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Second order parameter-uniform numerical method for a partially singularly perturbed linear system of reaction-diffusion type

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Abstract. A partially singularly perturbed linear system of second order ordinary differential equations of reaction-diffusion type with given boundary conditions is considered. The leading terms of first m equations are multiplied by small positive singular perturbation parameters which are assumed to be distinct. The rest of the equations are not singularly perturbed. The first m components of the solution exhibit overlapping layers and the remaining $n-m$ components have less-severe overlapping layers. Shishkin piecewise-uniform meshes are used in conjunction with a classical finite difference discretisation, to construct a numerical method for solving this problem. It is proved that the numerical approximation obtained by this method is essentially second order convergent uniformly with respect to all the parameters. Numerical illustrations are presented in support of the theory.
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1. Introduction

The following two-point boundary value problem is considered for the partially singularly perturbed linear system of second order differential equations

$$-E\vec{u}''(x) + A(x)\vec{u}(x) = \vec{f}(x) \text{ on } \Omega, \vec{u} \text{ given on } \Gamma, \tag{1}$$

where $\Omega = \{x : 0 < x < 1\}$, $\overline{\Omega} = \Omega \cup \Gamma, \Gamma = \{0, 1\}$. Here, for all $x \in \overline{\Omega}, \vec{u}(x)$ and $\vec{f}(x)$ are column $n-$ vectors , E and $A(x)$ are $n \times n$ matrices, $E =$ diag $(\vec{\varepsilon}), \vec{\varepsilon} = (\varepsilon_1, \ldots, \varepsilon_n)$. The parameters $\varepsilon_i, i = 1, \ldots, m, m < n$, are assumed to be distinct and, for convenience, the ordering $0 < \varepsilon_1 < \cdots < \varepsilon_m < \varepsilon_{m+1} = \cdots = \varepsilon_n = 1$ is assumed. Clearly there are at most m singularly perturbed equations. The problem can also be written in the operator form $\vec{L}\vec{u} = \vec{f}$ on $\Omega, \vec{u}$ given on Γ, where the operator $\vec{L}$ is

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defined by $\vec{L} = -ED^2 + A$ and $D^2 = d^2/dx^2$.
For all $x \in \overline{\Omega}$, it is assumed that the components $a_{ij}(x)$ of $A(x)$ satisfy the inequalities

$$a_{ii}(x) > \sum_{\substack{j\neq i\\ j=1}}^{n} |a_{ij}(x)|, \text{ for } 1 \le i \le n \text{ and } a_{ij}(x) \le 0, \text{ for } i \ne j \tag{2}$$

and, for some α

$$0 < \alpha < \min_{\substack{x\in\overline{\Omega}\\ 1\le i\le n}} (\sum_{j=1}^{n} a_{ij}(x)). \tag{3}$$

It is assumed that $a_{ij}, f_i \in C^{(2)}(\overline{\Omega})$ for $i,j = 1,\dots,n$. Then (1) has a solution $\vec{u} \in C(\overline{\Omega}) \cap C^{(4)}(\Omega)$. It is also assumed that $\sqrt{\varepsilon_m} \le \sqrt{\alpha}/6$.

The norms $\parallel \vec{V} \parallel = \max_{1\le k\le n} |V_k|$ for any n-vector $\vec{V}$, $\parallel y \parallel = \sup_{0\le x\le 1} |y(x)|$ for any scalar-valued function y and $\parallel \vec{y} \parallel = \max_{1\le k\le n} \parallel y_k \parallel$ for any vector-valued function $\vec{y}$ are introduced. Throughout the paper C denotes a generic positive constant, which is independent of x and of all singular perturbation and discretization parameters. Furthermore, inequalities between vectors are understood in the componentwise sense.
In [8], the authors have suggested a parameter-uniform numerical method to solve a system of n singularly perturbed ordinary differential equations of second order with given boundary conditions. Motivated by [6] and [8], we have suggested a similar method to solve the problem (1). Significant contributions to the development of the techniques used here may be found in [1, 3, 4, 5, 6].

2. Standard analytical results

The operator $\vec{L}$ satisfies the following maximum principle.

Lemma 1. *Let $A(x)$ satisfy (2) and (3). Let $\vec{\psi}$ be any vector-valued function in the domain of $\vec{L}$ such that $\vec{\psi} \ge \vec{0}$ on Γ. Then $\vec{L}\vec{\psi}(x) \ge \vec{0}$ on Ω implies that $\vec{\psi}(x) \ge \vec{0}$ on $\overline{\Omega}$.*

Proof. Let i^*, x^* be such that $\psi_{i^*}(x^*) = \min_{i,x} \psi_i(x)$ and assume that the lemma is false. Then $\psi_{i^*}(x^*) < 0$. From the hypotheses we have $x^* \notin \Gamma$ and $\psi''_{i^*}(x^*) \ge 0$. Thus

$$(\vec{L}\vec{\psi})_{i^*}(x^*) = -\varepsilon_{i^*}\psi''_{i^*}(x^*) + \sum_{j=1}^{n} a_{i^*,j}(x^*)\psi_j(x^*) < 0,$$

which contradicts the assumption and proves the result for $\vec{L}$. □

Let $\tilde{A}(x)$ be any principal sub-matrix of $A(x)$ and $\tilde{\vec{L}}$ the corresponding operator. To see that any $\tilde{\vec{L}}$ satisfies the same maximum principle as $\vec{L}$, it suffices to observe that the elements of $\tilde{A}(x)$ satisfy *a fortiori* the same inequalities as those of $A(x)$.

Lemma 2. *Let $A(x)$ satisfy (2) and (3). If $\vec{\psi}$ is any vector-valued function in the domain of $\vec{L}$, then, for each $i, 1 \le i \le n$ and $x \in \overline{\Omega}$,*

$$|\psi_i(x)| \le \max\{\| \vec{\psi} \|_\Gamma, (1/\alpha) \| \vec{L}\vec{\psi} \|\}.$$

Proof. Define the two functions

$$\vec{\theta}^{\pm}(x) = \max\left\{\| \vec{\psi} \|_\Gamma, \frac{1}{\alpha} \| \vec{L}\vec{\psi} \|\right\} \vec{e} \pm \vec{\psi}(x)$$

where $\vec{e} = (1, \ldots, 1)^T$ is the unit column $n-$vector. Using the properties of A it is not hard to verify that $\vec{\theta}^{\pm} \ge \vec{0}$ on Γ and $\vec{L}\vec{\theta}^{\pm} \ge \vec{0}$ on Ω. It follows from Lemma 1 that $\vec{\theta}^{\pm} \ge \vec{0}$ on $\overline{\Omega}$ as required. □

Standard estimates of the exact solution and its derivatives are contained in the following lemma.

Lemma 3. *Let $A(x)$ satisfy (2) and (3) and let $\vec{u}$ be the exact solution of (1). Then, for all $x \in \overline{\Omega}$ and each $i = 1, \ldots, n$,*

$$|u_i(x)| \le C(||\vec{u}||_\Gamma + ||\vec{f}||),$$
$$|u_i^{(k)}(x)| \le C\varepsilon_i^{-\frac{k}{2}}(||\vec{u}|| + ||\vec{f}||), \quad \text{for } k = 1, 2,$$

and

$$|u_i^{(k)}(x)| \le C\varepsilon_1^{-(k-2)/2}\varepsilon_i^{-1}(||\vec{u}|| + ||\vec{f}|| + \varepsilon_1^{(k-2)/2}||\vec{f}^{(k-2)}||), \quad \text{for } k = 3, 4.$$

Proof. The bound on $\vec{u}$ is an immediate consequence of Lemma 2 and the differential equation. Rewriting the differential equation (1) gives $\vec{u}'' = E^{-1}(A\vec{u} - \vec{f})$ and it is not hard to see that the bounds on u_i'' follow.

The bound of $u_i'(x)$, for $i = 1, \ldots, m$, can be derived as in [5, 8]. To bound $u_i'(x)$, for $i = m+1, \ldots, n$ and any x, consider an interval $N_x = [a, a+t]$ where $a \ge 0$ and $0 \le t \le 1-a$ such that $x \in N_x$. Then, by the mean value theorem, for some $y \in N_x$,

$$u_i'(y) = \frac{u_i(a+t) - u_i(a)}{t}$$

and it follows that

$$|u_i'(y)| \le \frac{2}{t}||u_i||.$$

Now

$$u_i'(x) = u_i'(y) + \int_y^x u_i''(s)ds = u_i'(y) + \varepsilon_i^{-1}\int_y^x (-f_i(s) + \sum_{j=1}^n a_{ij}(s)u_j(s))ds$$

and so, for $i = m+1, \ldots, n$,

$$\begin{aligned} |u_i'(x)| &\le |u_i'(y)| + C(\| f_i \| + \| \vec{u} \|)\int_y^x ds \\ &\le \frac{2}{t}||u_i|| + Ct(\| f_i \| + \| \vec{u} \|) \end{aligned}$$

from which the required bound follows. Differentiating (1) once and twice give

$$\vec{u}^{(3)} = E^{-1}(A\vec{u}' + A'\vec{u} - \vec{f}'),$$
$$\vec{u}^{(4)} = E^{-1}(A\vec{u}'' + 2A'\vec{u}' + A''\vec{u} - \vec{f}'')$$

and the bounds on $u_i^{(3)}$, $u_i^{(4)}$ follow from those on u_i' and u_i''. □

The reduced solution $u_{0,i}, i = 1,\dots,n$, of (1) is the solution of the reduced problem

$$\begin{aligned} &\textstyle\sum_{j=1}^n a_{ij}(x)u_{0,j}(x) = f_i(x), i = 1,\dots,m, \\ &\left.\begin{aligned} -u_{0,i}''(x) + \textstyle\sum_{j=1}^n a_{ij}(x)u_{0,j}(x) &= f_i(x), \\ u_{0,i} &= u_i \text{ on } \Gamma, \end{aligned}\right\} i = m+1,\dots,n. \end{aligned} \tag{4}$$

