determines the ground state degeneracy. It is given by

$$G[\mathcal{R}, \mathcal{R}'] = \frac{\sqrt{\mathrm{Vol}(\Lambda[\mathcal{R}])\,\mathrm{Vol}(\Lambda[\mathcal{R}'])}}{\mathrm{Vol}(\Lambda[\mathcal{R}, \mathcal{R}'])} \sqrt{\det{}'(\mathbb{1} - \mathcal{R}^T \mathcal{R}')} \tag{2.29}$$

As a sanity check, note that if we put the same boundary conditions on each end, then we generically have a unique ground state: $G[\mathcal{R}, \mathcal{R}] = 1$. We will give a number of more intricate examples in Section 2.3. This formula bears a tantalising similarity to a result by Kapustin on the ground state degeneracy of Abelian quantum Hall states with topological order on the boundary [75]; it would be interesting to understand this relation better.

The number of ground states of the system should be an integer. Indeed, this is one of the key requirements of the boundary conformal field theory approach. It is not at all obvious that $G[\mathcal{R}, \mathcal{R}']$, defined in (2.29), is integer-valued. We claim that it almost is.

Specifically, we show that – under certain circumstances that we detail below – the matrices $\mathcal{R}$ fall into two separate classes which, following the introduction, we call vector-like $\mathcal{V}$ and axial-like $\mathcal{A}$. When $\mathcal{R}$ and $\mathcal{R}'$ are both taken from the same class, the ground state degeneracy is indeed an integer as it should be. However, if $\mathcal{R} \in \mathcal{V}$ and $\mathcal{R}' \in \mathcal{A}$, we find $G[\mathcal{R}, \mathcal{R}'] \in \sqrt{2}\,\mathbb{Z}$. The interpretation of this is that the two classes of ground states are mutually incompatible since they give rise to a Majorana zero mode.

2.2.2 The Two Classes of Boundary States

We conjecture that $G[\mathcal{R}, \mathcal{R}']$ takes values in $\mathbb{Z} \cup \sqrt{2}\,\mathbb{Z}$. Further, we conjecture the existence of two classes $\mathcal{V}$ and $\mathcal{A}$ such that the presence of a $\sqrt{2}$ is dictated by whether $\mathcal{R}$ and $\mathcal{R}'$ lie in different classes.

These conjectures do not seem easy to prove in full generality. We have been able to demonstrate that they hold in large classes of examples. In this section, we will show the following.

- Task 1: For a large class of examples, we prove the above conjectures, and, in the process, extract a criterion that determines which of the two classes $\mathcal{V}$ and $\mathcal{A}$ a given symmetry $\mathcal{R}$ falls into.
- Task 2: For an even larger class of examples, we prove a weaker version with $\mathbb{Z} \cup \sqrt{2}\,\mathbb{Z}$ replaced with $\mathbb{Q} \cup \sqrt{2}\,\mathbb{Q}$, again extracting a criterion for the classes $\mathcal{V}$ and $\mathcal{A}$.
- Task 3: By assuming the conjecture holds, we obtain a concrete criterion for the classes $\mathcal{V}$ and $\mathcal{A}$ in the general case.

Furthermore, in randomised numerical experiments, it is found that in all cases, the classes $\mathcal{V}, \mathcal{A}$ derived in the third line correctly predict whether $G[\mathcal{R}, \mathcal{R}']$ lies in $\mathbb{Z}$ or $\sqrt{2}\,\mathbb{Z}$, with no other values possible. We feel that this is convincing evidence in favour of the conjectures.

Task 1

In this section, we limit ourselves to choices of $\mathcal{R}$ and $\mathcal{R}'$ obeying the following two properties:

i) $\Lambda[\mathcal{R}, \mathcal{R}'] = \{0\}$ or, equivalently, $n_+ = 0$. This ensures that $\mathcal{R} - \mathcal{R}'$ is non-singular. Under this assumption, the number of ground states (2.29) takes the simplified form

$$G[\mathcal{R}, \mathcal{R}'] = \sqrt{\mathrm{Vol}(\Lambda[\mathcal{R}])\,\mathrm{Vol}(\Lambda[\mathcal{R}'])\,|\det(\mathcal{R} - \mathcal{R}')|} \tag{2.30}$$

ii) Neither $\mathcal{R}$ nor $\mathcal{R}'$ have eigenvalue -1. This allows the Cayley parametrisations

$$\mathcal{R} = \frac{\mathbb{1} - A}{\mathbb{1} + A} \quad \text{and} \quad \mathcal{R}' = \frac{\mathbb{1} - A'}{\mathbb{1} + A'}$$

This gives a one-to-one correspondence between the rational orthogonal matrix $\mathcal{R}$ with no -1 eigenvalues, and the rational anti-symmetric matrix A. The ground state degeneracy can then be written as

$$G[\mathcal{R}, \mathcal{R}'] = 2^{N/2}\,|\mathrm{Pf}(A - A')|\sqrt{\frac{\mathrm{Vol}(\Lambda[\mathcal{R}])\,\mathrm{Vol}(\Lambda[\mathcal{R}'])}{\det(\mathbb{1} + A)\det(\mathbb{1} + A')}}$$

Note that the combined requirements of i) and ii) mean that this proof holds only for rotation matrices with N even, but other than that, these assumptions are generic.

A simple warm-up

To begin with, we add one more assumption, namely that A, A' are integer-valued rather than merely rational-valued. This is straightforward to relax, and we will do so shortly. With these assumptions in place, we now associate an integer $n \in \{0, \ldots, N\}$ to the matrix $\mathcal{R}$,

$$n = \text{nullity}_{\mathbb{F}_2}(\mathbb{1} + A)$$

That is, n is the dimension of the kernel of the $N \times N$ matrix $\mathbb{1} + A$, regarded over the finite field $\mathbb{F}_2$. (Equivalently, n is the number of linearly independent vectors, with integer elements defined mod 2, which map to even-integer vectors under $\mathbb{1} + A$.) We then have, as shown in Appendix 2.C,

$$\text{Vol}(\Lambda[\mathcal{R}]) = 2^{-n} \det(\mathbb{1} + A)$$

Similarly, we can define the integer n' associated to $\mathcal{R}'$. The ground state degeneracy can then be written as

$$G[\mathcal{R}, \mathcal{R}'] = |\text{Pf}(A - A')| \, (\sqrt{2}\,)^{N-n-n'}$$

This is sufficient to prove the result we want, provided that $N \geq n + n'$. However, if $N < n + n'$ then we seemingly have a negative power of $\sqrt{2}$ and have to work a little harder. In fact, this situation has a nice linear-algebraic interpretation, since it guarantees that the two kernels intersect,

$$\dim_{\mathbb{F}_2}\Big(\ker_{\mathbb{F}_2}(\mathbb{1} + A) \cap \ker_{\mathbb{F}_2}(\mathbb{1} + A')\Big) \geq n + n' - N$$

That certainly implies

$$\text{nullity}_{\mathbb{F}_2}(A - A') \geq n + n' - N \tag{2.31}$$

We now utilise the fact that $A - A'$, being an antisymmetric integer matrix, has a Smith-like normal form,

$$U(A - A')U^T = \bigoplus_{i=1}^{N/2} \begin{pmatrix} 0 & \nu_i \\ -\nu_i & 0 \end{pmatrix}$$

where U is unimodular and the ν_i are integers. The nullity in equation (2.31) is then given in terms of this data by

$$\text{nullity}_{\mathbb{F}_2}(A - A') = 2 \cdot \#(\text{even } \nu_i)$$

We can conclude that there are at least $\lceil (n + n' - N)/2 \rceil$ even ν_i. Then, since

$$\text{Pf}(A - A') = \prod_{i=1}^{N/2} \nu_i$$

it follows that the pfaffian is divisible by $2^{\lceil \frac{1}{2}(n+n'-N) \rceil}$, which is just enough to offset the dangerous negative power of $(\sqrt{2}\,)^{\frac{1}{2}(N-n-n')}$. This ensures that, in all cases, $G[\mathcal{R}, \mathcal{R}']$ is an integer, or an integer times $\sqrt{2}$, as promised.

The derivation above also provides the criterion for whether a given boundary condition sits in class $\mathcal{V}$ or class $\mathcal{A}$. Since N is even, the irrational part of $G[\mathcal{R}, \mathcal{R}']$ is given by $(\sqrt{2}\,)^{n+n'}$. The ground state degeneracy fails to be an integer if $n \neq n' \bmod 2$. In other words, the class of boundary condition $\mathcal{R}$ is determined by $n \bmod 2$.

The rational case

With a little extra work, we can re-run the arguments of the last section in the case where A, A' are rational-valued. Once again, we start from

$$G[\mathcal{R}, \mathcal{R}'] = 2^{N/2}\, |\text{Pf}(A - A')| \sqrt{\frac{\text{Vol}(\Lambda[\mathcal{R}])\, \text{Vol}(\Lambda[\mathcal{R}'])}{\det(\mathbb{1} + A)\, \det(\mathbb{1} + A')}}$$

The difference now is that the pfaffian may only be rational, and therefore its denominator has to emerge out of the second expression in order to cancel it. To see how this works, we first need to construct a bunch of auxiliary data associated to $\mathcal{R}$:

- Write $A = \tilde{A}/g$ where $\tilde{A}$ is an integer matrix.

- Compute the Smith-like decomposition of $\tilde{A}$,

$$\tilde{A} = UDU^T \qquad D = J\,\mathrm{ddiag}(\nu_i) \qquad J = \bigoplus_{i=1}^{N/2} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

  where by 'ddiag$(\nu_1, \nu_2, \dots)$' we mean the diagonal matrix with each entry repeated twice, that is diag$(\nu_1, \nu_1, \nu_2, \nu_2, \dots)$.

- Define integers $g_i = \gcd(g, \nu_i)$.

- Define an integer matrix

$$X = U^{T,-1}\mathrm{ddiag}(g/g_i) + UJ\,\mathrm{ddiag}(\nu_i/g_i)$$

The analog of the integer n from the previous section is then defined to be

$$n = \mathrm{nullity}_{\mathbb{F}_2}(X)$$

It is shown in Appendix 2.C that

$$\mathrm{Vol}(\Lambda[\mathcal{R}]) = 2^{-n}\det(\mathbb{1} + A)\prod_{i=1}^{N/2}(g/g_i)^2$$

The new part of this expression is the product on the right. This will turn out to be precisely the integer required to cancel the denominator of the pfaffian. To see this, we plug the above result into the ground state degeneracy, yielding the result

$$G[\mathcal{R}, \mathcal{R}'] = \frac{|\mathrm{Pf}(g'\tilde{A} - g\tilde{A}')|}{\prod_{i=1}^{N/2} g_i\, g_i'}\,(\sqrt{2}\,)^{N-n-n'} \tag{2.32}$$

It's not hard to show that the fraction is integer-valued. For example, one can simply write it as

$$\frac{|\mathrm{Pf}(g'\tilde{A} - g\tilde{A}')|}{\prod_{i=1}^{N/2} g_i\, g_i'} = \det\Big[[UJ\mathrm{ddiag}(\tfrac{\nu_i}{g_i})]^T[U'^{,T,-1}\mathrm{ddiag}(\tfrac{g'}{g_i'})] \\ + [U^{T,-1}\mathrm{ddiag}(\tfrac{g}{g_i})]^T[U'J\mathrm{ddiag}(\tfrac{\nu_i'}{g_i'})]\Big]^{1/2} \tag{2.33}$$

Since the matrix involved on the right is an integer-valued one, the right side is manifestly the square root of an integer. Unfortunately, it's no longer manifestly rational. However, the left side is, so putting the two pieces of information together shows that the whole thing is indeed an integer.

The final piece of the argument is to show that the power of $\sqrt{2}$ in (2.32), if it ever goes negative, is compensated for by (2.33) becoming divisible by a power of 2. In the previous section, this went via an argument involving the intersection of two kernels. Similarly, here it follows from the linear-algebraic fact that for $N \times N$ matrices A, B, A', B',

$$\text{nullity}(A^T B' - B^T A') \geq \text{nullity}(A - B) + \text{nullity}(A' - B') - N$$

Applied to our situation, this says that the matrix on the right hand side of (2.33) has $\mathbb{F}_2$-nullity at least $n + n' - N$, and therefore that its determinant is divisible by $2^{n+n'-N}$. Then, since (2.33) is an integer, it follows that (2.33) is divisible by $2^{\lceil \frac{1}{2}(n+n'-N) \rceil}$. This establishes the claimed integrality property of $G[\mathcal{R}, \mathcal{R}']$.

Task 2

In this section we will concern ourselves purely with the irrational part of $G[\mathcal{R}, \mathcal{R}']$. By freeing us of the burden of having to show that the rational part is actually an integer, we will be able to establish the rest of the conjecture in greater generality.

This time we will only rely on the assumption that $\mathcal{R}$ and $\mathcal{R}'$ have a Cayley parametrisation. This assumption restricts us to rotation matrices, but is otherwise generic. We start from the general expression (2.29),

$$G[\mathcal{R}, \mathcal{R}'] = \frac{\sqrt{\text{Vol}(\Lambda[\mathcal{R}]) \, \text{Vol}(\Lambda[\mathcal{R}'])}}{\text{Vol}(\Lambda[\mathcal{R}, \mathcal{R}'])} \sqrt{\det{}'(\mathbb{1} - \mathcal{R}^T \mathcal{R}')}$$

The new ingredients are the volume of the intersection lattice, and the replacement of det with det′. As we shall see, these complications cancel one another. Let us deal with the latter complication first. Substituting in the Cayley parametrisations, we have

$$\det{}'(\mathbb{1} - \mathcal{R}^T \mathcal{R}') = \det{}'\left((-2) \frac{1}{(\mathbb{1} - A)(\mathbb{1} + A')} (A - A') \right)$$

In the previous sections, we could simply pull out the factors of $\frac{1}{\mathbb{1}-A}$ and $\frac{1}{\mathbb{1}-A'}$ from the determinant. However, for det′, this is no longer an allowed operation. Instead, we must invoke the Smith-like decomposition of $A - A'$,

$$U(A - A')U^T = D = \bigoplus_{i=1}^{(N-k)/2} \begin{pmatrix} 0 & \nu_i \\ -\nu_i & 0 \end{pmatrix} \oplus \bigoplus_{i=1}^{k} (0)$$

where U is unimodular, the ν_i are nonzero rationals, and k is the nullity of $A - A'$. Inserting this decomposition into the previous expression, we may then separate it into two factors as follows:

$$\begin{aligned} \det{}'(\mathbb{1} - \mathcal{R}^T\mathcal{R}') &= 2^{N-k} \det{}'\left(\frac{1}{U(\mathbb{1} - A)(\mathbb{1} + A')U^T} D\right) \\ &= 2^{N-k} \det\left(\frac{1}{U(\mathbb{1} - A)(\mathbb{1} + A')U^T}\Bigg|_{\boxplus}\right) \det{}'(D) \end{aligned}$$

Here, the symbol in front of the matrix in the second line instructs us to restrict to the top-left $(N - k) \times (N - k)$ block of that matrix. For the identity we have just used to be valid, this block must be invertible; one can check that this is indeed the case.

We now shift our attention to the term $\mathrm{Vol}(\Lambda[\mathcal{R}, \mathcal{R}'])$. To deal with this, we need to find a parametrisation of the lattice $\Lambda[\mathcal{R}, \mathcal{R}']$. Recalling definition (2.26),

$$\Lambda[\mathcal{R}, \mathcal{R}'] = \left\{ \lambda \in \mathbb{Z}^N \ : \ \mathcal{R}\lambda = \mathcal{R}'\lambda \in \mathbb{Z}^N \right\}$$

we see that λ is necessarily an element of $\ker(\mathcal{R} - \mathcal{R}')$. All elements of this kernel can be parametrised as $(\mathbb{1} + A')U^T v$, where v is a vector of the form $v = (0, \ldots, 0, v_1, \ldots, v_k)$, i.e. only its last k components are nonzero. We will use $\mathbb{R}^k$ to denote such vectors. On top of this, v is constrained by the fact that both λ and $\mathcal{R}'\lambda$ must be integer vectors, which forces v to lie in the sublattice

$$\Lambda_v = \left\{ v \in \mathbb{R}^k \ : \ (\mathbb{1} + A')U^T v \in \mathbb{Z}^N \ , \ (\mathbb{1} - A')U^T v \in \mathbb{Z}^N \right\} \tag{2.34}$$

It follows that the lattice volume we are interested in is

$$\mathrm{Vol}(\Lambda[\mathcal{R}, \mathcal{R}']) = \det\left(U(\mathbb{1} - A)(\mathbb{1} + A')U^T\Big|_{\boxplus}\right)^{1/2} \mathrm{Vol}(\Lambda_v)$$

where this time, the symbol in front of the matrix instructs us to restrict to its lower-right $k \times k$ block.

Let us return to the ground state degeneracy. Inserting the results so far, we have

$$G[\mathcal{R},\mathcal{R}'] = 2^{(N-k)/2}\,\frac{\sqrt{\mathrm{Vol}(\Lambda[\mathcal{R}])\,\mathrm{Vol}(\Lambda[\mathcal{R}'])}}{\mathrm{Vol}(\Lambda_v)}\sqrt{\frac{\det\Big(\big(U(\mathbb{1}-A)(\mathbb{1}+A')U^T\big)^{-1}\big|_{\boxplus}\Big)}{\det\Big(U(\mathbb{1}-A)(\mathbb{1}+A')U^T\big|_{\boxminus}\Big)}}\sqrt{\det{}'(D)}$$

As remarked at the start, we shall be content to focus only on the irrational part of this expression. To this end, we may immediately drop certain factors. For example, the term

$$2^{(N-k)/2}$$

is rational, since $N-k$ is an even number. So also is the term

$$\mathrm{Vol}(\Lambda_v)$$

as Λ_v is a rank-k sublattice of $\mathbb{R}^k$, whose elements v are defined by the conditions (2.34) that certain integer-linear combinations of their components v_i are integers. Similarly,

$$\sqrt{\det{}'(D)} = \prod_{i=1}^{(N-k)/2} \nu_i$$

where each of the ν_i is rational. Finally, we may invoke the linear-algebraic fact that

$$\sqrt{\frac{\det\Big(\big(U(\mathbb{1}-A)(\mathbb{1}+A')U^T\big)^{-1}\big|_{\boxplus}\Big)}{\det\Big(U(\mathbb{1}-A)(\mathbb{1}+A')U^T\big|_{\boxminus}\Big)}} = \frac{1}{\sqrt{\det(\mathbb{1}+A)\,\det(\mathbb{1}+A')}}$$

to rewrite the remaining irrational part as

$$G[\mathcal{R},\mathcal{R}']_{\text{irrational}} = \sqrt{\frac{\mathrm{Vol}(\Lambda[\mathcal{R}])\,\mathrm{Vol}(\Lambda[\mathcal{R}'])}{\det(\mathbb{1}+A)\,\det(\mathbb{1}+A')}}$$

This is something we have calculated before. Indeed, when N is even, we have already seen how to associate to a matrix $\mathcal{R}$ integers n, g, g_i such that

$$\mathrm{Vol}(\Lambda[\mathcal{R}]) = 2^{-n}\det(\mathbb{1}+A)\,\prod_{i=1}^{N/2}(g/g_i)^2$$

Here we need the extension of this result to matrices with odd N. As before, one first constructs a set of auxiliary data:

- Write $A = \tilde{A}/g$ where $\tilde{A}$ is an integer matrix.
- Compute the Smith-like decomposition of $\tilde{A}$,

$$\tilde{A} = UDU^T \qquad D = [J\,\mathrm{ddiag}(\nu_i)] \oplus (0) \qquad J = \bigoplus_{i=1}^{(N-1)/2} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

- Define integers $g_i = \gcd(g, \nu_i)$.
- Define an integer matrix

$$X = U^{T,-1}[\mathrm{ddiag}(g/g_i) \oplus (1)] + U J\,[\mathrm{ddiag}(\nu_i/g_i) \oplus (0)]$$

The integer n associated to $\mathcal{R}$ is then $n = \mathrm{nullity}_{\mathbb{F}_2}(X)$. The analogous result for the lattice volume is

$$\mathrm{Vol}(\Lambda[\mathcal{R}]) = 2^{-n}\det(\mathbb{1} + A) \prod_{i=1}^{(N-1)/2} (g/g_i)^2$$

The upshot of all this is that the irrational part of the ground state degeneracy is simply given by

$$G[\mathcal{R}, \mathcal{R}']_{\text{irrational}} = (\sqrt{2}\,)^{n+n'}$$

We thus conclude, as before, that the class of $\mathcal{R}$ is dictated by the value of n mod 2.

Task 3

In the last section, we saw how to associate an integer $n \in \{0, \ldots, N\}$ to any matrix $\mathcal{R}$ that admits a Cayley parametrisation, such that the two classes of boundary states are labelled by n mod 2.

Here, we would like to cast away the final crutch of the existence of a Cayley parametrisation. To do this, we appeal to a classical result of Liebeck-Osborne [76], which states that any rational orthogonal matrix $\mathcal{R}$ can be written as

$$\mathcal{R} = D\,\mathcal{R}_0$$

where $D = \text{diag}(\pm 1, \ldots, \pm 1)$, and $\mathcal{R}_0$ admits a Cayley parametrisation. It is not hard to show that the irrational part of $G[\mathcal{R}, \mathcal{R}_0]$ is simply

$$(\sqrt{2}\,)^{n_-}$$

where n_- is the number of negative eigenvalues of D. This suggests the following definition of the integer $n(\mathcal{R})$ for a general matrix $\mathcal{R}$. First write $\mathcal{R} = D\,\mathcal{R}_0$ in the form above. Then set

$$n(\mathcal{R}) = n(\mathcal{R}_0) + n_- \mod 2 \tag{2.35}$$

where $n(\mathcal{R}_0)$ is calculated as in the previous section. As discussed at the start, numerical experiments then suggest that the conjecture

$$G[\mathcal{R}, \mathcal{R}'] \in \begin{cases} \mathbb{Z} & n(\mathcal{R}) = n(\mathcal{R}') \\ \mathbb{Z}\sqrt{2} & n(\mathcal{R}) \neq n(\mathcal{R}') \end{cases} \tag{2.36}$$

continues to hold even in the cases that remain unaddressed by our proof.

Properties of $n(\mathcal{R})$

It is natural to ask whether the map that we have defined,

$$n : O(N, \mathbb{Q}) \longrightarrow \mathbb{Z}_2$$

is a group homomorphism. Or perhaps the opposite quantity, $1 - n$? It turns out that for general N, both statements are false. However, as we shall see in the next section, in the special case of $N = 2$, n is a homomorphism. Indeed, in that case it is possible to define a mod-2 reduction map

$$O(2, \mathbb{Q}) \xrightarrow{\mathbb{F}_2} O(2, \mathbb{F}_2) \cong \mathbb{Z}_2$$

which, when multiplied by

$$O(2, \mathbb{Q}) \xrightarrow{\det} \{\pm 1\} \cong \mathbb{Z}_2$$

gives a homomorphism that coincides with our n. (We thank Holly Krieger for this observation.) However, a clean interpretation of $n(\mathcal{R})$ for $N > 2$ is not so obvious.

2.3 Examples

In this section, we describe a number of different examples of boundary states and the resulting ground state degeneracy.

2.3.1 Simple Boundary States

The two simplest boundary conditions are the generalisations of the vector and axial conditions described in the introduction, now imposed independently on each of the N fermions. These are given by

- Vector: $Q = \bar{Q} = \mathbb{1}$. This gives $\mathcal{R} = \mathbb{1}$ and $\text{Vol}(\Lambda) = 1$.
- Axial: $Q = \mathbb{1}$ and $\bar{Q} = -\mathbb{1}$. This gives $\mathcal{R} = -\mathbb{1}$ and $\text{Vol}(\Lambda) = 1$.

If we impose the same boundary conditions at both ends, the generic ground state degeneracy is $G[\mathcal{R}, \mathcal{R}] = 1$. (As we have seen, this can be enhanced for special values of the phases.)

In contrast, if we impose vector boundary conditions at one end and axial boundary conditions at the other, we have $\Lambda[\mathcal{R}, \mathcal{R}'] = \{0\}$. In this case $\mathcal{R}^T\mathcal{R}' = -\mathbb{1}$ and the formula (2.29) gives

$$\text{Vector-Axial:} \quad G[\mathcal{R}, \mathcal{R}'] = 2^{N/2}$$

This is the expected answer since, as explained in the introduction, this system has N Majorana zero modes. This means that the vector and axial boundary conditions sit in the same class for N even, but different classes for N odd.

There is a third interesting boundary condition which arises in the study of fermions scattering off monopoles [15, 18]. Following [21], we refer to it as the *dyon boundary condition*. It is given by

- Dyon: It is simplest to specify the boundary condition in terms of the orthogonal matrix $\mathcal{R} = \bar{Q}^{-1}Q$, which is given by

$$\mathcal{R}_{ij} = \delta_{ij} - \frac{2}{N}$$

The charge lattice has $\text{Vol}(\Lambda) = N/2$ for N even and $\text{Vol}(\Lambda) = N$ for N odd. The corresponding charge matrices contain only ± 1. For $N = 4$ they are given by

$$Q_{\alpha i} = \begin{pmatrix} + & + & + & + \\ + & - & & \\ & + & - & \\ & & + & - \end{pmatrix} \qquad \text{and} \qquad \bar{Q}_{\alpha i} = \begin{pmatrix} - & - & - & - \\ + & - & & \\ & + & - & \\ & & + & - \end{pmatrix}$$

with the obvious extension to general N.

We now have two further pairings to consider:

The case of vector-dyon boundary conditions was considered in [21]. Here the matrix $\mathcal{R}^T\mathcal{R}'$ acts as a reflection along $\frac{1}{\sqrt{N}}(1\ldots1)$, which means that we have $n_- = 1$. The intersection of the charge lattices has $\text{Vol}(\Lambda[\mathcal{R},\mathcal{R}']) = \sqrt{N}$. The degeneracy of ground states is then

$$\text{Vector-Dyon}: \quad G[\mathcal{R},\mathcal{R}'] = \begin{cases} 1 & N \text{ even} \\ \sqrt{2} & N \text{ odd} \end{cases}$$

For axial-dyon boundary conditions, we again have $\text{Vol}(\Lambda[\mathcal{R},\mathcal{R}']) = \sqrt{N}$. Now, however, $\mathcal{R}^T\mathcal{R}'$ differs by a minus sign from the vector-dyon case which means that $n_- = N - 1$. This time, the ground state degeneracy is always an integer

$$\text{Axial-Dyon}: \quad G[\mathcal{R},\mathcal{R}'] = 2^{\lceil N/2 \rceil - 1}$$

We learn that the axial and dyon boundary condition lie in the same class.

2.3.2 Monopole Zero-Mode Counting

The examples above already allow us to make contact with monopole physics in four dimensions, in particular the counting of Jackiw-Rebbi zero modes. We will show that these are captured by the dyon boundary state.

To study monopoles, we return to the "second problem" of the Introduction, which featured the gauge theory

$$SU(2) \;+\; N \text{ Weyls in the fundamental representation} \;+\; \text{adjoint Higgs}$$

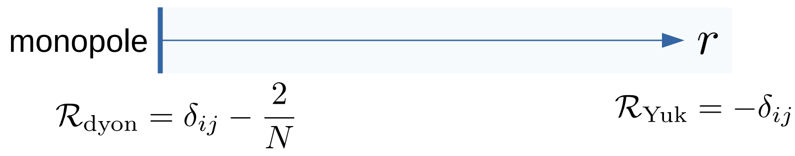


Figure 2.1: Effective boundary conditions generated by the monopole and Yukawa mass.

Putting an 't Hooft-Polyakov monopole at the origin, and turning on a Yukawa mass

$$\mathcal{L}_{\text{Yuk}} \sim \sum_{i=1}^{N} \psi_i \phi \psi_i$$

results in the appearance of N Jackiw-Rebbi zero modes [17]. These are real zero modes, so generate a Hilbert space of dimension $2^{N/2}$. Under the $SO(N)$ flavour symmetry left unbroken by the mass, the zero mode Hilbert space splits into the spinor and conjugate spinor of $SO(N)$, of equal dimension.

We would like to reproduce this zero-mode counting from boundary state calculations. To do this, we follow the ideas of [37], and deform the Higgs field ϕ so that it vanishes within some radius L of the monopole, then quickly ramps up to its asymptotic value outside of L. If L is taken large compared to the monopole core size, then the inner region is described by an effective 2d theory of N free Dirac fermions propagating between the dyon boundary state at $r = 0$ and an effective boundary state at $r = L$ set by the Yukawa mass term, shown in Figure 2.1. We expect the zero modes to be robust against this deformation, and therefore that the ground-state degeneracy $G[\mathcal{R}, \mathcal{R}']$ should count the number of zero modes. Below we will see that these expectations are borne out.

First we need to determine the relevant boundary states. As already explained, the dyon boundary state is known from [21] to be

$$(\mathcal{R}_{\text{dyon}})_{ij} = \delta_{ij} - \frac{2}{N}$$

Meanwhile, to determine the Yukawa mass boundary state, while this is possible by a detailed 4d to 2d translation of the mass term, it is equally valid to simply read it off from the symmetries preserved by $\mathcal{L}_{\text{Yuk}}$. We see that $\mathcal{L}_{\text{Yuk}}$ gives a mass to each flavour ψ_i separately, preserving $U(1)_A$. This uniquely fixes the boundary state to be

$$(\mathcal{R}_{\text{Yuk}})_{ij} = -\delta_{ij}$$

In other words, it is the same as the axial state of the previous section. Note that both boundary states preserve the $U(1)_A$ symmetry, as is necessary since $U(1)_A$ is the surviving gauge symmetry in the infra-red.

We have already seen that the ground state degeneracy between the dyon and Yukawa boundary states is given by $2^{N/2-1}$. (Here we restrict to N even to avoid a Witten anomaly in the 4d gauge theory.) At first sight, this does not agree with the answer $2^{N/2}$ expected on Jackiw-Rebbi zero mode grounds. However as we now show, the missing factor of two is supplied by the theta-angle dependent part of the degeneracy (2.28).

The dyon boundary state actually depends upon the $SU(2)$ theta-angle Θ of the 4d gauge theory, as emphasized in [18, 21]. We would like to determine how Θ manifests itself in the set of N phases θ_i appearing in the dyon boundary state. First, we recall that a shift in Θ is equivalent to a $U(1)_V$ transformation by angle $e^{i\Theta/N}$, where the factor of $1/N$ arises because there are N flavours. We apply this transformation to the dyon boundary state. Since the dyon boundary state preserves $U(1)_A$, we can equivalently rotate only the holomorphic fermions by twice the phase, $e^{2i\Theta/N}$. We therefore find $\theta_i = 2\Theta/N$.

Next we need to determine the orthogonal projection of the vector θ onto $\Lambda[\mathcal{R},\mathcal{R}']^*$. Since $\Lambda[\mathcal{R},\mathcal{R}']$ is one-dimensional and generated by $\lambda = (1,\ldots,1)$, no projection is necessary. The generator of the dual lattice is $\mu = \frac{1}{N}(1,\ldots,1)$. Putting these facts together, we find that the sum (2.28) takes the form[6]

$$\sum_{n\in\mathbb{Z}} q^{(n+\Theta/\pi+1/2)^2/2N} \tag{2.37}$$

Generically, the degeneracy of the lowest power of q in this expression is one. However, at the special values $\Theta = 0,\pi$, the degeneracy jumps to two. We therefore find the true degeneracy

$$\#\text{states} = \begin{cases} 2^{N/2} & \Theta = 0,\pi \\ 2^{N/2-1} & \Theta \neq 0,\pi \end{cases}$$

This is entirely consistent with the fact that the Hilbert space of Jackiw-Rebbi zero modes splits into $SO(N)$ representations as $\mathbf{s}\oplus\mathbf{c}$, with $\dim(\mathbf{s}) = \dim(\mathbf{c}) = 2^{N/2-1}$. For

[6] The formula (2.28) also includes other terms involving θ'_i and s, the latter being convention-dependent. Here we have simply used our freedom to shift the origin of Θ to eliminate these terms. The location of $\Theta = 0$ is then *defined* to be the location such that formula (2.37) takes the above form. As we shall see, this gives agreement with the 4d gauge theory.

generic Θ the energies of these representations are split, but cross at $\Theta = 0, \pi$ where the theory enjoys an additional CP symmetry.

2.3.3 Two Dirac Fermions

We now turn to the case of $N = 2$ fermions, where we can simply enumerate all possible boundary conditions and determine their class. This extends and completes the proof in Section 2.2.2, but only for this special case.

These boundary conditions include the example given in the introduction, where left-movers have charges $(3, 4)$ and right-movers have charges $(5, 0)$. However, our boundary state formalism require that a $U(1)^2$ symmetry is imposed on the boundary, which means that we must supplement the charges above with a second $U(1)$ symmetry. It is straightforward to find such symmetries: for example, we can take the left-movers to have charges $(-4, 3)$ and right-movers have charges $(0, 5)$. Alternatively, we could take the left-movers to have charges $(4, -3)$ and the right-movers to have charges $(0, 5)$. In what follows, we will see that all such boundary conditions can be associated to Pythagorean triples in this way. However, rather surprisingly, the choice of the minus signs in the second $U(1)$ can dramatically change the resulting physics.

We specify the boundary condition using the rational orthogonal matrix $\mathcal{R}$ defined in (2.12). Such matrices are either rotations or reflections and can be written accordingly as

$$\mathcal{R}_{\text{rot}} = \frac{1}{c}\begin{pmatrix} a & b \\ -b & a \end{pmatrix} \qquad \text{or} \quad \mathcal{R}_{\text{ref}} = \frac{1}{c}\begin{pmatrix} a & b \\ b & -a \end{pmatrix} \tag{2.38}$$

where a, b, c are co-prime integers with $a^2 + b^2 = c^2$ and $c > 0$.

It will be useful to first compute $\text{Vol}(\Lambda)$ for such boundary conditions. We have

Claim: $\text{Vol}(\Lambda) = c$

Proof: Consider rotation matrices. The charge lattice Λ consists of all integer-valued vectors $\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)$ such that $\mathcal{R}\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)$ is also integer-valued. In other words, we're looking for all integer solutions to

$$ax + by = cz \quad \text{and} \quad -bx + ay = cw$$

Since a, b are coprime, there exist integers λ, μ such that

$$a\lambda + b\mu = 1$$

Therefore any value of z can be attained by some (x, y), and for fixed z, the possible values of (x, y) are

$$(x, y) = cz(\lambda, \mu) + n(-b, a)$$

where n is a free integer. Plugging this into the second equation, we then find that w is automatically also an integer,

$$w = z(-b\lambda + a\mu) + cn$$

The lattice Λ is therefore spanned by $c(\lambda, \mu)$ and $(-b, a)$. The volume of the unit cell is

$$\mathrm{Vol}(\Lambda) = \det \begin{pmatrix} c\lambda & -b \\ c\mu & a \end{pmatrix} = c(\lambda a + \mu b) = c \tag{2.39}$$

The proof for reflection matrices proceeds in an identical fashion. □

Our next goal is to compute the ground state degeneracy (2.29) for an interval sandwiched between two boundaries $\mathcal{R}$ and $\mathcal{R}'$. As always, when $\mathcal{R} = \mathcal{R}'$, the ground state degeneracy is $G[\mathcal{R}, \mathcal{R}] = 1$. The remaining cases are less trivial.

First, it will prove useful to parameterise the Pythagorean triple (a, b, c) in (2.38) using Euclid's formula,

$$\mathcal{R}_{\mathrm{rot}}(p, q) = \frac{1}{p^2 + q^2} \begin{pmatrix} p^2 - q^2 & 2pq \\ -2pq & p^2 - q^2 \end{pmatrix} \tag{2.40}$$

and

$$\mathcal{R}_{\mathrm{ref}}(p, q) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \mathcal{R}_{\mathrm{rot}}(p, q) \tag{2.41}$$

with p, q co-prime.

