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HARMONIC MT-PREINVEX FUNCTIONS AND INTEGRAL INEQUALITIES

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Abstract. In this paper, we introduce a new class of harmonic preinvex functions, which is called harmonic MT-preinvex functions. Some new Hermite-Hadamard type inequalities for harmonic MT-preinvex functions are derived. Some special cases are also discussed. Results proved in this paper represent refinements and improvements of the known results.

1. Introduction

Convexity theory has played a fundamental role and has received special attention by many researchers in the development of various fields of pure and applied sciences. This theory provides us a natural, unified and general framework to study a wide class of unrelated problems. Due to its importance, the concepts of convex sets and convex functions have been extended in different directions using novel and innovative techniques, see [1, 9, 14-16, 19, 20].

Hanson [8] introduced a new class of generalized convex functions called invex functions, with the aim to extend the validity of the sufficiency of the Kuhn-Tucker conditions in nonlinear programming. Weir and Mond [26] introduced the preinvex functions, which is a significant generalization of convex functions and inspired many researchers to tackle complicated problems. It is known that that the differentiable preinvex functions are invex functions and the converse is also true under certain conditions. Every convex function is a preinvex function but the converse is not true, see [26]. Noor [13] has shown that a function f is a preinvex function, if and only if, it satisfies the Hermite-Hadamard type inequality. Tunc and Yildirim [25], introduced the class of MT-convex functions and established some Hermite-Hadamard inequalities for MT-convex functions. Anderson et al. [1] and Iscan [9] introduced and

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studied the harmonic convex functions, which is another important generalization of the convex functions. It is clear that preinvex functions, MT-convex and harmonic convex functions are different generalizations of the convex functions. It is natural to unify these classes of convex functions. Noor et. al. [17, 18] introduced the class of MT-harmonic convex functions and harmonic preinvex functions and obtained some Hermite-Hadamard inequalities.

Inspired and motivated by the ongoing research in this field, we introduce a new class of convex functions, which is called harmonic MT-preinvex function. It is shown that harmonic preinvex functions and MT-harmonic convex functions are special cases of this harmonic MT-preinvex functions. We also obtain some Hermite-Hadamard type inequalities. Some special cases are also discussed. The ideas and techniques of this paper may motivate further research.

2. Preliminaries

Let $I_\eta = [a, a + \eta(b, a)]$ be a nonempty closed set in $\mathbb{R}^n \setminus \{0\}$. Let $f : I_\eta \subseteq \mathbb{R} \setminus \{0\} \to \mathbb{R}$ be a continuous function and $\eta(\cdot, \cdot) : I_\eta \times I_\eta \to \mathbb{R}$ be a continuous bifunction. First of all, we recall the following well known concepts.

Definition 2.1. *[8] A set $I_\eta \subseteq \mathbb{R}$ is said to be invex set with respect to the bifunction $\eta(\cdot, \cdot)$ if*

$$x + t\eta(y, x) \in I_\eta, \quad \forall x, y \in I_\eta, t \in [0, 1].$$

The invex set I_η is also called η-connected set. If $\eta(y, x) = y - x$, then invex set reduces to the convex set. It is known that every convex set is an invex set, but the converse is not true.

Definition 2.2. *[26] A function $f : I_\eta \subseteq \mathbb{R} \to \mathbb{R}$ is said to be preinvex function with respect to the bifunction $\eta(\cdot, \cdot)$ if*

$$f\big(x + t\eta(y, x)\big) \leq (1 - t)f(x) + tf(y), \qquad \forall x, y \in I_\eta, t \in [0, 1].$$

Noor [12] proved that the minimum of a differentiable preinvex function on the invex set can be characterized by a class of variational inequalities, which is called variational-like inequality. For other aspects of the preinvex functions, see Noor and Noor [16] and the references therein.

If $\eta(y, x) = y - x$, then preinvex function becomes a convex function.

Definition 2.3. *A function $f : I_\eta \subseteq \mathbb{R} \to \mathbb{R}$ is said to be convex function if*

$$f\big(x + t(y - x)\big) \leq (1 - t)f(x) + tf(y), \qquad \forall x, y \in I_\eta, t \in [0, 1].$$

This shows that every convex function is a preinvex function, but the converse is not true, see [26]

Definition 2.4. *[2] A function $f : I_\eta \subseteq \mathbb{R} \to \mathbb{R}$ is said to be MT-preinvex function with respect to the bifunction $\eta(\cdot,\cdot)$ if*

$$f\big(x + t\eta(y,x)\big) \leq \frac{\sqrt{1-t}}{2\sqrt{t}} f(x) + \frac{\sqrt{t}}{2\sqrt{1-t}} f(y), \qquad \forall x, y \in I_\eta, t \in (0,1).$$

Definition 2.5. *[18] A set $I_\eta \subseteq \mathbb{R}\setminus\{0\}$ is said to be a harmonic preinvex set if*

$$\frac{x(x+\eta(y,x))}{x+(1-t)\eta(y,x)} \in I_\eta, \qquad \forall x, y \in I_\eta, t \in [0,1].$$

Definition 2.6. *[17] A function $f : I = [a,b] \subseteq \mathbb{R}\setminus\{0\} \to \mathbb{R}$ is said to be MT-harmonic convex function if it is nonnegative and satisfies the inequality*

$$f\left(\frac{xy}{tx+(1-t)y}\right) \leq \frac{\sqrt{1-t}}{2\sqrt{t}} f(x) + \frac{\sqrt{t}}{2\sqrt{1-t}} f(y), \qquad \forall x, y \in I, t \in (0,1).$$

Definition 2.7. *[18] A function $f : I_\eta \subseteq \mathbb{R} \setminus \{0\} \to \mathbb{R}$ is a harmonic preinvex function with respect to $\eta(\cdot,\cdot)$, if*

$$f\left(\frac{x(x+\eta(y,x))}{x+(1-t)\eta(y,x)}\right) \leq (1-t)f(x) + tf(y), \qquad \forall x, y \in I_\eta, t \in [0,1].$$

We now introduce a new concept of harmonic MT-preinvex functions.

Definition 2.8. *A function $f : I_\eta \subseteq \mathbb{R} \setminus \{0\} \to \mathbb{R}$ is said to be harmonic MT-preinvex function with respect to the bifunction $\eta(\cdot,\cdot)$ if it is nonnegative and satisfies the inequality*

$$f\left(\frac{x(x+\eta(y,x))}{x+(1-t)\eta(y,x)}\right) \leq \frac{\sqrt{1-t}}{2\sqrt{t}} f(x) + \frac{\sqrt{t}}{2\sqrt{1-t}} f(y), \quad \forall x, y \in I_\eta, t \in (0,1). \tag{2.1}$$

For $t = \frac{1}{2}$, harmonic MT-preinvex function collapses to

$$f\left(\frac{2x(x+\eta(y,x))}{2x+\eta(y,x)}\right) \leq \frac{f(x)+f(y)}{2}, \qquad \forall x, y \in I_\eta, \tag{2.2}$$

which is called Jensen type harmonic MT-preinvex function, see [18].

Definition 2.9. *[20] Two functions f and g are said to be similarly ordered (f is g-monotone) if*

$$\langle f(x) - f(y), g(x) - g(y)\rangle \geq 0, \qquad \forall x, y \in \mathbb{R}^n.$$

For the bifunction $\eta(.,.)$, we recall the well know Condition C, which is due to Mohan and Neogy [10]. This Conditions C plays an important role in the studies of the variational-like inequalities and other optimization problems. We use the Condition C to derive the left hand side of the Hermite-Hadamard inequalities for harmonic MT-preinvex functions, see Theorem 3.1 and Theorem 3.2.

Condition *C*. Let $I_\eta \subset \mathbb{R}$ be an invex set with respect to bifunction $\eta(\cdot,\cdot) : I_\eta \times I_\eta \to \mathbb{R}$. For any $x, y \in I_\eta$ and any $t \in [0,1]$, we have

$$\begin{aligned} \eta(y, y + t\eta(x,y)) &= -t\eta(x,y), \\ \eta(x, y + t\eta(x,y)) &= (1-t)\eta(x,y). \end{aligned}$$

Note that for every $x, y \in I_\eta$, $t_1, t_2 \in [0,1]$ from condition C, we have

$$\eta(y + t_2\eta(x,y), y + t_1\eta(x,y)) = (t_2 - t_1)\eta(x,y).$$

We also use the simple, but important following fact, which plays a crucial part in the derivation of the main results of this paper.

