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GENERAL QUADRATURE FORMULAE BASED ON THE WEIGHTED MONTGOMERY IDENTITY AND RELATED INEQUALITIES

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Abstract. In this paper two families of general two-point and closed four-point weighted quadrature formulae are established. Obtained formulae are used to present several Hadamard type and Ostrowski type inequalities for α-L-Hölder functions. These results are applied to establish error estimates for the Gauss-Chebyshev quadratures.

1. Introduction

Let $f:[a,b]\to\mathbb{R}$ be a differentiable function on $[a,b]$ such that $f':[a,b]\to\mathbb{R}$ is integrable on $[a,b]$ and let $w:[a,b]\to[0,\infty\rangle$ be some probability density function. In [1] J. Pečarić proved a weighted generalization of the well known Montgomery identity (more about the Montgomery identity can be found for example in [2]):

$$f(x)=\int_a^b w(t)\,f(t)\,\mathrm{d}t+\int_a^b P_w(x,t)\,f'(t)\,\mathrm{d}t,$$

where the weighted Peano kernel is defined by

$$P_w(x,t)=\begin{cases} W(t), & a\le t\le x \\ W(t)-1, & x<t\le b \end{cases}.$$

This was used in the recent paper [3], where A. Aglić Aljinović and J. Pečarić introduced two new extensions of the weighted Montgomery identity.

In this paper we use one of those new weighted Montgomery identities to establish for each $x\in[a,(a+b)/2]$ a general two-point quadrature

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formula of the type

$$\int_a^b w(t)\, f(t)\, \mathrm{d}t = \frac{1}{2}\left[f(x) + f(a+b-x)\right] + R(f,w;x), \tag{1.1}$$

and also a general closed four-point quadrature formula of the type

$$\int_a^b w(t)\, f(t)\, \mathrm{d}t = \frac{1}{4}\left[f(a) + f(x) + f(a+b-x) + f(b)\right] + R(f,w;x), \tag{1.2}$$

with $R(f,w;x)$ being the reminder. In the special case $w(t) = \frac{1}{b-a}$, $t \in [a,b]$, (1.1) reduces to the family of two point quadrature formulae which were considered by Guessab and Schmeisser in [4]. Obtained two-point and four-point formulae are used to prove some Ostrowski-type and Hadamard-type inequalities for α-L-Hölder functions. At the end of the paper we show how these results can be applied to obtain some error estimates for Gauss-Chebyshev two-point quadrature rules.

Before we go further let us recall that a function $\varphi : [a,b] \to \mathbb{R}$ is said to be of α-L-Hölder type if $|\varphi(x) - \varphi(y)| \leq L|x-y|^{\alpha}$ for every $x, y \in [a,b]$, where $L > 0$ and $\alpha \in \langle 0, 1]$. We will also make use of the Beta function of Euler type which is for $x, y > 0$ defined by $B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\, \mathrm{d}t$.

2. General two-point quadrature formula

Let I be an open interval in $\mathbb{R}$, $[a,b] \subset I$ and let $f : I \to \mathbb{R}$ be such that $f^{(n-1)}$ is absolutely continuous for some $n \geq 2$. In the recent paper [3] the following extension of the Montgomery identity has been proved for each $x \in [a,b]$:

$$\begin{aligned} f(x) = {} & \int_a^b w(t)\, f(t)\, \mathrm{d}t + \sum_{i=0}^{n-1} \frac{f^{(i+1)}(a)}{i!} \int_a^x W(t)\,(t-a)^i\, \mathrm{d}t \\ & + \sum_{i=0}^{n-1} \frac{f^{(i+1)}(b)}{i!} \int_x^b (W(t)-1)\,(t-b)^i\, \mathrm{d}t \\ & + \frac{1}{(n-2)!}\left\{\int_a^x W(t)\left[\int_a^t \left(f^{(n)}(s) - f^{(n)}(a)\right)(t-s)^{n-2}\, \mathrm{d}s\right] \mathrm{d}t \right. \\ & \left. + \int_x^b (1-W(t))\left[\int_t^b \left(f^{(n)}(s) - f^{(n)}(b)\right)(t-s)^{n-2}\, \mathrm{d}s\right] \mathrm{d}t\right\} \end{aligned} \tag{2.1}$$

where $w : [a,b] \to [0, \infty\rangle$ is some probability density function.

