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Independent sets and vertex covers considered within the context of robust optimization*

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Abstract. This paper studies robust variants of the maximum weighted independent set problem and the minimum weighted vertex cover problem, respectively. Both problems are posed in a vertex-weighted graph. The paper explores whether the complement of a robustly optimal independent set must be a robustly optimal vertex cover, and vice-versa.
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1. Introduction

In this paper, we consider a graph whose vertices are given nonnegative integer weights. Within our graph we study independent sets with maximum weights, as well as vertex covers with minimum weights. It is well known that the complement of a maximum weighted independent set is a minimum weighted vertex cover, and vice-versa. Thus an algorithm for finding one type of optimal solution can be used to solve the other type. But such equivalence of optimization problems is granted only within the context of conventional optimization. We cannot be sure that analogue properties will hold when we move to robust optimization.

The aim of this paper is to explore relationships among robust variants of the two considered optimization problems. Or in other words, the aim is to find out whether the complement of a robustly optimal independent set must be a robustly optimal vertex cover, and vice-versa. We expect that the answer to this question might not be simple, e.g. it could depend on the chosen criterion of robustness.

In the paper, we will analyze robust problem variants based on a finite collection of scenarios for vertex weights. The scenarios can be listed explicitly according to discrete uncertainty representation, or given implicitly through interval uncertainty representation. We will apply three traditional criteria for robustness, i.e. absolute robustness, robust deviation, and relative robust deviation. The three criteria will

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appear in their simple or their extended (OWA) forms. Also, we will consider a more comprehensive method for solving robust problems based on finding the whole collection of Pareto-efficient solutions.

Apart from this introduction, the rest of the paper is organized as follows. Section 2 lists preliminaries about independent sets, vertex covers and robust optimization. Section 3 studies robust problem variants based on explicitly given scenarios and the absolute robustness criterion. Similarly, Section 4 is concerned with robust deviation, while Section 5 deals with relative robust deviation. Section 6 considers cases where scenarios are given implicitly through intervals. Section 7 applies OWA criteria for robustness. Section 8 is dedicated to the approach based on Pareto efficiency. The final Section 9 gives conclusions.

2. Preliminaries

Let us consider an *undirected graph* $G = (V, E)$ [7, 9], where V is the set of its *vertices*, and E is the set of its *edges*. Within G we consider two types of objects: independent sets and vertex covers. An *independent set* is a subset X of V such that no two vertices in X are joined by an edge from E (i.e. no two vertices in X are adjacent). A *vertex cover* is a subset Y of V such that at least one endpoint of every edge from E is in Y (i.e. every edge from E is incident on at least one vertex from Y). The relationship between independent sets and vertex covers is established by the following two claims, which are well known and easy to prove.

- *Let $X \subset V$ be an independent set. Then the complement $Y = V \setminus X$ is a vertex cover.*
- *Let $Y \subset V$ be a vertex cover. Then the complement $X = V \setminus Y$ is an independent set.*

From now on, we assume that vertices of our graph G have *weights*, and that all of those weights are nonnegative integers. The weight of an independent set X, denoted as $F(X)$, is defined as the sum of weights of all vertices belonging to X. Similarly, the weight of a vertex cover Y denoted as $\bar{F}(Y)$ is the sum of weights of all vertices belonging to Y. Let T be the total sum of weights of all vertices in G. If X and Y are complements, then it obviously holds that

$$F(X) + \bar{F}(Y) = T.$$

From the application point of view, interesting independent sets are those with large weights (interpreted as profits), while vertex covers should have small weights (interpreted as costs). This is a motivation for the following definitions. A *maximum weighted independent set* is an independent set whose weight is as large as possible. Analogously, a *minimum weighted vertex cover* is a vertex cover whose weight is as small as possible.

The problem of finding a maximum weighted independent set is called the (conventional) *maximum weighted independent set problem* (*MWIS problem*). Similarly, the problem of finding a minimum weighted vertex cover is called the (conventional) *minimum weighted vertex cover problem* (*MWVC problem*). The two problems are closely related. Namely, the following two assertions can easily be proved.

- *Let X^* be an optimal solution for the MWIS problem (i.e. an independent set with maximum weight). Then its complement $Y^* = V \setminus X^*$ is an optimal solution to the MWVC problem (i.e. a vertex cover with minimum weight).*
- *Let Y^* be an optimal solution for the MWVC problem (i.e. a vertex cover with minimum weight). Then its complement $X^* = V \setminus Y^*$ is an optimal solution to the MWIS problem (i.e. an independent set with maximum weight).*

Denote by F^* the optimal weight for the MWIS problem, and by $\bar{F}^*$ the optimal weight for the MWVC problem. Then the last two assertions guarantee that

$$F^* + \bar{F}^* = T.$$

As we can see, the conventional MWIS and MWVC problems are *equivalent*. Indeed, from the optimal solution of one problem, by computing a set complement, one can obtain the optimal solution of the other problem, and vice versa. Any algorithm [13] that solves one problem can be used for solving the other problem. However, it is known that both problems are NP-hard [6], which means that their large instances can be solved only approximately [10, 16, 17]. Still, some special problem cases exist that can be solved in polynomial time, e.g. the cases with *interval graphs* [11, 18] or with *apple-free graphs* [4].

Let us now say a few words about *robust optimization* [1, 2, 3, 12, 14]. It is a state-of-the-art approach to deal with *uncertainty* in problem parameters (e.g. vertex weights in our case). Such uncertainty occurs frequently in real-world applications where parameter values may depend on unpredictable future circumstances or perhaps cannot be measured accurately. According to the robust optimization approach, uncertain values are expressed through a collection of *scenarios*, which can be listed explicitly or described implicitly through some kind of an uncertainty set (e.g. a Cartesian product of intervals). Only solutions that are feasible for all scenarios are taken into account. The "behavior" of any solution under any scenario is measured in some way. As the "robustly optimal" solution, the one is chosen whose overall behavior, measured over all scenarios, is the best possible.

Depending on the chosen behavior measure, the above procedure can lead to the application of different *criteria of robustness*. There are three popular criteria, and according to [14], they are called absolute robustness, robust deviation, and relative robust deviation. In some other literature, e.g. [1], the same criteria are referred to as max-min (min-max), min-max-regret, and relative min-max regret, respectively. Here are the definitions.

