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Finite production rate model with backlogging, service level constraint, rework, and random breakdown

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SUMMARY

In most real-life production systems, both random machine breakdown and the production of nonconforming items are inevitable, and adopting a backlogging policy with a predetermined minimum acceptable service level can sometimes be an effective strategy to help the management reduce operating cost or smoothen the production schedule. With the aim of addressing the aforementioned practical situations in production, this study explores the optimal production runtime for the finite production rate (FPR) model with allowable backlogging and service level constraint, rework of defective products, and random machine breakdown. Mathematical modelling is employed along with optimization techniques to derive the optimal production runtime that minimizes the long-run average system costs for the proposed FPR model. The joint effects of the allowable backlogging with a planned service level, rework, and random machine breakdown on optimal runtime decision have been carefully investigated through a numerical example and sensitivity analysis. As a result, important insights regarding various system parameters are revealed in order to enable the management to better understand, plan, and control such a practical production system.

KEY WORDS: *production runtime, backlogging, service level constraint, breakdown, optimization, rework, mathematical modelling.*

1. INTRODUCTION

This study explores the optimal replenishment runtime for a finite production rate (FPR) model with allowable backlogging and service level constraint, rework of defective products, and random machine breakdown. The conventional FPR model [1], also known as the economic production quantity (EPQ) model, derived the most economic lot size for a manufacturing system with no backlogging permitted, and implicitly assumed a perfect condition in production process. However, in real production planning, due to internal orders of parts or materials and other operating conditions, adopting planned backlogging can be a strategy to effectively minimize the expected production-inventory system cost. While allowing backlogging, abusive shortage may cause an unacceptable service level and turn into possible losses of future sales (because of the loss of customer goodwill). Hence, the maximal allowable shortage level per replenishment cycle is often set as a business operating constraint in order to attain the minimal service level. Examples are surveyed as follows: Schneider [2] examined a (Q,s) model and determined the optimal order quantity Q and reorder point s in which the average annual costs of inventory and orders are minimal, provided that a certain service level is reached. De Kok [3] examined a single product inventory model with adjustable production rate that can cope with random fluctuations in demand. The demand process of the product is described by a compound Poisson process and excess demand is lost. They considered service measures as the average number of lost-sales occurrences per unit time and the fraction of demand that is lost. Using a two-critical-number control policy they derived practically useful approximations for the switch-over level in order to achieve the required service level. Yildirim et al. [4] considered a stochastic multi-period production planning and sourcing problem for a manufacturer with a number of plants (or subcontractors). Each source, has a different production cost, capacity, and lead time. The manufacturer has to meet the demand for different products according to the service level requirements set by its customers. The demand for each product in each period is random. They presented a methodology, based on mathematical programming, for the manufacturer to decide on the production quantity, when and where to produce, and the exact inventories to carry. Their approach yields the same result as the base stock policy for a single plant with stationary demand. Tsai and Zheng [5] used a simulation optimization algorithm to solve a two-echelon constrained inventory problem. Their goal was to determine the optimal setting of stocking levels that minimize total inventory investment costs, while satisfying the expected response time targets for each field depot. Their algorithm can be applied to any multi-item multi-echelon inventory system, where the cost structure and service level function resemble what they assumed. Empirical studies were performed to compare the efficiency of the proposed algorithms with other existing simulation algorithms. A considerable amount of research has been carried out to address the service level constraint [6–12].

In addition, this study considers random machine breakdown and production of nonconforming items. In most real-world manufacturing systems, due to process deterioration or various other uncontrollable factors, these quality issues are inevitable. Groenevelt et al. [13] studied two production control policies that deal with random machine breakdown. The first policy assumes that production of the interrupted lot is not resumed (called NR policy) after a breakdown. The second policy considers that production of the interrupted lot will be immediately resumed (called abort/resume (AR) policy) after the breakdown is fixed and if the current on-hand inventory is below a certain threshold level. They assumed the repair time is negligible and studied the effects of machine breakdowns and corrective maintenance on economic lot size decisions. Dohi, et al. [14] derived the minimal repair policies for an economic manufacturing process. Two models with and without an infinite number of minimal repairs were formulated; and the optimal EMQ policies which minimize the expected costs were derived, respectively. Sana [15] proposed a model to determine the optimal product reliability and production rate that achieves the largest total integrated profit for an imperfect production system. He provided an optimal control formulation to the problem and developed necessary and sufficient conditions for the optimality of dynamic variables. Then, the Euler–Lagrange method was used to obtain the optimal solutions for product reliability parameter and dynamic production rate. Chakraborty et al. [16] examined economic manufacturing quantity model subject to stochastic breakdown, repair and stock threshold level. They considered production rate as a decision variable. Since the stress of the machine depends on production rate, hence failure rate of the machine will be a function of the production rate. Extra capacity of the machine was considered to buffer against the possible uncertainties of the production process where machine capacity is predetermined. The basic model was developed under general failure and general repair time distributions. They suggested two computational algorithms for determining production rate and stock threshold level, which minimize the expected cost rate in the steady state. Widyadana and Wee [17] studied EPQ models for deteriorating items with preventive maintenance (PM), random breakdown and immediate corrective action. Corrective and PM times were assumed to be stochastic and the unfulfilled demands are lost sales. Two economic production quantity models of uniform distribution and exponential distribution of corrective and maintenance times were developed and examined. An example and sensitivity analysis was provided for the purpose of illustrating the models. For the exponential distribution model, they showed that the corrective time parameter is one of the most sensitive parameters to optimal total cost. Additional studies that addressed various aspects of production systems with machine breakdown, defective product, or product quality assurance issues can also be found elsewhere [18–26].

Since little attention has been paid to the investigation of joint effects of backlogging and service level constraint, rework, and random machine breakdown on the optimal replenishment runtime of the FPR model, this study is intended to bridge the gap. Details of the proposed model are provided in the following section.

2. THE PROPOSED FPR MODEL

This study derives optimal runtime for the FPR model with allowable backlogging and service level constraint, rework, and random machine breakdown. Consider the production rate of an imperfect FPR model as P and during the fabrication process; an x portion of nonconforming items may be randomly produced at rate d, yielding $d = Px$. All nonconforming items are assumed to be repairable through a rework process at a rate of P_1 right after the end of the regular production process in each replenishment cycle. During production uptime, the machine is subject to a random breakdown that follows the Poisson distribution. When a breakdown occurs, an abort/resume (AR) policy is adopted, wherein the machine repair is taken up immediately and the interrupted lot resumed right after the machine is repaired and restored. The machine repair time is assumed to be a constant (a spare machine is used if repair time exceeds the allowable time). Shortage is permitted and backordered in the proposed FPR model, and a unit shortage cost b per unit time is associated with it. In order to avoid an abusive backlogging situation, a minimum acceptable service level $(1 - \alpha)\%$ is predetermined.