In this section we use (2.1) to study for each number $x \in \left[a, \frac{a+b}{2}\right]$ the general two-point quadrature formula of the type (1.1).

Let $f:[a,b]\to\mathbb{R}$ be such that $f^{(n)}$ exists on $[a,b]$ for some $n\geq 2$. We introduce the following notation for each $x\in\left[a,\frac{a+b}{2}\right]$

$$D(x)=\frac{f(x)+f(a+b-x)}{2}.$$

Further, we define

$$\begin{aligned}
&t_n(x)\\
&=\frac{1}{2}\left\{\sum_{i=0}^{n-1}\frac{f^{(i+1)}(b)}{i!}\left[\int_x^b(1-W(t))(t-b)^i\,\mathrm{d}t\right.\right.\\
&\left.+\int_{a+b-x}^b(1-W(t))(t-b)^i\,\mathrm{d}t\right]\\
&\left.-\sum_{i=0}^{n-1}\frac{f^{(i+1)}(a)}{i!}\left[\int_a^x W(t)(t-a)^i\,\mathrm{d}t+\int_a^{a+b-x}W(t)(t-a)^i\,\mathrm{d}t\right]\right\}
\end{aligned}$$

and

$$T_n(x)=\frac{1}{2}\left[T_n^a(x)+T_n^b(x)+T_n^a(a+b-x)+T_n^b(a+b-x)\right],$$

where

$$\begin{aligned}
T_n^a(x)&=\frac{1}{(n-2)!}\int_a^x W(t)\left[\int_a^t\left(f^{(n)}(a)-f^{(n)}(s)\right)(t-s)^{n-2}\,\mathrm{d}s\right]\mathrm{d}t,\\
T_n^b(x)&\\
&=\frac{1}{(n-2)!}\int_x^b(1-W(t))\left[\int_t^b\left(f^{(n)}(b)-f^{(n)}(s)\right)(t-s)^{n-2}\,\mathrm{d}s\right]\mathrm{d}t.
\end{aligned}$$

In the next theorem we establish a general two-point quadrature formula based on the extended Montgomery identity which plays the key role in this section.

Theorem 1. *Let I be an open interval in $\mathbb{R}$, $[a,b]\subset I$, and let $w:[a,b]\to[0,\infty\rangle$ be some probability density function. Let $f:I\to\mathbb{R}$ be such that $f^{(n-1)}$ is absolutely continuous for some $n\geq 2$. Then for each $x\in\left[a,\frac{a+b}{2}\right]$ the following identity holds*

$$\int_a^b w(t)f(t)\,\mathrm{d}t=D(x)+t_n(x)+T_n(x) \tag{2.2}$$

Proof. We put $x\equiv x$ and then $x\equiv a+b-x$ in (2.1) to obtain two new formulae. After adding these two formulae and multiplying by $\frac{1}{2}$, we get (2.2). $\square$

Remark 1. *If in Theorem 1 we choose* $x = a, \frac{2a+b}{3}, \frac{3a+b}{4}, \frac{a+b}{2}$ *we obtain trapezoid, two-point Newton-Cotes, two-point MacLaurin and midpoint rule respectively.*

Theorem 2. *Suppose that all the assumptions of Theorem 1 hold and additionally assume that* $f^{(n)} : [a,b] \to \mathbb{R}$ *is an* α*-*L*-Hölder type function. Then for each* $x \in \left[a, \frac{a+b}{2}\right]$ *the following inequalities hold*

$$\begin{aligned}
&\left|\int_a^b w(t) f(t)\,\mathrm{d}t - D(x) - t_n(x)\right| \\
&\le \frac{B(\alpha+1, n-1)}{2(n-2)!} L \left[\int_a^b W(x,t) U_n(x,t)\,\mathrm{d}t\right. \\
&\left. + \int_a^b W(a+b-x,t) U_n(a+b-x,t)\,\mathrm{d}t\right] \\
&\le \frac{B(\alpha+1, n-1)}{(\alpha+n)(n-2)!} L \left[(x-a)^{\alpha+n} + (b-x)^{\alpha+n}\right],
\end{aligned}$$

where

$$W(x,t) = \begin{cases} W(t), & a \le t \le x, \\ 1 - W(t), & x < t \le b \end{cases},$$

$$U_n(x,t) = \begin{cases} (t-a)^{\alpha+n-1}, & a \le t \le x, \\ (b-t)^{\alpha+n-1}, & x < t \le b \end{cases}.$$