The Shishkin decomposition of the exact solution $\vec{u}$ of (1) is $\vec{u} = \vec{v} + \vec{w}$ where the smooth component $\vec{v}$ is the solution of $\vec{L}\vec{v} = \vec{f}$ in $\Omega, \vec{v} = \vec{u}_0$ on Γ and the singular component $\vec{w}$ is the solution of $\vec{L}\vec{w} = \vec{0}$ in $\Omega, \vec{w} = \vec{u} - \vec{v}$ on Γ. For convenience, the left and right boundary layers of $\vec{w}$ are separated using the further decomposition $\vec{w} = \vec{w}^L + \vec{w}^R$ where $\vec{L}\vec{w}^L = \vec{0}$ on $\Omega, \vec{w}^L(0) = \vec{u}(0) - \vec{v}(0), \vec{w}^L(1) = \vec{0}$ and $\vec{L}\vec{w}^R = \vec{0}$ on $\Omega, \vec{w}^R(0) = \vec{0}, \vec{w}^R(1) = \vec{u}(1) - \vec{v}(1)$.
Bounds on the smooth component and its derivatives are contained in

Lemma 4. *Let $A(x)$ satisfy (2) and (3). Then the smooth component $\vec{v}$ and its derivatives satisfy, for each $x \in \overline{\Omega}$ and $i = 1,\dots,n$,*

$$|v_i^{(k)}(x)| \le C, \quad \text{for } k = 0,1,2$$

and

$$|v_i^{(k)}(x)| \le C(1 + \varepsilon_i^{1-k/2}) \quad \text{for } k = 3,4.$$

Proof. The bound on $\vec{v}$ is an immediate consequence of the defining equations for $\vec{v}$ and Lemma 2. Differentiating twice the equation for $\vec{v}$, it is not hard to see that $\vec{v}''$ satisfies

$$\vec{L}\vec{v}'' = \vec{g}, \text{ where } \vec{g} = \vec{f}'' - A''\vec{v} - 2A'\vec{v}'. \tag{5}$$

Also the defining equations for $\vec{v}$ yield $v_i'' = 0$ on Γ for $i = 1,\dots,m$ and $v_i''(0) = s_i^0, v_i''(1) = s_i^1$ for $i = m+1,\dots,n$ where s_i^0 and s_i^1 are definite constants for each $i = m+1,\dots,n$. Using the same arguments as in [8], the estimates of $v_i^{(k)}(x)$, $k = 1,2,3$ and 4 follow. □

3. Improved estimates

The layer functions $B_i^L, B_i^R, B_i, i = 1,\dots,m,$, associated with the solution $\vec{u}$, are defined on $\overline{\Omega}$ by

$$B_i^L(x) = e^{-x\sqrt{\alpha/\varepsilon_i}}, B_i^R(x) = B_i^L(1-x), B_i(x) = B_i^L(x) + B_i^R(x).$$

The following elementary properties of these layer functions, for all $1 \leq i < j \leq m$ and $0 \leq x < y \leq 1$, should be noted:

$$B_i(x) = B_i(1-x).$$
$$B_i^L(x) < B_j^L(x), B_i^L(x) > B_i^L(y), 0 < B_i^L(x) \leq 1.$$
$$B_i^R(x) < B_j^R(x), B_i^R(x) < B_i^R(y), 0 < B_i^R(x) \leq 1. \tag{6}$$
$$B_i(x) \text{ is monotone decreasing for increasing } x \in [0, \frac{1}{2}]. \tag{7}$$
$$B_i(x) \text{ is monotone increasing for increasing } x \in [\frac{1}{2}, 1]. \tag{8}$$
$$B_i(x) \leq 2B_i^L(x) \text{ for } x \in [0, \frac{1}{2}], B_i(x) \leq 2B_i^R(x) \text{ for } x \in [\frac{1}{2}, 1].$$
$$B_i^L(2\frac{\sqrt{\varepsilon_i}}{\sqrt{\alpha}} \ln N) = N^{-2}. \tag{9}$$

The interesting points $x_{i,j}^{(s)}$ are now defined.

Definition 1. *For B_i^L, B_j^L, each $i, j, 1 \leq i \neq j \leq m$ and each $s, s > 0$, the point $x_{i,j}^{(s)}$ is defined by*

$$\frac{B_i^L(x_{i,j}^{(s)})}{\varepsilon_i^s} = \frac{B_j^L(x_{i,j}^{(s)})}{\varepsilon_j^s}. \tag{10}$$

It is remarked that

$$\frac{B_i^R(1-x_{i,j}^{(s)})}{\varepsilon_i^s} = \frac{B_j^R(1-x_{i,j}^{(s)})}{\varepsilon_j^s}. \tag{11}$$

In the next lemma the existence and uniqueness of the points $x_{i,j}^{(s)}$ are shown. Various properties are also established.

Lemma 5. *For all i, j, such that $1 \leq i < j \leq m$ and $0 < s \leq 3/2$, the points $x_{i,j}^{(s)}$ exist, are uniquely defined and satisfy the following inequalities*

$$\frac{B_i^L(x)}{\varepsilon_i^s} > \frac{B_j^L(x)}{\varepsilon_j^s}, x \in [0, x_{i,j}^{(s)}), \quad \frac{B_i^L(x)}{\varepsilon_i^s} < \frac{B_j^L(x)}{\varepsilon_j^s}, x \in (x_{i,j}^{(s)}, 1]. \tag{12}$$

Moreover,

$$x_{i,j}^{(s)} < x_{i+1,j}^{(s)}, \text{ if } i+1 < j \text{ and } x_{i,j}^{(s)} < x_{i,j+1}^{(s)}, \text{ if } i < j. \tag{13}$$

Also

$$x_{i,j}^{(s)} < 2s\frac{\sqrt{\varepsilon_j}}{\sqrt{\alpha}} \text{ and } x_{i,j}^{(s)} \in (0, \frac{1}{2}) \text{ if } i < j. \tag{14}$$

Analogous results hold for the B_i^R, B_j^R and the points $1 - x_{i,j}^{(s)}$.

Proof. The proof is similar to that in [8]. □

Bounds on the singular components $\vec{w}^L, \vec{w}^R$ of $\vec{u}$ and their derivatives are contained in

Lemma 6. *Let $A(x)$ satisfy (2) and (3). Then there exists a constant C, such that, for each $x \in \overline{\Omega}$ and $i = 1, \ldots, m$,*

$$\begin{aligned} |w_i^L(x)| &\leq C_1 B_m^L(x) + C_2\varepsilon_m(1 - B_m^L(x)), \\ |w_i^{L,(k)}(x)| &\leq C \sum_{q=i}^{m} (B_q^L(x)/\varepsilon_q^{k/2}), \quad k = 1, 2, \\ |w_i^{L,(3)}(x)| &\leq C \sum_{q=1}^{m} (B_q^L(x)/\varepsilon_q^{3/2}), \\ |\varepsilon_i w_i^{L,(4)}(x)| &\leq C \sum_{q=1}^{m} (B_q^L(x)/\varepsilon_q) \end{aligned}$$

and for $i = m + 1, \ldots, n$,

$$\begin{aligned} |w_i^L(x)| &\leq C_2\varepsilon_m(1 - B_m^L(x)), \\ |w_i^{L,(k)}(x)| &\leq C_1 B_m^L(x) + C_2\varepsilon_m(1 - B_m^L(x)), \quad k = 1, 2, \\ |w_i^{L,(k)}(x)| &\leq C \sum_{q=1}^{m} (B_q^L(x)/\varepsilon_q^{(k-2)/2}), \quad k = 3, 4. \end{aligned}$$

Analogous results hold for w_i^R and their derivatives.

Proof. The lemma is proved by induction. The initial case when $n = 2$ and $m = 1$ is dealt with in [6]. Assume the lemma to be true for a partially singularly perturbed system of $n-1$ equations where $m, 1 \leq m \leq n-2$, equations are singularly perturbed.

Now consider $\vec{L}\vec{w}^L = \vec{0}$ on Ω, where $\vec{w}^L$ is an $n-$vector and m equations where $0 \leq m \leq n - 1$ are singularly perturbed. Inspired by [6], we define

$$\theta_i^{\pm}(x) = C_1 B_m^L(x) + C_2\varepsilon_m(1 - B_m^L(x)) \pm w_i^L(x), \quad i = 1, \ldots, m$$

and

$$\theta_i^{\pm}(x) = C_2\varepsilon_m(1 - B_m^L(x)) \pm w_i^L(x), \quad i = m + 1, \ldots, n.$$

We have m singular perturbation parameters unlike in [6] where there is only one parameter and we choose the largest parameter and use it in the appropriate definition. Using Lemma 1 for $\vec{\theta}^{\pm}$, we have

$$\left|w_i^L(x)\right| \leq C_1 B_m^L(x) + C_2\varepsilon_m(1 - B_m^L(x)) \text{ for } i = 1, \ldots, m$$

and

$$\left|w_i^L(x)\right| \leq C_2\varepsilon_m(1 - B_m^L(x)) \text{ for } i = m + 1, \ldots, n,$$

with a suitable choice of the constants C_1 and C_2.