The annual production rate P is assumed to be larger than the sum of annual demand rate λ and production rate of nonconforming items, i.e. $(P - d - \lambda) > 0$. All items produced are screened and the unit inspection cost is included in the unit production cost C. Cost-related parameters also include setup cost K, unit holding cost h, unit reworking cost C_R, holding cost h_1 for each reworked item, unit cost C_1 and unit holding cost h_3 per unit of safety stock, unit delivery cost C_T; and machine repairing cost M per breakdown. Additional notations are listed as follows:

T_1 = production uptime, the decision variable of the proposed FPR model,

Q = replenishment lot size per production cycle,

t = production time before a random breakdown occurs,

t_r = time required for repairing the machine,

β = number of breakdowns per year, a random variable that follows the Poisson distribution,

t_1' = uptime when stock piles up,

t_2' = time to rework nonconforming items,

t_3' = time to consume all available perfect quality items,

t_4' = time in which backlogging accumulated,

t_5' = uptime in which backlogging being satisfied,

T' = cycle length in the case of machine breakdown taking place,

H = maximum level of on-hand inventory in units when the rework process finishes,

H_0 = level of backlogging when a machine breakdown occurs,

H_1 = the maximum level of on-hand inventory in units when regular production process ends,

H_2 = level of on-hand inventory when a machine breakdown occurs,

B = maximum level of backlogging,

t_1 = uptime when stock piles up – in the case of no breakdown occurrence,

t_2 = rework time – in the case of no breakdown occurrence,

t_3 = time required for depleting all available perfect items – in the case of no breakdown occurrence,

T = cycle length – in the case of no breakdown occurrence,

$\boldsymbol{T}$ = cycle length whether machine breaks down or not,

$TC_1(T_1)$ = total system costs per cycle in the case of breakdown taking place in the backlogging stage,

$TC_2(T_1)$ = total system costs per cycle in the case of breakdown taking place in the stock pileup stage,

$TC_3(T_1)$ = total system costs per cycle in the case of no breakdown occurrence,

$E[TC_1(T_1)]$ = the expected total system costs per cycle in the case of breakdown taking place in backlogging stage,

$E[TC_2(T_1)]$ = the expected total system costs per cycle in the case of breakdown taking place in the stock pileup stage,

$E[TC_3(T_1)]$ = the expected system costs per cycle in the case of no breakdown occurrence,

$TCU(T_1)$ = total system costs per unit time whether a breakdown takes place or not,

$E[TCU(T_1)]$ = the expected system costs per unit time whether a breakdown takes place or not.

Because the time before a breakdown during production uptime T_1 is random, we must examine the following three possible cases of random breakdown.

2.1 CASE 1: $t < t_5'$

In this case, a machine breakdown takes place in the backlogging stage, and as per the AR policy, the production of the interrupted lot is immediately resumed when the machine breakdown is fixed. The on-hand inventory level of perfect quality products in this case is illustrated in Figure 1. It is noted that when a breakdown occurs, the level of backlogging is H_0, and after the machine is repaired and restored, the level of backlogging continues to reduce, and then changes to having positive stocks in t_1'. At the end of production uptime, the level of on-hand inventory reaches H_1. Subsequently, the rework process starts and brings the on-hand perfect quality items to a maximum level of H, at the end of t_2'. It follows that all available products are consumed in t_3', followed by a shortage in t_4' until they accumulate to a maximum allowable level of backlogging B (i.e. a predetermined level based on the minimum acceptable service level constraint). Then, the uptime of the following replenishment cycle begins. The following formulae can be observed directly from Figure 1:

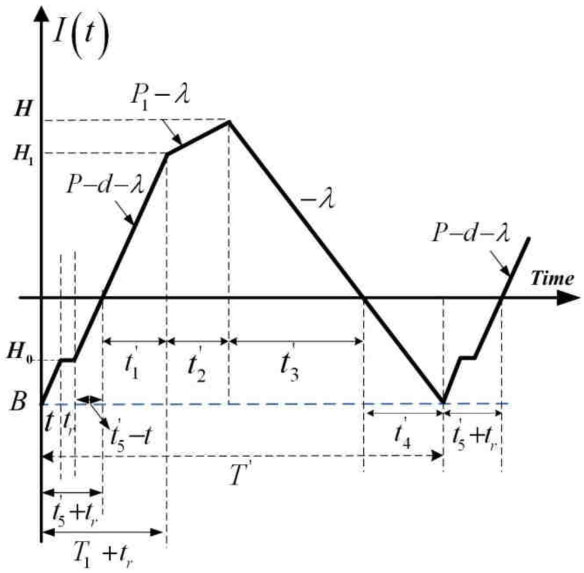

Fig. 1 *The on-hand inventory level of perfect quality products in the proposed FPR model when breakdown occurs in the backlogging stage*

$$T' = \sum_{i=1}^{5} t_i' + t_r \tag{1}$$

$$T_1 = t_5' + t_1' = \frac{Q}{P} \tag{2}$$

$$H_1 = (P-d-\lambda)\cdot t_1' \tag{3}$$

$$H = H_1 + (P_1-\lambda)\cdot t_2' \tag{4}$$

$$H_0 = B-(P-d-\lambda)\cdot t \tag{5}$$

$$t_1' = \frac{Q}{P} - t_5' \tag{6}$$

$$t_3' = \frac{H}{\lambda} \tag{7}$$

$$t_4^{'} = \frac{B}{\lambda} \tag{8}$$

$$t_5^{'} = \frac{B}{(P-d-\lambda)} \tag{9}$$

The on-hand inventory level of nonconforming items in the proposed FPR model is depicted in Figure 2. The maximum level of nonconforming products at the end of the uptime is d_1T_1 and the reworking time is:

$$t_2^{'} = \frac{dT_1}{P_1} \tag{10}$$

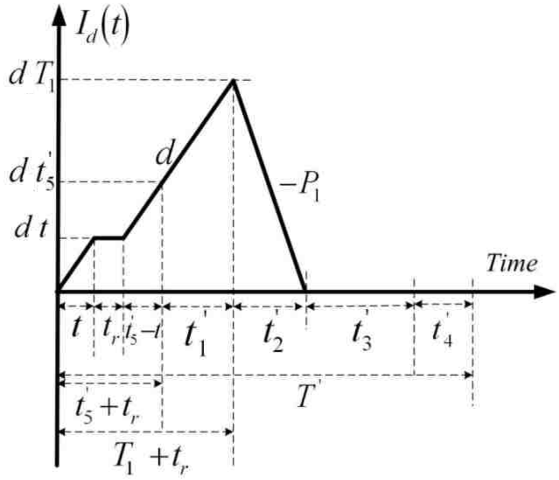

Fig. 2 *The on-hand inventory level of nonconforming items in the proposed FPR model when breakdown occurs in the backlogging stage*

