Is Deterministic Deployment Worse than Random Deployment for Wireless Sensor Networks?

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Abstract— Before a sensor network is deployed, it is important to determine how many sensors are required to achieve a certain coverage degree. The number of sensor required for maintaining k-coverage depends on the area of the monitored region, the probability that a node fails or powers off (to save energy), and the deployment strategy. In this paper, we derive the density required to maintain k-coverage under three deployment strategies: (i) nodes are deployed as a Poisson point process, (ii) nodes are uniformly randomly distributed, (iii) nodes are deployed on regular grids. Our results show that under most circumstances, grid deployment renders asymptotically lower node density than random deployment. These results override a previous conclusion that grid deployment may render higher node density than random node distributions.

I. Introduction

Research on wireless sensor networks has received tremendous attention in recent years due to the advances in MEMS technology, as well as their potentially wide applications in both civilian and military environments, such as environmental monitoring, industrial sensing and diagnostics, and information collecting for battlefield awareness [2], [3], [5], [7].

In most applications, a sensor network is used to monitor a certain region, and it is desirable that every point in a region R is monitored by k sensors, where k is often determined by applications. For example, for event detection, $k = 1$ may suffice, while for event localization and target tracking, multiple nodes are required to simultaneously monitor any point in a region R (so that the triangulation technique can be applied). An interesting and important question is then to determine how many sensors need to be deployed in an area in order to provide k-coverage. This has to figure in the fact that sensors are often considered not reliable, and they may also switch between the on and off states to save energy.

The problem of sensor deployment is often formulated as follows. Given a square region R with area A, how many sensors are needed to ensure k-coverage, assuming that (i) each sensor can cover a circular region centered at itself and with radius r; and (ii) each sensor has a probability p to be active. In addition, there are two fundamentally different ways to deploy sensors: random deployment (such as Poisson point process or uniform distribution), or deterministic deployment (such as grid deployment). Intuitively, deterministic deployment seems to require a fewer number of sensor nodes to achieve a given degree of coverage k than random deployment. In this paper, we would like to investigate whether or not this intuition holds true.

Zhang and Hou [10] first studied the density requirement for k-coverage of a square region with side length l. Assuming Torus convention [4] and that the sensors have sensing radius $r = 1/\sqrt{\pi}$, are distributed as a Poisson point process, and are always active, they showed that a necessary and sufficient condition for satisfying k-coverage with high probability (i.e., its probability tends to 1) is that the node density $\lambda = \log l^2 + (k+1) \log \log l^2 + c(l)$ and $c(l) \to \infty$. In particular, they avoided the boundary problem by assuming Torus convention.

Kumar *et al* [6] studied a similar problem in a square with unit area and considered the boundary conditions. In addition, three deployment strategies are considered: (a) grid deployment, (b) uniformly random node distribution, and (c) Poisson point process. Let n denote the number of sensors in the unit area, r the sensing range, p the probability that a node is active. They show that (i) for grid deployment, if there exists a slowly growing function $\phi(np)$ (i.e., $\phi(np)$ tends to infinity but grows slower than $\log \log n$) such that

$$np\pi r^2 \geq \log(np) + \phi(np)(1 + \sqrt{p \log(np)}) + k \log \log(np), \quad (1)$$

then the whole region R is k-covered with high probability; and (ii) for random deployment with uniform distribution, if

$$np\pi r^2 \geq \log(np) + k \log \log(np) + \phi(np), \quad (2)$$

then the region R is k-covered with high probability. Comparing Eqs. (1) and (2), it seems to conclude that *deterministic deployment requires more sensor nodes than random deployment to achieve the same level of coverage degree.*

The counter-intuitive results presented in [6] have motivated our study in this paper. We consider the case when the sensing range is fixed at $1/\sqrt{\pi}$ but the area A of the monitored square region R tends to infinity.[1] By performing rigorous analysis, we show that (i) for random deployment with uniform distribution, if

$$np/A = \log A + 2k \log \log A + c(A), \quad (3)$$

and $c(A) \to \infty$ as $A \to \infty$, the square region R is k-covered with high probability; (ii) for grid deployment, if

$$(-\log(1-p))n/A$$
$$= \log A + 2k \log \log A + 2\sqrt{-2\pi \log A \log(1-p)} + c(A), (4)$$

and $c(A) \to \infty$ as $A \to \infty$, the region R is k-covered with high probability. Comparing Eqs. (3) and (4), we find that the number of nodes required in the grid deployment is less than that in the random deployment if $0 < \epsilon \leq p \leq 1 - \epsilon < 1$

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[1] This simplifies our derivation and the results can be readily converted into the case presented in [6].

for some constant $\epsilon > 0$, since the most significant terms in the right hand sides of both Eqs. (3) and Eq. (4) are the same (i.e., $\log A$), and $p < -\log(1 - p)$. A more careful analysis shows that even for $p \to 0$ as $A \to \infty$, the number of nodes required for k-coverage using grid deployment is asymptotically less than or equal to that using random node distribution. Therefore, we conclude the intuition holds true if $p \leq 1 - \epsilon < 1$!

The seemingly contradictory results between the paper [6] and this paper result from three factors. First, all the above results are sufficient conditions. Therefore, we have obtained much weaker sufficient conditions on node density required to maintain k-coverage (especially for grid deployment). Second, although both Kumar *et al.*[6] and we have necessary conditions as well, the necessary condition for grid deployment in [6] is obtained under the assumption that $p \to 0$ as the number of nodes goes to 0, which is actually a special case of the results in this paper. In addition, there is a huge gap between the necessary condition and the sufficient condition in [6]. In our work, in the case that nodes are distributed randomly with uniform distribution or as a Poisson point process, there is no gap between the necessary and sufficient conditions. In the case of grid deployment, the gap between the necessary and sufficient conditions are much smaller than that in [6], and most of the gap is caused by the variance of the number of lattice points contained in a circle (which is called Gauss's Circle problem [1]). Moreover, for $p = O(1/\log A)$, the gap between the necessary and sufficient conditions diminishes. Third, we would like to point out that both the Eqs. (1) and (2) (from [6]) contain a minor mistake at the coefficient of the term of $\log \log np$, which will be further discussed in Section VI.

In addition to the difference in the conclusion, our proof technique is different from [6]. To prove the sufficient conditions, we use the linearity property of expectations and Markov inequality instead of Janson's inequality. The linearity property of expectations does not require any conditions on the random variables and simplifies the proof greatly. To prove the necessary conditions, we consider the k-vacancy area instead of counting a certain number of grid points, which enables us to obtain tighter conditions of the node density required to maintain k-coverage.

The rest of the paper is organized as follows. In Section II, we state the assumptions made throughout the paper and the system model. Then we present our derivation on the number of nodes required to provide k-coverage in Section III, IV, and V for the cases that nodes are distributed as a Poisson point process, with uniformly random distribution, and deployed on grids, respectively. Following that, we summarize related work in Section VI, and conclude the paper in Section VII.

II. MODEL ASSUMPTIONS

A. Assumptions on the System Model

We assume the region R to be monitored is a square region with area A and side length $\sqrt{A}$. Each sensor node can detect an event of interest within a distance of r, and this distance is termed as the *sensing range*. The disk centered at a sensor

node and with a radius of r is termed as the *coverage disk* of this node. Without loss of generality, we assume that each sensor node has a sensing range of $r = \frac{1}{\sqrt{\pi}}$ and thus each sensor node can cover a disk of unit area. We assume each sensor has an independent probability $p < 1$ to be active and p may be either a constant or dependent on A. When we say a region is k-covered, it means every point in the region is covered by at least k active nodes. We assume $A >> 1$.

We consider sensor nodes are deployed as one of the three models.

1. *Poisson point process:* the nodes form a Poisson point process with density D.
2. *uniform distribution:* n nodes are randomly, independently placed with uniform distributions. In this case, we define $D = n/A$.
3. *grid deployment:* $n = k^2$ nodes are regularly placed on $\sqrt{n} \times \sqrt{n}$ grids. In this case, we define $D = n/A$.

We assume k is a finite constant value, and consider most of the variables are functions (which may be constant) of A and investigate the asymptotic probability as $A \to \infty$. For two functions f and g, We denote $f = O(g)$ if $f(A) \leq Cg(A)$ for some constant $C > 0$ and sufficiently large A, and similarly, $f = \Omega(g)$ if $f(A) \geq Cg(A)$. We denote $f = o(g)$ if $f(A)/g(A) \to 0$ as $A \to \infty$, and $f \sim g$ if $f(A)/g(A) \to 1$ as $A \to \infty$. Therefore, $f = o(1)$ means f goes to 0 as $A \to \infty$, and $f = (1 + o(1))g$ means $f \sim g$. We also denote $f \lesssim g$ if $f(A) \leq g(A)(1 + o(1))$ and $f \gtrsim g$ if $f(A) \geq g(A)(1 + o(1))$. Throughout the paper, the logarithm is of the natural base.

B. Major Results

We obtain the following results. Assume D and p are defined as above.

1. In the case that nodes are deployed as a Poisson point process, Let $Dp = \log A + 2k \log \log A + c(A)$. If $c(A) \to \infty$, then the region R is k-covered with high probability, and if $c(A) \leq C$, then

$$P(\text{the region } R \text{ is } k\text{-covered})$$
$$\leq 1 - \frac{1}{1 + 32e^{C/2}2^{k-2}(k+1)!/\sqrt{\pi}} < 1. \quad (5)$$

2. In the case that nodes are deployed according to uniformly random distributions, the results are identical to those in the Poisson point process.

3. In the case of grid deployment, assume $0 < p \leq 1 - \epsilon < 1$ for some constant ϵ. If $-D \log(1 - p) = \log A + 2k \log \log A + 2\sqrt{-2\pi \log A \log(1 - p)} + c(A)$, and $c(A) \to \infty$ as $A \to \infty$, then the probability that the region R is k-covered tends to 1. If $-D \log(1 - p) = \log A + 2(k - 1) \log \log A - 2\sqrt{-2\pi \log A \log(1 - p)} - c(A)$, and $c(A) \to \infty$ as $A \to \infty$, then the probability that the region R is NOT k-covered tends to 1.

