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An improved stability result for a heat equation backward in time with nonlinear source

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Abstract. We consider a nonlinear backward heat conduction problem in a strip. The problem is ill-posed in the sense that the solution (if it exists) does not depend continuously on the data. We shall use a modified integral equation method to regularize the nonlinear problem. The error estimates of Hölder type of the regularized solutions are obtained.
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1. Introduction

Let T be a positive number. We consider the problem of finding the temperature $u(x,t), (x,t) \in R \times [0;T]$ such that

$$\begin{cases} u_t - u_{xx} = f(x,t,u(x,t)), & (x,t) \in R \times (0,T), \\ u(x,T) = \varphi(x), & x \in R, \end{cases} \tag{1}$$

where $\varphi(x)$ and $f(x,t,z)$ are given. This problem is well-known to be severely ill-posed and regularization methods for it are required. This problem is called backward heat problem, backward Cauchy problem, and final value problem.

As is known, if the initial temperature distribution in a heat conducting body is given, then the temperature distribution at a later time can be determined and the problem is well-posed. This is the direct problem. In geophysical exploration, one is often faced with the problem of determining the temperature distribution in the Earth or any part of the Earth at a time $t_0 > 0$ from the temperature measurement at a time $t_1 > t_0$. This is the backward heat problem. The type of a problem is severely ill-posed; i.e., solutions do not always exist, and in the case of existence, these do not depend continuously on the given data. In fact, from small noise contaminated physical measurements, the corresponding solutions have large errors.

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It makes difficult to perform numerical calculations. Hence, a regularization is in order. In the simplest case $f = 0$, problem (1) becomes

$$\begin{cases} u_t - u_{xx} = 0, & (x,t) \in R \times (0,T), \\ u(x,T) = \varphi(x), & x \in R. \end{cases} \tag{2}$$

Authors such as Lattes and Lions [7], Showalter [12] approximated problem (2) by a quasi-reversibility method. Tautenhahn and Schröter [13] established an optimal error estimate for (2). In [8], Liu introduced a group preserving scheme. A molification method was studied by [6]. Some papers [3, 5] approximated (1) by truncated methods. These methods were demonstrated to be very effective, and all of them were devoted to computational aspects.

Although there are many works on the homogeneous problem (2), the literature on the nonlinear case is quite scarce. In [10], Trong and Quan have established, under the hypothesis that f is a global Lipschitzian function, the existence of a unique solution for a well-posed problem as follows

$$\begin{aligned} u^\epsilon(x,t) = & \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \frac{e^{-tp^2}}{\epsilon + e^{-Tp^2}} \hat{\varphi}(p) e^{ipx} dp \\ & - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \int_t^T \frac{e^{-tp^2}}{\epsilon^{\frac{s}{T}} + e^{-sp^2}} \hat{f}(p,s,u^\epsilon) e^{ipx} dp ds, \end{aligned} \tag{3}$$

where ϵ is a positive parameter and

$$\hat{g}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} g(x) e^{-i\xi x} dx$$

is the Fourier transform of g. Under a strong smoothness assumption on the original solution, namely

$$\int_0^T \int_{-\infty}^{\infty} \left| \frac{\partial}{\partial t} \left(e^{sp^2} \hat{u}(p,t) \right) \right|^2 dp dt < \infty,$$

and

$$\int_{-\infty}^{\infty} \left| e^{Tp^2} \hat{\varphi}(p) \right|^2 dp < \infty,$$

they obtained the following error estimate

$$\|u(.,t) - u^\epsilon(.,t)\|_{L^2(R)} \le \sqrt{M} \exp\left(\frac{3k^2 T(T-t)}{2}\right) \epsilon^{\frac{t}{T}}, \tag{4}$$

where M is a constant dependent on u. The right-hand side of (4) is not close to zero if $\epsilon \to 0$ and $t = 0$. The convergence of the approximate solution is very slow

when t is in a neighborhood of zero. This is an open point of the paper [10].
In the present paper, we use a modified integral equation method in order to improve the results given in [10]. Under some assumptions on the exact solution, we obtain some faster convergence error estimates. In a sense, this is an improvement of the known result in [10], and it is our aim here is to obtain only a stability estimate.

This paper is organized as follows. In Section 2, we give some auxiliary results. In Section 3, we outline the main results while the proofs are given in Section 4.