The total system costs per cycle in the case of a breakdown during the backlogging stage consist of production setup cost, variable fabrication costs, variable reworking costs, fixed machine breakdown repairing cost, holding and purchasing costs of safety stock (to cope with breakdown occurrence), holding costs in the rework and regular process, variable shipping costs, and variable backordering costs. Therefore, $TC_1(T_1)$ is:

$$\begin{aligned} TC_1(T_1) &= K + CQ + C_R xQ + M + \left[h_3(\lambda t_r)\left(t + \frac{t_r}{2}\right) + C_1(\lambda t_r) \right] + h_1 \cdot \frac{P_1 t_2^{'}}{2}\left(t_2^{'}\right) + \\ &+ h\left[\frac{H_1}{2}\left(t_1^{'}\right) + \frac{(H_1+H)}{2}\left(t_2^{'}\right) + \frac{H}{2}\left(t_3^{'}\right) + (dt)t_r + \frac{d(T_1)}{2}(T_1) \right] + \\ &+ C_T\left[Q + (\lambda t_r)\right] + b\frac{B}{2}\left(t_4^{'} + t_5^{'}\right) + b(H_0)t_r \end{aligned} \tag{11}$$

By substituting Eqs. (1) to (10) in Eqs. (11), and with further derivations, we obtain $TC_1(T_1)$ as shown in Appendix A.

2.2 CASE 2: $t_5' < t < t_1$

In this case, a machine breakdown occurs in the stock pileup stage (see Figure 3). An additional formula can be observed directly from Figure 3 as follows:

$$H_2 = (P-d-\lambda)\cdot\left(t-t_5^{'}\right) \tag{12}$$

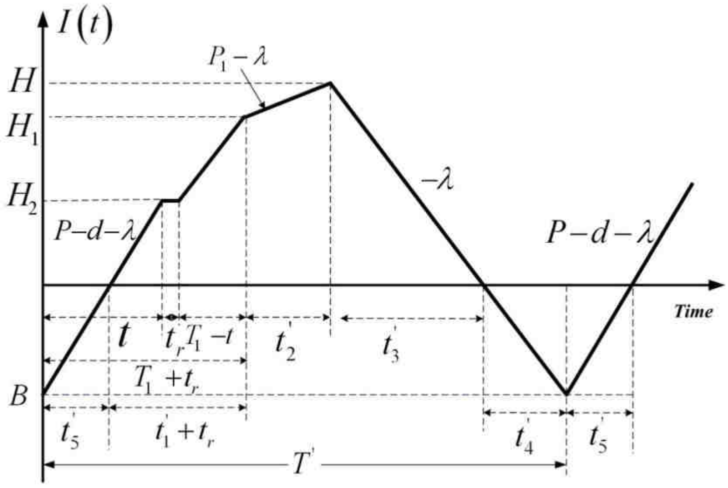

Fig. 3 *The on-hand inventory level of perfect quality products in the proposed FPR model when breakdown occurs in the stock pileup stage*

Similarly, total system costs per cycle in this case consist of production setup cost, variable fabrication costs, variable reworking costs, fixed machine breakdown repairing cost, holding and purchasing costs of safety stock, holding costs in the rework and regular process, variable shipping costs, and variable backordering costs. Hence, $TC_2(T_1)$ is:

$$\begin{aligned} TC_2(T_1) = {} & K + CQ + C_R xQ + M + \left[h_3(\lambda t_r)\left(t+\frac{t_r}{2}\right)+C_1(\lambda t_r)\right]+ \\ & +h\left[\begin{array}{l}\dfrac{H_2}{2}\left(t-t_5^{'}\right)+\dfrac{(H_2+H_1)}{2}(T_1-t)+\dfrac{(H_1+H)}{2}\left(t_2^{'}\right)+ \\ +\dfrac{H}{2}\left(t_3^{'}\right)+(H_2+d\cdot t)t_r+\dfrac{d\cdot T_1}{2}(T_1)\end{array}\right]+ \\ & +h_1\cdot\frac{P_1 t_2^{'}}{2}\left(t_2^{'}\right)+C_T\left[Q+(\lambda t_r)\right]+b\frac{B}{2}\left(t_4^{'}+t_5^{'}\right) \end{aligned} \tag{13}$$

By substituting Eqs. (1) to (4), (6) to (10), and (12) in Eq. (13), and with further derivations, we obtain $TC_2(T_1)$ as shown in Appendix A.

2.3 CASE 3: $t \geqq t_1$

In this case, no machine breakdown occurs during production uptime T_1 (see Figure 4). The following formula can be obtained directly from Figure 4:

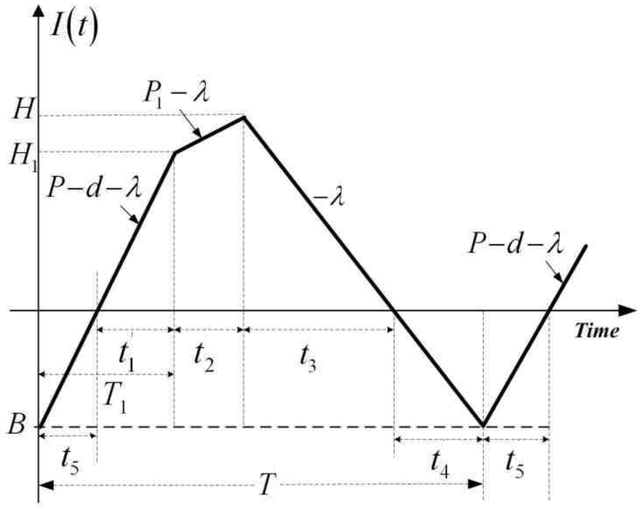

Fig. 4 *The on-hand inventory level of perfect quality products in the proposed FPR model when breakdown does not occur*

