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Incorporating machine reliability issue and backlogging into the EMQ model - Part II: Random breakdown occurring in inventory piling time

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SUMMARY

This paper presents the second part of a research which is concerned with incorporating machine reliability issues and backlogging into the economic manufacturing quantity (EMQ) model. It may be noted that in a production system when back-ordering is permitted, a random machine failure can take place in either backorder filling stage or in on-hand inventory piling time. The first part of the research investigates the effect of a machine failure occurring in backorder filling stage on the optimal lot-size; while this paper (the second part of the research) studies the effect of random breakdown happening in inventory piling time on the optimal batch size for such an imperfect EMQ model. The objective is to determine the optimal replenishment lot-size that minimizes the overall production-inventory costs. Mathematical modelling is used and the renewal reward theorem is employed to cope with the variable cycle length. Hessian matrix equations are utilized to prove convexity of the cost function. Then, the optimal lot size for such a real-life imperfect manufacturing system is derived. Practitioners and managers in the field can adopt these replenishment policies to establish their own robust production plan accordingly.

***Key words**: manufacturing, machine breakdown, EMQ model, rework, production, lot sizing, scrap, random defective rate.*

1. INTRODUCTION

The economic order quantity (EOQ) model was first introduced by Harris [1] to assist corporations in minimizing total purchase costs. As an extension to EOQ model, the economic manufacturing quantity (EMQ) model is often used in the manufacturing sector for determining the optimal production lot-size that minimizes the long-run average production-inventory costs [2-3]. Regardless to their simplicity, EOQ and EMQ models have still been applied nowadays and they remain to be the basis for analyzing the more complex systems [4-7].

Conventional EMQ model implicitly assumes that all manufactured items are of perfect quality. However, in real world production systems, owing to many unpredictable factors, generating nonconforming items seem inevitable. Studies have been carried out to address the imperfect quality issues of the EMQ model [8-13]. The defective items, in some circumstances such as the printed circuit board assembly (PCBA) in PCBA manufacturing, the plastic goods in plastic injection molding process, etc., can be reworked and repaired. Therefore, overall production-inventory costs can be reduced significantly [14-17]. Also, due to excess demands, stock-out situations may arise occasionally. Sometimes, shortages are permitted and they are backordered and satisfied in the very next replenishment. Hence, production inventory costs can be decreased substantially [14, 16].

Random machine breakdown is another common reliability factor that troubles mostly the production practitioners. Therefore, to effectively control and manage the disruption caused by the random breakdown in order to minimize overall production costs becomes the critical task to most production planners. It is not surprising that it has received extensive attention from researchers in the past decades [17-25]. Readers can refer to the Introduction Section of the first part of the research for more detailed literature reviews.

This research incorporates machine reliability issues and backlogging into the EMQ model for the reason that the classic EMQ model does not consider defective items produced during a production cycle, nor it deals with the machine breakdown situation. It may be noted that in a production system when backordering is permitted, a random machine failure can take place in either backorder filling stage or in on-hand inventory piling time. Therefore, the first part of the research investigates the effect of machine failure occurring in backorder filling stage on the optimal lot size; while this paper (the second phase of the research) studies the effect of machine breakdown happening in inventory piling time on the optimal batch size for such an imperfect EMQ model. Since little attention has been paid to the aforementioned area, this research intends to bridge the gap.

2. MATHEMATICAL MODELLING

As described in the first part of the research, let annual demand rate be λ and the production rate be P per year, where P is much larger than λ. All items produced are screened and unit inspection cost is included in the unit manufacturing cost C. Let x be the random defective rate and d be the production rate of defective items, where $d=Px$. Assuming that the production rate of perfect quality items must always be greater than or equal to the sum of the demand rate λ and the defective rate d. Hence, the following condition must hold: $(P-d-\lambda)\geq 0$ or $(1-x-\lambda/P)\geq 0$.

Shortages are allowed and backordered, they will be satisfied when the next replenishment production cycle starts. The imperfect quality items fall into two groups, a θ portion of them is the scrap and the other $(1-\theta)$ portion of the defective items is considered to be reworkable. When regular production ends, the rework process starts immediately at a rate of P_1, in each cycle. A θ_1 portion of the reworked items fails the repairing and becomes scrap. Further, according to the mean time between failures (MTBF) data, a machine breakdown may occur randomly in on-hand inventory piling time (see Figure 1), an abort/resume inventory control policy is adopted in this study. Under such a policy, when a breakdown takes place the machine is under corrective maintenance immediately, a constant repair time is assumed, and the interrupted lot will be resumed right after the restoration of machine.

It is also assumed that during the setup time, prior to the production uptime, the working status of machine is fully checked and confirmed. Hence, the chance of breakdown in a very short period of time when production begins is small. It is also assumed that due to tight preventive maintenance schedule, the probability of more than one machine breakdown occurrences in a production cycle is very small. However, if it does happen, safety stock will be used to satisfy the demand during machine repairing time. Therefore, multiple machine failures are assumed to have insignificant effect on the proposed model. Figure 1 depicts the level of on-hand inventory of perfect quality items in proposed EMQ model.