Proof. From (2.2) we have that

$$\begin{aligned}
&\left|\int_a^b w(t) f(t)\,\mathrm{d}t - D(x) - t_n(x)\right| \\
&\le \frac{1}{2}\left(\left|T_n^a(x)\right| + \left|T_n^a(a+b-x)\right| + \left|T_n^b(x)\right| + \left|T_n^b(a+b-x)\right|\right). \qquad (2.3)
\end{aligned}$$

Since $f^{(n)}$ is of α-L-Hölder type, from (2.3) we obtain

$$\begin{aligned}
&\left|\int_a^b w(t) f(t)\,\mathrm{d}t - D(x) - t_n(x)\right| \\
&\le \frac{L}{2(n-2)!}\left\{\int_a^x W(t)\left[\int_a^t (s-a)^\alpha (t-s)^{n-2}\,\mathrm{d}s\right]\mathrm{d}t\right.
\end{aligned}$$

$$+\int_a^{a+b-x} W(t)\left[\int_a^t (s-a)^\alpha (t-s)^{n-2}\,\mathrm{d}s\right]\mathrm{d}t$$
$$+\int_x^b (1-W(t))\left[\int_t^b (b-s)^\alpha (s-t)^{n-2}\,\mathrm{d}s\right]\mathrm{d}t$$
$$\left.+\int_{a+b-x}^b (1-W(t))\left[\int_t^b (b-s)^\alpha (s-t)^{n-2}\,\mathrm{d}s\right]\mathrm{d}t\right\}. \tag{2.4}$$

The first integral over $\mathrm{d}s$ in (2.4) can be written as

$$\int_a^t (s-a)^\alpha (t-s)^{n-2}\,\mathrm{d}s$$
$$=(t-a)^{\alpha+n-2}\int_a^t \left(\frac{s-a}{t-a}\right)^\alpha \left(\frac{t-s}{t-a}\right)^{n-2}\mathrm{d}s$$
$$=(t-a)^{\alpha+n-1}\int_0^1 u^\alpha (1-u)^{n-2}\,\mathrm{d}u=(t-a)^{\alpha+n-1}B(\alpha+1,n-1).$$

Similarly can be done with other integrals in (2.4), so we obtain

$$\left|\int_a^b w(t)f(t)\,\mathrm{d}t-D(x)-t_n(x)\right|$$
$$\leq\frac{B(\alpha+1,n-1)}{2(n-2)!}L\left[\int_a^b W(x,t)U_n(x,t)\,\mathrm{d}t\right. \tag{2.5}$$
$$\left.+\int_a^b W(a+b-x,t)U_n(a+b-x,t)\,\mathrm{d}t\right].$$

Since we have $0\leq W(t)\leq 1,\ t\in[a,b]$, from (2.5) we obtain

$$\frac{B(\alpha+1,n-1)}{2(n-2)!}L\left[\int_a^b W(x,t)U_n(x,t)\,\mathrm{d}t\right.$$
$$\left.+\int_a^b W(a+b-x,t)U_n(a+b-x,t)\,\mathrm{d}t\right]$$
$$\leq\frac{B(\alpha+1,n-1)}{(\alpha+n)(n-2)!}L\left[(x-a)^{\alpha+n}+(b-x)^{\alpha+n}\right],$$

which completes the proof. □

Corollary 1. *Let I be an open interval in $\mathbb{R}$, $[a,b]\subset I$, and let $f:I\to\mathbb{R}$ be such that $f^{(n-1)}$ is absolutely continuous and that $f^{(n)}:[a,b]\to\mathbb{R}$ is an α-L-Hölder type function for some $n\geq 2$. Then for each $x\in\left[a,\frac{a+b}{2}\right]$*