$$T = \sum_{i=1}^{5} t_i \tag{14}$$

$$T_1 = \left(t_5 + t_1\right) = \frac{Q}{P} \tag{15}$$

$$H_1 = (P - d - \lambda)\cdot t_1 \tag{16}$$

$$H = H_1 + (P_1 - \lambda)\cdot t_2 \tag{17}$$

$$t_1 = \frac{Q}{P} - t_5 \tag{18}$$

$$t_2 = \frac{dT_1}{P_1} \tag{19}$$

$$t_3 = \frac{H}{\lambda} \tag{20}$$

$$t_4 = \frac{B}{\lambda} \tag{21}$$

$$t_5 = \frac{B}{(P - d - \lambda)} \tag{22}$$

The total system costs per cycle in this case consist of production setup cost, variable fabrication costs, variable reworking costs, holding and purchasing costs of safety stock, holding costs in the rework and regular process, variable shipping costs, and variable backordering costs. Hence, $TC_3(T_1)$ is:

$$TC_3\left(T_1\right) = K + CQ + C_R xQ + \left[h_3(\lambda t_r)T + C_1\left(\lambda t_r\right)\right] + h_1\cdot\frac{P_1 t_2}{2}\left(t_2\right) + \\ + h\left[\frac{H_1}{2}\left(t_1\right) + \frac{\left(H_1 + H\right)}{2}\left(t_2\right) + \frac{H}{2}\left(t_3\right) + \frac{d\left(T_1\right)}{2}\left(T_1\right)\right] + C_T Q + b\frac{B}{2}\left(t_4 + t_5\right) \tag{23}$$

By substituting Eqs. (14) to (22) in Eq. (23), and with further derivations, we obtain $TC_3(T_1)$ as shown in Appendix A. As stated earlier, in order to avoid an abusive backlogging situation, a minimum acceptable service level $(1 - \alpha)\%$ is predetermined for the proposed study. From prior literature [10], we obtain the following relationship between the maximum backlogging level and the service level indicator α:

$$B = \alpha\left(1 - E[x] - \frac{\lambda}{P}\right)\left(\frac{1}{1 - E[x]}\right)T_1 P \tag{24}$$

2.4 INTEGRATION OF THE PROPOSED FPR MODEL WITH/WITHOUT BREAKDOWN

First, the machine breakdown is assumed to be a random variable that follows the Poisson distribution with mean equal to β per unit time. Therefore, the time to breakdown obeys the exponential distribution, with density function $f(t) = \beta e^{-\beta t}$ and cumulative density function $F(t) = (1 - e^{-\beta t})$. Then, the expected system costs per unit time $E[TCU(T_1)]$ (i.e., whether a breakdown occurs or not) is:

$$E\left[TCU(T_1)\right] = \frac{\int_0^{t_5'} E\left[TC_1(T_1)\right] f(t)dt + \int_{t_5'}^{T_1} E\left[TC_2(T_1)\right] f(t)dt + \int_{T_1}^{\infty} E\left[TC_3(T_1)\right] f(t)dt}{E[\boldsymbol{T}]} \tag{25}$$

where the expected cycle length $E[\boldsymbol{T}]$ is:

$$E[\boldsymbol{T}] = \int_0^{T_1} E[T'] f(t)dt + \int_{T_1}^{\infty} E[T] f(t)dt = \frac{PT_1}{\lambda} \tag{26}$$

We use the expected values of x to cope with the randomness of the nonconforming rate in the cost analysis first to obtain $E[TC_1(T_1)]$, $E[TC_2(T_1)]$, and $E[TC_3(T_1)]$. Then, in order to solve the integration of mean-time-to-breakdown in the expected system cost function $E[TCU(T_1)]$, we substitute $E[TC_1(T_1)]$, $E[TC_2(T_1)]$, $E[TC_3(T_1)]$, $f(t)$, and $E[\boldsymbol{T}]$ in Eq. (25), and with further derivations, $E[TCU(T_1)]$ can be obtained as follows:

$$\begin{aligned} E\left[TCU(T_1)\right] &= \lambda \cdot \left\{ \frac{z_1}{T_1} + T_1 \begin{bmatrix} \dfrac{v^2}{2P\lambda}(h+b) + \dfrac{v^2}{2P^2(1-E[x]-\frac{\lambda}{P})}(h+b) - \\ -\dfrac{h}{2} + (h_1 - h)\dfrac{E[x]^2 P}{2P_1} + \dfrac{1}{\lambda}\left(\dfrac{hP}{2} - hv\right) \end{bmatrix} \right\} + \\ &+ \lambda \cdot \left\{ \begin{aligned} &\left[C + C_R E[x] + \frac{v}{P}(bg - hg) + C_T + h_3 g\right] + \frac{w_1}{T_1} + w_2 e^{-\beta T_1} + \\ &+ \frac{w_3 e^{-\beta T_1}}{T_1} + \frac{w_4 e^{-\beta T_1 s}}{T_1} + w_5 e^{-\beta T_1(1-s)} + w_5 e^{-\beta T_1 s} \end{aligned} \right\} \end{aligned} \tag{27}$$

where v, z_1, Y_1, s, w_1, w_2, w_3, w_4, and w_5 denote the following:

$$v = \alpha\left(1 - E[x] - \frac{\lambda}{P}\right)\left(\frac{1}{1 - E[x]}\right)P; \qquad z_1 = \left[\frac{K}{P} + \frac{C_1 \lambda g}{P}\right] \tag{28}$$

$$Y_1 = \frac{B}{P - PE[x] - \lambda} = \frac{vT_1}{P - PE[x] - \lambda} = sT_1; \quad s = \frac{v}{P - PE[x] - \lambda} \tag{29}$$

$$w_1 = \left[\frac{M}{P} + \frac{h_3 \lambda g^2}{2P} + \frac{h_3 \lambda g}{\beta P} + \frac{C_T \lambda g}{P} + \frac{hE[x]g}{\beta} - \frac{bg(P - PE[x] - \lambda)}{\beta P} \right] \tag{30}$$

$$w_2 = \left[-\frac{h_3 \lambda g}{P} - hg + \frac{h \lambda g}{P} \right]; \; w_3 = \left[-\frac{M}{P} - \frac{h_3 \lambda g^2}{2P} - \frac{h_3 \lambda g}{\beta P} - \frac{C_T \lambda g}{P} - \frac{hg}{\beta} + \frac{h \lambda g}{\beta P} \right] \tag{31}$$

$$w_4 = \frac{g\left(1 - E[x] - \frac{\lambda}{P}\right)(h+b)}{\beta}; \; w_5 = \left[\frac{hvg}{P} \right] \tag{32}$$