Using arguments analogous to those used to bound $u_i'(x), i = m + 1, \ldots, n$, in Lemma 3, it can be proved that

$$|w_n^{L,\prime}(x)| \leq C_1 B_m^L(x) + C_2\varepsilon_m(1 - B_m^L(x)). \tag{15}$$

To bound the second order derivative $w_n^{L,\prime\prime}$, we consider

$$-w_n^{L,\prime\prime}(x) + \sum_{j=1}^{n} a_{nj}(x) w_j^L(x) = 0 \tag{16}$$

which implies that

$$|w_n^{L,\prime\prime}(x)| \le C_1 B_m^L(x) + C_2 \varepsilon_m (1 - B_m^L(x)). \tag{17}$$

Differentiating (16) once and twice and using the bounds of $w_i^L(x), i = 1, \ldots, n$, (15), (17) and the induction hypothesis for the bounds of $w_i^{L,(k)}(x), i = 1, \ldots, n-1$ and $k = 1, 2$, we have

$$|w_n^{L,(k)}(x)| \le C \sum_{q=1}^{m} (B_q^L(x)/\varepsilon_q^{(k-2)/2}), \quad k = 3, 4.$$

Consider the first $n-1$ equations satisfied by $\vec{w}^L$, it follows that $-\tilde{E}\vec{\tilde{w}}^{L,\prime\prime} + \tilde{A}\vec{\tilde{w}}^L = \vec{g}$, where $\tilde{E}, \tilde{A}$ are the matrices obtained by deleting the last row and column from E, A respectively, $\vec{\tilde{w}}^L = (w_1^L, \ldots, w_{n-1}^L)$ and the components of $\vec{g}$ are $g_i = -a_{i,n} w_n^L$ for $1 \le i \le n-1$.

Using the bounds already obtained for $w_n^{L,(k)}, k = 0, \ldots, 4$, it is seen that $\vec{g}$ is bounded by $C_2 \varepsilon_m (1 - B_m^L(x))$, $\vec{g}'$ and $\vec{g}''$ by $C_1 B_m^L(x) + C_2 \varepsilon_m (1 - B_m^L(x))$ and $\vec{g}^{(k)}$ by $C \sum_{q=1}^{m} (B_q^L(x)/\varepsilon_q^{(k-2)/2}), k = 3, 4$.

The boundary conditions for $\vec{\tilde{w}}^L$ are $\vec{\tilde{w}}^L(0) = \vec{\tilde{u}}(0) - \vec{\tilde{u}}_0(0)$, $\vec{\tilde{w}}^L(1) = \vec{\tilde{0}}$, where $\vec{\tilde{u}}_0$ is the solution of the reduced problem (4) and are bounded by $C(\| \vec{u}(0) \| + \| \vec{f}(0) \|)$ and $C(\| \vec{u}(1) \| + \| \vec{f}(1) \|)$.

Now decompose $\vec{\tilde{w}}^L$ into smooth and singular components to get $\vec{\tilde{w}}^L = \vec{z} + \vec{r}, \vec{\tilde{w}}^{L,\prime} = \vec{z}' + \vec{r}'$. Applying Lemma 4 to $\vec{z}$ and using the bounds on the inhomogeneous term $\vec{g}$ and its derivatives $\vec{g}^{(k)}, k = 1, \ldots, 4$, it follows that

$$|\vec{z}^{(k)}(x)| \le C_1 B_m^L(x) + C_2 \varepsilon_m (1 - B_m^L(x)), \quad k = 1, 2$$

and

$$|\vec{z}^{(k)}(x)| \le C \sum_{q=1}^{m} (B_q^L(x)/\varepsilon_q^{(k-2)/2}), \quad k = 3, 4.$$

As $\vec{r}$ is the singular component of the solution of a system of $n-1$ equations, consider the following two cases:

Case 1: All the $n-1$ equations are singularly perturbed:

Then from Lemma 7 of [8], the estimates are obtained as, for $i = 1, \ldots, n-1$,

$$|r_i^{(k)}(x)| \le C \sum_{q=i}^{n-1} (B_q^L(x)/\varepsilon_q^{k/2}), \quad k = 1, 2,$$

$$|r_i^{(3)}(x)| \le C \sum_{q=1}^{n-1} (B_q^L(x)/\varepsilon_q^{3/2}),$$

$$|\varepsilon_i r_i^{(4)}(x)| \le C \sum_{q=1}^{n-1} (B_q^L(x)/\varepsilon_q).$$

Case 2: The $n-1$ equations are partially singularly perturbed:
Then by induction on $\vec{r}$, the estimates are obtained as:
for $i = 1, \ldots, m$,

$$|r_i^{(k)}(x)| \le C \sum_{q=i}^{m} (B_q^L(x)/\varepsilon_q^{k/2}), \quad k = 1, 2,$$

$$|r_i^{(3)}(x)| \le C \sum_{q=1}^{m} (B_q^L(x)/\varepsilon_q^{3/2})$$

$$|\varepsilon_i r_i^{(4)}(x)| \le C \sum_{q=1}^{m} (B_q^L(x)/\varepsilon_q)$$

and for $i = m+1, \ldots, n-1$,

$$|r_i^{(k)}(x)| \le C_1 B_m^L(x) + C_2 \varepsilon_m (1 - B_m^L(x)), \quad k = 1, 2,$$

$$|r_i^{(k)}(x)| \le C \sum_{q=1}^{m} (B_q^L(x)/\varepsilon_q^{(k-2)/2}), \quad k = 3, 4.$$

Combining the bounds for the derivatives of z_i and r_i, $i = 1, \ldots, n-1$, the bounds of $w_i^{L,(k)}(x), i = 1, \ldots, n-1$ and $k = 1, \ldots, 4$, follow.
Thus, the bounds on $w_i^{L,(k)}(x), k = 1, \ldots, 4$, hold for a system with n equations, as required. A similar proof of the analogous results for the right boundary layer functions holds. □

In the following lemma sharper estimates of the smooth component are presented.

Lemma 7. *Let $A(x)$ satisfy (2) and (3). Then the smooth component $\vec{v}$ of the solution $\vec{u}$ of* (1) *satisfies for $k = 0, 1, 2$ and $x \in \overline{\Omega}$,*

$$|v_i^{(k)}(x)| \le C[1 + B_m(x)], \ \textit{for } i = 1, \ldots, n,$$

$$|v_i'''(x)| \le C(1 + \sum_{q=i}^{m} (B_q(x)/\sqrt{\varepsilon}_q)), \ \textit{for } i = 1, \ldots, m,$$

and

$$|v_i'''(x)| \le C[1 + B_m(x)], \ \textit{for} \quad i = m+1, \ldots, n.$$

Proof. Define barrier functions

$$\vec{\psi}^{\pm}(x) = C[1 + B_m(x)]\vec{e} \pm \vec{v}^{(k)}(x), k = 0, 1, 2 \text{ and } x \in \overline{\Omega}.$$

Using the estimates of $\vec{v}^{(k)}(x), k = 0, 1, 2$ from Lemma 4, the required bounds for $\vec{v}^{(k)}(x), k = 0, 1, 2$ are obtained using Lemma 1.

Consider (5) from Lemma 4 and note that $\| \vec{g}' \| \leq C$. For convenience let $\vec{p}$ denote $\vec{v}''$. If $\vec{z}$ and $\vec{r}$ are the smooth and singular components of $\vec{p}$, then for $x \in \overline{\Omega}$,

$$|v_i'''(x)| = |p_i'(x)| \leq |z_i'(x)| + |r_i'(x)|$$

and using Lemmas 4 and 6, the required bound for $\vec{v}'''$ follows. □

4. The Shishkin mesh

A piecewise uniform Shishkin mesh with N mesh-intervals is now constructed. Let

$$\Omega^N = \{x_j\}_{j=1}^{N-1}, \overline{\Omega}^N = \{x_j\}_{j=0}^{N}$$

and $\Gamma^N = \Gamma$. The mesh $\overline{\Omega}^N$ is a piecewise-uniform mesh on $[0,1]$ obtained by dividing $[0,1]$ into $2m+1$ mesh-intervals as follows

$$[0, \sigma_1] \cup \cdots \cup (\sigma_{m-1}, \sigma_m] \cup (\sigma_m, 1 - \sigma_m] \cup (1 - \sigma_m, 1 - \sigma_{m-1}] \cup \cdots \cup (1 - \sigma_1, 1].$$

The m parameters σ_r, which determine the points separating the uniform meshes, are defined by $\sigma_0 = 0$, $\sigma_{m+1} = \frac{1}{2}$ and, for $r = m, \ldots 1$,

$$\sigma_r = \min\left\{\frac{\sigma_{r+1}}{2}, 2\frac{\sqrt{\varepsilon_r}}{\sqrt{\alpha}} \ln N\right\}. \tag{18}$$

Clearly

$$0 < \sigma_1 < \cdots < \sigma_m \leq \frac{1}{4}, \qquad \frac{3}{4} \leq 1 - \sigma_m < \cdots < 1 - \sigma_1 < 1.$$

Then, on the sub-interval $(\sigma_m, 1 - \sigma_m]$ a uniform mesh with $N/2$ mesh-intervals is placed, on each of the sub-intervals $(\sigma_r, \sigma_{r+1}]$ and $(1 - \sigma_{r+1}, 1 - \sigma_r], r = 1, \ldots, m-1$, a uniform mesh of $N/2^{m-r+2}$ mesh-intervals is placed and on both of the sub-intervals $[0, \sigma_1]$ and $(1 - \sigma_1, 1]$ a uniform mesh of $N/2^{m+1}$ mesh-intervals is placed. In practice it is convenient to take

$$N = 2^{m+p+1} \tag{19}$$

for some natural number p. It follows that, for $2 \leq r \leq m$, in the sub-interval $[\sigma_{r-1}, \sigma_r]$ there are $N/2^{m-r+3} = 2^{r+p-2}$ mesh-intervals and in each of $[0, \sigma_1]$ and $[\sigma_1, \sigma_2]$ there are $N/2^{m+1} = 2^p$ mesh-intervals. This construction leads to a class of 2^m piecewise uniform Shishkin meshes $\overline{\Omega}^N$.