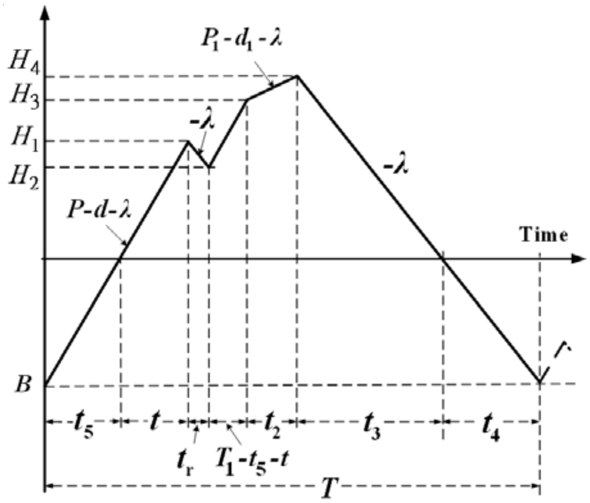

Fig. 1 On-hand inventory of perfect quality items in EMQ model with imperfect rework and breakdown occurring in stock piling time

Cost parameters considered in this study include: the cost for repairing and restoring machine M, unit manufacturing cost C, unit repair cost for each defective item reworked C_R, disposal cost for each scrap item C_S, setup cost K, unit holding cost h, unit shortage/backordered cost b, and unit holding cost per reworked item h_1. Additional variables used are listed as follows:

Q – production lot-size for each cycle, a decision variable to be determined by this study,
T_1 – production run time (i.e. uptime) to be determined by the proposed study,
B – the maximum backorder level allowed for each cycle,
T – the production cycle length,
t – production time before a random breakdown occurs,
t_r – time required for repairing and restoring the machine,
t_5 – time required for filling the backorder quantity B,
t_2 – time needed to rework the defective items,
t_3 – time required for depleting all available perfect quality on-hand items,
t_4 – shortage permitted time,
H_1 – level of on-hand inventory when machine breakdown occurs,
H_2 – level of on-hand inventory when machine is repaired and restored,

H_3 – level of on-hand inventory when regular production process ends,

H_4 – the maximum level of perfect quality inventory when rework finishes,

$TC(T_1,B)$ – total production-inventory costs per cycle,

$TCU(T_1,B)$ – total production-inventory costs per unit time (e.g. annual),

$E[TCU(T_1,B)]$ – the expected total production-inventory costs per unit time.

From Figure 1, one can obtain the following [14, 16]: production run time T_1; time for reworking defective items t_2; time required for depleting all available on-hand items t_3; shortage allowed time t_4, time for refilling B (maximum backorder quantity) t_5, the cycle length T; and the levels of on-hand inventory H_1, H_2, H_3 and H_4:

$$T_1 = \frac{Q}{P} \tag{1}$$

$$t_2 = \frac{d \cdot T_1 (1-\theta)}{P_1} \tag{2}$$

$$t_3 = \frac{H_4}{\lambda} \tag{3}$$

$$t_4 = \frac{B}{\lambda} \tag{4}$$

$$t_5 = \frac{B}{P-d-\lambda} \tag{5}$$

$$T = T_1 + t_2 + t_3 + t_4 + t_r \tag{6}$$

$$H_1 = (P-d-\lambda)t \tag{7}$$

$$H_2 = H_1 - t_r\lambda = H_1 - g\lambda \tag{8}$$

$$H_3 = H_2 + (P-d-\lambda)\cdot(T_1 - t_5 - t) \tag{9}$$

$$H_4 = H_3 + (P_1 - d_1 - \lambda)t_2 \tag{10}$$

where $d=Px$ and let the maximum machine repair time be a constant and $t_r=g$. In this study, it is conservatively assumed that if a failure of a machine cannot be fixed within a certain allowable amount of time, then a spare machine will be in place to avoid further delay of production. The level of on-hand defective items for the proposed EMQ model is shown in Figure 2. Total defective items produced during the production run time T_1 are:

$$d \cdot T_1 = x \cdot Q \tag{11}$$

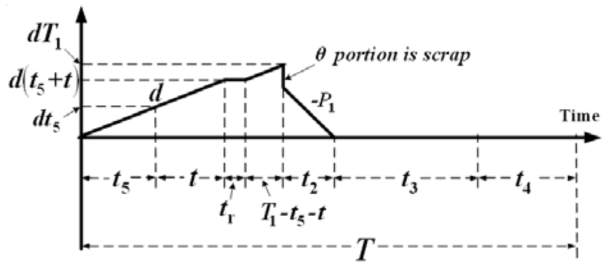

Fig. 2 On-hand inventory of defective items in the proposed EMQ model

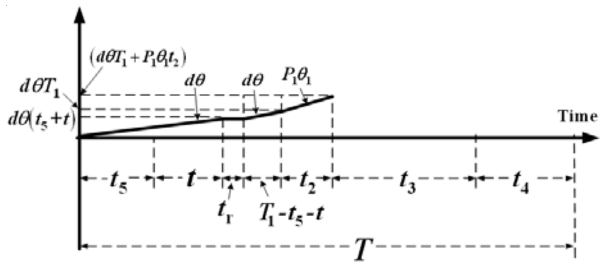

Fig. 3 On-hand inventory of scrap items in the proposed EMQ model

Figure 3 illustrates the on-hand inventory level of scrap items. During the rework process, the production rate of scrap items can be written as in Eq. (12) and the total scrap items produced can be computed by Eq. (13):

$$d_1 = P_1 \cdot \theta_1 \text{ ; where: } 0 \le \theta_1 \le 1 \tag{12}$$

$$d \cdot \theta \cdot T_1 + P_1 \cdot \theta_1 \cdot t_2 = Q \cdot x \cdot \left[\theta + (1-\theta)\cdot\theta_1\right] \tag{13}$$