2. Some auxiliary results.

Let $\hat{g}(\xi)$ denote the Fourier transform of function $g \in L^2(R)$ defined formally

$$\hat{g}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} g(x) e^{-i\xi x} dx. \tag{5}$$

Let $H^1 = W^{1,2}$, $H^2 = W^{2,2}$ be the Sobolev spaces defined by

$$\begin{aligned} H^1(R) &= \{g \in L^2(R), \xi\hat{g}(\xi) \in L^2(R)\}, \\ H^2(R) &= \{g \in L^2(R), \xi^2\hat{g}(\xi) \in L^2(R)\}. \end{aligned}$$

We denote by $\|.\|$, $\|.\|_{H^1}$, $\|.\|_{H^2}$ the norms in $L^2(R), H^1(R), H^2(R)$ respectively, namely

$$\begin{aligned} \|g\|^2_{H^1} &= \|g\|^2 + \|g_x\|^2 = \|(1+\xi^2)^{\frac{1}{2}}\hat{g}(\xi)\|^2, \\ \|g\|^2_{H^2} &= \|g\|^2 + \|g_x\|^2 + \|g_{xx}\|^2 = \|(1+\xi^2+\xi^4)^{\frac{1}{2}}\hat{g}(\xi)\|^2. \end{aligned}$$

Let us first make clear what a weak solution to problem (1) is.

Lemma 1. *Let $f \in L^\infty(R \times [0,T] \times R)$ be a function such that $f(x,y,0) = 0$ and*

$$|f(x,t,u) - f(x,t,v)| \le K|u - v|,$$

for all $(x,t) \in R \times [0,T]$ and for some constant $K > 0$ independent of x,t,u,v. Let $\varphi \in L^2(R)$. Assume that $u \in C([0,T], H^2(R)) \cap C^1([0,T], L^2(R))$ is a solution of the equation

$$\hat{u}(\xi,t) = e^{(T-t)\xi^2}\hat{\varphi}(\xi) - \int_t^T e^{-(t-s)\xi^2}\hat{f}(\xi,s,u)ds. \tag{6}$$

Then $u_t, u_{xx} \in C([0,T], L^2(R))$.

Proof. First, it is easy to see that the Fourier transform of f with respect to x belongs to $L^2(R)$. In fact, from the Lipschitzian of f, we get

$$|f(x,t,u(x,t)) - f(x,t,0)| \le K|u(x,t)|,$$

Since $f(x,t,0) = 0$, we obtain

$$|f(x,t,u(x,t))| \le K|u(x,t)|.$$

From u belonging to $L^2(R)$ with respect to x, we get f belonging to $L^2(R)$ with respect to x.

By letting $t = T$ in equation (6), we have immediately $\hat{u}(\xi, T) = \hat{\varphi}(\xi)$. Therefore, we get $u(x, T) = \varphi(x)$ in $L^2(R)$.
Multiplying the above equation by $e^{t\xi^2}$ we obtain

$$e^{t\xi^2}\hat{u}(\xi, t) = e^{T\xi^2}\hat{\varphi}(\xi) - \int_t^T e^{s\xi^2}\hat{f}(\xi, s, u)ds, \quad t \in [0, T].$$

Differentiating the latter equation w.r.t. the time variable t we get

$$e^{t\xi^2}\left(\xi^2\hat{u}(\xi, t) + \frac{d}{dt}\hat{u}(\xi, t)\right) = e^{t\xi^2}\hat{f}(\xi, t, u),$$

namely

$$\left(\xi^2\hat{u}(\xi, t) + \frac{d}{dt}\hat{u}(\xi, t)\right) = \hat{f}(\xi, t, u), \quad t \in [0, T].$$

Since $u \in C([0, T], H^2(R)) \cap C^1([0, T], L^2(R))$, we have $\xi^2\hat{u}(\xi, t) = \hat{u}_{xx}(\xi)$ and $\frac{d}{dt}\hat{u}(\xi, t)$ belongs to $C([0, T], L^2(R))$. This gives $u_t, u_{xx} \in C([0, T], L^2(R))$. □

3. The main results

Let

$$\hat{\varphi}(\xi) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\varphi(x)e^{-i\xi x}dx$$

be the Fourier transform of the function $\varphi \in L^2(R)$. As a solution of problem (1) we understand a function $u(x, t)$ satisfying (1) in the classical sense and for every fixed $t \in [0, T]$, the function $u(., t) \in L^2(R)$. In this class of functions, if the solution of problem (1) exists, then it must be unique (see [10]). In general, we have no guarantee that the solution to problem (1) exists. We do not know any general condition under which problem (1) is solvable. The main goal of this paper is to find a computation method on the exact solution when it exists. Hence, regularization techniques are required. Let $u(x, t)$ be a unique solution of (1) (if it exists). Using the Fourier transform technique to problem (1) with respect to the variable x, we can get the Fourier transform $\hat{u}(\xi, t)$ of the exact solution $u(x, t)$ of problem (1):

$$\hat{u}(\xi, t) = e^{(T-t)\xi^2}\hat{\varphi}(\xi) - \int_t^T e^{-(t-s)\xi^2}\hat{f}(\xi, s, u)ds. \tag{7}$$