- An *absolute robust solution* is the one whose worst (conventional) objective-function value, measured over all scenarios, is the best among all feasible solutions.
- A *robust deviation solution* is the one whose maximum deviation from the conventional optimum, measured over all scenarios, is the smallest among all feasible solutions.
- A *relative robust deviation solution* is the one whose maximum relative deviation from the conventional optimum, measured over all scenarios, is the smallest among all feasible solutions.

Each of the three robustness criteria shown above can further on be extended by using the corresponding ordered-weighted-averaging aggregation (OWA) criterion [12]. The idea is that the assessment of a given solution should not be based only on the worst measured value of the objective function or deviation. Instead, a convex combination of several vales should be taken into account. As an OWA solution, the one is chosen whose convex combination is the best.

As an alternative to satisfying the above robustness criteria, there is another method for solving robust problems, which is based on the so-called Pareto efficiency [5, 15]. According to that method, solving a problem instance means finding not only one robustly optimal solution, but the whole collection of efficient solutions. To explain the concept of efficiency, we must first explain the related concept of domination.

- A solution Z of a robust problem instance is *dominated* by another solution $\widetilde{Z}$ if Z is equally good or worse than $\widetilde{Z}$ under any scenario, and strictly worse than $\widetilde{Z}$ under at least one scenario.
- A solution Z is *efficient* if it is not dominated by any other solution.

It is easy to show that for any of the three robustness criteria, either in its simple or in its OWA form, there exists an efficient solution that is robustly optimal according to that criterion [1, 12]. Consequently, the method for solving a robust problem based on finding all efficient solutions can be regarded as more comprehensive than a method based on a particular criterion.

At this moment, there are only few papers on robust variants of the MWIS problem, e.g. [11] by Kasperski and Zielinski, or [18] by Talla Nobibon and Leus. These papers are mostly concerned with complexity or approximability issues. We are not aware of any publication dealing with robust variants of the MWVC problem.

3. Absolute robustness

From now on, it is assumed that vertex weights in our graph $G = (V, E)$ are uncertain, and that uncertainty is expressed through a finite and explicitly given collection S of scenarios. Each scenario $s \in S$ specifies its own list of weights.

In order to deal with more scenarios, it is necessary to extend the notation from the previous section. The weight of an independent set X under scenario s will now be denoted as $F(X, s)$. Similarly, the weight of a vertex cover Y under scenario s is denoted as $\bar{F}(Y, s)$. Let T_s be the total sum of weights of all vertices of G under scenario s. Suppose that X and Y are complements. Then for each $s \in S$ it holds that

$$F(X, s) + \bar{F}(Y, s) = T_s.$$

Due to more scenarios, it is also necessary to redefine the previously defined MWIS and MWVC problems, i.e. instead of their conventional variants their robust variants should be considered. However, such redefinition can be done in several ways. In this section, we restrict to the variants obtained by applying the absolute criterion of robustness. According to the general rule of absolute robustness from the previous section, the following definitions are obtained.

Definition 1. *An absolute robust solution for the MWIS problem is an independent set X_A that maximizes the function $\min_{s\in S} F(X,s)$ over the whole collection of possible independent sets X.*

Definition 2. *An absolute robust solution for the MWVC problem is a vertex cover Y_A that minimizes the function $\max_{s\in S} \bar{F}(Y,s)$ over the whole collection of possible vertex covers Y.*

A natural question one would like to answer is whether the absolute robust MWIS problem is equivalent to the absolute robust MWVC problem, as it was true in the conventional case. More precisely: is the complement of a robustly optimal independent set a robustly optimal vertex cover, and vice-versa? A partial answer to this question is given by the following proposition.

Proposition 1. *Suppose that all scenarios have the same sum of vertex weights, i.e. $T_s = T$ for all $s \in S$. Then the complement of an absolute robust solution for the MWIS problem is an absolute robust solution for the MWVC problem, and vice-versa.*

Proof. Let X_A be an absolute robust solution for the MWIS problem, and let Y_A be the complement of X_A. Then we have:

$$\begin{aligned}
\min_{s\in S} F(X,s) &\text{ achieves maximum for } X = X_A \quad \Longrightarrow \\
\min_{s\in S}(T - \bar{F}(Y,s)) &\text{ achieves maximum for } Y = Y_A \quad \Longrightarrow \\
T - \max_{s\in S} \bar{F}(Y,s) &\text{ achieves maximum for } Y = Y_A \quad \Longrightarrow \\
\max_{s\in S} \bar{F}(Y,s) &\text{ achieves minimum for } Y = Y_A.
\end{aligned}$$

Thus Y_A is by definition an absolute robust solution to the MWVC problem. The proof in opposite direction is conducted analogously. □

Note that Proposition 1 assures equivalence among the considered robust problems only in a special case, i.e. when all scenarios have the same total sum of vertex weights. Unfortunately, such equivalence does not hold in general, as demonstrated by the following example.

Example 1. *Let us consider the graph with nine vertices shown in Figure 1. There are three scenarios for weights, as indicated by triple labels assigned to vertices. Then the corresponding absolute robust solutions to the MWIS and MWVC problems, respectively, are presented in Table 1. The left-hand side of the table comprises all nontrivial independent sets, i.e. those that cannot be extended by adding more vertices. Similarly, the right-hand side of the table contains all nontrivial vertex covers, i.e. those that cannot be reduced by removing some of their vertices. For each independent set or vertex cover, there is a list of its weights under different scenarios. In each list, the worst weight is underlined. Robustly optimal solutions (i.e. those whose worst weight is the best) are shown in boldface. We can check that the complement of the robustly optimal independent set is not a robustly optimal vertex cover. Similarly, the complement of the robustly optimal vertex cover is not a* *robustly optimal independent set. The found absolute robust solutions are also shown in Figure 1 by shading. Black vertices belong to the optimal independent set, light grey vertices are from the optimal vertex cover, while dark gray vertices are common to both sets.*