Usually in applications of Euclid's formula, one further assumes that p and q are not both odd, which gives rise to a primitive Pythagorean triple. We do not insist on this condition here since it allows us to construct rotation matrices (2.38) with b odd. For example,

$$p = 2,\ q = 1 \quad \Rightarrow \quad \mathcal{R}_{\rm rot}(p,q) = \frac{1}{5}\left(\begin{array}{cc} 3 & 4 \\ -4 & 3 \end{array}\right)$$
$$p = 3,\ q = 1 \quad \Rightarrow \quad \mathcal{R}_{\rm rot}(p,q) = \frac{1}{5}\left(\begin{array}{cc} 4 & 3 \\ -3 & 4 \end{array}\right)$$

Nonetheless, as we go on, we will see that the distinction between p and q both odd, or one odd and one even, becomes more prominent. For example, the volume of the unit cell (2.39) is

$$\text{Vol}(\Lambda) = \begin{cases} p^2 + q^2 & \text{if either } p \text{ or } q \text{ is even} \\ \frac{1}{2}(p^2 + q^2) & \text{if } p \text{ and } q \text{ are both odd} \end{cases} \tag{2.42}$$

Indeed, ultimately we will see that the boundary conditions sit in one of two classes as follows:

$$\text{Class } \mathcal{V}: \quad \begin{cases} \text{Rotation matrices } \mathcal{R}_{\rm rot}(p,q) \text{ with either } p \text{ or } q \text{ even.} \\ \text{Reflection matrices } \mathcal{R}_{\rm ref}(p,q) \text{ with } p \text{ and } q \text{ both odd.} \end{cases}$$

and

$$\text{Class } \mathcal{A}: \quad \begin{cases} \text{Rotation matrices } \mathcal{R}_{\rm rot}(p,q) \text{ with } p \text{ and } q \text{ both odd.} \\ \text{Reflection matrices } \mathcal{R}_{\rm ref}(p,q) \text{ with either } p \text{ or } q \text{ even.} \end{cases}$$

To see this, we first consider two separate cases.

- Case 1: $\det(\mathcal{R}^T\mathcal{R}') = +1$ with $\mathcal{R} \neq \mathcal{R}'$.

  Here $\mathcal{R}$ and $\mathcal{R}'$ describe two different rotations or two different reflections. Either way, $\mathcal{R}^T\mathcal{R}'$ has no $+1$ eigenvalue and so $\Lambda[\mathcal{R},\mathcal{R}'] = \{0\}$. We can then use the simplified expression (2.30) for the ground state degeneracy. A direct evaluation, using the form of the matrices (2.38) gives

  $$G[\mathcal{R},\mathcal{R}'] = \sqrt{2(cc' - aa' - bb')}$$

  It is not at immediately obvious that this is an integer or $\sqrt{2}$ times an integer. However, invoking the Euclid form of the matrix (2.40) or (2.41), it is not hard to show that the ground state degeneracy can be written as

$$G[\mathcal{R},\mathcal{R}'] = \begin{cases} 2|pq'-qp'| & \text{if } (p,q) \text{ and } (p',q') \text{ lie in the same class} \\ \sqrt{2}\,|pq'-qp'| & \text{if } (p,q) \text{ and } (p',q') \text{ lie in different classes} \end{cases} \tag{2.43}$$

This confirms our classification if both matrices are rotations or both are reflections.

To illustrate this, consider the following three rotation matrices

$$\mathcal{R}_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \mathcal{R}_2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \quad \mathcal{R}_3 = \frac{1}{5}\begin{pmatrix} 3 & 4 \\ -4 & 3 \end{pmatrix} \quad \mathcal{R}_4 = \frac{1}{5}\begin{pmatrix} 4 & 3 \\ -3 & 4 \end{pmatrix}$$

From the discussion above, $\mathcal{R}_1$, $\mathcal{R}_2$ and $\mathcal{R}_3$ all lie in class $\mathcal{V}$ while $\mathcal{R}_4$ lies in class $\mathcal{A}$. The number of ground states in an interval with one of these boundary conditions imposed on each end is

	$\mathcal{R}_1$	$\mathcal{R}_2$	$\mathcal{R}_3$	$\mathcal{R}_4$
$\mathcal{R}_1$	1	2	2	$\sqrt{2}$
$\mathcal{R}_2$	2	1	4	$3\sqrt{2}$
$\mathcal{R}_3$	2	4	1	$\sqrt{2}$
$\mathcal{R}_4$	$\sqrt{2}$	$3\sqrt{2}$	$\sqrt{2}$	1

Although the number of ground states in class $\mathcal{V}$ in these examples have the form 2^n, as is familiar from quantising complex fermionic zero modes, it is clear from the general form (2.43) that we can get any even number of ground states in this class of examples.

The second case corresponds to one of the special cases not handled by the proof in Section 2.2.2, and requires a little more work. This is

- Case 2: $\det(\mathcal{R}^T\mathcal{R}') = -1$

  Here one of $\mathcal{R}$ and $\mathcal{R}'$ describes a rotation and the other a reflection. This means that the eigenvalues of $\mathcal{R}^T\mathcal{R}'$ are $+1$ and -1, and so $\det'(\mathbb{1} - \mathcal{R}^T\mathcal{R}') = +2$.

  The single $+1$ eigenvalue of $\mathcal{R}^T\mathcal{R}'$ implies that $\Lambda[\mathcal{R},\mathcal{R}']$ is one-dimensional. We must compute the volume of the unit cell of this lattice and this is a little involved. Without loss of generality, we take $\mathcal{R}_{\text{rot}}[p,q]$ and $\mathcal{R}'_{\text{ref}}(p',q')$. The unique $+1$ eigenvector of $\mathcal{R}^T\mathcal{R}'$ is, up to proportionality,

$$v = \begin{pmatrix} pp'-qq' \\ pq'+qp' \end{pmatrix} \quad \Rightarrow \quad \mathcal{R}_{\text{rot}}(p,q)v = \mathcal{R}_{\text{ref}}(p',q')v = \begin{pmatrix} pp'+qq' \\ pq'-qp' \end{pmatrix}$$

Clearly, both v and $\mathcal{R}_{\text{rot}}v = \mathcal{R}'_{\text{ref}}v$ are integer vectors. The trouble lies in the caveat of proportionality: it may be possible to divide v by some integer d so that the conclusion that we have an integer-valued eigenvector remains true. In fact, such a d is simply the greatest common divisor of the four components of v and $\mathcal{R}_{\text{rot}}v$,

$$d = \gcd(pp' - qq'\,;\, pq' + qp'\,;\, pp' + qq'\,;\, pq' - qp')$$

We have $d = 2$ if p, q, p', q' are all odd; otherwise $d = 1$. The one-dimensional lattice $\Lambda[\mathcal{R}, \mathcal{R}']$ is then spanned by the single vector v/d, and we have

$$\text{Vol}(\Lambda[\mathcal{R}, \mathcal{R}']) = \frac{|v|}{d} = \frac{\sqrt{(p^2+q^2)(p'^2+q'^2)}}{d}$$

We now have all the information we need to compute the ground state degeneracy (2.29). Using the expression (2.42) for the volume of the unit cells, we have

$$G[\mathcal{R}_{\text{rot}}, \mathcal{R}'_{\text{ref}}] = \begin{cases} 1 & \text{if } \mathcal{R}_{\text{rot}} \text{ and } \mathcal{R}'_{\text{ref}} \text{ belong to the same class} \\ \sqrt{2} & \text{if } \mathcal{R}_{\text{rot}} \text{ and } \mathcal{R}'_{\text{ref}} \text{ belong to different classes} \end{cases}$$

These results can be summarised in the following table, which shows the ground state degeneracy between all pairs of boundary states, with the red line reminding us that on the diagonal, the table should be ignored and the result taken to be one:

	$R_{\text{rot}}(p_2, q_2)$ not both odd	$R_{\text{ref}}(p_2, q_2)$ both odd	$R_{\text{rot}}(p_2, q_2)$ both odd	$R_{\text{ref}}(p_2, q_2)$ not both odd
$R_{\text{rot}}(p_1, q_1)$ not both odd	$2\lvert p_1q_2 - p_2q_1\rvert$	1	$\sqrt{2}\lvert p_1q_2 - p_2q_1\rvert$	$\sqrt{2}$
$R_{\text{ref}}(p_1, q_1)$ both odd	1	$2\lvert p_1q_2 - p_2q_1\rvert$	$\sqrt{2}$	$\sqrt{2}\lvert p_1q_2 - p_2q_1\rvert$
$R_{\text{rot}}(p_1, q_1)$ both odd	$\sqrt{2}\lvert p_1q_2 - p_2q_1\rvert$	$\sqrt{2}$	$2\lvert p_1q_2 - p_2q_1\rvert$	1
$R_{\text{ref}}(p_1, q_1)$ not both odd	$\sqrt{2}$	$\sqrt{2}\lvert p_1q_2 - p_2q_1\rvert$	1	$2\lvert p_1q_2 - p_2q_1\rvert$

1

Appendix 2.A Fermion Conventions

Our convention for a Majorana fermion in 1+1D is

$$S = \frac{i}{4\pi} \int dx dt \left(\chi_+ \partial_+ \chi_+ + \chi_- \partial_- \chi_- \right)$$

where $\partial_\pm = \partial_t \pm \partial_x$. This Euclideanises to the standard CFT action

$$S = \frac{1}{2\pi} \int dx d\tau \left(\chi \bar{\partial} \chi + \bar{\chi} \partial \bar{\chi} \right)$$

where $z = x + i\tau$, provided we define the new fermions by

$$\chi = e^{-i\pi/4} \chi_+ \quad \text{and} \quad \bar{\chi} = e^{+i\pi/4} \chi_-$$

The N Dirac fermions are built from $2N$ Majorana fermions via $\psi_i = \frac{1}{\sqrt{2}}(\chi_{2i-1} + i\chi_{2i})$. The corresponding $U(1)$ currents are

$$J_i = i \chi_{2i-1} \chi_{2i}$$

Appendix 2.B Clustering and the Cardy-Lewellen Sewing Conditions

In the rest of this appendix we describe a few subtleties that we felt were best avoided in the main text.

In Section 2.1.1, a set of ground states $|\lambda, \bar{\lambda}\rangle$ was introduced, but at the time we did not specify their phases. The easiest way to do this is via the bosonisation formula,[7]

$$\psi_i(z) = F_i \, t_i \, z^{-\lambda_i} \exp\left(-\sum_{n=1}^{\infty} \frac{z^n}{n} J_{i,-n} \right) \exp\left(\sum_{n=1}^{\infty} \frac{z^{-n}}{n} J_{i,n} \right)$$

Here, F_i is a ladder operator which moves between ground states as $F_i|\lambda, \bar{\lambda}\rangle = |\lambda - \hat{e}_i, \bar{\lambda}\rangle$, and t_i is a cocycle arising from Fermi statistics which acts by a phase on each ground state. The precise form of t_i (and its barred cousin $\bar{t}_i$) will depend on the phase convention

[7] Our handling of Klein factors takes inspiration from (though slightly differs from) [77].

chosen for the $|\lambda, \bar{\lambda}\rangle$. We stipulate them to be

$$t_i = (-1)^{\sum_{j=1}^{i-1} \lambda_j} \quad \text{and} \quad \bar{t}_i = (-1)^{\sum_{j=1}^{i-1} \bar{\lambda}_j + \sum_{j=1}^{N} \lambda_j} \tag{2.44}$$

and this then implicitly fixes the relative phases of the $|\lambda, \bar{\lambda}\rangle$.

In Section 2.1.2, it was claimed that the requirement of cluster decomposition in the presence of the boundary state $|a; \mathcal{R}\rangle$ dictates the form of the coefficients a_λ. Here we give more details.

To formulate the requirement of clustering, we start by placing the theory on the planar region $|z| \geq 1$, and impose the boundary condition $|a; \mathcal{R}\rangle$ at $|z| = 1$. Let $\mathcal{O}_i(z)$ be a list of all composite local operators built out of the fermions.[8] Then we demand that the following limit involving a ratio of normalised correlators is equal to one,

$$\lim_{|z|,|w| \to 1^+} \frac{\langle \mathcal{O}_i(z) \mathcal{O}_j(w) \rangle}{\langle \mathcal{O}_i(z) \rangle \langle \mathcal{O}_j(w) \rangle} = 1$$

where the limit is taken with $\arg(z)$ and $\arg(w)$ fixed.

The $\mathcal{O}_i(z)$ must have non-vanishing vev in the presence of the boundary. This condition will be met if our operator has compatible $U(1)^N$ charges $(q, \bar{q})$, in the sense that $\bar{q} = -\mathcal{R}q$. In particular, we are forced to take $q \in \Lambda[\mathcal{R}]$. To build an operator with these charges, we can take $|q_i|$ copies of each $\psi_i(z)$, and $|\bar{q}_i|$ copies of each $\bar{\psi}_i(z)$, and combine them into a composite operator $\mathcal{O}_q(z)$ using a suitable point-splitting regularisation. (If any of the charges q_i are negative, we should replace ψ_i with its complex conjugate, $\frac{1}{\sqrt{2}}(\chi_{2i-1} - i\chi_{2i})$.) The clustering requirement for $\mathcal{O}_q$ and $\mathcal{O}_p$ is then

$$\lim_{|z|,|w| \to 1^+} \frac{\langle 0,0|\mathcal{O}_q(z)\mathcal{O}_p(w)|a;\mathcal{R}\rangle \; \langle 0,0|a;\mathcal{R}\rangle}{\langle 0,0|\mathcal{O}_q(z)|a;\mathcal{R}\rangle \; \langle 0,0|\mathcal{O}_p(w)|a;\mathcal{R}\rangle} = 1$$

It turns out that the only interesting contribution to this expression comes from the $F_i\, t_i$ part of $\psi_i(z)$, and everything else can dropped. That is, we can make the replacement

$$\mathcal{O}_q(z) \longrightarrow \prod_{i=1}^{N} (F_i t_i)^{q_i} \prod_{i=1}^{N} (\bar{F}_i \bar{t}_i)^{\bar{q}_i}$$

[8] Here and throughout, we eschew the convention that operators should be written $\mathcal{O}(z, \bar{z})$, and instead prefer $\mathcal{O}(z)$ to emphasize the locality of the operator at z. (The latter convention is more helpful in the presence of boundaries.) Hence the $\mathcal{O}_i(z)$ are not implied to be holomorphic operators.

whereupon the clustering condition reduces to

$$\frac{a_{q+p}\,a_0}{a_q\,a_p}\frac{\langle q,\bar{q}|\mathcal{O}_p|q+p,\bar{q}+\bar{p}\rangle}{\langle 0,0|\mathcal{O}_p|p,\bar{p}\rangle}=1$$

The ratio of matrix elements in the above expression evaluates to $(-1)^{f_{\mathcal{R}}(q,p)}$ where

$$f_{\mathcal{R}}(q,p) \coloneqq \sum_{i=1}^{N} p_i \sum_{j=1}^{i-1} q_j + \sum_{i=1}^{N}(\mathcal{R}p)_i\left(\sum_{j=1}^{i-1}(\mathcal{R}q)_j + \sum_{j=1}^{N} q_j\right) \mod 2 \tag{2.45}$$

This is a symmetric bilinear form on $\Lambda[\mathcal{R}]$ taking values mod 2, whose corresponding quadratic form is fermion parity: $f_{\mathcal{R}}(\lambda,\lambda)=\lambda^2 \mod 2$. The clustering condition now takes the final form

$$\frac{a_{q+p}}{a_0}=\frac{a_q}{a_0}\frac{a_p}{a_0}(-1)^{f_{\mathcal{R}}(q,p)} \tag{2.46}$$

To solve it, let $\hat{f}_{\mathcal{R}}(q,p)$ be an arbitrary choice of lift of $f_{\mathcal{R}}(q,p)$ from a mod-2 to a mod-4 valued symmetric bilinear form. Then the general solution to (2.46) is

$$\frac{a_\lambda}{a_0}=e^{i\gamma_{\mathcal{R}}(\lambda)}e^{i\theta\cdot\lambda} \quad \text{where} \quad e^{i\gamma_{\mathcal{R}}(\lambda)} \coloneqq i^{\hat{f}_{\mathcal{R}}(\lambda,\lambda)} \tag{2.47}$$

Due to the freedom of choice in the lift $\hat{f}_{\mathcal{R}}(q,p)$, the reference solution $e^{i\gamma_{\mathcal{R}}(\lambda)}$ is actually ambiguous up to multiplication by $(-1)^{s\cdot\lambda}$ for any $s\in\Lambda[\mathcal{R}]^\star$. The ambiguity is equivalent to that of choosing a quadratic refinement of $(-1)^{f_{\mathcal{R}}(q,p)}$, and there is no canonical way to fix it. As a result, the origin of θ is also ambiguous up to shifts by $\pi\,\Lambda[\mathcal{R}]^\star$. On the other hand, the square of the reference solution is well-defined, and is equal to $(e^{i\gamma_{\mathcal{R}}(\lambda)})^2=(-1)^{f_{\mathcal{R}}(\lambda,\lambda)}=(-1)^{\lambda^2}$.

For completeness, we outline the proofs of the above properties of $f_{\mathcal{R}}$. We start by rewriting (no modulo)

$$f_{\mathcal{R}}(q,p)+f_{\mathcal{R}}(p,q)=(|p|+|\mathcal{R}p|)(|q|+|\mathcal{R}q|)-2p\cdot q$$

where we temporarily introduce $|p|$ to mean $\sum_{i=1}^N p_i$. Because $|p|=p^2$ mod 2, the first term on the RHS is a multiple of 4, hence

$$f_{\mathcal{R}}(q,p)+f_{\mathcal{R}}(p,q)=2p\cdot q \bmod 4$$

From this it follows both that $f_{\mathcal{R}}(q,p)=f_{\mathcal{R}}(p,q)$ mod 2 and $f_{\mathcal{R}}(q,q)=q^2$ mod 2.

Finally, in Section 2.2, we needed the fact that if $\mathcal{R}$ and $\mathcal{R}'$ are two different symmetries, then for all $\lambda \in \Lambda[\mathcal{R}, \mathcal{R}']$,

$$\frac{e^{i\gamma_{\mathcal{R}}(\lambda)}}{e^{i\gamma_{\mathcal{R}'}(\lambda)}} = (-1)^{s\cdot\lambda}$$

for some $s \in \Lambda[\mathcal{R}, \mathcal{R}']^{\star}$. (The precise value of s is actually ambiguous, for the reasons described above.[9]) To see this, first note that from (2.46),

$$\frac{e^{i\gamma_{\mathcal{R}}(q+p)}}{e^{i\gamma_{\mathcal{R}'}(q+p)}} = \frac{e^{i\gamma_{\mathcal{R}}(q)}}{e^{i\gamma_{\mathcal{R}'}(q)}} \frac{e^{i\gamma_{\mathcal{R}}(p)}}{e^{i\gamma_{\mathcal{R}'}(p)}} \frac{(-1)^{f_{\mathcal{R}}(q,p)}}{(-1)^{f_{\mathcal{R}'}(q,p)}}$$

and that $f_{\mathcal{R}}(q,p) = f_{\mathcal{R}'}(q,p)$ for $q, p \in \Lambda[\mathcal{R}, \mathcal{R}']$. This forces $\frac{e^{i\gamma_{\mathcal{R}}(\lambda)}}{e^{i\gamma_{\mathcal{R}'}(\lambda)}}$ to take the form $e^{i\theta\cdot\lambda}$. Since it also squares to $\frac{(-1)^{\lambda^2}}{(-1)^{\lambda^2}} = 1$, we must have $\theta \in \pi\, \Lambda[\mathcal{R}, \mathcal{R}']^{\star}$.

Appendix 2.C Lattice Calculations

We record here a technical calculation of lattice volumes that we used several times in Section 2.2.2. Let N be an even number, let A be a rational $N \times N$ antisymmetric matrix, and let $\mathcal{R} = \frac{\mathbb{1}-A}{\mathbb{1}+A}$. Then we claim that

$$\mathrm{Vol}(\Lambda[\mathcal{R}]) = 2^{-n} \det(\mathbb{1} + A) \prod_{i=1}^{N/2} (g/g_i)^2$$

where the integers n, g, g_i are constructed along the way during the proof.

Proof: To describe $\Lambda[\mathcal{R}]$, we need to find all integer solutions to

$$\mathcal{R}v = w$$

In terms of new variables $x = v - w$ and $y = v + w$, this reads

$$x = Ay$$

[9] One might hope that the ambiguities in $\gamma_{\mathcal{R}}$ could be chosen in such a way that s is always zero, but sadly this turns out not to be possible.

Let us pull out a common denominator from A by writing it as $A = \tilde{A}/g$ with $\tilde{A}$ an integer matrix. We also invoke the Smith-like decomposition of $\tilde{A}$,

$$\tilde{A} = UDU^T \qquad D = J\,\mathrm{ddiag}(\nu_i) \qquad J = \bigoplus_{i=1}^{N/2} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

with U unimodular and ν_i integers. Then in terms of further new variables $\tilde{x} = U^{-1}x$ and $\tilde{y} = U^T y$, which are still integer vectors, our equation becomes

$$g\tilde{x} = D\tilde{y}$$

which is now diagonal, hence trivial to solve. The set of all solutions can be parametrised, in terms of an integer vector z, via

$$\tilde{x} = J\,\mathrm{ddiag}(\nu_i/g_i)z \qquad \tilde{y} = \mathrm{ddiag}(g/g_i)z$$

with $g_i = \gcd(g, \nu_i)$. Returning to the original variable v, we have

$$2v = Xz \qquad X = U^{T,-1}\mathrm{ddiag}(g/g_i) + UJ\,\mathrm{ddiag}(d_i/g_i)$$

We are almost done, except for the requirement that v be integral, which places a constraint on the allowed values of z:

$$Xz = 0 \mod 2$$

By considering the SNF of X, one can show that this constraint forces z to lie in a certain sublattice $\Lambda_z \subseteq \mathbb{Z}^N$, whose volume is

$$\mathrm{Vol}(\Lambda_z) = 2^{N-n}$$

where $n = \mathrm{nullity}_{\mathbb{F}_2}(X)$. With z and v now integer vectors, w automatically is too, and so we have parametrised all integer solutions to $\mathcal{R}v = w$. The lattice $\Lambda[\mathcal{R}]$ is then the set of allowed values of v. To calculate its volume, we simply chain together several earlier equations, namely $v = \frac{1}{2}(\mathbb{1} + A)y$, $y = U^T\tilde{y}$, $\tilde{y} = \mathrm{ddiag}(g/g_i)z$, and $z \in \Lambda_z$, with the result

$$\mathrm{Vol}(\Lambda[\mathcal{R}]) = \det\Big(\tfrac{1}{2}(\mathbb{1} + A)\Big)\ \det\Big(U^T\Big)\ \det\Big(\mathrm{ddiag}(g/g_i)\Big)\ \mathrm{Vol}\Big(\Lambda_z\Big)$$

which gives the formula stated at the beginning. An entirely analogous result also holds for odd N, with an identical proof.

Chapter 3

Boundary RG Flows

In the previous chapter, we explored a large family of ways to place Dirac fermions on a manifold with boundary. Next, as further exploration of this family of boundary states, we work out the structure of boundary RG flows that connect different states. While this question has little to do with monopole physics, it does however form a natural and logical next question from the BCFT perspective.

The theory of RG for the d=0+1 dimensional boundary behaves, in many ways, the same as in any other quantum field theory. There are operators restricted to the boundary and these can be classified as relevant, irrelevant or marginal. Operators that are exactly marginal move among a continuous family of boundary conditions. Meanwhile, boundary operators that are relevant initiate an RG flow within the space of boundary conditions without endangering the gapless nature of the bulk modes. As in higher dimensional situations, the number of boundary degrees of freedom g necessarily decreases under RG flow [78, 79].

The purpose of this chapter is to study such RG flows between different boundary conditions for massless fermions. We will find a simple, and elegant story in which, with some reasonable assumptions, one can follow boundary RG flows from one fixed point to another. There are a number of parts to this story, not least the $\mathbb{Z}_2$ SPT classification of boundary states that emerged in the last chapter; here it plays a crucial role in ensuring the consistency of our story. The current chapter therefore serves as a nontrivial consistency check of our results in the last chapter.

Summary of Results

Our goal is to take a boundary state preserving a $U(1)^N$ symmetry, and perturb by a relevant bosonic boundary operator. Any such perturbation necessarily breaks one or more of the $U(1)^N$ symmetries. However, we propose that, while the RG flow breaks the symmetry, a new emergent $U(1)^N$ symmetry is restored at the end of the RG flow. It is not obvious that this is the case: one might have anticipated that, by flowing away from states preserving a full $U(1)^N$ symmetry, we would leave them for good. Instead we argue that, like the famous hotel, you can check out from these states, but you can never leave.

Assuming that a full $U(1)^N$ emerges in the infra-red allows us to track the RG flow. There are a number of interesting features that emerge from our analysis. First, one can ask: is it possible to flow from one class of boundary states to the other? Say, from vector-like boundary conditions to axial-like boundary conditions? Given the anomaly restrictions described above, one might have thought that such flows are forbidden. Instead, we find that they are very much allowed. However, whenever such a flow occurs, the resulting boundary state comes equipped with an extra Majorana mode λ, needed to cancel the anomaly.

Secondly, we find the following surprising and simple formula: if we initiate an RG flow by turning on a single, relevant boundary operator $\mathcal{O}$ with dimension $\dim(\mathcal{O})$, then the UV and IR central charges are related by

$$g_{IR} = g_{UV}\sqrt{\dim(\mathcal{O})} \tag{3.1}$$

Since a relevant boundary operator necessarily has $\dim(\mathcal{O}) < 1$, this relation is consistent with the g-theorem [69, 78, 79], which states that the boundary central charge g must decrease.

Plan

We start in Section 3.1 with a recap of the most important properties of the chiral boundary states that we will need. We also introduce some examples of such states, which we will use as a running illustration. In Section 3.2 we determine the spectrum of boundary operators for a given boundary state and, in particular, extract the ones that are relevant and bosonic.

The RG analysis is given in Section 3.3. We explain how, for each relevant bosonic boundary operator, there is a unique candidate for the endpoint of the flow, and elaborate on a number of subtleties that arise including the emergence of Majorana bound states and what string theorists refer to as Chan-Paton factors. The statements of the results are more straightforward than the proofs; these statements are placed front and centre, and we refrain as long as possible from wallowing in the glorious technicalities. The wallowing finally occurs in Section 3.4.

An Application to D-Branes

As far as we are aware, the kinds of chiral boundary conditions that we discuss do not have application to the fermions on the superstring worldsheet. However, there is a more indirect connection. We could consider bosonising our fermions so that the chiral boundary conditions now describe the end-point of a string moving on a torus $\mathbf{T}^N$, with radius of order the string length.

In this context, the chiral boundary conditions are nothing more than D-branes in bosonic string theory, wrapping $\mathbf{T}^N$ with fluxes. Even translated to this familiar context, our results appear novel. Things are simplest for $N = 2$ fermions, corresponding to a D2-brane wrapping $\mathbf{T}^2$. After a T-duality, the general chiral boundary state simply translates to a D-string wrapped (p, q) times around the two cycles of $\mathbf{T}^2$. We describe this in Appendix 3.C.

3.1 Some Examples

In this section we will introduce a useful set of examples of the chiral boundary states (2.24). We will pay particular attention to the values of their normalisation factor $g_{\mathcal{R}}$. In Chapter 2, we showed that this normalisation factor is given by

$$g_{\mathcal{R}} = \sqrt{\text{Vol}(\Lambda[\mathcal{R}])} \tag{3.2}$$

where $\text{Vol}(\Lambda[\mathcal{R}])$ is the volume of the primitive unit cell of the lattice $\Lambda[\mathcal{R}]$ defined in (2.16), which always takes integer values. The normalisation factor is important because it coincides with the Affleck-Ludwig central charge, defined by

$$g_{\mathcal{R}} = \langle 0 \,|\, \theta; \mathcal{R} \rangle$$

Hence, $g_{\mathcal{R}}$ should be thought of as a count of the number of boundary degrees of freedom. This number must strictly decrease in any boundary RG flow.

The trivial boundary conditions, corresponding to $\mathcal{R} = \pm\mathbb{1}$ (or, indeed, to any diagonal $\mathcal{R}$ with entries ± 1) has $g_{\mathcal{R}} = 1$. This is the smallest value of the central charge. Any chiral boundary condition necessarily has $g_{\mathcal{R}} > 1$. We now turn to some examples and their corresponding $g_{\mathcal{R}}$ values.

3.1.1 The Pythagorean States

With $N = 2$ Dirac fermions, there is a rather simple classification of boundary states, as shown in Section 2.3.3. A large class of these arise from taking co-prime integers (p, q) with one odd, one even, and setting

$$Q_{\alpha i} = \begin{pmatrix} p & q \\ -q & p \end{pmatrix} \quad , \quad \bar{Q}_{\alpha i} = \begin{pmatrix} p & -q \\ q & p \end{pmatrix} \quad \Rightarrow \quad \mathcal{R}_{ij} = \frac{1}{c}\begin{pmatrix} a & b \\ -b & a \end{pmatrix} \tag{3.3}$$

Here a, b and c form a Pythagorean triple $a^2 + b^2 = c^2$ with the Euclid parametrisation

$$a = p^2 - q^2 \quad , \quad b = 2pq \quad , \quad c = p^2 + q^2$$

The boundary central charge of these states is simply $g_{\mathcal{R}} = \sqrt{c}$.

The state (3.3) always lies in the vector class of boundary conditions. However, for any choice of central charge, it is not hard to find states that lie in either class. For example, after the trivial states, the simplest states have $g_{\mathcal{R}} = \sqrt{5}$. If we take $p = 2$ and $q = 1$, we get a vector-like boundary state with

$$\mathcal{R}_{ij} = \frac{1}{5}\begin{pmatrix} 3 & 4 \\ -4 & 3 \end{pmatrix}$$

However, flipping the sign of a single row, we get an axial-like boundary state with

$$\mathcal{R}_{ij} = \frac{1}{5}\begin{pmatrix} 3 & 4 \\ 4 & -3 \end{pmatrix}$$

As we proceed, many of the key ideas will be illustrated by this $g_{\mathcal{R}} = \sqrt{5}$ state. For now, there are a couple of points worth highlighting.

First, the fact that sign-flipping a row or column of $\mathcal{R}$ changes the topological class is a property of all boundary states. Meanwhile, permuting rows or columns leaves the class unchanged. In general, one can transform $\mathcal{R} \to P_R \mathcal{R} P_L$ where P_L and P_R are signed permutation matrices. This transformation corresponds to acting with a Weyl group element $(W_L, W_R) \in O(2N)_L \times O(2N)_R$ on the boundary state; the class then changes if $\det(W_L)\det(W_R) = -1$, while $g_{\mathcal{R}}$ always stays the same. This illustrates the fact that, for any given choice of $g_{\mathcal{R}}$, there are boundary states that lie in both classes.

Secondly, a number of different charges Q and $\bar{Q}$ share the same boundary state, characterised by $\mathcal{R}$. For example, we could also take

$$Q_{\alpha i} = \begin{pmatrix} 3 & 4 \\ -4 & 3 \end{pmatrix} \quad , \quad \bar{Q}_{\alpha i} = \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix} \quad \Rightarrow \quad \mathcal{R}_{ij} = \frac{1}{5}\begin{pmatrix} 3 & 4 \\ -4 & 3 \end{pmatrix}$$

In contrast to the charge matrices in (3.3), here the $U(1)^2$ symmetry does not act faithfully on the bulk fermions. The fermions are untouched by a discrete $\mathbb{Z}_5$ which acts on the left-movers as $\psi_i \to e^{i\beta_\alpha Q_{\alpha i}}\psi_i$ and on the right-movers as $\bar{\psi}_i \to e^{i\beta_\alpha \bar{Q}_{\alpha i}}\bar{\psi}_i$, with $\beta = (\frac{2\pi}{5}, \frac{4\pi}{5})$.

In what follows, the key physics will depend only on $\mathcal{R}$; for example, the collection of relevant boundary operators and their dimensions depend only on $\mathcal{R}$. Nonetheless, we will see that the labelling of the charges of these operators is inherited from Q and $\bar{Q}$ and so requires extra information beyond a knowledge of $\mathcal{R}$. This extra information corresponds to choosing a different basis $U(1)_{\alpha=1\ldots N}$ for the same group $U(1)^N$.

3.1.2 The Maldacena-Ludwig State

Our second example involves $N = 4$ Dirac fermions. The boundary conditions are, perhaps, most simply described by requiring an $SU(4) \times U(1)$ global symmetry under which the left-movers transform in the $\mathbf{4}_{+1}$ representation, while the right-movers transform as $\mathbf{4}_{-1}$. There is no linear boundary condition on the fermions that reflects one into another, a fact first noted in the context of monopole physics [18, 19]. Instead, the boundary condition is implemented by the boundary state with

$$Q_{\alpha i} = \begin{pmatrix} + & + & + & + \\ + & - & & \\ & + & - & \\ & & + & - \end{pmatrix} \quad , \quad \bar{Q}_{\alpha i} = \begin{pmatrix} - & - & - & - \\ + & - & & \\ & + & - & \\ & & + & - \end{pmatrix} \quad \Rightarrow \quad \mathcal{R}_{ij} = \delta_{ij} - \frac{1}{2} \qquad (3.4)$$

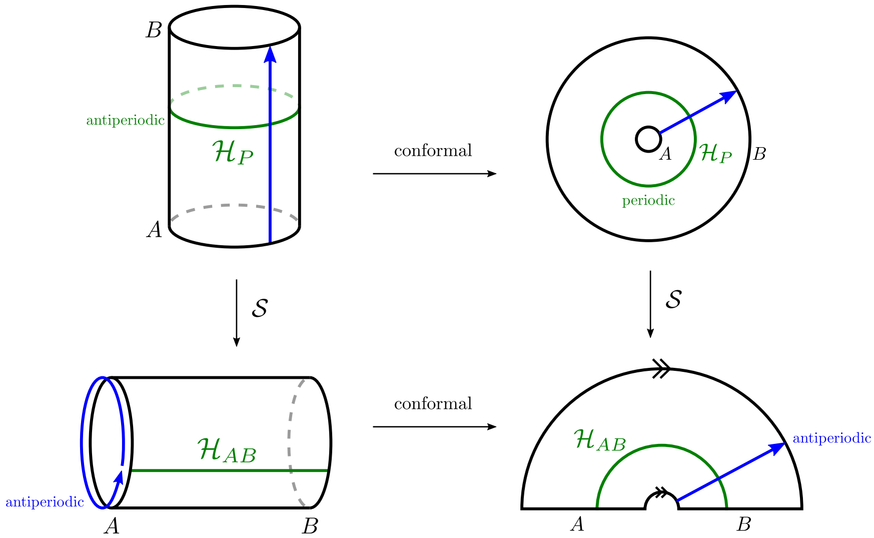


Figure 3.1: The various conformal identifications used, including those which correspond to an $\mathcal{S}$ transformation of the argument of the partition function.

This boundary state was previously introduced by Maldacena and Ludwig [20]. It manifestly implements the symmetry of the Cartan subalgebra $U(1)^4 \subset SU(4) \times U(1)$. Less manifestly, it also preserves the full $SU(4) \times U(1)$. Remarkably, in this special four-fermion case, it preserves yet a larger Spin(8) symmetry group, whose existence can be traced to triality. In Section 2.3.1 we showed this state has boundary central charge $g_{\mathcal{R}} = \sqrt{2}$. Once again, by acting with Weyl group transformations we have such states of either $\mathbb{Z}_2$ SPT class.