From the above construction it is clear that the transition points $\{\sigma_r, 1 - \sigma_r\}_{r=1}^{m}$ are the only points at which the mesh-size can change and that it does not necessarily change at each of these points. The following notation is introduced: if $x_j = \sigma_r$, then $h_r^- = x_j - x_{j-1}, h_r^+ = x_{j+1} - x_j$, $J = \{\sigma_r : h_r^+ \neq h_r^-\}$. In general, for each point x_j in the mesh-interval $(\sigma_{r-1}, \sigma_r]$,

$$x_j - x_{j-1} = 2^{m-r+3} N^{-1} (\sigma_r - \sigma_{r-1}). \tag{20}$$

Also, for $x_j \in (\sigma_m, \frac{1}{4}], x_j - x_{j-1} = N^{-1}(1 - 4\sigma_m)$ and for $x_j \in (0, \sigma_1], x_j - x_{j-1} = 2^{m+1} N^{-1} \sigma_1$. Thus, for $1 \leq r \leq m$, the change in the mesh-size at the point $x_j = \sigma_r$ is

$$h_r^+ - h_r^- = 2^{m-r+3} N^{-1} (d_r - d_{r-1}), \tag{21}$$

where

$$d_r = \frac{\sigma_{r+1}}{2} - \sigma_r \tag{22}$$

with the convention $d_0 = 0$. Notice that $d_r \geq 0$, that Ω^N is a classical uniform mesh when $d_r = 0$ for all $r = 1 \dots m$ and, from (18), that

$$\sigma_r \leq C\sqrt{\varepsilon_r} \ln N, 1 \leq r \leq m. \tag{23}$$

It follows from (20) and (23) that for $r = 1, \dots, m$,

$$h_r^- + h_r^+ \leq C\sqrt{\varepsilon_{r+1}} N^{-1} \ln N. \tag{24}$$

Also

$$\sigma_r = 2^{-(s-r+1)} \sigma_{s+1} \text{ when } d_r = \cdots = d_s = 0, 1 \leq r \leq s \leq m. \tag{25}$$

The results in the following lemma are used later.

Lemma 8. *Assume that $d_r > 0$ for some $r, 1 \leq r \leq m$. Then the following inequalities hold*

$$B_r^L(1 - \sigma_r) \leq B_r^L(\sigma_r) = N^{-2}, \tag{26}$$

$$x_{r-1,r}^{(s)} \leq \sigma_r - h_r^- \text{ for } 0 < s \leq 3/2, 1 < r \leq m, \tag{27}$$

$$B_q^L(\sigma_r - h_r^-) \leq C B_q^L(\sigma_r) \text{ for } 1 \leq r \leq q \leq m, \tag{28}$$

$$\frac{B_q^L(\sigma_r)}{\sqrt{\varepsilon_q}} \leq C \frac{1}{\sqrt{\varepsilon_r} \ln N} \text{ for } 1 \leq q \leq m, 1 \leq r \leq m. \tag{29}$$

Analogous results hold for B_r^R.

Proof. The proof is similar to that in [8]. □

Remark 1. *It is not hard to verify that the different properties of the geometry of the mesh and the relationship between the transition points σ_r's and the layer interaction points $x_{i,j}^{(s)}$, established here also hold good for the mesh considered in [3]. Hence, the entire numerical analysis holds good for the mesh in [3] also.*

5. The discrete problem

In this section a classical finite difference operator with an appropriate Shishkin mesh is used to construct a numerical method for (1), which is shown later to be essentially second order parameter-uniform convergent. It is assumed that the problem data satisfy whatever smoothness conditions are required.
The discrete two-point boundary value problem is now defined on any mesh by the finite difference method

$$-E\delta^2\vec{U}(x) + A(x)\vec{U}(x) = \vec{f}(x) \text{ on } \Omega^N, \vec{U} = \vec{u} \text{ on } \Gamma^N. \tag{30}$$

This is used to compute numerical approximations to the exact solution of (1). It is assumed henceforth that the mesh is a Shishkin mesh, as defined in the previous section. Note that (30) can also be written in the operator form

$$\vec{L}^N\vec{U} = \vec{f} \text{ on } \Omega^N, \vec{U} = \vec{u} \text{ on } \Gamma^N,$$

where

$$\vec{L}^N = -E\delta^2 + A$$

and δ^2, D^+ and D^- are the difference operators

$$\delta^2\vec{U}(x_j) = \frac{D^+\vec{U}(x_j) - D^-\vec{U}(x_j)}{(x_{j+1} - x_{j-1})/2},$$

$$D^+\vec{U}(x_j) = \frac{\vec{U}(x_{j+1}) - \vec{U}(x_j)}{x_{j+1} - x_j}$$

and

$$D^-\vec{U}(x_j) = \frac{\vec{U}(x_j) - \vec{U}(x_{j-1})}{x_j - x_{j-1}}.$$

For any function $\vec{Z}$ defined on the Shishkin mesh $\overline{\Omega}^N$, we define

$$||\vec{Z}|| = \max_{i,j} |Z_i(x_j)|.$$

The following discrete results are analogous to those for the continuous case.

Lemma 9. *Let $A(x)$ satisfy (2) and (3). Then, for any vector-valued mesh function $\vec{\Psi}$, the inequalities $\vec{\Psi} \geq \vec{0}$ on Γ^N and $\vec{L}^N\vec{\Psi} \geq \vec{0}$ on Ω^N imply that $\vec{\Psi} \geq \vec{0}$ on $\overline{\Omega}^N$.*

Proof. Let i^*, j^* be such that $\Psi_{i^*}(x_{j^*}) = \min_{i,j} \Psi_i(x_j)$ and assume that the lemma is false. Then $\Psi_{i^*}(x_{j^*}) < 0$. From the hypotheses we have $j^* \neq 0, N$ and

$$\Psi_{i^*}(x_{j^*}) - \Psi_{i^*}(x_{j^*-1}) \leq 0,$$
$$\Psi_{i^*}(x_{j^*+1}) - \Psi_{i^*}(x_{j^*}) \geq 0,$$

so $\delta^2\Psi_{i^*}(x_{j^*}) > 0$. It follows that

$$(\vec{L}^N\vec{\Psi})_{i^*}(x_{j^*}) = -\varepsilon_{i^*}\delta^2\Psi_{i^*}(x_{j^*}) + \sum_{k=1}^{n} a_{i^*,k}(x_{j^*})\Psi_k(x_{j^*}) < 0,$$

which is a contradiction, as required. □

An immediate consequence of this is the following discrete stability result.

Lemma 10. *Let $A(x)$ satisfy (2) and (3). Then, for any vector-valued mesh function $\vec{\Psi}$ on $\overline{\Omega}^N$ and $i = 1, \ldots, n$,*

$$|\Psi_i(x_j)| \leq \max\left\{||\vec{\Psi}||_{\Gamma^N}, \frac{1}{\alpha}||\vec{L}^N\vec{\Psi}||\right\}, \quad 0 \leq j \leq N.$$

Proof. Define the two functions

$$\vec{\Theta}^{\pm}(x_j) = \max\{||\vec{\Psi}||_{\Gamma^N}, \frac{1}{\alpha}||\vec{L}^N\vec{\Psi}||\}\vec{e} \pm \vec{\Psi}(x_j)$$

where $\vec{e} = (1, \ldots, 1)$ is the unit $n-$vector. Using the properties of A it is not hard to verify that $\vec{\Theta}^{\pm} \geq \vec{0}$ on Γ^N and $\vec{L}^N\vec{\Theta}^{\pm} \geq \vec{0}$ on Ω^N. It follows from Lemma 9 that $\vec{\Theta}^{\pm} \geq \vec{0}$ on $\overline{\Omega}^N$. □

The following comparison principle will be used in the proof of the error estimate.

Lemma 11. *Assume that, for each $i = 1, \ldots, n$, the vector-valued mesh functions $\vec{\Phi}$ and $\vec{Z}$ satisfy*

$$|Z_i| \leq \Phi_i \text{ on } \Gamma^N \text{ and } |(\vec{L}^N\vec{Z})_i| \leq (\vec{L}^N\vec{\Phi})_i \text{ on } \Omega^N.$$

Then, for each $i = 1, \ldots, n$,

$$|Z_i| \leq \Phi_i \text{ on } \overline{\Omega}^N.$$

Proof. Define the two mesh functions $\vec{\Psi}^{\pm}$ by

$$\vec{\Psi}^{\pm} = \vec{\Phi} \pm \vec{Z}.$$

Then, for each $i = 1, \ldots, n, \Psi_i^{\pm}$ satisfies

$$\Psi_i^{\pm} \geq 0 \text{ on } \Gamma^N \text{ and } (\vec{L}^N\vec{\Psi}^{\pm})_i \geq 0 \text{ on } \Omega^N.$$

The result follows from an application of Lemma 9. □

6. The local truncation error

From Lemma 10, it is seen that in order to bound the error $\vec{U} - \vec{u}$, it suffices to bound $\vec{L}^N(\vec{U} - \vec{u})$. But this expression satisfies, for $x_j \in \Omega^N$,

$$\begin{aligned}\vec{L}^N(\vec{U} - \vec{u}) &= \vec{L}^N(\vec{U}) - \vec{L}^N(\vec{u}) = \vec{f} - \vec{L}^N(\vec{u}) = \vec{L}(\vec{u}) - \vec{L}^N(\vec{u}) \\ &= (\vec{L} - \vec{L}^N)\vec{u} = -E(\delta^2 - D^2)\vec{u}\end{aligned}$$

which is the local truncation of the second derivative. Let $\vec{V}, \vec{W}^L, \vec{W}^R$ be the discrete analogues of $\vec{v}, \vec{w}^L, \vec{w}^R$ respectively. Then, similarly,

$$\begin{aligned}\vec{L}^N(\vec{V} - \vec{v}) &= -E(\delta^2 - D^2)\vec{v}, \\ \vec{L}^N(\vec{W}^L - \vec{w}^L) &= -E(\delta^2 - D^2)\vec{w}^L, \\ \vec{L}^N(\vec{W}^R - \vec{w}^R) &= -E(\delta^2 - D^2)\vec{w}^R.\end{aligned}$$

By the triangle inequality,

$$\| \vec{L}^N(\vec{U}-\vec{u}) \| \le \| \vec{L}^N(\vec{V}-\vec{v}) \| + \| \vec{L}^N(\vec{W}-\vec{w}) \| . \tag{31}$$

Thus, the smooth and singular components of the local truncation error can be treated separately. In view of this it is noted that, for any smooth function ψ and for each $x_j \in \Omega^N$, the following distinct estimates of the local truncation error hold:

$$|(\delta^2 - D^2)\psi(x_j)| \le C \max_{s\in I_j} |\psi''(s)|, \tag{32}$$

$$|(\delta^2 - D^2)\psi(x_j)| \le C(x_{j+1} - x_{j-1}) \max_{s\in I_j} |\psi^{(3)}(s)|. \tag{33}$$

Furthermore, if $x_j \notin J$, then

$$|(\delta^2 - D^2)\psi(x_j)| \le C(x_{j+1} - x_{j-1})^2 \max_{s\in I_j} |\psi^{(4)}(s)|. \tag{34}$$

Here $I_j = [x_{j-1}, x_{j+1}]$.