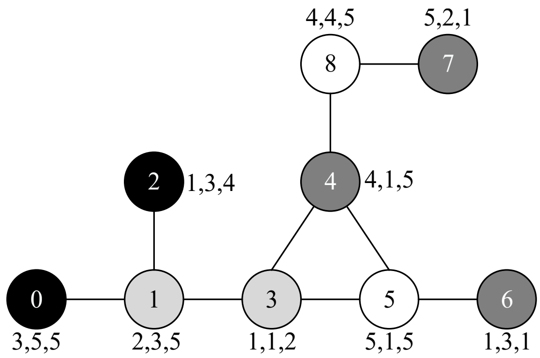

Figure 1: *A graph where the complement of an absolute robust solution to the MWIS problem is not an absolute robust solution to the MWVC problem*

Independent set	Weight for each scenario			Vertex cover	Weight for each scenario		
0,2,3,6,7	$\underline{11}$	14	13	1,4,5,8	15	9	$\underline{20}$
0,2,3,6,8	$\underline{10}$	16	17	1,4,5,7	$\underline{16}$	7	16
0,2,4,6,7	$\underline{14}$	14	16	1,3,5,8	12	9	$\underline{17}$
0,2,5,7	14	$\underline{11}$	15	1,3,4,6,8	12	12	$\underline{18}$
0,2,5,8	$\underline{13}$	13	19	**1,3,4,6,7**	13	10	$\underline{14}$
1,4,6,7	12	$\underline{9}$	12	0,2,3,5,8	14	14	$\underline{21}$
1,5,7	12	$\underline{6}$	11	0,2,3,4,6,8	14	17	$\underline{22}$
1,5,8	11	$\underline{8}$	15	0,2,3,4,6,7	15	15	$\underline{18}$

Table 1: *Finding absolute robust solutions for the graph shown in Figure 1*

In the remaining part of this section, we will explain how our Proposition 1 can be used to transfer some of the available complexity or approximation results from the robust MWIS to the robust MWVC context. This can be done although Proposition 1 establishes only a partial equivalence among absolutely robust MWIS and MWVC problems.

Indeed, in [18], there are two results on NP-hardness of the absolute robust MWIS problem on interval graphs. Both results can be converted to the corresponding MWVC problem. Conversion is done so that the MWIS problem instances used in the proofs are (polynomially) reduced to the corresponding MWVC instances. Such reduction is possible thanks to the fact that in both proofs the constructed instances satisfy the restriction regarding scenarios from Proposition 1.

In [18], there is also a pseudo-polynomial-time algorithm for solving the absolute robust MWIS problem on interval graphs. Obviously, the same algorithm can also be used to solve the corresponding MWVC problem. More precisely, for a given problem instance, a robustly optimal independent set is first found and then converted (in polynomial time) into the complementary vertex cover. Such computation will be correct if the given instance satisfies the restriction from Proposition 1.

In addition to the results from [18], there is a fairly general proposition in [1] dealing with approximability of robust solutions within the number of scenarios. The proposition is applicable to absolute robust variants originating from conventional minimization, so that it can be applied to the MWVC problem, but not to the MWIS problem.

Putting it all together, our Proposition 1 combined with the results from [1, 18] brings the following consequences. They deal with interval graphs, where the conventional MWVC problem (being equivalent to the conventional MWIS problem) is solvable in polynomial time.

Corollary 1. *We consider the absolute robust MWVC problem on interval graphs. Then the considered problem is NP-hard even with only two scenarios. An instance of the problem whose scenarios have the same sum of vertex weights can be solved in pseudo-polynomial time when the number of scenarios is bounded. The problem is strongly NP-hard when the number of scenarios is unbounded, but approximable within the number of scenarios.*

4. Robust deviation

Similarly to the previous section, we again study robust variants of the MWIS and MWVC problems with explicitly given scenarios. But now we apply the second criterion of robustness called robust deviation. Along with the notation introduced previously, two additional symbols are used. The symbol F^*_s denotes the optimal solution weight for the (conventional) MWIS problem instance with vertex weights set according to scenario $s \in S$. Similarly, $\bar{F}^*_s$ is the optimal solution weight for the (conventional) MWVC problem instance corresponding to scenario s. Obviously, it holds:

$$F^*_s + \bar{F}^*_s = T_s.$$

The general rule of robust deviation was stated in Section 2. By applying that rule to our problems, the following two definitions are obtained.

Definition 3. *A robust deviation solution to the MWIS problem is an independent set X_D that minimizes the function $\max_{s\in S}(F^*_s - F(X,s))$ over the whole collection of possible independent sets X.*

Definition 4. *A robust deviation solution to the MWVC problem is a vertex cover Y_D that minimizes the function $\max_{s\in S}(\bar{F}(Y,s) - \bar{F}^*_s)$ over the whole collection of possible vertex covers Y.*

Again, as in the previous section, an important question is whether the obtained robust MWIS and MWVC problems are equivalent or not. This time the answer is affirmative, and it is given by the following Proposition 2.

Proposition 2. *The complement of a robust deviation solution to the MWIS problem is a robust deviation solution for the MWVC problem, and vice-versa.*

Proof. Let X_D be a robust deviation solution to the MWIS problem, and let Y_D be the complement of X_D. Then:

$$\max_{s\in S}(F_s^* - F(X,s)) \text{ achieves minimum for } X = X_D \quad \Longrightarrow$$
$$\max_{s\in S}(T_s - \bar{F}_s^* - (T_s - \bar{F}(Y,s)) \text{ achieves minimum for } Y = Y_D \quad \Longrightarrow$$
$$\max_{s\in S}(\bar{F}(Y,s) - \bar{F}_s^*) \text{ achieves minimum for } Y = Y_D.$$

Thus Y_D is by definition a robust deviation solution for the MWVC problem. The claim in opposite direction is proved analogously. □

By using Proposition 2, many complexity or approximation results on the robust deviation MWIS problem can be reinterpreted for the MWVC problem. Indeed, [18] contains two NP-hardness results and one pseudo-polynomial-time algorithm for the robust deviation MWIS problem on interval graphs. In [11], there are additional approximability results dealing with the same problem again on interval graphs. All those results can be converted to the corresponding MWVC problem. Conversion is done in the same manner as explained in Section 3. Switching from MWIS to MWVC does not spoil accuracy of approximation thanks to the following fact (visible in the proof of Proposition 2): two complementary vertex sets (i.e. an independent set and the corresponding vertex cover) have the same "regret" over any scenario, and therefore their robust objective function values are also the same.

Putting it all together, by combining [18, 11] with Proposition 2, the following results can be established. They again deal with interval graphs where the conventional MWVC problem is solvable in polynomial time.