3.2 The Partition Function

Our goal in this section is to determine the relevant boundary operators, and their charges, for each choice of boundary condition $\mathcal{R}$. To do this, we compute the partition function of the theory on an interval, with boundary conditions $\mathcal{R}$ imposed on each end. We extend the computation of Section 2.2 by now including $U(1)^N$ symmetry defects. The partition function now encodes information about the charges of states in the Hilbert space on the interval. We then use the state-operator map to determine the spectrum of boundary operators.

3.2.1 Adding Fugacities

Recall that the partition function $\mathcal{Z}_{AB}$, for two distinct boundary conditions A and B at either end of the interval is defined as the trace over the Hilbert space, $\mathcal{H}_{AB}$. After implementing a conformal transformation to the half-annulus, as shown in Figure 3.1 along the bottom row, this partition function is given by

$$\mathcal{Z}_{AB}(q) = \mathrm{Tr}_{\mathcal{H}_{AB}}\left(q^{L_0 - c/24}\right)$$

Via the usual Cardy trick, this is related to the closed sector partition function of free fermions on a cylinder which, after the conformal map shown along the top of Figure 3.1, becomes the annulus

$$\mathcal{Z}_{\text{closed}}(q) = \langle B| q^{\frac{1}{2}(L_0 + \bar{L}_0 - c/12)} |A\rangle$$

The two partition functions are then related by a modular $\mathcal{S}$-transformation of q.

For two boundary states $A = |\theta, \mathcal{R}\rangle$ and $B = |\theta', \mathcal{R}\rangle$, sharing the same $\mathcal{R}$ matrix but differing in the theta angles, we already showed in Section 2.2 that the closed string partition function is given by

$$\mathcal{Z}_{\text{closed}}(q) = g_{\mathcal{R}}^2 \sum_{\lambda \in \Lambda[\mathcal{R}]} e^{i(\theta - \theta' + \pi)\cdot\lambda} \frac{q^{\lambda^2/2}}{\eta(\tau)^N} \tag{3.5}$$

where $q = e^{2\pi i \tau}$. The partition function for the theory on the interval is then found by applying a modular $\mathcal{S}$-transformation; it is

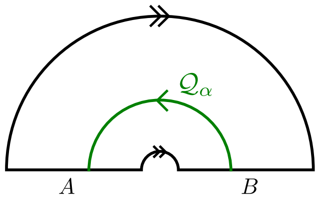
$$\mathcal{Z}_{AB}(q) = \sum_{\rho \in \Lambda[\mathcal{R}]^\star} \frac{q^{\frac{1}{2}(\rho + \frac{\theta - \theta'}{2\pi} + \frac{1}{2})^2}}{\eta(\tau)^N} \tag{3.6}$$

with $\Lambda[\mathcal{R}]^\star$ the dual lattice, defined by $\rho \cdot \lambda \in \mathbb{Z}$ for all $\lambda \in \Lambda[\mathcal{R}]$ and $\rho \in \Lambda[\mathcal{R}]^\star$.

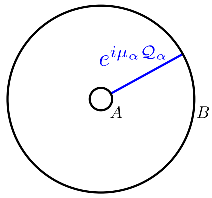
Here we wish to extend this computation to include fugacities for the $U(1)^N$ symmetry, providing information about the charges of the states. This means that we weight the contribution of states in the open-string partition function according to their charges under

$$\mathcal{Q}_\alpha = \frac{1}{2\pi i} \int_C dz\ \mathcal{J}_\alpha(z) - d\bar{z}\ \bar{\mathcal{J}}_\alpha(\bar{z})$$

where the contour C is the counter-clockwise semi-circle shown in Figure 3.2a, and the $U(1)_\alpha$ current $\mathcal{J}_\alpha$ was defined in (2.9). The partition function now depends both on the



(a) Contour C used to define $\mathcal{Q}_\alpha$.



(b) Corresponding defect operator.

Figure 3.2

modular parameter q and the chemical potentials μ_α,

$$\mathcal{Z}_{AB}(q;\mu) = \mathrm{Tr}_{\mathcal{H}_{AB}}\left(q^{L_0 - c/24} e^{i\mu_\alpha \mathcal{Q}_\alpha}\right)$$

Again, this object is simplest to compute in the closed-string picture. The operator $e^{i\mu_\alpha \mathcal{Q}_\alpha}$ is now a defect, oriented along the "temporal" or "thermal" direction, as shown in Figure 3.2b. Its role is to shift each fermion by a phase as we move around the spatial circle. The left-moving fermion ψ_i picks up a phase $e^{i\mu_\alpha Q_{\alpha i}}$, while the right-moving fermion $\bar{\psi}_i$ picks up $e^{i\mu_\alpha \bar{Q}_{\alpha i}}$.

This, in turn, affects the quantisation of the charges λ_i and $\bar{\lambda}_i$ defined in (2.13). Rather than living in the integer lattice $\mathbb{Z}^N$, we instead have

$$\lambda_i \in \mathbb{Z} + \frac{\mu_\alpha Q_{\alpha i}}{2\pi} \quad \text{and} \quad \bar{\lambda}_i \in \mathbb{Z} - \frac{\mu_\alpha \bar{Q}_{\alpha i}}{2\pi} \tag{3.7}$$

Note that left- and right-moving charges are shifted in opposite directions. (This computation leaves an ambiguity in the overall sign of the shifts, which is unimportant for what follows.)

The boundary condition (2.10) still requires that left- and right-moving charges are related by (2.15)

$$Q_{\alpha i}\lambda_i + \bar{Q}_{\alpha i}\bar{\lambda}_i = 0$$

which is only possible for all choices of μ if

$$\mu_\beta(Q_{\alpha i}Q_{\beta i} - \bar{Q}_{\alpha i}\bar{Q}_{\beta i}) = 0$$

Happily this follows from the condition for vanishing 't Hooft anomalies (2.11).

The closed string partition function is now easily computed by implementing the shift (3.7) in our previous result (3.5). The contribution from the $e^{i(\theta-\theta'+\pi)\cdot\lambda}$ term gives an overall phase which we ignore. We're then left with

$$\mathcal{Z}_{\text{closed}}(q;\mu) = g_{\mathcal{R}}^2 \sum_{\lambda\in\Lambda[\mathcal{R}]} e^{i(\theta-\theta'+\pi)\cdot\lambda} \frac{q^{\frac{1}{2}(\lambda_i+\mu_\alpha Q_{\alpha i}/2\pi)^2}}{\eta(\tau)^N}$$

We can now invoke the usual modular transformation to compute the open-string partition function of interest. We pull back the function Z_{closed} under a modular S-transformation of q, to find

$$\mathcal{Z}_{AB}(q;\mu) = \text{Vol}(\Lambda[\mathcal{R}]) \int d^N x\, e^{i\mu_\alpha Q_{\alpha i} x_i} \frac{q^{x^2/2}}{\eta(\tau)^N} \sum_{\lambda\in\Lambda[\mathcal{R}]} e^{i(\theta-\theta'+\pi+2\pi x)\cdot\lambda}$$

Upon doing the integral, we have

$$\mathcal{Z}_{AB}(q;\mu) = \sum_{\rho\in\Lambda[\mathcal{R};\Delta\theta]^\star} e^{i\mu^T Q\rho} \frac{q^{\rho^2/2}}{\eta(\tau)^N} \tag{3.8}$$

The difference from our previous result (3.6) lies in both the explicit μ dependent factor $e^{i\mu_\alpha Q_{\alpha i}\rho_i}$, and in the sum which now runs over the shifted dual lattice

$$\Lambda[\mathcal{R};\Delta\theta]^\star := \Lambda[\mathcal{R}]^\star + \frac{\theta'-\theta+\pi}{2\pi}$$

The highest weight states are labelled by vectors $\rho\in\Lambda[\mathcal{R};\Delta\theta]^\star$. From (3.8), we can read off their charges

$$\mathcal{Q}_\alpha = Q_{\alpha i}\rho_i \tag{3.9}$$

and energy

$$L_0 = \frac{1}{2}\rho^2 = \frac{1}{2}\mathcal{Q}_\alpha\, \mathcal{M}^{-1}_{\alpha\beta}\, \mathcal{Q}_\beta \tag{3.10}$$

where we have introduced the matrix $\mathcal{M}_{\alpha\beta} = Q_{\alpha i}Q_{\beta i} = \bar{Q}_{\alpha i}\bar{Q}_{\beta i}$. This latter equality, relating the charges to the energy, is consistent with the Sugawara construction.

3.2.2 Boundary Operators

The state-operator map means that the partition function also contains information about the spectrum of boundary operators. To extract this information, we set $\theta = \theta'$ and drop the contribution of π from the $(-1)^F$ factor. The boundary operators are then labelled by $\rho \in \Lambda[\mathcal{R}]^\star$. Like the states, the operators have charges $\mathcal{Q}_\alpha$ and dimension L_0, again given by (3.9) and (3.10).

Boundary operators also come in one of two classes: they are fermionic or bosonic. This fermion parity will play a key role in Section 3.3 where we discuss RG flows initiated by such operators. We pause here to discuss how to classify operators. As we now explain, it is possible to assign a fermion parity to the lattice vectors $\rho \in \Lambda[\mathcal{R}]^\star$.

First, recall that by definition, under a $U(1)^N_L \times U(1)^N_R$ transformation

$$(e^{i\mu_\alpha Q_{\alpha i}}, e^{i\mu_\alpha \bar{Q}_{\alpha i}})$$

belonging to the preserved $U(1)^N$ subgroup, the boundary operator labelled by ρ picks up a phase $e^{i\mu_\alpha \mathcal{Q}_\alpha} = e^{i\mu_\alpha Q_{\alpha i}\rho_i}$. Importantly, the bulk fermion parity operator $(-1)^{F+\bar{F}}$ is of the above form, as a special case of the result we prove in Section 4.2.1. That is, there exists a choice of μ_α for which the above transformation is

$$(e^{i\mu_\alpha Q_{\alpha i}}, e^{i\mu_\alpha \bar{Q}_{\alpha i}}) = (-1, \ldots, -1, -1, \ldots, -1)$$

It will be more convenient to work not with μ_α, but with the vector $f_i = \mu_\alpha Q_{\alpha i}/\pi$. We shall refer to this as the "fermion vector". The above condition can then be written

$$(e^{i\pi f}, e^{i\pi \mathcal{R} f}) = (-1, \ldots, -1, -1, \ldots, -1)$$

which shows that f is characterised by the requirement that both f and $\mathcal{R}f$ are odd-integer vectors. It therefore naturally lives in $\Lambda[\mathcal{R}]/2\Lambda[\mathcal{R}]$. With this notation in hand, the key point is then that since fermion parity lies within $U(1)^N$, the charge ρ dictates the fermion parity $(-1)^F$ of the boundary operator[1], through

$$(-1)^F = e^{i\mu_\alpha Q_{\alpha i}\rho_i} = (-1)^{f\cdot\rho} \tag{3.11}$$

[1]Just as for the Virasoro generators L_n, the notation $(-1)^F$ is ambiguous, and means something different depending on whether one is working in the open or closed sector.

We therefore classify vectors $\rho \in \Lambda[\mathcal{R}]^\star$ as bosonic or fermionic depending on whether $\rho \cdot f$ is even or odd, respectively.

Relevant boundary operators are associated to lattice vectors $\rho \in \Lambda[\mathcal{R}]^\star$ with $\rho^2 < 2$ and can be either bosonic or fermionic. These will be our primary focus in Section 3.3 where we discuss RG flows initiated by such operators. Here we describe the relevant operators in the two examples introduced in Section 3.1.

The First Example: $g = \sqrt{5}$

As we've seen, the simplest, non-trivial two fermion boundary state has

$$\mathcal{R}_{ij} = \begin{pmatrix} 3/5 & 4/5 \\ -4/5 & 3/5 \end{pmatrix}$$

and $g_\mathcal{R} = \sqrt{5}$. One possible choice of the fermion vector in this case is $f = (5, 5)$.

As we explained in Section 3.1, there are many choices of $Q_{\alpha i}$ and $\bar{Q}_{\alpha i}$ that give rise to this boundary state. The dimension of boundary operators depends only on $\mathcal{R}_{ij}$ while, as we see from (3.9), the charges of these operators depend on the choice of $\mathcal{Q}$. The operators are further distinguished by fermion number $(-1)^F$. The operators with $L_0 \leq 1$ are associated to the following lattice sites ρ,

L_0	$(-1)^F$	$\rho \in \Lambda[\mathcal{R}]^\star$
0	+	$(0, 0)$
$1/10$	−	$\pm(\frac{2}{5}, \frac{1}{5}),\ \ \pm(\frac{1}{5}, -\frac{2}{5})$
$1/5$	+	$\pm(\frac{1}{5}, \frac{3}{5}),\ \ \pm(\frac{3}{5}, -\frac{1}{5})$
$2/5$	+	$\pm(\frac{4}{5}, \frac{2}{5}),\ \ \pm(\frac{2}{5}, -\frac{4}{5})$
$1/2$	−	$\pm(\frac{3}{5}, \frac{4}{5}),\ \ \pm(\frac{4}{5}, -\frac{3}{5}),\ \ \pm(1, 0),\ \ \pm(0, 1)$
$4/5$	+	$\pm(\frac{2}{5}, \frac{6}{5}),\ \ \pm(\frac{6}{5}, -\frac{2}{5})$
$9/10$	−	$\pm(\frac{6}{5}, \frac{3}{5}),\ \ \pm(\frac{3}{5}, -\frac{6}{5})$
1	+	$\pm(\frac{7}{5}, \frac{1}{5}),\ \ \pm(\frac{1}{5}, -\frac{7}{5}),\ \ \pm(1, 1),\ \ \pm(1, -1)$

As we proceed, we'll see the interpretation of a number of these operators.

The Other Example: The Maldacena-Ludwig State

The relevant boundary operators for the Maldacena-Ludwig state (3.4) are

L_0	$(-1)^F$	$\rho \in \Lambda[\mathcal{R}]^\star$
0	$+$	$(0,0)$
$1/2$	$+$	$\pm(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2})$, $(\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2})$ (and all permutations)
$1/2$	$-$	$(\pm 1,0,0,0)$, $\pm(\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2})$ (and all permutations)
1	$+$	$(\pm 1,\pm 1,0,0)$ (and all permutations)

As we briefly mentioned previously, the Maldacena-Ludwig state represents the gapped Fidkowski-Kitaev state. This has the property that it preserves both left and right fermion parity $(-1)^F$ and $(-1)^{\bar{F}}$. Furthermore, it is the state with the smallest $g_{\mathcal{R}}$ with this property. This latter statement is reflected in the fact that the dimension $L_0 = \frac{1}{2}$ bosonic operators are charged under both of the two fermionic parities. We will return to these aspects of the boundary states in Chapter 4.

Marginal Operators

Marginal boundary operators have $L_0 = 1$. If such operators are exactly marginal, they give rise to continuous families of boundary states. As we now explain, marginal operators fall into a number of different categories.

First, we can take the vacuum module, $\rho = 0$, and form a level-1 descendent under the current algebra. From the perspective of the interval Hilbert space, these correspond to states $\mathcal{J}_{\alpha,-1}|0\rangle$. Similarly, the existence of the boundary operators follows on symmetry grounds: they are associated to the symmetries broken by the boundary in the reduction $U(1)^N_L \times U(1)^N_R \to U(1)^N$. Acting with these operators changes the θ-angles that, as we saw in (2.24), are needed to characterise the boundary state.

The second class of marginal operators are highest weight states associated to lattice vectors $\rho \in \Lambda[\mathcal{R}]^\star$ with $\rho^2 = 2$. We have listed these operators in the table above for the simple examples. Many of these operators also have an interpretation in terms of symmetries. But not all.

To understand this, first recall that the symmetry breaking pattern, as shown in (2.8), is generically

$$\mathfrak{so}(2N)_L \times \mathfrak{so}(2N)_R \to \mathfrak{u}(1)^N$$

The broken, off-diagonal elements of $\mathfrak{so}(2N)_L \times \mathfrak{so}(2N)_R$ will also give rise to marginal operators. Acting with them simply rotates the unbroken Cartan sub-algebra.

It is straightforward to identify these states. The off-diagonal elements of $so(2N)_L$ arise from vectors with $\rho^2 = 2$ that sit in $\rho \in \mathbb{Z}^N$. The off-diagonal elements of $so(2N)_R$ arise from vectors with $\rho^2 = 2$ that sit in $\rho \in \mathcal{R}^{-1}\mathbb{Z}^N$.

This pattern can be clearly seen in the two fermion boundary state with $g_{\mathcal{R}} = \sqrt{5}$. The final line of the table shows the 8 boundary operators that are associated to the off-diagonal elements of $SO(4)_L \times SO(4)_R$.

However, in other examples things may not be so straightforward. First, it may be that there is an overlap between the operators associated to $\mathfrak{so}(2N)_L$ and those associated to $\mathfrak{so}(2N)_R$. This occurs if there are lattice sites with $\rho^2 = 2$ that sit in $\rho \in \mathbb{Z}^N \cap \mathcal{R}^{-1}\mathbb{Z}^N$. But the intersection of the latter two lattices is simply

$$\Lambda[\mathcal{R}] = \mathbb{Z}^N \cap \mathcal{R}^{-1}\mathbb{Z}^N$$

This overlap has a very natural interpretation. As we explain later in Chapter 4, vectors $\rho \in \Lambda[\mathcal{R}]$ with $\rho^2 = 2$ correspond to enhanced symmetries of the boundary state. As expected, the presence of such hidden symmetries reduces the number of marginal boundary operators. For example, in the table for the Maldacena-Ludwig boundary state shown above, there are 24 marginal operators. This is lower than the number 48 of off-diagonal generators of $SO(8)_L \times SO(8)_R$. The difference can be accounted for by the enhanced Spin(8) symmetry, which eliminates 24 generators.

Finally, some boundary states have marginal operators that do not correspond to symmetries. These are lattice vectors with $\rho^2 = 2$ that sit in $\rho \in \Lambda[\mathcal{R}]^\star$ but with $\rho \notin \mathbb{Z}^N \cup \mathcal{R}^{-1}\mathbb{Z}^N$. In such cases, one must work harder to determine whether the boundary operator is exactly marginal, or marginally relevant or irrelevant. We do not explore this issue further here.

3.2.3 An Aside: The Unitarity "Paradox"

There is an interesting structure to the charges carried by states in the Hilbert space $\mathcal{H}_{AB}$. To illustrate our point, it's simplest if we ignore the shift of the lattice by the theta angles for now, so $\rho \in \Lambda[\mathcal{R}]^\star$. In this case, the states of the Hilbert space carry charges in the lattice (3.9)

$$\mathcal{Q} \in Q\Lambda[\mathcal{R}]^\star$$

We can compare this to the charges of states that we get by acting with left- and right-moving operators. Acting with the holomorphic fermions ψ_i produce states with charges in $Q\mathbb{Z}^N$, while acting with anti-holomorphic fermions $\bar{\psi}_i$ produce states with charges in $\bar{Q}\mathbb{Z}^N$. It is not hard to show that this accounts for the full charge lattice

$$Q\Lambda[\mathcal{R}]^\star = Q\mathbb{Z}^N + \bar{Q}\mathbb{Z}^N$$

However, there's a twist. It's not true that one can reach states of all charges by acting only with, say, holomorphic operators. This is, at heart, what it means for our boundary states to be chiral. Indeed, we have the following:

$$\left[Q\Lambda[\mathcal{R}]^\star : Q\mathbb{Z}^N\right] = \left[Q\Lambda[\mathcal{R}]^\star : \bar{Q}\mathbb{Z}^N\right] = \mathrm{Vol}(\Lambda[\mathcal{R}])$$

This means that, while one cannot access states of any charge by acting on the vacuum with only holomorphic operators, we can do so by acting on an appropriate choice of $g_{\mathcal{R}}^2 = \mathrm{Vol}(\Lambda[\mathcal{R}])$ states (one of which is the ground state). These can be viewed as holomorphic superselection sectors.

Similarly, there are a different set of $g_{\mathcal{R}}^2$ states in the Hilbert space, from which we can access states of any charge by acting with anti-holomorphic operators.

In the context of scattering off a single boundary, this leads to a seeming "unitarity paradox". It is not hard to set up situations in which a single left-moving fermion scatters off the boundary, but cannot return as any combination of right-moving fermions. This is captured by the vanishing correlation functions

$$\langle 0|\psi_i(z)\bar{\psi}_{j_1}(\bar{z}_1)\dots\bar{\psi}_{j_N}(\bar{z}_N)|0\rangle = 0 \quad \text{for all } N$$

Such behaviour was seen, for example, in [18–21]. Our general discussion above shows that the right-moving fermions are not excitations above the ground state, but instead above one of the other $\mathrm{Vol}(\Lambda[\mathcal{R}])$ superselection sectors.

The above discussion highlights the problem of IR divergences in the unitarity paradox. First, our resolution in terms of superselection sectors only applies when both boundaries are taken to be the same. Yet the unitarity paradox is a statement about scattering off a single boundary $\mathcal{R}$, so whether the other boundary is taken to be $\mathcal{R}$ or something entirely different should not matter. Second, the superselection sectors become degenerate as the length of the interval tends to infinity. It is therefore unclear what becomes of them in the infinite half-line limit.

3.3 RG Flows: Statements

We now turn to the main results of this chapter. We will follow the RG flow between different boundary states.

We start with a given UV boundary state, preserving the $U(1)^N$ symmetry characterised by the charge matrix $\mathcal{R}_{UV}$. As we have seen, relevant boundary operators are labelled by a vector $\rho \in \Lambda[\mathcal{R}_{UV}]^\star$ and carry charge

$$\mathcal{Q}_\alpha = Q_{\alpha i}\rho_i$$

We turn on a single, relevant, bosonic boundary operator of definite charge to initiate an RG flow. Along the flow, the symmetry is broken to

$$U(1)^N \to U(1)^{N-1}$$

In what follows, we make the following, important assumption: *At the end of the flow, an emergent $U(1)^N$ symmetry is again restored.* This means that, in the infra-red, the physics is again described by a boundary state of the form (2.24), now with a different charge matrix $\mathcal{R}_{IR}$.

There is, in fact, a unique choice for $\mathcal{R}_{IR}$ for each relevant operator labelled by ρ. This follows because of the $U(1)^{N-1}$ symmetry that exists along the RG flow. This symmetry must be preserved by the IR boundary state, a condition which translates into the simple requirement that

$$\mathcal{R}_{IR}\Big|_{\rho^\perp} = \mathcal{R}_{UV}\Big|_{\rho^\perp} \tag{3.12}$$

or in other words, that the two matrices must agree on the orthogonal complement of ρ. But for orthogonal matrices, this condition is highly constraining. In particular, there are only two options for $\mathcal{R}_{IR}$. One is $\mathcal{R}_{UV}$ itself, but this is quickly ruled out by the fact that $g_{IR} = g_{UV}$, in contravention of the g-theorem which states that the central charge must strictly decrease under relevant perturbations. This only leaves the second option, which is that the matrices differ by a reflection along the vector ρ:

$$(\mathcal{R}_{IR})_{ik} = (\mathcal{R}_{UV})_{ij}\left(\delta_{jk} - \frac{2}{\rho^2}\rho_j\rho_k\right) \tag{3.13}$$

The second factor is the matrix implementing the reflection along ρ.

One might think that the infra-red central charge is, following (3.2),

$$g_{\text{naive}} = \sqrt{\text{Vol}(\Lambda[\mathcal{R}_{IR}])} \tag{3.14}$$

And, for some of the RG flows, where no subtleties arise, this indeed the correct answer. However, it is not true in general. There are two rather interesting effects that may occur, both of which leave us with an infra-red central charge larger than (3.14). First, certain RG flows necessarily result in a Majorana zero mode stuck on the boundary. This phenomenon, which is explained in Section 3.3.1, increases the normalisation of the boundary state and its central charge by a factor of $\sqrt{2}$. Secondly, some RG flows result in a superposition of primitive boundary states, and larger central charge. This phenomenon is explained in Section 3.3.2.

Furthermore, we will see that the infra-red central always obeys the g-theorem, which states that the boundary central charge must always decrease [69, 78, 79]. This fact arises in a mathematically non-trivial manner for our boundary states, and presents a stringent test of the assumption a full $U(1)^N$ symmetry emerges in the infra-red.

3.3.1 Majorana Zero Modes

As we explained in Section 1.2, boundary conditions fall into two distinct topological classes, characterised by a mod 2 anomaly. One might have thought that RG flows would remain within a given class. However, as we now describe, our conjecture (3.13) does *not* have this property. It is not difficult to find RG flows that go from one class to another, and we present examples below. We will explain why this is not problematic.

First, we review the earlier result that determines the topological class in which a given boundary state, labelled by $\mathcal{R}$, sits. Given a CFT on an interval, we can impose different boundary conditions $\mathcal{R}$ and $\mathcal{R}'$ on either end. In Chapter 2, we found the number of ground states of this system to be

$$G[\mathcal{R},\mathcal{R}'] = \frac{\sqrt{\text{Vol}(\Lambda[\mathcal{R}])\,\text{Vol}(\Lambda[\mathcal{R}'])}}{\text{Vol}(\Lambda[\mathcal{R},\mathcal{R}'])}\sqrt{\det{}'(\mathbb{1}-\mathcal{R}^T\mathcal{R}')} \tag{3.15}$$

Here the intersection lattice $\Lambda[\mathcal{R},\mathcal{R}']$ is defined to be those integer vectors λ for which $\mathcal{R}\lambda = \mathcal{R}'\lambda \in \mathbb{Z}^N$. The notation $\det'$ denotes the product over non-vanishing eigenvalues.

The ground state degeneracy has an interesting property. If the two boundary states $\mathcal{R}$ and $\mathcal{R}'$ lie in the same class (i.e. either both vector, or both axial) then the number of ground states is integer as expected

$$G[\mathcal{R},\mathcal{R}'] \in \mathbb{Z}$$

In contrast, if the two boundary states lie in different classes, then

$$G[\mathcal{R},\mathcal{R}'] \in \sqrt{2}\,\mathbb{Z}$$

The $\sqrt{2}$ factor reflects the existence of a bulk Majorana zero mode. This is telling us that it is not consistent to put boundary conditions from different classes at the two ends of an interval.

What to make of the fact that RG flows take us from one class to another? Clearly, a consistent quantum system, with compatible boundary conditions on each end, cannot flow to an inconsistent quantum system. It must be that the bulk Majorana mode that appears in the infra-red is accompanied by a second, boundary Majorana mode. This boundary Majorana mode contributes a further factor of $\sqrt{2}$ to the partition function, as in (1.17), and hence to the boundary central charge. This means that, if there's no further subtlety, RG flows which interpolate between different classes have

$$g_{IR} = \sqrt{2\,\mathrm{Vol}(\Lambda[\mathcal{R}_{IR}])} \tag{3.16}$$

The condition for the appearance of a boundary Majorana mode is encoded in a simple property of ρ. First, we recall that although $\rho \in \Lambda[\mathcal{R}_{UV}]^\star$, it need not be *primitive* within this lattice. Instead, it may be possible to write it as some multiple $n \geq 1$ of an underlying primitive vector, which we denote as $\hat{\rho}$:

$$\rho = n\hat{\rho} \tag{3.17}$$

Since we must perturb by a bosonic relevant operator, ρ is always required to be bosonic. But there is no such condition on $\hat{\rho}$. In particular, it is perfectly acceptable for $\hat{\rho}$ to be fermionic provided that n is even. The property of ρ which determines the existence of a boundary mode is then the fermionic/bosonic nature of $\hat{\rho}$. This follows by computing the ground state degeneracy (3.15) between $\mathcal{R}_{IR}$ and $\mathcal{R}_{UV}$; as we show in Section 3.4, is given by

$$G[\mathcal{R}_{UV},\mathcal{R}_{IR}] = \begin{cases} 1 & \text{if } \hat{\rho} \text{ is bosonic} \\ \sqrt{2} & \text{if } \hat{\rho} \text{ is fermionic} \end{cases} \tag{3.18}$$

In other words, there is a bulk Majorana zero mode only if the relevant operator is associated to a lattice vector $\rho = n\hat{\rho}$ built on a fermionic primitive vector $\hat{\rho}$.

In Appendix 3.B, we give more details illustrating the coupling between the boundary mode and the bulk fermions using a simple model.

An Example

We can illustrate these ideas with the example that we met in Section 3.1: two fermions with

$$\mathcal{R}_{UV} = \begin{pmatrix} 3/5 & 4/5 \\ -4/5 & 3/5 \end{pmatrix}$$

The boundary central charge is $g_{UV} = \sqrt{5}$.

We listed the relevant and marginal operators for this boundary state in Section 3.2.2. Here we are interested only in the relevant, bosonic operators. For each of these, we can determine the infra-red charge matrix and whether or not there exists a boundary Majorana zero mode at the end of the flow.

ρ	L_0	$\mathcal{R}_{IR}$	Majorana?
$(\frac{1}{5}, \frac{3}{5})$	$\frac{1}{5}$	$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$	No
$(\frac{3}{5}, -\frac{1}{5})$	$\frac{1}{5}$	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	No
$(\frac{4}{5}, \frac{2}{5})$	$\frac{2}{5}$	$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$	Yes
$(\frac{2}{5}, -\frac{4}{5})$	$\frac{2}{5}$	$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$	Yes
$(\frac{2}{5}, \frac{6}{5})$	$\frac{4}{5}$	$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$	No
$(\frac{6}{5}, -\frac{2}{5})$	$\frac{4}{5}$	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	No

The middle two rows are built on the underlying fermionic vectors $\pm(\frac{2}{5}, \frac{1}{5})$ and $\pm(\frac{1}{5}, -\frac{2}{5})$, while the remaining rows are built on bosonic vectors. Note that the ρ-vectors for the operators with dimension $\frac{4}{5}$ are proportional to those with dimension $\frac{1}{5}$. We'll see the difference between these two RG flows in the next section.

An analogous table, for a more complicated example, is given in Appendix 3.A.

Flows with Fermionic Operators

RG flows are always initiated by bosonic, relevant operators. As we've seen, at the end of an RG flow we may end up with a localised Majorana fermion. We could also ask: what happens if we start from a boundary condition with such a Majorana mode?

The boundary state including such a Majorana mode is simply given by[2] $\sqrt{2}\,|\theta;\mathcal{R}_{UV}\rangle$, and has central charge

$$g_{UV} = \sqrt{2\,\mathrm{Vol}(\Lambda[\mathcal{R}_{UV}])}$$

Starting with such a state opens up a new possibility, because we could dress boundary fermionic operators with the Majorana mode to give a bosonic boundary operator, and then use this to initiate the RG flow.

Such fermionic boundary operators are characterised by $\rho = n\hat{\rho}$, as in (3.17), with $\hat{\rho}$ fermionic, n odd. Because $\hat{\rho}$ is fermionic, this means that such flows always flip the SPT class, and the Majorana mode is absorbed along the flow. The absorption of the Majorana mode means that the infra-red central charge is reduced by an extra factor of $\sqrt{2}$.

The Maldacena-Ludwig state serves as a good example of fermionic flows. Recall that this state has boundary central charge $g = \sqrt{2}$. If we further add a Majorana mode, the central charge is $g_{UV} = 2$. We can now perturb this state by relevant fermionic operators.

These operators were listed in the table in Section 3.2.2: there are two kinds, with charge given by permutations of

$$\rho = (1,0,0,0) \quad \text{and} \quad \rho = (\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},-\tfrac{1}{2})$$

These are primitive vectors, and both have dimension $L_0 = \frac{1}{2}$. Deforming by any of these operators gives us back the Maldacena-Ludwig state, up to a Weyl group transformation of $O(8)_L \times O(8)_R$. In other words, the sole effect of the flow is to eliminate the Majorana mode from the boundary.

In fact, this kind of flow, in which a Majorana mode is absorbed is possible for all boundary states. All such states have a boundary fermionic operator of dimension $\frac{1}{2}$, since this is simply the bulk fermion brought to the boundary. Deforming by this operator

[2]This normalisation for the axial boundary state was recently advocated in [45] to ensure compatibility with the vector-like boundary conditions, although the connection to the mod 2 anomaly was not made.

initiates an RG flow from $\sqrt{2}\,|\mathcal{R}\rangle$ to $|\mathcal{R}'\rangle$, where $\mathcal{R}'$ differs from $\mathcal{R}$ only by the sign flip of a row or column.

A particularly simple example of such a flow occurs for a single Dirac fermion. In Appendix 3.B, we show explicitly how the absorption of a boundary Majorana mode exchanges the vector and axial boundary conditions (2.1) and (2.2).

3.3.2 Non-Primitive Boundary States

We now turn to the second subtlety in the RG flows. We have seen that turning on a single, relevant operator in the UV breaks $U(1)^N \to U(1)^{N-1}$. However, this is not the full story. There is also a remnant discrete symmetry, so that

$$U(1)^N \to U(1)^{N-1} \times \mathbb{Z}_n$$

Here, the integer n is the same one introduced in (3.17), which measures the failure of ρ to be a primitive vector.

This discrete symmetry $\mathbb{Z}_n$ is preserved along the RG flow. However, one finds that the naïve IR boundary state is *not* invariant under the full $\mathbb{Z}_n$ symmetry. To rectify this, the infra-red boundary state must be a linear sum of states of the form (2.24) such that the overall sum is $\mathbb{Z}_n$ invariant. The different states in this sum have the same $\mathcal{R}_{IR}$ charge matrix, but differ in their theta angles. This then shows up in the infra-red central charge, with each state in the sum contributing a factor of $\sqrt{\text{Vol}(\Lambda[\mathcal{R}_{IR}])}$. We'll discuss this further in Section 3.3.3.

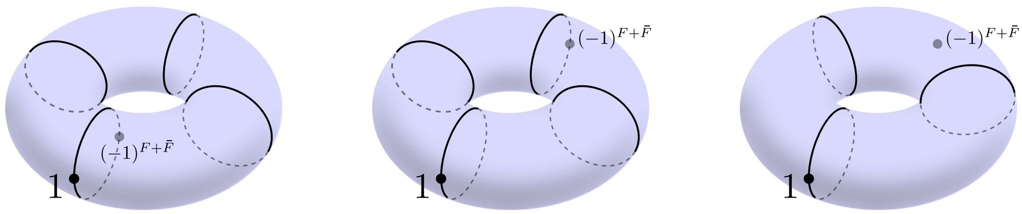
To put some flesh on these ideas, we will need to understand how the $\mathbb{Z}_n$ symmetry acts on our candidate infra-red boundary state,

$$|\theta;\mathcal{R}_{IR}\rangle = g_{\mathcal{R}} \sum_{\lambda\in\Lambda[\mathcal{R}_{IR}]} e^{i\gamma(\lambda)}\, e^{i\theta\cdot\lambda} \| \lambda, -\mathcal{R}_{IR}\lambda\,\rangle\!\rangle \tag{3.19}$$

Under a transformation by $k \in \mathbb{Z}_n$, the sole effect on the infra-red boundary state is to shift the theta angles θ_i by

$$\frac{\theta}{2\pi} \mapsto \frac{\theta}{2\pi} + \frac{2k}{\rho^2}\rho$$

The unbroken subgroup of $\mathbb{Z}_n$ will consist of those k for which this shift has no effect on the boundary state. To determine when this is the case, we note that the theta angles in (3.19) appear in the phase $e^{i\theta\cdot\lambda}$, which means that $\theta/2\pi$ is naturally defined mod $\Lambda[\mathcal{R}_{IR}]^\star$.



(a) $\hat{\rho}$ bosonic (b) $\hat{\rho}$ fermionic, n even (c) $\hat{\rho}$ fermionic, n odd

Figure 3.3: How $(-1)^{F+\bar{F}}$ sits in $U(1)^{N-1} \times \mathbb{Z}_n \subset U(1)^N$.