Now we show that using grid deployment requires asymptotically less or equal node density than using uniform random distribution in all cases of $0 < p \leq 1 - \epsilon < 1$ for any positive constant ϵ ($0 < \epsilon < 1$). If $p \geq \epsilon > 0$, since $p < -\log(1 - p)$, the result follows immediately. If $p = O(\log^{-1} A)$, the extra term $2\sqrt{-2\pi \log A \log(1 - p)}$ in grid deployment is in the

order of $\sqrt{p \log A}$ and bounded by a constant, so the result holds. In the last case, we consider $p \to 0$ but $p \log A \to \infty$ as $A \to \infty$. By Taylor series expansion, we have $-\log(1 - p) = p + \xi p^2$ for $p < \epsilon$ where $\xi > 0$ (since the second order derivative of $-\log(1 - p)$ at $p = 0$ is $1 > 0$). Hence, the sufficient condition for grid deployment is

$$Dp + \xi Dp^2$$
$$= \log A + 2k \log \log A + 2\sqrt{-2\pi \log A \log(1 - p)} + c(A). \quad (6)$$

Since $Dp^2 \sim (\log A)p \to \infty$ and $(\log A) \log(1 - p) \sim (\log A)p \to \infty$, $\xi Dp^2 >> 2\sqrt{-2\pi \log A \log(1 - p)}$. Therefore, the density requirement in grid deployment is again less than that in the uniform random distribution.

III. POISSON DISTRIBUTION

In this model, the nodes form a Poisson point process with density D and each node is active independently with probability p. By the property of Poisson point process, the active nodes form a Poisson point process with density $\lambda = Dp$. Therefore in this section, we simply consider a sensor network deployed as a Poisson point process with density λ and every node is active. In the following we establish a sufficient and a necessary condition for k-coverage in such a network. We denote $P(k\text{-coverage})$ as the probability that the monitored square region R is k-covered.

A. Sufficient condition

Theorem 1 *Let $Dp = \lambda = \log A + 2k \log \log A + c(A)$, where A is the area of the monitored square region. If $c(A) \to \infty$ as $A \to \infty$, then $P(k\text{-coverage}) \to 1$.*

Before we delve into the proof, we want to emphasize that we don't require how fast $c(A)$ should converge to infinity. The theorem is proved under the assumption that $c(A) = o(\log \log A)$ as $A \to \infty$. However, if it converges faster, the theorem still holds because $P(k\text{-coverage})$ is clearly an increasing function of λ and $c(A)$.

Proof. Let's first divide the area into small grids with side length $s = \sqrt{2}ur$ where $u = 1/(\log A)$. The area of each grid is s^2 and the number of grids is A/s^2. Denote X_i as the indicator function of whether grid i is NOT completely k-covered (i.e., $X_i = 1$ if grid i is NOT completely k-covered and 0 otherwise). Denote X as the number of grids that are not completely k-covered. Therefore $X = \sum_i X_i$. The key idea below is to show that as $A \to \infty$, $E[X] \to 0$, and therefore by Markov inequality, $P(X > 0) = P(X \geq 1) \leq E[X] \to 0$, and $P(k\text{-coverage}) = P(X = 0) = 1 - P(X > 0) \to 1$ as $A \to \infty$.

In order for a grid i to be completely k-covered, it is sufficient that there are at least k sensor nodes within a disk centered at the center of the grid and with radius $r - \frac{\sqrt{2}}{2}s = (1 - u)r$, denoted as $B_i((1 - u)r)$. Equivalently, if a grid i is not k-covered, then there are less than k nodes in the disk $B_i((1 - u)r)$. Note that we assume there are sensor nodes only inside the monitored square region R, and thus we shall only be interested in the region of $B_i((1 - u)r) \cap R$ and the nodes inside it. The area of the disk $B_i((1 - u)r)$ is $(1 - u)^2$.

We consider three types of grids: inner grids, side grids and corner grids, where inner grids are at least $r(= 1/\sqrt{\pi})$ distance away from any side of the square, side grids are at most r distance away from one side of the square and at least r distance away from any other three sides, corner grids are at most r distance away from two adjacent sides.

For an inner grid i, $B_i((1 - u)r)$ (the area of which is $(1 - u)^2$) is completely contained in the monitored square region, therefore,

$$
\begin{aligned}
E[X_i] &= P(\text{grid } i \text{ is not } k\text{-covered}) \\
&\leq P(\text{there are less than } k \text{ nodes inside } B_i((1 - u)r)) \\
&= e^{-\lambda(1-u)^2} \sum_{i=0}^{k-1} \frac{(\lambda(1 - u)^2)^i}{i!} \\
&= e^{-\lambda(1-u)^2} \frac{(\lambda(1 - u)^2)^{k-1}}{(k - 1)!} \cdot (1 + o(1)), \quad (7)
\end{aligned}
$$

where the last equality holds because $\lambda \to \infty$ and $i < k$ is assumed to be bounded, and thus the last term in the summation dominates all other (finitely many) terms.

Since there are at most A/s^2 inner grids, the expectation of total number, X^I, of un-k-covered inner grids, is

$$
\begin{aligned}
E[X^I] &\leq \frac{A}{s^2} E X_i \\
&= \frac{A}{2u^2 r^2} e^{-\lambda(1-u)^2} \frac{(\lambda(1 - u)^2)^{k-1}}{(k - 1)!} \cdot (1 + o(1)) \\
&= \frac{\pi A}{2u^2} e^{-\lambda(1-u)^2} \frac{(\lambda(1 - u)^2)^{k-1}}{(k - 1)!} \cdot (1 + o(1)),
\end{aligned}
$$

where the last equality holds because $\pi r^2 = 1$. We take the logarithm (with natural bases) on both sides, and obtain

$$
\begin{aligned}
\log E[X^I] \leq{} & \log A + \log(\pi/2) - 2 \log u - \lambda(1 - u)^2 + \\
& (k - 1)(\log \lambda + 2 \log(1 - u)) - \log(k - 1)! + o(1). \quad (8)
\end{aligned}
$$

Since $u = 1/\log A \to 0$ as $A \to \infty$, $\log(1 - u) \sim -u \to 0$. By the assumption, $\lambda = \log A + 2k \log \log A + c(A)$. Based on the discussion before the proof, we can assume $c(A) = o(\log \log A)$, so $\log \lambda = \log \log A + o(1)$. Plugging these results into Eq. (8), we obtain

$$
\begin{aligned}
\log E[X^I] \leq{} & (2u - u^2) \log A + (k + 1 - 2k(1 - u)^2) \log \log A \\
& -(1 - u)^2 c(A) + \log(\pi/2) - \log((k - 1)!) + o(1).
\end{aligned}
$$

So if $c(A) \to \infty$ (actually a weaker condition is OK in this case), $\log E[X^I] \to -\infty$ and $E[X^I] \to 0$.

For a side grid, part of the disk that is centered at the grid center and with radius $(1 - u)r$ is out of the monitored region. We need to estimate how much of the disk area is contained in the monitored square region. We assign each side grid a row index according to its distance to the side of the square region. The row closest to the side has index 0.

For a side grid g at row j, the distance from its center to the closest side is $x = (j + 1/2)s$. Denote $B_g(t)$ as the disk centered at the center of the grid g and with radius t. Let v denote the area of the part of the disk $B_g(t)$ that is contained in the monitored square region (assuming the disk $B_g(t)$ only intersects one side of the square). The bound of v is given in the following lemma.

Lemma 1 $(\pi t^2 + \pi x t)/2 \leq v \leq \pi t^2$, *and* $v \leq \pi t^2/2 + 2xt$.

The first inequality follows from the result on the area of $B_2 - B_1$ in [10], where B_1 and B_2 are two intersecting disks. The second inequality is because partial disk area is certainly not larger than the area of the whole disk, πt^2. The third inequality can be obtained by simple geometric analysis: if we draw a rectangle with side length $2r$ and x, it will clearly contain the intersection of R and the half disk that intersects with one side of the region R. We point out that a less stringent result $v \geq (\pi t^2 + 2xt)/2$ (for the first inequality) also suffices for the following proof. Since we are considering a disk centered at the center of a grid at row j with radius $r(1-u)$, the area of the part of the disk that is inside the square region is

$$
\begin{aligned}
v &\geq \frac{\pi r^2(1-u)^2 + \pi r(1-u)(j+1/2)s}{2} \\
&\geq \frac{(1-u)^2 + j\sqrt{2}u(1-u)}{2}
\end{aligned} \tag{9}
$$

since $s = \sqrt{2}ur$ and $\pi r^2 = 1$. If a side grid g at row j is not k-covered, then there are fewer than k nodes in $B_g(r(1-u)) \cap R$. Therefore (note $\sum_{i=0}^{k-1} e^{-x}\frac{x^i}{i!}$ is a decreasing function of x),

P(a side grid at row j is not k-covered)

$$
\begin{aligned}
&\leq \sum_{i=0}^{k-1} e^{-\lambda\frac{(1-u)^2+j\sqrt{2}u(1-u)}{2}} \frac{(\lambda((1-u)^2+j\sqrt{2}u(1-u))/2)^i}{i!} \\
&= e^{-\lambda\frac{(1-u)^2+j\sqrt{2}u(1-u)}{2}} \frac{(\lambda((1-u)^2+j\sqrt{2}u(1-u))/2)^{k-1}}{(k-1)!} \\
&\quad \cdot (1+o(1)) \\
&\leq e^{-\lambda\frac{(1-u)^2+j\sqrt{2}u(1-u)}{2}} \frac{(\lambda(1-u)^2)^{k-1}}{(k-1)!} \cdot (1+o(1))
\end{aligned} \tag{10}
$$

Since there are four side regions in the square and at most r/s rows in each side region and at most $\sqrt{A}/s$ grids in each row of a side region, the expectation of the number X^S of the side grids that are not k-covered can be written as

$$
\begin{aligned}
&E[X^S] \\
&= \frac{4\sqrt{A}}{s} \sum_{j=0}^{r/s} P(\text{a grid at row } j \text{ is not } k\text{-covered}) \\
&\leq \frac{4\sqrt{A}}{s} \sum_{j=0}^{r/s} e^{-\lambda\frac{(1-u)^2+j\sqrt{2}u(1-u)}{2}} \frac{(\lambda(1-u)^2)^{k-1}}{(k-1)!} \cdot (1+o(1)) \\
&\leq \frac{4\sqrt{A}}{s} e^{-\lambda\frac{(1-u)^2}{2}} \frac{(\lambda(1-u)^2)^{k-1}}{(k-1)!} \cdot \frac{1+o(1)}{1-e^{-\lambda\sqrt{2}u(1-u)/2}}
\end{aligned}
$$

Again, we take the logarithm on both sides, and obtain (notice $s = \sqrt{2}ur = \sqrt{2/\pi}/\log A$)

$$
\begin{aligned}
\log E[X^S] &\leq \log 4 + \frac{1}{2}\log A + \frac{1}{2}\log(\pi/2) + \log\log A \\
&\quad -\lambda\frac{(1-u)^2}{2} + (k-1)(\log\lambda + 2\log(1-u)) \\
&\quad -\log((k-1)!) - \log(1-e^{-\lambda\sqrt{2}u(1-u)/2}) + o(1)
\end{aligned}
$$

Since $\lambda = \log A + 2k\log\log A + c(A))$ and $c(A) = o(\log\log A)$, $\log\lambda = \log\log A + o(1)$. Therefore,

$$
\begin{aligned}
\log E[X^S] &\leq \frac{2u-u^2}{2}(\log A + 2k\log\log A) - \frac{(1-u)^2}{2}c(A) \\
&\quad + \frac{1}{2}\log(8\pi) + 2(k-1)\log(1-u) \\
&\quad -\log((k-1)!) - \log(1-e^{-\lambda\sqrt{2}u(1-u)/2}) + o(1).
\end{aligned}
$$

Since k is assumed to be a fixed integer and $u = 1/\log A$, as $A \to \infty$, most of the terms converge to a finite value except $-\frac{(1-u)^2}{2}c(A)$ which converges to $-\infty$. Therefore, $E[X^S] \to 0$ as $A \to \infty$.

