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On Bakhvalov-type meshes for a linear convection-diffusion problem in 2D

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Abstract. For singularly perturbed two-dimensional linear convection-diffusion problems, although optimal error estimates of an upwind finite difference scheme on Bakhvalov-type meshes are widely known, the analysis remains unanswered (Roos and Stynes in Comput. Meth. Appl. Math. 15 (2015), 531–550). In this short communication, by means of a new truncation error and barrier function based analysis, we address this open question for a generalization of Bakhvalov-type meshes in the sense of Boglaev and Kopteva. We prove that the upwind scheme on these mesh modifications is optimal first-order convergence, uniformly with respect to the perturbation parameter, in the discrete maximum norm. Furthermore, we derive a sufficient condition on the transition point choices to guarantee that our modified meshes can preserve the favorable properties of the original Bakhvalov mesh.
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1. Introduction

We consider an upwind difference method for solving the singularly perturbed convection-diffusion problem,

$$\begin{aligned} \mathcal{L}u := -\varepsilon\Delta u - b_1(x,y)u_x - b_2(x,y)u_y + c(x,y)u &= f(x,y) \quad \text{on} \quad \Omega = (0,1)^2, \\ u &= 0 \quad \text{on} \quad \Gamma = \partial\Omega, \end{aligned} \tag{1}$$

where ε is a small positive perturbation parameter, $0 < \varepsilon \ll 1$. We assume that the functions b_1, b_2, c and f are sufficiently smooth functions, and $b_1(x,y) > \beta_1 > 0$, $b_2(x,y) > \beta_2 > 0$, $c(x,y) \geq 0$ for all $(x,y) \in \bar{\Omega}$, and that they satisfy the compatibility conditions guaranteeing that problem (1) has a unique solution u in some suitable normed space [8, Theorem 5.1]. Then, u typically exhibits two exponential layers along the sides $x = 0$ and $y = 0$.

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It is well known that for higher-dimensional convection-dominated problems like (1), uniform convergence analysis of finite-difference and finite-element methods on Bakhvalov-type meshes is usually more challenging than on Shishkin-type meshes (cf. [14, Section 4] for a precise definition of these layer-fitted meshes and for related open problems). For example, in [13] and recently in [17, 18], new interpolation techniques are derived to prove optimal convergence results on Bakhvalov-type meshes when finite-element methods are applied to the one-dimensional analogue of (1); but no similar result is known in 2D [14, Question 7]. For finite-difference discretizations applied on Bakhvalov-type meshes, while optimal error estimates and analysis, proven by several different techniques (see, for instance, [3, 6, 10, 11, 16]), are well known in 1D, the following question [14, Question 6] remains open in 2D for certain Bakhvalov-type meshes:

> For an upwind finite-difference method applied on a Bakhvalov-type mesh to the convection-diffusion problem (1), can one prove a discrete maximum norm convergence result
>
> $$\max_{0\le i,j\le N} \left|u(x_i, y_j) - u_{ij}^N\right| \le CN^{-1}?$$

Here N mesh intervals are used in each coordinate direction, and u_{ij}^N denotes the computed solution at the point (x_i, y_j). From this point on, we use C to denote a generic positive constant independent of ε and N.

The very first attempt to answer the above challenge problem is given in [12] for which the Bakhvalov mesh [1] is considered. However, the construction of this mesh results in an implicitly defined transition point which is a main drawback in calculations. There are other modifications of the Bakhvalov mesh proposed in the research literature to resolve this issue with more relaxed transition points, and yet, the convergence order of the upwind scheme on these Bakhvalov-type meshes remains the same.

Our goal in this short article is to address the aforementioned open question for a mild generalization of Bakhvalov-type meshes of Boglaev [2] and Kopteva [3, 4]. These modifications replace an implicitly defined transition point of the original Bakhvalov mesh with an explicitly predefined one. The advantage of our modified transition point variants is the simplicity of calculation (and analysis as we shall show), while the grid obtained still enjoys the favorable properties of the Bakhvalov mesh.

In the next section, we list some preliminary facts about the solution u of problem (1) and introduce the upwind scheme for discretizing the problem. We propose a mild generalization of Boglaev and Kopteva's Bakhvalov-type meshes and discuss a sufficient condition to guarantee the grading properties for the proposed meshes in Section 3. Then, in Section 4, we apply the novel error analysis given in [12], which is specially tailored for the Bakhvalov mesh, to obtain the main convergence result. Finally, we finish with a concluding remark in the last section.

