Choukri Derbazi; Hadda Hammouche

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EXISTENCE AND UNIQUENESS OF INTEGRABLE SOLUTIONS TO FRACTIONAL LANGEVIN EQUATIONS INVOLVING TWO FRACTIONAL ORDERS WITH INITIAL VALUE PROBLEMS

Choukri Derbazi, Hadda Hammouche, Ghardaia

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Abstract. We study the existence and uniqueness of integrable solutions to fractional Langevin equations involving two fractional orders with initial value problems. Our results are based on Schauder's fixed point theorem and the Banach contraction principle fixed point theorem. Examples are provided to illustrate the main results.

Keywords: fractional Langevin equation; Caputo fractional derivative; integrable solution; existence; uniqueness; initial value problem; fixed point theorem

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1. Introduction

Fractional differential equations have been of increasing importance in the past decades due to their diverse applications in science and engineering, such as the memory of a variety of materials, signal identification and image processing, optical systems, thermal system materials and mechanical systems, control systems, etc., see [11], [13], [14], [15].

By the use of techniques of nonlinear analysis, many authors have studied the existence and uniqueness of solutions of nonlinear fractional differential equations involving different kinds of fractional derivatives under various boundary conditions, see [3], [5], [6], [12], [16], [20] and references therein.

In recent years, the nonlinear Langevin equation with two fractional orders has attracted a great deal of interest and attention from several researchers. For some developments on the existence results of the nonlinear Langevin equation with two fractional orders, we can refer to [1], [2], [4], [19], [21] and the references therein.

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However, as far as we know, few papers deal with the existence of integrable solutions for nonlinear fractional differential equations. We refer the readers to the papers [7], [9], [10], [17], [18] and references therein.

Motivated by the papers above, in this paper, we deal with the existence and uniqueness of integrable solutions for the following initial value problem (IVP for short) of the Langevin equation involving two fractional orders:

$$\begin{cases} {}^cD^{\beta}_{0+}({}^cD^{\alpha}_{0+}+\gamma)x(t)=f(t,x(t)), & t\in J:=[0,1],\\ x^{(k)}(0)=\mu_k, & 0\leqslant k<l,\\ x^{(\alpha+k)}(0)=\nu_k, & 0\leqslant k<n, \end{cases} \tag{1.1}$$

where ${}^cD^{\alpha}_{0+}$, ${}^cD^{\beta}_{0+}$ are the Caputo fractional derivatives $m-1<\alpha\leqslant m$, $n-1<\beta\leqslant n$, $l=\max\{m,n\}$, $m,n\in\mathbb{N}^*$, $\gamma\in\mathbb{R}$ and $f\colon J\times\mathbb{R}\to\mathbb{R}$ is a given function satisfying some assumptions that will be specified later.

This paper is organized as follows. The second section provides the definitions and preliminary results to be used in this paper. In Section 3, we give the proof of our main results by applying fixed point theorems such as Banach's contraction principle and Schauder's fixed point theorem. Finally, in Section 4, some illustrative examples are introduced to explain the applicability of the theory.

2. Preliminaries

Before proceeding to the statement of our main results, we set forth definitions, preliminaries, and hypotheses that will be used in our subsequent discussion. For more details, see [11], [14], [15].

Denote by $L^1(J,\mathbb{R})$ the class of Lebesgue integrable functions on the interval $J:=[0,1]$ with the norm

$$\|u\|_{L^1}=\int_0^1|u(t)|\,\mathrm{d}t.$$

Definition 2.1 ([14])**.** The Gamma function, or second order Euler integral, denoted $\Gamma(\cdot)$ is defined as:

$$\Gamma(\alpha)=\int_0^{\infty}\mathrm{e}^{-t}t^{\alpha-1}\,\mathrm{d}t,\quad \alpha>0. \tag{2.1}$$