Since $t < T$, we know from (7) that, when $|\xi|$ becomes large, $\exp\{(T-t)\xi^2\}$ and $\exp\{(s-t)\xi^2\}$ increase rather quickly. Thus, these terms are the unstability cause. Hence, to regularize the problem, we have to replace the terms by some better terms. In our idea, we shall replace them by $\frac{e^{-(t+m)\xi^2}}{\epsilon + e^{-(T+m)\xi^2}}$ and $\frac{e^{(s-t-T-m)\xi^2}}{\epsilon + e^{-(T+m)\xi^2}}$ $(m > 0)$, respectively. The main conclusion of this paper is:

Theorem 1. *Let f be as Lemma 1. Let $\varphi \in L^2(R)$ and let $\varphi_\epsilon \in L^2(R)$ be a measured data such that $\|\varphi_\epsilon - \varphi\| \le \epsilon$. Suppose that problem (1) has a unique solution $u \in C([0,T], H^2(R)) \cap C^1([0,T], L^2(R))$ such that*

$$\int_{-\infty}^{+\infty} \left| e^{(t+m)\xi^2} \hat{u}(\xi,t) \right|^2 d\xi < \infty. \tag{8}$$

Then, we construct a regularized solution w_ϵ such that

$$\|u(.,t) - w_\epsilon(.,t)\| \le C\epsilon^{\frac{t+m}{T+m}}, \forall t \in [0,T],$$

where w_ϵ is the function whose Fourier transform is defined by

$$\hat{w}_\epsilon(\xi,t) = \frac{e^{-(t+m)\xi^2}}{\epsilon + e^{-(T+m)\xi^2}} \hat{\varphi}_\epsilon(\xi) - \int_t^T \frac{e^{(s-t-T-m)\xi^2}}{\epsilon + e^{-(T+m)\xi^2}} \hat{f}(\xi,s,w_\epsilon) ds, \tag{9}$$

for $m \ge 0$ and

$$B = 2 \sup_{0 \le t \le T} \left(\int_{-\infty}^{+\infty} \left| e^{(t+m)\xi^2} \hat{u}(\xi,t) \right|^2 d\xi \right),$$
$$C = \sqrt{2} e^{K^2(T-t)^2} + \sqrt{B} e^{\frac{3T^2K^2}{2}}.$$

Remark 1. *a) Tautenhahn and Schröter [13] regularized the homogeneous problem ($f = 0$) and obtained the following error estimate*

$$\|u(.,t) - u^\beta(.,t)\| \le 2E^{1-\frac{t}{T}} \epsilon^{\frac{t}{T}},$$

where E is a positive constant such that

$$\|u(.,0)\| \le E. \tag{10}$$

They also proved that it is an order optimal stability estimate in $L^2(R)$. If $m = 0$ and $f = 0$, then we have $\int_{-\infty}^{+\infty} \left| e^{(t+m)\xi^2} \hat{u}(\xi,t) \right|^2 d\xi = \|u(.,0)\|^2$. Thus, condition (8) is similar to (10) and it may be acceptable. Moreover, in this case, our result is of the same order as the results of Tautenhahn with the same condition.

b) If $m = 0$ and $f = f(x,t,u)$, then the error is similar to the results obtained by Trong and Quan [10]. However, the order $\epsilon^{\frac{t}{T}}$ is useful at $t > 0$, but useless at $t = 0$. Moreover, when $t \to 0^+$, the accuracy of the regularized solution becomes progressively lower. To improve this, we choose $m > 0$, then the error is of order $\epsilon^{\frac{m}{T+m}}$. This error estimate is not introduced in the ealier work of Quan and Trong [10].

4. Proof of the main results

First, we consider the Lemma which is useful to this paper.

Lemma 2. *Let $s, t, T, \epsilon, m, \xi$ be real numbers such that $0 \le t \le s \le T$ and $\epsilon > 0, m \ge 0$. Then the following estimates hold*

$$a)\quad \frac{e^{-(t+m)\xi^2}}{\epsilon + e^{-(T+m)\xi^2}} \le \epsilon^{\frac{t-T}{T+m}}.$$

$$b)\quad \frac{e^{(s-t-T-m)\xi^2}}{\epsilon + e^{-(T+m)\xi^2}} \le \epsilon^{\frac{t-s}{T+m}}.$$