2. Preliminaries

The following decomposition of u [5, Lemma 1] is often used in error analysis of finite-difference methods for (1):

$$u = S + E_1 + E_2 + E_{12},$$

where, for all $(x,y) \in \bar{\Omega}$, we have

$$\|S\|_2 \le C, \tag{2}$$

$$\left|\frac{\partial^{k+\ell} E_1}{\partial x^k \partial y^\ell}(x,y)\right| \le C\varepsilon^{-k} e^{-\beta_1 x/\varepsilon}, \tag{3}$$

$$\left|\frac{\partial^{k+\ell} E_2}{\partial x^k \partial y^\ell}(x,y)\right| \le C\varepsilon^{-\ell} e^{-\beta_2 y/\varepsilon}, \tag{4}$$

and

$$\left|\frac{\partial^{k+\ell} E_{12}}{\partial x^k \partial y^\ell}(x,y)\right| \le C\varepsilon^{-(k+\ell)} e^{-(\beta_1 x + \beta_2 y)/\varepsilon}, \tag{5}$$

for $0 \le k+\ell \le 3$, and $\|\cdot\|_2$ denotes the supremum norm in $C^2(\Omega)$. Furthermore,

$$|\mathcal{L}E_1(x,y)| \le Ce^{-\beta_1 x/\varepsilon}, \tag{6}$$

$$|\mathcal{L}E_2(x,y)| \le Ce^{-\beta_2 y/\varepsilon}, \tag{7}$$

and

$$|\mathcal{L}E_{12}(x,y)| \le Ce^{-(\beta_1 x + \beta_2 y)/\varepsilon}. \tag{8}$$

Let N be an even positive integer and let $\Omega^N = \{(x_i, y_j) : \; i,j = 0,1,\ldots,N\}$ be the discretization mesh, where the mesh-point coordinates x_i and y_j satisfy

$$0 = x_0 < x_1 < \cdots < x_N = 1 \quad \text{and} \quad 0 = y_0 < y_1 < \cdots < y_N = 1.$$

We denote $\Gamma^N = \Gamma \bigcap \Omega^N$, and also set $h_{x,i} = x_i - x_{i-1}$, $\hbar_{x,i} = (h_{x,i+1} + h_{x,i})/2$ and $h_{y,j} = y_j - y_{j-1}$, $\hbar_{y,j} = (h_{y,j+1} + h_{y,j})/2$. We set $g_{ij} = g(x_i, y_j)$ for any function g, while g_{ij}^N denotes an approximation of g at the point (x_i, y_j). Given a mesh function $\{w_{ij}^N\}$ on Ω^N, we discretize problem (1) by the standard upwind scheme as follows:

$$\begin{aligned} \mathcal{L}^N w_{ij}^N := \left(-\varepsilon(D_x^2 + D_y^2) - b_{1,ij} D_x^+ - b_{2,ij} D_y^+ + c_{ij}\right) w_{ij}^N &= f_{ij} \quad \text{on} \quad \Omega^N \backslash \Gamma^N, \\ w_{ij}^N &= 0 \quad \text{on} \quad \Gamma^N, \end{aligned}$$

with

$$D_x^2 w_{ij}^N = \frac{1}{\hbar_{x,i}}\left(D_x^+ w_{ij}^N - D_x^- w_{ij}^N\right), \qquad D_y^2 w_{ij}^N = \frac{1}{\hbar_{y,j}}\left(D_y^+ w_{ij}^N - D_y^- w_{ij}^N\right),$$

$$D_x^- w_{ij}^N = \frac{w_{ij}^N - w_{i-1,j}^N}{h_{x,i}}, \qquad D_x^+ w_{ij}^N = \frac{w_{i+1,j}^N - w_{i,j}^N}{h_{x,i+1}},$$

$$D_y^- w_{ij}^N = \frac{w_{ij}^N - w_{i,j-1}^N}{h_{y,j}}, \qquad D_y^+ w_{ij}^N = \frac{w_{i,j+1}^N - w_{i,j}^N}{h_{y,j+1}}.$$

We split $\mathcal{L}^N$ into $\mathcal{L}_x^N + \mathcal{L}_y^N$, where

$$\mathcal{L}_x^N w_{ij}^N = \left(-\varepsilon D_x^2 - b_{1,ij} D_x^+ + c_{ij}\right) w_{ij}^N \quad \text{and} \quad \mathcal{L}_y^N w_{ij}^N = \left(-\varepsilon D_y^2 - b_{2,ij} D_y^+\right) w_{ij}^N.$$

The matrix associated with the discrete operator $\mathcal{L}^N$ is an M-matrix. Therefore, the discrete comparison principle holds true [5, Lemma 6]. We also need the following standard truncation-error bounds for the discrete operator $\mathcal{L}^N$.