Let $\varphi = \left[\theta + (1-\theta)\cdot\theta_1\right]$, then φ denotes the total scrap rate among the defective items. From Figures 1 to 3, one obtains the total production-inventory cost per cycle $TC(T_1,B)$ as follows:

$$\begin{aligned} TC(T_1,B) = {} & K + M + C\cdot(PT_1) + C_R\cdot\left[PT_1\cdot x(1-\theta)\right] + C_S\cdot(PT_1\cdot x)\varphi + \\ & + h\left[\frac{H_1(t)}{2} + \frac{H_1+H_2}{2}(t_r) + \frac{H_2+H_3}{2}(T_1-t_5-t) + \frac{H_3+H_4}{2}(t_2) + \frac{H_4(t_3)}{2}\right] + \\ & + h\left[\frac{d(t_5+t)}{2}(t_5+t) + (t_5+t)t_r + \frac{(t_5+t)+dT_1}{2}(T_1-t_5-t)\right] + h_1\left[\frac{P_1 t_2}{2}(t_2)\right] + b\left[\frac{B}{2}(t_5) + \frac{B}{2}(t_4)\right] \end{aligned} \tag{14}$$

Substituting all related parameters from Eqs. (1) to (13) in Eq. (14), $TC(T_1,B)$ becomes:

$$\begin{aligned} TC(T_1,B) &= C\cdot P\cdot T_1+K+M+C_R\cdot T_1\cdot P\cdot x\cdot(1-\theta)+C_S\cdot T_1\cdot P\cdot x\cdot\varphi+ \\ &+\frac{h}{2}\left\{\frac{P^2}{\lambda}\left(1-2x\varphi+x^2\varphi^2\right)T_1^2+\frac{(1-x)}{\lambda(1-x-\lambda/P)}B^2-PT_1^2+2Px\varphi T_1^2\right\}+ \\ &+\frac{b(1-x)}{2\lambda(1-x-\lambda/P)}B^2+\frac{P^2x^2(1-\theta)^2}{2P_1}\left[h_1-h(1-\theta_1)\right]T_1^2-h\frac{P}{\lambda}T_1B+ \\ &+h\frac{Px\varphi}{\lambda}T_1B+\frac{hg}{(1-x-\lambda/P)}B-hPgT_1+hPgx\varphi T_1+hPgt+\frac{hg^2\lambda}{(1-x-\lambda/P)} \end{aligned} \tag{15}$$

Due to random defective/scrap rates, the production cycle length is not constant. Therefore, to take the randomness of scrap rate into account, one can employ the renewal reward theorem in the production-inventory cost analysis to cope with the variable cycle length. Also, because a Poisson machine breakdown (with mean equals to β per unit time) is assumed to occur in inventory piling time, one can use the integration of $TC(T_1,B)$ to deal with random breakdown. The long-run expected costs per unit time $E[TCU(T_1,B)]$ can be calculated as follows:

$$E\left[TCU(T_1,B)\right]=\frac{E\left[\int_0^{T_1-t_5}TC(T_1,B)\cdot f(t)\,dt\right]}{E[T]}=\frac{E\left[\int_0^{T_1-t_5}TC(T_1,B)\cdot\left(\beta e^{-\beta t}\right)dt\right]}{\left[T_1P\left(1-\varphi E[x]\right)/\lambda\right]\cdot\left(1-e^{-\beta(T_1-t_5)}\right)} \tag{16}$$

Substituting all related parameters from Eqs. (1) to (15) in the numerator of Eq. (16), one has:

$$\begin{aligned} &E\left[\int_0^{T_1-t_5}TC(T_1,B)\,f(t)\,dt\right] \\ &=E\left[\begin{array}{l} (1-e^{-\beta(T_1-t_5)})\left\{\begin{array}{l} K+M+P\cdot T_1\cdot\left[C+C_R\cdot x\cdot(1-\theta)+C_S\cdot x\cdot\varphi\right] \\ +\frac{h}{2}\left[\frac{P^2}{\lambda}\left(1-2x\varphi+x^2\varphi^2\right)T_1^2+\frac{(1-x)}{\lambda(1-x-\lambda/P)}B^2-PT_1^2+2Px\varphi T_1^2\right] \\ +\frac{b(1-x)}{2\lambda(1-x-\lambda/P)}\cdot B^2+\frac{P^2x^2(1-\theta)^2}{2P_1}\left[h_1-h(1-\theta_1)\right]T_1^2-h\cdot\frac{P}{\lambda}T_1B+h\cdot\frac{Px\varphi}{\lambda}T_1B \\ +\frac{hg}{(1-x-\lambda/P)}B-hPgT_1+hPgx\varphi T_1+hPg\left(T_1-\frac{(g\lambda+B)}{P(1-x-\lambda/P)}+\frac{1}{\beta}\right)+\frac{hg^2\lambda}{(1-x-\lambda/P)} \end{array}\right\} \\ -hPg\left(T_1-\frac{(g\lambda+B)}{P(1-x-\lambda/P)}\right) \end{array}\right] \end{aligned} \tag{17}$$

With further derivations, one obtains the numerator of Eq. (16) as follows:

$$\begin{aligned} &E\left[\int_0^{T_1-t_5}TC(T_1,B)\,f(t)\,dt\right]=-hPgT_1+h\left(g^2\lambda+gB\right)E\left[\frac{1}{1-x-\lambda/P}\right]+ \\ &+\left[1-e^{-\beta(T_1-t_5)}\right]\cdot\left\{\begin{array}{l} K+M+P\cdot T_1\cdot\left[C+C_R\cdot E[x]\cdot(1-\theta)+C_S\cdot E[x]\cdot\varphi\right]+hPg\varphi T_1E[x]+hPg/\beta+ \\ +\frac{h}{2}\left[\frac{P^2}{\lambda}\left(1-2\varphi E[x]+\varphi^2\left(E[x]\right)^2\right)T_1^2-PT_1^2+2P\varphi T_1^2E[x]\right]-\frac{hPT_1B}{\lambda}\left(1-\varphi E[x]\right)+ \\ +\frac{B^2}{2\lambda}(b+h)E\left[\frac{1-x}{1-x-\lambda/P}\right]+\frac{P^2(1-\theta)^2T_1^2\left(E[x]\right)^2}{2P_1}\left[h_1-h(1-\theta_1)\right] \end{array}\right\} \end{aligned} \tag{18}$$