Therefore, the above shift is trivial whenever $(2k/\rho^2)\rho \in \Lambda[\mathcal{R}_{IR}]^\star$. We introduce the integer $m \geq 1$, defined as the least integer such that

$$\frac{2m}{\rho^2}\rho \in \Lambda[\mathcal{R}_{IR}]^\star \tag{3.20}$$

Then m divides n, and in the infra-red, the $\mathbb{Z}_n$ symmetry is spontaneously broken by the boundary state (3.19) to

$$\mathbb{Z}_n \to \mathbb{Z}_{n/m} \tag{3.21}$$

Just like the criterion for whether a boundary Majorana mode appears, the integer m can also be read off from ρ in a simple way. It is given by

$$m = \begin{cases} n & \text{if } \hat{\rho} \text{ is bosonic} \\ n/\gcd(n,2) & \text{if } \hat{\rho} \text{ is fermionic} \end{cases} \tag{3.22}$$

The upshot is that there are only two possibilities for the residual discrete symmetry:

$$\mathbb{Z}_n \to 1 \text{ or } \mathbb{Z}_2$$

The presence or absence of the unbroken $\mathbb{Z}_2$ has a simple physical explanation: it remains unbroken due to fermion parity. This is illustrated in Figure 3.3. Here the blue torus represents the whole $U(1)^N$ symmetry group preserved in the UV, while the black submanifold represents the $U(1)^{N-1} \times \mathbb{Z}_n$ subgroup left unbroken by the perturbation. It splits into a number n of disconnected components, or cosets, each of which is either fully preserved or fully broken by the naïve IR boundary state $|\theta; \mathcal{R}_{IR}\rangle$.

We have also labelled the location of fermion parity: it belongs to the coset $k = 0$ in case (a), to $k = n/2$ in case (b), and to no coset in case (c). But fermion parity is automatically preserved by any boundary state, and in particular by $|\theta; \mathcal{R}_{IR}\rangle$. This means that any coset containing fermion parity is automatically unbroken. The upshot is that aside from the trivial coset $k = 0$, only the coset $k = n/2$ in the case (b) is forced to be unbroken. The rest, as we have seen, are broken.

Finally, we should ask: what is the infra-red boundary state? Clearly the boundary state must be invariant under the $\mathbb{Z}_n$ symmetry. The obvious choice is to take a non-fundamental boundary state, consisting of a sum over the various theta angles

$$|\mathrm{IR}\rangle = \sum_{k=0}^{m-1} |\theta + \tfrac{2k}{\rho^2}\rho; \mathcal{R}_{IR}\rangle \tag{3.23}$$

This captures the symmetry breaking (3.21) in a minimal way, with the least possible number of fundamental boundary states in the sum. The boundary central charge picks up a contribution from each term in (3.23). Furthermore, it turns out that, in some examples, any attempt to add further boundary states to this sum results in a violation of the g-theorem. This gives credence to this minimalist conjecture. The result is that, if there is no emergent Majorana zero mode, then the infra-red central charge is given by

$$g_{IR} = m\,\mathrm{Vol}(\Lambda[\mathcal{R}_{IR}]) \tag{3.24}$$

If, in addition, there is an emergent Majorana mode then we have an additional factor of $\sqrt{2}$, as in (3.16).

An Example

The simplest example of a non-primitive boundary state can be found in the two fermion theory with $g_{\mathcal{R}} = \sqrt{5}$.

A glance at the table in Section 3.3.1 shows that there are two operators with dimension $L_0 = \frac{1}{5}$, characterised by the primitive vectors

$$\hat{\rho}_1 = \left(\frac{1}{5}, \frac{3}{5}\right) \quad \text{and} \quad \hat{\rho}_2 = \left(\frac{3}{5}, -\frac{1}{5}\right)$$

Deforming by either of these operators breaks $U(1)^2 \to U(1)$.

There are also two operators with dimension $L_0 = \frac{4}{5}$, which have $\rho_a = 2\hat{\rho}_a$, with $a = 1, 2$. Deforming by either of these operators breaks $U(1)^2 \to U(1) \times \mathbb{Z}_2$.

From the previous table, we see that deforming by either $\hat{\rho}_a$ or $\rho_a = 2\hat{\rho}_a$ results in the same infra-red charge matrix $\mathcal{R}_{IR}$. This is a trivial, non-chiral state with $\text{Vol}(\mathcal{R}_{IR}) = 1$. However, when we deform by the non-primitive vector, we must sum over two infra-red boundary states to preserve the $\mathbb{Z}_2$. The net result is that the two deformations give different infra-red central charges

$$\begin{aligned} \hat{\rho}_a \quad &\Rightarrow \quad g_{IR} = 1 \\ \rho_a = 2\hat{\rho}_a \quad &\Rightarrow \quad g_{IR} = 2 \end{aligned}$$

3.3.3 The Boundary Central Charge

All the ingredients are now in place to determine the boundary state in the infra-red and its central charge. We start from a UV boundary state $|\theta; \mathcal{R}_{UV}\rangle$, with

$$g_{UV} = \sqrt{\text{Vol}(\Lambda_{UV})}$$

where $\Lambda_{UV} = \Lambda[\mathcal{R}_{UV}]$. We deform by a relevant, bosonic, boundary operator characterised by $\rho \in \Lambda^\star_{UV}$. The IR boundary state is then determined by several factors:

- The infra-red charge matrix $\mathcal{R}_{IR}$, given by (3.13). It contributes a factor of $\sqrt{\text{Vol}(\Lambda_{IR})}$ to the central charge, where $\Lambda_{IR} = \Lambda[\mathcal{R}_{IR}]$.
- If the boundary state changes SPT class, as determined by (3.18), there is an emergent Majorana mode on the boundary. This increases the infra-red central charge by $\sqrt{2}$.
- If $\rho = n\hat{\rho}$ is not primitive, there is naïvely a discrete symmetry breaking pattern in which $\mathbb{Z}_n \to \mathbb{Z}_{n/m}$ with m determined by (3.22). To avoid spontaneous breaking of this symmetry, we must sum over m boundary states. This increases the central charge by m.

To compute the IR central charge, we need the relation between the volumes of the IR and UV charge lattices. This will be computed in Section 3.4: it turns out to be

$$\text{Vol}(\Lambda_{IR}) = \hat{\rho}^2\,\text{Vol}(\Lambda_{UV}) \times \begin{cases} \frac{1}{2} & \text{if } \hat{\rho} \text{ is bosonic} \\ 1 & \text{if } \hat{\rho} \text{ is fermionic} \end{cases} \tag{3.25}$$

We can now consider the following three types of flows.

- Bosonic flows that preserve the SPT class

  Flows that leave the SPT class unchanged are initiated by operators with charge $\rho = n\hat{\rho}$ with $\hat{\rho}$ bosonic, and n any integer. The discrete symmetry breaking pattern is $\mathbb{Z}_n \to 1$, and the boundary state takes the form

$$|\theta; \mathcal{R}_{UV}\rangle \ \to \ \sum_{k=0}^{n-1} |\theta + \tfrac{2k}{\rho^2}\rho; \mathcal{R}_{IR}\rangle$$

  In this case, the ratio of IR to UV central charges is given by

$$\frac{g_{IR}}{g_{UV}} = n\sqrt{\frac{\text{Vol}(\Lambda_{IR})}{\text{Vol}(\Lambda_{UV})}} = \sqrt{\rho^2/2}$$

- Bosonic flows that change the class

  Flows that flip the SPT class are initiated by operators with charge $\rho = n\hat{\rho}$ with $\hat{\rho}$ fermionic. If this operator is bosonic then n is even. This time the discrete symmetry breaking is $\mathbb{Z}_n \to \mathbb{Z}_2$, and

$$|\theta; \mathcal{R}_{UV}\rangle \ \to \ \sqrt{2}\sum_{k=0}^{\frac{n}{2}-1} |\theta + \tfrac{2k}{\rho^2}\rho; \mathcal{R}_{IR}\rangle$$

  The ratio of IR to UV central charge is now

$$\frac{g_{IR}}{g_{UV}} = \sqrt{2} \times \frac{n}{2}\sqrt{\frac{\text{Vol}(\Lambda_{IR})}{\text{Vol}(\Lambda_{UV})}} = \sqrt{\rho^2/2}$$

- Fermionic flows

  Finally, if we start in the UV with an extra Majorana mode then we can perturb by a fermionic operator with charge $\rho = n\hat{\rho}$ with $\hat{\rho}$ fermionic and n odd. The discrete symmetry breaking is $\mathbb{Z}_n \to 1$. We also know that the flow flips the SPT class, since $\hat{\rho}$ is fermionic. The flow of boundary states is now

$$\sqrt{2}\,|\theta; \mathcal{R}_{UV}\rangle \ \to \ \sum_{k=0}^{n-1} |\theta + \tfrac{2k}{\rho^2}\rho; \mathcal{R}_{IR}\rangle$$

  and the ratio of IR to UV central charges is

$$\frac{g_{IR}}{g_{UV}} = \frac{1}{\sqrt{2}} \times n\sqrt{\frac{\text{Vol}(\Lambda_{IR})}{\text{Vol}(\Lambda_{UV})}} = \sqrt{\rho^2/2}$$

The Central Charge Relation

Importantly, we find the same ratio of central charges for each of the three types of RG flows described above. Moreover, we recognise $L_0 = \rho^2/2$ as the dimension of the UV operator $\mathcal{O}$ that initiates the RG flow. We learn that

$$g_{IR} = g_{UV}\sqrt{\dim(\mathcal{O})}$$

This is the formula (3.1) advertised in the introduction. Since the UV operator is necessarily relevant, we have $\rho^2 < 2$. This ensures that $g_{IR} < g_{UV}$, and the g-theorem is obeyed.

More General RG Flows

In our discussion above, we have restricted attention to RG flows initiated by operators with a definite charge under $U(1)^N$. This ensures that the original symmetry is broken to $U(1)^{N-1}$, which allowed us to identify the infra-red state (3.13).

More generally, we could deform by turning on superpositions of such operators with different ρ. The resulting RG flows can be understood by following first one deformation, then the other. For certain UV boundary states, we can reach IR states this way which cannot be reached by turning on one operator alone.

An example is provided by the $g^2 = 9$ four-fermion state

$$\mathcal{R}_{UV} = \begin{pmatrix} 0 & -\frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ -\frac{2}{3} & 0 & -\frac{2}{3} & \frac{1}{3} \\ -\frac{1}{3} & -\frac{2}{3} & 0 & -\frac{2}{3} \\ \frac{2}{3} & -\frac{1}{3} & -\frac{2}{3} & 0 \end{pmatrix}$$

Deformations by charge eigenstates will take us to IR states with $g^2 = 9, 6, 3$. However, they will not take us the trivial state with $g = 1$. This can be reached by a more general perturbation, such as by chaining together the flows $9 \to 3 \to 1$.

We address one final point before moving on. Our analysis did not fix the theta-angle mapping from the UV to the IR boundary states. Yet in the above flows, we have used the same θ in both. Here we give justification. In fact, our picture of deforming by a single operator ρ is a little oversimplified. The requirement that the boundary perturbation must be real forces us to deform by a real combination of boundary operators with charges ρ and $-\rho$. This present no issue to our analysis, as both preserve the same symmetries. However, there is now the possibility of a relative phase between the two operators. This manifests itself as a shift in the theta-angles of the IR boundary state, and hence renders the question moot.

3.4 RG Flows: Proofs

In Section 3.3 we stated a number of results without proof. Here we give the proofs.

3.4.1 The UV Symmetry

Given the charge matrix $\mathcal{R}_{UV}$, the $U(1)^N$ symmetry group preserved in the UV is

$$U(1)^N = \left\{ (e^{2\pi i t_\alpha Q_{\alpha i}},\, e^{2\pi i t_\alpha \bar{Q}_{\alpha i}}) \,:\, t \in \mathbb{R}^N \right\}$$

where $Q_{\alpha i}$ and $\bar{Q}_{\alpha i}$ are the UV charge assignments. Using the definition $\mathcal{R}_{UV} = \bar{Q}^{-1}Q$, this group can be parametrised in the more useful form

$$U(1)^N = \left\{ (e^{2\pi i x},\, e^{2\pi i \mathcal{R}_{UV} x}) \,:\, x \in \mathbb{R}^N \right\}$$

The symmetry parameter x is naturally valued in $\mathbb{R}^N/\Lambda_{UV}$.

Given the boundary operator charge $\rho \in \Lambda^\star_{UV}$, we first wish to determine how much of $U(1)^N$ remains unbroken by the perturbation. Under the $U(1)^N$ transformation with parameter x, the boundary operator picks up a phase of $e^{2\pi i x\cdot\rho}$. This means that perturbing operator is invariant when

$$x \cdot \rho \in \mathbb{Z}$$

Let us write $\rho = n\hat{\rho}$ with $n \geq 1$ and $\hat{\rho}$ primitive in $\Lambda^\star_{UV}$. Because $\hat{\rho}$ is primitive, we can introduce a special basis for Λ_{UV} with

$$\begin{aligned} \Lambda_{UV} &= \text{span}\,\{\lambda_1, \ldots, \lambda_N\} \\ \lambda_1 \cdot \hat{\rho} &= 1 \\ \{\lambda_2, \ldots, \lambda_N\} \cdot \hat{\rho} &= 0 \end{aligned}$$

Writing x in components with respect to this basis, the above condition for invariance becomes

$$x_1 \in \tfrac{1}{n}\mathbb{Z} \qquad x_2, \ldots, x_N \in \mathbb{R}$$

Since the variables x_i are defined mod 1, we see that the first variable x_1 parametrises a discrete $\mathbb{Z}_n$, while the remaining variables $x_2, \ldots x_N$ parametrise a $U(1)^{N-1}$. In other words, the $U(1)^N$ is broken to $U(1)^{N-1} \times \mathbb{Z}_n$, with the coset corresponding to $k \in \mathbb{Z}_n$ being all those transformations with parameter x obeying

$$x \cdot \rho = k$$

This puts us in a position to justify the form of the IR charge matrix. The $U(1)^{N-1}$ corresponds to those transformations with

$$x \in \rho^{\perp}$$

The statement that these are also preserved by the IR boundary state is that

$$\mathcal{R}_{IR} x = \mathcal{R}_{UV} x$$

This immediately leads to (3.12).

3.4.2 The IR Lattice

Given that the IR charge matrix takes the form

$$\mathcal{R}_{IR} = \mathcal{R}_{UV} \mathrm{Ref}_{\rho}$$

where Ref_{ρ} denotes reflection along ρ, it follows immediately that both Λ_{IR} and Λ_{UV} share the same intersection with $\rho^{\perp}$, the hyperplane perpendicular to ρ:

$$\Lambda_{IR} \cap \rho^{\perp} = \Lambda_{UV} \cap \rho^{\perp} = \mathrm{span}\left\{\lambda_2, \ldots, \lambda_N\right\}$$

It follows that there is a basis for Λ_{IR} consisting of

$$\Lambda_{IR} = \mathrm{span}\left\{\tilde{\lambda}_1, \lambda_2, \ldots, \lambda_N\right\}$$

Here $\tilde{\lambda}_1$ is the single, remaining basis vector of Λ_{IR}, which remains to be determined. In fact, all we shall need to know about it is provided by the following claim:

Claim: The extra basis vector $\tilde{\lambda}_1$ of Λ_{IR} is of the form

$$\tilde{\lambda}_1 = \left\{ \begin{array}{ll} \frac{1}{2} & \text{if } \hat{\rho} \text{ is bosonic} \\ 1 & \text{if } \hat{\rho} \text{ is fermionic} \end{array} \right\} \hat{\rho} \mod \rho^{\perp}$$

Proof: A general vector $\lambda \in \mathbb{R}^N$ can be written in the form

$$\lambda = a\hat{\rho} + \eta \quad \text{with } a \in \mathbb{R} \text{ and } \eta \in \rho^{\perp}$$

We wish to determine the constraints on a and η that arise from insisting $\lambda \in \Lambda_{IR}$. In particular, we are particularly interested in the quantisation condition on a. The first constraint is that λ must be an integer vector, which we call x:

$$a\hat{\rho} + \eta = x \tag{3.26}$$

The second constraint is that $(\mathcal{R}_{UV}\text{Ref}_{\rho})\lambda$ must be an integer vector, which we call y. Using the fact that Ref_{ρ} flips $\hat{\rho}$ while leaving η unaffected, we have

$$\mathcal{R}_{UV}(a\hat{\rho} - \eta) = y \quad \Rightarrow \quad a\hat{\rho} - \eta = \mathcal{R}_{UV}^{-1} y \tag{3.27}$$

To proceed, we take the sum and difference of (3.26) and (3.27). First, the sum tells us that

$$2a\hat{\rho} = x + \mathcal{R}_{UV}^{-1} y \quad \text{with } x, y \in \mathbb{Z}^N \tag{3.28}$$

We take the inner product with the basis vector $\lambda_1 \in \Lambda_{UV}$, which obeys $\lambda_1 \cdot \hat{\rho} = 1$. On the right-hand side, we have $\lambda_1 \cdot x \in \mathbb{Z}$ since both λ_1 and x are integral. Furthermore, $\lambda_1 \cdot \mathcal{R}_{UV}^{-1} y \in \mathbb{Z}$ since this is equal to $\mathcal{R}_{UV}\lambda_1 \cdot y$ and $\mathcal{R}_{UV}\lambda_1$ is integral by definition of Λ_{UV}. We learn that

$$2a \in \mathbb{Z}$$

Next we invoke the fact that $\hat{\rho}$ lies in $\Lambda_{UV}^{\star} = \mathbb{Z}^N + \mathcal{R}_{UV}^{-1}\mathbb{Z}^N$. This means that $\hat{\rho}$ can be written in the form $\rho = v + \mathcal{R}_{UV}^{-1} w$ for two further integer vectors v and w. The equation

(3.28) then becomes

$$2a(v + \mathcal{R}_{UV}^{-1} w) = x + \mathcal{R}_{UV}^{-1} y \tag{3.29}$$

It is obvious that one solution to this equation for (x, y) is $x = 2av$ and $y = 2aw$. However, this is not the unique solution since we still have the freedom to shift by any integer solution to $x + \mathcal{R}_{UV}^{-1} y = 0$. These are precisely $(x, y) = (\zeta, -\mathcal{R}_{UV}\zeta)$ for $\zeta \in \Lambda_{UV}$. The general solution to (3.29) is then

$$x = 2av + \zeta \quad \text{and} \quad y = 2aw - \mathcal{R}_{UV}\zeta$$

The above equations were derived by taking the sum of (3.26) and (3.27). Next we take the difference. This gives

$$2\eta = x - \mathcal{R}_{UV}^{-1} y \quad \Rightarrow \quad \eta = a(v - \mathcal{R}_{UV}^{-1} w) + \zeta$$

The variables $a \in \frac{1}{2}\mathbb{Z}$ and $\zeta \in \Lambda_{UV}$ are further constrained by the requirement that $\eta \in \rho^{\perp}$. Taking the inner product with $\hat{\rho}$ and setting this to zero gives

$$\zeta \cdot \hat{\rho} = -a\left[(v - \mathcal{R}_{UV}^{-1}) \cdot \hat{\rho}\right] = -a(v^2 - w^2)$$

The left-hand side is an integer. But a can be either integer of half-integer. Clearly the half-integer values can only occur when $v^2 - w^2$ is even which, in turn, requires $\sum_{i=1}^{N}(v_i + w_i)$ to be even. But this is precisely the fermionic parity of $\hat{\rho}$.

To see this, note that $\Lambda[\mathcal{R}]^\star = \mathbb{Z}^N + \mathcal{R}^{-1}\mathbb{Z}^N$ has a simple physical interpretation: all boundary operators can be made by taking suitably regularised products of holomorphic and antiholomorphic fermion fields as they approach the boundary. A product of n_i copies of $\psi_i(z)$ and m_i copies of $\bar{\psi}_i(z)$ would give rise to a boundary operator with charge $\rho = n + \mathcal{R}^{-1}m$. It's clear that the fermion parity of this operator is

$$(-1)^{n_1 + \cdots + n_N + m_1 + \cdots + m_N} \tag{3.30}$$

Using the properties of the fermion vector f, defined in (3.11), this can easily be shown to agree with the earlier characterisation $(-1)^{f \cdot \rho}$. Using the fact that $\hat{\rho} = v + \mathcal{R}_{UV}^{-1} w$, we then learn that

$$a \in \begin{cases} \frac{1}{2}\mathbb{Z} & \text{if } \hat{\rho} \text{ is bosonic} \\ \mathbb{Z} & \text{if } \hat{\rho} \text{ is fermionic} \end{cases}$$

The conditions derived above are necessary for $\lambda = a\hat{\rho} + \eta$ to lie in Λ_{IR}. The same derivation can also be followed backwards to show they are sufficient. All of which means that we finally have an expression for our last remaining basis vector of Λ_{IR};

$$\tilde{\lambda}_1 = \left\{ \begin{array}{ll} \frac{1}{2} & \text{if } \hat{\rho} \text{ is bosonic} \\ 1 & \text{if } \hat{\rho} \text{ is fermionic} \end{array} \right\} \hat{\rho} + \eta \tag{3.31}$$

for some $\eta \in \rho^\perp$ whose value is unimportant. This completes the proof of the claim.

We are now in a position to compute the volume of Λ_{IR}. This is

$$\text{Vol}(\Lambda_{IR}) = \text{Vol}(\tilde{\lambda}_1, \lambda_2, \ldots, \lambda_N) = \text{Vol}((\hat{\rho}\cdot\tilde{\lambda}_1)\lambda_1, \lambda_2, \ldots, \lambda_N)$$

We therefore find

$$\text{Vol}(\Lambda_{IR}) = \hat{\rho}\cdot\tilde{\lambda}_1 \,\text{Vol}(\Lambda_{UV}) = \left\{ \begin{array}{ll} \frac{1}{2} & \text{if } \hat{\rho} \text{ is bosonic} \\ 1 & \text{if } \hat{\rho} \text{ is fermionic} \end{array} \right\} \hat{\rho}^2 \,\text{Vol}(\Lambda_{UV}) \tag{3.32}$$

This provides the justification for (3.25).

The above result also allows us to determine the integer m which governs the amount of discrete symmetry breaking. Under a general $U(1)^N$ transformation with parameter x, we have

$$|\theta; \mathcal{R}_{IR}\rangle \mapsto g_{\mathcal{R}} \sum_{\lambda\in\Lambda[\mathcal{R}_{IR}]} e^{i\gamma(\lambda)}\, e^{\theta\cdot\lambda} e^{2\pi i x\cdot\lambda} e^{2\pi i(\mathcal{R}_{UV}x)\cdot(-\mathcal{R}_{IR}\lambda)} \| \lambda, -\mathcal{R}_{IR}\lambda\,\rangle\!\rangle$$

We see that the effect of this is to shift the theta angles θ_i of the infra-red boundary state by

$$\frac{\theta}{2\pi} \mapsto \frac{\theta}{2\pi} + \left(\mathbb{1} - \mathcal{R}_{UV}^{-1}\mathcal{R}_{IR}\right) x = \frac{\theta}{2\pi} + \frac{2(x\cdot\rho)}{\rho^2}\rho$$

where, in the second equality, we have used the expression (3.13) for $\mathcal{R}_{IR}$. We see explicitly that the theta angles are invariant under the preserved $U(1)^{N-1}$ symmetry defined by those x with $x\cdot\rho = 0$. But what of the discrete $\mathbb{Z}_n$ symmetry? A transformation by $k \in \mathbb{Z}_n$ is enacted by any x for which $x\cdot\rho = k$, and shifts the theta angles by

$$\frac{\theta}{2\pi} \mapsto \frac{\theta}{2\pi} + \frac{2k}{\rho^2}\rho$$

The theta angles in (3.19) appear in the phase $e^{i\theta\cdot\lambda}$, which means that they are naturally valued mod $2\pi\Lambda^\star[\mathcal{R}_{IR}]$. Therefore the transformation above leaves the theta angles invariant whenever

$$\frac{2k}{\rho^2}\rho \in \Lambda^\star[\mathcal{R}_{IR}]$$

The above condition will be satisfied if the LHS gives an integer when dotted with every basis vector of Λ_{IR}. Of these, the last $N-1$ vectors $\lambda_2,\ldots,\lambda_N$ give zero. Thus a constraint only arises by dotting with $\tilde{\lambda}_1$. Recalling the definition (3.31) of $\tilde{\lambda}_1$, this gives

$$\left\{\begin{array}{ll} \frac{1}{2} & \text{if } \hat{\rho} \text{ is bosonic} \\ 1 & \text{if } \hat{\rho} \text{ is fermionic} \end{array}\right\} \cdot \frac{1}{n} \cdot 2k \in \mathbb{Z}$$

It is now straightforward to read off the quantisation condition on k. It must be a multiple of m, where m is defined by

$$m = \left\{\begin{array}{ll} n & \text{if } \hat{\rho} \text{ is bosonic} \\ n/\gcd(n,2) & \text{if } \hat{\rho} \text{ is fermionic} \end{array}\right.$$

This is the statement of (3.22).

3.4.3 The Emergent Majorana Mode

The final missing ingredient is to determine when a boundary Majorana mode arises. As explained in Section 3.3, this happens when the UV and IR charge matrices lie in different classes, which is detected by the ground degeneracy of bulk states (3.15),

$$G[\mathcal{R}_{UV},\mathcal{R}_{IR}] = \frac{\sqrt{\mathrm{Vol}(\Lambda_{UV})\,\mathrm{Vol}(\Lambda_{IR})}}{\mathrm{Vol}(\Lambda[\mathcal{R}_{UV},\mathcal{R}_{IR}])}\sqrt{2}$$

where the factor of $\sqrt{2}$ comes from the truncated determinant in (3.15), using the expression (3.13) for $\mathcal{R}_{IR}$. Clearly, we need to compute the volume of the intersection lattice

$$\Lambda[\mathcal{R}_{UV},\mathcal{R}_{IR}] = \left\{\lambda \in \mathbb{Z}^N : \mathcal{R}_{UV}\lambda = \mathcal{R}_{IR}\lambda \in \mathbb{Z}^N\right\}$$

First, we can write

$$\Lambda[\mathcal{R}_{UV},\mathcal{R}_{IR}] = \left\{\lambda \in \mathbb{Z}^N : \lambda\cdot\rho = 0 \text{ and } \mathcal{R}_{UV}\lambda \in \mathbb{Z}^N\right\} = \Lambda_{UV}\cap\rho^\perp$$

But using the basis of Λ_{UV}, the intersection lattice takes the particularly simple form

$$\Lambda[\mathcal{R}_{UV}, \mathcal{R}_{IR}] = \text{span}\,\{\lambda_2, \ldots, \lambda_N\}$$

To determine the volume of this intersection lattice, we need to take the above basis and add a unit vector orthogonal to them all. This vector is $\hat{\rho}/\sqrt{\hat{\rho}^2}$, so

$$\text{Vol}(\Lambda[\mathcal{R}_{UV}, \mathcal{R}_{IR}]) = \text{Vol}\left(\hat{\rho}/\sqrt{\hat{\rho}^2}, \lambda_2, \ldots, \lambda_N\right)$$

But we could equally well shift the first basis vector by any element in $\rho^\perp$. Using the property $\lambda_1 \cdot \hat{\rho} = 1$, we then have

$$\text{Vol}(\Lambda[\mathcal{R}_{UV}, \mathcal{R}_{IR}]) = \text{Vol}\left(\sqrt{\hat{\rho}^2}\lambda_1, \lambda_2, \ldots, \lambda_N\right) = \sqrt{\hat{\rho}^2}\,\text{Vol}(\Lambda_{UV})$$

If we now put this together with our expression (3.32) for the volume of Λ_{IR}, we have the simple result

$$G[\mathcal{R}_{UV}, \mathcal{R}_{IR}] = \begin{cases} 1 & \text{if } \hat{\rho} \text{ is bosonic} \\ \sqrt{2} & \text{if } \hat{\rho} \text{ is fermionic} \end{cases}$$

which establishes (3.18).

Appendix 3.A A Higher Pythagorean Triple

For $N = 2$ Dirac fermions, the chiral boundary conditions are in one-to-one correspondence with Pythagorean triples, from Section 2.3.3. With the Euclid parametrisation (3.3) with $p = 4$ and $q = 1$, we have the Pythagorean triple $8^2 + 15^2 = 17^2$. The charge matrix is

$$\mathcal{R}_{UV} = \frac{1}{17}\begin{pmatrix} 8 & 15 \\ -15 & 8 \end{pmatrix}$$

This boundary state has $g_{UV}^2 = 17$. Various RG flows initiated by bosonic operators are summarised in the following table:

ρ	L_0	$\mathcal{R}_{IR}$	Majorana?	g^2_{IR}
$(\frac{5}{17}, \frac{3}{17})$	$\frac{1}{17}$	$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$	No	1
$(\frac{3}{17}, -\frac{5}{17})$	$\frac{1}{17}$	$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$	No	1
$(\frac{8}{17}, -\frac{2}{17})$	$\frac{2}{17}$	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	Yes	2
$(\frac{2}{17}, \frac{8}{17})$	$\frac{2}{17}$	$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$	Yes	2
$(\frac{13}{17}, \frac{1}{17})$	$\frac{5}{17}$	$\frac{1}{5}\begin{pmatrix} -3 & 4 \\ 4 & 3 \end{pmatrix}$	No	5
$(\frac{11}{17}, -\frac{7}{17})$	$\frac{5}{17}$	$\frac{1}{5}\begin{pmatrix} 3 & 4 \\ 4 & -3 \end{pmatrix}$	No	5
$(\frac{7}{17}, \frac{11}{17})$	$\frac{5}{17}$	$\frac{1}{5}\begin{pmatrix} -3 & -4 \\ -4 & 3 \end{pmatrix}$	No	5
$(\frac{1}{17}, -\frac{13}{17})$	$\frac{5}{17}$	$\frac{1}{5}\begin{pmatrix} 3 & -4 \\ -4 & -3 \end{pmatrix}$	No	5
$(\frac{18}{17}, \frac{4}{17})$	$\frac{10}{17}$	$\frac{1}{5}\begin{pmatrix} -4 & 3 \\ 3 & 4 \end{pmatrix}$	Yes	10
$(\frac{14}{17}, -\frac{12}{17})$	$\frac{10}{17}$	$\frac{1}{5}\begin{pmatrix} 4 & 3 \\ 3 & -4 \end{pmatrix}$	Yes	10
$(\frac{12}{17}, \frac{14}{17})$	$\frac{10}{17}$	$\frac{1}{5}\begin{pmatrix} -4 & -3 \\ -3 & 4 \end{pmatrix}$	Yes	10
$(\frac{4}{17}, -\frac{18}{17})$	$\frac{10}{17}$	$\frac{1}{5}\begin{pmatrix} 4 & -3 \\ -3 & -4 \end{pmatrix}$	Yes	10
$(\frac{21}{17}, -\frac{1}{17})$	$\frac{13}{17}$	$\frac{1}{13}\begin{pmatrix} -5 & 12 \\ 12 & 5 \end{pmatrix}$	No	13
$(\frac{19}{17}, -\frac{9}{17})$	$\frac{13}{17}$	$\frac{1}{13}\begin{pmatrix} 5 & 12 \\ 12 & -5 \end{pmatrix}$	No	13
$(\frac{19}{17}, -\frac{9}{17})$	$\frac{13}{17}$	$\frac{1}{13}\begin{pmatrix} -5 & -12 \\ -12 & 5 \end{pmatrix}$	No	13
$(\frac{1}{17}, \frac{21}{17})$	$\frac{13}{17}$	$\frac{1}{13}\begin{pmatrix} 5 & -12 \\ -12 & -5 \end{pmatrix}$	No	13

This table lists relevant, bosonic operators and their end points under RG. For simplicity, we restrict to primitive ρ, so that there are no discrete symmetries and the infra-red central charge g_{IR} is determined solely by $\mathcal{R}_{IR}$ and the existence of a boundary Majorana fermion.

Note that the dimensions of the relevant operators take the form

$$L_0 = \frac{m^2 + n^2}{p^2 + q^2} \qquad p, q, m, n \in \mathbb{Z}$$

where, for us, $p = 4$ and $q = 1$. Turning on an operator with this dimension takes us to a new state with primitive charges m, n in (3.3). This same property holds for all boundary states with $N = 2$ fermions. We do not know of such a simple pattern for $N \geq 4$.

Appendix 3.B The Boundary Majorana Mode

In this appendix, we explain how a boundary Majorana mode interacts with the bulk fermions. Very similar calculations can be found in [64, 79] and related analysis in [80, 81].

A Fermion on a Half Line

We start with a single Majorana fermion ξ on a half-line, interacting with a quantum mechanical Majorana fermion χ sitting on the boundary. It is simplest if we unfold the system, leaving us with a single right-moving Majorana-Weyl fermion on a line, interacting with a Majorana impurity at the origin. The Lagrangian is

$$\mathcal{L} = \frac{i}{2}\chi\partial_t\chi + \int dx \left[\frac{i}{2}\xi\partial_+\xi + i\sqrt{2m}\delta(x)\xi(x)\,\chi\right]$$

The coupling between bulk and boundary is simply a quadratic term, set by a mass scale m. As we will see, only modes with momentum $k \ll m$ are significantly affected by the impurity.

To proceed, it is useful to temporarily smooth out the delta-function coupling. We replace the Lagrangian with

$$\mathcal{L} = \frac{i}{2}\chi\partial_t\chi + \int dx \left[\frac{i}{2}\xi\partial_+\xi + i\sqrt{2m}f(x)\xi\chi\right]$$

where $f(x)$ is some function localised around the origin, with support in $x \in [-\epsilon, +\epsilon]$, and with $\int dx\, f(x) = 1$. The equations of motion are:

$$\partial_t\chi = \sqrt{2m}\int dx\ f\xi$$
$$\partial_+\xi = -\sqrt{2m}f\chi$$

Modes with energy k have time dependence e^{-ikt}. (All fermions are subject to a reality condition, but the equations of motion are linear so we can work with complex objects and take the real part at the end.) The equations of motion become

$$-ik\chi = \sqrt{2m}\int dx\ f\xi$$
$$-ik\xi + \partial_x\xi = -\sqrt{2m}f\chi$$

We are interested in modes with $k \ll 1/\epsilon$, which ensures that they don't probe the microscopic details of the function $f(x)$. Near the origin, $|x| \leq \epsilon$, the second equation can then be replaced by $\partial_x\xi = -\sqrt{2m}f\chi$. We integrate the second equation in the asymptotic regions, and join them up to find

$$\xi(x) = \begin{cases} e^{ikx} & x < -\epsilon \\ 1 - \sqrt{2m}F(x)\chi & \text{otherwise} \\ (1-\sqrt{2m}\chi)e^{ikx} & x > \epsilon \end{cases} \tag{3.33}$$

where $F(x)$ is a step function that goes smoothly from 0 to 1, with $F'(x) = f(x)$. Substituting this into the equation for χ gives us a consistency condition,

$$-ik\chi = \sqrt{2m}\left(1 - \sqrt{m/2}\,\chi\right)$$

which has the solution

$$\chi = -\frac{\sqrt{2m}}{ik - m}$$

Inserting this back into (3.33) gives the required expression for a chiral Weyl fermion passing through a Majorana impurity. Taking the limit $\epsilon \to 0$, we find that ψ jumps by a phase as it passes through the origin

$$\xi(x) = e^{ikx}\begin{cases} 1 & x < 0 \\ \frac{ik+m}{ik-m} & x > 0 \end{cases}$$

High energy modes, with $k \gg m$, are unaffected by the impurity. Low energy modes, with $k \ll m$, suffer a sign flip.