7. Error estimate

The proof of the error estimate is split into two parts. In the first, a theorem concerning the smooth part of the error is proved. Then the singular part of the error is considered. A barrier function is now constructed, which is used in both parts of the proof.

For each $x_j = \sigma_r \in J$, introduce a piecewise linear polynomial θ_r on $\overline{\Omega}$, defined by

$$\theta_r(x) = \begin{cases} \dfrac{x}{\sigma_r}, \ 0 \le x \le \sigma_r, \\ 1, \ \sigma_r < x < 1 - \sigma_r. \\ \dfrac{1-x}{\sigma_r}, \ 1 - \sigma_r \le x \le 1. \end{cases}$$

It is not hard to verify that for any $x_j \in \Omega^N$

$$(\vec{L}^N \theta_r \vec{e})_i(x_j) \ge \begin{cases} \alpha\theta_r(x_j), \text{ if } x_j \notin J \\ \alpha + \dfrac{2\varepsilon_i}{\sigma_r(h_r^- + h_r^+)}, \text{ if } x_j \in J, x_j \in \{\sigma_r, 1-\sigma_r\}. \end{cases} \tag{35}$$

Now, define the barrier function $\vec{\Phi}$ by

$$\vec{\Phi}(x_j) = C[(N^{-1}\ln N)^2 + (N^{-1}\ln N)^2 \sum_{\{r:\sigma_r \in J\}} \theta_r(x_j)]\vec{e}, \tag{36}$$

where C is any sufficiently large constant.

Then, on Ω^N, $\vec{\Phi}$ satisfies

$$0 \le \Phi_i(x_j) \le C(N^{-1}\ln N)^2, 1 \le i \le n. \tag{37}$$

Also, for $x_j \notin J$,

$$(\vec{L}^N\vec{\Phi})_i(x_j) \geq C(N^{-1}\ln N)^2 \tag{38}$$

and, for $x_j \in J, x_j \in \{\sigma_r, 1-\sigma_r\}$, using (23), (24) and (35),

$$(\vec{L}^N\vec{\Phi})_i(x_j) \geq C((N^{-1}\ln N)^2 + \frac{\varepsilon_i}{\sqrt{\varepsilon_r}\sqrt{\varepsilon_{r+1}}}N^{-1}). \tag{39}$$

The following theorem gives the estimate for the smooth component of the error.

Theorem 1. *Let $A(x)$ satisfy (2) and (3). Let $\vec{v}$ denote the smooth component of the exact solution from (1) and $\vec{V}$ the smooth component of the discrete solution from (30). Then*

$$||\vec{V} - \vec{v}|| \leq C(N^{-1}\ln N)^2. \tag{40}$$

Proof. By the comparison principle in Lemma 11, it suffices to show that, for all i, j and some C,

$$|(\vec{L}^N(\vec{V} - \vec{v}))_i(x_j)| \leq (\vec{L}^N\vec{\Phi})_i(x_j). \tag{41}$$

For each mesh point x_j there are two possibilities: either $x_j \notin J$ or $x_j \in J$.

If $x_j \notin J$, apply Lemma 4 and (34) to get

$$\begin{aligned} |(\vec{L}^N(\vec{V} - \vec{v}))_i(x_j)| &\leq C(x_{j+1} - x_{j-1})^2 \\ &\leq C(N^{-1}\ln N)^2. \end{aligned} \tag{42}$$

Then (38) and (42) imply (41).

On the other hand, if $x_j \in J$, then $x_j \in \{\sigma_r, 1-\sigma_r\}$, for some $r, 1 \leq r \leq m$. Here the argument for $x_j = \sigma_r$ is given. For $x_j = 1 - \sigma_r$, it is analogous.

If $x_j = \sigma_r \in J$, apply Lemma 7 and (33) to get

$$|(\vec{L}^N(\vec{V} - \vec{v}))_i(x_j)| \leq C\varepsilon_i(x_{j+1} - x_{j-1})(1 + \sum_{q=i}^{m} \frac{B_q(x_{j-1})}{\sqrt{\varepsilon_q}}),$$

so, since $x_{j-1} = \sigma_r - h_r^-$,

$$|(\vec{L}^N(\vec{V} - \vec{v}))_i(x_j)| \leq C\varepsilon_i N^{-1}(1 + \sum_{q=i}^{m} \frac{B_q(\sigma_r - h_r^-)}{\sqrt{\varepsilon_q}}). \tag{43}$$

For each $r, 1 \leq r \leq m$, there are at most two possibilities: either $i \geq r$ or $i \leq r-1$. If $i \geq r$, then

$$\sum_{q=i}^{m} \frac{B_q(\sigma_r - h_r^-)}{\sqrt{\varepsilon_q}} \leq \frac{C}{\sqrt{\varepsilon_i}} \leq \frac{C}{\sqrt{\varepsilon_r}}.$$

Substituting this into (43) gives

$$|(\vec{L}^N(\vec{V} - \vec{v}))_i(x_j)| \leq C\frac{\varepsilon_i}{\sqrt{\varepsilon_r}}N^{-1}. \tag{44}$$

(39) and (44) imply (41).

If $i \le r-1$, which arises only if $r \ge 1$, there are two possibilities: either $d_r > 0$ or $d_r = 0$ and $d_{r-1} > 0$, because the case $d_r = d_{r-1} = 0$ cannot occur for $x_j = \sigma_r \in J$. Since $x_{j-1} = \sigma_r - h_r^-$ and $\sigma_r - h_r^- < \frac{1}{2}$,

$$B_q(x_{j-1}) = B_q(\sigma_r - h_r^-) = B_q^L(\sigma_r - h_r^-) + B_q^R(\sigma_r - h_r^-) \le 2B_q^L(\sigma_r - h_r^-).$$

Then

$$\sum_{q=i}^{m} \frac{B_q(\sigma_r - h_r^-)}{\sqrt{\varepsilon_q}} \le 2\sum_{q=i}^{m} \frac{B_q^L(\sigma_r - h_r^-)}{\sqrt{\varepsilon_q}}.$$

If $d_r > 0$, then using (12) in Lemma 5 and (27) in Lemma 8 give

$$\frac{B_q^L(\sigma_r - h_r^-)}{\sqrt{\varepsilon_q}} \le \frac{B_r^L(\sigma_r - h_r^-)}{\sqrt{\varepsilon_r}}$$

for $1 \le q \le r$. Hence

$$\sum_{q=i}^{m} \frac{B_q(\sigma_r - h_r^-)}{\sqrt{\varepsilon_q}} \le \frac{C}{\sqrt{\varepsilon_r}}.$$

Substituting this into (43) gives

$$|(\vec{L}^N(\vec{V} - \vec{v}))_i(x_j)| \le C\frac{\varepsilon_i}{\sqrt{\varepsilon_r}}N^{-1}. \tag{45}$$

(39) and (45) imply (41).

If $d_r = 0$ and $d_{r-1} > 0$ then using (12) and the fact that $\sigma_r - h_r^- \ge \sigma_{r-1} \ge x_{q,r-1}, 1 \le q \le r-2$ give

$$\frac{B_q^L(\sigma_r - h_r^-)}{\sqrt{\varepsilon_q}} \le \frac{B_{r-1}^L(\sigma_r - h_r^-)}{\sqrt{\varepsilon_{r-1}}}$$

for $1 \le q \le r-1$. Hence

$$\sum_{q=i}^{m} \frac{B_q^L(\sigma_r - h_r^-)}{\sqrt{\varepsilon_q}} \le C\sum_{q=r-1}^{m} \frac{B_q^L(\sigma_{r-1})}{\sqrt{\varepsilon_q}} \le C[\frac{B_{r-1}^L(\sigma_{r-1})}{\sqrt{\varepsilon_{r-1}}} + \frac{1}{\sqrt{\varepsilon_r}}]C[\frac{N^{-2}}{\sqrt{\varepsilon_{r-1}}} + \frac{1}{\sqrt{\varepsilon_r}}].$$

Substituting this into (43) gives

$$\begin{aligned} |(L^N(\vec{V} - \vec{v}))_i(x_j)| &\le C[\tfrac{\varepsilon_i}{\sqrt{\varepsilon_r}}N^{-1} + \tfrac{\varepsilon_i}{\sqrt{\varepsilon_{r-1}}}N^{-3}] \\ &\le C\tfrac{\varepsilon_i}{\sqrt{\varepsilon_{r-1}}}N^{-1}. \end{aligned} \tag{46}$$

(39) and (46) imply (41). This completes the proof. □

In order to estimate the singular component of the error the following four lemmas are required.