To determine the optimal replenishment runtime T_1*, the following theorem is proposed. Let $y(T_1)$ denote the following term:

$$y(T_1) = \frac{\alpha_2}{\alpha_3 + \alpha_4} \tag{33}$$

where:

$$\alpha_2 = \left[2z_1 + 2w_1 + 2w_3 e^{-\beta T_1} + 2w_4 e^{-\beta T_1 s} \right]$$

$$\alpha_3 = -T_1^2 \beta^2 \cdot \left[\begin{array}{l} w_2 e^{-\beta T_1} + w_5 e^{-\beta T_1} \cdot \left(e^{-\beta T_1 s}\right)^{-1} - 2sw_5 e^{-\beta T_1} \cdot \left(e^{-\beta T_1 s}\right)^{-1} \\ + s^2 w_5 \cdot e^{-\beta T_1} \left(e^{-\beta T_1 s}\right)^{-1} + s^2 w_5 e^{-\beta T_1 s} \end{array} \right]$$

$$\alpha_4 = \left[-T_1 \beta^2 w_3 e^{-\beta T_1} - 2\beta w_3 e^{-\beta T_1} - T_1 \beta^2 s^2 w_4 e^{-\beta T_1 s} - 2\beta s w_4 e^{-\beta T_1 s} \right]$$

Theorem 1: $E[TCU(T_1)]$ is convex if $0 < T_1 < y(T_1)$.

The first and second derivatives of $E[TCU(T_1)]$ are:

$$\frac{dE\left[TCU(T_1)\right]}{dT_1} = \lambda \cdot \left\{ \begin{array}{l} -\frac{1}{T_1^2}\left(\frac{K}{P} + \frac{C_1 \lambda g}{P}\right) + \left[\begin{array}{l} \frac{v^2}{2P\lambda}(h+b) + \frac{v^2}{2P^2\left(1 - E[x] - \frac{\lambda}{P}\right)}(h+b) - \frac{h}{2} + \\ +(h_1 - h)\frac{E[x]^2 P}{2P_1} + \frac{1}{\lambda}\left(\frac{hP}{2} - hv\right) \end{array} \right] - \\ -\frac{1}{T_1^2}\left[\frac{M}{P} + \frac{h_3 \lambda g^2}{2P} + \frac{h_3 \lambda g}{\beta P} + \frac{C_T \lambda g}{P} + \frac{hE[x]g}{\beta} - \frac{bg(P - PE[x] - \lambda)}{\beta P} \right] - \\ -\left[-\frac{h_3 \lambda g}{P} - hg + \frac{h\lambda g}{P} \right]\beta e^{-\beta T_1} - \left(\frac{hvg}{P}\right)\beta e^{-\beta T_1} \cdot \left(e^{-\beta T_1 s}\right)^{-1}(1 - s) - \\ -\left[-\frac{M}{P} - \frac{h_3 \lambda g^2}{2P} - \frac{h_3 \lambda g}{\beta P} - \frac{C_T \lambda g}{P} - \frac{hg}{\beta} + \frac{h\lambda g}{\beta P} \right]\left(\frac{\beta e^{-\beta T_1}}{T_1} + \frac{e^{-\beta T_1}}{T_1^2} \right) - \\ -\frac{g\left(1 - E[x] - \frac{\lambda}{P}\right)(h+b)}{\beta} \cdot \left(\frac{\beta s e^{-\beta T_1 s}}{T_1} + \frac{e^{-\beta T_1 s}}{T_1^2} \right) - \left(\frac{hvg}{P}\right)\beta s e^{-\beta T_1 s} \end{array} \right\} \tag{34}$$

$$\frac{d^2E\left[TCU(T_1)\right]}{dT_1^2}=$$
$$\lambda\cdot\left\{\begin{array}{l}\frac{2}{T_1^3}\left[\frac{K}{P}+\frac{C_1\lambda g}{P}\right]+\frac{2}{T_1^3}\left[\frac{M}{P}+\frac{h_3\lambda g^2}{2P}+\frac{h_3\lambda g}{\beta P}+\frac{C_T\lambda g}{P}+\frac{hE[x]g}{\beta}-\frac{bg(P-PE[x]-\lambda)}{\beta P}\right]+\\+\left[-\frac{h_3\lambda g}{P}-hg+\frac{h\lambda g}{P}\right]\beta^2e^{-\beta T_1}+\left(\frac{hvg}{P}\right)\beta^2e^{-\beta T_1}\cdot\left(e^{-\beta T_1 s}\right)^{-1}\left[1-2s+s^2\right]+\left(\frac{hvg}{P}\right)\beta^2s^2e^{-\beta T_1 s}+\\+\left[-\frac{M}{P}-\frac{h_3\lambda g^2}{2P}-\frac{h_3\lambda g}{\beta P}-\frac{C_T\lambda g}{P}-\frac{hg}{\beta}+\frac{h\lambda g}{\beta P}\right]\cdot\left(\frac{\beta^2e^{-\beta T_1}}{T_1}+\frac{2\beta e^{-\beta T_1}}{T_1^2}+\frac{2e^{-\beta T_1}}{T_1^3}\right)+\\+\frac{g\left(1-E[x]-\frac{\lambda}{P}\right)}{\beta}(h+b)\cdot\left[\frac{\beta^2s^2e^{-\beta T_1 s}}{T_1}+\frac{2\beta se^{-\beta T_1 s}}{T_1^2}+\frac{2e^{-\beta T_1 s}}{T_1^3}\right]\end{array}\right\} \quad \textbf{(35)}$$

If the second derivative of $E[TCU(T_1)]$ is greater than zero, then $E[TCU(T_1)]$ is convex. From Eq. (35), with further derivations, we obtain the following:

$$\frac{2z_1+2w_1+2w_3e^{-\beta T_1}+2w_4e^{-\beta T_1 s}}{\left\{\begin{array}{l}-T_1^2\beta^2\left[\begin{array}{l}w_2e^{-\beta T_1}+w_5e^{-\beta T_1}\cdot\left(e^{-\beta T_1 s}\right)-2sw_5e^{-\beta T_1}\cdot\left(e^{-\beta T_1 s}\right)^{-1}+\\+s^2w_5\cdot e^{-\beta T_1}\left(e^{-\beta T_1 s}\right)^{-1}+s^2w_5e^{-\beta T_1 s}\end{array}\right]-\\-T_1\beta^2s^2w_4e^{-\beta T_1 s}-2\beta w_3e^{-\beta T_1}-T_1\beta^2w_3e^{-\beta T_1}-2\beta sw_4e^{-\beta T_1 s-1}\end{array}\right\}}>T_1>0 \quad \textbf{(36)}$$

$$\therefore\ \frac{d^2\ E\left[TCU(T_1)\right]}{dT_1^2}>0\quad if\ 0\ <\ T_1<\ \frac{\alpha_2}{\alpha_3+\alpha_4}=y(T_1) \quad \textbf{(37)}$$