For a corner grid, if we draw a disk centered at the center of the grid and with radius $r(1-u)$, at least a quarter of the disk is inside the monitored square region. Therefore,

P(a corner grid is not k-covered)

$$
\begin{aligned}
&\leq \sum_{i=0}^{k-1} \frac{e^{-\lambda(1-u)^2/4}(\lambda(1-u)^2/4)^i}{i!} \\
&= \frac{e^{-\lambda(1-u)^2/4}(\lambda(1-u)^2/4)^{k-1}}{(k-1)!}(1+o(1))
\end{aligned} \tag{11}
$$

The number of corner grids is $4\frac{r^2}{s^2} = \frac{2}{u^2}$. Hence, the expectation of the number X^C of the corner grids that are not k-covered is

$$
E[X^C] \leq \frac{2}{u^2} e^{-\lambda(1-u)^2/4} \frac{(\lambda(1-u)^2/4)^{k-1}}{(k-1)!}(1+o(1)). \tag{12}
$$

Since $1/u = \log A$ is in the same order of λ, $E[X^C] \to 0$ as long as $\lambda \to \infty$.

Therefore, the expectation of the total number of un-k-covered grids $X = X^I + X^S + X^C$ converges to 0 as the area $A \to \infty$ if λ is given as in the theorem. This is all we needed for the proof of the theorem. $\qquad \square$

B. Necessary condition

Let the k-vacancy V_k denote the area of the region that is covered by less than k nodes, and $\chi_k(Z)$ denote the indicator function of whether a point Z is covered by less than k nodes, i.e.,

$$
\chi_k(Z) = \begin{cases} 1, & \text{if less than } k \text{ nodes cover the point } Z; \\ 0, & \text{otherwise.} \end{cases} \tag{13}
$$

To derive the necessary condition for k-coverage, we first derive the bounds on $E[V_k]$ and $E[V_k^2]$.

Proposition 1 *If $Dp = \lambda = \log A + 2k\log\log A + C$, where C is a constant, then*

$$
E[V_k](\log A)^2 \geq \frac{\sqrt{\pi}}{e^{C/2}2^{k-2}(k-1)!},
$$

as $A \to \infty$.

Proof. Since we are interested in deriving a lower bound, we only consider the k-vacancy in the side area of the square, i.e., those locations which are at most r distance away from one side and at least r distance away from all other sides. Without loss of generality, we consider a point Z in the side area with coordinate (x, y), where $0 \leq x \leq r, r \leq y \leq \sqrt{A} - r$. Now the expectation of the k-vacancy indicator function $\chi_k((x,y))$ is

$$
\begin{aligned}
&E[\chi_k((x,y))] \\
&= P((x,y) \text{ is not } k\text{-covered}) \\
&= P(B_{(x,y)}(r) \cap R \text{ contains less than } k \text{ nodes }),
\end{aligned} \tag{14}
$$

where $B_{(x,y)}(r)$ denotes the disk centered at (x, y) with radius r. Since the area of $B_{(x,y)}(r) \cap R$ is not larger than $\frac{1}{2} + 2xr$ by Lemma 1,

$$
\begin{aligned}
E[\chi_k((x,y))] &\geq e^{-\lambda(1/2+2xr)} \sum_{i=0}^{k-1} \frac{(\lambda(1/2+2xr))^i}{i!} \\
&\geq e^{-\lambda(1/2+2xr)} \sum_{i=0}^{k-1} \frac{(\lambda/2)^i}{i!} \quad (15)
\end{aligned}
$$

Since there are four side regions in the square,

$$
\begin{aligned}
E[V_k] &\geq 4 \int_0^r \int_r^{\sqrt{A}-r} E[\chi_k(x,y)]dydx \quad (16) \\
&\geq 4 \int_0^r \int_r^{\sqrt{A}-r} e^{-\lambda(1/2+2xr)} \sum_{i=0}^{k-1} \frac{(\lambda/2)^i}{i!} \\
&= 4(\sqrt{A}-2r) \sum_{i=0}^{k-1} \frac{(\lambda/2)^i}{i!} e^{-\lambda/2} \frac{1-e^{-2\lambda r^2}}{2\lambda r} \\
&\geq 4(\sqrt{A}-2r) \frac{(\lambda/2)^{k-1}}{(k-1)!} e^{-\lambda/2} \frac{1-e^{-2\lambda/\pi}}{2\lambda r} \quad (17)
\end{aligned}
$$

We take logarithm on both sides. Since $\log(\sqrt{A}-2r) = \frac{1}{2}\log A + o(1)$, $\lambda = \log A + 2k \log\log A + C$, and $\log\lambda = \log\log A + o(1)$, we have

$$
\begin{aligned}
&\log E[V_k] \\
&\geq \log 4 + \frac{1}{2}\log A + (k-1)\log(\lambda/2) - \log((k-1)!) \\
&\quad -\frac{\lambda}{2} + \log(1-e^{-2\lambda/\pi}) - \log\lambda - \log(2r) + o(1) \\
&= -2\log\log A - \frac{C}{2} - \log((k-1)!) - (k-2)\log 2 \\
&\quad + \log\sqrt{\pi} + \log(1-e^{-2\lambda/\pi}) + o(1). \quad (18)
\end{aligned}
$$

As $A \to \infty$, $\lambda \to \infty$, so $\log(1-e^{-2\lambda/\pi}) \to 0$. Therefore,

$$
E[V_k](\log A)^2 \geq \frac{\sqrt{\pi}}{e^{C/2} 2^{k-2}(k-1)!} \quad (19)
$$

as $A \to \infty$. $\qquad \square$

Proposition 2 *If* $\lambda = \log A + 2k \log\log A + C$, *where* C *is a constant, then*

$$
\frac{E[V_k^2]}{E[V_k]^2} \leq 1 + \frac{32k(k+1)}{\lambda^2 E[V_k]}, \quad (20)
$$

as $A \to \infty$.

Proof. A few results derived in [10] will be utilized here and we summarize them in the following lemmas.

Lemma 2

$$
E[V_k^2] \leq E[V_k]^2 + \int\int_{R^2 \cap \{|Z_1-Z_2|\leq 2r\}} E[\chi_k(Z_1)\chi_k(Z_2)]dZ_1dZ_2 \quad (21)
$$

Lemma 3 *Let* B_1 *and* B_2 *denote the disks with radius* r, *centered at* Z_1 *and* Z_2, *respectively. If* $|Z_1 - Z_2| = x \leq 2r$, *then the area of* $B_2 - B_1$ *is*

$$
||B_2 - B_1|| \geq x/(2r). \quad (22)
$$

Lemma 4

$$
\int_0^\infty e^{-\lambda u} \sum_{i=0}^{k-1} \frac{(\lambda u)^i}{i!} \cdot u\, du = \frac{1}{2}k(k+1)\lambda^{-2} \quad (23)
$$

Lemma 2 follows from the fact that if the distance of Z_1 and Z_2 is greater than $2r$, $\chi_k(Z_1)$ and $\chi_k(Z_2)$ (i.e., the indicator functions of whether Z_1 and Z_2 are covered by less than k nodes, respectively) are independent random variables. By Lemma 2, we only need to evaluate the second term in Eq. (21).

The challenger when we consider the boundary conditions is that now the area of $(B_2 - B_1) \cap R$ may be zero if B_2 is close to the boundary. We overcome this difficulty by exploiting the symmetric relation between Z_1 and Z_2. Let $Z_1 = (x_1, y_1), Z_2 = (x_2, y_2)$, and define the ∞-norm distance $d_\infty(Z_1, Z_2)$ of the two points Z_1, Z_2 as

$$
d_\infty(Z_1, Z_2) = \max(|x_1 - x_2|, |y_1 - y_2|). \quad (24)
$$

Denote the center of the square region as O. Consider the event $Q = \{d_\infty(Z_1, O) \geq d_\infty(Z_2, O)\}$. Intuitively, if we draw a square centered at O and the boundary of the square goes through Z_1, then Q is the event that Z_2 is inside this square. For each pair of points $(Z_1, Z_2) \notin Q$, there is a unique pair of symmetric points $(Z_1', Z_2')(= (Z_2, Z_1)) \in Q$, and these two pairs of points contribute exactly the same to the integral in Eq. (21). Therefore, the second term in Eq. (21), denoted as I_0, can be written as

$$
\begin{aligned}
I_0 &\triangleq \int\int_{R^2 \cap \{|Z_1-Z_2|\leq 2r\}} E[\chi_k(Z_1)\chi_k(Z_2)]dZ_1dZ_2 \\
&= 2 \int\int_{R^2 \cap \{|Z_1-Z_2|\leq 2r\} \cap Q} E[\chi_k(Z_1)\chi_k(Z_2)]dZ_1dZ_2 \quad (25)
\end{aligned}
$$

We define the central, side, corner regions similar to those in Section III-A. We further define the extended corner region C_E as those points which are within distance r from one side and within distance $3r$ from another side. We consider two possible cases: (i) $Z_1 \in C_E$ and (ii) $Z_1 \notin C_E$.