Lemma 1 (see [5], Lemma 8)**.** *Let $g(x,y)$ be a smooth function defined on Ω. Then the following estimates for the truncation error hold:*

$$\left|\mathcal{L}_x^N g_{ij} - (\mathcal{L}_x g)_{ij}\right| \le C\left(\varepsilon + h_{x,i} + h_{x,i+1}\right) \max_{\zeta\in[x_{i-1},x_{i+1}]} \left|\frac{\partial^2 g}{\partial x^2}(\zeta, y_j)\right|,$$

and

$$\left|\mathcal{L}_x^N g_{ij} - (\mathcal{L}_x g)_{ij}\right| \le C\left(\varepsilon \int_{x_{i-1}}^{x_{i+1}} \left|\frac{\partial^3 g}{\partial x^3}(\zeta, y_j)\right| d\zeta + \int_{x_{i-1}}^{x_i} \left|\frac{\partial^2 g}{\partial x^2}(\zeta, y_j)\right| d\zeta\right),$$

for $0 < i,\ j < N$, with analogous estimates for $\left|\mathcal{L}_y^N g_{ij} - (\mathcal{L}_y g)_{ij}\right|$.

3. Bakhvalov-type meshes

The mesh-generating function λ of Bakhvalov [1] is defined as follows:

$$\lambda(t) = \begin{cases} \psi(t) := a\varepsilon\phi(t), & t \in [0,\alpha], \\ \psi(\alpha) + \psi'(\alpha)(t-\alpha), & t \in [\alpha, 1], \end{cases} \tag{9}$$

with $\phi(t) := -\ln(1-2t)$ for $t \in [0, 1/2)$, where a is a fixed positive mesh-parameter and $\psi'(0) < 1$, which is equivalent to $a\varepsilon^* < 1/2$. The transition point α is defined implicitly by

$$\psi(\alpha) + \psi'(\alpha)(1-\alpha) = 1.$$

In [3, 4], Kopteva replaces α with $\tilde{\alpha} = 1/2 - a\varepsilon$, which yields $\lambda(\tilde{\alpha}) = a\varepsilon \ln\left[1/(2a\varepsilon)\right]$.

In this article, we consider a mild generalization of the Bakhvalov-type meshes of Boglaev [2, 13] and Kopteva which can be described jointly as below. Define the mesh points:

$$x_i = \hat{\lambda}(t_i) = \begin{cases} a\varepsilon\hat{\phi}(t_i), & i = 0, 1, \ldots, N/2, \\ \hat{\lambda}(\hat{\alpha}) + 2(t_i - 1/2)\left(1 - \hat{\lambda}(\hat{\alpha})\right), & i = N/2+1, \ldots, N, \end{cases} \tag{10}$$

with a positive constant κ satisfying $\kappa \ge \varepsilon$, $\hat{\phi}(t) = -\ln\left[1 - 2(1-\varepsilon/\kappa)t\right]$ for $t \in [0, 1/2]$, $t_i = i/N$, and

$$\hat{\alpha} = 1/2, \quad \hat{\lambda}(\hat{\alpha}) = x_{N/2} = a\varepsilon \ln(\kappa/\varepsilon). \tag{11}$$

When $\kappa = 1$, it gives Boglaev's transition point. In particular, when we are in the singularly perturbed regime, the value of $\tilde{\alpha}$ is very close to $1/2$. Thus, we relax

Kopteva's transition point in the t-coordinate to $\hat{\alpha} = 1/2$, but we keep the same transition point of Kopteva in the x-coordinate, which is attained when $\kappa = 1/(2a)$. That is, our variant shrinks the very first mesh point in the x-coordinate in the regular region to be equal to $\lambda(\tilde{\alpha})$, whereas the function $\hat{\phi}(t)$ re-distributes $(N/2+1)$ mesh points, $x_0, \dots, x_{N/2}$, in such a way that the desirable properties of the original Bakhvalov mesh (with the function $\phi(t)$) can be preserved as shown in the following lemma.