Lemma 12. *Assume that $x_j \notin J$. Let $A(x)$ satisfy (2) and (3). Then, on Ω^N, for each $1 \le i \le n$, the following estimates hold*

$$|(\vec{L}^N(\vec{W}^L - \vec{w}^L))_i(x_j)| \le C\frac{(x_{j+1} - x_{j-1})^2}{\varepsilon_1}. \tag{47}$$

An analogous result holds for the $\vec{W}^R - \vec{w}^R$.

Proof. Since $x_j \notin J$, from (34) and Lemma 6, it follows that

$$\begin{aligned}|(\vec{L}^N(\vec{W}^L - \vec{w}^L))_i(x_j)| &= |\varepsilon_i(\delta^2 - D^2)w_i^L(x_j)| \\ &\le C(x_{j+1} - x_{j-1})^2 \max_{s \in I_j} \sum_{q=1}^{m} \frac{B_q^L(s)}{\varepsilon_q} \\ &\le C\frac{(x_{j+1} - x_{j-1})^2}{\varepsilon_1}\end{aligned}$$

as required. □

The following decompositions of the singular components w_i^L are used in the next lemma

$$w_i^L = \sum_{q=1}^{r+1} w_{i,q}, \tag{48}$$

where the components $w_{i,q}$ are defined by

$$w_{i,r+1} = \begin{cases} p_i^{(s)}, & \text{on } [0, x_{r,r+1}^{(s)}) \\ w_i^L, & \text{otherwise} \end{cases}$$

and, for each q, $r \ge q \ge 2$,

$$w_{i,q} = \begin{cases} p_i^{(s)}, & \text{on } [0, x_{q-1,q}^{(s)}) \\ w_i^L - \displaystyle\sum_{k=q+1}^{r+1} w_{i,k}, & \text{otherwise} \end{cases}$$

and

$$w_{i,1} = w_i^L - \sum_{k=2}^{r+1} w_{i,k} \text{ on } [0,1].$$

Here the polynomials $p_i^{(s)}$, for $s = 3/2$ and $s = 1$, are defined by

$$p_i^{(3/2)}(x) = \sum_{k=0}^{3} w_i^{L,(k)}(x_{r,r+1}^{(3/2)})\frac{(x - x_{r,r+1}^{(3/2)})^k}{k!}$$

and

$$p_i^{(1)}(x) = \sum_{k=0}^{4} w_i^{L,(k)}(x_{r,r+1}^{(1)})\frac{(x - x_{r,r+1}^{(1)})^k}{k!}.$$

Notice that the decomposition (48) depends on the choice of the polynomials $p_i^{(s)}$ and that the $x_{i,j}^{(s)}$ are defined by (10). The following lemma provides estimates of the derivatives of the components in the decomposition (48).

Lemma 13. *Assume that $d_r > 0$ for some r, $1 \le r \le m$. Let $A(x)$ satisfy (2) and (3). Then, for each q and r, $1 \le q \le r$, and all $x_j \in \Omega^N$, the components in the decomposition* (48) *satisfy the following estimates for each $1 \le i \le m$,*

$$|w''_{i,q}(x_j)| \le C \min\{\frac{1}{\varepsilon_i}, \frac{1}{\varepsilon_q}\} B_q^L(x_j),$$
$$|w'''_{i,q}(x_j)| \le C \min\{\frac{1}{\varepsilon_i\sqrt{\varepsilon_q}}, \frac{1}{\varepsilon_q^{3/2}}\} B_q^L(x_j),$$
$$|w'''_{i,r+1}(x_j)| \le C \min\{\sum_{k=r+1}^{m} \frac{B_k^L(x_j)}{\varepsilon_i\sqrt{\varepsilon_k}}, \sum_{k=r+1}^{m} \frac{B_k^L(x_j)}{\varepsilon_k^{3/2}}\},$$
$$|w^{(4)}_{i,q}(x_j)| \le C \frac{B_q^L(x_j)}{\varepsilon_i\varepsilon_q},$$
$$|w^{(4)}_{i,r+1}(x_j)| \le C \sum_{k=r+1}^{m} \frac{B_k^L(x_j)}{\varepsilon_i\varepsilon_k}$$

and, for each $m+1 \le i \le n$,

$$|w''_{i,q}(x_j)| \le C B_q^L(x_j),$$
$$|w'''_{i,q}(x_j)| \le C \frac{B_q^L(x_j)}{\sqrt{\varepsilon_q}},$$
$$|w'''_{i,r+1}(x_j)| \le C \sum_{k=r+1}^{m} \frac{B_k^L(x_j)}{\sqrt{\varepsilon_k}},$$
$$|w^{(4)}_{i,q}(x_j)| \le C \frac{B_q^L(x_j)}{\varepsilon_q},$$
$$|w^{(4)}_{i,r+1}(x_j)| \le C \sum_{k=r+1}^{m} \frac{B_k^L(x_j)}{\varepsilon_k}.$$

Analogous results hold for the w_i^R and their derivatives.

Proof. Consider first the decomposition (48) corresponding to the polynomials $p_i^{(3/2)}$. From the above definitions it follows that, for each q, $1 \le q \le r$,

$$w_{i,q} = 0 \text{ on } [x_{q,q+1}^{(3/2)}, 1].$$

To establish the bounds on the third derivatives, for $i = 1, \ldots, m$, it is seen that:

for $x \in [x_{r,r+1}^{(3/2)}, 1]$, Lemma 6 and $x \ge x_{r,r+1}^{(3/2)}$ imply that

$$|w'''_{i,r+1}(x)| = |w_i^{L,'''}(x)| \le C \sum_{k=1}^{m} \frac{B_k^L(x)}{\varepsilon_k^{3/2}} \le C \sum_{k=r+1}^{m} \frac{B_k^L(x)}{\varepsilon_k^{3/2}};$$

for $x \in [0, x^{(3/2)}_{r,r+1}]$, Lemma 6 and $x \le x^{(3/2)}_{r,r+1}$ imply that

$$|w'''_{i,r+1}(x)| = |w^{L,'''}_i(x^{(3/2)}_{r,r+1})| \le C\sum_{k=1}^m \frac{B^L_k(x^{(3/2)}_{r,r+1})}{\varepsilon_k^{3/2}} \le C\sum_{k=r+1}^m \frac{B^L_k(x^{(3/2)}_{r,r+1})}{\varepsilon_k^{3/2}} \le C\sum_{k=r+1}^m \frac{B^L_k(x)}{\varepsilon_k^{3/2}};$$

and for each $q = r, \dots, 2$, it follows that

for $x \in [x^{(3/2)}_{q,q+1}, 1]$, $w'''_{i,q} = 0$;

for $x \in [x^{(3/2)}_{q-1,q}, x^{(3/2)}_{q,q+1}]$, Lemma 6 implies that

$$|w'''_{i,q}(x)| \le |w^{L,'''}_i(x)| + \sum_{k=q+1}^{r+1} |w'''_{i,k}(x)| \le C\sum_{k=1}^m \frac{B^L_k(x)}{\varepsilon_k^{3/2}} \le C\frac{B^L_q(x)}{\varepsilon_q^{3/2}}, \text{ using (12);}$$

for $x \in [0, x^{(3/2)}_{q-1,q}]$, Lemma 6 and $x \le x^{(3/2)}_{q-1,q}$ imply that

$$|w'''_{i,q}(x)| = |w^{L,'''}_i(x^{(3/2)}_{q-1,q})| \le C\sum_{k=1}^m \frac{B^L_k(x^{(3/2)}_{q-1,q})}{\varepsilon_k^{3/2}} \le C\frac{B^L_q(x^{(3/2)}_{q-1,q})}{\varepsilon_q^{3/2}} \le C\frac{B^L_q(x)}{\varepsilon_q^{3/2}}, \text{ using (10) and (12);}$$

for $x \in [x^{(3/2)}_{1,2}, 1], w'''_{i,1} = 0$;

for $x \in [0, x^{(3/2)}_{1,2}]$, Lemma 6 implies that

$$|w'''_{i,1}(x)| \le |w^{L,'''}_i(x)| + \sum_{k=2}^{r+1} |w'''_{i,k}(x)| \le C\sum_{k=1}^m \frac{B^L_k(x)}{\varepsilon_k^{3/2}} \le C\frac{B^L_1(x)}{\varepsilon_1^{3/2}}.$$

The required bounds for $|w'''_{i,q}(x)|$, for $i = m+1, \dots, n$ and $q = r+1, \dots, 1$, are obtained using the above steps with the appropriate bound of $|w^{L,'''}_i(x)|$, $i = m+1, \dots, n$, from Lemma 6.

For the bounds on the second derivatives note that, for $i = 1, \dots, m$ and each q, $1 \le q \le r$:

for $x \in [x^{(3/2)}_{q,q+1}, 1], w''_{i,q} = 0$;

for $x \in [0, x^{(3/2)}_{q,q+1}]$,

$$\int_x^{x^{(3/2)}_{q,q+1}} w'''_{i,q}(\psi)d\psi = w''_{i,q}(x^{(3/2)}_{q,q+1}) - w''_{i,q}(x) = -w''_{i,q}(x)$$

and so

$$|w''_{i,q}(x)| \le \int_x^{x^{(3/2)}_{q,q+1}} |w'''_{i,q}(\psi)|d\psi \le \frac{C}{\varepsilon_q^{3/2}}\int_x^{x^{(3/2)}_{q,q+1}} B^L_q(\psi)d\psi \le C\frac{B^L_q(x)}{\varepsilon_q}.$$

Similarly, for $i = m+1, \dots, n$ and each q, $1 \le q \le r$, $|w''_{i,q}(x)| \le CB^L_q(x)$. This completes the proof of the estimates for $s = 3/2$.