Eq. (37) must be satisfied so that $E[TCU(T_1)]$ has a minimum value. Now, searching for the optimal value of T_1* that yields minimum cost, one can set the first derivative of $E[TCU(T_1)]$ equal to 0. From Eq. (34), with further derivations, we obtain:

$$\delta_aT_1^2+\delta_bT_1+\delta_c=0 \quad \textbf{(38)}$$

where δ_a, δ_b, and δ_c denote the following:

$$\delta_a=\left[\begin{array}{l}-h+\frac{E[x]^2P}{P_1}\left[h_1-h\right]-2\beta(w_2)e^{-\beta T_1}+\frac{v^2}{P\lambda}(h+b)+\\+\frac{v^2}{P^2(1-E[x]-\frac{\lambda}{P})}(h+b)+\frac{1}{\lambda}(hP-2hv)-\\-2\beta(w_5)e^{-\beta T_1}\left(e^{-\beta T_1 s}\right)^{-1}+2\beta s(w_5)e^{-\beta T_1}\left(e^{-\beta T_1 s}\right)^{-1}-2\beta s(w_5)e^{-\beta T_1 s}\end{array}\right] \quad \textbf{(39)}$$

$$\delta_b=\left[-2\beta(w_3)e^{-\beta T_1}-2\beta s(w_4)e^{-\beta T_1 s}\right] \quad \textbf{(40)}$$

$$\delta_c=\left[-2z_1-2w_1-2(w_3)e^{-\beta T_1}-2(w_4)e^{-\beta T_1 s}\right] \quad \textbf{(41)}$$

By applying the square root solution of Eq. (38) we obtain the following:

$$T_1^* = \frac{-\delta_b \pm \sqrt{(\delta_b)^2 - 4\delta_a\delta_c}}{2\delta_a} \tag{42}$$

In addition, by rearranging Eq. (34), we obtain:

$$e^{-\beta T_1} = \frac{-2z_1 - 2w_1 - 2w_4(e^{-\beta T_1})^s}{T_1^2\left\{2\beta w_2 + [2\beta w_5(e^{-\beta T_1})^{-1}]^s - [2\beta s w_5(e^{-\beta T_1})^{-1}]^s\right\} + T_1(2\beta w_3) + 2w_3} \tag{43}$$

Although the optimal runtime T_1^* cannot be expressed in a closed form, it can be located through the use of a searching algorithm based on the existence of bounds for $e^{-\beta T_1}$ and T_1^*.

Since $e^{-\beta T_1}$ is the complement of the cumulative density function, $F(T_1) = (1-e^{-\beta T_1})$ and $0 \leqq F(T_1) \leqq 1$. Hence, $0 \leqq e^{-\beta T_1} \leqq 1$. If $e^{-\beta T_1} = 0$ and $e^{-\beta T_1} = 1$, we can find the initial upper bound (i.e. T_{1U}) and the lower bound (i.e., T_{1L}) for the replenishment runtime, and use them to find the updated value of $e^{-\beta T_{1U}}$ and $e^{-\beta T_{1L}}$. Back and forth, we repeatedly compute Eqs. (42) and (43) until there is no significant difference between the upper bound T_{1U} and lower bound T_{1L}. Subsequently, the optimal production runtime T_1^* is derived.

3. NUMERICAL EXAMPLE

Suppose a manufacturing firm can fabricate a product at an annual rate of $P = 10{,}000$ units in order to meet its annual demand rate $\lambda = 4{,}000$ units. A FPR-based model with backlogging and a predetermined minimum acceptable 80% (i.e., $(1 - \alpha)\%$) service level is adopted by the firm. During the production, random nonconforming rate is assumed to be uniformly distributed over the interval $[0, 0.2]$. All nonconforming products can be repaired through a rework process, which starts when regular production ends, at a rate of $P_1 = 5{,}000$ units per year. Moreover, the production equipment is subject to a random breakdown that follows a Poisson distribution with mean $\beta = 0.5$ times per year. An AR policy is used when a breakdown takes place.

Other values of parameters used by this example include: $C = \$2$ per unit; $K = \$450$ per production run; $h = \$0.8$ per item per unit time; $b = \$0.1$ for each backordering item; $C_R = \$0.5$ repaired cost for each reworked item; $h_1 = \$0.8$ per reworked item per unit time; $h_3 = \$0.6$ per unit per unit time and $C_1 = \$2$ per unit of safety stock; $C_T = \$0.01$ per unit; $M = \$500$ repair cost for each machine breakdown; $g = 0.018$ years (i.e. t_r, the fixed machine repair time).

For convexity of $E[TCU(T_1)]$ (Eq. (37)), at $\beta = 0.5$, by verifying both of the upper and lower bounds (from Eqs. (42) and (43)) of T_1^*, we found that $T_{1U}^* = 0.5441 < y(T_{1U}^*) = 2.4355$ and $T_{1L}^* = 0.3396 < y(T_{1L}^*) = 2.1339$ (see Table 1). Therefore, Eq. (37) holds and $E[TCU(T_1)]$ is convex. A further investigation utilizing different β values to test the satisfaction of Eq. (37) is presented in Table 1.

For determining the optimal production runtime T_1^*, we first let $e^{-\beta T_1} = 0$ and $e^{-\beta T_1} = 1$, and by using Eq. (42), we obtain the initial upper bound $T_{1U} = 0.5441$ and lower bound $T_{1L} = 0.3396$. Then, by applying Eq. (27), we obtain $E[TCU(T_{1U})] = \$9{,}688.73$ and $E[TCU(T_{1L})] = \$9{,}625.20$. By substituting the initial values of T_{1U} and T_{1L} in Eq. (43), we obtain $e^{-\beta T_{1U}} = 0.7618$ and $e^{-\beta T_{1L}} = 0.8438$ as the starting exponential values for the second iteration.

By repeatedly applying Eqs. (42) and (27), we get a new set of $T_{1U} = 0.4015$ and $T_{1L} = 0.3809$; and obtain $E[TCU(T_{1U})] = \$9{,}616.15$ and $E[TCU(T_{1L})] = \$9{,}615.27$. It is noted that the difference between $E[TCU(T_{1U})]$ and $E[TCU(T_{1L})]$ in the second iteration becomes smaller. Table 2 displays the results of this recursive searching algorithm for T_1* after a few iterations, at $\beta = 0.5$ and $(1 - \alpha)\% = 80\%$.