Case (i): $Z_1 \in C_E$:

$$
\begin{aligned}
&\int\int_{R^2 \cap \{|Z_1-Z_2|\leq 2r\} \cap Q \cap \{Z_1 \in C_E\}} E[\chi_k(Z_1)\chi_k(Z_2)]dZ_1dZ_2 \\
&\leq \int\int_{R^2 \cap \{|Z_1-Z_2|\leq 2r\} \cap Q \cap \{Z_1 \in C_E\}} E[\chi_k(Z_1)]dZ_1dZ_2 \\
&\leq 20r^2 \cdot \pi(2r)^2 \cdot e^{-\lambda/4} \sum_{i=0}^{k-1} \frac{(\lambda/4)^i}{i!} \\
&= \frac{80}{\pi} e^{-\lambda/4} \sum_{i=0}^{k-1} \frac{(\lambda/4)^i}{i!}, \quad (26)
\end{aligned}
$$

where in the third equation, $20r^2$ is the total area of C_E, $\pi(2r)^2$ is the maximum possible area of Z_2 for a given point Z_1, and the last factor is an upper bound of $E[\chi_k(Z_1)]$, since at least a quarter of the disk $B_{Z_1}(r)$ is inside the monitored region R.

Case (ii): $Z_1 \notin C_E$: If Z_1 is not in the extended corner, Z_2 is within distance $2r$ from Z_1, and Z_2 is inside the square centered at O whose boundary passes through Z_1, then Z_2 cannot be in the corner area, and moreover, at least half of the

area $B_2 - B_1$ is inside the region R. This is the key to the proof! Now for any given $Z_1 \notin C_E$, let $Z_2 \in R^2 \cap \{|Z_1 - Z_2| \leq 2r\} \cap Q$, and $x = |Z_1 - Z_2| \leq 2r$. By lemma 3, the area of $(B_2 - B_1) \cap R$ is at least $x/(4r)$. Therefore,

$$
\begin{aligned}
& E[\chi_k(Z_1)\chi_k(Z_2)] \\
=\ & P(\text{there are less than } k \text{ nodes in } B_1 \cap R \\
& \text{and there are less than } k \text{ nodes in } B_2 \cap R) \\
\leq\ & P(\text{there are less than } k \text{ nodes in } B_1 \cap R \\
& \text{and there are less than } k \text{ nodes in } (B_2 - B_1) \cap R) \\
\leq\ & P(\text{there are less than } k \text{ nodes in } B_1 \cap R) \\
& \cdot P(\text{there are less than } k \text{ nodes in } (B_2 - B_1) \cap R) \\
\leq\ & E(\chi_k(Z_1)) \cdot e^{-\lambda x/(4r)} \sum_{i=0}^{k-1} \frac{(\lambda x/(4r))^i}{i!},
\end{aligned}
\tag{27}
$$

where the third equation is because the number of nodes in $B_1 \cap R$ and that in $B_2 - B_1 \cap R$ are independent. So,

$$
\begin{aligned}
& \int\int_{R^2 \cap \{|Z_1-Z_2|\leq 2r\} \cap Q \cap \{Z_1 \notin C_E\}} E[\chi_k(Z_1)\chi_k(Z_2)]dZ_1 dZ_2 \\
\leq\ & \int_R E[\chi_k(Z_1)]dZ_1 \int_0^{2r} e^{-\lambda x/(4r)} \sum_{i=0}^{k-1} \frac{(\lambda x/(4r))^i}{i!} 2\pi x dx, \\
=\ & \int_R E[\chi_k(Z_1)]dZ_1 \int_0^{1/2} e^{-\lambda u} \sum_{i=0}^{k-1} \frac{(\lambda u)^i}{i!} 32u du, \\
\leq\ & E[V_k] \cdot \int_0^\infty e^{-\lambda u} \sum_{i=0}^{k-1} \frac{(\lambda u)^i}{i!} 32u du, \\
=\ & E[V_k] \cdot 16k(k+1)\lambda^{-2}
\end{aligned}
\tag{28}
$$

where the factor $2\pi x$ in the second equation comes from the conversion from a Cartesian coordinate system to a polar coordinate system, the third equation is obtained by changing variable $u = x/(4r)$, and the last equation comes from Lemma 4.

Note that the value in case (i), which converges to 0 in the exponential rate of λ, is dominated by the value in case (ii), which converges to 0 in the polynomial rate of λ (since $E[V_k]$ is in the same order of λ^{-2} from Proposition 1). Combining the results in the above two cases, as well as Eqs. (21) and (25), we obtain Eq. (20). $\qquad\square$

It has been proved in [10] using Cauchy-Schwartz inequality,

$$
P(V_k > 0) \geq \frac{E[V_k]^2}{E[V_k^2]}.
\tag{29}
$$

By Proposition 2, we have

$$
P(V_k > 0) \geq \frac{1}{1 + 32k(k+1)/(\lambda^2 E[V_k])}.
\tag{30}
$$

By Proposition 1,

$$
E[V_k](\log A)^2 \geq \frac{\sqrt{\pi}}{e^{C/2} 2^{k-2}(k-1)!},
\tag{31}
$$

and $\lambda = (1 + o(1))\log A$,

$$
P(V_k > 0) \geq \frac{1}{1 + 32e^{C/2} 2^{k-2}(k+1)!/\sqrt{\pi}}
\tag{32}
$$

as $A \to \infty$.

Based on the above derivations, we can establish the following necessary condition for k-coverage, which also leads to a sufficient condition on un-k-coverage.

Theorem 2 *Let $\lambda = \log A + 2k \log \log A + c(A)$. If $c(A) \leq C$, where C is a constant, then, as $A \to \infty$,*

$$
\begin{aligned}
& P(\text{the monitored square region } R \text{ is } k\text{-covered}) \\
& \leq\ 1 - \frac{1}{1 + 32e^{C/2} 2^{k-2}(k+1)!/\sqrt{\pi}} < 1.
\end{aligned}
\tag{33}
$$

In addition, if $c(A) \to -\infty$, P(the monitored square region is k-covered) tends to 0.

Proof. The proof follows by two observations. First, that the region is k-covered implies $V_k = 0$. Therefore, P(the region is k-covered) $\leq P(V_k = 0) = 1 - P(V_k > 0)$. Second, $P(V_k > 0)$ is a non-increasing function of λ. Since we have proved the conclusion holds for $c(A) = C$, it also holds for $c(A) \leq C$. The second part of the theorem is obtained by letting $C \to -\infty$. $\qquad\square$

IV. UNIFORM DISTRIBUTION

In this model, we assume there are n nodes in the square region R with area A and each node's location is identically, independently distributed with uniform distribution. Each node has an independent probability p to be active. In the following we establish a sufficient and a necessary condition for k-coverage under such a model.

A. Sufficient condition

Theorem 3 *Under the uniform distribution model, let $np/A = \log A + 2k \log \log A + c(A)$, where A is the area of the deployment square region R. If $c(A) \to \infty$, as $A \to \infty$, then P(the region R is k-covered) $\to 1$.*

Proof. We still divide the area into small grids with side length $s = \sqrt{2}ur$ where $u = 1/\log A$. Still denote X_i as the indicator function of whether a grid is NOT k-covered and X as the total number of the un-k-covered grids (a grid is un-k-covered if it is not completely k-covered). Again we proceed to compute the expectation of the number of grids that are not k-covered in the three types of regions: inner, side and corner, respectively.

For an inner grid i to be k-covered, it is sufficient that there are k active nodes inside the disk $B_i((1-u)r)$ (since the whole disk is in the region R). Let p_1 denote the probability of a node to be inside $B_i((1-u)r)$ and to be active, i.e., $p_1 = p\pi((1-u)r)^2/A = p(1-u)^2/A$. The number of active nodes inside the disk $B_i((1-u)r)$ follows a binomial distribution with parameter n and p_1. If an inner grid i is not k-covered, the number of active nodes inside the disk $B_i((1-u)r)$ is less than k. Hence,

$$
\begin{aligned}
E[X_i] &=\ P(\text{grid } i \text{ is not } k\text{-covered}) \\
&\leq\ \sum_{i=0}^{k-1} \binom{n}{i} p_1^i (1 - p_1)^{n-i}
\end{aligned}
$$

Again, since $n/A \to \infty$ as $A \to \infty$, the term with $i = k - 1$ dominates all others. Also since $\binom{n}{i} \leq n^i/i!$ and $(1-p_1)^{n-i} \leq e^{-p_1(n-i)}$,

$$
\begin{aligned}
E[X_i] \\
\leq & \binom{n}{k-1} p_1^{k-1}(1-p_1)^{n-k+1}(1+o(1)) \\
\leq & \frac{(np_1)^{k-1}}{(k-1)!} e^{-p_1(n-k+1)}(1+o(1)) \\
= & \frac{(np(1-u)^2/A)^{k-1}}{(k-1)!} e^{-(n-k+1)p(1-u)^2/A}(1+o(1))
\end{aligned}
$$

If we replace np/A with λ, the above equation is identical to Eq. (7) except for a factor $e^{(k-1)p(1-u)^2/A}$, which converges to 1 as $A \to \infty$. Therefore, follow the same derivation in Section III-A, we can obtain the expectation of the number X^I of the un-k-covered inner grids converges to 0 as $A \to \infty$ and $c(A) \to \infty$.

For a corner grid, if we draw a disk centered at the center of the grid, and with radius $(1 - u)r$, at least a quarter of the disk is inside the region R. If the corner grid is not k-covered, then it is necessary that the quarter of the disk that is inside the region R has less than k active nodes. Again, the number of the active nodes inside the quarter of the disk is a binomial random variable with parameter n and $p_2 = p(\frac{1}{4}\pi r^2(1-u)^2)/A = p(1-u)^2/(4A)$. Therefore,

$$
\begin{aligned}
& P(\text{a corner grid is not } k\text{-covered}) \\
\leq & \sum_{i=0}^{k-1} \binom{n}{i} p_2^i (1-p_2)^{n-i} \\
= & (1+o(1)) \binom{n}{k-1} p_2^{k-1}(1-p_2)^{n-k+1} \\
\leq & (1+o(1)) \frac{(np_2)^{k-1}}{(k-1)!} e^{-p_2(n-k+1)}
\end{aligned}
$$

The number of corner grids is at most $4r^2/s^2 = 2/u^2$ since there are 4 corners. Therefore, the expectation of the number X^C of the corner grids that are not k-covered, is

$$
E[X^C] \leq \frac{2}{u^2} \frac{(np_2)^{k-1}}{(k-1)!} e^{-p_2(n-k+1)}(1+o(1)) \tag{34}
$$

Since $u = 1/\log A$, and $np_2 = np(1-u)^2/(4A) \to \infty$, it is not hard to verify that $E[X^C] \to 0$ as $A \to \infty$ and $c(A) \to \infty$.