Remark 2.10. Let $I_\eta \subseteq \mathbb{R} \setminus \{0\}$ and consider the function $g : \left[\frac{1}{a+\eta(b,a)}, \frac{1}{a}\right] \to \mathbb{R}$ defined by $g(t) = f\left(\frac{1}{t}\right)$. Then f is harmonic MT-preinvex on $[a, a+\eta(b,a)]$ if and only if g is MT-preinvex function in the usual sense on $\left[\frac{1}{a+\eta(b,a)}, \frac{1}{a}\right]$.

3. Main results

In this section, we obtain several Hermite-Hadamard inequalities for harmonic MT-preinvex functions, which is the main motivation of this paper.

Theorem 3.1. *Let $f : I_\eta \subseteq \mathbb{R} \setminus \{0\} \to \mathbb{R}$ be harmonic MT-preinvex function with $\eta(b,a) > 0$. If $f \in L[a, a+\eta(b,a)]$ and condition C holds, then*

$$\frac{\pi}{2} f\left(\frac{2a(a+\eta(b,a))}{2a+\eta(b,a)}\right) \le f(a) + f(b).$$

Proof. Let f be harmonic MT-preinvex function. Then, taking $x = \frac{a(a+\eta(b,a))}{a+(1-t)\eta(b,a)}$ and $y = \frac{a(a+\eta(b,a))}{a+t\eta(b,a)}$ in (4.3), and using condition C, we have

$$\begin{aligned} f\left(\frac{2a(a+\eta(b,a))}{2a+\eta(b,a)}\right) &\le \frac{1}{2}\left[f\left(\frac{a(a+\eta(b,a))}{a+(1-t)\eta(b,a)}\right) + f\left(\frac{a(a+\eta(b,a))}{a+t\eta(b,a)}\right)\right] \\ &\le \frac{1}{2}\left(\frac{\sqrt{1-t}}{2\sqrt{t}} + \frac{\sqrt{t}}{2\sqrt{1-t}}\right)(f(a)+f(b)) \\ &= \frac{1}{4\sqrt{t(1-t)}}(f(a)+f(b)). \end{aligned}$$

Thus

$$4\sqrt{t(1-t)}\, f\left(\frac{2a(a+\eta(b,a))}{2a+\eta(b,a)}\right) \le f(a) + f(b).$$

By integration with respect to t over $[0,1]$, we obtain

$$4f\left(\frac{2a(a+\eta(b,a))}{2a+\eta(b,a)}\right)\int_0^1 t^{\frac{1}{2}}(1-t)^{\frac{1}{2}}\mathrm{d}t=\frac{\pi}{2}f\left(\frac{2a(a+\eta(b,a))}{2a+\eta(b,a)}\right)\le f(a)+f(b),$$

the required result. ∎

THEOREM 3.2. *Let $f: I_\eta \subseteq \mathbb{R}\setminus\{0\}\to\mathbb{R}$ be harmonic MT-preinvex function with $\eta(b,a)>0$. If $f\in L[a,a+\eta(b,a)]$ and condition C holds, then*

(3.1)

$$f\left(\frac{2a(a+\eta(b,a))}{2a+\eta(b,a)}\right)\le\frac{a(a+\eta(b,a))}{\eta(b,a)}\int_a^{a+\eta(b,a)}\frac{f(x)}{x^2}\,\mathrm{d}x\le\frac{\pi(f(a)+f(b))}{4},$$

and

$$\frac{a(a+\eta(b,a))}{\eta(b,a)}\int_{\frac{1}{a+\eta(b,a)}}^{\frac{1}{a}}\xi(x)f\left(\frac{1}{x}\right)\mathrm{d}x\le\frac{f(a)+f(b)}{4}, \tag{3.2}$$

where

$$\xi(x)=\frac{a(a+\eta(b,a))\sqrt{(\frac{1}{a}-x)(x-\frac{1}{a+\eta(b,a)})}}{\eta(b,a)}. \tag{3.3}$$

PROOF. Let f be harmonic MT-preinvex function. Then, taking $x=\frac{a(a+\eta(b,a))}{a+(1-t)\eta(b,a)}$ and $y=\frac{a(a+\eta(b,a))}{a+t\eta(b,a)}$ in (4.3) and using condition C, we have

$$\begin{aligned}
f\left(\frac{2a(a+\eta(b,a))}{2a+\eta(b,a)}\right)&\le\frac{1}{2}\left[f\left(\frac{a(a+\eta(b,a))}{a+(1-t)\eta(b,a)}\right)+f\left(\frac{a(a+\eta(b,a))}{a+t\eta(b,a)}\right)\right]\\
&=\frac{1}{2}\left[\int_0^1 f\left(\frac{a(a+\eta(b,a))}{a+(1-t)\eta(b,a)}\right)\mathrm{d}t+\int_0^1 f\left(\frac{a(a+\eta(b,a))}{a+t\eta(b,a)}\right)\mathrm{d}t\right]\\
&=\frac{a(a+\eta(b,a))}{\eta(b,a)}\int_a^{a+\eta(b,a)}\frac{f(x)}{x^2}\mathrm{d}x\\
&\le\frac{1}{2}\int_0^1\left[\frac{\sqrt{1-t}}{2\sqrt{t}}+\frac{\sqrt{t}}{2\sqrt{1-t}}\right][f(a)+f(b)]\mathrm{d}t\\
&=\frac{1}{4}[f(a)+f(b)]\int_0^1 t^{-\frac{1}{2}}(1-t)^{-\frac{1}{2}}\mathrm{d}t\\
&=\frac{\pi}{4}[f(a)+f(b)].
\end{aligned}$$

For the proof of (3.2) we first note that, if f is harmonic MT-preinvex function, then

$$2\sqrt{t(1-t)}f\left(\frac{a(a+\eta(b,a))}{a+(1-t)\eta(b,a)}\right)\le(1-t)f(a)+tf(b),\quad\forall a,b\in I_\eta,t\in(0,1),$$

and

$$2\sqrt{t(1-t)}f\left(\frac{a(a+\eta(b,a))}{a+t\eta(b,a)}\right)\le tf(a)+(1-t)f(b),\quad\forall a,b\in I_\eta,t\in(0,1).$$

Adding these inequalities and integrating the resultant with respect to t over $[0,1]$, we obtain

$$\int_0^1 \sqrt{t(1-t)}\left[f\left(\frac{a(a+\eta(b,a))}{a+(1-t)\eta(b,a)}\right)+f\left(\frac{a(a+\eta(b,a))}{a+t\eta(b,a)}\right)\right]\mathrm{d}t \leq \frac{f(a)+f(b)}{2}.$$

This implies

$$\frac{a(a+\eta(b,a))}{\eta(b,a)}\int_{\frac{1}{a+\eta(b,a)}}^{\frac{1}{a}} \frac{a(a+\eta(b,a))\sqrt{(\frac{1}{a}-x)(x-\frac{1}{a+\eta(b,a)})}}{\eta(b,a)} f\left(\frac{1}{x}\right)\mathrm{d}x$$
$$\leq \frac{f(a)+f(b)}{4}.$$

This completes the proof. ∎

Theorem 3.3. *Let $f,g: I_\eta \subseteq \mathbb{R}\setminus\{0\} \to \mathbb{R}$ be harmonic MT-preinvex functions with $\eta(b,a)>0$. If $fg \in L[a,a+\eta(b,a)]$, then*

$$\frac{a(a+\eta(b,a))}{\eta(b,a)}\int_{\frac{1}{a+\eta(b,a)}}^{\frac{1}{a}} \xi^2(x) f\left(\frac{1}{x}\right) g\left(\frac{1}{x}\right)\mathrm{d}x \leq \frac{1}{12}M(a,b)+\frac{1}{24}N(a,b), \tag{3.4}$$

where $\xi(x)$ is given by (3.3) and

$$M(a,b)=f(a)g(a)+f(b)g(b), \tag{3.5}$$

$$N(a,b)=f(a)g(b)+f(b)g(a). \tag{3.6}$$

Proof. Let f,g be harmonic MT-preinvex functions. Then

$$f\left(\frac{a(a+\eta(b,a))}{a+(1-t)\eta(b,a)}\right) \leq \frac{\sqrt{1-t}}{2\sqrt{t}}f(a)+\frac{\sqrt{t}}{2\sqrt{1-t}}f(b),$$
$$g\left(\frac{a(a+\eta(b,a))}{a+(1-t)\eta(b,a)}\right) \leq \frac{\sqrt{1-t}}{2\sqrt{t}}g(a)+\frac{\sqrt{t}}{2\sqrt{1-t}}g(b).$$