Substituting Eqs. (18) in Eq. (16) one has $E[TCU(T_1,B)]$ as follows:

$$E\left[TCU\left(T_1,B\right)\right]=\frac{h\lambda\left(g^2\lambda+gB\right)}{T_1P(1-e^{-\beta\left(T_1-t_5\right)})\left(1-\varphi E\left[x\right]\right)}E\left[\frac{1}{1-x-\lambda/P}\right]-\frac{hg\lambda}{(1-e^{-\beta\left(T_1-t_5\right)})\left(1-\varphi E\left[x\right]\right)}+$$
$$+\left\{\begin{array}{l}\frac{\lambda\left(K+M\right)}{T_1P\left(1-\varphi E\left[x\right]\right)}+\frac{\lambda\cdot\left[C+C_R\cdot E\left[x\right]\cdot(1-\theta)+C_S\cdot E\left[x\right]\cdot\varphi\right]}{\left(1-\varphi E\left[x\right]\right)}+\frac{hg\lambda}{T_1\beta\left(1-\varphi E\left[x\right]\right)}+\\ +\frac{h}{2}\left[PT_1\left(1-\varphi E\left[x\right]\right)-\frac{T_1\lambda}{1-\varphi E\left[x\right]}+2\varphi T_1\lambda\frac{E\left[x\right]}{1-\varphi E\left[x\right]}\right]-hB+hg\varphi\lambda\frac{E\left[x\right]}{1-\varphi E\left[x\right]}+\\ +\frac{B^2}{2PT_1\left(1-\varphi E\left[x\right]\right)}\left(b+h\right)E\left[\frac{1-x}{1-x-\lambda/P}\right]+\frac{PT_1\lambda\left(1-\theta\right)^2\left(E\left[x\right]\right)^2}{2P_1\left(1-\varphi E\left[x\right]\right)}\left[h_1-h\left(1-\theta_1\right)\right]\end{array}\right\} \tag{19}$$

$$\text{Let: } t_1'=\left(T_1-t_5\right);\quad E_0=\frac{1}{1-\varphi E\left[x\right]};\quad E_1=\frac{E\left[x\right]}{1-\varphi E\left[x\right]};\quad E_2=\frac{\left(E\left[x\right]\right)^2}{1-\varphi E\left[x\right]};$$
$$E_3=\frac{1}{1-\varphi E\left[x\right]}E\left[\frac{1-x}{1-x-\lambda/P}\right];\quad E_4=\frac{1}{1-\varphi E\left[x\right]}E\left[\frac{1}{1-x-\lambda/P}\right];\quad E_5=1-\varphi E\left[x\right] \tag{20}$$

Equation (19) now becomes:

$$E\left[TCU\left(T_1,B\right)\right]=\frac{\lambda\left(K+M\right)}{T_1P}E_0+\lambda\cdot\left[C\cdot E_0+C_R\cdot(1-\theta)\cdot E_1+C_S\cdot\varphi\cdot E_1\right]-hB+hg\varphi\lambda\cdot E_1+$$
$$+\frac{hT_1}{2}\left(P\cdot E_5-\lambda\cdot E_0+2\varphi\lambda\cdot E_1\right)+\frac{B^2}{2PT_1}\left(b+h\right)E_3-\frac{hg\lambda}{(1-e^{-\beta t_1'})}E_0+$$
$$+\frac{PT_1\lambda\left(1-\theta\right)^2}{2P_1}\left[h_1-h\left(1-\theta_1\right)\right]\cdot E_2+\frac{hg\lambda}{T_1\beta}\cdot E_0+\frac{h\lambda\left(g^2\lambda+gB\right)}{T_1P(1-e^{-\beta t_1'})}E_4 \tag{21}$$

3. CONVEXITY AND THE OPTIMAL SOLUTIONS

In order to find the optimal production lot-size, one should first prove the convexity of $E[TCU(T_1,B)]$. Hessian matrix equations [26] can be employed for proof as follows:

$$\left[T_1\quad B\right]\cdot\begin{pmatrix}\frac{\partial^2E\left[TCU\left(T_1,B\right)\right]}{\partial T_1^2} & \frac{\partial^2E\left[TCU\left(T_1,B\right)\right]}{\partial T_1\partial B}\\ \frac{\partial^2E\left[TCU\left(T_1,B\right)\right]}{\partial T_1\partial B} & \frac{\partial^2E\left[TCU\left(T_1,B\right)\right]}{\partial B^2}\end{pmatrix}\cdot\begin{bmatrix}T_1\\ B\end{bmatrix}>0 \tag{22}$$