Table 1 *The effect of variations in β on $T_{1U}*$, $y(T_{1U}*)$, $T_{1L}*$, and $y(T_{1L}*)$*

β	$1/\beta$	$T_{1U}*$	$y(T_{1U}*)$	$T_{1L}*$	$y(T_{1L}*)$
10.00	0.10	0.5226	11.6176	0.1049	0.3371
9.00	0.11	0.5227	7.9549	0.1143	0.3673
8.00	0.13	0.5229	5.6054	0.1253	0.4031
7.00	0.14	0.5231	4.0639	0.1385	0.4462
6.00	0.17	0.5234	3.0387	0.1545	0.4993
5.00	0.20	0.5237	2.3556	0.1739	0.5663
4.00	0.25	0.5243	1.9107	0.1978	0.6547
3.00	0.33	0.5253	1.6491	0.2276	0.7802
2.00	0.50	0.5272	1.5701	0.2650	0.9860
1.00	1.00	0.5329	1.8345	0.3119	1.4581
0.50	2.00	0.5441	2.4355	0.3396	2.1339
:	:	:	:	:	:
0.01	100.00	1.2159	6.3508	0.3698	5.4807

Table 2 *Results of iterations of the proposed recursive searching algorithm for T_1**

β	Iteration	$e^{-\beta T_{1U}}$	T_{1U}	$e^{-\beta T_{1L}}$	$T_{1L}*$	Difference between T_{1U} & T_{1L}	$E[TCU(T_{1U})]$	$E[TCU(T_{1L})]$	Diff. b/w $E[TCU(T_{1U})]$ & $E[TCU(T_{1L})]$
0.5	initial	0.000	0.5441	1.000	0.3396	0.2045	\$9,688.73	\$9,625.20	\$63.53
	2nd	0.7618	0.4015	0.8438	0.3809	0.0206	\$9,616.15	\$9,615.27	\$0.88
	3rd	0.8181	0.3874	0.8266	0.3853	0.0021	\$9,615.18	\$9,615.17	\$0.01
	4th	0.8239	0.3859	0.8248	0.3857	0.0002	\$9,615.17	\$9,615.17	\$0.00
	5th	0.8245	0.3858	0.8246	0.3858	0.0000	\$9,615.17	\$9,615.17	\$0.00

From Table 2, it is noted that $E[TCU(T_1)]$ is convex and the optimal run time T_1* falls within the interval of $[T_{1L}*, T_{1U}*]$. By using the proposed recursive searching algorithm for T_1*, for $\beta = 0.5$ and $(1 - \alpha)\% = 80\%$, we can locate optimal replenishment runtime $T_1* = 0.3858$ and the expected system costs per unit time $E[TCU(T_1*)] = \$9{,}615.17$. The behaviour of $E[TCU(T_1)]$ with respect to run time T_1 is illustrated in Figure 5.

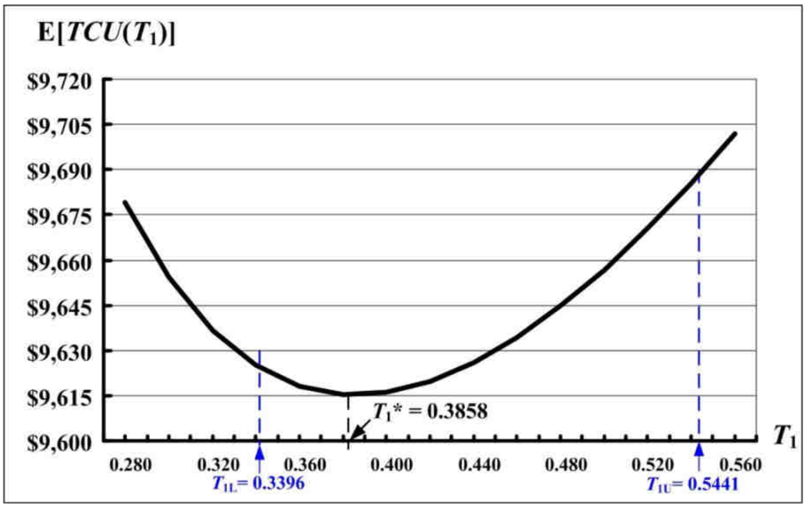

Fig. 5 *The behaviour of E[TCU(T_1)] with respect to run time T_1*

The joint effects of the backlogging level B and the service level $(1 - \alpha)\%$ on the expected system costs $E[TCU(T_1)]$ are depicted in Figure 6. It is noted that as backlogging level B increases, the expected system costs $E[TCU(T_1)]$ notably reduces; and as service level $(1 - \alpha)\%$ raises, the expected $E[TCU(T_1*)]$ significantly boosts.

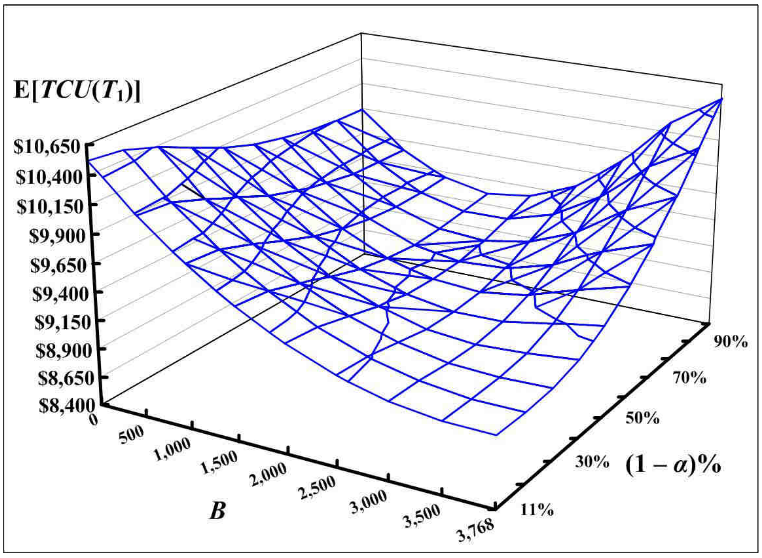

Fig. 6 *The joint effects of the backlogging level B and the service level (1 – α)% on the expected system costs E[TCU(T_1*)]*

Analytical results of the effects of variations in service levels on maximum inventory holding level H, annual expected holding cost, maximum backlogging level B, annual expected backordering cost, the optimal runtime T_1*, and annual expected system costs $E[TCU(T_1*)]$, respectively, are exhibited in Table 3. It is noted that the lowest system cost falls to *11%* of the service level (i.e. *89%* of time the system runs out of stock). If the manufacturing firm decides to keep the service level at greater than or equal to *80%* to cope with customer satisfaction in purchasing, then the analytical result (see Table 3) indicates that there is an increase in the cost of *\$594.67* (i.e. *\$9,615.17 – \$9,020.50*) or *6.56%* of the annual system cost increases for raising the service level from *20%* to *80%*.