Lemma 2. *The mesh widths of the Bahkvalov-type mesh defined by* (10) *and* (11) *satisfy*

$$h_{x,i-1} \le h_{x,i} \le 2a\kappa N^{-1}, \quad i = 2, 3, \dots, N/2, \tag{12}$$

$$h_{x,i} \le a\varepsilon, \quad i = 1, 2, \dots, N/2-1, \tag{13}$$

and in particular

$$h_{x,i-1} \le h_{x,i} \le 2N^{-1}, \quad i = 2, 3, \dots, N, \quad \textit{when} \quad \kappa \le 1/(2a). \tag{14}$$

Furthermore,

$$e^{-\beta_1 x_{N/2-1}/\varepsilon} = \left(\varepsilon/\kappa + 2(1-\varepsilon/\kappa)N^{-1}\right)^{a\beta_1} \le C\left(\varepsilon + N^{-1}\right)^{a\beta_1} \tag{15}$$

and

$$e^{-\beta_1 x_{N/2}/\varepsilon} = (\varepsilon/\kappa)^{a\beta_1} \le (\varepsilon)^{a\beta_1}. \tag{16}$$

Proof. By the definition of $\hat{\phi}(t)$, we have $\hat{\phi}'(t) = \dfrac{2(1-\varepsilon/\kappa)}{1-2(1-\varepsilon/\kappa)t}$, thus $\hat{\phi}(t)$ is monotonically increasing for $t \in [0, 1/2]$. Therefore, $h_{x,i-1} \le h_{x,i},\ i = 2, 3, \dots, N/2-1$, and

$$\begin{aligned} h_{x,N/2} &= a\varepsilon \int_{t_{N/2-1}}^{t_{N/2}} \phi'(s)\, ds \le \frac{a\varepsilon}{N} \phi'(t_{N/2}) = \frac{a\varepsilon}{N} \cdot \frac{2(1-\varepsilon/\kappa)}{1-2(1-\varepsilon/\kappa)t_{N/2}} \\ &\le \frac{a\varepsilon}{N} \cdot \frac{2\kappa}{\varepsilon} = 2a\kappa N^{-1}, \end{aligned}$$

which gives (12).

For the estimate in (13), we have

$$\begin{aligned} h_{x,i} &= a\varepsilon \int_{t_{i-1}}^{t_i} \phi'(s)\, ds \le \frac{a\varepsilon}{N} \max_{t \in [t_{i-1}, t_i]} \phi'(t) = \frac{a\varepsilon}{N} \cdot \frac{2(1-\varepsilon/\kappa)}{1-2(1-\varepsilon/\kappa)t_i} \\ &\le \frac{a\varepsilon}{N} \cdot \frac{2}{1-2t_{N/2-1}} \le \frac{a\varepsilon}{N} \cdot N = a\varepsilon, \end{aligned}$$

whereas estimate (14) results from (12) and the fact that $N^{-1} \le h_{x,i-1} \le h_{x,i} \le 2N^{-1},\ i = N/2+2, \dots, N$, by mesh definition (10). It is an easy calculation to get inequalities (15) and (16). □

Remark 1. *The user-chosen parameter κ allows the mesh to preserve the reasonable properties of the original Bakhvalov mesh and Kopteva's modification as shown in* (14)*; that is $\max_i h_{x,i} = \mathcal{O}(N^{-1})$ and $h_{x,i-1} \le h_{x,i}$ for all i, see [4, page 181].*

4. Error analysis

The numerical solution u_{ij}^N of the upwind finite-difference discretization is decomposed analogously to its continuous counterpart:

$$u_{ij}^N = S_{ij}^N + E_{1,ij}^N + E_{2,ij}^N + E_{12,ij}^N,$$

for which

$$\mathcal{L}^N S_{ij}^N = (\mathcal{L}S)_{ij}\,, \quad \mathcal{L}^N E_{1,ij}^N = (\mathcal{L}E_1)_{ij}\,, \quad \mathcal{L}^N E_{2,ij}^N = (\mathcal{L}E_2)_{ij}\,,$$
$$\mathcal{L}^N E_{12,ij}^N = (\mathcal{L}E_{12})_{ij} \text{ on } \Omega^N \setminus \Gamma^N,$$

and

$$S_{ij}^N = S_{ij},\; E_{1,ij}^N = E_{1,ij},\;\; E_{2,ij}^N = E_{2,ij},\; E_{12,ij}^N = E_{12,ij} \text{ on } \Gamma^N.$$