The Spectrum on a Circle

To further understand the role played by the Majorana impurity, let us now consider a right-moving Majorana-Weyl fermion on a spatial circle, which we take to have length L.

We will impose periodic boundary conditions on this fermion, which means that it has a single Majorana zero mode. Such a system is anomalous and to rectify the situation we must add an odd number of extra Majorana modes. We do this by including $2n-1$ Majorana impurities, at locations x_i with couplings m_i. Periodicity of ξ then imposes a quantisation condition on the momentum k which is

$$\prod_{i=1}^{2n-1} 2\tan^{-1}\left(\frac{m_i}{k}\right) = kL \mod 2\pi$$

When $m_i \ll 1/L$, the impurities pair up with the bulk zero mode to form n independent complex zero modes. This results in a ground state degeneracy of 2^n. Further modes are then quantised as $\sim 2\pi/L$.

Now consider increasing the interaction of a single impurity, say $m_1 \gg 1/L$. All bulk modes with $k < m_1$, including the bulk zero mode, undergo a sign flip, which means that their energy increases by π/L, corresponding to a spectral flow of $+1/2$. There are $n-1$ remaining complex zero modes, and 2^{n-1} degenerate ground states.

Something a little different happens when we increase a second impurity coupling, say $m_2 \gg 1/L$. Once again, there is a spectral flow of $+1/2$. But instead of an impurity zero mode being lifted, it now mixes with a new bulk zero mode. Once again there are 2^{n-1} degenerate ground states. Clearly this pattern now repeats as further impurity couplings are increased.

Absorbing Majorana Fermions into the Boundary State

The ideas described above help build intuition for how Majorana boundary modes can be incorporated in a boundary state. To illustrate this, consider a single Dirac fermion ψ on an interval of length L. We impose vector boundary conditions at one end

$$\psi_L = \psi_R \quad \text{at } x = 0 \tag{3.34}$$

and axial boundary conditions at the other,

$$\psi_L = \psi_R^\dagger \quad \text{at } x = L \tag{3.35}$$

As explained in detail in Chapter 2, these two boundary conditions are mutually inconsistent in the sense that they result in a single Majorana zero mode in the bulk. Indeed, if we write

$$\psi = \xi_1 + i\xi_2$$

Then ξ_1 has a zero mode, while ξ_2 does not.

We now invoke the doubling trick, and view both fermions as chiral, living on a circle of length $2L$. The boundary conditions mean that ξ_1 is periodic, while ξ_2 is anti-periodic. To make the theory consistent, we add a single Majorana impurity, χ, at $x = 0$. Now we have two options:

- We could couple χ to ξ_1. As we've seen above, the resulting spectral flow renders ξ_1 anti-periodic. The net effect is that the right-most boundary condition (3.34) is shifted from axial, to vector, but with a theta angle $\theta = \pi$, so that $\psi_L = -\psi_R$ at $x = L$. In this case, the ground state is non-degenerate. This shift of the theta angle due to a boundary fermion was also found in [79].
- If, instead, we couple χ to ξ_2, then the spectral flow renders ξ_2 periodic, with vanishing theta angle, so that $\psi_L = \psi_R$ at $x = L$. Now both ξ_1 and ξ_2 admit a Majorana zero mode, and there are two ground states.

Appendix 3.C A D-Brane Perspective

The chiral boundary conditions have a more familiar interpretation in terms of boundary states for D-branes. Details of such states can be found, for example, in [34] or the textbook [35].

The geometric viewpoint arises after bosonisation. This relates the N Dirac fermions to N periodic scalars, ϕ_i with the currents mapped as

$$\partial_+\phi_i = \psi_i^\dagger\psi_i \quad , \quad \partial_-\phi_i = \bar{\psi}_i^\dagger\bar{\psi}_i$$

where $\partial_\pm = \frac{1}{2}(\partial_t \pm \partial_x)$. The chiral boundary conditions require that there is no net flow of the left- and right-moving currents $\mathcal{J}_\alpha$ and $\bar{\mathcal{J}}_\alpha$, defined in (2.9), into the boundary. In the bosonic picture, these become simple, linear boundary conditions on the periodic scalars

$$(Q_{\alpha i} + \bar{Q}_{\alpha i})\partial_x\phi_i = (Q_{\alpha i} - \bar{Q}_{\alpha i})\partial_t\phi_i \qquad (3.36)$$

The trivial boundary condition $\mathcal{R} = \mathbb{1}$ gives Neumann boundary conditions $\partial_x \phi_i = 0$ in each direction, corresponding to a D-brane that wraps the full torus $\mathbf{T}^N$. Meanwhile, the other trivial boundary condition $\mathcal{R} = -\mathbb{1}$ gives a D0-brane, with $\phi_i =$ constant. Clearly by taking $\mathcal{R} = \text{diag}(+1, \ldots, -1, \ldots)$ we have any Dp-brane for $p = 0, \ldots, N$.

A general boundary state can be interpreted as a D-brane with flux, whose boundary conditions are written as

$$g_{\mu\nu} \partial_x \phi^\nu = B_{\mu\nu} \partial_t \phi^\nu$$

with g the metric and B the NS-NS 2-form.

The D-brane interpretation is particularly straightforward when $N = 2$ and we can consider the charge matrices (3.3) labelled by co-prime integers p and q. The boundary conditions (3.36) are then

$$p\phi_1 = q\dot{\phi}_2 \quad \text{and} \quad p\phi_2' = -q\dot{\phi}_1$$

This is simpler to interpret if we perform a T-duality on ϕ_2, introducing $\partial_\mu \tilde{\phi}_2 = \epsilon_{\mu\nu} \partial^\nu \phi_2$. The boundary conditions then become

$$p\phi_1' = q\tilde{\phi}_2' \quad \text{and} \quad q\dot{\phi}_1 = -p\dot{\tilde{\phi}}_2$$

This describes a D-string wrapping (p, q) times around the two cycles of the torus $\mathbf{T}^2$. Aspects of the boundary states for such a D-string, including the boundary central charge, were previously discussed in [82].

As described in Appendix 3.A, the relevant boundary operators have dimension $L_0 = (m^2 + n^2)/(p^2 + q^2)$ for pairs of integers m, n. The associated RG flow describes the decay of a D-brane wrapping (p, q) times around the torus to one wrapping (m, n) times.

Chapter 4

Symmetries and SPTs

Having defined and explored the chiral boundary states (2.24), the question we would like to answer now is: given a chiral boundary state specified by some choice of charges $Q, \bar{Q}$, what symmetries does it preserve, both continuous and discrete?

The first of these questions has obvious motivations from the problem of defining magnetic line operators, the 'first problem' discussed in the Introduction. There, our goal was to determine boundary states preserving interesting chiral subgroups of $SO(2N)_L \times SO(2N)_R$. This goal therefore motivates a careful study of the continuous symmetry preserved by a generic boundary state $|\mathcal{R};\theta\rangle$. If we can determine a criterion for the unbroken subgroup of $SO(2N)_L \times SO(2N)_R$, then it will allow us to determine whether the boundary state $|\mathcal{R};\theta\rangle$ is suitable for defining a given magnetic line operator. Our first objective in this chapter is to derive such a criterion.

Secondly, it is natural to also ask about the discrete symmetries preserved by $|\mathcal{R};\theta\rangle$. If a boundary state preserving certain discrete symmetries exists, then this implies the vanishing of the corresponding anomaly and triviality of the 3d SPT phase. We can therefore make contact with known facts about 3d SPT phases from the perspective of boundary states.

We will focus particularly on the discrete symmetry $(-1)^{F_L} \times (-1)^{F_R}$, or chiral fermion parity. Usually, a left-moving fermion reflects off a boundary to become a right-moving fermion, so that while overall fermion parity $(-1)^F$ is conserved, chiral fermion parity for left- and right-movers individually is not. Remarkably, there are boundary conditions that do preserve chiral fermion parity, but only when the number of Majorana fermions is a multiple of 8. This fact is reproduced by the chiral boundary states, where it translates to an interesting property of charge lattices.

Summary of Results

The answers to our questions above are straightforward to state, and somewhat fiddly to prove. The enhanced symmetries are determined by the rational, orthogonal matrix,

$$\mathcal{R}_{ij} = (\bar{Q}^{-1})_{i\alpha} Q_{\alpha j}$$

and the associated charge lattice, defined by

$$\Lambda[\mathcal{R}] \coloneqq \mathbb{Z}^N \cap \mathcal{R}^{-1}\mathbb{Z}^N \tag{4.1}$$

Lattices of this kind, which are the intersection of a lattice with a rotated version of itself, are sometimes referred to as *coincidence site lattices* [83]. In the present context, the lattice $\Lambda[\mathcal{R}]$ captures the difference between the charges carried by the left- and right-moving fermions. For example, when the charges are equal, with $Q = \bar{Q}$ so $\mathcal{R} = \mathbb{1}$, the lattice is simply $\Lambda[\mathcal{R}] = \mathbb{Z}^N$. For boundary states in which the left- and right-moving charges differ, the associated lattice $\Lambda[\mathcal{R}]$ becomes sparser.

As advertised above, we would like to understand the enhanced non-abelian symmetry. We will show that any such symmetry can be observed by its root system $\Delta[\mathcal{R}]$ nestled inside $\Lambda[\mathcal{R}]$ such that

$$\Delta[\mathcal{R}] \coloneqq \left\{ \lambda \in \Lambda[\mathcal{R}] \; : \; |\lambda|^2 = 2 \right\} \tag{4.2}$$

The simplest, trivial example occurs for a non-chiral boundary condition, which has $Q = \bar{Q}$ and so $\mathcal{R} = \mathbb{1}$. In this case, the enhanced symmetry is $SO(2N)_V \subset SO(2N)_L \times SO(2N)_R$. There are, however, a number of less trivial examples. For example the Maldacena-Ludwig boundary state, which preserves $\mathbb{Z}_2 \times \mathbb{Z}_2$, has $\mathcal{R} \neq \mathbb{1}$ but also preserves a Spin(8). (Indeed, the state was originally constructed to have this property using triality of $\mathfrak{so}(8)$.)

Our other goal is to determine the choices of Q and $\bar{Q}$ that preserve $\mathbb{Z}_2 \times \mathbb{Z}_2$ chiral fermion number. We first show that the boundary condition has such a property if and only if $\Lambda[\mathcal{R}]$ is an even lattice, i.e. the length-squared of any lattice vector is an even integer.

We further show that coincidence site lattices (4.1) can be even only when N is divisible by 4. In this way, we reproduce the $\mathbb{Z}_8$ classification of interacting SPT phases in d=2+1 dimensions [24, 84–86], albeit from a rather unconventional perspective.

The simplest boundary state that preserves $\mathbb{Z}_2 \times \mathbb{Z}_2$ chiral symmetry has $2N = 8$ Majorana fermions and was constructed long ago by Maldacena and Ludwig to describe the scattering of fermions off a monopole. We show how to construct all boundary states $\mathcal{R}$ with this property, and provide a general formula for them in the special case $N = 4$. This shows that although triality originally played an important role in the symmetric gapping of fermions preserving $\mathbb{Z}_2 \times \mathbb{Z}_2$ [24], it is not essential.

Finally, we determine how $\mathbb{Z}_2 \times \mathbb{Z}_2$ invariance restricts the spectrum of the Affleck-Ludwig central charge g. While essentially all possible values of g can be attained by generic boundary states, we find that those preserving chiral fermion parity form a strict subset defined by certain algebraic constraints.

Plan

We start in Section 4.1 by deriving the criterion (4.2) for the emergence of non-abelian symmetries preserved by the boundary state. In Section 4.2, we turn to the question of $\mathbb{Z}_2 \times \mathbb{Z}_2$ chiral fermion parity. We will first show that this is only a symmetry if it is already a subgroup of the original $U(1)^N$ preserved by the boundary state. Using this result, we then deduce that N must be divisible by 4, and construct all sets of charges Q, $\bar{Q}$ with this property. We then determine the spectrum of allowed central charges g.

4.1 Continuous Symmetries

In this section we derive the criterion (4.2) for the emergence of a larger non-abelian symmetry, and apply it to some simple examples. We continue to work with the CFT of $2N$ Majorana fermions

$$S = \frac{1}{4\pi} \int dz d\bar{z} \left(\chi_i \bar{\partial} \chi_i + \bar{\chi}_i \partial \bar{\chi}_i \right)$$

written down in Appendix 2.A. All of our work will be done in the closed sector, where we consider a single boundary state $|B\rangle$ living on the unit circle $|z| = 1$, as in Figure 4.1. For us the boundary state $|B\rangle$ will be one of the states $|\theta; \mathcal{R}\rangle$.

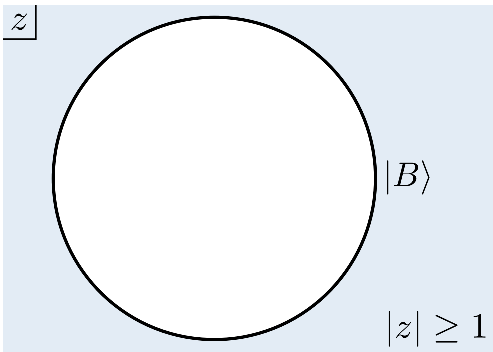


Figure 4.1: The bulk region $|z| \geq 1$, and boundary state $|B\rangle$ at $|z| = 1$.

4.1.1 Enhanced Continuous Symmetries

To say that an infinitesimal symmetry is unbroken at a boundary is to say that its corresponding Noether current has vanishing flux into the boundary. We start by considering a single symmetry $(A, \bar{A}) \in \mathfrak{so}(2N)_L \times \mathfrak{so}(2N)_R$. The condition for preservation reads

$$\left(zJ_A(z) + \bar{z}\bar{J}_{\bar{A}}(\bar{z})\right)|\theta; \mathcal{R}\rangle = 0 \qquad \text{for all } |z| = 1 \tag{4.3}$$

where the $J_A(z)$ is the $\mathfrak{so}(2N)_L$ current associated to the generator A, and $\bar{J}_{\bar{A}}(\bar{z})$ is the $\mathfrak{so}(2N)_R$ current associated to the generator $\bar{A}$.[1] Note that these non-abelian currents are distinguished from their Cartan counterparts J_i and $\bar{J}_i$ only by their index.

Our goal is to find all solutions $(A, \bar{A})$ to the above equation. However, doing this directly by computing the action of the currents on the boundary state turns out to be a bit of a pain. Instead, we will take a slightly different approach. We will first show, using algebraic arguments and anomalies, that the problem of checking (4.3) for general $(A, \bar{A})$ can be reduced to checking it for a certain finite set of special generators. Then we carry out the check for these special generators directly.

There is one other simplification that is important to make before we start. Instead of looking for solutions with $A, \bar{A}$ in $\mathfrak{so}(2N)$, we will take them to lie in $\mathfrak{so}(2N)_{\mathbb{C}}$. Solving the complexified problem turns out to be technically easier, and besides, we can always recover the answer to the original question by imposing a reality condition on $A, \bar{A}$.

[1]The currents are defined by $J_A(z) \coloneqq \frac{1}{2}A_{ij}\chi_i(z)\chi_j(z)$, where the generator A is regarded as a real, $2N \times 2N$ antisymmetric matrix A_{ij}.

Anomalies

Our first job is to show that the set of solutions $(A, \bar{A})$ to (4.3) forms an anomaly-free subalgebra. We begin by recasting (4.3) as an equivalent algebraic condition, written in terms of modes,

$$\left(J_{A,n} + \bar{J}_{\bar{A},-n}\right)|\theta;\mathcal{R}\rangle = 0 \qquad \text{for all } n \in \mathbb{Z} \tag{4.4}$$

Suppose that we have two solutions $(A, \bar{A})$ and $(B, \bar{B})$. Clearly, we have

$$[J_{A,n} + \bar{J}_{\bar{A},-n}, J_{B,m} + \bar{J}_{\bar{B},-m}]\,|\theta;\mathcal{R}\rangle = 0 \qquad \text{for all } n, m \in \mathbb{Z}$$

Using the fact that the modes obey the $\widehat{\mathfrak{so}}(2N)_1$ algebra

$$[J_{A,n}, J_{B,m}] = J_{[A,B],n+m} + n\delta_{n+m,0}\, K(A,B)$$

with $K(A,B) = \frac{1}{2}\mathrm{Tr}(AB)$ the correctly-normalised Killing form, we can simplify the previous equation to

$$\left(\underline{J_{[A,B],n+m} + \bar{J}_{[\bar{A},\bar{B}],-(n+m)}} + n\delta_{n+m,0}\underline{(K(A,B) - K(\bar{A},\bar{B}))}\right)|\theta;\mathcal{R}\rangle = 0$$

Since this must hold for all n, m, both underlined terms must vanish separately. The first shows that the solutions close into an algebra, while the second forces the vanishing of the mixed 't Hooft anomaly between any two solutions

$$K(A,B) - K(\bar{A},\bar{B}) = 0$$

which is what we wanted to show.

Next, we show that the set of solutions to (4.3), or the *unbroken subalgebra*, has a second important property: whenever $(A, \bar{A})$ is a solution, so is its complex conjugate $(A^*, \bar{A}^*)$. To see this, we cook up a judicious choice of antilinear operation $\mathcal{T}$ defined by the stipulations

$$\begin{aligned} \mathcal{T}|\lambda,\bar{\lambda}\rangle &= (-1)^{\lambda^2}|-\lambda,-\bar{\lambda}\rangle \\ \mathcal{T}J_{i,n}\mathcal{T}^{-1} &= -J_{i,n} \\ \mathcal{T}\bar{J}_{i,n}\mathcal{T}^{-1} &= -\bar{J}_{i,n} \end{aligned}$$

Under this operation, the boundary states are invariant,

$$\mathcal{T}|\theta;\mathcal{R}\rangle = |\theta;\mathcal{R}\rangle$$

while the non-abelian currents transform as

$$\mathcal{T} J_A(z)\mathcal{T}^{-1} = J_{A^*}(z^*) \quad \text{and} \quad \mathcal{T}\bar{J}_A(\bar{z})\mathcal{T}^{-1} = \bar{J}_{A^*}(\bar{z}^*)$$

Acting on (4.3) with $\mathcal{T}$, this then establishes the claim[2].

The previous two results place strong constraints on the structure of the unbroken subalgebra, and ultimately force it to take the form

$$\{\, (A,\ \bar{A} = \phi(A)) \ : \ A \in \mathfrak{g} \,\} \tag{4.5}$$

Here $\mathfrak{g}$ is some subalgebra of $\mathfrak{so}(2N)_{\mathbb{C}}$, describing the allowed holomorphic parts A of the symmetries, while ϕ is a map $\mathfrak{g} \to \mathfrak{so}(2N)_{\mathbb{C}}$ which sends them to their corresponding antiholomorphic parts $\bar{A}$. The map ϕ must be both a homomorphism of Lie algebras, and an isometry with respect to the Killing form K, in order that (4.5) describe an anomaly-free subalgebra.

To demonstrate the assertion of the last paragraph, note that the unbroken subalgebra cannot contain any purely chiral elements $(A, 0)$. For if it did, then it would also contain $(A^*, 0)$, and then the anomaly condition $K(A, A^*) = 0$ would force $A = 0$, since the Killing form is negative definite. Therefore, for every generator $A \in \mathfrak{so}(2N)_{\mathbb{C}}$, there can be at most a single $\bar{A} \in \mathfrak{so}(2N)_{\mathbb{C}}$ such that $(A, \bar{A})$ is in the unbroken subalgebra. This forces the algebra to take the form (4.5), as claimed.

The Existing Abelian Symmetry

Until now, we haven't actually used any properties of the boundary state $|\theta;\mathcal{R}\rangle$ itself. Here we take into account the property that it preserves a $U(1)^N$ symmetry. This is encoded by the identity

$$(\mathcal{R}_{ij} J_{j,n} + \bar{J}_{i,-n})\, |\theta;\mathcal{R}\rangle = 0$$

[2]To give more details, the first transformation uses (2.24) and the identities $e^{-i\gamma_{\mathcal{R}}(-\lambda)} = e^{-i\gamma_{\mathcal{R}}(\lambda)} = (-1)^{\lambda^2} e^{i\gamma_{\mathcal{R}}(\lambda)}$ which follow from (2.47). The last two transformations follow from the bosonised form of the currents, both diagonal (4.6) and off-diagonal (4.7).

as can easily be verified starting from the definition (2.24). Our goal is to determine the resulting constraints that this imposes on the algebra (4.5). To do this, we use the relation

$$J_i(z) = J_{H_i}(z) \tag{4.6}$$

between the $\mathfrak{u}(1)^N$ and $\mathfrak{so}(2N)$ currents, where the generators H_i are the following choice of basis for the Cartan subalgebra,

$$H_1 = \begin{pmatrix} 0 & i & \\ -i & 0 & \\ & & \ddots \end{pmatrix} \qquad \dots \qquad H_N = \begin{pmatrix} \ddots & & \\ & 0 & i \\ & -i & 0 \end{pmatrix}$$

The existence of the $U(1)^N$ symmetry can then be recast as

$$(\mathcal{R}_{ij} J_{H_j,n} + \bar{J}_{H_i,-n}) \, |\theta; \mathcal{R}\rangle = 0$$

This equation now takes the same form as (4.4), and states that the symmetry $(A, \bar{A}) = (\mathcal{R}_{ij} H_j, H_i)$ is preserved for each i, or equivalently, that $(H_i, \mathcal{R}_{ji} H_j)$ is. We can now read off the constraints on the algebra (4.5) that we were looking for:

1. $\mathfrak{g}$ contains the Cartan subalgebra: $H_i \in \mathfrak{g}$ for all i.

2. ϕ is uniquely determined on the Cartan subalgebra: $\phi(H_i) = \mathcal{R}_{ji} H_j$.

Algebraic Constraints

The above two constraints allow us to reduce the task of solving (4.3) for $(A, \bar{A})$ to checking a finite list of candidates. The candidates are

$$(A, \bar{A}) = (E_\alpha, e^{i\chi} E_{R\alpha})$$

where α ranges over the finite set $\Delta[\mathcal{R}]$ introduced in (4.2), and χ is a phase to be determined as part of the test. The enhanced symmetry is then given by the linear span of all the generators of the above kind that pass the test[3].

[3]This statement makes free use of the identification between the roots of $\mathfrak{so}(2N)$ and the vectors in $\mathbb{Z}^N$ of length-squared 2. Indeed, given a root α, its components $(\alpha_1, \dots, \alpha_N)$ with respect to the Cartan generators H_i are exactly a vector of this form. The subset $\Delta[\mathcal{R}] \subset \mathbb{Z}^N$ has the property that, for any member α, both α and $\mathcal{R}\alpha$ are roots; this ensures the expression for $(A, \bar{A})$ makes sense.

To prove this claim, let A be any generator in $\mathfrak{g}$. We can decompose A in the Chevalley basis as

$$A = \sum_{i=1}^{N} x_i H_i + \sum_{\alpha \in \Delta} x_\alpha E_\alpha$$

where Δ is the set of roots of $\mathfrak{so}(2N)$, and the x_i and x_α are sets of complex coefficients. For any α such that $x_\alpha \neq 0$, we can repeatedly take commutators with H_i and form linear combinations to project A onto the single generator E_α. Because these operations do not take us outside $\mathfrak{g}$, the result E_α must lie in $\mathfrak{g}$. We learn that $\mathfrak{g}$ is fully determined by asking whether it contains E_α for each $\alpha \in \Delta$, and is given by the span of all such generators.

Next we restrict the possible α that can occur to the set $\Delta[\mathcal{R}]$. Suppose that $E_\alpha \in \mathfrak{g}$. Using the homomorphism property $[\phi(H_i), \phi(E_\alpha)] = \phi([H_i, E_\alpha])$ together with the action on the Cartan generators $\phi(H_i) = \mathcal{R}_{ji} H_j$, we deduce that

$$[H_i, \phi(E_\alpha)] = \mathcal{R}_{ij} \alpha_j \, \phi(E_\alpha)$$

This states that $\phi(E_\alpha)$ is proportional to $E_{\mathcal{R}\alpha}$. For this to be possible at all, both α and $\mathcal{R}\alpha$ must be roots. To see what this says about α, first note that since both α and $\mathcal{R}\alpha$ are integer vectors, α must lie in $\Lambda[\mathcal{R}]$. The additional requirement they be roots is that $|\alpha|^2 = 2$, which further restricts α to lie in $\Delta[\mathcal{R}]$, as we wanted to show.

Finally, given $\alpha \in \Delta[\mathcal{R}]$, the proportionality constant in $\phi(E_\alpha) \propto E_{\mathcal{R}\alpha}$ must actually be a phase. This follows because ϕ is an isometry with respect to the Killing form. Denoting this phase by $e^{i\chi}$, we recover the result claimed at the start.

Explicitly Checking the Generators

The only remaining task is to test the generators identified at the start of the last section to see whether they satisfy (4.3). We will find that in fact they all do. That is, there are no further obstructions to the existence of enhanced symmetries beyond those we have already identified.

This section is necessarily slightly more technical than the rest. Before proceeding, we first need to describe the root system Δ of $\mathfrak{so}(2N)$ a little more explicitly. Each root of $\mathfrak{so}(2N)$ is labelled by a pair of integers $1 \leq i < j \leq N$ and a pair of signs $s, t = \pm 1$, and is represented by the N-component vector

$$\alpha_{i,j,s,t} = (0,\ldots,0,\overset{\overset{i}{\downarrow}}{s},0,\ldots,0,\overset{\overset{j}{\downarrow}}{t},0,\ldots,0)$$

The corresponding generator is the $2N \times 2N$ matrix

$$E_{i,j,s,t} = \frac{1}{2}\begin{array}{c} \begin{array}{cccccccc} \ldots & 2i-1 & 2i & \ldots & 2j-1 & 2j & \ldots & \end{array} \\ \left(\begin{array}{ccccccc} & & & & & & \\ & & & & 1 & it & \\ & & & & is & -st & \\ & & & & & & \\ & -1 & -is & & & & \\ & -it & st & & & & \\ & & & & & & \end{array}\right)\begin{array}{l} \vdots \\ 2i-1 \\ 2i \\ \vdots \\ 2j-1 \\ 2j \\ \vdots \end{array} \end{array}$$

and the corresponding bosonised currents are

$$J_{i,j,s,t}(z) = (F_i)^s (F_j)^t \, t_i t_j \, z^{-\alpha_k \lambda_k} \exp\left(-\alpha_k \sum_{n=1}^{\infty} \frac{z^n}{n} J_{k,-n}\right) \exp\left(\alpha_k \sum_{n=1}^{\infty} \frac{z^{-n}}{n} J_{k,n}\right) \tag{4.7}$$

with a similar expression for the antiholomorphic currents $\bar{J}_{i,j,s,t}(\bar{z})$.

We now proceed to carry out the check. Let α be any root in $\Delta[\mathcal{R}]$. Since α and $\mathcal{R}\alpha$ are both roots, they can be written in the form

$$\alpha = \alpha_{i,j,s,t} \qquad \mathcal{R}\alpha = \alpha_{i',j',s',t'}$$

for some choices of the various indices. The condition (4.3) for our candidate generator $(E_\alpha, e^{i\chi} E_{\mathcal{R}\alpha})$ to be preserved then takes the form

$$\Big(J_{i,j,s,t}(z) + e^{i\chi} z^{-2} \bar{J}_{i',j',s',t'}(\bar{z})\Big)|\theta;\mathcal{R}\rangle \overset{?}{=} 0$$

where $|z| = 1$. Next, we plug in the definition of the boundary state (2.24) and bosonised currents (4.7). The condition becomes

$$\sum_{\lambda\in\Lambda[\mathcal{R}]} (t_i t_j|_{\lambda,-\mathcal{R}\lambda}) e^{i\gamma_{\mathcal{R}}(\lambda)} e^{i\theta\cdot\lambda} z^{-\alpha\cdot\lambda} \exp\left(-\alpha_k \sum_{n=1}^{\infty} \frac{z^n}{n} J_{k,-n}\right)$$
$$\exp\left(\alpha_k \sum_{n=1}^{\infty} \frac{z^{-n}}{n} J_{k,n}\right) \exp\left(-\sum_{n=1}^{\infty} \frac{1}{n} \mathcal{R}_{ij}\, \bar{J}_{i,-n} J_{j,-n}\right)|\lambda-\alpha, -\mathcal{R}\lambda\rangle$$

$$+ \; e^{i\chi} \sum_{\lambda \in \Lambda[\mathcal{R}]} (\bar{t}_{i'}\bar{t}_{j'}|_{\lambda,-\mathcal{R}\lambda}) e^{i\gamma_{\mathcal{R}}(\lambda)} e^{i\theta\cdot\lambda} z^{-\alpha\cdot\lambda-2} \exp\Big(-(\mathcal{R}\alpha)_k \sum_{n=1}^{\infty} \frac{z^{-n}}{n} \bar{J}_{k,-n}\Big)$$
$$\exp\Big((\mathcal{R}\alpha)_k \sum_{n=1}^{\infty} \frac{z^{n}}{n} \bar{J}_{k,n}\Big) \exp\Big(-\sum_{n=1}^{\infty} \frac{1}{n} \mathcal{R}_{ij}\, \bar{J}_{i,-n} J_{j,-n}\Big) |\lambda, -\mathcal{R}(\lambda+\alpha)\rangle \;\overset{?}{=}\; 0$$

A short calculation shows that the oscillator parts of the above two sums are equal. This means that the exponentials of oscillators can be dropped. We can also shift the argument of the second sum by $\lambda \to \lambda - \alpha$ to make the sums look more alike. This is allowed because $\alpha \in \Lambda[\mathcal{R}]$. After the dust has settled, we are left with

$$(t_i t_j|_{\lambda,-\mathcal{R}\lambda})\, e^{i\gamma_{\mathcal{R}}(\lambda)} + e^{i(\chi-\theta\cdot\alpha)}\, (\bar{t}_{i'}\bar{t}_{j'}|_{\lambda-\alpha,-\mathcal{R}(\lambda-\alpha)})\, e^{i\gamma_{\mathcal{R}}(\lambda-\alpha)} \;\overset{?}{=}\; 0$$

This equation is required to hold for each $\lambda \in \Lambda[\mathcal{R}]$.

Verifying that it actually holds is the part unique to fermionic CFTs. It is possible that the presence of the cocycles t_i might render the list of symmetries smaller than what is naïvely expected. We will show that this does not happen. To do this, we appeal to a bunch of identities satisfied by the various objects that arise:

- From definition (2.44), and the fact $\mathcal{R}\alpha = \alpha_{i',j',s',t'}$,

$$(\bar{t}_{i'}\bar{t}_{j'}|_{\lambda-\alpha,-\mathcal{R}(\lambda-\alpha)}) = -(\bar{t}_{i'}\bar{t}_{j'}|_{\lambda,-\mathcal{R}\lambda})$$

- From definition (2.47), bilinearity of $\hat{f}$, and the fact $\hat{f}_{\mathcal{R}} = f_{\mathcal{R}}$ mod 2,

$$e^{i\gamma_{\mathcal{R}}(\lambda-\alpha)} = e^{i\gamma_{\mathcal{R}}(\lambda)} e^{i\gamma_{\mathcal{R}}(\alpha)} (-1)^{f_{\mathcal{R}}(\lambda,\alpha)}$$

- From definitions (2.44) and (2.45),

$$(-1)^{f_{\mathcal{R}}(\lambda,\alpha)} = (t_i t_j|_{\lambda,-\mathcal{R}\lambda})(\bar{t}_{i'}\bar{t}_{j'}|_{\lambda,-\mathcal{R}\lambda})$$

Combining the above three identities shows that the condition is satisfied, with the phase χ given by $e^{i\chi} = e^{i\theta\cdot\alpha} e^{-i\gamma_{\mathcal{R}}(\alpha)}$. The factor $e^{i\gamma_{\mathcal{R}}(\alpha)}$ can be set to one using the multiplicative ambiguity by $(-1)^{s\cdot\lambda}$ inherent in its definition. To see this, first note that $(e^{i\gamma_{\mathcal{R}}(\alpha)})^2 = (-1)^{|\alpha|^2=2} = 1$, so $e^{i\gamma_{\mathcal{R}}(\alpha)}$ is a sign. It can be argued that these signs fit together in a consistent way, in the sense that $e^{i\gamma_{\mathcal{R}}(\alpha)} = (-1)^{s\cdot\alpha}$ for some s. This is precisely of the form that can be absorbed by a redefinition. So without loss of generality, $e^{i\chi} = e^{i\theta\cdot\alpha}$. This shows how changing the theta-angles rotates the preserved subalgebra within $\mathfrak{so}(2N)_L \times \mathfrak{so}(2N)_R$.

Summary

The upshot of this section is that the unbroken subalgebra for $|\theta; \mathcal{R}\rangle$ is spanned by the abelian generators, $(H_i, \mathcal{R}_{ij}H_j)$, with $i = 1, \ldots, N$, together with the off-diagonal enhanced generators

$$(E_\alpha, e^{i\theta\cdot\alpha}E_{\mathcal{R}\alpha}) \quad : \quad \alpha \in \Delta[\mathcal{R}] \tag{4.8}$$

with $\Delta[\mathcal{R}]$ consisting of the points $\alpha \in \Lambda[\mathcal{R}]$ with length-squared $\alpha^2 = 2$.

4.1.2 Some Examples

In this section we give a few examples of the criterion. First, we show how it can be used to classify all states with maximal symmetry (i.e. with a preserved subalgebra isomorphic to $\mathfrak{so}(2N)$). Second, we explore a family of states which previously arose in the context of fermions scattering off monopoles and, for that reason, are called *dyon states*. These will go on to play a starring role in Section 4.2.

To begin, suppose that $|\theta; \mathcal{R}\rangle$ has unbroken subalgebra isomorphic to $\mathfrak{so}(2N)$. By the criterion, this happens whenever

$$\mathcal{R}\Delta = \Delta$$

where Δ is the root system of $\mathfrak{so}(2N)$. We will classify such matrices $\mathcal{R}$ up to the freedom to shift $\mathcal{R} \to \mathcal{W}_R\mathcal{R}\mathcal{W}_L$ where $\mathcal{W}_L, \mathcal{W}_R$ are $N \times N$ signed permutation matrices. (Transforming $\mathcal{R}$ in this way corresponds to acting on the boundary state with an $O(2N)_L \times O(2N)_R$ Weyl group transformation.) Using this freedom, we may assume that $\mathcal{R}$ maps the simple roots to a permutation of themselves. We can then classify $\mathcal{R}$ by considering its induced Dynkin diagram automorphism:

- For $N \neq 4$, the Dynkin diagram has a $\mathbb{Z}_2$ automorphism group, whose generator exchanges the two spinor representations. But this automorphism can be undone by acting with a $O(2N)$ Weyl transformation $\mathcal{W}_R$ that is not a $SO(2N)$ Weyl transformation. Therefore, the only possibility in this case is the trivial state

$$\mathcal{R} = \mathbb{1} \quad \text{with} \quad g_{\mathcal{R}} = 1$$

- For $N = 4$, the Dynkin diagram of $\mathfrak{so}(8)$ exhibits a famous triality, and the automorphism group is $\mathbb{S}_3$. In addition to the $\mathbb{Z}_2$ above there are 4 nontrivial triality automorphisms. Under the identifications $\mathcal{R} \sim \mathcal{W}_R \mathcal{R} \mathcal{W}_L$ these collapse to just one,

$$\mathcal{R} = \frac{1}{2}\begin{pmatrix} +1 & -1 & -1 & -1 \\ -1 & +1 & -1 & -1 \\ -1 & -1 & +1 & -1 \\ -1 & -1 & -1 & +1 \end{pmatrix} \quad \text{with} \quad g_{\mathcal{R}} = \sqrt{2} \tag{4.9}$$

This state was first constructed by Maldacena and Ludwig [20] in the context of fermion-monopole scattering.