the following inequality holds

$$\begin{aligned}&\left|\frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t - D(x) - \widehat{t}_n(x)\right| \\ &\leq \frac{B(\alpha+1,n-1)}{(b-a)(\alpha+n+1)(n-2)!}L\left[(x-a)^{\alpha+n+1}+(b-x)^{\alpha+n+1}\right],\end{aligned}$$

where

$$\widehat{t}_n(x)=\frac{1}{2}\sum_{i=0}^{n-1}\left[(-1)^i f^{(i+1)}(b)-f^{(i+1)}(a)\right]\frac{(x-a)^{i+2}+(b-x)^{i+2}}{i!\,(i+2)(b-a)}$$

Proof. This is a special case of Theorem 2 for $w(t)=\frac{1}{b-a}$, $t\in[a,b]$. □

Theorem 3. *Suppose that all the assumptions of Theorem 1 hold for some $n=2k-1$, $k\geq 2$. If f is $(2k)$-convex, then for each $x\in\left[a,\frac{a+b}{2}\right]$ the following inequality holds*

$$\int_a^b w(t)f(t)\,\mathrm{d}t-\frac{f(x)+f(a+b-x)}{2}-t_n(x)\leq 0. \tag{2.6}$$

If f is $(2k)$-concave, then the inequality (2.6) is reversed.

Proof. First note that if f is $(2k)$-convex the derivative $f^{(2k-1)}=f^{(n)}$ is nondecreasing, and if f is $(2k)$-concave the derivative $f^{(n)}$ is nonincreasing.

From (2.2) we have that

$$\begin{aligned}&\int_a^b w(t)f(t)\,\mathrm{d}t-D(x)-t_n(x)\\ &=\frac{1}{2}\left[T_n^a(x)+T_n^a(a+b-x)+T_n^b(x)+T_n^b(a+b-x)\right].\end{aligned}$$

Let us consider the sign of the integral

$$T_n^a(x)=\frac{1}{(n-2)!}\int_a^x W(t)\left[\int_a^t\left(f^{(n)}(a)-f^{(n)}(s)\right)(t-s)^{n-2}\,\mathrm{d}s\right]\mathrm{d}t$$

when $f^{(n)}$ is nondecreasing. We have

$$\int_a^t\left(f^{(n)}(a)-f^{(n)}(s)\right)(t-s)^{n-2}\,\mathrm{d}s\leq 0,$$

hence we may conclude that $T_n^a(x)\leq 0$. Analogously we obtain $T_n^a(a+b-x)\leq 0$. On the other hand, the sign of the integral

$$T_n^b(x)$$
$$= \frac{1}{(n-2)!}\int_x^b (1-W(t))\left[\int_t^b \left(f^{(n)}(b)-f^{(n)}(s)\right)(t-s)^{n-2}\,\mathrm{d}s\right]\mathrm{d}t$$

depends on the parity of n : if n is odd, that is $n = 2k-1$, then $T_n^b(x) \leq 0$ and analogously $T_n^b(a+b-x) \leq 0$. Hence, if $n = 2k-1$ and $f^{(n)}$ is nondecreasing we have

$$\int_a^b w(t) f(t)\,\mathrm{d}t - D(x) - t_n(x) \leq 0.$$

The reversed inequality in (2.6) can be obtained analogously. □

Corollary 2. *Let I be an open interval in $\mathbb{R}$, $[a,b] \subset I$, and let $f : I \to \mathbb{R}$ be such that $f^{(n-1)}$ is absolutely continuous for some $n = 2k-1$, $k \geq 2$. If f is $(2k)$-convex, then for each $x \in \left[a, \frac{a+b}{2}\right]$ the following inequality holds*

$$\frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t - \frac{f(x)+f(a+b-x)}{2} - \widehat{t}_n(x) \leq 0. \tag{2.7}$$

If f is $(2k)$-concave, then the inequality (2.7) is reversed.