Let

$$\mathcal{L}^N (u_{ij}) - (\mathcal{L}u)_{ij} = \mathcal{L}^N \left(u_{ij} - u_{ij}^N\right), \quad 1 \le i,j \le N-1,$$

be the truncation error of the upwind discretization of problem (1) on the Bakhvalov-type mesh (10). We establish the upper bounds for the truncation error by using

$$\begin{aligned}\left|\mathcal{L}^N \left(u_{ij} - u_{ij}^N\right)\right| \le& \left|\mathcal{L}^N \left(S_{ij} - S_{ij}^N\right)\right| + \left|\mathcal{L}^N \left(E_{1,ij} - E_{1,ij}^N\right)\right| \\ &+ \left|\mathcal{L}^N \left(E_{2,ij} - E_{2,ij}^N\right)\right| + \left|\mathcal{L}^N \left(E_{12,ij} - E_{12,ij}^N\right)\right| \\ \le& \theta_{ij}^x + \theta_{ij}^y,\end{aligned}$$

where we set

$$\begin{aligned}\theta_{ij}^x :=& \left|\mathcal{L}_x^N \left(S_{ij} - S_{ij}^N\right)\right| + \left|\mathcal{L}_x^N \left(E_{1,ij} - E_{1,ij}^N\right)\right| \\ &+ \left|\mathcal{L}_x^N \left(E_{2,ij} - E_{2,ij}^N\right)\right| + \left|\mathcal{L}_x^N \left(E_{12,ij} - E_{12,ij}^N\right)\right|\end{aligned} \tag{17}$$

and

$$\begin{aligned}\theta_{ij}^y :=& \left|\mathcal{L}_y^N \left(S_{ij} - S_{ij}^N\right)\right| + \left|\mathcal{L}_y^N \left(E_{1,ij} - E_{1,ij}^N\right)\right| \\ &+ \left|\mathcal{L}_y^N \left(E_{2,ij} - E_{2,ij}^N\right)\right| + \left|\mathcal{L}_y^N \left(E_{12,ij} - E_{12,ij}^N\right)\right|.\end{aligned}$$

Also, let

$$\bar{E}_{ij}^x = \prod_{k=1}^{i} \left(1 + \frac{\beta_1 h_{x,k}}{2\varepsilon}\right)^{-1} \quad \text{and} \quad \bar{E}_{ij}^y = \prod_{k=1}^{j} \left(1 + \frac{\beta_2 h_{y,k}}{2\varepsilon}\right)^{-1}.$$

We also make an assumption on the mesh parameter a such that $\min\{a\beta_1, a\beta_2\} \ge 2$ for the rest of our analysis. In the following lemmas, we follow the proof techniques introduced in [12, Theorem 1] to obtain the truncation-error estimate, and [12, Lemma 9] to form the appropriate barrier functions. It is worth noting that because the mesh modification of Boglaev and Kopteva simplifies the original Bakhvalov mesh by its explicit transition point, the analysis here can be significantly simplified compared to that of [12].

Lemma 3. *The truncation error of the upwind discretization of problem* (1) *on the Bakhvalov-type mesh defined in* (10) *and* (11) *satisfies the following:*

$$\left|\mathcal{L}^N\left(u_{ij}\right)-\left(\mathcal{L}u\right)_{ij}\right| \le \theta_{ij}^x+\theta_{ij}^y,$$

where for θ_{ij}^x *and any* $j=1,2,\ldots,N-1$, *we have*

$$\theta_{ij}^x \le \begin{cases} CN^{-1}, & i \ge N/2,\\ C\left(N^{-1}+\varepsilon^{-1}\bar{E}_{ij}^x N^{-1}\right), & i=N/2-1 \ \&\ h_{x,i+1}\le\varepsilon,\\ C\left(N^{-1}+h_{x,i+1}^{-1}\bar{E}_{ij}^x N^{-1}\right), & i=N/2-1 \ \&\ h_{x,i+1}>\varepsilon,\\ C\left(N^{-1}+\varepsilon^{-1}\bar{E}_{ij}^x N^{-1}\right), & i\le N/2-2,\end{cases}$$

whereas θ_{ij}^y, *for any* $i=1,2,\ldots,N-1$, *can be bounded in the same way with* $\bar{E}_{ij}^y$, *instead of* $\bar{E}_{ij}^x$, *and* $h_{y,j}$, *instead of* $h_{x,i}$.