We now consider

$$
\begin{aligned}
(3.7)\qquad & f\left(\frac{a(a+\eta(b,a))}{a+(1-t)\eta(b,a)}\right)g\left(\frac{a(a+\eta(b,a))}{a+(1-t)\eta(b,a)}\right)\\
\le\ & \left(\frac{\sqrt{1-t}}{2\sqrt{t}}f(a)+\frac{\sqrt{t}}{2\sqrt{1-t}}f(b)\right)\left(\frac{\sqrt{1-t}}{2\sqrt{t}}g(a)+\frac{\sqrt{t}}{2\sqrt{1-t}}g(b)\right)\\
=\ & \frac{1-t}{4t}[f(a)g(a)]+\frac{1}{4}[f(a)g(b)+f(b)g(a)]+\frac{t}{4(1-t)}[f(b)g(b)]\\
=\ & \frac{(1-t)^2}{4t(1-t)}[f(a)g(a)]+\frac{t(1-t)}{4t(1-t)}[f(a)g(b)+f(b)g(a)]\\
& +\frac{t^2}{4t(1-t)}[f(b)g(b)]\\
\le\ & \frac{1}{4t(1-t)}\Big[(1-t)^2[f(a)g(a)]+t(1-t)[f(a)g(b)+f(b)g(a)]\\
& +t^2[f(b)g(b)]\Big].
\end{aligned}
$$

Thus

$$
\begin{aligned}
& 4t(1-t)f\left(\frac{a(a+\eta(b,a))}{a+(1-t)\eta(b,a)}\right)g\left(\frac{a(a+\eta(b,a))}{a+(1-t)\eta(b,a)}\right)\\
& \quad\le\ (1-t)^2[f(a)g(a)]+t(1-t)[f(a)g(b)+f(b)g(a)]+t^2[f(b)g(b)].
\end{aligned}
$$

By integrating the above inequality, we have

$$
\begin{aligned}
& \frac{4a(a+\eta(b,a))}{\eta(b,a)}\int_{\frac{1}{a+\eta(b,a)}}^{\frac{1}{a}}\xi^2(x)f\left(\frac{1}{x}\right)g\left(\frac{1}{x}\right)\mathrm{d}x\\
=\ & 4\int_0^1 t(1-t)f\left(\frac{a(a+\eta(b,a))}{a+(1-t)\eta(b,a)}\right)g\left(\frac{a(a+\eta(b,a))}{a+(1-t)\eta(b,a)}\right)\mathrm{d}t\\
\le\ & \frac{1}{3}[f(a)g(a)+f(b)g(b)]+\frac{1}{6}[f(a)g(b)+f(b)g(a)]\\
=\ & \frac{1}{3}M(a,b)+\frac{1}{6}N(a,b).
\end{aligned}
$$

This completes the proof. ∎

If we choose $x=\frac{2a+\eta(b,a)}{2a(a+\eta(b,a))}$ in the inequality (3.4), we obtain the special case of the inequality (3.4) which appears to be a new one.

Corollary 3.4. *Let $f,g:I_\eta\subseteq\mathbb{R}\setminus\{0\}\to\mathbb{R}$ be harmonic MT-preinvex functions with $\eta(b,a)>0$. If $fg\in L[a,a+\eta(b,a)]$, then*

$$
f\left(\frac{2a(a+\eta(b,a))}{2a+\eta(b,a)}\right)g\left(\frac{2a(a+\eta(b,a))}{2a+\eta(b,a)}\right)\le\frac{1}{3}M(a,b)+\frac{1}{6}N(a,b),
$$

where $M(a,b)$ and $N(a,b)$ are given by (3.5) *and* (3.6) *respectively.*

Theorem 3.5. *Let $f, g : I_\eta \subseteq \mathbb{R} \setminus \{0\} \to \mathbb{R}$ be similarly ordered harmonic MT-preinvex functions with $\eta(b,a) > 0$. If $fg \in L[a, a+\eta(b,a)]$, then*

$$\frac{a(a+\eta(b,a))}{\eta(b,a)} \int_{\frac{1}{a+\eta(b,a)}}^{\frac{1}{a}} \xi^2(x) f\left(\frac{1}{x}\right) g\left(\frac{1}{x}\right) \mathrm{d}x \leq \frac{1}{8} M(a,b), \tag{3.8}$$

where $\xi(x)$ and $M(a,b)$ are given by (3.3) and (3.5) respectively.

Proof. Let f, g be harmonic MT-preinvex functions. Then, from (3.7), we have

$$\begin{aligned} &f\left(\frac{a(a+\eta(b,a))}{a+(1-t)\eta(b,a)}\right) g\left(\frac{a(a+\eta(b,a))}{a+(1-t)\eta(b,a)}\right) \\ &\quad \leq \quad \frac{1}{4t(1-t)} \Bigg[(1-t)^2[f(a)g(a)] + t(1-t)[f(a)g(b) \\ &\qquad\quad + f(b)g(a)] + t^2[f(b)g(b)]\Bigg]. \end{aligned}$$

This implies that

$$\begin{aligned} &4t(1-t) f\left(\frac{a(a+\eta(b,a))}{a+(1-t)\eta(b,a)}\right) g\left(\frac{a(a+\eta(b,a))}{a+(1-t)\eta(b,a)}\right) \\ &\leq \quad (1-t)^2[f(a)g(a)] + t(1-t)[f(a)g(b) + f(b)g(a)] \\ &\qquad + t^2[f(b)g(b)](1-t)[f(a)g(a)] + t[f(b)g(b)] + (1-t)^2[f(a)g(a)] \\ &\qquad + t(1-t)[f(a)g(b) + f(b)g(a)] + t^2[f(b)g(b)] - (1-t)[f(a)g(a)] \\ &\qquad - t[f(b)g(b)] \\ &= \quad (1-t)[f(a)g(a)] + tf[(b)g(b)] - t(1-t)[(f(a) - f(b))(g(a) - g(b))] \\ &\leq \quad (1-t)[f(a)g(a)] + t[f(b)g(b)], \end{aligned}$$

where we have used the fact that the functions f and g are similarly ordered.

Now integrating the above inequality, we have

$$\frac{a(a+\eta(b,a))}{\eta(b,a)} \int_{\frac{1}{a+\eta(b,a)}}^{\frac{1}{a}} \xi^2(x) f\left(\frac{1}{x}\right) g\left(\frac{1}{x}\right) \mathrm{d}x \quad \leq \quad \frac{1}{8} M(a,b),$$

which is the required result. ∎

If $x = \frac{2a+\eta(b,a)}{2a(a+\eta(b,a))}$ in the inequality (3.8), then we obtain a special case of the inequality (3.8), which appears to be a new one.

Corollary 3.6. *Let $f, g : I_\eta \subseteq \mathbb{R} \setminus \{0\} \to \mathbb{R}$ be similarly ordered harmonic MT-preinvex functions with $\eta(b,a) > 0$. If $fg \in L[a, a+\eta(b,a)]$, then*

$$f\left(\frac{2a(a+\eta(b,a))}{2a+\eta(b,a)}\right) g\left(\frac{2a(a+\eta(b,a))}{2a+\eta(b,a)}\right) \leq \frac{1}{2} M(a,b),$$

where $M(a,b)$ is given by (3.5).