$E[TCU(T_1,B)]$ is strictly convex only if Eq. (22) is satisfied, for all T_1 and B different from zero. By further derivation one obtains (see Appendix A for detailed computations):

$$\left[T_1\quad B\right]\cdot\begin{pmatrix}\frac{\partial^2E\left[TCU\left(T_1,B\right)\right]}{\partial T_1^2} & \frac{\partial^2E\left[TCU\left(T_1,B\right)\right]}{\partial T_1\partial B}\\ \frac{\partial^2E\left[TCU\left(T_1,B\right)\right]}{\partial T_1\partial B} & \frac{\partial^2E\left[TCU\left(T_1,B\right)\right]}{\partial B^2}\end{pmatrix}\cdot\begin{bmatrix}T_1\\ B\end{bmatrix}=\frac{2\left(K+M\right)\lambda}{T_1P}E_0+\frac{2hg\lambda}{T_1\beta}E_0+\frac{2hg^2\lambda^2E_4}{T_1P(1-e^{-\beta t_1'})}>0 \tag{23}$$

The Eq. (23) is resulting positive because all parameters are positive. Hence, $E[TCU(T_1,B)]$ is a strictly convex function. It follows that for the optimal uptime T_1 and maximal backorder level B, one can differentiate $E[TCU(T_1,B)]$ with respect to T_1 and with respect to B, and solve linear systems of Eqs. (24) and (25) by setting these partial derivatives equal to zero:

$$\frac{\partial E\left[TCU\left(T_1,B\right)\right]}{\partial T_1}=\left\{\begin{array}{l} -\dfrac{\lambda\left(K+M\right)}{T_1{}^2P}E_0+\dfrac{h}{2}\left(P\cdot E_5-\lambda\cdot E_0+2\varphi\lambda\cdot E_1\right)-\dfrac{B^2}{2PT_1{}^2}\left(b+h\right)E_3+ \\ +\dfrac{P\lambda\left(1-\theta\right)^2}{2P_1}\left[h_1-h\left(1-\theta_1\right)\right]\cdot E_2-\dfrac{hg\lambda}{T_1{}^2\beta}\cdot E_0-\dfrac{h\lambda\left(g^2\lambda+gB\right)}{T_1{}^2P(1-e^{-\beta t_1'})}E_4 \end{array}\right\} \tag{24}$$

$$\frac{\partial E\left[TCU\left(T_1,B\right)\right]}{\partial B}=\left\{\frac{B}{T_1P}\left(b+h\right)E_3-h+\frac{hg\lambda}{T_1P(1-e^{-\beta t_1'})}E_4\right\} \tag{25}$$

From Eq. (25) one has:

$$\therefore\ B^*=\left(\frac{h}{b+h}\right)\left(\frac{1}{E_3}\right)\cdot\left[PT_1-\frac{g\lambda E_4}{(1-e^{-\beta t_1'})}\right] \tag{26}$$

With further derivations, Eq. (24) becomes:

$$\begin{aligned}&\frac{1}{T_1{}^2}\left[\frac{\lambda\left(K+M\right)}{P}E_0+\frac{B^2}{2P}\left(b+h\right)E_3+\frac{hg\lambda}{\beta}\cdot E_0+\frac{h\lambda\left(g^2\lambda+gB\right)}{P(1-e^{-\beta t_1'})}E_4\right]=\\&=\frac{h}{2}\left(P\cdot E_5-\lambda\cdot E_0+2\varphi\lambda\cdot E_1\right)+\frac{P\lambda\left(1-\theta\right)^2}{2P_1}\left[h_1-h\left(1-\theta_1\right)\right]\cdot E_2\end{aligned} \tag{27}$$

Substitute Eq. (26) in Eq. (27) one has:

$$\begin{aligned}&\frac{1}{T_1{}^2}\left[\begin{array}{l}\dfrac{\lambda\left(K+M\right)}{P}E_0+\dfrac{1}{2P}\left(b+h\right)E_3\left[\dfrac{hPT_1}{\left(b+h\right)E_3}-\dfrac{hg\lambda E_4}{\left(b+h\right)E_3(1-e^{-\beta t_1'})}\right]^2+\dfrac{hg\lambda}{\beta}\cdot E_0+\\+\dfrac{hg^2\lambda^2}{P(1-e^{-\beta t_1'})}E_4+\dfrac{hg\lambda}{P(1-e^{-\beta t_1'})}E_4\left[\dfrac{hPT_1}{\left(b+h\right)E_3}-\dfrac{hg\lambda E_4}{\left(b+h\right)E_3(1-e^{-\beta t_1'})}\right]\end{array}\right]=\\&=\frac{h}{2}\left(P\cdot E_5-\lambda\cdot E_0+2\varphi\lambda\cdot E_1\right)+\frac{P\lambda\left(1-\theta\right)^2}{2P_1}\left[h_1-h\left(1-\theta_1\right)\right]\cdot E_2\end{aligned} \tag{28}$$

or:

$$T_1{}^2=\frac{\dfrac{1}{P}\left[2\lambda\left(K+M\right)E_0-\dfrac{h^2g^2\lambda^2E_4{}^2}{\left(b+h\right)E_3(1-e^{-\beta t_1'})^2}+\dfrac{2hg^2\lambda^2}{(1-e^{-\beta t_1'})}E_4+\dfrac{2Phg\lambda}{\beta}\cdot E_0\right]}{P\left\{h\left(E_5-\dfrac{\lambda}{P}\cdot E_0+\dfrac{2\varphi\lambda}{P}\cdot E_1\right)+\dfrac{\lambda\left(1-\theta\right)^2}{P_1}\left[h_1-h\left(1-\theta_1\right)\right]\cdot E_2-\dfrac{h^2}{\left(b+h\right)E_3}\right\}} \tag{29}$$