Proof. We shall prove the estimates for θ_{ij}^x only, since the bounds for θ_{ij}^y can be proved analogously. From (17), we will bound each right-hand-side term separately. Let $1\le j\le N-1$ throughout the proof. By (2), (4), (12), and Lemma 1, we easily get the bounds $\left|\mathcal{L}_x^N\left(S_{ij}-S_{ij}^N\right)\right|\le CN^{-1}$ and $\left|\mathcal{L}_x^N\left(E_{2,ij}-E_{2,ij}^N\right)\right|\le CN^{-1}$ for $1\le i\le N-1$.

We consider cases for the indices i to bound the layer component E_1. First, for $i\ge N/2+1$, we apply Lemma 1 to E_1. Then we have

$$\left|\mathcal{L}_x^N\left(E_{1,ij}-E_{1,ij}^N\right)\right|\le CN^{-1}\varepsilon^{-2}e^{-\beta_1 x_{N/2}/\varepsilon}\le CN^{-1},$$

where we have used (16) and $a\beta_1\ge 2$ in the last inequality.

For $i=N/2$, we consider two subcases, $\varepsilon\le N^{-1}$ and $\varepsilon>N^{-1}$. For $\varepsilon\le N^{-1}$, our approach is

$$\left|\mathcal{L}_x^N\left(E_{1,ij}-E_{1,ij}^N\right)\right|\le\left|\mathcal{L}_x^N\left(E_{1,ij}\right)\right|+\left|\left(\mathcal{L}_xE_1\right)_{ij}\right|. \tag{18}$$

Then, we can bound both $\left|\mathcal{L}_x^N\left(E_{1,ij}\right)\right|\le CN^{-1}$ and $\left|\left(\mathcal{L}_xE_1\right)_{ij}\right|\le CN^{-1}$ by invoking (3), (6), and (15). For $\varepsilon>N^{-1}$, we get that $h_{x,N/2}\le C\varepsilon$ because of (12). Therefore, we use again Lemma 1 to get:

$$\left|\mathcal{L}_x^N\left(E_{1,ij}-E_{1,ij}^N\right)\right|\le CN^{-1}\varepsilon^{-2}e^{-\beta_1 x_{N/2-1}/\varepsilon}\le CN^{-1}\varepsilon^{-2}e^{-\beta_1 x_{N/2}/\varepsilon}\le CN^{-1},$$

where (16) is used in the last step.

Next, we combine the cases when $i\le N/2-2$ and $i=N/2-1$ when $h_{x,N/2}\le\varepsilon$ together. Indeed, for $i\le N/2-1$, we have $h_{x,i}\le a\varepsilon$ because of (13). Hence,

$$\left|\mathcal{L}_x^N\left(E_{1,ij}-E_{1,ij}^N\right)\right|\le CN^{-1}\varepsilon^{-1}e^{-\beta_1 x_{i-1}/\varepsilon}\le C\varepsilon^{-1}N^{-1}e^{-\beta_1 x_i/2\varepsilon}\le C\varepsilon^{-1}N^{-1}\bar{E}_{ij}^x.$$

Lastly, when $i=N/2-1$ and $h_{x,N/2}>\varepsilon$, this means that $\max\{\varepsilon,h_{x,N/2}\}=h_{x,N/2}$. Then, $\varepsilon\le CN^{-1}$ because of (12), and we can modify the approach in (18) and use (15) to get

$$\begin{aligned}\left|\mathcal{L}_x^N\left(E_{1,ij}\right)\right|+\left|\left(\mathcal{L}_x E_1\right)_{ij}\right| &\leq C h_{x,N/2}^{-1} e^{-\beta_1 x_{N/2-1}/(2\varepsilon)} e^{-\beta_1 x_{N/2-1}/(2\varepsilon)} \\ &\leq C h_{x,N/2}^{-1} \bar{E}_{N/2-1,j}^x \left(\varepsilon+N^{-1}\right)^{a\beta_1/2} \\ &\leq C h_{x,N/2}^{-1} \bar{E}_{N/2-1,j}^x N^{-1}.\end{aligned} \tag{19}$$

The proof is completed when we apply the above argument analogously to bound $\left|\mathcal{L}_x^N\left(E_{12,ij}-E_{12,ij}^N\right)\right|$ by invoking (5) and (8). □

Next, we form the barrier function

$$\gamma_{ij}=\gamma_{ij}^x+\gamma_{ij}^y,\quad 1\leq i,j\leq N-1,$$

with

$$\gamma_{ij}^x=C_1(1-x_i)N^{-1}+C_2\bar{E}_{ij}^x N^{-1}\quad\text{and}\quad \gamma_{ij}^y=C_3(1-y_j)N^{-1}+C_4\bar{E}_{ij}^y N^{-1},$$

where $C_k,\ k=1,2,3,4$, are appropriately chosen positive constants independent of both ε and N.