THEOREM 3.7. *Let $f, g : I_\eta \subseteq \mathbb{R} \setminus \{0\} \to \mathbb{R}$ be harmonic MT-preinvex functions with $\eta(b,a) > 0$. If $fg \in L[a, a+\eta(b,a)]$, then*

$$\left(\frac{a(a+\eta(b,a))}{\eta(b,a)}\right)^2 \int_{\frac{1}{a+\eta(b,a)}}^{\frac{1}{a}} \xi(x)\left(x - \frac{1}{a+\eta(b,a)}\right)\left[g(a)f\left(\frac{1}{x}\right)+f(a)g\left(\frac{1}{x}\right)\right]\mathrm{d}x$$
$$+\left(\frac{a(a+\eta(b,a))}{\eta(b,a)}\right)^2 \int_{\frac{1}{a+\eta(b,a)}}^{\frac{1}{a}} \xi(x)\left(\frac{1}{a} - x\right)\left[g(b)f\left(\frac{1}{x}\right)+f(b)g\left(\frac{1}{x}\right)\right]\mathrm{d}x$$
$$\leq \frac{1}{6}M(a,b) + \frac{1}{12}N(a,b) + \frac{2a(a+\eta(b,a))}{\eta(b,a)} \int_{\frac{1}{a+\eta(b,a)}}^{\frac{1}{a}} \xi^2(x) f\left(\frac{1}{x}\right) g\left(\frac{1}{x}\right)\mathrm{d}x,$$

where $\xi(x)$, $M(a,b)$ and $N(a,b)$ are given by (3.3), (3.5) *and* (3.6) *respectively.*

PROOF. Let f, g be harmonic MT-preinvex functions. Then

$$f\left(\frac{a(a+\eta(b,a))}{a+(1-t)\eta(b,a)}\right) \leq \frac{\sqrt{1-t}}{2\sqrt{t}}f(a) + \frac{\sqrt{t}}{2\sqrt{1-t}}f(b),$$
$$g\left(\frac{a(a+\eta(b,a))}{a+(1-t)\eta(b,a)}\right) \leq \frac{\sqrt{1-t}}{2\sqrt{t}}g(a) + \frac{\sqrt{t}}{2\sqrt{1-t}}g(b).$$

Now, using $\langle x_1 - x_2, x_3 - x_4\rangle \geq 0, (x_1, x_2, x_3, x_4 \in \mathbb{R})$ and $x_1 < x_2$, $x_3 < x_4$, we have

$$f\left(\frac{a(a+\eta(b,a))}{a+(1-t)\eta(b,a)}\right)\left(\frac{\sqrt{1-t}}{2\sqrt{t}}g(a) + \frac{\sqrt{t}}{2\sqrt{1-t}}g(b)\right)$$
$$+ g\left(\frac{a(a+\eta(b,a))}{a+(1-t)\eta(b,a)}\right)\left(\frac{\sqrt{1-t}}{2\sqrt{t}}f(a) + \frac{\sqrt{t}}{2\sqrt{1-t}}f(b)\right)$$
$$\leq \left(\frac{\sqrt{1-t}}{2\sqrt{t}}f(a) + \frac{\sqrt{t}}{2\sqrt{1-t}}f(b)\right)\left(\frac{\sqrt{1-t}}{2\sqrt{t}}g(a) + \frac{\sqrt{t}}{2\sqrt{1-t}}g(b)\right)$$
$$+ f\left(\frac{a(a+\eta(b,a))}{a+(1-t)\eta(b,a)}\right)g\left(\frac{a(a+\eta(b,a))}{a+(1-t)\eta(b,a)}\right).$$

Thus

$$g(a)\frac{\sqrt{1-t}}{2\sqrt{t}}f\left(\frac{a(a+\eta(b,a))}{a+(1-t)\eta(b,a)}\right) + g(b)\frac{\sqrt{t}}{2\sqrt{1-t}}f\left(\frac{a(a+\eta(b,a))}{a+(1-t)\eta(b,a)}\right)$$
$$+ f(a)\frac{\sqrt{1-t}}{2\sqrt{t}}g\left(\frac{a(a+\eta(b,a))}{a+(1-t)\eta(b,a)}\right) + f(b)\frac{\sqrt{t}}{2\sqrt{1-t}}g\left(\frac{a(a+\eta(b,a))}{a+(1-t)\eta(b,a)}\right)$$
$$\leq \frac{1-t}{4t}[f(a)+g(a)] + \frac{1}{4}[f(a)g(b)+f(b)g(a)] + \frac{t}{4(1-t)}[f(b)+g(a)]$$
$$+ f\left(\frac{a(a+\eta(b,a))}{a+(1-t)\eta(b,a)}\right)g\left(\frac{a(a+\eta(b,a))}{a+(1-t)\eta(b,a)}\right).$$

This implies that

$$\begin{aligned}
&g(a)(1-t)\sqrt{t(1-t)}f\left(\frac{a(a+\eta(b,a))}{a+(1-t)\eta(b,a)}\right)+g(b)t\sqrt{t(1-t)}f\left(\frac{a(a+\eta(b,a))}{a+(1-t)\eta(b,a)}\right)\\
&+f(a)(1-t)\sqrt{t(1-t)}g\left(\frac{a(a+\eta(b,a))}{a+(1-t)\eta(b,a)}\right)+f(b)t\sqrt{t(1-t)}g\left(\frac{a(a+\eta(b,a))}{a+(1-t)\eta(b,a)}\right)\\
\leq\;& \frac{1}{2}\left\{(1-t)^2[f(a)g(a)]+t(1-t)[f(a)g(b)+f(b)g(a)]+t^2[f(b)g(a)]\right\}\\
&+2t(1-t)f\left(\frac{a(a+\eta(b,a))}{a+(1-t)\eta(b,a)}\right)g\left(\frac{a(a+\eta(b,a))}{a+(1-t)\eta(b,a)}\right).
\end{aligned}$$

Integrating the above inequality with respect to t over $[0,1]$, we have

$$\begin{aligned}
&\left(\frac{a(a+\eta(b,a))}{\eta(b,a)}\right)^2\int_{\frac{1}{a+\eta(b,a)}}^{\frac{1}{a}}\xi(x)\left(x-\frac{1}{a+\eta(b,a)}\right)\left[g(a)f\left(\frac{1}{x}\right)+f(a)g\left(\frac{1}{x}\right)\right]\mathrm{d}x\\
&+\left(\frac{a(a+\eta(b,a))}{\eta(b,a)}\right)^2\int_{\frac{1}{a+\eta(b,a)}}^{\frac{1}{a}}\xi(x)\left(\frac{1}{a}-x\right)\left[g(b)f\left(\frac{1}{x}\right)+f(b)g\left(\frac{1}{x}\right)\right]\mathrm{d}x\\
\leq\;& \frac{1}{6}[f(a)g(a)+f(b)g(a)]+\frac{1}{12}[f(a)g(b)+f(b)g(a)]\\
&+\frac{2a(a+\eta(b,a))}{\eta(b,a)}\int_{\frac{1}{a+\eta(b,a)}}^{\frac{1}{a}}\xi^2(x)f\left(\frac{1}{x}\right)g\left(\frac{1}{x}\right)\mathrm{d}x,
\end{aligned}$$

the required result. □

Theorem 3.8. *Let $f,g: I_\eta \subseteq \mathbb{R}\setminus\{0\} \to \mathbb{R}$ be harmonic MT-preinvex functions with $\eta(b,a)>0$. If $fg \in L[a,a+\eta(b,a)]$, then*

$$\begin{aligned}
&f\left(\frac{2a(a+\eta(b,a))}{2a+\eta(b,a)}\right)[g(a)+g(b)]+g\left(\frac{2a(a+\eta(b,a))}{2a+\eta(b,a)}\right)[f(a)+f(b)]\\
\leq\;& \frac{2}{\pi}[M(a,b)+N(a,b)]+\frac{16}{3\pi}f\left(\frac{2a(a+\eta(b,a))}{2a+\eta(b,a)}\right)g\left(\frac{2a(a+\eta(b,a))}{2a+\eta(b,a)}\right),
\end{aligned}$$

where $M(a,b)$ and $N(a,b)$ are given by (3.5) *and* (3.6) *respectively.*

Proof. Let f, g be harmonic MT-preinvex functions. Then

$$\begin{aligned}
f\left(\frac{2a(a+\eta(b,a))}{2a+\eta(b,a)}\right)&\leq\frac{1}{2}\left(\frac{\sqrt{1-t}}{2\sqrt{t}}+\frac{\sqrt{t}}{2\sqrt{1-t}}\right)[f(a)+f(b)],\\
g\left(\frac{2a(a+\eta(b,a))}{2a+\eta(b,a)}\right)&\leq\frac{1}{2}\left(\frac{\sqrt{1-t}}{2\sqrt{t}}+\frac{\sqrt{t}}{2\sqrt{1-t}}\right)[g(a)+g(b)].
\end{aligned}$$

Now, using $\langle x_1 - x_2, x_3 - x_4\rangle \geq 0, (x_1, x_2, x_3, x_4 \in \mathbb{R})$ and $x_1 < x_2$, $x_3 < x_4$, we have

$$\begin{aligned}
&\frac{1}{4}[g(a)+g(b)]f\left(\frac{2a(a+\eta(b,a))}{2a+\eta(b,a)}\right)\left(\frac{\sqrt{1-t}}{\sqrt{t}}+\frac{\sqrt{t}}{\sqrt{1-t}}\right)\\
&\qquad+\frac{1}{4}g\left(\frac{2a(a+\eta(b,a))}{2a+\eta(b,a)}\right)[f(a)+f(b)]\left(\frac{\sqrt{1-t}}{\sqrt{t}}+\frac{\sqrt{t}}{\sqrt{1-t}}\right)\\
&\leq \frac{1}{16}\frac{1}{t(1-t)}[f(a)+f(b)][g(a)+g(b)]\\
&\qquad+f\left(\frac{2a(a+\eta(b,a))}{2a+\eta(b,a)}\right)g\left(\frac{2a(a+\eta(b,a))}{2a+\eta(b,a)}\right).
\end{aligned}$$