The Euler Gamma function is an extension of the factorial function to real numbers and is considered the most important Eulerian function used in fractional calculus because it appears in almost every fractional integral and derivative definitions. For positive integer values n, the Gamma function becomes $\Gamma(n)=(n-1)!$ and thus can be seen as an extension of the factorial function to real values. An important property of the Gamma function $\Gamma(\alpha)$ is that it satisfies:

$$\Gamma(\alpha+1)=\alpha\Gamma(\alpha), \quad \alpha>0.$$

Definition 2.2 ([14]). The Beta function, or the first order Euler function, can be defined as

$$\mathrm{B}(p,q)=\int_0^1 t^{p-1}(1-t)^{q-1}\,\mathrm{d}t, \quad p,q>0.$$

The following formula expresses the Beta function in terms of the Gamma function:

$$\mathrm{B}(p,q)=\frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}, \quad p,q>0.$$

We recall the well-known Cauchy formula for n-fold integrals:

$$\int_0^t \mathrm{d}t_1 \int_0^{t_1} \mathrm{d}t_2 \int_0^{t_2} \mathrm{d}t_3 \ldots \int_0^{t_{n-1}} f(t_n)\,\mathrm{d}t_n = \frac{1}{(n-1)!}\int_0^t (t-s)^{\alpha-1} f(s)\,\mathrm{d}s. \tag{2.2}$$

Focusing on the formula (2.2) we see it as an inspiration to define the Riemann-Liouville fractional integral; we generalize this formula by letting n take values other than the non-negative integers and note at the same time that the factorial function is a special case of the Gamma function $\Gamma(\cdot)$.

Definition 2.3 ([11]). The Riemann-Liouville fractional integral of order $\alpha>0$ of a function $f\in L^1(J,\mathbb{R})$ is defined by

$$I_{0+}^{\alpha} f(t)=\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s)\,\mathrm{d}s, \quad t>0,\ \alpha>0.$$

Moreover, for $\alpha=0$, we set $I_{0+}^{\alpha} f:=f$. It is obvious that the Riemann-Liouville fractional integral coincides with the classical definition of I_{0+}^{n} in the case $n\in\mathbb{N}$.

Lemma 2.4 ([11]). *The following basic properties of the Riemann-Liouville integrals hold:*

(1) *The integral operator I_{0+}^{α} is linear;*
(2) *The semigroup property of the fractional integration operator I_{0+}^{α} is given by the following result*

$$I_{0+}^{\alpha}(I_{0+}^{\beta} f(t))=I_{0+}^{\alpha+\beta} f(t), \quad \alpha,\beta>0,$$

holds at every point if $f\in C([0,1])$ and holds almost everywhere if $f\in L^1(J,\mathbb{R})$;

(3) *Commutativity*

$$I_{0+}^{\alpha}(I_{0+}^{\beta}f(t)) = I_{0+}^{\beta}(I_{0+}^{\alpha}f(t)), \quad \alpha,\beta > 0;$$

(4) *The fractional integration operator* I_{0+}^{α} *is bounded in* $L^p(J,\mathbb{R})$, $1 \leqslant p \leqslant \infty$,

$$\|I_{0+}^{\alpha}f\|_{L^p} \leqslant \frac{1}{\Gamma(\alpha+1)}\|f\|_{L^p}.$$

Example 2.5 ([11]). Let $\alpha > 0$ and $\beta > -1$. Then the Riemann-Liouville fractional integral of the power function is given by

$$I_{0+}^{\alpha}t^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)}t^{\alpha+\beta}.$$

From the definition of the Riemann-Liouville fractional integral, the fractional derivative is obtained not by replacing α with $-\alpha$ because the integral $\int_0^t (t-s)^{-\alpha-1} \times f(s)\,\mathrm{d}s$ is, in general, divergent. Instead, differentiation of arbitrary order is defined as the composition of ordinary differentiation and fractional integration.

Definition 2.6 ([11], [14]). The Caputo fractional derivative ${}^cD_{0+}^{\alpha}$ of order α of a function $f \in AC^n(J,\mathbb{R})$ is represented by

$${}^cD_{0+}^{\alpha}f(t) = \begin{cases} \dfrac{1}{\Gamma(n-\alpha)}\displaystyle\int_0^t (t-s)^{n-\alpha-1}f^{(n)}(s)\,\mathrm{d}s & \text{if } \alpha \notin \mathbb{N}, \\ f^{(n)}(t) & \text{if } \alpha \in \mathbb{N}, \end{cases}$$

where $f^{(n)}(t) = \mathrm{d}^n f(t)/\,\mathrm{d}t^n$, $\alpha > 0$, $n = [\alpha]+1$ and $[\alpha]$ denotes the integer part of the real number α.