Finally the optimal run time is obtained as follows:

$$T_1^*=\frac{1}{P}\sqrt{\frac{2\lambda\left(K+M\right)E_0-\dfrac{h^2g^2\lambda^2E_4{}^2}{\left(b+h\right)E_3(1-e^{-\beta t_1'})^2}+\dfrac{2hg^2\lambda^2}{(1-e^{-\beta t_1'})}E_4+\dfrac{2Phg\lambda}{\beta}\cdot E_0}{h\left(E_5-\dfrac{\lambda}{P}\cdot E_0+\dfrac{2\varphi\lambda}{P}\cdot E_1\right)+\dfrac{\lambda\left(1-\theta\right)^2}{P_1}\left[h_1-h\left(1-\theta_1\right)\right]\cdot E_2-\dfrac{h^2}{\left(b+h\right)E_3}}} \tag{30}$$

Substitute Eq. (20) in Eq. (30) and let:

$$\pi_1 = E\left[1/\left(1-x-\lambda/P\right)\right] \text{ and } \pi_2 = \left(b+h\right)\cdot E\left[\left(1-x\right)/\left(1-x-\lambda/P\right)\right] \tag{31}$$

The optimal solutions in terms of production run time and lot-size are obtained as follows:

$$T_1^* = \frac{1}{P}\sqrt{\frac{2\lambda\left(K+M\right)-\left(h^2g^2\lambda^2\pi_1^2\right)/\left[(1-e^{-\beta t_1'})^2\pi_2\right]+\left(2hg^2\lambda^2\pi_1\right)/(1-e^{-\beta t_1'})+\left(2Phg\lambda\right)/\beta}{h\left[\left(1-\varphi E[x]\right)^2-\frac{\lambda}{P}\left(1-2\varphi E[x]\right)\right]+\frac{\lambda(1-\theta)^2}{P_1}\left[h_1-h\left(1-\theta_1\right)\right]\cdot\left(E[x]\right)^2-\frac{h^2\left(1-\varphi E[x]\right)^2}{\pi_2}}} \tag{32}$$

$$Q^* = \sqrt{\frac{2\lambda\left(K+M\right)-\left(h^2g^2\lambda^2\pi_1^2\right)/\left[(1-e^{-\beta t_1'})^2\pi_2\right]+\left(2hg^2\lambda^2\pi_1\right)/(1-e^{-\beta t_1'})+\left(2Phg\lambda\right)/\beta}{h\left[\left(1-\varphi E[x]\right)^2-\frac{\lambda}{P}\left(1-2\varphi E[x]\right)\right]+\frac{\lambda(1-\theta)^2}{P_1}\left[h_1-h\left(1-\theta_1\right)\right]\cdot\left(E[x]\right)^2-\frac{h^2\left(1-\varphi E[x]\right)^2}{\pi_2}}} \tag{33}$$

3.1 Solutions verification

Suppose the machine breakdown factor is not considered, then the machine repair cost and time are both zero, i.e. $M=0$ and $g=0$; Eqs. (33) and (26) become the same equations as were given by Chiu [14]:

$$\therefore Q^* = \sqrt{\frac{2K\lambda}{h\left(1-\frac{\lambda}{P}\right)+\left\{\frac{\lambda(1-\theta)^2}{P_1}\left[h_1-h\left(1-\theta_1\right)\right]+h\varphi^2\right\}E\left[x^2\right]-2h\varphi\left(1-\frac{\lambda}{P}\right)\cdot E[x]-\frac{h^2\left[1-E[x]\varphi\right]^2}{\left(b+h\right)\cdot E\left[\frac{1-x}{1-x-\lambda/P}\right]}}} \tag{34}$$

$$B^* = \left(\frac{h}{b+h}\right)\left(\frac{1-E[x]\varphi}{E\left[\frac{1-x}{1-x-\lambda/P}\right]}\right)Q^* \tag{35}$$

Further, suppose that the defective rate is not considered, i.e. $x=0$, then Eqs. (34) and (35) become the same equations as they were presented by classic EMQ model with shortages permitted and backordered [27]:

$$Q^* = \sqrt{\frac{2K\lambda}{h\left(1-\frac{\lambda}{P}\right)}}\cdot\sqrt{\frac{b+h}{b}} \tag{36}$$

$$B^* = \left[\frac{h}{(b+h)}\left(1-\frac{\lambda}{P}\right)\right]\cdot Q^* \tag{37}$$

4. EXAMPLE AND DISCUSSION

Consider the production rate of a manufactured product is *11500* units per year and its annual demand is *4600* units. Manufacturing equipment is subject to a random breakdown that follows a Poisson distribution with mean $\beta=2$ times per year and according to the MTBF data from the maintenance department, a breakdown is expected to occur in inventory piling time. Abort/resume (AR) policy is used when a random breakdown takes place. The percentage x of defective items produced follows a uniform distribution over the interval $[0, 0.2]$. Among all nonconforming items, a $\theta=0.15$ portion is the scrap and the other $(1-\theta)$ portion of the defective items is reworked at a rate of $P_1= 600$ units per year. During the reworking time, a $\theta_1=0.15$ portion of the reworked items fails and becomes scrap. Therefore, after the rework process, overall scrap rate $\varphi=[\theta+(1-\theta)\theta_1]=0.2775$. Additional values of related variables are summarized as follows:

K = *\$450* for each production run,
M = *\$500* repair cost for each breakdown,
C = *\$2* per item,
C_R = *\$0.5* for each item reworked,
C_S = *\$0.3* disposal cost for each scrap item,
h = *\$0.6* per item per unit time,
h_1 = *\$0.8* per item per unit time,
b = *\$0.2* per item backordered per unit time,
g = *0.018* years, time needed to repair and restore the machine.