Thus

$$\begin{aligned}
&\frac{1}{4}[g(a)+g(b)]f\left(\frac{2a(a+\eta(b,a))}{2a+\eta(b,a)}\right)\left(\sqrt{t}\sqrt{1-t}\right)\\
&\qquad+\frac{1}{4}g\left(\frac{2a(a+\eta(b,a))}{2a+\eta(b,a)}\right)[f(a)+f(b)]\left(\sqrt{t}\sqrt{1-t}\right)\\
&\leq \frac{1}{16}[f(a)+f(b)][g(a)+g(b)]\\
&\qquad+t(1-t)f\left(\frac{2a(a+\eta(b,a))}{2a+\eta(b,a)}\right)g\left(\frac{2a(a+\eta(b,a))}{2a+\eta(b,a)}\right).
\end{aligned}$$

Integrating the above inequality with respect to t over $[0,1]$, we have

$$\begin{aligned}
&f\left(\frac{2a(a+\eta(b,a))}{2a+\eta(b,a)}\right)[g(a)+g(b)]+g\left(\frac{2a(a+\eta(b,a))}{2a+\eta(b,a)}\right)[f(a)+f(b)]\\
&\leq \frac{2}{\pi}[f(a)+f(b)][g(a)+g(b)]+\frac{16}{3\pi}f\left(\frac{2a(a+\eta(b,a))}{2a+\eta(b,a)}\right)g\left(\frac{2a(a+\eta(b,a))}{2a+\eta(b,a)}\right)\\
&= \frac{2}{\pi}[M(a,b)+N(a,b)]+\frac{16}{3\pi}f\left(\frac{2a(a+\eta(b,a))}{2a+\eta(b,a)}\right)g\left(\frac{2a(a+\eta(b,a))}{2a+\eta(b,a)}\right),
\end{aligned}$$

which is the required result. □

We now obtain several integral inequalities via harmonic MT-preinvexity for midpoint, two point Trapezoidal and Simpson rule and three points Trapezoidal rule respectively. For this purpose, we need the following result.

LEMMA 3.9. *[19] Let $f : I_\eta \subset \mathbb{R}\setminus\{0\} \to \mathbb{R}$ be a differentiable function on the interior I_η^o of I_η. If $f' \in L[a, a+\eta(b,a)]$ and $\lambda \in [0,1]$, then*

$$\begin{aligned}
&(1-\lambda)f\left(\frac{2a(a+\eta(b,a))}{2a+\eta(b,a)}\right)+\lambda\left(\frac{f(a)+f(a+\eta(b,a))}{2}\right)\\
&\quad-\frac{a(a+\eta(b,a))}{\eta(b,a)}\int_a^{a+\eta(b,a)}\frac{f(x)}{x^2}\mathrm{d}x\\
&= \frac{a(a+\eta(b,a))\eta(b,a)}{2}\left[\int_0^{\frac{1}{2}}\frac{2t-\lambda}{A_t^2}f'\left(\frac{a(a+\eta(b,a))}{A_t}\right)\mathrm{d}t\right.\\
&\quad\left.+\int_{\frac{1}{2}}^1\frac{2t-2+\lambda}{A_t^2}f'\left(\frac{a(a+\eta(b,a))}{A_t}\right)\mathrm{d}t\right],
\end{aligned}$$

where

$$A_t = a+(1-t)\eta(b,a).$$

THEOREM 3.10. *Let $f : I_\eta \subset \mathbb{R}\setminus\{0\} \to \mathbb{R}$ be a differentiable function on the interior I_η^o of I_η. If $f' \in [a, a+\eta(b,a)]$ and $|f'|^q$ is harmonic MT-preinvex function on I_η for $q \geq 1$ and $\lambda \in [0,1]$, then*

$$\begin{aligned}
&\left|(1-\lambda)f\left(\frac{2a(a+\eta(b,a))}{2a+\eta(b,a)}\right)+\lambda\left(\frac{f(a)+f(a+\eta(b,a))}{2}\right)\right.\\
&\quad\left.-\frac{a(a+\eta(b,a))}{\eta(b,a)}\int_a^{a+\eta(b,a)}\frac{f(x)}{x^2}\mathrm{d}x\right|\\
&\leq \frac{a(a+\eta(b,a))\eta(b,a)}{2^{\frac{1}{q}+1}}\left\{(\xi_1(\lambda,a,b))^{1-\frac{1}{q}}[\xi_2(\lambda;a,b)|f'(a)|^q+\xi_3(\lambda;a,b)|f'(b)|^q]^{\frac{1}{q}}\right.\\
&\quad\left.+(\xi_1^*(\lambda,b,a))^{1-\frac{1}{q}}[\xi_3^*(\lambda;b,a)|f'(a)|^q+\xi_2^*(\lambda;b,a)|f'(b)|^q]^{\frac{1}{q}}\right\},
\end{aligned}$$

where

$$\xi_1(\lambda;a,b) = \int_0^{\frac{1}{2}}\frac{|2t-\lambda|}{A_t^2}\mathrm{d}t,$$

$$\xi_1^*(\lambda;b,a) = \int_{\frac{1}{2}}^1\frac{|2t-2+\lambda|}{A_t^2}\mathrm{d}t,$$

$$\xi_2(\lambda;a,b) = \int_0^{\frac{1}{2}}\frac{|2t-\lambda|\frac{\sqrt{1-t}}{\sqrt{t}}}{A_t^2}\mathrm{d}t, \tag{3.9}$$

$$\xi_2^*(\lambda;b,a) = \int_{\frac{1}{2}}^1\frac{|2t-2+\lambda|\frac{\sqrt{t}}{\sqrt{1-t}}}{A_t^2}\mathrm{d}t, \tag{3.10}$$

$$\xi_3(\lambda; a, b) = \int_0^{\frac{1}{2}} \frac{|2t-\lambda|\frac{\sqrt{t}}{\sqrt{1-t}}}{A_t^2}\mathrm{d}t, \tag{3.11}$$

$$\xi_3^*(\lambda; b, a) = \int_{\frac{1}{2}}^1 \frac{|2t-2+\lambda|\frac{\sqrt{1-t}}{\sqrt{t}}}{A_t^2}\mathrm{d}t. \tag{3.12}$$