The Maldacena-Ludwig state lies in a sequence of so called *dyon states* that exist for general N. They have preserved charges and boundary central charge given by

$$\text{dyon}_N: \qquad \mathcal{R}_{ij} = \delta_{ij} - \frac{2}{N} \quad \text{with} \quad g_{\mathcal{R}} = \sqrt{\tfrac{N}{\gcd(N,2)}} \tag{4.10}$$

For most values of N, the enhanced symmetry of the dyon state is $SU(N)_V \times U(1)_A$ and no larger. This symmetry is chiral: the left-moving fermions transform in $\mathbf{N}_{+1}$, while the right-movers transform in $\mathbf{N}_{-1}$.

There are three exceptions to the statement above. When $N = 1$ and $N = 2$, the dyon states are trivial and preserve the subgroup $SO(2N)_V$. When $N = 4$, the dyon state coincides with the Maldacena-Ludwig state, which preserves a Spin(8) subgroup nontrivially embedded within $SO(8) \times SO(8)$.

4.2 Discrete Symmetries

In this section we turn to our second goal: to determine which charges Q and $\bar{Q}$ admit a $\mathbb{Z}_2 \times \mathbb{Z}_2$ preserving boundary condition.

It is easy to derive a criterion for when this symmetry is preserved. The $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry acts via left and right fermion parity which, in Euclidean space, we continue to denote as $(-1)^{F_L}$ and $(-1)^{F_R}$. Under these symmetries, the ground states transform with charges

$$(-1)^{F_L}|\lambda, \bar{\lambda}\rangle = (-1)^{\lambda_1 + \cdots + \lambda_N}|\lambda, \bar{\lambda}\rangle = (-1)^{\lambda^2}|\lambda, \bar{\lambda}\rangle$$
$$(-1)^{F_R}|\lambda, \bar{\lambda}\rangle = (-1)^{\bar{\lambda}_1 + \cdots + \bar{\lambda}_N}|\lambda, \bar{\lambda}\rangle = (-1)^{\bar{\lambda}^2}|\lambda, \bar{\lambda}\rangle$$

while the abelian currents are uncharged. The condition for a boundary state $|\theta; \mathcal{R}\rangle$ to be neutral under both symmetries is simply the requirement that the allowed charge sectors $|\lambda, -\mathcal{R}\lambda\rangle$, with $\lambda \in \Lambda[\mathcal{R}]$, are all neutral. This in turn will hold if the lattice $\Lambda[\mathcal{R}]$ contains only vectors of even length-squared. We can equivalently write this as

$$\Lambda[\mathcal{R}] \subseteq D_N$$

with D_N the root lattice of $\mathfrak{so}(2N)$. Our goal is to investigate the set of all $\mathcal{R}$ for which this condition holds.

4.2.1 Discrete Symmetries are Continuous

Our starting point is the following lemma, which applies to the global structure of the symmetry group of the boundary state. We restrict attention to the $U(1)^N \times U(1)^N$ maximal torus of $SO(2N)_L \times SO(2N)_R$, which we parametrise as

$$U(1)^N \times U(1)^N = \Big\{ (e^{2\pi i x}, e^{2\pi i \bar{x}}) \,:\, x, \bar{x} \in \mathbb{R}^N/\mathbb{Z}^N \Big\}$$

Of this, the boundary state $|\theta; \mathcal{R}\rangle$ was designed to preserve the subgroup

$$U(1)^N_{\mathcal{R}} := \Big\{ (e^{2\pi i x}, e^{2\pi i \mathcal{R} x}) \,:\, x \in \mathbb{R}^N/\Lambda[\mathcal{R}] \Big\}$$

In principle, it could be the case that the boundary state also preserves some extra discrete symmetries that lie in $U(1)^N \times U(1)^N$ but not in $U(1)^N_{\mathcal{R}}$. We will show that this does not happen.

Claim: Any element of $U(1)^N \times U(1)^N$ that leaves $|\theta; \mathcal{R}\rangle$ invariant lies in $U(1)^N_{\mathcal{R}}$.

Proof: We begin with the obvious exact sequence of abelian groups

$$0 \longrightarrow \mathbb{R}^N \xrightarrow[\left(\begin{smallmatrix} \mathbb{1} \\ \mathcal{R} \end{smallmatrix}\right)]{} \mathbb{R}^N \oplus \mathbb{R}^N \xrightarrow[\left(\begin{smallmatrix} \mathbb{1} & -\mathcal{R}^{-1} \end{smallmatrix}\right)]{} \mathbb{R}^N \longrightarrow 0$$

This contains the sub-exact-sequence

$$0 \longrightarrow \mathbb{Z}^N \cap \mathcal{R}^{-1}\mathbb{Z}^N \longrightarrow \mathbb{Z}^N \oplus \mathbb{Z}^N \longrightarrow \mathbb{Z}^N + \mathcal{R}^{-1}\mathbb{Z}^N \longrightarrow 0$$

Taking the quotient of these yields the further exact sequence

$$0 \longrightarrow \frac{\mathbb{R}^N}{\mathbb{Z}^N \cap \mathcal{R}^{-1}\mathbb{Z}^N} \longrightarrow \frac{\mathbb{R}^N}{\mathbb{Z}^N} \oplus \frac{\mathbb{R}^N}{\mathbb{Z}^N} \longrightarrow \frac{\mathbb{R}^N}{\mathbb{Z}^N + \mathcal{R}^{-1}\mathbb{Z}^N} \longrightarrow 0$$

In the denominator of the left term, we see the definition $\mathbb{Z}^N \cap \mathcal{R}^{-1}\mathbb{Z}^N = \Lambda[\mathcal{R}]$. Furthermore, taking the dual of this definition yields $\mathbb{Z}^N + \mathcal{R}^T\mathbb{Z}^N = \Lambda[\mathcal{R}]^\star$. Because $\mathcal{R}$ is orthogonal, we can replace in this equation $\mathcal{R}^T = \mathcal{R}^{-1}$. The above exact sequence then takes the final form

$$0 \longrightarrow \frac{\mathbb{R}^N}{\Lambda[\mathcal{R}]} \longrightarrow \frac{\mathbb{R}^N}{\mathbb{Z}^N} \oplus \frac{\mathbb{R}^N}{\mathbb{Z}^N} \longrightarrow \frac{\mathbb{R}^N}{\Lambda[\mathcal{R}]^\star} \longrightarrow 0 \tag{4.11}$$

This exact sequence also appeared in a different guise in [87]. The various groups appearing in this sequence all have simple interpretations:

- The middle group is the group of all $U(1)^N \times U(1)^N$ transformations.
- The left group is the preserved $U(1)^N_{\mathcal{R}}$ subgroup, with the map into $U(1)^N \times U(1)^N$ simply corresponding to the inclusion map.
- The right group is the obstruction of a $U(1)^N \times U(1)^N$ transformation to be a symmetry of $|\theta;\mathcal{R}\rangle$. To see this, we recall the fact that such a transformation acts on the ground states as

  $$(e^{2\pi i x}, e^{2\pi i \bar{x}}) \;:\; |\lambda,\bar{\lambda}\rangle \;\to\; e^{2\pi i(x\cdot\lambda + \bar{x}\cdot\bar{\lambda})}|\lambda,\bar{\lambda}\rangle$$

  In order for $|\theta;\mathcal{R}\rangle$ to be invariant, all the ground states that occur in it must be neutral. These ground states are $|\lambda, -\mathcal{R}\lambda\rangle$ for $\lambda \in \Lambda[\mathcal{R}]$. Therefore, we require

  $$(x - \mathcal{R}^T\bar{x})\cdot\lambda \in \mathbb{Z} \qquad \text{for all } \lambda \in \Lambda[\mathcal{R}]$$

  This failure of the quantity $x - \mathcal{R}^T\bar{x}$ to lie in $\Lambda[\mathcal{R}]^\star$ therefore measures the obstruction for the transformation to be a symmetry, as claimed.

Exactness of (4.11) at the middle group then gives the desired conclusion, that a $U(1)^N \times U(1)^N$ transformation is a symmetry if and only if it lives in $U(1)^N_{\mathcal{R}}$.

4.2.2 The $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry

From now on, we specialise to the $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry. As we already have seen, the condition for this to be preserved is

$$\lambda^2 = 0 \bmod 2 \qquad \text{for all } \lambda \in \Lambda[\mathcal{R}] \tag{4.12}$$

Applying the result of the previous section, we see that this is equivalent to either of the following two stronger statements:

$$\exists\ \lambda \in \Lambda[\mathcal{R}] \qquad \text{s.t.} \qquad \begin{aligned} \lambda_i &= 1 \bmod 2 \\ (\mathcal{R}\lambda)_i &= 0 \bmod 2 \end{aligned}$$

and

$$\exists\ \lambda \in \Lambda[\mathcal{R}] \qquad \text{s.t.} \qquad \begin{aligned} \lambda_i &= 0 \bmod 2 \\ (\mathcal{R}\lambda)_i &= 1 \bmod 2 \end{aligned}$$

These are respectively the statements that $(-1)^{F_L}$ and $(-1)^{F_R}$ lie within the $U(1)^N_{\mathcal{R}}$ group. For example, if λ is a solution to the first condition, then $(-1)^{F_L}$ is given by the $U(1)^N_{\mathcal{R}}$ symmetry transformation with parameter $x = \lambda/2$.

Recovering the $\mathbb{Z}_8$ Index

Using the first of the conditions above, we can easily show that a boundary state preserving $\mathbb{Z}_2 \times \mathbb{Z}_2$ can only exist when the number of Dirac fermions N is a multiple of 4. Since each Dirac fermion comprises two Majorana fermions, we recover the result stated in the introduction that the number of Majorana fermions must be a multiple of 8.

For the proof, we need only to examine the length-squared of λ. Since $\mathcal{R}$ is orthogonal, we have

$$\lambda^2 = (\mathcal{R}\lambda)^2$$

The vector λ on the left is an N-component vector with odd components, so its length-squared is N mod 8. Meanwhile, on the right hand side, $\mathcal{R}\lambda$ has even components, hence has length-squared 0 mod 4. Equating the two, we learn that

$$N = 0 \bmod 4$$

as claimed.

4.2.3 The Minimal Case: 4 Dirac Fermions

We now turn to some examples. We start by constructing all $\mathbb{Z}_2 \times \mathbb{Z}_2$ preserving states with $N = 4$ Dirac fermions. We then discuss how to construct such states for higher N, their allowed values of g, and characterise the states that attain the lowest values of g.

For the case of $N = 4$ Dirac fermions, a parametrisation of the chiral-parity preserving boundary states can be given using the results of [83, 88].

First, we briefly recount how a 4×4 rational orthogonal matrix can be parametrised in terms of a pair of integer quaternions, based on the isomorphism $SO(4) = (SU(2) \times SU(2))/\mathbb{Z}_2$, following [83]. To each pair of integer quaternions $(\mathbf{p}, \mathbf{q})$, we associate an $SO(4)$ matrix $\mathcal{R}$, implicitly defined by

$$\mathcal{R}\mathbf{x} = \frac{1}{|\mathbf{pq}|}\,\mathbf{q}\,\mathbf{x}\,\bar{\mathbf{p}}$$

where on the left $\mathbf{x}$ is regarded as a 4-component vector, acted on by the matrix $\mathcal{R}$, while on the right it is regarded as a quaternion. In order that $\mathcal{R}$ be rational, we must impose the constraint $|\mathbf{pq}| \in \mathbb{Z}$. Without loss of generality, we may also assume that $\mathbf{p}$ and $\mathbf{q}$ are *primitive*, meaning their four components are coprime. Then the set of all such pairs $(\mathbf{p}, \mathbf{q})$ gives a parametrisation of all 4×4 rational, special-orthogonal matrices $\mathcal{R}$, uniquely up to an overall sign redundancy $(\mathbf{p}, \mathbf{q}) \to (-\mathbf{p}, -\mathbf{q})$. (The remaining matrices with $\det(\mathcal{R}) = -1$ can be parametrised in an identical way, simply by multiplying the above expression by a reflection in the first coordinate.)

Next, we turn to the conditions on $\mathbf{p}$ and $\mathbf{q}$ for the $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry to be preserved. This question is addressed in [88], but to connect with their work we need to recast the condition for $\mathbb{Z}_2 \times \mathbb{Z}_2$ invariance in a slightly different form. It is straightforward to show that the following ratio of indexes can only take on the two possible values

$$\frac{[\mathbb{Z}^N : \mathbb{Z}^N \cap \mathcal{R}^{-1}\mathbb{Z}^N]}{[D_N : D_N \cap \mathcal{R}^{-1} D_N]} = 1 \text{ or } 2$$

and that the value 2 is attained precisely when the $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry is preserved. Proposition 11 of [88] then tells us that this happens if and only if

- Exactly one of $|\mathbf{p}|^2$, $|\mathbf{q}|^2$ is a multiple of 4.
- Both $|\mathbf{p}|^2$ and $|\mathbf{q}|^2$ are 0 mod 4, and $\mathbf{p} \cdot \mathbf{q} \neq 0$ mod 4.
- Both $|\mathbf{p}|^2$ and $|\mathbf{q}|^2$ are 2 mod 4, and $\mathbf{p} \cdot \mathbf{q}$ is odd.

In Section 4.1.2, we first met the 4-fermion dyon state, whose importance was first emphasised by Maldacena and Ludwig [20] as an example of a particularly symmetric boundary state. It is also the simplest example of a $\mathbb{Z}_2 \times \mathbb{Z}_2$-preserving boundary state, a connection that was first made in [36]. To see how it sits within the above parametrisation, we choose $p = (1, 1, 1, 1)$ and $q = (1, -1, -1, -1)$, and apply a reflection in the first coordinate to account for the fact that the dyon has determinant -1. This then gives us back the matrix $\mathcal{R}$ for the dyon that we met earlier,

$$\text{dyon}_4 \;=\; \frac{1}{2}\begin{pmatrix} +1 & -1 & -1 & -1 \\ -1 & +1 & -1 & -1 \\ -1 & -1 & +1 & -1 \\ -1 & -1 & -1 & +1 \end{pmatrix}$$

The dyon is actually the simplest of an infinite tower of $\mathbb{Z}_2 \times \mathbb{Z}_2$-preserving boundary states, in the sense that it has the lowest Affleck-Ludwig central charge $g_{\mathcal{R}}$. The full spectrum of allowed $g_{\mathcal{R}}$ values of all such states, together with the number of matrices supporting each one, reads:

$(g_{\mathcal{R}})^2$	2	6	10	14	18	22	26	30	34	38	...
$\#\mathcal{R}\,/\,2^5 4!$	1	16	36	64	168	144	196	576	324	400	...

The first entry corresponds to the venerable dyon state. Along the top, we see that the possible $(g_{\mathcal{R}})^2$ values are $2 + 4k$ for $k \geq 0$, with their multiplicities in the bottom row given by sequence A031360 in OEIS [89]. In particular this sequence never vanishes, showing that all values $(g_{\mathcal{R}})^2 = 2 + 4k$ are actually attained.

4.2.4 The Higher Cases: $4k$ Dirac Fermions

We have seen that for 4 Dirac fermions, demanding $\mathbb{Z}_2 \times \mathbb{Z}_2$ invariance restricts the possible values of $g_{\mathcal{R}}$, and the lowest value of $g_{\mathcal{R}}$ is attained by the dyon_4 state. Here we explore how this story generalises for higher numbers of fermions.

For a larger number $N = 4k$ of Dirac fermions, our options for boundary states are considerably increased. For example, it is always possible simply to group the fermions into bunches of 4, and write down the dyon state for each one. This corresponds to choosing the matrix

$$\mathcal{R} = \underbrace{\text{dyon}_4 \oplus \cdots \oplus \text{dyon}_4}_{k} \quad \text{with} \quad (g_{\mathcal{R}})^2 = 2^k$$

On the other hand, we also have the option of writing down a dyon state for all $4k$ fermions. This is also a $\mathbb{Z}_2 \times \mathbb{Z}_2$ preserving boundary state[4], and has central charge

$$\mathcal{R} = \text{dyon}_{4k} \quad \text{with} \quad (g_{\mathcal{R}})^2 = 2k$$

For all k, the central charge of the dyon is always at least as low as that of the first state. We conjecture that more generally, the dyon has the lowest central charge among *all* $\mathbb{Z}_2 \times \mathbb{Z}_2$ preserving boundary states. If true, then the dyon would be the maximally stable state under boundary RG flow by perturbations that preserve chiral fermion parity, and would therefore be the generic boundary state realised in the infra-red if all one demands of the boundary is $\mathbb{Z}_2 \times \mathbb{Z}_2$ invariance.

To give evidence for this conjecture, we will devise a suitable numerical experiment. The most obvious such experiment is simply to generate as many $\mathcal{R}$ matrices as possible, reject those which violate condition (4.12), and calculate their central charges $g_{\mathcal{R}}$. We would then hope to find a spectrum of allowed values of $(g_{\mathcal{R}})^2$ that extends all the way down to $2k$, corresponding to the dyon, but no further. In practice, however, this approach runs into a problem. As k increases, the probability of generating a matrix $\mathcal{R}$ that obeys conditions (4.12) becomes vanishingly small. This means we need an alternative way to generate matrices that obey the condition directly.

Such a method can be found in the "null-vector construction" of [70] which, as more recently emphasised in [87], can be exploited for generating sets of charges $Q_{\alpha i}$ and $\bar{Q}_{\alpha i}$ that are free of anomalies. Using our result in Section 4.2.2, we will show that this construction can be modified so as to produce only charges for which the corresponding $\mathcal{R}$ matrix obeys (4.12).

In the original null-vector construction of [87], one constructs the charges $Q_{\alpha i}$ and $\bar{Q}_{\alpha i}$ iteratively. First one chooses the charges for $\alpha = 1$ to be random values obeying

$$\sum_{i=1}^{N} (Q_{\alpha i})^2 = \sum_{i=1}^{N} (\bar{Q}_{\alpha i})^2 \tag{4.13}$$

Then one chooses the charges for $\alpha = 2$ to obey a similar equation, as well as a further linear equation expressing the fact they have no mixed anomaly with the charges for $\alpha = 1$. It turns out there is an efficient way to generate all such solutions; one then picks

[4] This is because $\Lambda[\mathcal{R}]$ consists of all integer vectors with component-sum a multiple of $2k$. As $2k$ is even, the boundary state satisfies the condition for $\mathbb{Z}_2 \times \mathbb{Z}_2$ invariance.

a random one. This process continues iteratively until the charges for all $\alpha = 1 \ldots N$ have been chosen.

To modify this construction to produce only the desired boundary states, we recall the result of Section 4.2.2 which states that such boundary states admit a choice of $Q_{\alpha i}$ and $\bar{Q}_{\alpha i}$ for which the first vector $\alpha = 1$ obeys

$$Q_{\alpha i} = \text{odd} \qquad \bar{Q}_{\alpha i} = \text{even} \tag{4.14}$$

for all i. Therefore, we can simply ensure our solution to (4.13) also obeys (4.14), and randomly choosing the remaining charges will eventually produce every chiral parity preserving boundary state.

We used the above recipe to explore boundary states for $N = 8, 12, 16, \ldots$ Dirac fermions. We combined them with the analytic results for $N = 4$ Dirac fermions that we saw in Section 4.2.3, and came to the following conclusions:

- When $N = 4$ or $N = 8, 16, 24, \ldots$, the possible central charges are

$$(g_{\mathcal{R}})^2 \in \{\, N/2 + 4r : r \geq 0 \,\}$$

- When $N = 12, 20, 28, \ldots$, the possible central charges are

$$(g_{\mathcal{R}})^2 \in \{\, N/2 + 2r : r \geq 0 \,\}$$

Not only did we find no values of $(g_{\mathcal{R}})^2$ inconsistent with these statements, all values consistent with them also occurred many times. In both cases the minimum allowed $(g_{\mathcal{R}})^2$ is $N/2$. This is the central charge of the dyon state. The above results therefore support our conjecture, as well as showing additional algebraic structure of the spectrum of allowed $g_{\mathcal{R}}$ values. Whether these statements hold in full generality remains to be proven.

Chapter 5

Fermionic Minimal Models

5.1 Introduction

In this final chapter, we would like to extend the results we have found for fermionic boundary states in a different direction. So far, we have focused on N Dirac fermions, or $2N$ copies of the Majorana fermion. But rather than increasing the number of copies of the system to find richer behaviour, there is a different knob we can tun. The Majorana fermion lives at the start of an infinite sequence of models called the fermionic minimal models [45, 63, 90–92], and we can instead choose to move along this family while keeping the number of copies the same. We would like to ask how much of our story remains intact under this extension.

The above questions are made more interesting by the fact that the fermionic CFTs remain far less well explored than their bosonic cousins. Despite this, they can exhibit new fermionic phenomena that are not seen in the ordinary bosonic minimal models. First, as we saw in Chapters 2 and 3, the fact that $\Omega_2^{\mathrm{SO}}(\mathrm{pt}) = 0$ is trivial while $\Omega_2^{\mathrm{Spin}}(\mathrm{pt}) = \mathbb{Z}_2$ is not [49, 50] means that boundary states in fermionic theories fall into one of two SPT classes characterised by the appearance of Majorana zero modes.

Second, there is also the possibility of supporting symmetries with nontrivial 't Hooft anomalies. For the bosonic minimal models, this is not an option: all global symmetries are necessarily non-anomalous [93]. But for fermionic theories, it is a possibility. We will limit ourselves to $\mathbb{Z}_2$ global symmetries. In fermionic theories, such symmetries can carry a mod-8 valued anomaly. As was explained in [84, 85], this anomaly is related to SPT phases in three dimensions, and the fact it arises only for fermionic theories is encoded in the facts $\Omega_3^{\text{SO}}(B\mathbb{Z}_2) = 0$ while $\Omega_3^{\text{Spin}}(B\mathbb{Z}_2) = \mathbb{Z}_8$.

The above two phenomena are easy to exhibit for the simplest fermionic minimal model, the Majorana fermion. Here, one can simply use free-field techniques to explicitly show everything one could possibly want to show. But for the other fermionic minimal models, which are all interacting, things are not so transparent. Our goal is to extend the analysis of boundary conditions and anomalous symmetries to these remaining fermionic minimal models. We would like to see which features of the Majorana fermion generalise to the family as a whole, and which are artefacts of a free theory. Therefore, to set the scene, it will be useful to first review the basic facts about the Majorana fermion we want to generalise. A summary of our results follows, and after that the organisation of the remaining sections.

5.1.1 A Simple Illustration

We begin by reviewing our earlier discussion of SPT phases from Section 1.2, now in the setting of Virasoro minimal models. This will allow us to introduce various RCFT conventions we will use later on. We will also need to make the mapping of boundary states to SPT phases from Section 1.2.2 slightly more precise.

A Majorana fermion of mass m in d=1+1 dimensions has action

$$S = \frac{i}{2}\int dtdx\, \chi_+\partial_+\chi_+ + \chi_-\partial_-\chi_- + m\chi_+\chi_-$$

Recall that the two possible gapped phases with $m > 0$ and $m < 0$ provide an example of two distinct SPT phases. When placed side by side, the interface hosts an unpaired Majorana zero mode which exhibits an anomaly for $(-1)^F$, as discussed in Section 1.2.2.

As stressed in [36, 37, 46, 47], this story has an alternative guise in the language of boundary conditions. Recall that for a massless Majorana fermion there are two possible boundary conditions, which we denote as

$$\begin{aligned} +: &\quad \chi_+ = +\chi_- \\ -: &\quad \chi_+ = -\chi_- \end{aligned}$$

Consider a mass profile $m(x)$ that interpolates from 0 to $\pm M$. On the left side lives a massless Majorana fermion, while the right side is gapped at a scale M. At low energies, the massless fermion experiences this set-up as a boundary condition. Pictorially, the map from gapped phases to boundary conditions is

Similarly when the gapped phase sits on the left, one obtains a left boundary condition. However, due to an annoying technicality, the outcome is now reversed:

$(\star)$

The analog of the previous story is now that on a spatial interval with $++$ boundary conditions at both ends, the spectrum again includes an unpaired Majorana zero mode. A lattice version of this mechanism also exists, and was the subject of [55]. We note that the sign flip in $(\star)$ is also the reason we had to insert the $(-1)^F$ factor in Chapter 2, when we wished to find the dual boundary state representing a given boundary condition.

The above story changes, but only slightly, when mapped into the language of boundary states. To each right boundary condition, there is an associated boundary state, which lives in the NS sector of the theory:

$$\chi_+ = \pm\chi_- \text{ on right} \quad \to \quad |\pm\rangle$$

Meanwhile, boundary conditions on the left are described by dual states. One might have thought that the correct dual state that describes boundary condition A on the left is simply the dual of the state that describes boundary condition A on the right. But this isn't quite right. Annoyingly, there is an extra sign flip, and the correct mapping of boundary conditions to dual states is actually

$$\chi_+ = \pm\chi_- \text{ on left} \quad \to \quad \langle\mp|$$

Because of the similar sign flip in $(\star)$, however, a gapped phase $m = \pm M$ corresponds to the same boundary state $\langle\pm|$ or $|\pm\rangle$ regardless of whether we sit at a left or a right boundary. It is for this reason that boundary states more precisely correspond to SPT phases than boundary conditions. In any case, once the boundary states are known, we can use the procedure of Section 1.1 to calculate the partition function for states on an interval. Doing this for the boundary states $\langle -|$ and $|+\rangle$ yields the partition function

$$\mathcal{Z}_{-+}(\tau) = \sqrt{2}\,\chi_{1/16}(\tau)$$

where $\chi_{1/16}(\tau)$ is a Virasoro character. The most important feature of this partition function is the overall factor of $\sqrt{2}$ which, as is now familiar, signals the unpaired Majorana mode and hence the distinctness of the SPT phases underlying $\langle -|$ and $|+\rangle$.

For the second half of our story, we will be interested in the $\mathbb{Z}_2$ global symmetry known as chiral fermion parity. It acts by flipping the sign of only one of the fermions, which without loss of generality we can take to be the left-movers:

$$\mathbb{Z}_2: \quad \chi_+ \to \chi_+ \qquad \chi_- \to -\chi_-$$

Under this symmetry, the mass parameter m is odd. The symmetry therefore exchanges the two SPT phases. One can check that the symmetry also exchanges the corresponding boundary states

$$\mathbb{Z}_2: \quad |\pm\rangle \to |\mp\rangle$$

as expected.

Importantly, the symmetry also carries an anomaly whose strength is 1 mod 8. To see this, one places the theory on a background with a defect line for the $\mathbb{Z}_2$ symmetry. It will be sufficient for us to consider a torus. We will denote the partition function on such a background by

$$\mathcal{Z}\left[\tau;\ {}^{\text{AP}}\underset{\text{P}}{\boxed{\,\vdots\,}}\right]$$

where τ is the modular parameter describing a choice of flat metric on the torus, the diagram labels the spin structure, and the dashed line, if present, labels the defect. Under the shift $\tau \to \tau + 2$, we have the transformation law

$$\mathcal{Z}\left[\tau+2;\ {}^{\text{AP}}\underset{\text{P}}{\boxed{\,\vdots\,}}\right] \;=\; e^{2\pi i/8}\,\mathcal{Z}\left[\tau;\ {}^{\text{AP}}\underset{\text{P}}{\boxed{\,\vdots\,}}\right]$$

The presence of the phase $e^{2\pi i/8}$ indicates the anomaly. In general, it could be any eighth root of unity $e^{2\pi i k/8}$; the fact that for the Majorana fermion $k = 1$ indicates that the strength of the anomaly is 1 mod 8, as claimed [48, 95]. The anomaly also manifests itself in a far more obvious way, which becomes clear if we look at the partition function. It turns out to be

$$\mathcal{Z}\left[\tau;\ {}^{\text{AP}}\underset{\text{P}}{\boxed{\,\vdots\,}}\right] \;=\; \sqrt{2}\;\overline{(\chi_0+\chi_{1/2})(\tau)}\;\chi_{1/16}(\tau)$$

Once again we find a notorious factor of $\sqrt{2}$ in the partition function. This time it is telling us that the R sector, when frustrated by a $\mathbb{Z}_2$ symmetry defect, contains an

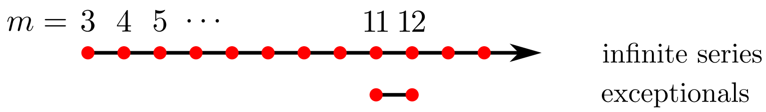


Figure 5.1: The classification of fermionic minimal models. Both the infinite series and exceptionals are labelled by a choice of integer m.

unpaired Majorana zero mode. (This statement would also have been true of the NS sector.)

5.1.2 Summary of Results

We now consider generalising these facts to the other models in the family. Each model should have a complete set of conformal boundary states. We expect that all of these states will arise from a deformation to a gapped phase. If so, then each boundary state will fall into one of two distinct classes depending on which SPT class its gapped phase sits in.

Our first main claim is that this expectation is borne out. We determine, for each fermionic minimal model, the complete list of conformal boundary states. We show how these naturally fall into two classes. When boundary conditions a and b are taken from the same class, we find the partition function

$$\mathcal{Z}_{ab}(\tau) = \sum_{i \in \mathrm{KT}} n^i_{ab}\, \chi_i(\tau)$$

where KT denotes the Kac table, as we review in Section 5.2. The coefficients n^i_{ab} are non-negative integers, so this is a manifestly sensible partition function. In contrast, when a and b are from different classes, we find

$$\mathcal{Z}_{ab}(\tau) = \sqrt{2} \sum_{i \in \mathrm{KT}} n^i_{ab}\, \chi_i(\tau)$$

which contains a factor of $\sqrt{2}$, signalling an unpaired Majorana mode, but other than that the partition function is perfectly sensible, with the coefficients n^i_{ab} again given by non-negative integers, generalising what we saw in Chapter 2. All partition functions are explicitly presented in Section 5.3.

To describe these results in a little more detail, we recall that there are two kinds of fermionic minimal models: an infinite series, and two exceptionals. Both kinds are labelled by an integer m; for the infinite series, this integer takes the values $m \geq 3$,

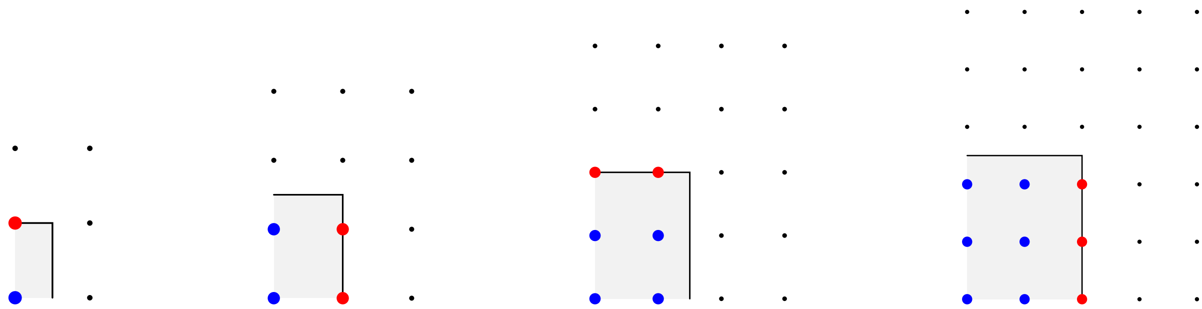


Figure 5.2: Boundary states for the infinite series of models are labelled by points in the bottom-left quadrant of the Kac table. The two classes are shown in blue and red. Note that for $m = 3, 4$ but no other values, the classes have equal sizes.

while for the exceptionals, it takes the values $m = 11, 12$. This situation is depicted in Figure 5.1. We explain our results for each kind of model in turn.

Infinite series

Here the boundary states are labelled by pairs (r, s) in the bottom-left quadrant of the Kac table, defined by the inequalities

$$r \leq m/2 \quad \text{and} \quad s \leq (m+1)/2$$

These states are shown in Figure 5.2 for the first four models in the series. The states fall into two classes, with class 1 consisting of the points in the interior of the region defined by the above inequalities, and class 2 consisting of the border. It is straightforward to count the number of states in the two classes. They have sizes

$$\begin{aligned} &\text{class 1:} \qquad \lfloor (m-1)^2/4 \rfloor \\ &\text{class 2:} \qquad \lfloor m/2 \rfloor \end{aligned}$$

Exceptionals

In this case the labelling of the boundary states is a little more complicated and will be left to Section 5.3. However, the counting is straightforward. Now both classes have the same size. For the $m = 11$ exceptional, both classes have size 10, while for the $m = 12$ exceptional, both have size 12.

Our second main result is a classification of which fermionic minimal models have a global $\mathbb{Z}_2$ symmetry, modulo $(-1)^F$. We find that the first two models in the infinite

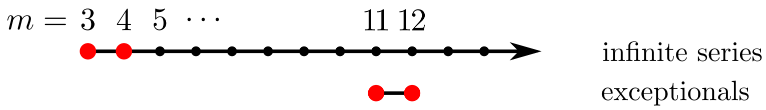


Figure 5.3: The models with an extra $\mathbb{Z}_2$ global symmetry, shown in red. Note that these are the same models which had matching class sizes earlier, as well as the models with vanishing RR sector.

series and both exceptionals have a unique such symmetry, generalising the chiral fermion parity of the Majorana fermion, while the remaining models have none. This situation is depicted in Figure 5.3.

In the four models where the symmetry exists, we also find that the strength of the anomaly takes the same value 1 mod 8 for all models; that the twisted-sector partition functions contain a Majorana zero mode whenever there is a symmetry defect crossing the equal-time contour; and that the symmetry exchanges the two classes of boundary states. This forces them to have equal sizes, which is indeed what we found earlier. We also notice that these models are the same as the ones that have a vanishing RR sector partition function.

In view of the above results, the four special models form a close generalisation of the Majorana fermion, with all the facts we saw in Section 5.1.1 continuing to hold. The remaining models, on the other hand, do not form such a close generalisation, and the only fact that survives is the existence of two incompatible classes of boundary states.

Finally we mention earlier related research. The 'chiral fermion parity' in the $m = 3, 4$ models is already well known. The paper [45] also obtains some of our results on boundary states for a subset of the infinite series, with a different choice of normalisation. The main novelties of our approach are: a uniform treatment of all fermionic minimal models, the normalisation, the interplay between boundary states and symmetries, and the perspective of SPT phases and discrete anomalies. As was only very recently pointed out, the paper [91] also derives the same partition functions as in Section 5.4 by a totally different means. Our approach however is novel; rather than proceeding via an analysis of submodular partition functions, we pay attention to the transformation of the partition function under the whole modular (or metaplectic) group.

Plan

In Section 5.2 we review a handful of facts about the minimal models and their fermionic counterparts that we will need to use, for the purposes of self-containment. In Section 5.3, we write down the two classes of boundary states, and explicitly compute all interval partition functions. In Section 5.4, we determine all $\mathbb{Z}_2$ global symmetries, and in doing so compute their anomalies and all twisted-sector partition functions. We also compute their action on the boundary states found earlier. Finally, in Section 5.5, we argue various implications between our results.

5.2 Review of Fermionic Minimal Models

In this section we review the barest essentials from the theory of minimal models that we will have need to use. We will not review this material from scratch; instead, for reviews of minimal models see for example [96], whose conventions we have strived to match, while for those of the fermionic ones see [63, 92].

The fermionic minimal models are defined to be the set of all unitary spin-CFTs that are rational with respect to the Virasoro algebra. Their classification has been carried out, and the results are as in Figure 5.1. The classification consists of

- An infinite series, with $m \geq 3$.
- Two exceptionals, with $m = 11, 12$.