Example 2.7 ([11]). The Caputo fractional derivative of order $n-1 < \alpha < n$ for t^{β} is given by

$$(2.3) \qquad {}^cD_{0+}^{\alpha}t^{\beta} = \begin{cases} \dfrac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}t^{\beta-\alpha}, & \beta \in \mathbb{N} \wedge \beta \geqslant n \text{ or } \beta \notin \mathbb{N} \wedge \beta > n-1, \\ 0, & \beta \in \{0,\ldots,n-1\}. \end{cases}$$

Lemma 2.8 ([11], [14]). *Let* $\alpha > 0$ *and* $n = [\alpha]+1$. *Then the differential equation*

$${}^cD_{0+}^{\alpha}f(t) = 0,$$

has solutions

$$f(t) = \sum_{j=0}^{n-1} c_j t^j, \quad c_j \in \mathbb{R},\ j = 0,\ldots,n-1.$$

Lemma 2.9 ([11], [14]). *Let $\alpha > \beta > 0$, and $f \in L^1(J,\mathbb{R})$. Then we have:*

(1) *The Caputo fractional derivative is linear;*

(2) *The Caputo fractional derivative obeys the following property:*

$$I^{\alpha}_{0+}\,{}^{c}D^{\alpha}_{0+} f(t) = f(t) + \sum_{j=0}^{n-1} c_j t^j$$

for some $c_j \in \mathbb{R}$, $j = 0,1,2,\ldots,n-1$, where $n = [\alpha]+1$;

(3) ${}^{c}D^{\alpha}_{0+} I^{\alpha}_{0+} f(t) = f(t)$;

(4) ${}^{c}D^{\beta}_{0+} I^{\alpha}_{0+} f(t) = I^{\alpha-\beta}_{0+} f(t)$.

Theorem 2.10 (Schauder's fixed point theorem, [8]). *Let (E,d) be a complete metric space, let U be a closed convex subset of E, and let $\mathcal{T}\colon U \to U$ be a mapping such that the set $\{\mathcal{T}u\colon u \in U\}$ is relatively compact in E. Then $\mathcal{T}$ has at least one fixed point.*

Theorem 2.11 (Kolmogorov compactness criterion, [8]). *Let $\Omega \subset L^p(J,\mathbb{R})$, $1 \leqslant p < \infty$. If*

(a) *Ω is bounded in $L^p(J,\mathbb{R})$,*

(b) *$x_h \to x$ as $h \to 0$ uniformly with respect to $x \in \Omega$, where*

$$x_h(t) = \frac{1}{h}\int_t^{t+h} x(s)\,\mathrm{d}s,$$

then Ω is relatively compact in $L^p(J,\mathbb{R})$.

3. Main results

Before starting and proving our main results we introduce the following auxiliary lemma:

Lemma 3.1 ([19]). *$x(t)$ is a solution of the initial problem (1.1) if and only if $x(t)$ is a solution of the integral equation:*

$$x(t) = I^{\alpha+\beta}_{0+} f(t,x(t)) - \gamma I^{\alpha}_{0+} x(t) + Q(t), \tag{3.1}$$

where

$$Q(t) = \sum_{i=0}^{n-1} \frac{\nu_i + \gamma\mu_i}{\Gamma(\alpha+i+1)} t^{\alpha+i} + \sum_{j=0}^{m-1} \frac{\mu_i}{\Gamma(j+1)} t^j.$$

In view of Lemma 3.1 we define the integral operator $\mathcal{T}\colon L^1(J,\mathbb{R}) \to L^1(J,\mathbb{R})$ by

$$
\begin{aligned}
\mathcal{T}x(t) &= I_{0^+}^{\alpha+\beta} f(t,x(t)) - \gamma I_{0^+}^{\alpha} x(t) + Q(t) \\
&= \frac{1}{\Gamma(\alpha+\beta)} \int_0^t (t-s)^{\alpha+\beta-1} f(s,x(s))\,\mathrm{d}s \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} x(s)\,\mathrm{d}s + Q(t).
\end{aligned} \tag{3.2}
$$

In this section we shall present and prove our main results. First, consider the following hypotheses:

(H1) $f\colon J \times \mathbb{R} \to \mathbb{R}$ is measurable with respect to t on J and is continuous with respect to x on $\mathbb{R}$.