Proof. This is a special case of Theorem 3 for $w(t) = \frac{1}{b-a}$, $t \in [a,b]$. □

3. General four-point quadrature formula

In this section we use (2.1) to study for each number $x \in \left[a, \frac{a+b}{2}\right]$ the general four-point quadrature formula of the type (1.2) . For $x \in \left[a, \frac{a+b}{2}\right]$ and $f,\ w,\ D(x)\ ,\ T_n(x),\ t_n(x)$ as in Section 2 we introduce the following notation:

$$\widetilde{D}(x) = \frac{D(x)+D(a)}{2} = \frac{f(a)+f(x)+f(a+b-x)+f(b)}{4},$$
$$\widetilde{T}_n(x) = \frac{T_n(x)+T_n(a)}{2}, \qquad \widetilde{t}_n(x) = \frac{t_n(x)+t_n(a)}{2}.$$

Theorem 4. *Suppose that all the assumptions of Theorem 1 hold. Then for each $x \in \left[a, \frac{a+b}{2}\right]$ the following identity holds*

$$\int_a^b w(t) f(t)\,\mathrm{d}t = \widetilde{D}(x) + \widetilde{t}_n(x) + \widetilde{T}_n(x). \tag{3.1}$$

Proof. We put $x \equiv x$, $x \equiv a+b-x$, $x \equiv a$ and $x \equiv b$ in (2.1) to obtain four new formulae. After adding these four formulae and multiplying by $\frac{1}{4}$, we obtain (3.1). □

Theorem 5. *Suppose that all the assumptions of Theorem 2 hold. Then for each $x \in \left[a, \frac{a+b}{2}\right]$ the following inequalities hold*

$$
\begin{aligned}
&\left|\int_a^b w(t) f(t)\,\mathrm{d}t - \widetilde{D}(x) - \widetilde{t}_n(x)\right| \\
&\leq \frac{B(\alpha+1, n-1)}{4(n-2)!} L \left[\int_a^b W(x,t) U_n(x,t)\,\mathrm{d}t + \int_a^b W(a,t) U_n(a,t)\,\mathrm{d}t\right. \\
&\left. + \int_a^b W(a+b-x,t) U_n(a+b-x,t)\,\mathrm{d}t + \int_a^b W(b,t) U_n(b,t)\,\mathrm{d}t\right] \\
&\leq \frac{B(\alpha+1, n-1)}{2(\alpha+n)(n-2)!} L \left[(x-a)^{\alpha+n} + (b-x)^{\alpha+n} + (b-a)^{\alpha+n}\right].
\end{aligned}
$$

Proof. Similarly as in Theorem 2. □

Theorem 6. *Suppose that all the assumptions of Theorem 1 hold for some $n = 2k-1,\ k \geq 2$. If f is $(2k)$-convex, then for each $x \in \left[a, \frac{a+b}{2}\right]$ the following inequality holds*

$$
\begin{aligned}
&\int_a^b w(t) f(t)\,\mathrm{d}t - \frac{f(x) + f(a+b-x)}{2} - t_n(x) \\
&\leq \frac{f(a) + f(b)}{2} - \int_a^b w(t) f(t)\,\mathrm{d}t + t_n(a). \qquad (3.2)
\end{aligned}
$$

If f is $(2k)$-concave, then the inequality (3.2) is reversed.

Proof. Similarly as in Theorem 3 starting from (3.1). □

Corollary 3. *Let I be an open interval in $\mathbb{R}$, $[a,b] \subset I$, and let $f : I \to \mathbb{R}$ be such that $f^{(n-1)}$ is absolutely continuous for some $n = 2k-1,\ k \geq 2$. If f is $(2k)$-convex, then for each $x \in \left[a, \frac{a+b}{2}\right]$ the following inequality holds*

$$
\begin{aligned}
&\frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t - \frac{f(x) + f(a+b-x)}{2} - \widehat{t}_n(x) \\
&\leq \frac{f(a) + f(b)}{2} - \frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t + \widehat{t}_n(a). \qquad (3.3)
\end{aligned}
$$

If f is $(2k)$-concave, then the inequality (3.3) is reversed.