Proof. We have

$$\begin{aligned}\frac{e^{-(t+m)\xi^2}}{\epsilon + e^{-(T+m)\xi^2}} &= \frac{e^{-(t+m)\xi^2}}{\left(\epsilon + e^{-(T+m)\xi^2}\right)^{\frac{t+m}{T+m}}\left(\epsilon + e^{-(T+m)\xi^2}\right)^{\frac{T-t}{T+m}}}\\ &\le \frac{1}{\left(\epsilon + e^{-(T+m)\xi^2}\right)^{\frac{T-t}{T+m}}}\\ &\le \epsilon^{\frac{t-T}{T+m}}.\end{aligned}$$

and

$$\begin{aligned}\frac{e^{(s-t-T-m)\xi^2}}{\epsilon + e^{-(T+m)\xi^2}} &= \frac{e^{(s-t-T-m)\xi^2}}{\left(\epsilon + e^{-(T+m)\xi^2}\right)^{\frac{T+m-s+t}{T+m}}\left(\epsilon + e^{-(T+m)\xi^2}\right)^{\frac{s-t}{T+m}}}\\ &\le \frac{1}{\left(\epsilon + e^{-(T+m)\xi^2}\right)^{\frac{s-t}{T+m}}}\\ &\le \epsilon^{\frac{t-s}{T+m}}.\end{aligned}$$

□

Next, we continue to prove the main Theorem. We divide the proof into three steps.

Step 1. Construct a regularized solution w_ϵ.
We consider the following problem

$$\hat{w}_\epsilon(\xi, t) = \frac{e^{-(t+m)\xi^2}}{\epsilon + e^{-(T+m)\xi^2}}\hat{\varphi}_\epsilon(\xi) - \int_t^T \frac{e^{(s-t-T-m)\xi^2}}{\epsilon + e^{-(T+m)\xi^2}}\hat{f}(\xi, s, w_\epsilon)ds,$$

or

$$\begin{aligned}w_\epsilon(x, t) = &\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} \frac{e^{-(t+m)\xi^2}}{\epsilon + e^{-(T+m)\xi^2}}\hat{\varphi}_\epsilon(\xi)e^{i\xi x}d\xi\\ &-\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\int_t^T \frac{e^{(s-t-T-m)\xi^2}}{\epsilon + e^{-(T+m)\xi^2}}\hat{f}(\xi, s, w_\epsilon)e^{i\xi x}dsd\xi. \qquad (11)\end{aligned}$$

First, we prove problem (11) has a unique solution $w_\epsilon \in C([0,T];L^2(R))$. Denote

$$G(w)(x,t) = \frac{1}{\sqrt{2\pi}}\psi(x,t) - \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\int_t^T \frac{e^{(s-t-T-m)\xi^2}}{\epsilon + e^{-(T+m)\xi^2}}\hat{f}(\xi,s,w)e^{i\xi x}dsd\xi$$

for all $w \in C([0,T];L^2(R))$ and

$$\psi(x,t) = \int_{-\infty}^{+\infty} \frac{e^{-(t+m)\xi^2}}{\epsilon + e^{-(T+m)\xi^2}}\hat{\varphi}_\epsilon(\xi)e^{i\xi x}d\xi.$$

Since $f(x,y,0) = 0$, and the Lipschitzian property of $f(x,y,w)$ with respect to w, we get $G(w) \in C([0,T];L^2(R))$ for every $w \in C([0,T];L^2(R))$. We claim that, for every $w, v \in C([0,T];L^2(R)), n \geq 1$, we have

$$\|G^n(w)(.,t) - G^n(v)(.,t)\|^2 \leq \left(\frac{K}{\epsilon}\right)^{2n} \frac{(T-t)^n C^n}{n!}|||w-v|||^2, \tag{12}$$

where $C = \max\{T,1\}$ and $|||.|||$ is the sup norm in $C([0,T];L^2(R))$. We shall prove the latter inequality by induction.
When $n = 1$, we have

$$\begin{aligned}
\|G(w)(.,t) - G(v)(.,t)\|^2 &= \|\hat{G}(w)(.,t) - \hat{G}(v)(.,t)\|^2 \\
&= \int_{-\infty}^{+\infty}\left|\int_t^T \frac{e^{(s-t-T-m)\xi^2}}{\epsilon + e^{-(T+m)\xi^2}}\Big(\hat{f}(\xi,s,w) - \hat{f}(\xi,s,v)\Big)ds\right|^2 d\xi \\
&\leq \int_{-\infty}^{+\infty}\left(\int_t^T\left(\frac{e^{(s-t-T-m)\xi^2}}{\epsilon + e^{-(T+m)\xi^2}}\right)^2 ds \int_t^T \left|\hat{f}(\xi,s,w) - \hat{f}(\xi,s,v)\right|^2 ds\right)^2 d\xi \\
&\leq \frac{1}{\epsilon^2}(T-t)\int_t^T \|\hat{f}(.,s,w(.,s)) - \hat{f}(.,s,v(.,s))\|^2 ds = \\
&= \frac{1}{\epsilon^2}(T-t)\int_t^T \|f(.,s,w(.,s)) - f(.,s,v(.,s))\|^2 ds \\
&= \frac{K^2}{\epsilon^2}(T-t)\int_t^T \|w(.,s) - v(.,s)\|^2 ds \\
&\leq C\frac{K^2}{\epsilon^2}(T-t)|||w-v|||^2.
\end{aligned}$$