(H2) There exist constant $L > 0$ such that

$$|f(t,x) - f(t,y)| \leqslant L|x-y|, \quad t \in J,\ x,y \in \mathbb{R}.$$

(H3) There exists a positive function $a \in L^1([0,1],\mathbb{R}_+)$ and a constant $b > 0$ such that:

$$|f(t,x)| \leqslant a(t) + b|x| \quad \forall\,(t,x) \in [0,1] \times \mathbb{R}.$$

Now we are able to establish the main results.

Our first result is based on the Banach contraction principle.

Theorem 3.2. *Assume that* (H1) *and* (H2) *hold. If*

$$\frac{L}{\Gamma(\alpha+\beta+1)} + \frac{|\gamma|}{\Gamma(\alpha+1)} < 1, \tag{3.3}$$

then the IVP (1.1) *has a unique solution* $x \in L^1(J,\mathbb{R})$.

Proof. Transform the problem (1.1) into a fixed point problem. Clearly, the fixed points of the operator $\mathcal{T}$ defined by (3.2) are solutions of the problem (1.1). Let $x,y \in L^1(J,\mathbb{R})$ and $t \in J$. Then we have

$$
\begin{aligned}
\|\mathcal{T}x - \mathcal{T}y\|_{L^1} &= \int_0^1 |\mathcal{T}x(t) - \mathcal{T}y(t)|\,\mathrm{d}t \\
&= \int_0^1 |I_{0^+}^{\alpha+\beta}(f(t,x(t)) - f(t,y(t))) - \gamma I_{0^+}^{\alpha}(x(t)-y(t))|\,\mathrm{d}t. \\
&\leqslant \int_0^1 |I_{0^+}^{\alpha+\beta}(f(t,x(t)) - f(t,y(t)))|\,\mathrm{d}t \\
&\quad + |\gamma| \int_0^1 |I_{0^+}^{\alpha}(x(t)-y(t))|\,\mathrm{d}t.
\end{aligned}
$$

Using hypothese (H2) we get

$$\|\mathcal{T}x - \mathcal{T}y\|_{L^1} \leqslant L \int_0^1 (I_{0^+}^{\alpha+\beta} |x(t) - y(t)|) \, \mathrm{d}t + |\gamma| \int_0^1 (I_{0^+}^{\alpha} |x(t) - y(t)|) \, \mathrm{d}t.$$

According to the Lemma 2.4, we have

$$\begin{aligned} \|\mathcal{T}x - \mathcal{T}y\|_{L^1} &\leqslant \frac{L}{\Gamma(\alpha+\beta+1)} \int_0^1 |x(t) - y(t)| \, \mathrm{d}t + \frac{|\gamma|}{\Gamma(\alpha+1)} \int_0^1 |x(t) - y(t)| \, \mathrm{d}t \\ &\leqslant \Big(\frac{L}{\Gamma(\alpha+\beta+1)} + \frac{|\gamma|}{\Gamma(\alpha+1)}\Big) \|x - y\|_{L^1}. \end{aligned}$$

In view of the given condition $(L/\Gamma(\alpha+\beta+1)) + (|\gamma|/\Gamma(\alpha+1)) < 1$, it follows that the mapping $\mathcal{T}$ is a contraction. Hence, by the Banach fixed point theorem, $\mathcal{T}$ has a unique fixed point which is a unique solution of problem (1.1). This completes the proof. □

Our next result is upon the Schauder's fixed point theorem.

Theorem 3.3. *Assume that the assumptions* (H1) *and* (H3) *are satisfied. If*

$$\frac{b}{\Gamma(\alpha+\beta+1)} + \frac{|\gamma|}{\Gamma(\alpha+1)} < 1. \tag{3.4}$$

Then the IVP (1.1) *has at least one solution* $x \in L^1(J, \mathbb{R})$.