Secondly, consider the decomposition (48) corresponding to the polynomials $p_i^{(1)}$. From the above definitions it follows that, for each q, $1 \le q \le r$, $w_{i,q} = 0$ on $[x^{(1)}_{q,q+1}, 1]$. To establish the bounds on the fourth derivatives it is seen that:

for $x \in [x^{(1)}_{r,r+1}, 1]$, Lemma 6 and $x \ge x^{(1)}_{r,r+1}$ imply that

$$|\varepsilon_i w^{(4)}_{i,r+1}(x)| = |\varepsilon_i w_i^{L,(4)}(x)| \le C \sum_{k=1}^{m} \frac{B^L_k(x)}{\varepsilon_k} \le C \sum_{k=r+1}^{m} \frac{B^L_k(x)}{\varepsilon_k};$$

for $x \in [0, x^{(1)}_{r,r+1}]$, Lemma 6 and $x \le x^{(1)}_{r,r+1}$ imply that

$$|\varepsilon_i w^{(4)}_{i,r+1}(x)| = |\varepsilon_i w_i^{L,(4)}(x^{(1)}_{r,r+1})| \le C \sum_{k=1}^{m} \frac{B^L_k(x^{(1)}_{r,r+1})}{\varepsilon_k}$$
$$\le C \sum_{k=r+1}^{m} \frac{B^L_k(x^{(1)}_{r,r+1})}{\varepsilon_k} \le C \sum_{k=r+1}^{m} \frac{B^L_k(x)}{\varepsilon_k};$$

and for each $q = r, \dots, 2$, it follows that

for $x \in [x^{(1)}_{q,q+1}, 1]$, $w^{(4)}_{i,q} = 0$;

for $x \in [x^{(1)}_{q-1,q}, x^{(1)}_{q,q+1}]$, Lemma 6 implies that

$$|\varepsilon_i w^{(4)}_{i,q}(x)| \le |\varepsilon_i w_i^{L,(4)}(x)| + \sum_{k=q+1}^{r+1} |\varepsilon_i w^{(4)}_{i,k}(x)| \le C \sum_{k=1}^{m} \frac{B^L_k(x)}{\varepsilon_k} \le C \frac{B^L_q(x)}{\varepsilon_q}, \text{ using (12)};$$

for $x \in [0, x^{(1)}_{q-1,q}]$, Lemma 6 and $x \le x^{(1)}_{q-1,q}$ imply that

$$|\varepsilon_i w^{(4)}_{i,q}(x)| = |\varepsilon_i w_i^{L,(4)}(x^{(1)}_{q-1,q})| \le C \sum_{k=1}^{m} \frac{B^L_k(x^{(1)}_{q-1,q})}{\varepsilon_k}$$
$$\le C \frac{B^L_q(x^{(1)}_{q-1,q})}{\varepsilon_q} \le C \frac{B^L_q(x)}{\varepsilon_q}, \text{ using (10) and (12)};$$

for $x \in [x^{(1)}_{1,2}, 1], w^{(4)}_{i,1} = 0$;

for $x \in [0, x^{(1)}_{1,2}]$, Lemma 6 implies that

$$|\varepsilon_i w^{(4)}_{i,1}(x)| \le |\varepsilon_i w_i^{L,(4)}(x)| + \sum_{k=2}^{r+1} |\varepsilon_i w^{(4)}_{i,k}(x)| \le C \sum_{k=1}^{m} \frac{B^L_k(x)}{\varepsilon_k} \le C \frac{B^L_1(x)}{\varepsilon_1}.$$

For the bounds on the second and third derivatives note that, for each q, $1 \le q \le r$:

for $x \in [x^{(1)}_{q,q+1}, 1], w''_{i,q} = 0 = w'''_{i,q}$;

for $x \in [0, x^{(1)}_{q,q+1}]$,

$$\int_x^{x^{(1)}_{q,q+1}} \varepsilon_i w^{(4)}_{i,q}(\psi) d\psi = \varepsilon_i w'''_{i,q}(x^{(1)}_{q,q+1}) - \varepsilon_i w'''_{i,q}(x) = -\varepsilon_i w'''_{i,q}(x)$$

and so

$$|\varepsilon_i w'''_{i,q}(x)| \leq \int_x^{x^{(1)}_{q,q+1}} |\varepsilon_i w^{(4)}_{i,q}(\psi)| d\psi \leq \frac{C}{\varepsilon_q} \int_x^{x^{(1)}_{q,q+1}} B^L_q(\psi) d\psi \leq C \frac{B^L_q(x)}{\sqrt{\varepsilon_q}}.$$

In a similar way, it can be shown that

$$|\varepsilon_i w''_{i,q}(x)| \leq C B^L_q(x).$$

The proof for the w^R_i and their derivatives is similar. □

Lemma 14. *Assume that $d_r > 0$ for some r, $1 \leq r \leq m$. Let $A(x)$ satisfy (2) and (3). Then, if $x_j \notin J$,*

$$|(\vec{L}^N(\vec{W}^L - \vec{w}^L))_i(x_j)| \leq C[B^L_r(x_{j-1}) + \frac{(x_{j+1} - x_{j-1})^2}{\varepsilon_{r+1}}] \tag{49}$$

and if $x_j \in J$

$$|(\vec{L}^N(\vec{W}^L - \vec{w}^L))_i(x_j)| \leq C[N^{-2} + \frac{\varepsilon_i}{\sqrt{\varepsilon_r}\sqrt{\varepsilon_{r+1}}} N^{-1}]. \tag{50}$$

Analogous results hold for the $\vec{W}^R - \vec{w}^R$.

Proof. Suppose first that $x_j \notin J$. Then, by (32), (34) and Lemma 13

$$\begin{aligned} &|\varepsilon_i(\delta^2 - D^2) w^L_i(x_j)| \\ &\quad \leq C[\textstyle\sum_{q=1}^r \max\limits_{s \in I_j} |\varepsilon_i w''_{i,q}(s)| + (x_{j+1} - x_{j-1})^2 \max\limits_{s \in I_j} |\varepsilon_i w^{(4)}_{i,r+1}(s)|] \\ &\quad \leq C[\textstyle\sum_{q=1}^r \min\{1, \frac{\varepsilon_i}{\varepsilon_q}\} B^L_q(x_{j-1}) + (x_{j+1} - x_{j-1})^2 \sum_{q=r+1}^m \frac{B^L_q(x_{j-1})}{\varepsilon_q}] \\ &\quad \leq C[B^L_r(x_{j-1}) + \frac{(x_{j+1} - x_{j-1})^2}{\varepsilon_{r+1}}]. \end{aligned} \tag{51}$$

Suppose now that $x_j = \sigma_r \in J$ (an analogous argument holds if $x_j = 1 - \sigma_r \in J$). Then, by Lemma 13 and expressions (32) and (33),

$$\begin{aligned} &|\varepsilon_i(\delta^2 - D^2) w^L_i(x_j)| \\ &\quad \leq C[\textstyle\sum_{q=1}^r \max\limits_{s \in I_j} |\varepsilon_i w''_{i,q}(s)| + (x_{j+1} - x_{j-1}) \max\limits_{s \in I_j} |\varepsilon_i w'''_{i,r+1}(s)|] \\ &\quad \leq C[\textstyle\sum_{q=1}^r \min\{1, \frac{\varepsilon_i}{\varepsilon_q}\} B^L_q(\sigma_r - h^-_r) + (h^-_r + h^+_r) \sum_{q=r+1}^m \min\{1, \frac{\varepsilon_i}{\varepsilon_q}\} \frac{B^L_q(\sigma_r - h^-_r)}{\sqrt{\varepsilon_q}}]. \end{aligned}$$

When $i \geq r+1$ replace both minima by the upper bound 1 and get, using (6), (28),(29),(26) and (24),

$$\begin{aligned}|\varepsilon_i(\delta^2 - D^2)w_i^L(x_j)| &\leq C[B_r^L(\sigma_r - h_r^-) + (h_r^- + h_r^+)\sum_{q=r+1}^{m} \frac{B_q^L(\sigma_r)}{\sqrt{\varepsilon_q}}] \\ &\leq C[B_r^L(\sigma_r) + \frac{h_r^- + h_r^+}{\sqrt{\varepsilon_r}\ln N}] \leq C[N^{-2} + \frac{\sqrt{\varepsilon_{r+1}}}{\sqrt{\varepsilon_r}}N^{-1}] \\ &\leq C[N^{-2} + \frac{\varepsilon_i}{\sqrt{\varepsilon_r}\sqrt{\varepsilon_{r+1}}}N^{-1}],\end{aligned}$$

which is (50) for this case. On the other hand, when $i \leq r$ replace both minima by the upper bound $\frac{\varepsilon_i}{\varepsilon_q}$ and get, using Lemma 8,

$$\begin{aligned}&|\varepsilon_i(\delta^2 - D^2)w_i^L(x_j)| \\ &\leq C[\varepsilon_i \tfrac{B_r^L(\sigma_r - h_r^-)}{\varepsilon_r} + (h_r^- + h_r^+)\varepsilon_i \textstyle\sum_{q=r+1}^{m} \tfrac{B_q^L(\sigma_r)}{\varepsilon_q^{3/2}}] \\ &\leq C[\tfrac{\varepsilon_i}{\varepsilon_r}N^{-2} + \tfrac{\varepsilon_i}{\sqrt{\varepsilon_r}\sqrt{\varepsilon_{r+1}}}N^{-1}] \leq C[N^{-2} + \tfrac{\varepsilon_i}{\sqrt{\varepsilon_r}\sqrt{\varepsilon_{r+1}}}N^{-1}],\end{aligned}$$

which is (50) for this case. The proof for $\vec{W}^R - \vec{w}^R$ is similar. □

Lemma 15. *Let $A(x)$ satisfy (2) and (3). Then, on Ω^N, for each $i = 1, \ldots, n$, the following estimates hold*

$$|(\vec{L}^N(\vec{W}^L - \vec{w}^L))_i(x_j)| \leq CB_m^L(x_{j-1}). \tag{52}$$

An analogous result holds for $\vec{W}^R - \vec{w}^R$.