The integer m dictates the chiral data of the theory. For example, the central charge is determined via

$$c = 1 - \frac{6}{m(m+1)}$$

Of special importance is the Kac table, which is defined as a set of pairs of integers modulo an equivalence relation,

$$\mathrm{KT} \; = \; \frac{\Big\{ (r,s) \; : \; 1 \leq r \leq m-1, \, 1 \leq s \leq m \Big\}}{(r,s) \sim (m-r, m+1-s)}$$

The relevance of the Kac table is that it labels the available Virasoro characters at this central charge. We will denote them by $\chi_{r,s}(\tau)$, and their conformal dimensions by $h_{r,s}$. Sometimes, we will find it economical to compress an element of KT down to a single composite index, which we will denote by a letter such as $i, j, k, \ldots$.

For the special values $m = 11, 12$, one also faces a dichotomy between the infinite series and the exceptionals. This choice affects the way the characters are combined in the partition function. We will follow the presentation of [63]. We arrange the states of even/odd fermion parity on an antiperiodic/periodic[1] circle into a 2×2 table. For the infinite series, this table is[2]

	even	odd
AP	$\sum_{1\le s\le r<m}^{(m+1)r+ms \text{ odd}} \|\chi_{r,s}\|^2$	$\sum_{1\le s\le r<m}^{(m+1)r+ms+m(m+1)/2 \text{ odd}} \overline{\chi_{m-r,s}}\chi_{r,s}$
P	$\sum_{1\le s\le r<m}^{(m+1)r+ms+m(m+1)/2 \text{ even}} \overline{\chi_{m-r,s}}\chi_{r,s}$	$\sum_{1\le s\le r<m}^{(m+1)r+ms \text{ even}} \|\chi_{r,s}\|^2$

For the $m = 11$ exceptional, it is

	even	odd
AP	$\sum_{r=1,\text{odd}}^{9} \|\chi_{r,1} + \chi_{r,7}\|^2 + \|\chi_{r,5} + \chi_{r,11}\|^2$	$\sum_{r=1,\text{odd}}^{9} (\overline{\chi_{r,1} + \chi_{r,7}})(\chi_{r,5} + \chi_{r,11}) + \text{c.c.}$
P	$\sum_{r=1,\text{odd}}^{9} \|\chi_{r,4} + \chi_{r,8}\|^2$	$\sum_{r=1,\text{odd}}^{9} \|\chi_{r,4} + \chi_{r,8}\|^2$

while for the $m = 12$ exceptional, it is

	even	odd
AP	$\sum_{s=1,\text{odd}}^{11} \|\chi_{1,s} + \chi_{7,s}\|^2 + \|\chi_{5,s} + \chi_{11,s}\|^2$	$\sum_{s=1,\text{odd}}^{11} (\overline{\chi_{1,s} + \chi_{7,s}})(\chi_{5,s} + \chi_{11,s}) + \text{c.c.}$
P	$\sum_{s=1,\text{odd}}^{11} \|\chi_{4,s} + \chi_{8,s}\|^2$	$\sum_{s=1,\text{odd}}^{11} \|\chi_{4,s} + \chi_{8,s}\|^2$

5.3 Boundary States

In this section we write down complete sets of boundary states for the fermionic minimal models. We compute their interval partition functions to confirm their consistency, and use them to show which states lie in which class.

First we review the general formalism that we will use. For our purposes, boundary states live in the AP sector of the theory. As they preserve the conformal symmetry, and this includes $(-1)^F$, they in fact belong to the AP-even sector. This Hilbert space decomposes into a direct sum of Verma modules

$$\mathcal{H}_{\text{AP}}^{\text{even}} = \bigoplus_{i,j\in \text{KT}} M_{ij} \overline{\mathcal{V}_i} \otimes \mathcal{V}_j$$

where the multiplicities M_{ij} can be read off from the partition function tables listed in Section 5.2. To preserve the conformal symmetry, boundary states can only contain

[1] Some alternative common synonyms are AP/P, NS/R, and bounding/non-bounding.

[2] Here we regard $\mathcal{Z}$ and $(-1)^{\text{Arf}}\mathcal{Z}$ as equivalent theories, so we only show a single table for both of them. We revisit their distinction more carefully in Section 5.5.

states coming from terms with $i = j$, so must lie in the subspace

$$\mathcal{H}_{\text{AP}}^{\text{even}}\Big|_{\text{diagonal}} = \bigoplus_{i\in\text{KT}} M_{ii}\overline{\mathcal{V}_i}\otimes\mathcal{V}_i = \bigoplus_{\substack{i\in\text{KT}\\ M_{ii}=1}} \overline{\mathcal{V}_i}\otimes\mathcal{V}_i$$

where in the last step, we've taken advantage of the fact that $M_{ii} = 0$ or 1 to simplify our answer. In each $\overline{\mathcal{V}_i}\otimes\mathcal{V}_i$, there is a unique conformally-invariant state $\|i\rangle\rangle$ whose inner product with the vacuum state is 1, called an Ishibashi state [41]. Any boundary state must therefore take the form

$$|a\rangle = \sum_{\substack{i\in\text{KT}\\ M_{ii}=1}} a_i \|i\rangle\rangle$$

for some set of coefficients a_i, which we assume to be real.[3]

When we impose boundary states $\langle a|$ and $|b\rangle$ on an interval, the partition function that counts states on the interval is

$$\text{Tr}_{\mathcal{H}_{ab}}(q^{(L/\pi)H}) = \sum_{i\in\text{KT}} \chi_i(\tau) \sum_{\substack{j\in\text{KT}\\ M_{jj}=1}} \mathcal{S}_{ij}a_j b_j$$

where $q = e^{2\pi i\tau}$ and L is the length of the interval. We demand that the coefficients

$$n_{ab}^i = \sum_j \mathcal{S}_{ij}a_j b_j \tag{5.1}$$

occurring in this expression are either positive integers or $\sqrt{2}$ times positive integers. This is a weakened version of Cardy's condition [30], motivated by Chapter 2, that allows for the presence of unpaired Majorana modes. Finally, we will look for a basis of such solutions, known as fundamental boundary states [44], defined by imposing the additional requirement that

$$n_{ab}^0 = \delta_{ab}$$

Here $i = 0 \in \text{KT}$ denotes the identity module, $(r, s) = (1, 1)$. All other solutions can then be expressed as a linear combination of the fundamental ones with nonnegative-integer coefficients.

[3]The phases of the $\|i\rangle\rangle$ are actually ambiguous. We assume that these choices of phase have been fixed to make the a_i real. As we will see, this can always be consistently done.

Our goal in what follows will be to simply write down a complete set of fundamental boundary states consistent with all of the above properties. We do this for the infinite series and the exceptionals in turn.

5.3.1 Infinite Series

For these models, the quickest way to get the boundary states is to start with those of the underlying diagonal bosonic minimal model, and project onto their even part under the model's unique $\mathbb{Z}_2$ global symmetry. After discarding duplicates, and suitably adjusting the normalisation, we will have our answer.

We begin by recalling that the boundary states of the mth diagonal bosonic minimal model are given by the Cardy ansatz [30]

$$|i\rangle_{\rm B} = \sum_{j\in{\rm KT}} \frac{\mathcal{S}_{ij}}{\sqrt{\mathcal{S}_{0j}}} \|j\rangle\rangle \tag{5.2}$$

where $i \in \text{KT}$ labels the states, and the 'B' stands for bosonic. As it stands, these are *not* valid boundary states of the fermionic theory, because the sum includes all states $\|j\rangle\rangle$, whereas it should only include those with $M_{jj} = 1$. Looking back at the partition function tables in Section 5.2, we see the latter condition is equivalent to

$$(m+1)r + ms = 1 \mod 2$$

where $j = (r, s)$. But this is the same as the condition for $\|j\rangle\rangle$ to be even under the $\mathbb{Z}_2$ global symmetry of the bosonic model [96]. Indeed, this symmetry acts as

$$U\|j\rangle\rangle = (-1)^{(m+1)r+ms+1}\|j\rangle\rangle$$

It follows that if we simply define our fermionic boundary states to be the $\mathbb{Z}_2$-even projections of the bosonic ones, then the undesirable states $\|j\rangle\rangle$ disappear from the sum, and we are left with

$$|i\rangle_F \; := \; \frac{1+U}{2}\,|i\rangle_B \; = \; \sum_{\substack{j\in{\rm KT}\\ M_{jj}=1}} \frac{\mathcal{S}_{ij}}{\sqrt{\mathcal{S}_{0j}}} \|j\rangle\rangle \tag{5.3}$$

which, for each $i \in \text{KT}$, defines a valid fermionic boundary state.

Next, we discard duplicates. Currently the states $|i\rangle_F$ are overcomplete. We wish to restrict the range of i to eliminate this redundancy. This can be done by considering the action of the $\mathbb{Z}_2$ symmetry on the bosonic boundary states (5.2), which is

$$U|i\rangle_B = |i'\rangle_B$$

Here $i \to i'$ is the involution of the Kac Table defined by

$$(r, s) \to (r, s)' = (m - r, s) \tag{5.4}$$

which corresponds to fusion with the special primary $(m-1, 1)$. From (5.3), we see that $|i\rangle_F = |i'\rangle_F$. It follows that we can eliminate the redundancy by restricting i to lie in a set of representatives for the equivalence classes of KT$/(i \sim i')$. One possible choice of a set of representatives, which we will use, is to take the bottom-left quadrant of the Kac Table, defined by the inequalities

$$r \leq m/2 \quad \text{and} \quad s \leq (m+1)/2$$

This is because the involution (5.4) acts as a horizontal reflection of the Kac Table, while the Kac Table is itself defined modulo a combined horizontal + vertical reflection.

Finally we adjust the normalisation to ensure $n^0_{ab} = \delta_{ab}$ between all pairs of states. To do this it will first prove useful to write the boundary states in the form

$$|i\rangle_F = \frac{|i\rangle_B + |i'\rangle_B}{2}$$

This makes it easy to calculate the coefficients n^k_{ij} between the fermionic boundary states ${}_F\langle i|$ and $|j\rangle_F$ using known results for the bosonic boundary states [30]. We find

$$n^k_{ij} = \frac{\mathcal{N}_{ijk} + \mathcal{N}_{i'jk} + \mathcal{N}_{ij'k} + \mathcal{N}_{i'j'k}}{4} = \frac{\mathcal{N}_{ijk} + \mathcal{N}_{ij'k}}{2}$$

From this we can extract the multiplicity of the identity module,

$$n^0_{ij} = \delta_{ij} \begin{cases} 1 & i = i' \\ 1/2 & i \neq i' \end{cases}$$

This is telling us that in order to achieve $n^0_{ij} = \delta_{ij}$, we must rescale the boundary states corresponding to the second case by $\sqrt{2}$. After performing this rescaling, we arrive at our final result

$$|i\rangle_F = \begin{cases} |i\rangle_B & i = i' \\ \frac{|i\rangle_B + |i'\rangle_B}{\sqrt{2}} & i \neq i' \end{cases}$$

Not surprisingly, the two cases will turn out to correspond to the two classes of boundary states. We also take the opportunity to remark that the two cases have a very simple graphical interpretation: the first corresponds to the border of the quadrant, the second to the interior.

We are now in a position where we can list the two classes of boundary states, and demonstrate the consistency of their interval partition functions.

- Class 1 consists of the points (r, s) in the interior of the bottom-left quadrant of the Kac Table, defined by the inequalities $1 \leq r < m/2$ and $1 \leq s < (m+1)/2$. The states take the form

$$|(r,s)\rangle_F = \sqrt{2} \sum_{\substack{1 \leq s' \leq r' < m \\ (m+1)r' + ms' \text{ odd}}} \frac{\mathcal{S}_{(r,s),(r',s')}}{\sqrt{\mathcal{S}_{(1,1),(r',s')}}} \|(r', s')\rangle\rangle$$

- Class 2 consists of the points (r, s) on the border of the bottom-left quadrant of the Kac Table, defined by the inequalities $1 \leq r \leq m/2$ and $1 \leq s \leq (m+1)/2$ with one of the upper bounds saturated. The states take an almost identical form, differing only in normalisation:

$$|(r,s)\rangle_F = \sum_{\substack{1 \leq s' \leq r' < m \\ (m+1)r' + ms' \text{ odd}}} \frac{\mathcal{S}_{(r,s),(r',s')}}{\sqrt{\mathcal{S}_{(1,1),(r',s')}}} \|(r', s')\rangle\rangle$$

On an interval with boundary states i and j, the answer for the coefficients n_{ij}^k depends on the classes of i and j. There are three possible combinations to consider. From our earlier results, they are

$$\begin{aligned} \text{1–1:} \quad & n_{ij}^k = \mathcal{N}_{ijk} + \mathcal{N}_{ijk'} \\ \text{1–2 or 2–1:} \quad & n_{ij}^k = \sqrt{2}\,\mathcal{N}_{ijk} \\ \text{2–2:} \quad & n_{ij}^k = \mathcal{N}_{ijk} \end{aligned}$$

As promised, we see that since the fusion numbers $\mathcal{N}_{ijk}$ are nonnegative integers (in fact 0 or 1), these answers have the desired integrality or $\sqrt{2}$-integrality properties.

5.3.2 Exceptionals

For the exceptional models, instead of starting from the underlying bosonic model, it is perhaps simplest to write down the boundary states directly. We do this by choosing a pair of seed boundary states and applying fusion to generate the rest. Most of the details will be the same between the $m = 11$ and the $m = 12$ models, so we show the details only for the $m = 11$ model.

We start by consulting the table in Section 5.2 to see when $M_{ii} = 1$. We see that this is true when $i = (r, s)$ with $s = 1, 5, 7, 11$ and r odd. It follows that the admissible boundary states of the fermionic model must lie in the span

$$|a\rangle \in \text{span}\Big\{ \|(r,1)\rangle\!\rangle,\ \|(r,5)\rangle\!\rangle,\ \|(r,7)\rangle\!\rangle,\ \|(r,11)\rangle\!\rangle : r \text{ odd} \Big\} \tag{5.5}$$

Our first goal will be to write down the simplest consistent boundary state obeying this property. To do this, we assume that we already know in advance what the coefficients n^i_{aa} will be. We can then reverse-engineer a boundary state that does the job:

$$a_i = \sqrt{\mathcal{S}_{ij} n^j_{aa}}$$

For most choices of the coefficients n^i_{aa}, the resulting boundary state $|a\rangle$ will not be allowed, due to containing extra states $\|i\rangle\!\rangle$ not in the span (5.5). We would like to make the simplest choice of n^i_{aa} such that it is allowed. Our claim is that this choice is

$$n^i_{aa} = \delta_{i,(1,1)} + \delta_{i,(1,5)} + \delta_{i,(1,7)} + \delta_{i,(1,11)}$$

Proof To demonstrate the above assertion, we need to show that

$$\mathcal{S}_{(r,s),(1,1)} + \mathcal{S}_{(r,s),(1,5)} + \mathcal{S}_{(r,s),(1,7)} + \mathcal{S}_{(r,s),(1,11)} = 0 \quad \text{for } s \notin \{1, 5, 7, 11\}$$

For this we invoke the explicit formula for the modular $\mathcal{S}$-matrix

$$\mathcal{S}_{(r,s),(r',s')} = \sqrt{\frac{8}{m(m+1)}} \sin\left(\frac{\pi r r'}{m}\right) \sin\left(\frac{\pi s s'}{m+1}\right) (-1)^{(r+s)(r'+s')}$$

and well as the trigonometric identity

$$\sum_{s'=1,5,7,11} \sin\left(\frac{\pi s s'}{12}\right) = \begin{cases} \sqrt{6} & s \in \{1, 5, 7, 11\} \\ 0 & \text{otherwise} \end{cases}$$

Putting these together gives the promised result. □

So far we have obtained a single consistent seed state $|a\rangle$ with coefficients

$$a_i = \sqrt{\mathcal{S}_{i,(1,1)} + \mathcal{S}_{i,(1,5)} + \mathcal{S}_{i,(1,7)} + \mathcal{S}_{i,(1,11)}}$$

Next we cook up a second seed state $|b\rangle$ by making a different set of sign choices in the coefficients. We choose

$$b_{(r,s)} = a_{(r,s)} \begin{cases} +1 & s = 1,7 \\ -1 & s = 5,11 \end{cases}$$

where we recall when we discuss the coefficients of a fermionic boundary state, r is odd and $s = 1, 5, 7, 11$. Our next claim is that the state $|b\rangle$ gives rise to the interval partition functions

$$n^i_{bb} = n^i_{aa} \quad \text{and} \quad n^i_{ab} = \sqrt{2}\,(\delta_{i,(1,4)} + \delta_{i,(1,8)}) \tag{5.6}$$

This is telling us that the two seed states $|a\rangle$, $|b\rangle$ lie in different SPT classes, but are otherwise consistent.

Proof The first equation in (5.6) is trivial since a_i and b_i differ only by signs. The computation of n^i_{ab} is somewhat trickier. This time the key fact we need is

$$(-1)^{[s=5,11]} \sum_{s'=1,5,7,11} \sin\left(\frac{\pi s s'}{12}\right) = \sqrt{2} \sum_{s'=4,8} \sin\left(\frac{\pi s s'}{12}\right)$$

Using the above identity, we compute

$$a_{(r,s)} b_{(r,s)} = (-1)^{[s=5,11]} \sum_{s'=1,5,7,11} \mathcal{S}_{(r,s),(1,s')} = \sqrt{2} \sum_{s'=4,8} \mathcal{S}_{(r,s),(1,s')}$$

By (5.1) the claimed result for n^i_{ab} immediately follows. □

With the two seed states in hand, all remaining boundary states can now be generated by fusion [43, 97]. This is a recipe which takes as input a boundary state, which for us will be either $|a\rangle$ or $|b\rangle$, as well as a primary operator $i \in \text{KT}$, and produces a new consistent boundary state. The recipe for the fused boundary states is

$$|a;i\rangle = \sum_{\substack{j\in\text{KT} \\ M_{jj}=1}} a_j \frac{\mathcal{S}_{ij}}{\mathcal{S}_{0j}} \,\|j\rangle\!\rangle \qquad |b;i\rangle = \sum_{\substack{j\in\text{KT} \\ M_{jj}=1}} b_j \frac{\mathcal{S}_{ij}}{\mathcal{S}_{0j}} \,\|j\rangle\!\rangle$$

which reduce back to the seed states when $i = 0$. Our final claim is that to obtain a complete basis of fundamental boundary states, we should let i range over

$$i = (r, s) \quad \text{where} \quad r = 1, 3, 5, 7, 9 \text{ and } s = 1, 2$$

This assertion will become evident in a moment when we list the interval partition functions between all pairs of boundary states. For now, we simply note that the counting works out: the above family contains $2 \cdot 5 \cdot 2 = 20$ states, which coincides with the number of allowed Ishibashi states $\|j\rangle\rangle$, as these correspond to $j = (r, s)$ with $r = 1, 3, 5, 7, 9$ and $s = 1, 5, 7, 11$.

We are now in a position to list the states for the $m = 11$ exceptional model, and their interval partition functions:

- Both classes are labelled by pairs (r, s) with $r = 1, 3, 5, 7, 9$ and $s = 1, 2$. Regarded as elements of the Kac Table, these are all distinct elements. The two classes of states, after some algebra, are

$$|(r,s)\rangle_1 = \sum_{\substack{r'=1,3,5,7,9 \\ s'=1,5,7,11}} \begin{Bmatrix} \alpha : s'=1 \\ \beta : s'=5 \\ \beta : s'=7 \\ \alpha : s'=11 \end{Bmatrix} \frac{\mathcal{S}_{(r,s),(r',s')}}{\sqrt{\mathcal{S}_{(1,1),(r',s')}}} \|(r', s')\rangle\rangle$$

$$|(r,s)\rangle_2 = \sum_{\substack{r'=1,3,5,7,9 \\ s'=1,5,7,11}} \begin{Bmatrix} \alpha : s'=1 \\ -\beta : s'=5 \\ \beta : s'=7 \\ -\alpha : s'=11 \end{Bmatrix} \frac{\mathcal{S}_{(r,s),(r',s')}}{\sqrt{\mathcal{S}_{(1,1),(r',s')}}} \|(r', s')\rangle\rangle$$

  where $\alpha = \sqrt{2(3+\sqrt{3})}$ and $\beta = \sqrt{2(3-\sqrt{3})}$.

- The interval partition functions are

$$\begin{aligned} &\text{1–1 or 2–2:} \qquad n_{ij}^k = \mathcal{N}_{ij}^x \left[\mathcal{N}_{x,(1,1)}^k + \mathcal{N}_{x,(1,5)}^k + \mathcal{N}_{x,(1,7)}^k + \mathcal{N}_{x,(1,11)}^k \right] \\ &\text{1–2 or 2–1:} \qquad n_{ij}^k = \sqrt{2}\, \mathcal{N}_{ij}^x \left[\mathcal{N}_{x,(1,4)}^k + \mathcal{N}_{x,(1,8)}^k \right] \end{aligned}$$

  where the appearance of the fusion numbers is from Verlinde's formula [98].

Before we go on, we return to an earlier point and verify the completeness of the states. Using the above formulas, the multiplicity of the identity module is

$$\begin{aligned} &\text{1–1 or 2–2:} \qquad n_{ij}^0 = \delta_{ij} + \mathcal{N}_{ij}^{(1,5)} + \mathcal{N}_{ij}^{(1,7)} + \mathcal{N}_{ij}^{(1,11)} \\ &\text{1–2 or 2–1:} \qquad n_{ij}^0 = \sqrt{2} \left[\mathcal{N}_{ij}^{(1,4)} + \mathcal{N}_{ij}^{(1,8)} \right] \end{aligned}$$

One can show that for the range of values of i and j allowed, all the fusion numbers in the above expression vanish identically, and only the δ_{ij} term survives. This shows that $n^0_{ab} = \delta_{ab}$ between all pairs of states a and b, establishing completeness.

We now briefly turn to the $m = 12$ exceptional. Here, the results are virtually identical, except with the roles of r and s swapped around:

- Both classes are labelled by pairs (r, s) with $r = 1, 2$ and $s = 1, 3, 5, 7, 9, 11$. Regarded as elements of the Kac Table, these are again distinct. The states are

$$|(r,s)\rangle_1 = \sum_{\substack{r'=1,5,7,11 \\ s'=1,3,5,7,9,11}} \begin{Bmatrix} \alpha : r'=1 \\ \beta : r'=5 \\ \beta : r'=7 \\ \alpha : r'=11 \end{Bmatrix} \frac{\mathcal{S}_{(r,s),(r',s')}}{\sqrt{\mathcal{S}_{(1,1),(r',s')}}} \, \|(r', s')\rangle\rangle$$

$$|(r,s)\rangle_2 = \sum_{\substack{r'=1,5,7,11 \\ s'=1,3,5,7,9,11}} \begin{Bmatrix} \alpha & : r'=1 \\ -\beta & : r'=5 \\ \beta & : r'=7 \\ -\alpha & : r'=11 \end{Bmatrix} \frac{\mathcal{S}_{(r,s),(r',s')}}{\sqrt{\mathcal{S}_{(1,1),(r',s')}}} \, \|(r', s')\rangle\rangle$$

  with the same constants α and β as before.

- The interval partition functions are

$$\text{1–1 or 2–2:} \qquad n^k_{ij} = \mathcal{N}^x_{ij} \, [\, \mathcal{N}^k_{x,(1,1)} + \mathcal{N}^k_{x,(5,1)} + \mathcal{N}^k_{x,(7,1)} + \mathcal{N}^k_{x,(11,1)} \,]$$

$$\text{1–2 or 2–1:} \qquad n^k_{ij} = \sqrt{2} \, \mathcal{N}^x_{ij} \, [\, \mathcal{N}^k_{x,(4,1)} + \mathcal{N}^k_{x,(8,1)} \,]$$

5.4 Anomalous Symmetries

Here we turn to our second main goal, which is to list all $\mathbb{Z}_2$ global symmetries of the fermionic minimal models – including potentially those with an anomaly. The motivation for performing this task comes from looking at the boundary states of the four special models for which the classes have equal sizes. In Section 5.3, we found these states to be

- Infinite series, $m = 3$:

$$\|0\rangle\rangle \pm \|\tfrac{1}{2}\rangle\rangle$$

  (Here and in the next example, we label Ishibashi states by conformal dimensions rather than elements of the Kac Table.)

- Infinite series, $m = 4$:

$$\frac{(5-\sqrt{5})^{1/4}}{10^{1/4}}\left(\|0\rangle\!\rangle \pm \|\tfrac{3}{2}\rangle\!\rangle\right) + \frac{(5+\sqrt{5})^{1/4}}{10^{1/4}}\left(\|\tfrac{3}{5}\rangle\!\rangle \pm \|\tfrac{1}{10}\rangle\!\rangle\right)$$
$$\frac{(5+\sqrt{5})^{3/4}}{2^{3/4}\sqrt{5}}\left(\|0\rangle\!\rangle \pm \|\tfrac{3}{2}\rangle\!\rangle\right) - \frac{(5-\sqrt{5})^{3/4}}{2^{3/4}\sqrt{5}}\left(\|\tfrac{3}{5}\rangle\!\rangle \pm \|\tfrac{1}{10}\rangle\!\rangle\right)$$

- Exceptional, $m = 11$:

$$\sum_{\substack{r'=1\ldots 9,\text{odd}\\ s'=1,5,7,11}} \begin{Bmatrix} \alpha & : s'=1 \\ \pm\beta & : s'=5 \\ \beta & : s'=7 \\ \pm\alpha & : s'=11 \end{Bmatrix} \frac{\mathcal{S}_{(r,s),(r',s')}}{\sqrt{\mathcal{S}_{(1,1),(r',s')}}} \|(r',s')\rangle\!\rangle$$

- Exceptional, $m = 12$:

$$\sum_{\substack{r'=1,5,7,11\\ s'=1\ldots 11,\text{odd}}} \begin{Bmatrix} \alpha & : r'=1 \\ \pm\beta & : r'=5 \\ \beta & : r'=7 \\ \pm\alpha & : r'=11 \end{Bmatrix} \frac{\mathcal{S}_{(r,s),(r',s')}}{\sqrt{\mathcal{S}_{(1,1),(r',s')}}} \|(r',s')\rangle\!\rangle$$

where, in all cases, the upper choice of sign for the $\pm$ corresponds to the first class and the lower choice to the second. The above formulas give rise to an important observation: the two classes differ only by flipping the sign of a certain subset of the Ishibashi states. This strongly suggests that the classes are related by the action of a $\mathbb{Z}_2$ global symmetry. Our goal in this section will be to show that these models do indeed have a unique $\mathbb{Z}_2$ symmetry that exchanges the classes, while the remaining models have no such symmetry.

Anomalous $\mathbb{Z}_2$ Symmetries

We start by reviewing some basic formalism about $\mathbb{Z}_2$ symmetries. Our perspective is similar to that described in [99]. The Hilbert space in the AP sector is

$$\mathcal{H}_{\text{AP}} = \mathcal{H}_{\text{AP}}^{\text{even}} \oplus \mathcal{H}_{\text{AP}}^{\text{odd}} = \bigoplus_{i,j\in\text{KT}} N_{ij}\overline{\mathcal{V}_i}\otimes\mathcal{V}_j = \bigoplus_{\substack{i,j\in\text{KT}\\ N_{ij}=1}} \overline{\mathcal{V}_i}\otimes\mathcal{V}_j$$

where the multiplicities N_{ij} can be read off from the partition function tables in Section 5.2. Again, these only take the values $N_{ij} = 0, 1$, allowing us to make the final step. A $\mathbb{Z}_2$ symmetry must act on each $\overline{\mathcal{V}_i}\otimes\mathcal{V}_j$ by a sign, which we will denote as $s_{ij} = \pm 1$. These signs determine the partition function on a background with an AP-AP spin structure and a symmetry defect along the space direction:

$$\mathcal{Z}\left[\tau;\ {}^{\mathrm{AP}}\underset{\mathrm{AP}}{\boxminus}\right] = \sum_{\substack{i,j\in \mathrm{KT} \\ N_{ij}=1}} s_{ij}\,\overline{\chi_i(\tau)}\,\chi_j(\tau)$$

Knowledge of this one partition function then allows further partition functions to be determined. This is because the partition functions on various backgrounds are related among each other by acting on τ with modular transformations $\mathcal{S}(\tau) = -1/\tau$ and $\mathcal{T}(\tau) = \tau + 1$. The relationships we need are encoded by the diagram

$$\begin{array}{ccccc}
\mathcal{Z}\left[\tau;\ {}^{\mathrm{AP}}\underset{\mathrm{AP}}{\boxminus}\right] & \overset{\mathcal{S}}{\text{———}} & \mathcal{Z}\left[\tau;\ {}^{\mathrm{AP}}\underset{\mathrm{AP}}{\boxbar}\right] & \overset{\mathcal{T}}{\text{———}} & \mathcal{Z}\left[\tau;\ {}^{\mathrm{P}}\underset{\mathrm{AP}}{\boxslash}\right] \\
\Big|\,\mathcal{T} & & & & \Big|\,\mathcal{S} \\
\mathcal{Z}\left[\tau;\ {}^{\mathrm{P}}\underset{\mathrm{AP}}{\boxminus}\right] & \overset{\mathcal{S}}{\text{———}} & \mathcal{Z}\left[\tau;\ {}^{\mathrm{AP}}\underset{\mathrm{P}}{\boxbar}\right] & \overset{\mathcal{T}}{\text{———}} & \mathcal{Z}\left[\tau;\ {}^{\mathrm{AP}}\underset{\mathrm{P}}{\boxslash}\right]
\end{array}$$

which allows all partition functions to be determined from the one on the top-left. A consistent $\mathbb{Z}_2$ symmetry is one for which all these partition functions admit a sensible expansion into Virasoro characters. For most of the backgrounds,

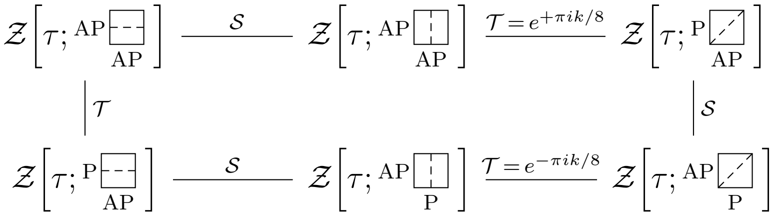
$${}^{\mathrm{AP}}\underset{\mathrm{AP}}{\boxminus} \qquad {}^{\mathrm{P}}\underset{\mathrm{AP}}{\boxminus} \qquad {}^{\mathrm{P}}\underset{\mathrm{AP}}{\boxslash} \qquad {}^{\mathrm{AP}}\underset{\mathrm{P}}{\boxslash}$$

this means a sum weighted by integers. However for the two special backgrounds

$${}^{\mathrm{AP}}\underset{\mathrm{AP}}{\boxbar} \qquad {}^{\mathrm{AP}}\underset{\mathrm{P}}{\boxbar}$$

the partition function is untwisted by any symmetries, and so we impose the stronger constraint that the weights be *nonnegative* integers.

Actually, what we have described is not quite correct for anomalous symmetries. In this case we must weaken the above requirements in several ways. First, the diagram need only hold projectively, meaning some of the relationships expressed by the edges are violated by a phase. Second, the partition functions themselves are ambiguously defined up to a phase. By adjusting these phases if necessary, which amounts to making a choice of gauge for the diagram, we can always cast the diagram into the form



(5.7)

where the phase violations are expressed by the notation $\mathcal{Z}[\tau; A] \xrightarrow{\mathcal{T}=e^{i\theta}} \mathcal{Z}[\tau; B]$, which means that $\mathcal{Z}[\tau+1; A] = e^{i\theta}\mathcal{Z}[\tau; B]$ and vice-versa.[4] The integer k is the strength of the anomaly, and is valued mod 8.[5]

To use the diagram, we start with the top-left partition function, and determine the other partition functions and the integer k by insisting that the diagram commutes. If no such integer k can be found, we do not have a consistent symmetry. Otherwise we demand that the partition functions are sensible, as before. But because we may have needed to adjust their phases to gauge-fix the diagram, it only makes sense to demand that they have a sensible expansion into characters up to an overall phase.

There is one final consequence of the anomaly, and that is the possible appearance of an unpaired Majorana mode in the frustrated sectors. This means that for the backgrounds

with a symmetry defect that wraps vertically, we should allow a possible overall factor of $\sqrt{2}$ in the partition function. We previously saw an example of this phenomenon in Section 5.1.1.

[4]The argument for why this pattern of phases is universal is that the holonomy of a closed loop in the diagram is the value of $\exp(-\frac{i\pi}{2}\eta(\mathcal{D}))^k$ on a suitable mapping torus, where $\mathcal{D}$ is a certain 3d Dirac operator. Then because the phases are universal, they can be read off from the Majorana fermion. For more details, see [48], and for this particular example, also [95]. Another, more concrete way to see the pattern of phases is to use the identities $\mathcal{S}^2 = (\mathcal{S}\mathcal{T})^3 = 1$, and the fact that the top-left partition function is invariant under $\mathcal{T}^2$ – though with this approach the quantisation of k is less obvious.

[5]Although k appears as the exponent of a 16th root of unity, the shift $k \to k+8$ is a gauge transformation, so k is valued mod 8 not mod 16. We could have made k manifestly mod-8 valued by making a different gauge choice. But the one we have chosen is more convenient in the long run.

Solving the Constraints

We turn now to the task of solving the above constraints. To do this, we will recast a subset of the constraints as a set of matrix equations, solve them, and then check that the resulting solutions satisfy the remaining constraints.

We begin by deriving the set of matrix equations. The partition function on ${}^{\text{AP}}\square_{\text{AP}}$ is encoded by the matrix

$$A_{ij} = N_{ij} s_{ij}$$

where we recall that N_{ij} is a known matrix of 0s and 1s encoding the partition function on ${}^{\text{AP}}\square_{\text{AP}}$, and the s_{ij} are a set of unknown signs – unknown except for that of the identity operator, which we set to be $s_{11} = +1$. Meanwhile, the corresponding matrix for the ${}^{\text{AP}}\square_{\text{AP}}$ partition function must take the form

$$[\sqrt{2}\,]\, e^{i\theta} B_{ij}$$

where B is an unknown matrix of nonnegative integers, the phase θ is arbitrary, and the $\sqrt{2}$ may or may not be present. Our first equation arises from the fact that ${}^{\text{AP}}\square_{\text{AP}}$ and ${}^{\text{AP}}\square_{\text{AP}}$ are related by an $\mathcal{S}$-transformation. This fact is expressed by

$$\mathcal{S} A\, \mathcal{S} = [\sqrt{2}\,]\, e^{i\theta} B$$

Actually, since $\mathcal{S}$ is real, the phase $e^{i\theta}$ must equal ± 1, so we obtain

$$\mathcal{S} A\, \mathcal{S} = \pm[\sqrt{2}\,] B \tag{5.8}$$

A second equation arises by considering a $\mathcal{T}^2$ transformation of ${}^{\text{AP}}\square_{\text{AP}}$. Such a transformation preserves the background, but contributes an anomalous phase of $e^{2\pi i k/8}$. This fact is expressed by

$$\mathcal{T}^{-2} B\, \mathcal{T}^2 = e^{2\pi i k/8} B \tag{5.9}$$

Equations (5.8) and (5.9) will be all we need to determine the symmetries. We shall analyse them separately depending on whether the $\sqrt{2}$ is present or absent from (5.8).

5.4.1 The Case of No Majorana Mode

If (5.8) contains no $\sqrt{2}$, then we can easily show that the only solutions are the trivial symmetry and fermion parity. To do this, we will make use of various Galois-theoretic results that were used extensively to solve the bosonic version of this problem [96]. As these results will play an important role both in this section and the next, we begin with a brief review of these ideas.