Applying Eqs. (32), (33) and (26), one obtains the optimal production run time T_1*= *0.7621* (years), the optimal lot-size Q*= *8,753*, optimal backorder level B*=*3,433*, and optimal $E[TCU(T1^*,B^*)]$= *\$10,681.52*. Figure 4 demonstrates convexity of the long-run average cost function $E[TCU(Q^*,B^*)]$.

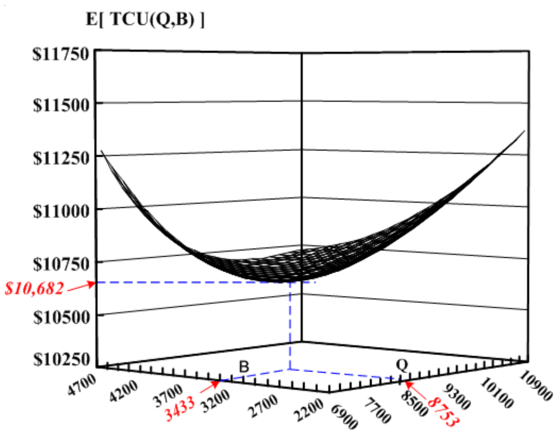

Fig. 4 Convexity of the long-run cost function E[TCU(Q,B)]

Variation of the expected values of defective rate $E[x]$ and overall scrap rate φ effects on the optimal lot-size Q^* is depicted in Figure 5. It is noted that as φ increase, the values of Q^* increase; and for different $E[x]$ values, as $E[x]$ increases the value of the optimal production lot-size Q^* decreases significantly.

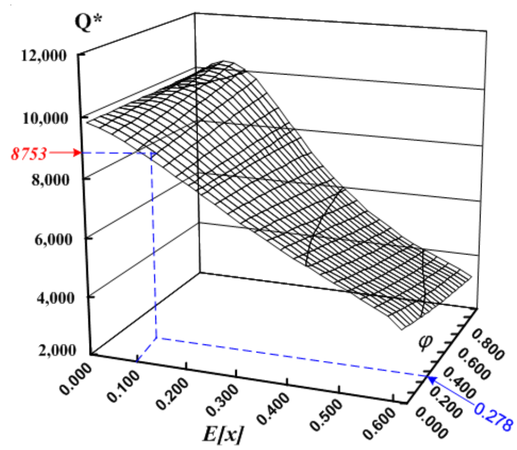

*Fig. 5 Variation of the expected values of defective rate E[x] and overall scrap rate φ effects on the optimal lot-size Q**

Figure 6 depicts variation of the expected values of defective rate $E[x]$ effects on the long-run average cost function $E[TCU(Q^*,B^*)]$. It may be noted that as $E[x]$ increases, the value of the long-run average cost function $E[TCU(Q^*,B^*)]$ increases significantly.

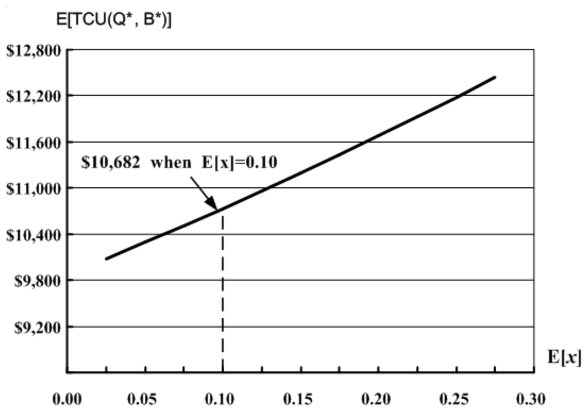

Fig. 6 Variation of the expected values of defective rate E[x] effects on the long-run average cost function E[TCU(Q,B*)]*

As discussed in the first part of the research, suppose that a random breakdown is expected to take place and the result of the present study is not available, one can probably use the most related lot-size solutions given by Chiu [14] to solve such an unreliable EMQ model and obtain $Q=5{,}606$ (or $T_1=0.4875$) and $B= 2{,}257$. Then by plugging T_1 and B in Eq. (21) one obtains $E[TCU(Q,B)]=\$10{,}774.14$. Excluding *\$9,703.60* (i.e. total manufacturing, rework, and scrap costs) and the expected annual machine repair cost *\$421.97* from $E[TCU(Q,B)]$, our optimal cost $E[TCU(Q^*,B^*)]$ is *16.6%* less (on total setup and holding costs) in comparison with the corresponding costs computed without using the exact model as presented in this paper.

5. CONCLUSION

This study incorporates machine reliability issues and backlogging into the EMQ model. It is divided into two parts. The first part investigates the effect of machine failure occurring in backorder filling stage on the optimal lot size; while the second part (this paper) studies the effect of machine breakdown happening in inventory piling time on the optimal batch size for such an imperfect EMQ model.

Classic EMQ model does not consider defective items produced during a production cycle, nor it deals with the machine breakdown situation. However, in the real-life manufacturing environments, generation of the defective items and random machine breakdown are inevitable. Without an in-depth investigation and robust analysis of such a realistic system, the optimal lot size solution which minimizes total production-inventory costs cannot be obtained. Since little attention has been paid to the aforementioned area, this research intends to bridge the gap.

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APPENDIX A

Proof of convexity of $E[TCU(T_1,B)]$.