Proof. Using Lemma 3.9 and the power mean inequality, we have

$$\begin{aligned}
&\Bigg|(1-\lambda)f\left(\frac{2a(a+\eta(b,a))}{2a+\eta(b,a)}\right)+\lambda\left(\frac{f(a)+f(a+\eta(b,a))}{2}\right)\\
&-\frac{a(a+\eta(b,a))}{\eta(b,a)}\int_a^{a+\eta(b,a)}\frac{f(x)}{x^2}\mathrm{d}x\Bigg|\\
\le\ &\frac{a(a+\eta(b,a))\eta(b,a)}{2}\left[\int_0^{\frac{1}{2}}\frac{|2t-\lambda|}{A_t^2}\left|f'\left(\frac{a(a+\eta(b,a))}{A_t}\right)\right|\mathrm{d}t\right.\\
&\left.+\int_{\frac{1}{2}}^1\frac{|2t-2+\lambda|}{A_t^2}\left|f'\left(\frac{a(a+\eta(b,a))}{A_t}\right)\right|\mathrm{d}t\right]\\
\le\ &\frac{a(a+\eta(b,a))\eta(b,a)}{2}\left\{\left(\int_0^{\frac{1}{2}}\frac{|2t-\lambda|}{A_t^2}\mathrm{d}t\right)^{1-\frac{1}{q}}\left(\int_0^{\frac{1}{2}}\frac{|2t-\lambda|}{A_t^2}\left|f'\left(\frac{a(a+\eta(b,a))}{A_t}\right)\right|^q\mathrm{d}t\right)^{\frac{1}{q}}\right.\\
&\left.+\left(\int_{\frac{1}{2}}^1\frac{|2t-2+\lambda|}{A_t^2}\mathrm{d}t\right)^{1-\frac{1}{q}}\left(\int_{\frac{1}{2}}^1\frac{|2t-2+\lambda|}{A_t^2}\left|f'\left(\frac{a(a+\eta(b,a))}{A_t}\right)\right|^q\mathrm{d}t\right)^{\frac{1}{q}}\right\}\\
\le\ &\frac{a(a+\eta(b,a))\eta(b,a)}{2}\left\{\left(\int_0^{\frac{1}{2}}\frac{|2t-\lambda|}{A_t^2}\mathrm{d}t\right)^{1-\frac{1}{q}}\right.\\
&\times\left(\int_0^{\frac{1}{2}}\frac{|2t-\lambda|\left[\frac{\sqrt{1-t}}{2\sqrt{t}}|f'(a)|^q+\frac{\sqrt{t}}{2\sqrt{1-t}}|f'(b)|^q\right]}{A_t^2}\mathrm{d}t\right)^{\frac{1}{q}}\\
&\left.+\left(\int_{\frac{1}{2}}^1\frac{|2t-2+\lambda|}{A_t^2}\mathrm{d}t\right)^{1-\frac{1}{q}}\left(\int_{\frac{1}{2}}^1\frac{|2t-2+\lambda|\left[\frac{\sqrt{1-t}}{2\sqrt{t}}|f'(a)|^q+\frac{\sqrt{t}}{2\sqrt{1-t}}|f'(b)|^q\right]}{A_t^2}\mathrm{d}t\right)^{\frac{1}{q}}\right\}\\
=\ &\frac{a(a+\eta(b,a))\eta(b,a)}{2}\left\{\left(\int_0^{\frac{1}{2}}\frac{|2t-\lambda|}{A_t^2}\mathrm{d}t\right)^{1-\frac{1}{q}}\left(\int_0^{\frac{1}{2}}\frac{|2t-\lambda|\frac{\sqrt{1-t}}{2\sqrt{t}}}{A_t^2}|f'(a)|^q\mathrm{d}t\right.\right.\\
&\left.+\int_0^{\frac{1}{2}}\frac{|2t-\lambda|\frac{\sqrt{t}}{2\sqrt{1-t}}}{A_t^2}|f'(b)|^q\mathrm{d}t\right)^{\frac{1}{q}}+\left(\int_{\frac{1}{2}}^1\frac{|2t-2+\lambda|}{A_t^2}\mathrm{d}t\right)^{1-\frac{1}{q}}\\
&\left.\times\left(\int_{\frac{1}{2}}^1\frac{|2t-2+\lambda|\frac{\sqrt{1-t}}{2\sqrt{t}}}{A_t^2}|f'(a)|^q\mathrm{d}t+\int_{\frac{1}{2}}^1\frac{|2t-2+\lambda|\frac{\sqrt{t}}{2\sqrt{1-t}}}{A_t^2}|f'(b)|^q\mathrm{d}t\right)^{\frac{1}{q}}\right\}\\
=\ &\frac{a(a+\eta(b,a))\eta(b,a)}{2^{\frac{1}{q}+1}}\left\{(\xi_1(\lambda,a,b))^{1-\frac{1}{q}}[\xi_2(\lambda;a,b)|f'(a)|^q+\xi_3(\lambda;a,b)|f'(b)|^q]^{\frac{1}{q}}\right.\\
&\left.+(\xi_1^*(\lambda,b,a))^{1-\frac{1}{q}}[\xi_3^*(\lambda;b,a)|f'(a)|^q+\xi_2^*(\lambda;b,a)|f'(b)|^q]^{\frac{1}{q}}\right\},
\end{aligned}$$

which is the required result. ◻

If $q = 1$, then Theorem 3.10 reduces to following result, which appears to be a new one.

Corollary 3.11. *Let $f : I_\eta \subset \mathbb{R}\backslash\{0\} \to \mathbb{R}$ be a differentiable function on the interior I_η^o of I_η. If $f' \in [a, a+\eta(b,a)]$ and $|f'|$ is harmonic MT-preinvex function on I_η and $\lambda \in [0,1]$, then*

$$\begin{aligned}
&\Bigg|(1-\lambda)f\left(\frac{2a(a+\eta(b,a))}{2a+\eta(b,a)}\right)+\lambda\left(\frac{f(a)+f(a+\eta(b,a))}{2}\right)\\
&\quad-\frac{a(a+\eta(b,a))}{\eta(b,a)}\int_a^{a+\eta(b,a)}\frac{f(x)}{x^2}\mathrm{d}x\Bigg|\\
\leq&\quad\frac{a(a+\eta(b,a))\eta(b,a)}{2}[\xi_2(\lambda;a,b)|f'(a)|+\xi_3(\lambda;a,b)|f'(b)|]\\
&\quad+[\xi_3^*(\lambda;b,a)|f'(a)|+\xi_2^*(\lambda;b,a)|f'(b)|],
\end{aligned}$$

where $\xi_2(\lambda;a,b)$, $\xi_2^*(\lambda;b,a)$, $\xi_3(\lambda;a,b)$ and $\xi_3^*(\lambda;b,a)$ are given by (3.9)*,* (3.10)*,* (3.11) *and* (3.12) *respectively.*

Theorem 3.12. *Let $f : I_\eta \subset \mathbb{R}\setminus\{0\} \to \mathbb{R}$ be a differentiable function on the interior I_η^o of I_η. If $f' \in [a, a+\eta(b,a)]$ and $|f'|^q$ is harmonic MT-preinvex function on I_η for $p, q > 1$, $\frac{1}{p}+\frac{1}{q} = 1$ and $\lambda \in [0,1]$, then*

$$\begin{aligned}
&\Bigg|(1-\lambda)f\left(\frac{2a(a+\eta(b,a))}{2a+\eta(b,a)}\right)+\lambda\left(\frac{f(a)+f(a+\eta(b,a))}{2}\right)\\
&\quad-\frac{a(a+\eta(b,a))}{\eta(b,a)}\int_a^{a+\eta(b,a)}\frac{f(x)}{x^2}\mathrm{d}x\Bigg|\\
&\leq\frac{a(a+\eta(b,a))\eta(b,a)}{2}\Bigg\{(\xi_4(\lambda,p;a,b))^{\frac{1}{p}}\left(\frac{\pi[|f'(a)|^q+|f'\left(\frac{2a(a+\eta(b,a))}{2a+\eta(b,a)}\right)|^q]}{8}\right)^{\frac{1}{q}}\\
&\quad+\quad(\xi_4^*(\lambda,p;b,a))^{\frac{1}{p}}\left(\frac{\pi[|f'\left(\frac{2a(a+\eta(b,a))}{2a+\eta(b,a)}\right)|^q+|f'(b)|^q]}{8}\right)^{\frac{1}{q}}\Bigg\},
\end{aligned}\tag{3.13}$$

where

$$\xi_4(\lambda,p;a,b)=\int_0^{\frac{1}{2}}\frac{|2t-\lambda|^p}{A_t^{2p}}\mathrm{d}t,\tag{3.14}$$

$$\xi_4^*(\lambda,p;b,a)=\int_{\frac{1}{2}}^1\frac{|2t-2+\lambda|^p}{A_t^{2p}}\mathrm{d}t.\tag{3.15}$$