Therefore (12) holds.

Suppose that (12) holds for $n = p$. We prove that (12) holds for $n = p+1$. We have

$$
\begin{aligned}
\|G^{p+1}(w)(.,t) \,-\, G^{p+1}(v)(.,t)\|^2 &= \|\hat{G}(G^p(w))(.,t) - \hat{G}(G^p(v))(.,t)\|^2 \\
&= \int_{-\infty}^{+\infty} \left| \int_t^T \frac{e^{(s-t-T-m)\xi^2}}{\epsilon + e^{-(T+m)\xi^2}} \left(\hat{f}(\xi,s,G^p(w)) - \hat{f}(\xi,s,G^p(v)) \right) ds \right|^2 d\xi \\
&\le \int_{-\infty}^{+\infty} \left(\int_t^T \left(\frac{e^{(s-t-T-m)\xi^2}}{\epsilon + e^{-(T+m)\xi^2}} \right)^2 ds \right. \\
&\quad \times \left. \int_t^T \left| \hat{f}(\xi,s,G^p(w)) - \hat{f}(\xi,s,G^p(v)) \right|^2 ds \right)^2 d\xi.
\end{aligned}
$$

Hence

$$
\begin{aligned}
&\|G^{p+1}(w)(.,t) \,-\, G^{p+1}(v)(.,t)\|^2 \\
&\qquad \le \frac{1}{\epsilon^2}(T-t) \int_t^T \|f(.,s,G^p(w)(.,s)) - f(.,s,G^p(v)(.,s))\|^2 ds \\
&\qquad \le \frac{K^2}{\epsilon^2}(T-t) \int_t^T \|G^p(w)(.,s) - G^p(v)(.,s)\|^2 ds \\
&\qquad \le \frac{K^2}{\epsilon^2}(T-t) \left(\frac{K}{\epsilon}\right)^{2p} \int_t^T \frac{(T-s)^p}{p!} ds C^p |||w-v|||^2 \\
&\qquad \le \left(\frac{K}{\epsilon}\right)^{2(p+1)} \frac{(T-t)^{(p+1)} C^{(p+1)}}{(p+1)!} |||w-v|||^2.
\end{aligned}
$$

Therefore, by the induction principle, (12) holds for every m.

$$
|||G^m(w) - G^m(v)||| \le \left(\frac{K}{\epsilon}\right)^m \frac{T^{m/2}}{\sqrt{m!}} C^m |||w-v|||,
$$

for every $w, v \in C([0,T];L^2(R))$. Consider $G : C([0,T];L^2(R)) \to C([0,T];L^2(R))$.

Since

$$
\lim_{m\to\infty} \left(\frac{K}{\epsilon}\right)^m \frac{T^{m/2} C^m}{\sqrt{m!}} = 0,
$$

there exists a positive integer number m_0 such that G^{m_0} is a contraction. It follows that $G^{m_0}(w) = w$ has a unique solution $w_\epsilon \in C([0,T];L^2(R))$.

We claim that $G(w_\epsilon) = w_\epsilon$. In fact, one has $G(G^{m_0}(w_\epsilon)) = G(w_\epsilon)$. Hence $G^{m_0}(G(w_\epsilon)) = G(w_\epsilon)$. By the uniqueness of the fixed point of G^{m_0}, one has $G(w_\epsilon) = w_\epsilon$, i.e., the equation $G(w) = w$ has a unique solution $w_\epsilon \in C([0,T];L^2(R))$. The main purpose of this paper is to estimate the error $\|w_\epsilon - u\|$. To do this, we consider two next steps.

Step 2. Let u_ϵ be the solution of problem (11) corresponding to the final value φ. We shall estimate the error $\|w_\epsilon - u_\epsilon\|$.