Table 3 *Analytical results of the effects of variations in service levels on different system parameters and their related costs*

$(1-\alpha)\%$	H	Annual expected holding cost	B	Annual expected backordering cost	T_1^*	$E[TCU(T_1^*)]$
100%	1640	$746	0	$0	0.3154	$9,889.89
90%	1614	$675	193	$2	0.3475	$9,750.71
80%	**1577**	**$603**	**429**	**$4**	**0.3858**	**$9,615.17**
70%	1525	$530	719	$11	0.4315	$9,484.62
60%	1447	$456	1080	$22	0.4860	$9,361.05
50%	1332	$381	1527	$38	0.5498	$9,247.50
40%	1159	$305	2070	$62	0.6209	$9,148.44
30%	906	$232	2689	$94	0.6914	$9,070.19
20%	563	$167	3311	$133	0.7450	$9,020.50
11%	195	$124	3768	$168	0.7620	$9,005.94

Further analysis reveals the joint effects of the mean-time-to-breakdown $1/\beta$ and service level $(1-\alpha)\%$ on the expected system costs $E[TCU(T_1^*)]$ as depicted in Figure 7. It is noted that as the service level $(1-\alpha)\%$ raises, annual expected system costs $E[TCU(T_1^*)]$ increases; and as the mean-time-to-breakdown $1/\beta$ increases, $E[TCU(T_1^*)]$ decreases. When $1/\beta$ reaches the infinite value, the proposed FPR model becomes the same as the FPR model without machine breakdown [10]; and if the service level $(1-\alpha)\%$ increases to *100%*, the proposed FPR model becomes the same as the FPR model without backlogging [22].

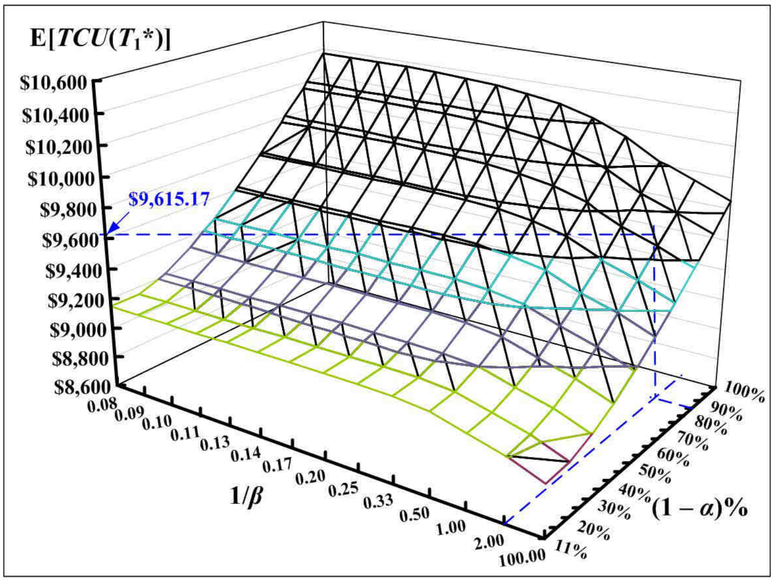

Fig. 7 *The joint effects of the mean-time-to-breakdown 1/β and service level (1 – α)% on the expected system costs E[TCU(T₁*)]*

4. CONCLUSIONS

This study determines the optimal replenishment runtime for the FPR model with allowable backlogging and service level constraint, rework, and random machine breakdown. As a result, we provide a complete solution procedure for such a practical manufacturing system, which includes applying a mathematical model to the problem, deriving the system cost function from integration of cost functions of three separate sub-problems, proposing a way to verify the convexity of system cost function, presenting a recursive runtime searching procedure, and providing a numerical demonstration with sensitivity analysis to confirm the applicability of our research results.

Without an in-depth investigation of the problem, the optimal replenishment runtime, the relationship between backlogging level and the service level, the effect of variations in mean time to breakdown on optimal runtime and expected system cost, and various critical system information, etc. cannot be revealed. For future research, an interesting topic will be to explore the effect of discontinuous inventory issuing policy on the same model.

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6. APPENDIX – A

Results of derivations for $TC_1(T_1)$, $TC_2(T_1)$ and $TC_3(T_1)$.

$$TC_1(T_1) = K + CT_1P + C_R\left[xT_1P\right] + M + \left[h_3\left(\lambda t_r\right)\left(t + \frac{t_r}{2}\right) + C_1\left(\lambda t_r\right)\right] + \\ + h\left[-\frac{T_1PB}{\lambda} - \frac{T_1^2P}{2} + \frac{T_1^2P^2}{2\lambda}\right] + (h+b)\left[\frac{B^2}{2(P-Px-\lambda)} + \frac{B^2}{2\lambda}\right] + \\ + (h_1 - h)\frac{x^2T_1^2P^2}{2P_1} + C_T\left[T_1P + \lambda t_r\right] + bt_r\left(B - Pt + \lambda t\right) + Pxtt_r(h+b) \tag{A-1}$$

$$TC_2(T_1) = K + CT_1P + C_R\left[xT_1P\right] + M + \left[h_3\left(\lambda t_r\right)\left(t + \frac{t_r}{2}\right) + C_1\left(\lambda t_r\right)\right] + \\ + h\left(-\frac{T_1PB}{\lambda} + \frac{T_1^2P^2}{2\lambda} - \frac{T_1^2P}{2}\right) + (h+b)\left[\frac{B^2}{2(P-Px-\lambda)} + \frac{B^2}{2\lambda}\right] + \\ + (h_1 - h)\frac{x^2T_1^2P^2}{2P_1} + C_T\left(T_1P + \lambda t_r\right) + htt_rP - htt_r\lambda - hBt_r \tag{A-2}$$

$$TC_3(T_1) = K + CT_1P + C_R\left[xT_1P\right] + \left[h_3\left(\lambda t_r\right)T + C_1\left(\lambda t_r\right)\right] + (h_1 - h)\frac{x^2T_1^2P^2}{2P_1}Đ + \\ + C_TT_1P + h\left[-\frac{T_1PB}{\lambda} - \frac{T_1^2P}{2} + \frac{T_1^2P^2}{2\lambda}\right] + (h+b)\left[\frac{B^2}{2(P-Px-\lambda)} + \frac{B^2}{2\lambda}\right] \tag{A-3}$$

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