PROOF. Using Lemma 3.9 and the Holder's integral inequality, we have

$$\begin{aligned}
&\left|(1-\lambda)f\left(\frac{2a(a+\eta(b,a))}{2a+\eta(b,a)}\right)+\lambda\left(\frac{f(a)+f(a+\eta(b,a))}{2}\right)\right.\\
&\left.-\frac{a(a+\eta(b,a))}{\eta(b,a)}\int_a^{a+\eta(b,a)}\frac{f(x)}{x^2}\mathrm{d}x\right|\\
\leq\;&\frac{a(a+\eta(b,a))\eta(b,a)}{2}\left[\int_0^{\frac{1}{2}}\frac{\left|2t-\lambda\right|}{A_t^2}\left|f'\left(\frac{a(a+\eta(b,a))}{A_t}\right)\right|\mathrm{d}t\right.\\
&\left.+\int_{\frac{1}{2}}^1\frac{\left|2t-2+\lambda\right|}{A_t^2}\left|f'\left(\frac{a(a+\eta(b,a))}{A_t}\right)\right|\mathrm{d}t\right]\\
\leq\;&\frac{a(a+\eta(b,a))\eta(b,a)}{2}\left\{\left(\int_0^{\frac{1}{2}}\frac{\left|2t-\lambda\right|^p}{A_t^{2p}}\mathrm{d}t\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{2}}\left|f'\left(\frac{a(a+\eta(b,a))}{A_t}\right)\right|^q\mathrm{d}t\right)^{\frac{1}{q}}\right.\\
&\left.+\left(\int_{\frac{1}{2}}^1\frac{\left|2t-2+\lambda\right|^p}{A_t^{2p}}\mathrm{d}t\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^1\left|f'\left(\frac{a(a+\eta(b,a))}{A_t}\right)\right|^q\mathrm{d}t\right)^{\frac{1}{q}}\right\}\\
=\;&\frac{a(a+\eta(b,a))\eta(b,a)}{2}\left\{\left(\int_0^{\frac{1}{2}}\frac{\left|2t-\lambda\right|^p}{A_t^{2p}}\mathrm{d}t\right)^{\frac{1}{p}}\right.\\
&\times\left(\frac{a(a+\eta(b,a))}{\eta(b,a)}\int_a^{\frac{2a(a+\eta(b,a))}{2a+\eta(b,a)}}\frac{\left|f'(x)\right|^q}{x^2}\mathrm{d}x\right)^{\frac{1}{q}}\\
&\left.+\left(\int_{\frac{1}{2}}^1\frac{\left|2t-2+\lambda\right|^p}{A_t^{2p}}\mathrm{d}t\right)^{\frac{1}{p}}\left(\frac{a(a+\eta(b,a))}{\eta(b,a)}\int_{\frac{2a(a+\eta(b,a))}{2a+\eta(b,a)}}^b\frac{\left|f'(x)\right|^q}{x^2}\mathrm{d}x\right)^{\frac{1}{q}}\right\}.
\end{aligned}\tag{3.16}$$

Using the harmonic MT-preinvexity of $|f'|^q$, we obtain the following inequalities from inequality (3.1)

$$\begin{aligned}
&\frac{2a(a+\eta(b,a))}{\eta(b,a)}\int_a^{\frac{2a(a+\eta(b,a))}{2a+\eta(b,a)}}\frac{|f'(x)|^q}{x^2}\mathrm{d}x\\
&\quad\leq\frac{\pi\left[|f'(a)|^q+\left|f'\left(\frac{2a(a+\eta(b,a))}{2a+\eta(b,a)}\right)\right|^q\right]}{4},
\end{aligned}\tag{3.17}$$

and

$$\begin{aligned}
&\frac{2a(a+\eta(b,a))}{\eta(b,a)}\int_{\frac{2a(a+\eta(b,a))}{2a+\eta(b,a)}}^b\frac{|f'(x)|^q}{x^2}\mathrm{d}x\\
&\quad\leq\frac{\pi\left[\left|f'\left(\frac{2a(a+\eta(b,a))}{2a+\eta(b,a)}\right)\right|^q+|f'(b)|^q\right]}{4}.
\end{aligned}\tag{3.18}$$

A combination of (3.16)–(3.18) gives the required inequality (3.13). ∎

THEOREM 3.13. *Let $f : I_\eta \subset \mathbb{R}\setminus\{0\} \to \mathbb{R}$ be a differentiable function on the interior I_η^o of I_η. If $f' \in L[a, a+\eta(b,a)]$ and $|f'|^q$ is harmonic MT-preinvex function on I_η for $p, q > 1$, $\frac{1}{p}+\frac{1}{q}=1$ and $\lambda \in [0,1]$, then*

$$\begin{aligned}
&\left|(1-\lambda)f\left(\frac{2a(a+\eta(b,a))}{2a+\eta(b,a)}\right)+\lambda\left(\frac{f(a)+f(a+\eta(b,a))}{2}\right)\right.\\
&\qquad\left.-\frac{a(a+\eta(b,a))}{\eta(b,a)}\int_a^{a+\eta(b,a)}\frac{f(x)}{x^2}\mathrm{d}x\right|\\
\leq\ & \frac{a(a+\eta(b,a))\eta(b,a)}{2^{\frac{1}{q}+1}}\times\left\{\left(\frac{\lambda^{p+1}+(1-\lambda)^{p+1}}{2(p+1)}\right)^{\frac{1}{p}}\left\{(\xi_5(q;a,b)|f'(a)|^q\right.\right.\\
&\qquad\left.\left.+\xi_6(q;a,b)|f'(b)|^q)^{\frac{1}{q}}+(\xi_6^*(q;b,a)|f'(a)|^q+\xi_5^*(q;b,a)|f'(b)|^q)^{\frac{1}{q}}\right\}\right.,
\end{aligned}$$

where

$$\begin{aligned}
\xi_5(q;a,b) &= \int_0^{\frac{1}{2}}\frac{\sqrt{1-t}}{A_t^{2q}\sqrt{t}}\mathrm{d}t, \qquad \xi_5^*(q;b,a)=\int_{\frac{1}{2}}^1\frac{\sqrt{t}}{A_t^{2q}\sqrt{1-t}}\mathrm{d}t,\\
\xi_6(q;a,b) &= \int_0^{\frac{1}{2}}\frac{\sqrt{t}}{A_t^{2q}\sqrt{1-t}}\mathrm{d}t, \qquad \xi_6^*(q;b,a)=\int_{\frac{1}{2}}^1\frac{\sqrt{1-t}}{A_t^{2q}\sqrt{t}}\mathrm{d}t.
\end{aligned}$$

PROOF. Using Lemma 3.9 and the Holder's integral inequality, we have

$$\begin{aligned}
&\left|(1-\lambda)f\left(\frac{2a(a+\eta(b,a))}{2a+\eta(b,a)}\right)+\lambda\left(\frac{f(a)+f(a+\eta(b,a))}{2}\right)\right.\\
&\qquad\left.-\frac{a(a+\eta(b,a))}{\eta(b,a)}\int_a^{a+\eta(b,a)}\frac{f(x)}{x^2}\mathrm{d}x\right|\\
\leq\ & \frac{a(a+\eta(b,a))\eta(b,a)}{2}\left[\int_0^{\frac{1}{2}}\left|2t-\lambda\right|\left|\frac{1}{A_t^2}f'\left(\frac{a(a+\eta(b,a))}{A_t}\right)\right|\mathrm{d}t\right.\\
&\qquad\left.+\int_{\frac{1}{2}}^1\left|2t-2+\lambda\right|\left|\frac{1}{A_t^2}f'\left(\frac{a(a+\eta(b,a))}{A_t}\right)\right|\mathrm{d}t\right]\\
\leq\ & \frac{a(a+\eta(b,a))\eta(b,a)}{2}\left\{\left(\int_0^{\frac{1}{2}}\left|2t-\lambda\right|^p\mathrm{d}t\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{2}}\left|\frac{1}{A_t^2}f'\left(\frac{a(a+\eta(b,a))}{A_t}\right)\right|^q\mathrm{d}t\right)^{\frac{1}{q}}\right.\\
&\qquad\left.+\left(\int_{\frac{1}{2}}^1\left|2t-2+\lambda\right|^p\mathrm{d}t\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^1\left|\frac{1}{A_t^2}f'\left(\frac{a(a+\eta(b,a))}{A_t}\right)\right|^q\mathrm{d}t\right)^{\frac{1}{q}}\right\}\\
=\ & \frac{a(a+\eta(b,a))\eta(b,a)}{2}\left\{\left(\int_0^{\frac{1}{2}}\left|2t-\lambda\right|^p\mathrm{d}t\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{2}}\frac{1}{A_t^{2q}}\left|f'\left(\frac{a(a+\eta(b,a))}{A_t}\right)\right|^q\mathrm{d}t\right)^{\frac{1}{q}}\right.\\
&\qquad\left.+\left(\int_{\frac{1}{2}}^1\left|2t-2+\lambda\right|^p\mathrm{d}t\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^1\frac{1}{A_t^{2q}}\left|f'\left(\frac{a(a+\eta(b,a))}{A_t}\right)\right|^q\mathrm{d}t\right)^{\frac{1}{q}}\right\}
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{a(a+\eta(b,a))\eta(b,a)}{2}\Bigg\{\bigg(\int_0^{\frac{1}{2}}\big|2t-\lambda\big|^p\mathrm{d}t\bigg)^{\frac{1}{p}} \\
&\quad\times\bigg(\int_0^{\frac{1}{2}}\frac{1}{A_t^{2q}}\Big[\frac{\sqrt{1-t}}{2\sqrt{t}}|f'(a)|^q+\frac{\sqrt{t}}{2\sqrt{1-t}}|f'(b)|^q\Big]\mathrm{d}t\bigg)^{\frac{1}{q}} \\
&\quad+\bigg(\int_{\frac{1}{2}}^1\big|2t-2+\lambda\big|^p\mathrm{d}t\bigg)^{\frac{1}{p}}\bigg(\int_{\frac{1}{2}}^1\frac{1}{A_t^{2q}}\Big[\frac{\sqrt{1-t}}{2\sqrt{t}}|f'(a)|^q+\frac{\sqrt{t}}{2\sqrt{1-t}}|f'(b)|^q\Big]\mathrm{d}t\bigg)^{\frac{1}{q}}\Bigg\} \\
&= \frac{a(a+\eta(b,a))\eta(b,a)}{2}\Bigg\{\bigg(\frac{\lambda^{p+1}+(1-\lambda)^{p+1}}{2(p+1)}\bigg)^{\frac{1}{p}}\Bigg[\bigg(\int_0^{\frac{1}{2}}\frac{\sqrt{1-t}}{2A_t^{2q}\sqrt{t}}|f'(a)|^q \\
&\quad+\int_0^{\frac{1}{2}}\frac{\sqrt{t}}{2A_t^{2q}\sqrt{1-t}}|f'(b)|^q\Big]\bigg)^{\frac{1}{q}} \\
&\quad+\bigg(\int_{\frac{1}{2}}^1\frac{\sqrt{1-t}}{2A_t^{2q}\sqrt{t}}|f'(a)|^q+\int_{\frac{1}{2}}^1\frac{\sqrt{t}}{2A_t^{2q}\sqrt{1-t}}|f'(b)|^q\bigg)^{\frac{1}{q}}\Bigg]\Bigg\} \\
&= \frac{a(a+\eta(b,a))\eta(b,a)}{2^{\frac{1}{q}+1}}\times\Bigg\{\bigg(\frac{\lambda^{p+1}+(1-\lambda)^{p+1}}{2(p+1)}\bigg)^{\frac{1}{p}}\big\{(\xi_5(q;a,b)|f'(a)|^q \\
&\quad+\xi_6(q;a,b)|f'(b)|^q)^{\frac{1}{q}}+(\xi_6^*(q;b,a)|f'(a)|^q+\xi_5^*(q;b,a)|f'(b)|^q)^{\frac{1}{q}}\big\},
\end{aligned}$$