Corollary 2. *We consider the robust deviation MWVC problem on interval graphs. Then the considered problem is NP-hard even with only two scenarios, and it can be solved in pseudo-polynomial time when the number of scenarios is bounded. The problem also admits a fully polynomial approximation scheme if the number of scenarios is bounded. The problem is strongly NP-hard when the number of scenarios is unbounded, but approximable within the number of scenarios.*

5. Relative robust deviation

In this section, we explore robust variants of the MWIS and MWVC problems based on explicitly given scenarios and on the third criterion of robustness called relative robust deviation. The same notation is used as in the previous sections. By applying the general formulation of the criterion from Section 2 to our problems, the next two definitions are obtained.

Definition 5. *A relative robust deviation solution to the MWIS problem is an independent set X_R that minimizes the function $\max_{s\in S}((F_s^* - F(X,s))/F_s^*)$ over the whole collection of possible independent sets X.*

Definition 6. *A relative robust deviation solution to the MWVC problem is a vertex cover* Y_R *that minimizes the function* $\max_{s\in S}((\bar{F}(Y,s) - \bar{F}_s^*)/\bar{F}_s^*)$ *over the whole collection of possible vertex covers* Y.

It is assumed here that both F_s^* and $\bar{F}_s^*$ are > 0. This is in fact always the case except for trivial problem instances.

Again, as in the previous sections, it would be interesting to determine whether the newly obtained robust MWIS and MWVC problems are equivalent or not. The following proposition establishes a sufficient condition for equivalence.

Proposition 3. *Suppose that the ratio among optimal solution weights for the conventional MWIS and MWVC problems, respectively, is the same for all scenarios. Or in other words, suppose that* $\bar{F}_s^*/F_s^* = Q$ *for all* $s \in S$. *Then the complement of a relative robust deviation solution to the MWIS problem is a relative robust deviation solution to the MWVC problem, and vice-versa.*

Proof. Let X_R be a relative robust deviation solution to the MWIS problem, and let Y_R be the complement of X_R. Then:

$$\max_{s\in S}\left(\frac{F_s^* - F(X,s)}{F_s^*}\right) \text{ achieves minimum for } X = X_R \quad \Longrightarrow$$

$$\max_{s\in S}\left(\frac{T_s - \bar{F}_s^* - T_s + \bar{F}(Y,s)}{F_s^*}\right) \text{ achieves minimum for } Y = Y_R \quad \Longrightarrow$$

$$\max_{s\in S}\left(\frac{\bar{F}_s^*}{F_s^*} \cdot \frac{\bar{F}(Y,s) - \bar{F}_s^*}{\bar{F}_s^*}\right) \text{ achieves minimum for } Y = Y_R \quad \Longrightarrow$$

$$Q \cdot \max_{s\in S}\left(\frac{\bar{F}(Y,s) - \bar{F}_s^*}{\bar{F}_s^*}\right) \text{ achieves minimum for } Y = Y_R \quad \Longrightarrow$$

$$\max_{s\in S}\left(\frac{\bar{F}(Y,s) - \bar{F}_s^*}{\bar{F}_s^*}\right) \text{ achieves minimum for } Y = Y_R.$$

Thus Y_R is by definition a relative robust deviation solution to the MWVC problem. The proof in opposite direction is analogous. □

Note that Proposition 3 guarantees equivalence among the considered robust problems only under some very special conditions. Unfortunately, equivalence is not assured in general. It is also not assured by the condition from Proposition 1 (equal total sum of weights for all scenarios). Indeed, here follows an example.

Example 2. *We consider the graph shown in Figure 2. The corresponding absolute robust solutions are presented in Table 2, which is organized analogously to Table 1. We can see that the two solutions are complementary one to another - this is in accordance with Proposition 1, which can be applied since all scenarios have the same total sum of weights. By scanning and recomputing the data from Table 2, we obtain Table 3, where the corresponding relative robust deviation solutions are determined. For each independent set or vertex cover, Table 3 shows its relative deviations from the conventional optimum depending on scenarios. The largest relative deviations are underlined. Robustly optimal solutions (i.e. those whose largest relative deviation* *is minimal) are shown in boldface. We can check that the complement of the robustly optimal independent set is not a robustly optimal vertex cover. Similarly, the complement of the robustly optimal vertex cover is not a robustly optimal independent set. The found relative robust deviation solutions are shown in Figure 2 by shading. The same colors are used as in Example 1.*

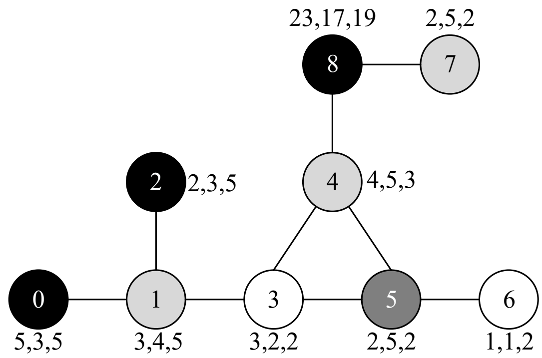

Figure 2: *A graph where the complement of a relative robust deviation solution to the MWIS problem is not a relative robust deviation solution to the MWVC problem*

Independent set	Weight for each scenario			Vertex cover	Weight for each scenario		
0,2,3,6,7	$\underline{13}$	14	16	1,4,5,8	$\underline{32}$	31	29
0,2,3,6,8	34	$\underline{26}$	33	1,4,5,7	11	$\underline{19}$	12
0,2,4,6,7	$\underline{14}$	17	17	1,3,5,8	$\underline{31}$	28	28
0,2,5,7	$\underline{11}$	16	14	1,3,4,6,8	$\underline{34}$	29	31
0,2,5,8	32	$\underline{28}$	31	**1,3,4,6,7**	13	$\underline{17}$	14
1,4,6,7	$\underline{10}$	15	12	0,2,3,5,8	$\underline{35}$	30	33
1,5,7	$\underline{7}$	14	9	0,2,3,4,6,8	$\underline{38}$	31	36
1,5,8	28	$\underline{26}$	26	0,2,3,4,6,7	17	$\underline{19}$	19

Table 2: *Finding absolute robust solutions for the graph shown in Figure 2*

6. Interval uncertainty

This section studies situations where uncertainty in vertex weights is expressed by intervals. More precisely, we assume that the weight of a vertex v_i can take any value from a given integer interval $I_i = [l_i, u_i]$. Vertex weights are chosen independently from each other. Thus the set of scenarios S is implicitly given as the full Cartesian product of all intervals I_i. Such scenario set can be combined with any of the previously considered robustness criteria. In this way, robust variants of the MWIS and MWVC problem are obtained, which are special cases of those from Sections 3, 4 and 5.