Now consider a side grid at row j (row zero is the one closest to the boundary of the square region). Again, if we draw a disk centered at the center of the grid and with radius $r(1-u)^2$, the area of the disk that falls in the square region R is at least $((1-u)^2 + j\sqrt{2}u(1-u))/2$ by Eq. (9). The probability that a node falls in this area and is active is $p_3 = p((1-u)^2 + j\sqrt{2}u(1-u))/(2A)$, and the number of the active nodes in this area follows a binomial distribution with parameter n and p_3. Therefore, we have

$$
\begin{aligned}
& P(\text{a side grid at row } j \text{ is not } k\text{-covered}) \\
\leq & \sum_i \binom{n}{i} p_3^i (1-p_3)^{n-i} \\
\leq & \binom{n}{k-1} p_3^{k-1}(1-p_3)^{n-k+1}(1+o(1))
\end{aligned}
$$

$$
\begin{aligned}
\leq & \frac{n^{k-1}}{(k-1)!} \left(\frac{p((1-u)^2 + j\sqrt{2}u(1-u))}{2A} \right)^{k-1} \\
& \cdot e^{-(n-k+1)p((1-u)^2+j\sqrt{2}u(1-u))/(2A)}(1+o(1)) \\
\leq & \frac{(np(1-u)^2/A)^{k-1}}{(k-1)!} e^{-np((1-u)^2+j\sqrt{2}u(1-u))/(2A)} \\
& \cdot e^{(k-1)p((1-u)^2+j\sqrt{2}u(1-u))/(2A)}(1+o(1)) \tag{35}
\end{aligned}
$$

Since $e^{(k-1)p((1-u)^2+j\sqrt{2}u(1-u))/(2A)}$ converges to 1 as $A \to \infty$, the remaining factors are identical to Eq. (10) if we replace np/A with λ. Since we have chosen the same value of np/A as that of λ in Section III-A, we have obtained the same upper bound of $P(\text{a grid at row } j \text{ is not } k\text{-covered})$ as that in Eq. (10). Therefore, following the same derivation as that in Section III-A, we can obtain $E[X^S] \to 0$ (where X^S is the number of un-k-covered side grids) as $A \to \infty$ and $c(A) \to \infty$. Therefore, the expectation of the total number of un-k-covered grids tends to 0 and $P(\text{the region } R \text{ is } k\text{-covered}) \to 1$ as $A \to \infty$. $\square$

B. Necessary condition

The necessary condition in the uniform distribution is obtained by approximating the uniform random node distribution with a Poisson point process. In this section, we use $E_U[Y]$ and $E_P[Y]$ to denote the expectation of the quantity Y in a uniform random node distribution and that in a Poisson point process. In particular, we shall show $E_U[V_k] \sim E_P[V_k]$ and $E_U[V_k^2] \sim E_P[V_k^2]$ with appropriately chosen n, A, p in the uniform random node distribution and the node density λ in the Poisson point process.

Proposition 3 *If $np/A = \lambda = \log A + 2k \log \log A + C$, where C is a constant, then*

$$
E_U[V_k](\log A)^2 \geq \frac{\sqrt{\pi}}{e^{C/2}2^{k-2}(k-1)!} \tag{36}
$$

Proof. Still use $\chi_k((x,y))$ to denote the indicator function of whether a point (x, y) is covered by less than k nodes. Again, we only consider the case that (x, y) is in the side area of the square to obtain a lower bound. Without loss of generality, we assume $0 \leq x \leq r, r \leq y \leq \sqrt{A} - r$. Thus the disk $B_{(x,y)}(r)$ has at most area $\frac{1}{2} + 2xr$ inside the square region R. Denote $p_1 \triangleq p(\frac{1}{2} + 2xr)/A$ as the probability that one node falls in a region with area $\frac{1}{2} + 2xr$ and is active. Therefore,

$$
\begin{aligned}
& E_U[\chi_k((x,y))] \\
= & P((x,y) \text{ is not } k\text{-covered}) \\
= & P(B_{(x,y)}(r) \cap R \text{ contains less than } k \text{ active nodes}) \\
\geq & \sum_{i=0}^{k-1} \binom{n}{i} p_1^i (1-p_1)^{n-i} \\
\sim & \sum_{i=0}^{k-1} \frac{n^i}{i!} \left(\frac{p(\frac{1}{2} + 2xr)}{A} \right)^i e^{(n-i)\log(1-p_1)} \\
= & \sum_{i=0}^{k-1} \frac{(\lambda(\frac{1}{2} + 2xr))^i}{i!} e^{(n-i)\log(1-p_1)} \tag{37}
\end{aligned}
$$

Since $i < k$ is bounded, $np/A = \lambda$, $np_1 = \lambda(\frac{1}{2} + 2xr)$, and $p_1 = p(\frac{1}{2} + 2xr)/A < 2/A \to 0$ as $A \to \infty$, we have

$$
(n-i)\log(1-p_1)
$$

$$
\begin{aligned}
&= (n-i)(-p_1 + O(p_1^2)) \\
&= -np_1 + O(np_1^2) + i(p_1 - O(p_1^2)). \quad (38)
\end{aligned}
$$

As $A \to \infty$, $np_1^2 = \lambda(\frac{1}{2}+2xr)p_1 \lesssim (\frac{1}{2}+2xr)(\log A)2/A \to 0$, and $i(p_1 - O(p_1^2)) \to 0$. Therefore, by Eq. (38),

$$
e^{(n-i)\log(1-p_1)} \sim e^{-np_1} = e^{-\lambda(\frac{1}{2}+2xr)}. \quad (39)
$$

Putting this back to Eq. (37), we obtain

$$
\begin{aligned}
E_U[\chi_k((x,y))] & \\
\geq \sum_{i=0}^{k-1} & \frac{(\lambda(\frac{1}{2}+2xr))^i}{i!} e^{-\lambda(\frac{1}{2}+2xr)}(1+o(1)) \\
\geq e^{-\lambda(\frac{1}{2}+2xr)} & \sum_{i=0}^{k-1} \frac{(\lambda/2)^i}{i!}(1+o(1)) \quad (40)
\end{aligned}
$$

Comparing with Eqs. (15) and (16), we can easily obtain

$$
E_U[V_k](\log A)^2 \geq \frac{\sqrt{\pi}}{e^{C/2}2^{k-2}(k-1)!} \quad (41)
$$

as $A \to \infty$. $\qquad \square$

Proposition 4 *If $np/A = \lambda = \log A + 2k \log \log A + C$, where C is a constant, then*

$$
\frac{E[V_k^2]}{(E[V_k])^2} \leq 1 + \frac{32k(k+1)}{\lambda^2 E[V_k]}, \quad (42)
$$

as $A \to \infty$.

Proof. Still use $\chi_k(Z_1)$ to denote the indicator function of whether a point Z_1 is not k-covered. We have $V_k = \int_R \chi_k(Z_1)dZ_1$. Therefore,

$$
\begin{aligned}
E_U[V_k^2] &= E_U\left[\iint_R \chi_k(Z_1)\chi_k(Z_2)dZ_1dZ_2\right] \\
&= \iint_R E[\chi_k(Z_1)\chi_k(Z_2)]dZ_1dZ_2 \quad (43)
\end{aligned}
$$

We still consider two cases in the integration. In the first case, $|Z_1 - Z_2| > 2r$,

$$
E_U[\chi_k(Z_1)\chi_k(Z_2)] = \sum_{i=0}^{k-1}\sum_{j=0}^{k-1}\binom{n}{i,j}p_1^i p_2^j (1-p_1-p_2)^{n-i-j}, \quad (44)
$$

where p_1 and p_2 are the probability that a node is active and falls in an area within range r from Z_1 and Z_2, respectively. Notice

$$
\begin{aligned}
E_U[\chi_k(Z_1)] & E_U[\chi_k(Z_2)] \\
&= \sum_{i=0}^{k-1}\binom{n}{i}p_1^i(1-p_1)^{n-i}\sum_{j=0}^{k-1}\binom{n}{j}p_2^j(1-p_2)^{n-j} \\
&= \sum_{i=0}^{k-1}\sum_{j=0}^{k-1}\binom{n}{i}\binom{n}{j}p_1^i p_2^j (1-p_1)^{n-i}(1-p_2)^{n-j} (45)
\end{aligned}
$$

Since $1/(4A) \leq p_1, p_2 \leq 1/A$, $np \sim A\log A$, and $i,j < k$ is bounded,

$$
(1-p_1-p_2)^{n-i-j} \sim (1-p_1)^{n-i}(1-p_2)^{n-j}, \quad (46)
$$

and

$$
\binom{n}{i,j} = \frac{n!}{i!j!(n-i-j)!} \sim \frac{n!n!}{i!(n-i)!j!(n-j)!} = \binom{n}{i}\binom{n}{j}.
$$

Hence, in the case of $|Z_1 - Z_2| > 2r$,

$$
E_U[\chi_k(Z_1)\chi_k(Z_2)] \sim E_U[\chi_k(Z_1)]E_U[\chi_k(Z_2)] \quad (47)
$$

Therefore, by Eq. (43),

$$
\begin{aligned}
E_U[V_k^2] &= E_U[V_k]^2(1+o(1)) \\
&+ \iint_{R^2 \cap \{|Z_1-Z_2| \leq 2r\}} E[\chi_k(Z_1)\chi_k(Z_2)]dZ_1dZ_2 (48)
\end{aligned}
$$

