MATHEMATICAL COMMUNICATIONS 283
Math. Commun., Vol. **14**, No. 2, pp. 283-286 (2009)

Weak-compactness of some tight vector measures of bounded variations

Surjit Singh Khurana[1,*]

[1] *Department of Mathematics, University of Iowa, Iowa City, Iowa 52242-1419, U.S.A.*

Received December 2, 2007; accepted September 2, 2009

Abstract. Let X be a completely regular Hausdorff space, E a Banach space, and $M_{tbv}(X, E)$ the space of all E-valued tight measures of bounded variations. We prove some result about the weak compactness of some special subsets of $M_{tbv}(X, E)$.

AMS subject classifications: Primary 60B05,46G10, 28C15; Secondary 46E10, 28A05

Key words: strict topologies, vector measures, weak compactness and sequential compactness of sets of vector measures

1. Introduction and notations

When X is a separable metric space, in [3] an interesting result is proved about the sequential compactness for some special subsets of $M_{tbv}(X, E)$. The proof is quite sophisticated and long. In this paper we give a rather short proof of a more general result and then derive the result of [3] from this general result without the assumption of separability. Our methods of proofs are very different from the ones given in [3].

In this paper K will always denote the field of real or complex numbers (we will call them scalars), X a completely regular Hausdorff space and $C_b(X)$ all bounded continuous scalar-valued functions on X. E will denote a Banach space over K and E', E'' will be its continuous dual and bidual, respectively. $C_b(X, E)$ will stand for all E-valued, bounded continuous functions on X. For locally convex spaces, the notations and results of [10] will be used and for measures, notations and results of [4, 7, 8, 11] will be used. For a locally convex space F, F' will be its continuous dual.

Denoting by $\mathcal{B}(X)$ the class of Borel subsets of X, a countably additive measure, of finite variation, $\mu : \mathcal{B}(X) \to E$ is called tight if the total variation measure $|\mu|$ is inner regular by compact subsets of X.

On $C_b(X)$, the topology β_t is defined as the finest locally convex, agreeing with the topology of uniform convergence on the compact subsets of X, on norm-bounded subsets of $C_b(X)$ (see [11]). For a Banach space F, this topology β_t is defined in a similar way on $C_b(X, F)$ (see [5]); we have $(C_b(X, F), \beta_t)' = M_{tbv}(X, F')$ and $H \subset M_{tbv}(X, F')$ is β_t-equicontinuous iff H is uniformly bounded in total variation norm and uniformly tight (see [4, 5, 11]).

*Corresponding author. *Email address:* khurana@math.uiowa.edu (S. S. Khurana)

All measures in this paper are countably additive and integration with respect to vector measures is taken in the sense of [4, 7, 8].

2. Weak compactness in $M_{tbv}(X,E)$

The weak topology on $M_{tbv}(X,E)$ is the induced topology when $M_{tbv}(X,E)$ is considered as a subspace of $E^{C_b(X)}$ with weak topology on E (see [2, 3]).

We start with a general theorem:

Theorem 1. *Suppose $H \subset M_{tbv}(X,E)$ is uniformly bounded in total variation norm, uniformly tight, and also has the property that for every compact $K \subset X$, $\{\nu(K) : \nu \in H\} \subset C_K$, a relatively weakly compact subset of E. Then H is relatively compact in $M_{tbv}(X,E)$ with weak topology.*

Proof. We can assume that $|\nu|(X) \leq 1, \forall \nu \in H$. Considering E to be embedded in E'', H can be considered a subset of $M_{tbv}(X,E'') = (C_b(X,E'),\beta_t)'$ (see [4]). Using the fact that H is uniformly bounded in total variation norm and uniformly tight, we get that H is equicontinuous (see [5, Lemma 2]) and so its closure in

$$(M_{tbv}(X,E''), \sigma(M_{tbv}(X,E''), C_b(X,E')))$$

is relatively compact. Take a net $\{\mu_\alpha\} \subset H$. This means there exists a subnet, which again we denote by $\{\mu_\alpha\}$, such that

$$\mu_\alpha \to \mu \in (M_{tbv}(X,E''), \sigma(M_{tbv}(X,E''), C_b(X,E'))).$$

We claim that $\mu(f) \in E$, $\forall f \in C_b(X)$:
Take an $f \in C_b(X)$, $0 \leq f \leq 1$, and assume that $\mu(f) \in E'' \setminus E$. There is an $\eta > 0$ such that