which is the required result. ∎

Remarks: With different choices of λ, we obtain the following integral inequalities:

I. If $\lambda = 0$, then from Theorem 3.12, we obtain the midpoint inequality:

$$\begin{aligned}
&\bigg|f\bigg(\frac{2a(a+\eta(b,a))}{2a+\eta(b,a)}\bigg)-\frac{a(a+\eta(b,a))}{\eta(b,a)}\int_a^{a+\eta(b,a)}\frac{f(x)}{x^2}\mathrm{d}x\bigg| \\
&\leq \frac{a(a+\eta(b,a))\eta(b,a)}{2}\Bigg\{(\xi_4(0,p;a,b))^{\frac{1}{p}}\bigg(\frac{\pi[|f'(a)|^q+|f'\big(\frac{2a(a+\eta(b,a))}{2a+\eta(b,a)}\big)|^q]}{8}\bigg)^{\frac{1}{q}} \\
&\quad+(\xi_4^*(0,p;b,a))^{\frac{1}{p}}\bigg(\frac{\pi[|f'\big(\frac{2a(a+\eta(b,a))}{2a+\eta(b,a)}\big)|^q+|f'(b)|^q]}{8}\bigg)^{\frac{1}{q}}\Bigg\},
\end{aligned}$$

where $\xi_4(0,p;a,b)$ and $\xi_4^*(0,p;b,a)$ can be deduced from (3.14) and (3.15) respectively.

II. If $\lambda = 1$, then, from Theorem 3.12, we obtain the two point Trapezoidal inequality:

$$\begin{aligned}&\left|\frac{f(a)+f(a+\eta(b,a))}{2} - \frac{a(a+\eta(b,a))}{\eta(b,a)}\int_a^{a+\eta(b,a)}\frac{f(x)}{x^2}\mathrm{d}x\right|\\ &\leq \frac{a(a+\eta(b,a))\eta(b,a)}{2}\Bigg\{(\xi_4(1,p;a,b))^{\frac{1}{p}}\left(\frac{\pi[|f'(a)|^q + |f'\big(\frac{2a(a+\eta(b,a))}{2a+\eta(b,a)}\big)|^q]}{8}\right)^{\frac{1}{q}}\\ &\quad + (\xi_4^*(1,p;b,a))^{\frac{1}{p}}\left(\frac{\pi[|f'\big(\frac{2a(a+\eta(b,a))}{2a+\eta(b,a)}\big)|^q + |f'(b)|^q]}{8}\right)^{\frac{1}{q}}\Bigg\},\end{aligned}$$

where $\xi_4(1,p;a,b)$ and $\xi_4^*(1,p;b,a)$ can be deduced from (3.14) and (3.15) respectively.

III. If $\lambda = \frac{1}{2}$, then, from Theorem 3.12, we obtain the three point Trapezoidal inequality:

$$\begin{aligned}&\Bigg|\frac{1}{2}f\left(\frac{2a(a+\eta(b,a))}{2a+\eta(b,a)}\right) + \frac{f(a)+f(a+\eta(b,a))}{4}\\ &\quad - \frac{a(a+\eta(b,a))}{\eta(b,a)}\int_a^{a+\eta(b,a)}\frac{f(x)}{x^2}\mathrm{d}x\Bigg|\\ &\leq \frac{a(a+\eta(b,a))\eta(b,a)}{2}\Bigg\{(\xi_4(1/2,p;a,b))^{\frac{1}{p}}\left(\frac{\pi[|f'(a)|^q + |f'\big(\frac{2a(a+\eta(b,a))}{2a+\eta(b,a)}\big)|^q]}{8}\right)^{\frac{1}{q}}\\ &\quad + (\xi_4^*(1/2,p;b,a))^{\frac{1}{p}}\left(\frac{\pi[|f'\big(\frac{2a(a+\eta(b,a))}{2a+\eta(b,a)}\big)|^q + |f'(b)|^q]}{8}\right)^{\frac{1}{q}}\Bigg\},\end{aligned}$$

where $\xi_4(1/2,p;a,b)$ and $\xi_4^*(1/2,p;b,a)$ can be deduced from (3.14) and (3.15) respectively.

IV. If $\lambda = \frac{1}{3}$, then, from Theorem 3.12, we obtain Simpson's inequality:

$$\begin{aligned}&\Bigg|\frac{1}{6}\left[f(a) + 4f\left(\frac{2a(a+\eta(b,a))}{2a+\eta(b,a)}\right) + f(a+\eta(b,a))\right]\\ &\quad - \frac{a(a+\eta(b,a))}{\eta(b,a)}\int_a^{a+\eta(b,a)}\frac{f(x)}{x^2}\mathrm{d}x\Bigg|\\ &\leq \frac{a(a+\eta(b,a))\eta(b,a)}{2}\Bigg\{(\xi_4(1/3,p;a,b))^{\frac{1}{p}}\left(\frac{\pi[|f'(a)|^q + |f'\big(\frac{2a(a+\eta(b,a))}{2a+\eta(b,a)}\big)|^q]}{8}\right)^{\frac{1}{q}}\\ &\quad + (\xi_4^*(1/3,p;b,a))^{\frac{1}{p}}\left(\frac{\pi[|f'\big(\frac{2a(a+\eta(b,a))}{2a+\eta(b,a)}\big)|^q + |f'(b)|^q]}{8}\right)^{\frac{1}{q}}\Bigg\},\end{aligned}$$

where $\xi_4(1/3, p; a, b)$ and $\xi_4^*(1/3, p; b, a)$ can be deduced from (3.14) and (3.15) respectively.

In a similar way, one can obtain several other integral inequalities for harmonic MT-preinvex functions for suitable and appropriate choice of the parameter λ. Interested readers are encouraged to derive these integral inequalities.

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Harmonijske MT-preinveksne funkcije i integralne nejednakosti

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Sažetak. U ovom članku uvodimo novu klasu harmonijskih preinveksnih funkcija, koje nazivamo harmonijske MT-preinveksne funkcije. Izvode se neke nove nejednakosti Hermite-Hadamardovog tipa za harmonijske MT-preinveksne funkcije. Diskutiraju se također neki specijalni slučajevi. Rezultati dokazani u ovom članku predstavljaju profinjenja i poboljšanja poznatih rezultata.

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