Again, we could ask ourselves whether the obtained robust problem variants are equivalent in the sense that the complement of a robustly optimal independent set

Independent set	Relative deviation for each scenario			Vertex cover	Relative deviation for each scenario		
0,2,3,6,7	0.618	0.500	0.515	1,4,5,8	1.909	0.824	1.417
0,2,3,6,8	0.000	0.071	0.000	**1,4,5,7**	0.000	0.118	0.000
0,2,4,6,7	0.588	0.393	0.485	1,3,5,8	1.818	0.647	1.333
0,2,5,7	0.676	0.429	0.576	1,3,4,6,8	2.091	0.706	1.583
0,2,5,8	0.059	0.000	0.061	1,3,4,6,7	0.182	0.000	0.167
1,4,6,7	0.706	0.464	0.636	0,2,3,5,8	2.182	0.765	1.750
1,5,7	0.794	0.500	0.727	0,2,3,4,6,8	2.455	0.824	2.000
1,5,8	0.176	0.071	0.212	0,2,3,4,6,7	0.545	0.118	0.583

Table 3: *Finding relative robust deviation solutions for the graph shown in Figure 2*

is a robustly optimal vertex cover, and vice-versa. Obviously, the answer depends on the chosen robustness criteria, and it should probably be the same as in Sections 3, 4 or 5 or similar. However, since the scenario set is rather regular, it is possible that the answer could be somewhat different of more specific. In this section, we will explore such possibilities.

Let us first consider the MWIS and MWVC problem variants based on interval uncertainty and *absolute robustness*. Let us identify two special scenarios. The *minimum scenario* is the one where each vertex v_i has the minimum possible weight l_i. Similarly, the *maximum scenario* is the one where each vertex v_i has the maximum possible weight u_i. It is easy to see that the following claim is valid.

Proposition 4. *Suppose that uncertainty in vertex weights is given by intervals. Then an absolute robust solution to the MWIS problem is obtained by solving the conventional MWIS problem according to the minimum scenario. Similarly, an absolute robust solution to the MWVC problem is obtained by solving the conventional MWVC problem according to the maximum scenario.*

Proof. The claim is an obvious consequence of the way how absolute robustness is defined, combined with the fact that the minimum and maximum scenarios are available. The same claim in a more general setting is also mentioned in [1]. A more formal proof for the MWIS problem is given in [18]. The same proof can easily be adjusted to the MWVC problem. □

According to Proposition 4, the considered absolute robust problem variants with interval uncertainty can be solved relatively easily, i.e. as conventional variants. But note that those conventional variants are in general still NP-hard. However, there are some graph types that allow polynomial-time algorithms [4, 11, 18]. Thus the following consequence of Proposition 4 can be stated.

Corollary 3. *Let the involved graph be an apple-free graph or an interval graph. Then an absolute robust solution to the MWIS or MWVC problem with interval uncertainty can be obtained in polynomial time.*

Note also that Proposition 4 does not claim that solutions to the considered problems are equivalent in the sense that one of them is the complement of the other. Indeed, according to Proposition 4, each problem should be solved separately by using a different scenario. This point is illustrated by the following example.

Example 3. *Let us consider the graph in Figure 3 whose vertex weights are given by intervals. Then the corresponding absolute robust solutions to the MWIS and MWVC problems, respectively, are shown in Table 4. For each independent set Table 4 shows its weight according to the minimum scenario. The robustly optimal independent set (i.e. the one with the largest weight) is shown in boldface. Similarly, for each vertex cover Table 4 shows its weight according to the maximum scenario. The robustly optimal vertex cover (i.e. the one with the smallest weight) is shown in boldface. We see that the complement of the optimal independent set is not an optimal vertex cover. Similarly, the complement of the optimal vertex cover is not an optimal independent set. The found solutions are shown in Figure 3 by shading.*

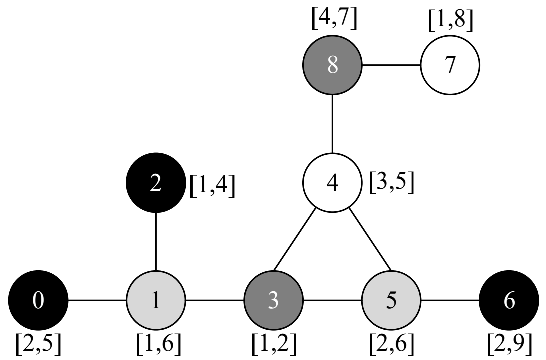

Figure 3: *A graph where the complement of an optimal solution to the MWIS problem under the minimum scenario is not an optimal solution to the MWVC problem under the maximum scenario*

Independent set	Weight under minimum scenario	Vertex cover	Weight under maximum scenario
0,2,3,6,7	7	1,4,5,8	24
0,2,3,6,8	10	1,4,5,7	25
0,2,4,6,7	9	**1,3,5,8**	21
0,2,5,7	6	1,3,4,6,8	29
0,2,5,8	9	1,3,4,6,7	30
1,4,6,7	7	0,2,3,5,8	24
1,5,7	4	0,2,3,4,6,8	32
1,5,8	7	0,2,3,4,6,7	33

Table 4: *Finding absolute robust solutions for the graph shown in Figure 3*

One way how to enforce equivalence among the above considered MWIS and MWVC problem variants would be to impose an additional constraint. Such constraint would require that the sum of vertex weights in each scenario should be equal to a predefined value T. The obtained (restricted) uncertainty set can be visualized as an intersection of a hyper-parallelepiped (Cartesian product) and a hyperplane.

The restriction makes sense in applications where scenarios describe different possibilities of how to distribute a fixed amount of some resource. With the restricted uncertainty set, equivalence of the two problems is assured by Proposition 1. On the other hand, the solutions from Proposition 4 become infeasible since the hyperplane cuts off both the minimum and the maximum scenario.