From the formula of w_ϵ and u_ϵ, we have

$$\hat{w}_\epsilon(\xi,t) = \frac{e^{-(t+m)\xi^2}}{\epsilon + e^{-(T+m)\xi^2}}\hat{\varphi}_\epsilon(\xi) - \int_t^T \frac{e^{(s-t-T-m)\xi^2}}{\epsilon + e^{-(T+m)\xi^2}}\hat{f}(\xi,s,w_\epsilon)ds, \tag{13}$$

and

$$\hat{u}_\epsilon(\xi,t) = \frac{e^{-(t+m)\xi^2}}{\epsilon + e^{-(T+m)\xi^2}}\hat{\varphi}(\xi) - \int_t^T \frac{e^{(s-t-T-m)\xi^2}}{\epsilon + e^{-(T+m)\xi^2}}\hat{f}(\xi,s,u_\epsilon)ds, \tag{14}$$

Using the Parseval equality and the inequality $(a+b)^2 \le 2a^2 + 2b^2$, we get

$$\begin{aligned}\|w_\epsilon(.,t) - u_\epsilon(.,t)\|^2 &= \|\hat{w}_\epsilon(.,t) - \hat{u}_\epsilon(.,t)\|^2 \\ &\le 2\int_{-\infty}^{+\infty}\left|\frac{e^{-(t+m)\xi^2}}{\epsilon + e^{-(T+m)\xi^2}}\left(\hat{\varphi}_\epsilon(\xi) - \hat{\varphi}(\xi)\right)\right|^2 d\xi \\ &\quad +2\int_{-\infty}^{+\infty}\left|\int_t^T \frac{e^{(s-t-T-m)\xi^2}}{\epsilon + e^{-(T+m)\xi^2}}\left(\hat{f}(\xi,s,w_\epsilon) - \hat{f}(\xi,s,u_\epsilon)\right)ds\right|^2 d\xi.\end{aligned} \tag{15}$$

Term (15) can be estimated as follows

$$\begin{aligned}J_1 &= 2\int_{-\infty}^{+\infty}\left|\frac{e^{-(t+m)\xi^2}}{\epsilon + e^{-(T+m)\xi^2}}\left(\hat{\varphi}_\epsilon(\xi) - \hat{\varphi}(\xi)\right)\right|^2 d\xi \\ &\le 2\epsilon^{\frac{2t-2T}{T+m}}\|\hat{\varphi}_\epsilon - \hat{\varphi}\|^2 \le 2\epsilon^{\frac{2t-2T}{T+m}}\|\varphi_\epsilon - \varphi\|^2.\end{aligned} \tag{16}$$

Term (15) can be estimated as follows

$$\begin{aligned}J_2 &= 2\int_{-\infty}^{+\infty}\left|\int_t^T \frac{e^{(s-t-T-m)\xi^2}}{\epsilon + e^{-(T+m)\xi^2}}\left(\hat{f}(\xi,s,w_\epsilon) - \hat{f}(\xi,s,u_\epsilon)\right)ds\right|^2 d\xi \\ &\le 2(T-t)\int_{-\infty}^{+\infty}\int_t^T \epsilon^{\frac{2t-2s}{T+m}}\left|\hat{f}(\xi,s,w_\epsilon) - \hat{f}(\xi,s,u_\epsilon)\right|^2 ds d\xi \\ &\le 2(T-t)K^2\int_t^T \epsilon^{\frac{2t-2s}{T+m}}\|w_\epsilon(.,s) - u_\epsilon(.,s)\|^2 ds.\end{aligned} \tag{17}$$

Combining (15), (16) and (17) we have

$$\|w_\epsilon(.,t)-u_\epsilon(.,t)\|^2 \le 2\epsilon^{\frac{2t-2T}{T+m}}\|\varphi_\epsilon-\varphi\|^2 + 2(T-t)K^2\int_t^T \epsilon^{\frac{2t-2s}{T+m}}\|w_\epsilon(.,s)-u_\epsilon(.,s)\|^2\,ds.$$

Hence

$$\epsilon^{\frac{-2t}{T+m}}\|w_\epsilon(.,t)-u_\epsilon(.,t)\|^2 \le 2\epsilon^{\frac{-2T}{T+m}}\|\varphi_\epsilon-\varphi\|^2 + 2K^2(T-t)\int_t^T \epsilon^{\frac{-2s}{T+m}}\|w_\epsilon(.,s)-u_\epsilon(.,s)\|^2 ds.$$

Using the Gronwall inequality, we obtain

$$\epsilon^{\frac{-2t}{T+m}}\|w_\epsilon(.,t)-u_\epsilon(.,t)\|^2 \le 2e^{2K^2(T-t)^2}\epsilon^{\frac{-2T}{T+m}}\|\varphi_\epsilon-\varphi\|^2.$$