Proof. This is a special case of Theorem 6 for $w(t) = \frac{1}{b-a},\ t \in [a,b]$. □

Remark 2. *Inequalities*

$$
\frac{f(a) + f(b)}{2} - \frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t \geq \frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t - f\left(\frac{a+b}{2}\right) \geq 0,
$$

which hold for any convex function f defined on $[a,b]$, were first proved by Bullen in [5]*. His results were generalized for $(2k)$-convex functions ($k \in \mathbb{N}$) in* [6]*. Further generalizations for $(2k)$-convex functions ($k \in \mathbb{N}$) and $x \in \left[a, \frac{a+b}{2} - \frac{b-a}{4\sqrt{6}}\right] \cup \left\{\frac{a+b}{2}\right\}$ (of the same type as in Corollary 3) were obtained in* [7]*.*

4. Applications

Gaussian quadrature rules are formulae of the following type

$$\int_a^b \varpi(t) f(t)\,\mathrm{d}t \approx \sum_{i=1}^{k} A_i f(x_i), \quad k \in \mathbb{N}. \tag{4.1}$$

Without any loss of generality we may restrict ourselves to the special case $[a,b] = [-1,1]$. Further, if in (4.1) the function ϖ is defined by

$$\varpi(t) = \frac{1}{\sqrt{1-t^2}}, \quad t \in (-1,1)$$

we obtain **Gauss-Chebyshev quadrature rule** of the first kind. In this case

$$\int_{-1}^{1} \frac{1}{\sqrt{1-t^2}} f(t)\,\mathrm{d}t \approx \sum_{i=1}^{k} A_i f(x_i), \tag{4.2}$$

where the weights A_i are defined by $A_i = \pi/k$, $i = 1, \ldots, k$ and x_i are zeros of the Chebyshev polynomials of the first kind defined by $C_k(x) = \cos(k \arccos(x))$. Each $C_k(x)$ has exactly k distinct zeros

$$x_i = \cos\left(\frac{(2i-1)\pi}{2k}\right)$$

all of which lie in the interval $(-1,1)$ (see for instance [8]).

Error of the approximation formula (4.2) is

$$E_k(f) = \frac{\pi}{2^{2k-1}(2k)!} f^{(2k)}(\xi), \quad \xi \in (-1,1).$$

In case $k = 2$ (4.1) reduces to

$$\begin{aligned}
&\int_{-1}^{1} \frac{1}{\sqrt{1-t^2}} f(t)\,\mathrm{d}t \\
&= \frac{\pi}{2}\left[f\left(-\frac{\sqrt{2}}{2}\right) + f\left(\frac{\sqrt{2}}{2}\right)\right] + \frac{\pi}{192} f^{(4)}(\xi), \quad \xi \in (-1,1).
\end{aligned}$$

Next we show how to apply the results of Section 2 to obtain some error estimates for Gauss-Chebyshev quadrature rules of the first kind involving α-L-Hölder type functions.

Theorem 7. *Let I be an open interval in $\mathbb{R}$, $[-1,1] \subset I$, and let $f : I \to \mathbb{R}$ be such that for some $n \geq 2$ the derivative $f^{(n-1)}$ is absolutely continuous and $f^{(n)}$ is an α-L-Hölder function. Then*

$$\left| \int_{-1}^{1} \frac{1}{\pi\sqrt{1-t^2}} f(t)\,\mathrm{d}t - \frac{1}{2}\left[f\left(-\frac{\sqrt{2}}{2}\right) + f\left(\frac{\sqrt{2}}{2}\right)\right] - t_n\left(-\frac{\sqrt{2}}{2}\right)\right|$$
$$\leq \frac{B(\alpha+1,n-1)}{(\alpha+n)(n-2)!} L\left[\left(1-\frac{\sqrt{2}}{2}\right)^{\alpha+n} + \left(1+\frac{\sqrt{2}}{2}\right)^{\alpha+n}\right],$$

where t_n is defined as in Section 2 and $W(t) = \frac{1}{\pi}\left(\arcsin t + \frac{\pi}{2}\right)$.

Proof. This is a special case of Theorem 2 for $[a,b] = [-1,1]$, $x = -\sqrt{2}/2$ and

$$w(t) = \frac{1}{\pi\sqrt{1-t^2}}, \quad t \in (-1,1).$$

□

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