P r o o f. In order to use the Schauder fixed-point theorem to prove our main result, we define a subset B_r of $L^1(J, \mathbb{R})$ defined by

$$B_r = \{x \in L^1(J, \mathbb{R}) \colon \|x\|_{L^1} \leqslant r\},$$

where r satisfies the inequality

$$r \geqslant \Big(\frac{\|a\|_{L^1}}{\Gamma(\alpha+\beta+1)} + \|Q\|_{L^1}\Big) \Big/ \Big(1 - \frac{b}{\Gamma(\alpha+\beta+1)} - \frac{|\gamma|}{\Gamma(\alpha+1)}\Big). \tag{3.5}$$

And

$$\|Q\|_{L^1} = \sum_{i=0}^{n-1} \frac{|\nu_i| + |\gamma\mu_i|}{\Gamma(\alpha+i+2)} + \sum_{j=0}^{m-1} \frac{|\mu_j|}{\Gamma(j+2)}.$$

Notice that B_r is a closed, convex and bounded subset of the Banach space $L^1(J, \mathbb{R})$. From the assumption (H1) we can deduce that the operator $\mathcal{T}$ is continuous.

We will show that $\mathcal{T}(B_r) \subset B_r$. Suppose that x is an arbitrary element in B_r. Then we have

$$\begin{aligned}
\|\mathcal{T}x\|_{L^1} &= \int_0^1 |\mathcal{T}x(t)|\,\mathrm{d}t \\
&= \int_0^1 |I_{0^+}^{\alpha+\beta} f(t,x(t)) - \gamma I_{0^+}^{\alpha} x(t) + Q(t)|\,\mathrm{d}t \\
&\leqslant \int_0^1 |I_{0^+}^{\alpha+\beta} f(t,x(t))|\,\mathrm{d}t + |\gamma| \int_0^1 |I_{0^+}^{\alpha} x(t)|\,\mathrm{d}t + \int_0^1 |Q(t)|\,\mathrm{d}t.
\end{aligned}$$

Using hypothese (H3) we get

$$\begin{aligned}
\|\mathcal{T}x\|_{L^1} &\leqslant \int_0^1 |I_{0^+}^{\alpha+\beta}(a(t) + b|x(t)|)|\,\mathrm{d}t \\
&\quad + |\gamma| \int_0^1 I_{0^+}^{\alpha} |x(t)|\,\mathrm{d}t + \int_0^1 |Q(t)|\,\mathrm{d}t \\
&\leqslant \|I_{0^+}^{\alpha+\beta} a\|_{L^1} + b \int_0^1 |I_{0^+}^{\alpha+\beta}(|x(t)|)|\,\mathrm{d}t \\
&\quad + |\gamma| \int_0^1 |I_{0^+}^{\alpha}(|x(t)|)|\,\mathrm{d}t + \|Q\|_{L^1}.
\end{aligned}$$

According to the Lemma 2.4 and (H3), we have

$$\begin{aligned}
\|\mathcal{T}x\|_{L^1} &\leqslant \frac{\|a\|_{L^1}}{\Gamma(\alpha+\beta+1)} + \frac{b}{\Gamma(\alpha+\beta+1)} \int_0^1 |x(t)|\,\mathrm{d}t \\
&\quad + \frac{|\gamma|}{\Gamma(\alpha+1)} \int_0^1 |x(t)|\,\mathrm{d}t + \|Q\|_{L^1} \\
&\leqslant \frac{\|a\|_{L^1}}{\Gamma(\alpha+\beta+1)} + \Big(\frac{b}{\Gamma(\alpha+\beta+1)} + \frac{|\gamma|}{\Gamma(\alpha+1)}\Big)\|x\|_{L^1} + \|Q\|_{L^1} \\
&\leqslant \frac{\|a\|_{L^1}}{\Gamma(\alpha+\beta+1)} + \Big(\frac{b}{\Gamma(\alpha+\beta+1)} + \frac{|\gamma|}{\Gamma(\alpha+1)}\Big) r + \|Q\|_{L^1}.
\end{aligned}$$

By the condition (3.5), we deduce that

$$\|\mathcal{T}x\|_{L^1} \leqslant r,$$

which implies that $\mathcal{T}(B_r) \subset B_r$.