Therefore

$$\begin{aligned}\|w_\epsilon(.,t)-u_\epsilon(.,t)\| &\le \sqrt{2}\epsilon^{\frac{t-T}{T+m}}e^{K^2(T-t)^2}\|\varphi_\epsilon-\varphi\| \\ &\le \sqrt{2}\epsilon^{\frac{t-T}{T+m}}e^{K^2(T-t)^2}\epsilon \\ &= \sqrt{2}e^{K^2(T-t)^2}\epsilon^{\frac{t+m}{T+m}}.\end{aligned}$$

Step 3. Let u be the exact solution of problem (1) corresponding to the final value φ. We shall estimate the error $\|u_\epsilon - u\|$.
Let u_ϵ be the function which is defined in Step 2. We recall the Fourier transform of u and u_ϵ from (7) and (14)

$$\hat{u}(\xi,t) = e^{(T-t)\xi^2}\hat{\varphi}(\xi) - \int_t^T e^{(s-t)\xi^2}\hat{f}(\xi,s,u)ds. \tag{18}$$

and

$$\hat{u}_\epsilon(\xi,t) = \frac{e^{-(t+m)\xi^2}}{\epsilon+e^{-(T+m)\xi^2}}\hat{\varphi}(\xi) - \int_t^T \frac{e^{(s-t-T-m)\xi^2}}{\epsilon+e^{-(T+m)\xi^2}}\hat{f}(\xi,s,u_\epsilon)ds, \tag{19}$$

Combining (18) and (19) and by direct transform, we get

$$\begin{aligned}\hat{u}(\xi,t)-\hat{u}_\epsilon(\xi,t) = &\left(e^{(T-t)\xi^2} - \frac{e^{-(t+m)\xi^2}}{\epsilon+e^{-(T+m)\xi^2}}\right)\hat{\varphi}(\xi) - \int_t^T e^{-(t-s)\xi^2}\hat{f}(\xi,s,u)ds \\ &+ \int_t^T \frac{e^{(s-t-T-m)\xi^2}}{\epsilon+e^{-(T+m)\xi^2}}\hat{f}(\xi,s,u_\epsilon)ds\end{aligned}$$

$$= \frac{\epsilon e^{(T-t)\xi^2}}{\epsilon + e^{-(T+m)\xi^2}}\hat{\varphi}(\xi) + \int_t^T \frac{e^{(s-t-T-m)\xi^2}}{\epsilon + e^{-(T+m)\xi^2}}\left(\hat{f}(\xi,s,u) - \hat{f}(\xi,s,u_\epsilon)\right)ds$$
$$- \int_t^T \frac{e^{(s-t)\xi^2}\epsilon}{\epsilon + e^{-(T+m)\xi^2}}\hat{f}(\xi,s,u)ds.$$

Using the Parseval equality, we obtain

$$\|u(.,t) - u_\epsilon(.,t)\|^2 = \int_{-\infty}^{+\infty} |\hat{u}(\xi,t) - \hat{u}_\epsilon(\xi,t)|^2 d\xi$$
$$= \int_{-\infty}^{+\infty}\Bigg| \frac{\epsilon e^{(T-t)\xi^2}}{\epsilon + e^{-(T+m)\xi^2}}\hat{\varphi}(\xi) + \int_t^T \frac{e^{(s-t-T-m)\xi^2}}{\epsilon + e^{-(T+m)\xi^2}}(\hat{f}(\xi,s,u) - \hat{f}(\xi,s,u_\epsilon))ds$$
$$- \int_t^T \frac{e^{(s-t)\xi^2}\epsilon}{\epsilon + e^{-(T+m)\xi^2}}\hat{f}(\xi,s,u)ds\Bigg|^2 d\xi$$
$$= \int_{-\infty}^{+\infty}\left| \frac{\epsilon}{\epsilon + e^{-(T+m)\xi^2}}\hat{u}(\xi,t) + \int_t^T \frac{e^{(s-t-T-m)\xi^2}}{\epsilon + e^{-(T+m)\xi^2}}(\hat{f}(\xi,s,u) - \hat{f}(\xi,s,u_\epsilon))ds\right|^2 d\xi$$
$$\le 2\int_{-\infty}^{+\infty}\left| \frac{\epsilon e^{-(t+m)\xi^2}}{\epsilon + e^{-(T+m)\xi^2}} e^{(t+m)\xi^2}\hat{u}(\xi,t)\right|^2 d\xi$$
$$+2\int_{-\infty}^{+\infty}\left| \int_t^T \frac{e^{(s-t-T-m)\xi^2}}{\epsilon + e^{-(T+m)\xi^2}}|\hat{f}(\xi,s,u) - \hat{f}(\xi,s,u_\epsilon)|ds\right|^2 d\xi.$$