Let us now consider the MWIS and MWVC problem variants based on interval uncertainty and *robust deviation*. Again, we have to identify a special type of scenario. An *extremal scenario* is such scenario where each vertex v_i has either minimal or maximal value (either l_i or u_i). The importance of extremal scenarios is stressed by the following proposition.

Proposition 5. *Suppose that uncertainty in vertex weights is expressed by intervals. Then a robust deviation solution to the MWIS problem can be obtained by using a reduced scenario set consisting only of extremal scenarios. The same claim is also valid for the MWVC problem. Moreover, the complement of a robust deviation solution to the MWIS problem is a robust deviation solution to the MWVC problem, and vice-versa.*

Proof. The first claim dealing with the MWIS problem has been proved in [18]. The second claim dealing with the MWVC problem then follows from Proposition 2, which is applicable in the considered situation. The third claim only repeats the statement of Proposition 2. □

Proposition 5 assures that the considered problem variants based on interval uncertainty and robust deviation can be solved a little bit more efficiently than expected, i.e. with a reduced set of scenarios. But note that the reduced set is still fairly large. Indeed, for a graph with n vertices, there can be as many as 2^n extremal scenarios.

Proposition 5 can also serve for transforming some of the available results on the robust MWIS problem into similar results for the MWVC problem. Indeed, [11] contains an NP-hardness and an approximability theorem that both refer to the robust deviation MWIS problem with interval uncertainty on interval graphs. Both results can be converted to the corresponding MWVC problem. Such conversion is correct for the same reasons as already explained in Section 4. More precisely, we obtain the following corollary.

Corollary 4. *We consider the robust deviation MWVC problem on interval graphs. Uncertainty in vertex weights is expressed by interval representation. Then the considered problem is NP-hard. Also, the problem is approximable within 2.*

At the end of this section, let us say few words about the MWIS and MWVC problem variants based on interval uncertainty and *relative robust deviation*. According to Section 5, one would expect that such variants are not equivalent, i.e. that their solutions are not complementary one to another. This is indeed true. In order to check this, we have constructed an additional example, which is similar to Example 2 but based on interval uncertainty. In our example, the graph consisted of 9 vertices, and each interval consisted of 2 integers, so that the total number of implicitly given scenarios was $2^9 = 512$. To find robust solutions with so many scenarios, we employed the CPLEX software package [8]. The obtained results confirmed our expectations, e.g. it turned out that the complement of the computed robustly optimal independent set is not a robustly optimal vertex cover. More details can be obtained from the first author by e-mail.

7. OWA criteria

Suppose that our collection of scenarios S consists of p scenarios denoted by s_1, $s_2, \ldots, s_p$. Any OWA criterion is based on a vector of real coefficients $a_1, a_2, \ldots, a_p$ given in advance, such that $0 \leq a_i \leq 1$ for all $i = 1, 2, \ldots, p$, $\sum_{i=1}^{p} a_i = 1$. In this section, we will put emphasis on the OWA criterion that extends absolute robustness. According to [12], the criterion is constructed as follows.

For a chosen independent set X the weights $F(X, s)$ are sorted in ascending order, i.e. a permutation σ is found such that

$$F(X, s_{\sigma(1)}) \leq F(X, s_{\sigma(2)}) \leq \ldots \leq F(X, s_{\sigma(p)}).$$

The *OWA cost* for X is computed as:

$$O(X) = \sum_{i=1}^{p} a_i \cdot F(X, s_{\sigma(i)}).$$

This cost should be maximized for independent sets.

Definition 7. *An OWA solution to the MWIS problem is an independent set X_O that maximizes the function $O(X)$ over the whole collection of possible independent sets X.*

For a chosen vertex cover Y the weights $\bar{F}(Y, s)$ are sorted in descending order, i.e. a permutation ψ is found such that

$$\bar{F}(Y, s_{\psi(1)}) \geq \bar{F}(Y, s_{\psi(2)}) \geq \ldots \geq \bar{F}(Y, s_{\psi(p)}).$$

The *OWA cost* for Y is evaluated as:

$$\bar{O}(Y) = \sum_{i=1}^{p} a_i \cdot \bar{F}(Y, s_{\psi(i)}).$$

This cost should be minimized for vertex covers.

Definition 8. *An OWA solution to the MWVC problem is a vertex cover Y_O that minimizes the function $\bar{O}(Y)$ over the whole collection of possible vertex covers Y.*

Let us present two special cases of our OWA criterion.

- With $a_1 = 1$, $a_2 = a_3 = \cdots = a_p = 0$, the traditional absolute robustness criterion (max-min or min-max) is obtained.

- With $a_1 = a_2 = \cdots = a_p = 1/p$, a criterion is obtained that optimizes the average weight of X or Y. Averaging is done over all scenarios.

Similarly to the previous sections, we are concerned with the following question: is the OWA variant of the MWIS problem equivalent to the OWA variant of the MWVC problem? Or more precisely: is the complement of an OWA-optimal independent set an OWA-optimal vertex cover, and vice-versa? We assume that in both problems the coefficients $a_1, a_2, \ldots, a_p$ are chosen in the same way.

Of course, we already know that the equivalence cannot hold in general - this has been shown by Example 1, which can be interpreted as an OWA example, where $a_1 = 1$ and $a_2 = a_3 = \cdots = a_p = 0$. Still, there are some special cases where equivalence holds, as described by the following two propositions.