In the second case: $|Z_1 - Z_2| \leq 2r$, we follow the derivations similar to those in Section III-B. First,

$$
\begin{aligned}
\iint_{R^2 \cap \{|Z_1-Z_2| \leq 2r\}} & E[\chi_k(Z_1)\chi_k(Z_2)]dZ_1dZ_2 \\
&= 2\iint_{R^2 \cap \{|Z_1-Z_2| \leq 2r\} \cap Q} E[\chi_k(Z_1)\chi_k(Z_2)]dZ_1dZ_2, \quad (49)
\end{aligned}
$$

where Q is the event that Z_2 is inside the square centered at the center of region and whose boundary goes through Z_1. Second, we only consider the dominating subcase when Z_1 is not in the extended corner region C_E. Still denote B_1, B_2 as the unit-area disks centered at Z_1, Z_2, respectively. Under all the above conditions ($R^2 \cap \{|Z_1 - Z_2| \leq 2r\} \cap Q$),

$$
\begin{aligned}
E[\chi_k(Z_1)&\chi_k(Z_2)] \\
= \ & P(\text{there are less than } k \text{ active nodes in } B_1 \cap R \text{ and} \\
& \text{there are less than } k \text{ active nodes in } B_2 \cap R) \\
\leq \ & P(\text{there are less than } k \text{ active nodes in } B_1 \cap R \text{ and} \\
& \text{there are less than } k \text{ active nodes in } (B_2 - B_1) \cap R)
\end{aligned}
$$

As have been proved in Section III-B, the area of $(B_2 - B_1) \cap R$ is at least $x/(4r)$ where $x = |Z_1 - Z_2|$ and $r = 1/\sqrt{\pi}$ is the radius of the disk B_1, B_2. Denote p_1 as the probability that a node falls in $B_1 \cap R$ and is active, p_2 as the probability that a node falls in $(B_2 - B_1) \cap R$ and is active. Therefore,

$$
\begin{aligned}
E_U[\chi_k(Z_1)&\chi_k(Z_2)] \\
=\ & \sum_{i=0}^{k-1}\sum_{j=0}^{k-1}\binom{n}{i,j}p_1^i p_2^j(1-p_1-p_2)^{n-i-j} \\
\sim\ & \sum_{i=0}^{k-1}\binom{n}{i}p_1^i(1-p_1)^{n-i}\sum_{j=0}^{k-1}\binom{n}{j}p_2^j(1-p_2)^{n-j} \\
=\ & E_U[\chi_k(Z_1)]\sum_{j=0}^{k-1}\binom{n}{j}p_2^j(1-p_2)^{n-j} \\
\leq\ & E_U[\chi_k(Z_1)]\sum_{j=0}^{k-1}\frac{(np_2)^j}{j!}e^{-(n-j)p_2} \\
\sim\ & E_U[\chi_k(Z_1)]e^{-np_2}\sum_{j=0}^{k-1}\frac{(np_2)^j}{j!} \\
\leq\ & E_U[\chi_k(Z_1)]e^{-\lambda x/(4r)}\sum_{j=0}^{k-1}\frac{(\lambda x/(4r))^j}{j!} \quad (50)
\end{aligned}
$$

where the second equation follows from the derivations to those in the case of $|Z_1 - Z_2| > 2r$, and the last equation is because $np_2 \geq np(x/(4r))/A = \lambda x/(4r)$, and the function $\sum_{j=0}^{k-1}\frac{x^j}{j!}e^{-x}$ is monotonically decreasing.

Now comparing Eqs. (50) and (27), and by Eq. (28), we can obtain,

$$
\iint_{R^2 \cap \{|Z_1-Z_2| \leq 2r\} \cap Q} E_U[\chi_k(Z_1)\chi_k(Z_2)
$$

$$\leq E_U[V_k] \cdot 16k(k+1)\lambda^{-2}(1+o(1)) \quad (51)$$

Combining Eqs. (48) (49)and (51), we have

$$
\begin{aligned}
E_U[V_k^2] = {} & E_U[V_k]^2(1+o(1)) \\
& + 32k(k+1)\lambda^{-2}E_U[V_k](1+o(1)). \quad (52)
\end{aligned}
$$

Eq. (42) follows by taking $A \to \infty$.

The following theorem follows immediately from Propositions 3 and 4. The proof is identical to that of Theorem 2.

Theorem 4 *Let* $np/A = \log A + 2k\log\log A + c(A)$. *If* $c(A) \leq C$, *where* C *is a constant value, then, as* $A \to \infty$,

$$
\begin{aligned}
& P(\textit{the monitored square region is k-covered}) \\
& \leq 1 - \frac{1}{1 + 32e^{C/2}2^{k-2}(k+1)!/\sqrt{\pi}} \quad (53)
\end{aligned}
$$

In addition, if $c(A) \to -\infty$, *P(the monitored square region is k-covered) tends to 0 as* $A \to \infty$. $\quad\square$

V. GRID DEPLOYMENT

In this section, we consider the deployment of grid distributions where $n = k^2$ nodes form regular square grids inside the square region R with area A. Each node has probability p to be active. We use $D = n/A$ to denote the node density. We show a lemma first.

Lemma 5 *for* $0 \leq p < 1$,

$$p \leq -\log(1-p) \leq \frac{p}{1-p} \quad (54)$$

Proof. Since $1 - p \leq e^{-p}$, taking logarithm, we have $\log(1 - p) \leq -p$ and hence the first inequality. To prove the second inequality, let $f(p) = p + (1-p)\log(1-p)$. It is simple to verify $f(0) = 0$ and $f'(p) \geq 0$ for $0 \leq p < 1$. So $f(p) \geq 0$ for $0 \leq p < 1$. Rearranging the equation, we can obtain the second inequality. $\quad\square$

A. Sufficient condition

Theorem 5 *Assume* $p \leq 1 - \epsilon < 1$ *for some constant* ϵ. *Let*

$$
\begin{aligned}
& (-\log(1-p))n/A = -D\log(1-p) \\
& = \log A + 2k\log\log A + 2\sqrt{-2\pi\log A\log(1-p)} + c(A) \quad (55)
\end{aligned}
$$

and $c(A) \to \infty$ *as* $A \to \infty$, *then P(the region R is k-covered)* $\to 1$.

Proof. We still divide the square region into grids with side length $s = \sqrt{2}ur$, where $u = 1/\log A$. We now calculate the expected number of grids that are not k-covered. We shall only consider the grids at side area of the region R since this area contributes most un-k-covered grids. Again, consider a side grid g at j rows away from the side (there are j other rows between the grid and the side of the square region R). Let m_j denote the number of nodes that are contained in the disk centered at the center of grid g with radius $r - s/\sqrt{2}$. Therefore, any one of the m_j nodes can completely cover the grid g if it is active (notice a node has sensing range r).

We first estimate the value m_j. If we draw a disk S_j with radius $t = r - s/\sqrt{2}$ centered at the center of grid g, the distance between the center of the disk and the closest side

of the square R is $j(s + 1/2) \geq js$. The area of the part of the disk S_j inside R is at least $\pi t^2/2 + \pi jst/2$ by Lemma 1. Therefore, the number m_j of nodes inside the area is roughly $D(\pi t^2/2 + \pi jst/2)$, but not exactly. To obtain a bound of m_j, we envision all nodes are at the centers of disjoint small squares of side length $d = 1/\sqrt{D} = \sqrt{A/n}$. If we draw a disk S_j' with radius $t - d/\sqrt{2}$, then any point in the disk S_j' must belong to some square whose center is covered by the disk S_j. Hence, the area of S_j' is less than the total area of all the squares whose center is covered by the disk S_j, i.e., $\pi(t - d/\sqrt{2})^2/2 + \pi js(t - d/\sqrt{2})/2 \leq m_j d^2$. Therefore,

$$
\begin{aligned}
m_j \geq {} & D(\pi(t - d/\sqrt{2})^2/2 + \pi js(t - d/\sqrt{2})/2) \\
= {} & D(\pi t^2/2 - \pi td/\sqrt{2} + \pi d^2/4 + \pi jst/2 - \pi jsd/(2\sqrt{2})) \\
\geq {} & D\pi t^2/2 - \pi t\sqrt{D/2} + D\pi jst/2 - D\pi js\sqrt{D/2}/2) \\
\geq {} & D(1-u)^2/2 + D\sqrt{\pi}js(1-u)/2 - \sqrt{2\pi D} \\
\triangleq {} & m_{j0} \quad (56)
\end{aligned}
$$

where the third equation is because $D = 1/d^2$, and the fourth equation is because $t = r - s/\sqrt{2} = r(1-u)$, $\pi t^2 = \pi r^2(1-u)^2 = (1-u)^2$, and $js \leq r = 1/\sqrt{\pi}$. Notice that $m_{j0} \leq D(1-u)^2$.

Let p_j denote the probability that the grid g at row j is not k-covered. Clearly, p_j increase if m_j decrease (because p_j is the probability that out of m_j nodes, less than k of them are active),

$$
\begin{aligned}
p_j = {} & \sum_{i=0}^{k-1} \binom{m_j}{i} p^i(1-p)^{m_j-i} \\
\leq {} & \sum_{i=0}^{k-1} \binom{m_{j0}}{i} p^i(1-p)^{m_{j0}-i}. \quad (57)
\end{aligned}
$$

Denote the ith item in the above summation as T_i. We have

$$\frac{T_{i+1}}{T_i} = \frac{m_{j0} - i}{i+1} \cdot \frac{p}{1-p} \geq \frac{(m_{j0} - k)p}{k(1-p)}. \quad (58)$$

Since

$$
\begin{aligned}
(m_{j0} - k)p/(1-p) \sim {} & m_{j0}p/(1-p) \\
\geq {} & D(1-u)^2/2 \cdot (-\log(1-p))
\end{aligned}
$$

(where the second equation is from Lemma 5 and $m_{j0} \geq D(1-u)^2/2$ by Eq. (56)) tends to infinity, T_i is dominated by T_{i+1} for $i < k$. Therefore,

$$
\begin{aligned}
p_j \lesssim {} & \binom{m_{j0}}{k-1} p^{k-1}(1-p)^{m_{j0}-k+1} \\
\leq {} & \frac{(m_{j0}p/(1-p))^{k-1}}{(k-1)!}(1-p)^{m_{j0}} \\
\leq {} & \frac{(Dp(1-u)^2/(1-p))^{k-1}}{(k-1)!}(1-p)^{m_{j0}}, \quad (59)
\end{aligned}
$$

where the last equation is because $m_{j0} \leq D(1-u)^2$.