The entries of the modular $\mathcal{S}$-matrix belong to the cyclotomic field $\mathbb{Q}(\zeta_n)$ where $n = 2m(m+1)$, and $\zeta_n = e^{2\pi i/n}$ is an nth root of unity. This field is acted on by Galois transformations σ_h, labelled by elements $h \in \mathbb{Z}_n^*$, via

$$\sigma_h(\zeta_n) = \zeta_n^h$$

The action of σ_h on the modular $\mathcal{S}$-matrix is [100]

$$\sigma_h(\mathcal{S}_{ij}) = \epsilon_h(i)\,\mathcal{S}_{\sigma_h(i)j}$$

where $\epsilon_h(i)$ is a sign given by

$$\epsilon_h(r,s) = \eta_h\,\epsilon_m(hr)\,\epsilon_{m+1}(hs)$$

while σ_h is a permutation that will not be of interest to us. Here we have also introduced $\eta_h = \sigma_h(\sqrt{n})/\sqrt{n}$, a computable but irrelevant sign, as well as another sign

$$\epsilon_m(x) = \operatorname{sign}\sin\left(\frac{\pi x}{m}\right)$$

known as an $\mathfrak{su}(2)$ affine parity, defined for all $x \neq 0$ mod m.

We can use these facts to derive a useful constraint on B_{ij}. Starting from (5.8), applying σ_h, and comparing the result back to (5.8) gives

$$B_{ij} = \epsilon_h(i)\,\epsilon_h(j)\,B_{\sigma_h(i)\sigma_h(j)}$$

Crucially, all the entries of B_{ij} are nonnegative. This means that if ever the product of signs $\epsilon_h(i)\epsilon_h(j)$ equals -1 for any h, then both sides must be zero. We learn that B_{ij} obeys the *parity rule*

$$B_{ij} \neq 0 \quad \text{only if} \quad \epsilon_h(i) = \epsilon_h(j)\ \forall h \tag{5.10}$$

The set of all pairs (i,j) obeying this condition was determined in [96], and by Result 4 of that paper, they have conformal dimensions satisfying

$$h_i - h_j \in \tfrac{1}{2}\mathbb{Z} \cup \tfrac{1}{3}\mathbb{Z} \cup \tfrac{1}{5}\mathbb{Z}$$

Now we recall (5.9), whose (i,j)th component reads

$$e^{4\pi i(h_j-h_i)}B_{ij}=e^{2\pi ik/8}B_{ij}$$

The phase on the left hand side can never be a nontrivial power of an eighth root of unity. We learn that $k=0$, or in other words, that the symmetry is non-anomalous.

The existence of a non-anomalous symmetry is a powerful statement. It implies that if we perform a GSO projection, then the symmetry survives in the resulting bosonic theory [58, 101–103]. Let us call the original symmetry α, and denote by $\iota(\alpha)$ its image in the bosonic theory. Then $\iota(\alpha)$ commutes with $\iota((-1)^F)$. In the bosonic minimal models, the only such symmetry is $\iota((-1)^F)$ itself, hence $\iota(\alpha)=\iota((-1)^F)$, which in turn forces $\alpha=(-1)^F$. We therefore learn that the original symmetry must have been either trivial or fermion parity, as claimed.

We note that it would have also been possible to derive this conclusion directly from equations (5.8) and (5.9), using technical arguments entirely parallel to those in [96]. However the above argument, similar to those used in [66], is more transparent.

5.4.2 The Case of a Majorana Mode

In the more novel case that (5.8) contains a $\sqrt{2}$, we show that solutions exist only for $m=3,4$ in the infinite series and $m=11,12$ in the exceptionals.

As a first step we show that $m=3,4$ mod 8. Our starting point is (5.8), from which we can immediately deduce

$$\sqrt{2}\in\mathbb{Q}(\zeta_n)$$

This is true if and only if n is a multiple of 8, which in turn requires

$$m=0,3 \bmod 4$$

Assuming from now on that this is the case, we can return to (5.8) and repeat our earlier derivation of the parity rule (5.10) obeyed by B_{ij}. We find that it is modified by the presence of the $\sqrt{2}$, and now takes the form

$$B_{ij}\neq 0 \quad \text{only if} \quad \epsilon_h(i)=f(h)\,\epsilon_h(j)\ \forall h \tag{5.11}$$

Here $f(h) = \sigma_h(\sqrt{2})/\sqrt{2}$ is a sign which comes from the Galois transformation of the factor of $\sqrt{2}$, and takes the explicit form

$$f(h) = \begin{cases} +1 & h = 1, 7 \bmod 8 \\ -1 & h = 3, 5 \bmod 8 \end{cases}$$

To solve condition (5.11) for (i, j), we write it in terms of affine parities as

$$\epsilon_m(hr_i)\, \epsilon_{m+1}(hs_i) = f(h)\, \epsilon_m(hr_j)\, \epsilon_{m+1}(hs_j) \quad \forall h \in \mathbb{Z}^*_{2n}$$

Let us assume that $m = 0 \bmod 4$, since the case $m = 3 \bmod 4$ can be treated almost identically. Then the above condition is equivalent to

$$\begin{aligned} \epsilon_m(hr_i) &= f(h)\, \epsilon_m(hr_j) \quad \forall h \in \mathbb{Z}^*_{2m} \\ \epsilon_{m+1}(hs_i) &= \epsilon_{m+1}(hs_j) \quad \forall h \in \mathbb{Z}^*_{2(m+1)} \end{aligned}$$

The solutions to the second equation were determined in [96], where Result 3 states that $s_i = s_j$ or $s_i = m + 1 - s_j$. Meanwhile, the solutions to the first equation are determined by the following conjecture:

Conjecture Suppose n is a multiple of 4, $1 \le x, y \le n - 1$, and $\epsilon_n(hx) = f(h)\, \epsilon_n(hy)$ for all $h \in \mathbb{Z}^*_{2n}$. Then the only solutions are

- $n = 4$ and $(x, y) = (1, 2)$,
- $n = 8$ and $(x, y) = (1, 3)$,
- $n = 12$ and $(x, y) = (1, 4), (2, 3), (4, 5)$,
- $n = 24$ and $(x, y) = (1, 7), (5, 11)$,

and those that can be obtained from them by the solution-preserving transformations

$$(n, x, y) \;\to\; (n, y, x),\, (n, n - x, y),\, (n, x, n - y),\, (kn, kx, ky) \tag{5.12}$$

It is computationally trivial to verify this conjecture up to $n = 1000$, and we expect, but have not proved, that it holds for all n.

We now know all solutions to (5.11). Without loss of generality, $s_i = s_j$. (This uses the equivalence $(r, s) \sim (m - r, m + 1 - s)$ of the Kac Table.) The values of r_i and r_j are given by

- if $m = 4a$, $(r_i, r_j) = (a, 2a)$,
- if $m = 8a$, $(r_i, r_j) = (a, 3a)$,
- if $m = 12a$, $(r_i, r_j) = (a, 4a), (2a, 3a), (4a, 5a)$,
- if $m = 24a$, $(r_i, r_j) = (a, 7a), (5a, 11a)$,

and those related to them under $(r_i, r_j) \to (r_j, r_i), (m - r_i, r_j), (r_i, m - r_j)$.

Next, we use this result to compute the possible values of the phase $e^{4\pi i(h_i - h_j)}$ among all solutions. Using the identity $2(h_i - h_j) = (m+1)\frac{r_i^2 - r_j^2}{2m}$ mod 1, we find the following contributions to the set of values taken by $e^{4\pi i(h_i - h_j)}$:

- $m = 4a \;\to\; \zeta_8^{\pm a}$
- $m = 8a \;\to\; (-1)^a$
- $m = 12a \;\to\; \zeta_8^{\pm a}, \zeta_{24}^{\pm 7a}$
- $m = 24a \;\to\; 1$

The payoff of these computations comes from considering the identity (5.9), which in components takes the form

$$e^{4\pi i(h_j - h_i)} B_{ij} = e^{2\pi i k/8} B_{ij}$$

In view of the possible phases that can arise on the left hand side when $B_{ij} \neq 0$, it is easy to see that the anomaly k must satisfy

$$\begin{aligned} m = 4 \bmod 8 \quad &\Longrightarrow \quad k = \text{odd} \\ m = 0 \bmod 8 \quad &\Longrightarrow \quad k = \text{even} \end{aligned}$$

All of the above analysis also goes through unchanged if instead $m = 3 \bmod 4$. Combining the results from the two cases, we conclude that

$$\begin{aligned} m = 3, 4 \bmod 8 \quad &\Longrightarrow \quad k = \text{odd} \\ m = 0, 7 \bmod 8 \quad &\Longrightarrow \quad k = \text{even} \end{aligned}$$

An even value of k is incompatible with our assumption of a $\sqrt{2}$ in the frustrated partition function [94]. Indeed, if k were even, then we could stack with an even number $8 - k$ of copies of the Majorana fermion, obtaining a system with no anomaly yet still with an unpaired Majorana zero in the frustrated sector. This gives a contradiction, since the presence of an unpaired zero mode is an exclusive feature of anomalous theories. The only way to avoid this contradiction is if

$$m = 3, 4 \bmod 8$$

which is what we wanted to show.

While we have just ruled out $m = 0, 7 \bmod 8$ by a physical argument, one can also ask how the mathematical requirements (5.8) and (5.9) fail in this case. The answer is that (5.8) is generically violated, with B_{ij} failing to be a nonnegative-integer matrix. The only exceptions are $m = 7, 8$, where there is a single solution for B_{ij}, but this then goes on to fail (5.9).

To make further progress with the remaining cases, we deal with the infinite series and the exceptional models in turn.

Infinite Series

For the infinite series when $m = 3, 4 \bmod 8$, the matrix N_{ij} takes the form

$$N = \bigoplus_{\substack{i \in \mathrm{KT}/' \\ \xi_i = 1}} \begin{array}{c} \\ i \\ i' \end{array} \!\!\begin{array}{c} \begin{array}{cc} i & i' \end{array} \\ \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \end{array}$$

This notation requires a little explanation. To each weight $i = (r, s) \in \mathrm{KT}$ there is an associated sign

$$\xi_i = (-1)^{(m+1)r + ms + 1}$$

One can show that $\xi_i = \xi_{i'}$, and that $\xi_i = +1 \implies i \neq i'$. (Here $(r, s)' = (m - r, s)$ is the involution (5.4) introduced in Section 5.3.) These facts are necessary to ensure the above block decomposition makes sense.

First we use the consistency conditions to constrain the form of A_{ij}, the matrix of unknown signs corresponding to ${}^{\mathrm{AP}}\boxminus_{\mathrm{AP}}$, using arguments analogous to [96]. Because $\mathcal{S} N \mathcal{S} = N$, which follows from $\mathcal{S}$-invariance of ${}^{\mathrm{AP}}\square_{\mathrm{AP}}$, N_{ij} obeys the parity rule (5.10). Since $A_{ij} = N_{ij} s_{ij}$, so too does A_{ij}. By acting on (5.8) with a Galois transformation, comparing it back to (5.8) and invoking the previous fact, we find that A_{ij} obeys a *modified permutation rule*

$$A_{\sigma_h(i)\sigma_h(j)} = f(h) A_{ij}$$

This states that the many unknown signs in A_{ij} are in fact related. Rather than using the full power of this equation, though, we shall only use it for the special case $h = m(m+1) - 1$. In this case $\sigma_h(i) = i'$ and $f(h) = -1$, so we obtain

$$A_{i'j'} = -A_{ij}$$

This states that the unknown signs within each 2×2 block of A_{ij} are related, and that A_{ij} takes the form

$$A = \bigoplus_{\substack{i \in \mathrm{KT}/' \\ \xi_i = 1}} \begin{array}{c} \\ i \\ i' \end{array} \!\!\begin{array}{c} \begin{array}{cc} i & i' \end{array} \\ \begin{pmatrix} \epsilon_i & \eta_i \\ -\eta_i & -\epsilon_i \end{pmatrix} \end{array} \tag{5.13}$$

where ϵ_i and η_i are a set of unknown signs associated to the elements of $\mathrm{KT}/'$, and the sign corresponding to the identity operator is $\epsilon_1 = +1$.

We are now in a position to rule out all but a finite number of models as having no symmetries. Recall that B_{ij} obeys the modified parity rule (5.11). By the conjecture, this implies that unless $m \in \{3, 4, 11, 12\}$, we have

$$B_{1i} = B_{1i} = 0$$

By (5.8) this tells us that A annihilates the vector $\mathcal{S}_{1i}$ from both sides. Using (5.13) and the fact $\mathcal{S}_{1i} = \mathcal{S}_{1i'} > 0$, we obtain the contradictory equations

$$\epsilon_i = \pm \eta_i$$

showing immediately there are no solutions.

Finally we return to the special cases $m \in \{3, 4, 11, 12\}$ that were exempt from the above no-go analysis. Listing the solutions for these cases is a purely finite problem, which can easily be done manually. We find the following results:

- $\underline{m = 3}$ The $m = 3$ model has two symmetries. We specify them by writing out the partition function diagram (5.7) explicitly. For the first one, we have

$$\begin{array}{ccccc} \overline{(\chi_0 + \chi_{\frac{1}{2}})}(\chi_0 - \chi_{\frac{1}{2}}) & \text{——} & \sqrt{2}\,\overline{(\chi_0 + \chi_{\frac{1}{2}})}\,\chi_{\frac{1}{16}} & \text{——} & \sqrt{2}\,\overline{(\chi_0 - \chi_{\frac{1}{2}})}\,\chi_{\frac{1}{16}} \\ | & & & & | \\ \overline{(\chi_0 - \chi_{\frac{1}{2}})}(\chi_0 + \chi_{\frac{1}{2}}) & \text{——} & \sqrt{2}\,\overline{\chi_{\frac{1}{16}}}(\chi_0 + \chi_{\frac{1}{2}}) & \text{——} & \sqrt{2}\,\overline{\chi_{\frac{1}{16}}}(\chi_0 - \chi_{\frac{1}{2}}) \end{array}$$

with anomaly $k = 1$. The other symmetry is given by flipping the diagram upside down, an operation which corresponds to composing with $(-1)^F$, and has the opposite anomaly $k = -1$. These symmetries are of course simply left and right chiral fermion parity of the Majorana fermion.

- $\underline{m = 4}$ The $m = 4$ model has a similar structure, with two symmetries related by $(-1)^F$ and opposite anomalies. We shall therefore only give details of the first one. The partition function diagram (5.7) in this case takes the form

$$
\begin{array}{ccccc}
\begin{array}{r} \overline{(\chi_0 + \chi_{\frac{3}{2}})}(\chi_0 - \chi_{\frac{3}{2}}) \\ + \overline{(\chi_{\frac{3}{5}} + \chi_{\frac{1}{10}})}(\chi_{\frac{3}{5}} - \chi_{\frac{1}{10}}) \end{array} & \text{---} & \begin{array}{r} \sqrt{2}\,[\,\overline{(\chi_0 + \chi_{\frac{3}{2}})}\,\chi_{\frac{7}{16}} \\ + \overline{(\chi_{\frac{3}{5}} + \chi_{\frac{1}{10}})}\,\chi_{\frac{3}{80}}\,] \end{array} & \text{---} & \begin{array}{r} \sqrt{2}\,[\,\overline{(\chi_0 - \chi_{\frac{3}{2}})}\,\chi_{\frac{7}{16}} \\ + \overline{(\chi_{\frac{3}{5}} - \chi_{\frac{1}{10}})}\,\chi_{\frac{3}{80}}\,] \end{array} \\
| & & & & | \\
\begin{array}{r} \overline{(\chi_0 - \chi_{\frac{3}{2}})}(\chi_0 + \chi_{\frac{3}{2}}) \\ + \overline{(\chi_{\frac{3}{5}} - \chi_{\frac{1}{10}})}(\chi_{\frac{3}{5}} + \chi_{\frac{1}{10}}) \end{array} & \text{---} & \begin{array}{r} \sqrt{2}\,[\,\overline{\chi_{\frac{7}{16}}}\,(\chi_0 + \chi_{\frac{3}{2}}) \\ + \overline{\chi_{\frac{3}{80}}}\,(\chi_{\frac{3}{5}} + \chi_{\frac{1}{10}})\,] \end{array} & \text{---} & \begin{array}{r} \sqrt{2}\,[\,\overline{\chi_{\frac{7}{16}}}\,(\chi_0 - \chi_{\frac{3}{2}}) \\ + \overline{\chi_{\frac{3}{80}}}\,(\chi_{\frac{3}{5}} - \chi_{\frac{1}{10}})\,] \end{array}
\end{array}
$$

and the anomaly is $k = -1$.

- $\underline{m = 11, 12}$ The models with $m = 11, 12$ turn out to have no symmetries.

Exceptionals

The above analysis can also be carried out for the two exceptional models at $m = 11, 12$. The results have the same structure as for $m = 3, 4$ earlier: there are two symmetries, related to each other by $(-1)^F$, with opposite anomalies. Below we list the symmetries for the two models. This time we will only list the partition functions on ${}^{\mathrm{AP}}\boxminus_{\mathrm{AP}}$ and ${}^{\mathrm{AP}}\boxed{\vdots}_{\mathrm{AP}}$, as all others are related to these by the same pattern as in previous cases.

- $\underline{m = 11}$ For the $m = 11$ exceptional, we have

$$
\begin{aligned}
\mathcal{Z}\left[\tau; {}^{\mathrm{AP}}\boxminus_{\mathrm{AP}}\right] &= \sum_{r=1,\mathrm{odd}}^{9} \overline{(\chi_{r,1} + \chi_{r,7} + \chi_{r,5} + \chi_{r,11})}(\chi_{r,1} + \chi_{r,7} - \chi_{r,5} - \chi_{r,11}) \\
\mathcal{Z}\left[\tau; {}^{\mathrm{AP}}\boxed{\vdots}_{\mathrm{AP}}\right] &= \sqrt{2} \sum_{r=1,\mathrm{odd}}^{9} \overline{(\chi_{r,1} + \chi_{r,7} + \chi_{r,5} + \chi_{r,11})}(\chi_{r,4} + \chi_{r,8})
\end{aligned}
$$

with anomaly $k = -1$.

- $\underline{m=12}$ The $m=12$ exceptional is similar, but with the roles of r and s reversed

$$\mathcal{Z}\left[\tau;\, {}^{\mathrm{AP}}\boxminus_{\mathrm{AP}}\right] = \sum_{s=1,\mathrm{odd}}^{11} \overline{(\chi_{1,s}+\chi_{7,s}+\chi_{5,s}+\chi_{11,s})}(\chi_{1,s}+\chi_{7,s}-\chi_{5,s}-\chi_{11,s})$$

$$\mathcal{Z}\left[\tau;\, {}^{\mathrm{AP}}\boxbar_{\mathrm{AP}}\right] = \sqrt{2}\sum_{s=1,\mathrm{odd}}^{11} \overline{(\chi_{1,s}+\chi_{7,s}+\chi_{5,s}+\chi_{11,s})}(\chi_{4,s}+\chi_{8,s})$$

and anomaly $k=1$.

Before going on, we pause to note that the earlier results for the infinite series can also be rewritten to look more like the results above. For $m=3$, we have

$$\mathcal{Z}\left[\tau;\, {}^{\mathrm{AP}}\boxminus_{\mathrm{AP}}\right] = \sum_{r=1,\mathrm{odd}}^{1} \overline{(\chi_{r,1}+\chi_{r,3})}(\chi_{r,1}-\chi_{r,3})$$

$$\mathcal{Z}\left[\tau;\, {}^{\mathrm{AP}}\boxbar_{\mathrm{AP}}\right] = \sqrt{2}\sum_{r=1,\mathrm{odd}}^{1} \overline{(\chi_{r,1}+\chi_{r,3})}\,\chi_{r,2}$$

while for $m=4$ we have

$$\mathcal{Z}\left[\tau;\, {}^{\mathrm{AP}}\boxminus_{\mathrm{AP}}\right] = \sum_{s=1,\mathrm{odd}}^{3} \overline{(\chi_{1,s}+\chi_{3,s})}(\chi_{1,s}-\chi_{3,s})$$

$$\mathcal{Z}\left[\tau;\, {}^{\mathrm{AP}}\boxbar_{\mathrm{AP}}\right] = \sqrt{2}\sum_{s=1,\mathrm{odd}}^{3} \overline{(\chi_{1,s}+\chi_{3,s})}\,\chi_{2,s}$$

5.5 Discussion

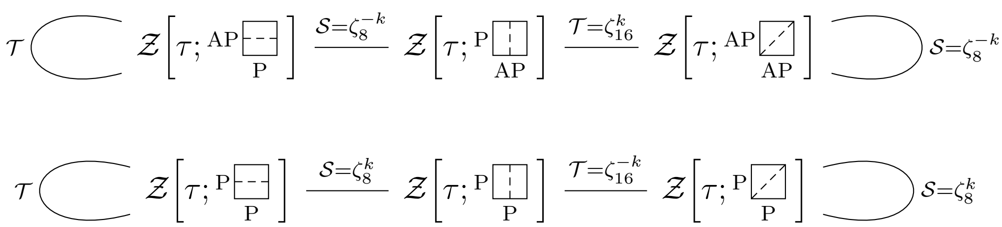
We conclude with some comments on our results. First we return to the earlier claim that the anomalous symmetries, where they exist, exchange the two classes of boundary states. This follows at a glance from the ${}^{\mathrm{AP}}\boxminus_{\mathrm{AP}}$ partition functions of Section 5.4.2. Indeed the coefficient of $|\chi_i|^2$ in this partition function determines the sign with which the symmetry acts on the Ishibashi state $\|i\rangle\rangle$. For example, in the $m=11$ exceptional, the charges of Ishibashi states are

$$\begin{array}{cccc} \|(r,1)\rangle\rangle & \|(r,5)\rangle\rangle & \|(r,7)\rangle\rangle & \|(r,11)\rangle\rangle \\ +1 & -1 & +1 & -1 \end{array}$$

These are precisely the charges needed to exchange the boundary states we wrote down in Section 5.3. The same conclusion also holds in the other models.

Independently of the details of any particular model, the fact the symmetry exchanges the classes can also be understood from the anomaly k being odd. This follows by a stacking argument. Suppose a theory has a boundary state $|a\rangle$ and a symmetry U with odd anomaly k. Then if we stack the theory with $-k$ mod 8 copies of the Majorana fermion, the resulting theory has a boundary state $|a\rangle \otimes |+\rangle^{-k}$ and a non-anomalous symmetry $U \otimes (-1)^{F_L}$. Acting on the state with the symmetry gives a new state $U|a\rangle \otimes |-\rangle^{-k}$, which must lie in the same class since the symmetry is non-anomalous. From Section 5.1.1, we know that $|+\rangle^{-k}$ and $|-\rangle^{-k}$ are in different classes when k is odd. Therefore so too must $|a\rangle$ and $U|a\rangle$, as claimed.

We would also like to return to a subtlety we felt was best left unaddressed in Section 5.4, but are now in a position to close. A theory with a $\mathbb{Z}_2$ symmetry actually has six more partition functions that we did not consider. These fit into an orbit diagram, analogous to (5.7), which looks like



In the models with an anomalous symmetry, all these partition functions are zero. This is because when $k = \pm 1$, the diagram violates the relations $S^2 = (ST)^3 = 1$ that must be satisfied by any minimal model partition functions, so there are no nonzero solutions. With these partition functions now in hand, one might ask why we did not demand the consistency of symmetry-projected traces, such as

$$\frac{1}{2}\left(\mathcal{Z}\left[\tau;\, {}^{\mathrm{AP}}\boxed{\,\vdots\,}_{\,\mathrm{P}}\right] + \mathcal{Z}\left[\tau;\, {}^{\mathrm{P}}\boxed{\,\vdots\,}_{\,\mathrm{P}}\right]\right)$$

which counts bosonic states on a periodic circle frustrated by a symmetry defect. The answer becomes clear if we look at the Majorana fermion. Here, the answer would be

$$\frac{1}{\sqrt{2}}\,\overline{(\chi_0 + \chi_{1/2})(\tau)}\,\chi_{1/16}(\tau)$$

which is inconsistent even for a fermionic theory. The issue is of course that with an unpaired Majorana mode, there is no separation into fermionic and bosonic states, and such symmetry-projected traces are meaningless. We conclude that there is no need to demand consistency of the symmetry-projected traces, and the constraints we have imposed on our symmetries are all that there are.

Next we attempt to shed some light on the observation that, from our results, the following properties of fermionic minimal models appear to be equivalent:

1. equal class sizes	$\Longleftrightarrow$	2. existence of an anomalous symmetry	$\Longleftrightarrow$	3. vanishing of the PP sector partition function

Indeed, all three conditions are satisfied by $m = 3, 4$ from the infinite series and $m = 11, 12$ from the exceptionals. It is natural to ask whether the above superficial equivalences are in fact honest equivalences. Below we will outline some arguments that show that for some for them at least, the equivalences are indeed honest.

$2 \Rightarrow 1$

As we have seen, when an anomalous symmetry is present it exchanges the two classes of boundary states. This trivially implies they have equal sizes.

$3 \Rightarrow 1$

Here we can argue the contrapositive as follows. SPT classes form an affine space: they can be compared, but there is no preferred choice of one phase as trivial. If however a model has boundary state classes of different sizes, then we seemingly have a way to distinguish one class over the other. But this is not a contradiction, as we should remember that for every fermionic minimal model listed in Section 5.2, there is another related by stacking with $(-1)^{\text{Arf}}$ [58, 101, 102], and whose boundary state class sizes are reversed. So it is acceptable for a theory to have different class sizes as long as $\mathcal{Z}$ and $\mathcal{Z}(-1)^{\text{Arf}}$ are different theories. The condition for this is that $\mathcal{Z}$ does not vanish in the PP sector.

$2 \Rightarrow 3$

This observation, that anomalies can force the vanishing of the Ramond sector, has been noted in various places in the literature. See for example [94], where it was explained using the algebra obeyed by $(-1)^{F_L}$, $(-1)^{F_R}$ and $(-1)^F$.

For the above implications, we do not know of arguments in the other direction.

Finally we comment on our Conjecture in Section 5.4.2. In earlier work on the bosonic minimal models, the role of this conjecture was played by Result 3 of [96]. This result was proved by relating it to a theorem about the classification of simple factors of Jacobians of Fermat Curves [104]. Given the close connection between the fermionic and bosonic minimal models, it is natural to ask if our conjecture has a similar interpretation in terms of Fermat curves. Whether or not it does remains an open question.

Chapter 6

Summary and Outlook

We have defined and studied boundary states in various fermionic CFTs, with the goal of connecting them to the fermion-monopole problem and to symmetry-protected topological phases. Below, we review our main novel results and how they relate to these goals.

We began by introducing a large family of boundary states (2.24) preserving abelian chiral symmetries for N Dirac fermions. Such boundary states correspond to Haldane's abelian gapped phases [70], as had previously been shown in [37]. Our first contribution was to investigate the consistency of this family as a whole. This problem had not previously been addressed due to both falling outside the scope of RCFT, and due to the extra complications that arise from working with fermionic CFTs. We found the family is indeed consistent, but that this comes about in an intricate way from the formula (2.29),

$$G[\mathcal{R}, \mathcal{R}'] = \frac{\sqrt{\mathrm{Vol}(\Lambda[\mathcal{R}])\,\mathrm{Vol}(\Lambda[\mathcal{R}'])}}{\mathrm{Vol}(\Lambda[\mathcal{R}, \mathcal{R}'])} \sqrt{\det{}'(\mathbb{1} - \mathcal{R}^T \mathcal{R}')}$$

provided that we also revise our notion of a consistent set of boundary states in fermionic CFTs to include unpaired Majorana zero modes. The formula (2.29) also yielded a mapping from boundary states to the $\mathbb{Z}_2$ classification of into SPT phases preserving $(-1)^F$ in two dimensions. This mapping, which hinged upon $\mathbb{F}_2$ linear-algebraic properties of the matrix $\mathcal{R}$, appears novel and still needs to be better understood.

Next we explored boundary RG flows starting from these states. Our motivation was to potentially show the existence of further boundary states *not* of the form (2.24), by proving that they must arise at the ends of certain RG flows. Instead, we found the opposite. Our results suggest a simple picture in which any[1] RG flow from one of the states (2.24) always ends at another such state, in a way consistent with symmetries, SPT class, and the g-theorem. This consistency is captured by the relation (3.1),

$$g_{IR} = g_{UV}\sqrt{\dim(\mathcal{O})}$$

which states that the simplest IR state with the right symmetry and SPT class also obeys the g-theorem. These results could be interpreted as weak evidence that no other boundary states exist. However there are several reasons to be more cautious. First, on the general grounds of Section 1.2, we expect such states do exist. Second, the states (2.24) span a tiny subspace of the span of all Ishibashi states, whereas in all CFTs where the full set of boundary states is known, their spans are the same. (We acknowledge that this could be a case of the Streetlight Effect.) Our conclusion is therefore that these other states are merely more evasive.

We then derived a simple criterion for the non-abelian enhanced symmetry of a boundary state (2.24). Its root system is given by (4.2),

$$\Delta[\mathcal{R}] \coloneqq \left\{ \lambda \in \Lambda[\mathcal{R}] \; : \; |\lambda|^2 = 2 \right\}$$

and it sits inside $\mathfrak{so}(2N)_L \times \mathfrak{so}(2N)_R$ according to (4.8). Our main motivation for studying this property came from the fermion-monopole problem, in which it is important to know exactly what non-abelian symmetries are and are not preserved. Our results however do not allow us to answer the “first question” posed in the introduction! For our example chiral gauge theory (1.2) with charges $\{1, 5, -7, -8, 9\}$, the $SU(2) \times U(1)$ symmetry needed of the boundary state is simply not of the form (4.2). None of the boundary states (2.24) are therefore suitable for defining a magnetic line operator in this gauge theory. This situation is in fact generic among chiral gauge theories. We learn that to solve this problem, we need to venture away from the “quasi-rational” regime $c = \mathrm{rk}(G)$ and into the non-rational regime $c > \mathrm{rk}(G)$. Doing so will likely require radically different techniques to the ones used here, and may not even be possible analytically.

At the same time we investigated when a boundary state can preserve chiral fermion parity. Here our main result is the proof of a lattices theorem that

$$\Lambda[\mathcal{R}] \subseteq D_N \quad \Longrightarrow \quad 4|N$$

[1]Any that is, except for flows initiated by marginal non-symmetry operators. These remain an intriguing mystery. If exactly marginal, then there is a novel extended symmetry connecting states with different $\mathcal{R}$ matrices. If marginally relevant, they may lead to novel boundary states. It is also entirely possible they are all marginally irrelevant, in which case there is simply nothing to find.

This theorem is a manifestation of the $\mathbb{Z}_8$ classification of SPT phases preserving $(-1)^{F_L} \times (-1)^{F_R}$ symmetry in three dimensions, with vanishing gravitational anomaly, from the perspective of chiral boundary states. We also formulated conjectures on the spectrum of allowed $g_{\mathcal{R}}$ values of states preserving this symmetry.

In the final section we expanded our scope to include the fermionic minimal models. These form a natural generalisation of the fermionic models we have considered so far, and one can ask the same questions as before. Our first result here is the set of boundary states for these models, given in Section 5.3. We showed that our earlier $\mathbb{Z}_2$ classification of boundary states into SPT phases generalises to this case. Second, we performed a careful application of the modular bootstrap, taking into account the possible complications of fermionic anomalies, to determine the symmetries of the fermionic minimal models. These are given in Section 5.4, and are consistent with [91]. The essential novelty of our approach is in the interplay between Galois-theoretic techniques and the hallmark factors of $\sqrt{2}$ and $e^{2\pi i/8}$ that arise from fermions and anomalies. We used this to formulate the combinatorial conjecture (5.12), which should have some interpretation in terms of Fermat curves; the precise connection still remains a mystery.

We close with a discussion of the limitations of our techniques, and some ways in which these limitations could be addressed in the future. The central difficulty is that even for as simple a theory as a pair of Dirac fermions, determining all the boundary states is an open problem.

The above difficulty can be well illustrated by a simple count of the Ishibashi states for N Diracs when one imposes only conformal symmetry. From the partition function $\sum_{\lambda \in \mathbb{Z}^N} q^{\lambda^2/2}/\eta(\tau)^N$ of the holomorphic sector, and the fact that a single Verma module with $h > 0$ has partition function $q^h/\eta(\tau)$, we can read off the generating function for highest weight states in the holomorphic sector,

$$\sum_{\lambda \in \mathbb{Z}^N} q^{\lambda^2/2} \quad \times \quad \sum_{n \geq 0} q^n \, p_{N-1}(n)$$

where $p_k(n)$ is the k-coloured partition counting function. If the coefficient of q^n in the above series is a_n, then for each $n \geq 0$ there are a_n^2 Ishibashi states, each with an undetermined coefficient in the boundary state.[2] The partition numbers $p_k(n)$ grow rapidly with n if $k \geq 1$: schematically, as fast as $e^{\sqrt{n}}$. This means that for any number $N \geq 2$ of Dirac fermions, there is an exponential explosion in the amount of unknown data to be determined. This is the reason the standard boundary states programme is unworkable in this case. The same issue is present more generally in any theory where the amount of symmetry imposed is less than the central charge, as expressed by $c > \mathrm{rk}(G)$; we term this situation the 'irrational case'. Progress has only been made in the 'quasi-rational' case, meaning an irrational theory with $c = \mathrm{rk}(G)$, and the rational case, meaning the theory is an RCFT.

These issues highlight our profound ignorance as to the landscape of boundary states in the irrational case. Is the spectrum of g discrete, as in the quasi-rational case, or continuous? Do boundary states form moduli spaces with dimension equal to the number of symmetries they break, as in the quasi-rational case, or moduli spaces of much larger dimension? Does the set of boundary states use up all available Ishibashi states, as in the rational case, or only a subset? All these questions are related, and remain to be answered.

A promising avenue appears to be the correspondence of boundary conditions with gapped phases. Recall from Section 1.2 that in $d \geq 3$ dimensions, the question of symmetric mass generation is essentially solved; any theory can be symmetrically gapped preserving any non-anomalous global symmetry. This allows boundaries to be defined with the same property. In d=2 dimensions things are less clear. The only known examples of symmetric mass generation are Haldane potentials [70], corresponding to the chiral boundary states we have studied, and the marginally relevant current-current interactions we reviewed in Section 1.2.2 that give rise to the Maldacena-Ludwig state [20, 37], also of the form that we have studied. Interactions capable of gapping preserving more interesting non-abelian symmetries are not known, but may be given by suitable combinations of current-current terms and cosine potentials. If so, then the relation of [36] allowing BCFT data to be extracted from the entanglement spectrum of these gapped phases may be useful to get a handle on the corresponding boundary states.

The boundary conformal bootstrap may also prove useful, especially if it can be combined with data gleaned from the previous approach. Technical obstacles in implementing such a programme however include the fact boundary states always form large moduli spaces corresponding to the breaking of $SO(2N)_L \times SO(2N)_R$ symmetry, and the existence of so-called Janik states [105], boundary states with pathological continuous spectrum that appear to have no sensible correspondence with SPT phases. These states must be ruled out of any search, but how to precisely characterise such states is unclear.

Finally we mention models in which interacting degrees of freedom on the boundary are taken directly into account. Such models are as old as the 'second' fermion-monopole

[2] For the purposes of this illustration, we have ignored the fact that at $h = 0$, the partition function $q^h/\eta(\tau)$ decomposes further into irreducible Verma modules. This will make the count even larger.

problem discussed in Chapter 1, for example the model of [18], which succeeds in reproducing the dyon boundary state using a rotor degree of freedom. These models also bear a tantalising similarity to the notion of a cMPS state introduced within the DMRG community. It would be useful to understand the meaning of the Affleck-Ludwig central charge g from these other points of view.

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