Proposition 6. *Suppose that all scenarios have the same sum of vertex weights, i.e. $T_s = T$ for all $s \in S$. Then the complement of an OWA solution to the MWIS problem is an OWA solution to the MWVC problem, and vice-versa.*

Proof. Let X be any independent set and Y its complement. Then it holds that $F(X,s) + \bar{F}(Y,s) = T$ for any scenario $s \in S$. Also, from

$$F(X, s_{\sigma(1)}) \leq F(X, s_{\sigma(2)}) \leq \ldots \leq F(X, s_{\sigma(p)}),$$

or equivalently

$$T - \bar{F}(Y, s_{\sigma(1)}) \leq T - \bar{F}(Y, s_{\sigma(2)}) \leq \ldots \leq T - \bar{F}(Y, s_{\sigma(p)}),$$

if follows that

$$\bar{F}(Y, s_{\sigma(1)}) \geq \bar{F}(Y, s_{\sigma(2)}) \geq \ldots \geq \bar{F}(Y, s_{\sigma(p)}).$$

Or in other words, the permutation σ used by the OWA criterion for an independent set coincides with the permutation ψ needed for the complementary vertex cover. Consequently, if X and Y are complements, then it holds:

$$\begin{aligned} O(X) &= \sum_{i=1}^{p} a_i \cdot F(X, s_{\sigma(i)}) = \sum_{i=1}^{p} a_i (T - \bar{F}(Y, s_{\sigma(i)}) \\ &= T - \sum_{i=1}^{p} a_i \cdot \bar{F}(Y, s_{\psi(i)}) = T - \bar{O}(Y). \end{aligned}$$

Assume now that X_O is an OWA solution to the MWIS problem, and that Y_O is the complement of X_O. Then:

$$\begin{aligned} O(X) \text{ achieves maximum for } X = X_O &\implies \\ T - \bar{O}(Y) \text{ achieves maximum for } Y = Y_O &\implies \\ \bar{O}(Y) \text{ achieves minimum for } Y = Y_O&. \end{aligned}$$

Thus Y_O is by definition an OWA solution to the MWVC problem. The proof in opposite direction is conducted analogously. □

Proposition 7. *Suppose that all coefficients $a_1, a_2, \dots, a_p$ are equal, i.e. $a_1 = a_2 = \dots = a_p = 1/p$. Then the complement of an OWA solution to the MWIS problem is an OWA solution to the MWVC problem, and vice-versa.*

Proof. Let X be any independent set and Y its complement. Then $F(X, s) + \bar{F}(Y, s) = T_s$ for any scenario $s \in S$. Denote the expression $(1/p) \sum_{i=1}^{p} T_i$ by $\tilde{T}$. Next, it holds:

$$\begin{aligned} O(X) &= \frac{1}{p} \sum_{i=1}^{p} F(X, s_{\sigma(i)}) = \frac{1}{p} \sum_{i=1}^{p} (T_{s_{\sigma(i)}} - \bar{F}(Y, s_{\sigma(i)})) \\ &= \frac{1}{p} \sum_{i=1}^{p} T_{s_{\sigma(i)}} - \frac{1}{p} \sum_{i=1}^{p} \bar{F}(Y, s_{\sigma(i)}) \\ &= \tilde{T} - \frac{1}{p} \sum_{i=1}^{p} \bar{F}(Y, s_{\psi(i)}) \\ &= \tilde{T} - \bar{O}(Y). \end{aligned}$$

The rest of the proof is similar to Proposition 6. □

Note that Proposition 6 is a generalization of Proposition 1 from Section 3. Note also that Proposition 7 is not a surprise since it refers to the case where the OWA variants of both problems reduce to their conventional variants over an *average* scenario. It is a scenario where the weight of any vertex is equal to the average of its weights over all scenarios. Reduction is correct because $O(X)$ (or $\bar{O}(Y)$) is a sum of sums, and the order of summation can be reversed. The involved conventional variants are surely equivalent even with non-integer vertex weights.

So far we have considered the OWA criterion that extends absolute robustness. However, a criterion corresponding to robust deviation can also be considered - we will call it the OWA-D criterion. The only difference with respect to OWA is that instead of objective function values $F(X, s)$ and $\bar{F}(Y, s)$ the corresponding regrets $F_s^* - F(X, s)$ and $\bar{F}(Y, s) - \bar{F}_s^*$, respectively, are used. In this way, the OWA-D cost is defined as a convex combination of regrets over all scenarios sorted in descending order. Also, an OWA-D solution is defined as the one that minimizes the OWA-D cost over the whole collection of possible independent sets or vertex covers. Obviously, the OWA-D criterion is a generalization of the traditional robust deviation criterion, i.e. OWA-D with $a_1 = 1$, $a_2 = a_3 = \dots = a_p = 0$ reduces to the traditional min-max regret. It can easily be seen that the OWA-D variant of the MWIS problem is equivalent to the OWA-D variant of the MWVC problem. More precisely, Proposition 2 from Section 4 can be generalized to cover OWA-D instead of simple robust deviation. The proof is similar to Propositions 2 and 6.

The third OWA-type criterion that can also be considered is the OWA-R criterion, which is a generalization of traditional relative robustness. It is similar to OWA-D, except that relative regrets $(F_s^* - F(X, s))/F_s^*$ and $(\bar{F}(Y, s) - \bar{F}_s^*)/\bar{F}_s^*$, respectively, are used instead of ordinary regrets. It is clear that the OWA-R variant of the MWIS problem cannot in general be equivalent to the OWA-R variant of the MWVC problem. As a counterexample, Example 2 from Section 5 can again be used. However, the sufficient condition for equivalence specified by Proposition 3 still holds, i.e. Proposition 3 can easily be generalized from the traditional relative min-max regret to the OWA-R criterion. The proof for such generalization is similar to the proofs for Propositions 3 and 6.

8. Pareto efficiency

In this section, we consider solving robust MWIS and MWVC problems by the method based on Pareto efficiency. Let us denote again the whole list of scenarios as $s_1, s_2, \ldots, s_p$. By applying the general ideas of dominance and efficiency from Section 2 to our problems, the following more concrete definitions are obtained.

Definition 9. *An independent set X is dominated by another independent set $\widetilde{X}$ if it holds that:*

$$(F(X, s_1), F(X, s_2), \ldots, F(X, s_p)) < (F(\widetilde{X}, s_1), F(\widetilde{X}, s_2), \ldots, F(\widetilde{X}, s_p)).$$

An independent set X is efficient if it is not dominated by any other independent set.

Definition 10. *A vertex cover Y is dominated by another vertex cover $\widetilde{Y}$ if it holds that:*

$$(\bar{F}(Y, s_1), \bar{F}(Y, s_2), \ldots, \bar{F}(Y, s_p)) > (\bar{F}(\widetilde{Y}, s_1), \bar{F}(\widetilde{Y}, s_2), \ldots, \bar{F}(\widetilde{Y}, s_p)).$$

A vertex cover Y is efficient if it is not dominated by any other vertex cover.