Now, we will show that $\mathcal{T}$ is compact, that is, $\mathcal{T}(B_r)$ is relatively compact. Clearly $\mathcal{T}(B_r)$ is bounded in $L^1(J,\mathbb{R})$, i.e. condition (a) of the Kolmogorov compactness criterion is satisfied. It remains to show $(\mathcal{T}x)_h \to (\mathcal{T}x)$ in $L^1(J,\mathbb{R})$, for each $x \in B_r$. Let $x \in B_r$. Then we have

$$\begin{aligned}
\|(\mathcal{T}x)_h - (\mathcal{T}x)\|_{L^1} &= \int_0^1 |(\mathcal{T}x)_h(t) - (\mathcal{T}x)(t)| \, \mathrm{d}t \\
&= \int_0^1 \Big| \frac{1}{h} \int_t^{t+h} (\mathcal{T}x)(s) \, \mathrm{d}s - (\mathcal{T}x)(t) \Big| \, \mathrm{d}t \\
&\leqslant \int_0^1 \Big(\frac{1}{h} \int_t^{t+h} |(\mathcal{T}x)(s) - (\mathcal{T}x)(t)| \, \mathrm{d}s \Big) \, \mathrm{d}t \\
&\leqslant \int_0^1 \frac{1}{h} \int_t^{t+h} |I_{0^+}^{\alpha+\beta} f(s, x(s)) - I_{0^+}^{\alpha+\beta} f(t, x(t))| \, \mathrm{d}s \, \mathrm{d}t \\
&\quad + |\gamma| \int_0^1 \frac{1}{h} \int_t^{t+h} |I_{0+}^{\alpha} x(s) - I_{0+}^{\alpha} x(t)| \, \mathrm{d}s \, \mathrm{d}t \\
&\quad + \int_0^1 \frac{1}{h} \int_t^{t+h} |Q(s) - Q(t)| \, \mathrm{d}s \, \mathrm{d}t.
\end{aligned}$$

Since $f, x \in L^1(J, \mathbb{R})$, we get that $I_{0^+}^{\alpha+\beta} f, I_{0+}^{\alpha} x \in L^1(J, \mathbb{R})$. Moreover $Q \in L^1(J, \mathbb{R})$. So, we have

$$\begin{aligned}
\frac{1}{h} \int_t^{t+h} |I_{0^+}^{\alpha+\beta} f(s, x(s)) - I_{0^+}^{\alpha+\beta} f(t, x(t))| \, \mathrm{d}s &\to 0, \\
|\gamma| \frac{1}{h} \int_t^{t+h} |I_{0+}^{\alpha} f(s, x(s)) - I_{0+}^{\alpha} x(t)| \, \mathrm{d}s &\to 0, \\
\frac{1}{h} \int_t^{t+h} |Q(s) - Q(t)| \, \mathrm{d}s &\to 0 \quad \text{as } h \to 0, \ t \in J.
\end{aligned}$$

Hence

$$(\mathcal{T}x)_h \to (\mathcal{T}x) \quad \text{uniformly as } h \to 0.$$

Then by Kolmogorov compactness criterion, $\mathcal{T}(B_r)$ is relatively compact. As a consequence of Schauder's fixed point theorem, the IVP (1.1) has at least one solution in B_r. □

4. Examples

In this section, in order to illustrate our results, we consider two examples.

Example 4.1. Consider the following fractional Langevin IVP:

$$\begin{cases} {}^cD_{0^+}^{1/4} \Big({}^cD_{0^+}^{1/2} + \dfrac{1}{8} \Big) x(t) = \dfrac{\mathrm{e}^{-t}}{20(9 + \mathrm{e}^t)} \big(x(t) + \sqrt{1 + x^2(t)} \big), \quad t \in J := [0, 1], \\ x(0) = 1, \\ x^{(1/2)}(0) = 1. \end{cases} \tag{4.1}$$

In this case we take

$$\alpha = \frac{1}{2}, \quad \beta = \frac{1}{4}, \quad \gamma = \frac{1}{8}, \quad \mu_0 = 1, \quad \nu_0 = 1, \quad f(t,x) = \frac{\mathrm{e}^{-t}}{20(9+\mathrm{e}^t)}\big(x + \sqrt{1+x^2}\big).$$