The expected number X^S of un-k-covered side grids is

$$
\begin{aligned}
& E[X^S] \\
& \leq \frac{4\sqrt{A}}{s} \sum_{j=0}^{r/s} p_j \\
& \lesssim \frac{4\sqrt{A}}{s} \sum_{j=0}^{r/s} \frac{(D(1-u)^2 p/(1-p))^{k-1}}{(k-1)!}.
\end{aligned}
$$

$$\leq \frac{4\sqrt{A}}{s} \frac{(1-p)^{\frac{D(1-u)^2}{2} - \sqrt{2\pi D} + j\frac{D\sqrt{\pi}s(1-u)}{2}} (D(1-u)^2 p/(1-p))^{k-1}}{(k-1)!} \cdot$$
$$(1-p)^{\frac{D(1-u)^2}{2} - \sqrt{2\pi D}} \cdot \frac{1}{1-(1-p)^{D\sqrt{\pi}s(1-u)/2}} \quad (60)$$

The last factor $\frac{1}{1-(1-p)^{D\sqrt{\pi}s(1-u)/2}}$ in Eq. (60) converges to a constant since $D\sqrt{\pi}s(1-u)/2 \sim 1/(-\log(1-p)\sqrt{2})$ by Eq. (55), $(1-p)^{D\sqrt{\pi}s(1-u)/2} = (1-p)^{1/(-\log(1-p)\sqrt{2}(1+o(1)))} = e^{-1/\sqrt{2}(1+o(1))}$.

Notice $s = \sqrt{2}ur = \sqrt{2/\pi}/\log A$. We take logarithm on both sides, put all constant and $o(1)$ terms into C_1, and get

$$\log E[X^S]$$
$$\leq \frac{1}{2}\log A + \log\log A + (k-1)\log(\frac{Dp(1-u)^2}{1-p})$$
$$+ (D\frac{(1-u)^2}{2} - \sqrt{2\pi D})\log(1-p) + C_1$$
$$= \frac{1}{2}\log A + \log\log A + (k-1)\log(\frac{Dp(1-u)^2}{1-p})$$
$$- \frac{(1-u)^2}{2}(\log A + 2k\log\log A + 2\sqrt{-2\pi\log A\log(1-p)}$$
$$+ c(A)) - \sqrt{2\pi D}\log(1-p) + C_1, \quad (61)$$

where the second equation is obtained by plugging in Eq. (55). Since $\log(1-p) \leq -p$, $Dp \leq -D\log(1-p) \sim \log A$, so $(k-1)\log Dp \leq (k-1)\log\log A + o(1)$. In addition, since $p < 1 - \epsilon$, $\log\frac{(1-u)^2}{1-p} \leq \log\frac{1}{\epsilon}$ is bounded. Therefore,

$$\log E[X^S]$$
$$\leq (u - \frac{u^2}{2})(\log A + 2k\log\log A) - \frac{(1-u)^2}{2}c(A) + C_2$$
$$- (1-u)^2\sqrt{-2\pi\log A\log(1-p)} - \sqrt{2\pi D}\log(1-p). \quad (62)$$

Since

$$-\sqrt{2\pi D}\log(1-p) - \sqrt{-2\pi\log A\log(1-p)}$$
$$= \sqrt{-2\pi\log(1-p)}(\sqrt{-D\log(1-p)} - \sqrt{\log A})$$

and

$$\sqrt{-D\log(1-p)} - \sqrt{\log A}$$
$$= \frac{-D\log(1-p) - \log A}{\sqrt{-D\log(1-p)} + \sqrt{\log A}}$$
$$\leq \frac{2k\log\log A + 2\sqrt{-2\pi\log A\log(1-p)} + c(A)}{\sqrt{-D\log(1-p)} + \sqrt{\log A}}$$
$$\leq \text{constant},$$

the terms $-(1-u)^2\sqrt{-2\pi\log A\log(1-p)} - \sqrt{2\pi D}\log(1-p)$ in Eq. (62) are bounded. Additionally, since $u = 1/\log A$, $(u - u^2/2)(\log A + 2k\log\log A)$ converges to 1. Therefore, if $c(A) \to \infty$, $\log E[X^S] \to -\infty$ and $E[X^S] \to 0$. Since the number of un-k-covered side grids dominates that of un-k-covered grids in the inner and corner region, the expected number X of total un-k-covered grids converges to 0 as $A \to \infty$. By Markov inequality again, we obtain that P(the whole region is completely k-covered) $\to 1$ as $A \to \infty$ if the number of nodes is given as in Eq. (55). $\square$

B. Necessary condition

The derivation of the necessary condition in the grid deployment follows the procedure similar to that in uniform node distributions and Poisson point process. Again we need to estimate the bounds on $E[V_k]$ and $E[V_k^2]$.

Proposition 5 *Assume $p \leq 1 - \epsilon$, where $\epsilon > 0$. If*

$$(-\log(1-p))n/A = -D\log(1-p)$$
$$= \log A + 2(k-1)\log\log A$$
$$- 2\sqrt{-2\pi\log A\log(1-p)} - c(A), \quad (63)$$

where $c(A)$ is slowly growing (i.e., $c(A) \to \infty$ and $c(A) = o(\log\log A)$) as $A \to \infty$ and k is fixed, then

$$E[V_k](\log A) \geq C_3 e^{c(A)/2}, \quad (64)$$

where C_3 is a constant.

Proof. We still use $\chi_k((x,y))$ to denote the indicator function of whether a point (x,y) is covered by less than k nodes. Again, we only consider the case that (x,y) is in the side area of the square R to obtain a lower bound. Without loss of generality, we assume $0 \leq x \leq r, r \leq y \leq \sqrt{A} - r$. Thus the disk $B_{(x,y)}(r)$ has at most area $\frac{1}{2} + 2xr$ inside the square region R. The region $B_{(x,y)}(r) \cap R$ contains at most $M_x = D(\pi r^2/2 + 2xr + 2\pi rd/\sqrt{2}) = D(\frac{1}{2} + 2xr) + \sqrt{2\pi D}$ nodes. Therefore,

$$E[\chi_k((x,y))]$$
$$\geq \sum_{i=0}^{k-1}\binom{M_x}{i}p^i(1-p)^{M_x-i}$$
$$\sim \sum_{i=0}^{k-1}\frac{(M_x p/(1-p))^i}{i!}(1-p)^{M_x}$$
$$= \sum_{i=0}^{k-1}\frac{(M_x p/(1-p))^i}{i!}e^{M_x\log(1-p)}. \quad (65)$$

Since $M_x \geq D/2$, and $p/(1-p) \geq -\log(1-p)$, we have

$$E[\chi_k((x,y))]$$
$$\gtrsim \sum_{i=0}^{k-1}\frac{(-\frac{D}{2}\log(1-p))^i}{i!}e^{M_x\log(1-p)}$$
$$\geq \sum_{i=0}^{k-1}\frac{(-\frac{D}{2}\log(1-p))^i}{i!}e^{(D/2+2xrD+\sqrt{2\pi D})\log(1-p)}$$
$$\geq \frac{(-\frac{D}{2}\log(1-p))^{k-1}}{(k-1)!}e^{(D/2+2xrD+\sqrt{2\pi D})\log(1-p)} \quad (66)$$

Therefore,

$$E[V_k]$$
$$\geq 4\int_0^r\int_r^{\sqrt{A}-r}E[\chi_k((x,y))]dydx$$
$$\gtrsim 4\int_0^r\int_r^{\sqrt{A}-r}\frac{(-\frac{D}{2}\log(1-p))^{k-1}}{(k-1)!}$$
$$\cdot e^{(D/2+2xrD+\sqrt{2\pi D})\log(1-p)}dydx$$
$$\geq 4(\sqrt{A} - 2r)\frac{(-\frac{D}{2}\log(1-p))^{k-1}}{(k-1)!}$$
$$\cdot e^{(D/2+\sqrt{2\pi D})\log(1-p)}\frac{1 - e^{2r^2 D\log(1-p)}}{-2rD\log(1-p)} \quad (67)$$

Since $-D\log(1-p) \sim \log A$ and $r^2 = 1/\pi$, $e^{2r^2 D\log(1-p)} \to 0$ as $A \to \infty$. We take logarithm on both sides, and obtain

$$\log E[V_k]$$
$$\geq \frac{\log A}{2} + (k-1)\log(-\frac{D}{2}\log(1-p)) - \log((k-1)!)$$
$$+(\frac{D}{2} + \sqrt{2\pi D})\log(1-p) - \log(-2rD\log(1-p)) + o(1)$$
$$\geq -1\log\log A + C_4 + c(A)/2. \tag{68}$$

where C_4 contains all constant and $o(1)$ terms. Therefore, we conclude that

$$E[V_k](\log A) \geq C_3 e^{c(A)/2}. \tag{69}$$

$\square$

Proposition 6 *Given the same conditions as in Proposition (5),*

$$\frac{E[V_k^2]}{E[V_k]^2} \leq 1 + \frac{C_5(p + (\log A)^{-1})}{E[V_k]\log A} \tag{70}$$

Proof. Similar to the proofs in Section III-B,

$$E[V_k^2] = E[\int_R \int_R \chi_k(Z_1)\chi_k(Z_2)dZ_1 dZ_2]$$
$$= \int_R \int_R E[\chi_k(Z_1)\chi_k(Z_2)]dZ_1 dZ_2 \tag{71}$$

We still consider two cases in the integral: case (i) $|Z_1 - Z_2| > 2r$; case (ii) $|Z_1 - Z_2| \leq 2r$. In the first case, $\chi_k(Z_1)$ and $\chi_k(Z_2)$ are independent. Therefore,

$$\int\int_{R^2 \cap \{|Z_1-Z_2|>2r\}} E[\chi_k(Z_1)\chi_k(Z_2)]dZ_1 dZ_2$$
$$\leq \int\int_{R^2} E[\chi_k(Z_1)]E[\chi_k(Z_2)]dZ_1 dZ_2$$
$$= E[V_k]^2 \tag{72}$$

In the second case, $|Z_1 - Z_2| \leq 2r$, we follow the derivations similar to those in Section III-B.

$$\int\int_{R^2 \cap \{|Z_1-Z_2|\leq 2r\}} E[\chi_k(Z_1)\chi_k(Z_2)]dZ_1 dZ_2$$
$$\leq 2\int\int_{R^2 \cap \{|Z_1-Z_2|\leq 2r\} \cap Q} E[\chi_k(Z_1)\chi_k(Z_2)]dZ_1 dZ_2 \tag{73}$$

where Q is the event that Z_2 is inside the square centered at the center of R and whose boundary crosses Z_1. Again, we only consider the dominating subcase when Z_1 is not in the extended corner region C_E (the extended corner region is the corner region and the side region that is at most $3r$ away from a second side of the region R). Still denote B_1, B_2 as the unit-area disks centered at Z_1, Z_2, respectively. Under the above conditions (i.e., $R^2 \cap \{|Z_1 - Z_2| \leq 2r\} \cap Q$),