In the above definitions, vector notation has been used. Also, the vectors have been written as rows. The ordering of vectors is defined in the standard way, i.e. componentwise. Indeed, for two vectors $\vec{a} = (a_1, a_2, \ldots, a_p)$ and $\vec{b} = (b_1, b_2, \ldots, b_p)$ it holds that $\vec{a} \le \vec{b}$ if $a_i \le b_i$ for all $i = 1, 2, \ldots, p$. Next, $\vec{a} < \vec{b}$ means that $\vec{a} \le \vec{b}$ and $\vec{a} \ne \vec{b}$.

The method for solving the robust MWIS problem analyzed in this section consists of finding the whole collection of efficient independent sets, i.e. those independent sets that are not dominated by some other independent set. The analogous method for solving the robust MWVC problem consists of finding the whole collection of efficient vertex covers.

As always before, we would like to know whether the proposed ways of solving the MWVIS and MWVC problem are equivalent in the sense that the solution to one problem can easily be transformed into the solution to the other problem. Once more, the answer is positive, as guaranteed by the following proposition.

Proposition 8. *The collection of complements of all efficient independent sets coincides with the collection of all efficient vertex covers, and vice-versa.*

Proof. Let X be an efficient independent set. Let Y be the complement of X. We claim that Y must be an efficient vertex cover. Indeed, if Y is not efficient, then there exists another vertex cover $\widetilde{Y}$ that dominates over Y. Denote by $\widetilde{X}$ the complement of $\widetilde{Y}$. Then it holds

$$(\bar{F}(Y, s_1), \bar{F}(Y, s_2), \ldots, \bar{F}(Y, s_p)) > (\bar{F}(\widetilde{Y}, s_1), \bar{F}(\widetilde{Y}, s_2), \ldots, \bar{F}(\widetilde{Y}, s_p)),$$

or equivalently

$$(T_{s_1} - F(X, s_1), T_{s_2} - F(X, s_2), \ldots, T_{s_p} - F(X, s_p)) > \\ (T_{s_1} - F(\widetilde{X}, s_1), T_{s_2} - F(\widetilde{X}, s_2), \ldots, T_{s_p} - F(\widetilde{X}, s_p)),$$

or componentwise

$T_{s_i} - F(X, s_i) \geq T_{s_i} - F(\widetilde{X}, s_i)$, $i = 1, 2, \ldots, p$ (inequality is strict for at least one i),

which is equivalent to

$$F(X, s_i) \leq F(\widetilde{X}, s_i), \ \ i = 1, 2, \ldots, p \quad \text{(inequality is strict for at least one } i\text{)},$$

or in vector notation

$$(F(X, s_1), F(X, s_2), \ldots, F(X, s_p)) < (F(\widetilde{X}, s_1), F(\widetilde{X}, s_2), \ldots, F(\widetilde{X}, s_p)).$$

So X is dominated by $\widetilde{X}$, which is a contradiction to the initial assumption that X is efficient. Thus Y must also be efficient.

So far we have proved that the collection of complements of all efficient independent sets must be a sub-collection of the collection of all efficient vertex covers. However, the same proof can be conducted in opposite direction, thus proving that the collection of complements of all efficient vertex covers must be a sub-collection of the collection of all efficient independent sets. Thanks to such bi-directionality, it is obvious that the considered collections must coincide, i.e. the mentioned inclusions are in fact equalities. □

According to the above proposition, any algorithm [5, 15] that determines all efficient independent sets could be used as an algorithm for determining all efficient vertex covers, and vice-versa.

In the remaining part of this section, we will analyze relationships among particular robust solutions from Sections 3-7 and efficient solutions. Such relationships are summarized by the following two propositions. Both of them are well known [1, 12], and therefore their proofs will be omitted.

Proposition 9. *An absolute robust solution to the MWIS problem can be chosen so that it is also an efficient solution to the MWIS problem. The same claim is also true regarding a (relative) robust deviation solution or OWA solutions to the MWIS problem. Analogous claims are valid for the MWVC problem as well.*

Proposition 10. *Suppose that all coefficients $s_1, s_2, \ldots, s_p$ within an OWA criterion are nonzero. Then any OWA solution to the MWIS problem must be an efficient solution to the MWIS problem. An analogous claim is also valid for the MWVC problem.*

The following two corollaries are obtained as simple consequences of Proposition 8 combined with Proposition 9 and 10, respectively.

Corollary 5. *An absolute robust solution to the MWIS problem can be chosen so that its complement is an efficient solution to the MWVC problem. Also, an absolute robust solution to the MWVC problem can be chosen so that its complement is an efficient solution to the MWIS problem. The same claims are also true regarding (relative) robust deviation solutions or OWA solutions.*

Corollary 6. *Suppose that all coefficients $s_1, s_2, \ldots, s_p$ within an OWA criterion are nonzero. Then the complement of any OWA solution to the MWIS problem must be an efficient solution to the MWVC problem. Also, the complement of any OWA solution to the MWVC problem must be an efficient solution to the MWIS problem.*

Roughly speaking, Corollary 5, and specially Corollary 6, say the following. Although the complement of a robust solution for the MWIS problem may not necessarily be an equivalent robust solution for the MWVC problem, it should still be an efficient solution for the MWVC problem. Analogous interpretation the other way around is also valid.

9. Conclusions

In this paper, we have explored relationships among maximum weighted independent sets (MWIS) and minimum weighted vertex covers (MWVC) within the context of robust optimization. More precisely, we have tried to find the answer to the following question: is the complement of a robustly optimal independent set a robustly optimal vertex cover, and vice-versa?

The answer to the above question is not straightforward - it depends on the chosen robustness criterion. Indeed, the answer is positive if the robust deviation criterion is used, and also if the whole collection of efficient solutions is considered. For absolute robustness or relative robust deviation the answer is in general negative, although some special cases exist when it is still positive.

In the recent literature, there are some articles dealing with complexity or approximability of robust MWIS problem variants. On the other hand, there are no similar works dealing with robust MWVC variants. This paper clearly indicates that the available results on the MWIS problem cannot automatically be transferred to the MWVC problem. However, the paper identifies several situations where such transfer is still possible.

The results of this paper are interesting because they emphasize some important differences between conventional and robust variants of the considered optimization problems. The paper clearly shows that our intuition and assumptions about problem relationships, acquired through conventional optimization, cannot be taken for granted when dealing with robust optimization.

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