From Eq. (21), applying the Hessian matrix Eqs. [26], one has:

$$\frac{\partial^2 E\left[TCU\left(T_1,B\right)\right]}{\partial T_1{}^2}=\frac{2\lambda(K+M)}{T_1{}^3 P}E_0+\frac{B^2}{T_1{}^3 P}\left(b+h\right)\cdot E_3+\frac{2hg\lambda}{T_1{}^3\beta}\cdot E_0+\frac{2h\lambda\left(g^2\lambda+gB\right)}{T_1{}^3 P(1-e^{-\beta t_1'})}E_4 \tag{A.1}$$

$$\frac{\partial^2 E\left[TCU\left(T_1,B\right)\right]}{\partial T_1 \partial B} = -\frac{B}{T_1^{\,2} P}\left(b+h\right)\cdot E_3 - \frac{hg\lambda}{T_1^{\,2} P(1-e^{-\beta t_1^{'}})}\cdot E_4 \tag{A.2}$$

$$\frac{\partial^2 E\left[TCU\left(T_1,B\right)\right]}{\partial B^2} = \frac{1}{T_1 P}\left(b+h\right)\cdot E_3 \tag{A.3}$$

then:

$$\begin{bmatrix} T_1 & B \end{bmatrix}\cdot\begin{pmatrix} \dfrac{\partial^2 E\left[TCU_A\left(T_1,B\right)\right]}{\partial T_1^{\,2}} & \dfrac{\partial^2 E\left[TCU_A\left(T_1,B\right)\right]}{\partial T_1 \partial B} \\ \dfrac{\partial^2 E\left[TCU_A\left(T_1,B\right)\right]}{\partial T_1 \partial B} & \dfrac{\partial^2 E\left[TCU_A\left(T_1,B\right)\right]}{\partial B^2} \end{pmatrix}\cdot\begin{bmatrix} T_1 \\ B \end{bmatrix} =$$

$$= \left\{ \begin{array}{l} \dfrac{2\left(K+M\right)\lambda}{T_1 P} E_0 + \dfrac{B^2}{T_1 P}\left(b+h\right)E_3 + \dfrac{2hg\lambda}{T_1 \beta} E_0 + \dfrac{2hg^2\lambda^2 E_4}{T_1 P(1-e^{-\beta t_1^{'}})} + \dfrac{2hg\lambda B E_4}{T_1 P(1-e^{-\beta t_1^{'}})} \\ -\dfrac{2B^2}{T_1 P}\left(b+h\right)E_3 - \dfrac{2Bhg\lambda E_4}{T_1 P(1-e^{-\beta t_1^{'}})} + \dfrac{B^2}{T_1 P}\left(b+h\right)E_3 \end{array} \right\} =$$

$$= \frac{2\left(K+M\right)\lambda}{T_1 P} E_0 + \frac{2hg\lambda}{T_1 \beta} E_0 + \frac{2hg^2\lambda^2 E_4}{T_1 P(1-e^{-\beta t_1^{'}})} \tag{A.4}$$

One obtains Eq. (23) as follows:

$$\begin{bmatrix} T_1 & B \end{bmatrix}\cdot\begin{pmatrix} \dfrac{\partial^2 E\left[TCU\left(T_1,B\right)\right]}{\partial T_1^{\,2}} & \dfrac{\partial^2 E\left[TCU\left(T_1,B\right)\right]}{\partial T_1 \partial B} \\ \dfrac{\partial^2 E\left[TCU\left(T_1,B\right)\right]}{\partial T_1 \partial B} & \dfrac{\partial^2 E\left[TCU\left(T_1,B\right)\right]}{\partial B^2} \end{pmatrix}\cdot\begin{bmatrix} T_1 \\ B \end{bmatrix} =$$

$$= \frac{2\left(K+M\right)\lambda}{T_1 P} E_0 + \frac{2hg\lambda}{T_1 \beta} E_0 + \frac{2hg^2\lambda^2 E_4}{T_1 P(1-e^{-\beta t_1^{'}})} > 0 \tag{23}$$

UKLJUČIVANJE PROBLEMA POUZDANOSTI OPREME I IZOSTALIH POTRAŽIVANJA U EMQ MODEL - II. DIO: SLUČAJNI PREKID U VREMENSKOM INTERVALU GOMILANJA INVENTARA

SAŽETAK

Ovaj rad predstavlja drugi dio istraživanja koji se bavi uključivanjem problema pouzdanosti opreme i izostalih potraživanja u modelu ekonomične količine proizvodnje (EMQ). Uočeno je da se u proizvodnom sustavu, kada je dozvoljeno popunjavanje zaostalih narudžbi, slučajni kvar opreme može pojaviti u tom vremenskom intervalu, ali i tijekom gomilanja inventara u skladištu. Prvi dio istraživanja ispituje utjecaj kvara opreme u vremenskom intervalu popunjavanja zaostalih narudžbi obzirom na optimalnu količinu robe, dok ovaj rad (drugi dio istraživanja) ispituje djelovanje slučajnog kvara opreme u vremenskom intervalu gomilanja inventara na optimalnu sveukupnu količinu za jedan takav nesavršeni EMQ model. Svrha istraživanja je odrediti optimalno popunjavanje zaostalih narudžbi što će smanjiti troškove proizvodnje i skladištenja. Da bi se savladala promjenjiva duljina ciklusa korišteni su matematičko modeliranje i "renewal reward" teorem. Za dokazivanje konveksnosti funkcije troška korištene su Hesseove matrice. Dobiveno je optimalno određivanje količine robe za stvarni nesavršeni proizvodni sustav. Stručnjaci iz tog područja mogu usvojiti ovu metodu nadopunjavanja kako bi prema tome formirali svoj vlastiti plan proizvodnje.

Ključne riječi: *proizvodnja, kvar opreme, EMQ model, ponovni rad, određivanje količine robe, otpadak, prirast slučajnih oštećenja.*