$E[\chi_k(Z_1)\chi_k(Z_2)]$
$= P$(there are less than k active nodes in $B_1 \cap R$ and there are less than k active nodes in $B_2 \cap R$)
$\leq P$(there are less than k active nodes in $B_1 \cap R$)
$\cdot P$(there are less than k active nodes in $(B_2 - B_1) \cap R$)

As has been proved in Section III-B, the area of $(B_2 - B_1) \cap R$ is at least $x/(4r)$ and its perimeter length is at most $2\pi r$ (noticing the shape $B_2 - B_1$ has the same perimeter length as

B_2 or B_1), Therefore, the number of nodes inside $(B_2 - B_1) \cap R$ is at least $m_x = \max(0, Dx/(4r) - \sqrt{2\pi D})$. Therefore, conditioning on $(R^2 \cap \{|Z_1 - Z_2| \leq 2r\} \cap Q)$,

$$E[\chi_k(Z_1)\chi_k(Z_2)]$$
$$\leq E[\chi_k(Z_1)] \sum_{j=0}^{k-1} \binom{m_x}{j} p^j (1-p)^{m_x-j} \tag{74}$$

Now we integrate over the space $(R^2 \cap \{|Z_1 - Z_2| \leq 2r\} \cap Q)$, and obtain

$$\int\int_{R^2 \cap \{|Z_1-Z_2|\leq 2r\} \cap Q} E[\chi_k(Z_1)\chi_k(Z_2)]dZ_1 dZ_2$$
$$\leq \int_R E[\chi_k(Z_1)] \int_{Z_2:R\cap\{|Z_1-Z_2|\leq 2r\}\cap Q}$$
$$\sum_{j=0}^{k-1} \binom{m_x}{j} p^j (1-p)^{m_x-j} dZ_2 dZ_1$$
$$\leq \int_R E[\chi_k(Z_1)]dZ_1 \int_0^{2r} \sum_{j=0}^{k-1} \binom{m_x}{j} p^j (1-p)^{m_x-j} 2\pi x dx \tag{75}$$

where the $2\pi x$ in the last equation comes from the conversion from a Cartesian coordinate system to a polar coordinate system. Notice a convention of $\binom{n}{m} = 0$ if $n < m$. Recall $r = 1/\sqrt{\pi}$ and $1/\sqrt{D}$ is the distance between two adjacent nodes. If $x \leq 4\sqrt{2/D}$, $m_x = 0$. Therefore, the integration in Eq. (75) can be divided into two parts.

$$\int_0^{2r} \sum_{j=0}^{k-1} \binom{m_x}{j} p^j (1-p)^{m_x-j} 2\pi x dx$$
$$\leq \left(\int_0^{4\sqrt{2/D}} + \int_{4\sqrt{2/D}}^{2r}\right) \sum_{j=0}^{k-1} \binom{m_x}{j} p^j (1-p)^{m_x-j} 2\pi x dx$$
$$= \int_0^{4\sqrt{2/D}} 2\pi x dx + \int_{4\sqrt{2/D}}^{2r} \sum_{j=0}^{k-1} \binom{m_x}{j} p^j (1-p)^{m_x-j} 2\pi x dx$$
$$\leq 32\pi/D + \int_{4\sqrt{2/D}}^{2r} \sum_{j=0}^{k-1} \frac{(m_x p/(1-p))^j}{j!} (1-p)^{m_x} 2\pi x dx$$
$$= 32\pi/D + \int_{4\sqrt{2/D}}^{2r} \sum_{j=0}^{k-1} \frac{((Dx/(4r) - \sqrt{2\pi D})p/(1-p))^j}{j!}$$
$$\cdot (1-p)^{(Dx/(4r)-\sqrt{2\pi D})} 2\pi x dx$$
(let $u = x/(4r) - \sqrt{2\pi/D}$)
$$= 32\pi/D + \int_0^{1/2-\sqrt{2\pi/D}} \sum_{j=0}^{k-1} \frac{(uDp/(1-p))^j}{j!}$$
$$\cdot (1-p)^{uD} 2\pi 4r(u + \sqrt{2\pi/D})4r du$$
(since $p \leq -\log(1-p)$)
$$\leq 32\pi/D + (1-p)^{-k+1} \int_0^\infty \sum_{j=0}^{k-1} \frac{(-uD\log(1-p))^j}{j!}$$
$$\cdot e^{uD\log(1-p)} 32(u + \sqrt{2\pi/D}) du \tag{76}$$

Treat $-D\log(1-p)$ as λ, by Lemma 4 we can obtain that

$$\int_0^\infty \sum_{j=0}^{k-1} \frac{(-uD\log(1-p))^j}{j!} e^{uD\log(1-p)} 32u du$$
$$= 16k(k+1)(-D\log(1-p))^{-2}, \tag{77}$$

and similarly

$$\int_0^\infty \sum_{j=0}^{k-1} \frac{(-uD\log(1-p))^j}{j!} e^{uD\log(1-p)} 32\sqrt{2\pi/D}\, du$$
$$= 32k\sqrt{2\pi/D}(-D\log(1-p))^{-1}. \tag{78}$$

Hence, combining Eqs. (76), (77) and (78),

$$\int_0^{2r} \sum_{j=0}^{k-1} \binom{m_x}{j} p^j (1-p)^{m_x-j} 2\pi x\, dx$$
$$\leq 32\pi/D + \epsilon^{-k+1}(16k(k+1)(-D\log(1-p))^{-2}$$
$$\quad + 32k\sqrt{2\pi/D}(-D\log(1-p))^{-1})$$
$$\leq (32\pi p + C_5(\log A)^{-1} + C_6\sqrt{p/\log A})(\log A)^{-1}$$
$$\leq C_7(p + (\log A)^{-1})(\log A)^{-1}, \tag{79}$$

for some constant C_7.

Plugging Eq. (79) into Eq. (75), we obtain

$$\iint_{R^2\cap\{|Z_1-Z_2|\leq 2r\}\cap Q} E[\chi_k(Z_1)\chi_k(Z_2)]dZ_1 dZ_2$$
$$\leq E[V_k]C_7(p + (\log A)^{-1})(\log A)^{-1} \tag{80}$$

Combining this equation with Eqs. (71), (72), and (73), we have

$$E[V_k^2] \leq E[V_k]^2 + 2C_7 E[V_k](p + (\log A)^{-1})(\log A)^{-1} \tag{81}$$

Therefore,

$$\frac{E[V_k^2]}{E[V_k]^2} \leq 1 + \frac{2C_7(p + (\log A)^{-1})}{E[V_k](\log A)} \tag{82}$$

Choosing $C_5 = 2C_7$ completes the proof. $\qquad\square$

We are now ready to show the following necessary condition of complete k-coverage in the case of grid deployment. The proof is almost identical to that of theorem 2 and is thus omitted.

Theorem 6 *Given the same conditions as in Proposition 5 except that $c(A)$ need not be $o(\log\log A)$ (it still needs to go to ∞), P(the region R is not k-covered) $\rightarrow 1$.*

Comments: There is a gap between the necessary condition and the sufficient condition on the density requirement for k-coverage. However, the gap with the term $2\sqrt{-2\pi\log A\log(1-p)}$ is caused by the uncertainty of the number of lattice points contained in a circle (called Gauss's circle problem [1]), which is most probably not closable. We are not clear whether the gap with the term $\log\log A$ is closable or not. However, if $p = O((\log A)^{-1})$, the gaps with both of the two terms diminish. The first gap diminishes obviously. The second gap diminishes because the conclusion in Proposition 6 reduces to

$$\frac{E[V_k^2]}{E[V_k]^2} \leq 1 + \frac{C_5}{E[V_k](\log A)^2}, \tag{83}$$

which is essential to close the gap on the term $\log\log A$.

VI. RELATED WORK

Early research on density requirement for coverage focused on 1-coverage. In [8], Philips showed that $\pi r^2\lambda \sim \log A$ is a necessary and sufficient condition for coverage and a necessary condition for connectivity in a random network where $r(n)$ is the radius of sensing (communication), and nodes are distributed according to a Poisson point process with density λ in a region of area A. In [4], Hall showed if the nodes are distributed as a Poisson point process with density λ in a unit-area square, then $0.05\min\{1, (1+\lambda^2\pi r^2)e^{-\lambda\pi r^2}\} < P(V_1 > 0) < 3\min\{1, (1+\lambda^2\pi r^2)e^{-\lambda\pi r^2}\}$, where r is the sensing range and V_1 denotes the 1-vacancy area. Both of the above results are consistent with our results in the special case of $k = 1$. In particular, for $k = 1$, boundary conditions do not cause extra density requirement. However, for $k > 1$, adding boundary conditions does require more density for complete k-coverage.

Recently, Shakkottai [9] derived necessary and sufficient conditions for 1-coverage and 1-connectivity when n sensors are deployed in a $\sqrt{n} \times \sqrt{n}$ grids and each sensor is active with probability p.

Some more recent works studied the density requirement for k-coverage. In particular, Zhang and Hou [10] derived the density requirement for k-coverage. Assuming that (i) nodes are distributed as a Poisson point process with density λ in a square region with side length l, (ii) each node covers a unit-area disk centered at itself, they proved that $\lambda = \log l^2 + (k+1)\log\log l^2 + c(l)$ and $c(l) \rightarrow \infty$ is necessary and sufficient for k-coverage of the monitored region.

Kumar *et al.* studied the issue of k-coverage under three different deployment strategies: grid deployment, uniform distribution, and Poisson point process. They also considered the boundary issues. However, in their derivation for k-coverage, they only showed their conclusion holds for the inner regions. As a result, they obtained the same density requirement for k-coverage as in [10] in the case that nodes are distributed as a Poisson point process. We have corrected the mistake in this paper, and moreover, we have proved much sharper bounds on the density requirement in all cases.

VII. CONCLUSION

In this paper we have studied the problem of determining the critical node density for maintaining k-coverage of a given square region. We have considered three different deployment strategies: Poisson point process, uniform random distribution, and grid deployment. We have showed that the two random strategies have identical density requirement for k-coverage, and that grid deployment requires less node density than the two random deployment strategies in order to achieve the same level of coverage degree if the probability that a node is active does not converge to 1 or 0. If the probability that a node is active tends to 0, all strategies require the same order of magnitude of node density to achieve a certain coverage degree. Our results overrule a previous counter-intuitive conclusion that grid deployment may require more node density than random deployment strategies.

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