Using Lemma 1, we get

$$\begin{aligned}\|u(.,t) - u_\epsilon(.,t)\|^2 &\le 2\epsilon^{\frac{2t+2m}{T+m}}\int_{-\infty}^{+\infty}\left|e^{(t+m)\xi^2}\hat{u}(\xi,t)\right|^2 d\xi + \\ &\quad +2\int_{-\infty}^{+\infty}\left|\int_t^T \epsilon^{\frac{t-s}{T+m}}\left|\hat{f}(\xi,s,u) - \hat{f}(\xi,s,u_\epsilon)\right|ds\right|^2 d\xi \\ &= 2\widetilde{A_1} + 2\widetilde{A_2},\end{aligned}$$

where the term $\widetilde{A_1}$ is equal to

$$\widetilde{A_1} = \epsilon^{\frac{2t+2m}{T+m}}\int_{-\infty}^{+\infty}\left|e^{(t+m)\xi^2}\hat{u}(\xi,t)\right|^2 d\xi.$$

We estimate $\widetilde{A_2}$ as follows

$$\begin{aligned}
\widetilde{A_2} &= \int_{-\infty}^{+\infty} \left| \int_t^T \epsilon^{\frac{t-s}{T+m}} \left| \hat{f}(\xi,s,u) - \hat{f}(\xi,s,u_\epsilon) \right| ds \right|^2 d\xi \\
&\le \epsilon^{\frac{2t+2m}{T+m}} (T-t) \int_{-\infty}^{+\infty} \int_t^T \epsilon^{\frac{-2s-2m}{T+m}} \left| \hat{f}(\xi,s,u) - \hat{f}(\xi,s,u_\epsilon) ds \right|^2 d\xi \\
&= \epsilon^{\frac{2t+2m}{T+m}} (T-t) \int_t^T \epsilon^{\frac{-2s-2m}{T+m}} \left\| f(.,s,u(.,s)) - f(.,s,u_\epsilon(.,s)) \right\|^2 ds \\
&\le \epsilon^{\frac{2t+2m}{T+m}} K^2 (T-t) \int_t^T \epsilon^{\frac{-2s-2m}{T+m}} \left\| u(.,s) - u_\epsilon(.,s) \right\|^2 ds.
\end{aligned}$$

Hence

$$\begin{aligned}
\left\| u(.,t) - u_\epsilon(.,t) \right\|^2 \le\ & 2\epsilon^{\frac{2t+2m}{T+m}} \int_{-\infty}^{+\infty} \left| e^{(t+m)\xi^2} \hat{u}(\xi,t) \right|^2 d\xi \\
&+ 2\epsilon^{\frac{2t+2m}{T+m}} K^2 (T-t) \int_t^T \epsilon^{\frac{-2s-2m}{T+m}} \left\| u(.,s) - u_\epsilon(.,s) \right\|^2 ds.
\end{aligned}$$

Thus

$$\epsilon^{\frac{-2t-2m}{T+m}} \left\| u(.,t) - u_\epsilon(.,t) \right\|^2 \le B + 3K^2 T \int_t^T \epsilon^{\frac{-2s-2m}{T+m}} \left\| u(.,s) - u_\epsilon(.,s) \right\|^2 ds.$$

Applying the Gronwall inequality, we obtain

$$\epsilon^{\frac{-2t-2m}{T+m}} \left\| u(.,t) - u_\epsilon(.,t) \right\|^2 \le B e^{3K^2 T(T-t)}.$$

Hence, we conclude that

$$\left\| u(.,t) - u_\epsilon(.,t) \right\| \le \sqrt{B}.e^{\frac{3T^2K^2}{2}} \epsilon^{\frac{t+m}{T+m}}.$$

Due to Step 2 and Step 3, we get the following estimate by using the triangle inequality

$$\begin{aligned}
\left\| w_\epsilon(.,t) - u(.,t) \right\| &\le \left\| w_\epsilon(.,t) - u_\epsilon(.,t) \right\| + \left\| u_\epsilon(.,t) - u(.,t) \right\| \\
&\le \sqrt{2} e^{K^2(T-t)^2} \epsilon^{\frac{t+m}{T+m}} + \sqrt{B}.e^{\frac{3T^2K^2}{2}} \epsilon^{\frac{t+m}{T+m}} \\
&\le C \epsilon^{\frac{t+m}{T+m}}
\end{aligned}$$

where

$$C = \sqrt{2} e^{K^2(T-t)^2} + \sqrt{B} \exp\{\frac{3T^2K^2}{2}\},$$

for all $t \in [0,T]$. This completes the proof of the theorem.

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