It is clear that assumption (H1) of the Theorem 3.2 is satisfied. On the other hand, for any $t \in J$, $x, y \in \mathbb{R}$ we have

$$\begin{aligned} |f(t,x) - f(t,y)| &= \frac{\mathrm{e}^{-t}}{10(9+\mathrm{e}^t)}\Big|\frac{1}{2}\big(x - y + \sqrt{1+x^2} - \sqrt{1+y^2}\big)\Big| \\ &= \frac{\mathrm{e}^{-t}}{10(9+\mathrm{e}^t)}\Big|\frac{1}{2}(x-y)\Big(1 + \frac{x+y}{\sqrt{1+x^2}+\sqrt{1+y^2}}\Big)\Big| \\ &\leqslant \frac{\mathrm{e}^{-t}}{10(9+\mathrm{e}^t)}|x-y| \leqslant \frac{1}{100}|x-y|. \end{aligned}$$

Hence condition (H2) holds with $L = \frac{1}{100}$. We shall check that condition (3.3) is satisfied. Indeed

$$\frac{L}{\Gamma(\alpha+\beta+1)} + \frac{|\gamma|}{\Gamma(\alpha+1)} = 0.1519 < 1.$$

Then by Theorem 3.2, the IVP (4.1) has a unique integrable solution on $[0,1]$.

Example 4.2. Consider the following fractional Langevin IVP:

$$\text{(4.2)} \qquad \begin{cases} {}^cD_{0^+}^{\frac{3}{2}}\Big({}^cD_{0^+}^{5/2} + \frac{1}{2}\Big)x(t) = \frac{1}{(t^2+2)^2}\sin x(t) + \frac{t^3}{5}, & t \in J := [0,1], \\ x^{(k)}(0) = \mu_k = k, & 0 \leqslant k < 3, \\ x^{(\alpha+k)}(0) = \nu_k = \frac{k}{2}, & 0 \leqslant k < 2. \end{cases}$$

In this case we take

$$\alpha = \frac{5}{2}, \quad \beta = \frac{3}{2}, \quad \gamma = \frac{1}{2}, \quad f(t,x) = \frac{1}{(t^2+2)^2}\sin x + \frac{t^3}{5},$$

$$Q(t) = \sum_{i=0}^{1} \frac{5i}{2\Gamma(\alpha+i+1)} t^{\alpha+i} + \sum_{j=0}^{2} \frac{j}{\Gamma(j+1)} t^j.$$

Simple calculus gives

$$\|Q\|_{L^1} = 0.8811.$$

It is clear that assumption (H1) of the Theorem 3.3 is satisfied. On the other hand, for any $t \in J$, $x \in \mathbb{R}$ we have

$$|f(t,x)| \leqslant \frac{1}{4}|x| + \frac{t^3}{5}.$$

Hence condition (H3) holds with $a(t) = \frac{1}{5}t^3$, $b = \frac{1}{4}$. We get easily that $\|a\|_{L^1} = \frac{1}{20}$. We shall check that condition (3.3) is satisfied. Indeed

$$\frac{b}{\Gamma(\alpha+\beta+1)} + \frac{|\gamma|}{\Gamma(\alpha+1)} = 0.1609 < 1,$$

and

$$\Big(\frac{\|a\|_{L^1}}{\Gamma(\alpha+\beta+1)} + \|Q\|_{L^1}\Big)\Big/\Big(1 - \frac{b}{\Gamma(\alpha+\beta+1)} - \frac{|\gamma|}{\Gamma(\alpha+1)}\Big) = 1.0624.$$

Then r can be chosen as $r = 1.5 > 1.0624$. Thus, by Theorem 3.3, the IVP (4.2) has at least one solution on $[0,1]$.

5. Conclusions

We have presented the existence and uniqueness of integrable solutions to fractional Langevin equations involving two fractional orders with initial value problems. The proof of the existence results is based on the Schauder fixed point theorem, while the uniqueness of the solution is proved by applying the Banach contraction principle. Moreover, two examples are presented to illustrate the validity of our main results. In the future, we will extend the results to other fractional derivatives and boundary value problems.

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Authors' address: Choukri Derbazi, Hadda Hammouche, Department of Mathematics, Faculty of Sciences, University of Ghardaia, Scientific Zone, P.O. Box 455, Ghardaia 47000, Algeria, e-mail: `choukriedp@yahoo.com`, `h.hammouche@yahoo.fr`.