Lemma 4. *There exist sufficiently large constants C_k, $k=1,2,3,4$, such that*

$$\mathcal{L}^N\gamma_{ij}=\mathcal{L}^N\gamma_{ij}^x+\mathcal{L}^N\gamma_{ij}^y\geq\chi_{ij}^x+\chi_{ij}^y\geq\theta_{ij}^x+\theta_{ij}^y,\quad 1\leq i,j\leq N-1,$$

where

$$\chi_{ij}^x=C_1N^{-1}+C_2[\max\{\varepsilon,h_{x,i+1}\}]^{-1}\bar{E}_{ij}^x N^{-1},$$

and

$$\chi_{ij}^y=C_3N^{-1}+C_4[\max\{\varepsilon,h_{y,j+1}\}]^{-1}\bar{E}_{ij}^y N^{-1}.$$

Proof. We shall prove that $\mathcal{L}_x^N\gamma_{ij}\geq\chi_{ij}^x\geq\theta_{ij}^x$. It is easy to verify that $\mathcal{L}_x^N\gamma_{ij}\geq\chi_{ij}^x$ (see, for instance, [9, 15]). Therefore, by Lemma 3, we will show that $\theta_{ij}^x\leq\chi_{ij}^x, 1\leq j\leq N-1$, for various values of the indices i.

Let $1\leq j\leq N-1$ throughout the proof again. It is clear from Lemma 3 that

$$\theta_{ij}^x\leq CN^{-1}\leq C_1N^{-1}\leq\chi_{ij}^x,\quad i=N/2,N/2+1,\ldots,N-1.$$

We are left to prove that $\theta_{ij}^x\leq\chi_{ij}^x$ for $i\leq N/2-1$. For $i\leq N/2-2$, we have from (13) that $h_{x,i+1}\leq a\varepsilon$ and

$$\theta_{ij}^x\leq C\left(N^{-1}+\varepsilon^{-1}\bar{E}_{ij}^x N^{-1}\right)\leq C_1N^{-1}+C_2\varepsilon^{-1}\bar{E}_{ij}^x N^{-1}\leq\chi_{ij}^x. \tag{20}$$

It remains to consider $i=N/2-1$. If $h_{x,N/2}\leq\varepsilon$, we have the same situation as above and estimate (20) is achieved for $i=N/2-1$. On the other hand, when $h_{x,N/2}>\varepsilon$, we apply a similar argument from (19) to get

$$\theta_{ij}^x\leq C\left(N^{-1}+h_{x,N/2}^{-1}\bar{E}_{N/2-1,j}^x N^{-1}\right)\leq C_1N^{-1}+C_2h_{x,N/2}^{-1}\bar{E}_{N/2-1,j}^x N^{-1}\leq\chi_{N/2-1,j}^x.$$

Analogous reasoning is used to show that $\mathcal{L}_y^N\gamma_{ij}\geq\chi_{ij}^y\geq\theta_{ij}^y$, which completes the proof. □

Combining Lemmas 3 and 4 together with the discrete maximum principle (cf. [5, Lemma 6]), we get the main result.

Theorem 1. *For the upwind finite-difference method applied on the Bakhvalov-type mesh defined in* (10) *and* (11) *to the convection-diffusion problem* (1)*, the error satisfies*

$$\left|u_{ij} - u_{ij}^N\right| \leq CN^{-1}, \quad \textit{for} \quad 0 \leq i,\ j \leq N.$$

5. Concluding remark

Our analysis fills an important theoretical gap for Bakhvalov-type meshes with explicitly defined transition points discretized by an upwind difference scheme of two-dimensional convection-diffusion problems. The proposed mesh mildly generalizes Boglaev and Kopteva's choices of the Bakhvalov-type transition points in the sense that the mesh not only enjoys the preferred mesh properties of the famous Bakhvalov mesh, but also simplifies the computational effort theoretically and practically.

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