Proof. From (32) and Lemma 6, for each $i = 1, \ldots, m$, it follows that on Ω^N,

$$\begin{aligned}|(\vec{L}^N(\vec{W}^L - \vec{w}^L))_i(x_j)| &= |\varepsilon_i(\delta^2 - D^2)w_i^L(x_j)| \\ &\leq C\varepsilon_i \sum_{q=i}^{m} \frac{B_q^L(x_{j-1})}{\varepsilon_q}) \\ &\leq CB_m^L(x_{j-1})\end{aligned}$$

and for each $i = m+1, \ldots, n$, it follows that on Ω^N,

$$\begin{aligned}|(\vec{L}^N(\vec{W}^L - \vec{w}^L))_i(x_j)| &= |(\delta^2 - D^2)w_i^L(x_j)| \\ &\leq CB_m^L(x_{j-1}).\end{aligned}$$

The proof for $\vec{W}^R - \vec{w}^R$ is similar. □

The following theorem gives the estimate of the singular component of the error.

Theorem 2. *Let $A(x)$ satisfy (2) and (3). Let $\vec{w}$ denote the singular component of the exact solution from (1) and $\vec{W}$ the singular component of the discrete solution from (30). Then*

$$||\vec{W} - \vec{w}|| \leq C(N^{-1}\ln N)^2. \tag{53}$$

Proof. Since $\vec{w} = \vec{w}^L + \vec{w}^R$, it suffices to prove the result for $\vec{w}^L$ and $\vec{w}^R$ separately. Here it is proved for $\vec{w}^L$; a similar proof holds for $\vec{w}^R$.

By the comparison principle in Lemma 11 it suffices to show that, for all i, j and some constant C,

$$|(\vec{L}^N(\vec{W}^L - \vec{w}^L))_i(x_j)| \le (\vec{L}^N\vec{\Phi})_i(x_j). \tag{54}$$

This is proved for each mesh point $x_j \in (0,1)$ by considering separately the 4 kinds of subintervals (a) $(0,\sigma_1)$, (b) $[\sigma_1,\sigma_2)$, (c) $[\sigma_q,\sigma_{q+1})$ for some $q, 2 \le q \le m-1$ and (d) $[\sigma_m, 1)$.
(a) Clearly $x_j \notin J$ and

$$x_{j+1} - x_{j-1} \le C\sqrt{\varepsilon_1}N^{-1}\ln N.$$

Then, Lemma 12 and (38) give (54).
(b) There are 2 possibilities: (b1) $d_1 = 0$ and (b2) $d_1 > 0$.
(b1) Since $\sigma_1 = \sigma_2/2$ and the mesh is uniform in $(0,\sigma_2)$ it follows that $x_j \notin J$, and

$$x_{j+1} - x_{j-1} \le C\sqrt{\varepsilon_1}N^{-1}\ln N.$$

Then Lemma 12 and (38) give (54).
(b2) Either $x_j \notin J$ or $x_j \in J$. If $x_j \notin J$ then

$$x_{j+1} - x_{j-1} \le C\sqrt{\varepsilon_2}N^{-1}\ln N$$

and by Lemma 8

$$B_1^L(x_{j-1}) \le B_1^L(\sigma_1 - h_1^-) \le CN^{-2},$$

so Lemma 14 (49) with $r = 1$ and (38) give (54). On the other hand, if $x_j \in J$, then Lemma 14 (50) with $r = 1$ and (39) give (54).
(c) There are 3 possibilities: (c1) $d_1 = d_2 = \cdots = d_q = 0$, (c2) $d_r > 0$ and $d_{r+1} = \cdots = d_q = 0$ for some $r, 1 \le r \le q-1$ and (c3) $d_q > 0$.
(c1) Since $\sigma_1 = C\sigma_{q+1}$ and the mesh is uniform in $(0,\sigma_{q+1})$, it follows that $x_j \notin J$ and

$$x_{j+1} - x_{j-1} \le C\sqrt{\varepsilon_1}N^{-1}\ln N.$$

Then Lemma 12 and (38) give (54).
(c2) Either $x_j \notin J$ or $x_j \in J$. If $x_j \notin J$ then

$$\sigma_{r+1} = C\sigma_{q+1}, x_{j+1} - x_{j-1} \le C\sqrt{\varepsilon_{q+1}}N^{-1}\ln N$$

and by Lemma 8

$$B_r^L(x_{j-1}) \le B_r^L(\sigma_q - h_q^-) \le B_r^L(\sigma_r - h_r^-) \le CN^{-2}.$$

Thus Lemma 14 (49) and (38) give (54). On the other hand, if $x_j \in J$, then $x_j = \sigma_q$, so Lemma 14 (50) with $r = q$ and (39) give (54).
(c3) Either $x_j \notin J$ or $x_j \in J$. If $x_j \notin J$ then

$$x_{j+1} - x_{j-1} \le C\sqrt{\varepsilon_{q+1}}N^{-1}\ln N$$

and by Lemma 8

$$B_q^L(x_{j-1}) \le B_q^L(\sigma_q - h_q^-) \le CN^{-2},$$

so Lemma 14 (49) with $r = q$ and (38) give (54). On the other hand, if $x_j = \sigma_q$, so Lemma 14 (50) with $r = q$ and (39) give (54).
(d) There are 3 possibilities: (d1) $d_1 = \cdots = d_m = 0$, (d2) $d_r > 0$ and $d_{r+1} = \cdots = d_m = 0$ for some $r, 1 \le r \le m-1$ and (d3) $d_m > 0$.
(d1) Since $\sigma_1 = C$ and the mesh is uniform in $(0,1)$, it follows that $x_j \notin J$,

$$\frac{1}{\sqrt{\varepsilon_1}} \le C \ln N$$

and

$$x_{j+1} - x_{j-1} \le CN^{-1}.$$

Then Lemma 12 and (38) give (54).
(d2) Either $x_j \notin J$ or $x_j \in J$. If $x_j \notin J$ then

$$\sigma_{r+1} = C,$$
$$\frac{1}{\sqrt{\varepsilon_{r+1}}} \le C \ln N,$$
$$x_{j+1} - x_{j-1} \le CN^{-1}$$

and, by Lemma 8,

$$B_r^L(x_{j-1}) \le B_r^L(\sigma_m - h_m^-) \le B_r^L(\sigma_r - h_r^-) \le CN^{-2}.$$

Thus Lemma 14 (49) and (38) give (54). On the other hand, if $x_j \in J$, then $x_j \in \{\sigma_m, 1-\sigma_m, \ldots, 1-\sigma_1\}$. Thus, Lemma 14 (50) and (39) give (54).
(d3) By Lemma 8 with $r = m$,

$$B_m^L(x_{j-1}) \le B_m^L(\sigma_m - h_m^-) \le CN^{-2}.$$

Then Lemma 15 and (38) give (54). □

The following theorem gives the required essentially second order parameter-uniform error estimate.

Theorem 3. *Let $A(x)$ satisfy (2) and (3). Let $\vec{u}$ denote the exact solution of (1) and $\vec{U}$ the discrete solution of (30). Then*

$$||\vec{U} - \vec{u}|| \le C(N^{-1} \ln N)^2. \tag{55}$$

Proof. An application of the triangle inequality and the results of Theorems 1 and 2 lead immediately to the required result. □

8. Numerical results

The above numerical method is applied to the following partially singularly perturbed boundary value problem

Example 1.

$$\left.\begin{array}{r} -\varepsilon_1 u_1''(x) + 5u_1(x) - u_2(x) - u_3(x) = x^2 \\ -\varepsilon_2 u_2''(x) - u_1(x) + (5+x)u_2(x) - u_3(x) = \exp^{-x} \\ -u_3''(x) - (1+x)u_1(x) - u_2(x) + (5+x)u_3(x) = 1+x \end{array}\right\}$$

for $x \in \Omega$ *and* $\vec{u} = \vec{0}$ *on* Γ. *For various values of* ε_1 *and* ε_2 *with* $\alpha = 2.0$ *and* $N = 2^r, r = 8, \ldots, 13$, *the computed order of* $\vec{\varepsilon}$*–uniform convergence and the computed* $\vec{\varepsilon}$*–uniform error constant are found using the general methodology from [2]. The results are presented in Table 1.*

η	Number of mesh points N				
	256	512	1024	2048	4096
0.100E+01	0.284E-05	0.709E-06	0.177E-06	0.431E-07	0.291E-06
0.100E-02	0.485E-03	0.235E-03	0.106E-03	0.408E-04	0.145E-04
0.100E-05	0.485E-03	0.235E-03	0.106E-03	0.408E-04	0.145E-04
0.100E-08	0.485E-03	0.235E-03	0.106E-03	0.408E-04	0.145E-04
0.100E-11	0.485E-03	0.235E-03	0.106E-03	0.408E-04	0.145E-04
0.100E-14	0.485E-03	0.235E-03	0.106E-03	0.408E-04	0.145E-04
0.100E-17	0.485E-03	0.235E-03	0.106E-03	0.407E-04	0.145E-04
D^N	0.485E-03	0.235E-03	0.106E-03	0.408E-04	0.145E-04
p^N	0.105E+01	0.115E+01	0.138E+01	0.149E+01	
C_p^N	0.311E+00	0.311E+00	0.290E+00	0.231E+00	0.169E+00
Computed order of $\vec{\varepsilon}$-uniform convergence $= 0.105E+01$					
Computed $\vec{\varepsilon}$-uniform error constant $= 0.311E+00$					

Table 1: *Numerical results of Example 1 for* $\varepsilon_1 = \dfrac{\eta}{16}, \varepsilon_2 = \eta$

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