$$\inf\{\|\mu(f) - x\| : x \in E\} > 4\eta. \tag{1}$$

Take a compact $K \subset X$ such that $|\nu|(X \setminus K) < \eta$, $\forall \nu \in H$. Fix a large positive integer $k > \frac{1}{\eta}$. For i, $1 \leq i \leq k$, let $Z_i = f^{-1}[\frac{i}{k}, 1]$. On X we get $\frac{1}{k}\sum_{i=1}^k \chi_{Z_i} \leq f \leq \frac{1}{k} + \frac{1}{k}\sum_{i=1}^k \chi_{Z_i}$. From this we get, on X, $0 \leq f - s \leq \frac{1}{k}$, where $s = \frac{1}{k}\sum_{i=1}^k \chi_{Z_i}$. For any $\nu \in H$, $\int s d\nu = \int_K s d\nu + \int_{X\setminus K} s d\nu$, and $\|\int_{X\setminus K} s d\nu\| \leq \eta$. Also $\int_K s d\nu = \frac{1}{k}\sum_{i=1}^k \nu(Z_i \cap K)$ is, by the hypothesis, contained in some weakly compact, absolutely convex subset C of E. Further $\|\int (f-s) d\nu\| \leq \frac{1}{k}$. Combining these results, $\mu_\alpha(f) \in C + \eta B + \frac{1}{k}B \subset C + 2\eta B, \forall \alpha$, B being the closed unit ball of E''. Taking limits in $\sigma(E'', E')$-topology, we get $\mu(f) \in C + 2\eta B$. This contradicts (1). This proves the claim. Thus H is relatively compact in $(M_{tbv}(X,E), \sigma(M_{tbv}(X,E), C_b(X,E')))$. From this it easily follows that H is relatively compact in $M_{tbv}(X,E)$ with weak topology.

□

Now we consider some sufficient conditions for sequential compactness.

Theorem 2. *Suppose X is a metric space, $H \subset M_{tbv}(X,E)$, uniformy bounded in total variation norm, and also has the property that for any compact $K \subset X$,*

$\{\nu(K) : \nu \in H\} \subset C_K$, a relatively weakly compact subset of E. If every sequence in H has a uniformly tight subsequence, then H is relatively sequentially compact in weak topology. If X is a complete metric space, then, in addition, H is also relatively compact in weak topology.

Proof. Take a sequence $\{\mu_n : 1 \leq n < \infty\} \subset H$. By hypothesis, there is a subsequence, which again we denote by $H_0 = \{\mu_n : 1 \leq n < \infty\}$ such that H_0 is uniformly tight. By [5, Lemma 2], H_0 is β_t-equicontinuous in

$$(M_{tbv}(X, E''), \sigma(M_{tbv}(X, E''), C_b(X, E'))),$$

and so its closure $\bar{H}_0$, in $(M_{tbv}(X, E''), \sigma(M_{tbv}(X, E''), C_b(X, E')))$, is compact and also β_t-equicontinuous. By Theorem 1, $\bar{H}_0 \subset M_{tbv}(X, E)$ and it is compact in $(M_{tbv}(X, E), \sigma(M_{tbv}(X, E), C_b(X, E')))$. Let $\{Q_n\}$ be an increasing sequence of compact subsets of X such that $|\nu|(X \setminus Q_n) < \frac{1}{n}$, $\forall \nu \in \bar{H}_0$. Since Q_n are compact metric spaces, $C(Q_n)$ are norm separable; for every k, choose sequence $\{g_n^k\} \subset C_b(X)$ such that $\{g_n^k\}$, when restricted to Q_k, are norm-dense in each of $C(Q_k)$. Put $P = \{g_n^k : 1 \leq n,\ k < \infty\}$ and $B = \{g \in E' : \|g\| \leq 1\}$. With discrete topology on P and $\sigma(E', E)$ topology on B, elements of $M_{tbv}(X, E)$ can be considered as continuous functions on the product topological space $P \times B$ which has a σ-compact dense subset. Now it is a simple verification that $P \times B$ separates the points of $\bar{H}_0$ and the topology, on $\bar{H}_0$, of pointwise convergence on $P \times B$ is weaker than $\sigma(M_{tbv}(X, E), C_b(X, E'))$. Thus these two topologies coincide on $\bar{H}_0$. By [9], there is a subsequence of $\{\mu_n\}$ which is convergent. From this it follows that H is relatively sequentially compact in weak topology.

If X is a complete metric space, then $\beta_t = \beta_\infty = \beta_\tau = \beta_g$ on $C_b(X, E)$ and so

$$(M_{tbv}(X, E), \tau(M_{tbv}(X, E), C_b(X, E')))$$

is complete (see [6]); this means H will be $\sigma(M_{tbv}(X, E), C_b(X, E'))$ relatively compact if every sequence in H has a cluster point (see [10, Theorem 11.2]). The result easily follows from this. □

Acknowledgement

We are very thankful to the referees for pointing out some typographical errors and also making some very useful suggestions which have improved the paper.

References

[1] J. Diestel, J. J. Uhl, *Vector Measures*, American Mathematical Society, Providence, 1977.

[2] M. Dekiert, *Kompaktheit, Fortsetzbarkeit und Konvergenz von Vectormassen*, Dissertation, University of Essen, 1991, in German.

[3] M. C. Isidori, *Weak convergence and Prokhorov's Theorem for measures in a Banach space*, J. Concr. Appl. Math. **4**(2006), 325–347.

[4] S. S. Khurana, *Topologies on spaces of continuous vector-valued functions*, Trans. Amer. Math. Soc. **241**(1978), 195–211.

[5] S. S. KHURANA, S. A. CHOO, *Strict Topology and P-spaces*, Proc. Amer. Math. Soc. **61**(1976), 280–284.

[6] S. S. KHURANA, *Completeness and sequential completeness in certain spaces of measures*, Math. Slovaca **45**(1995), 163–170.

[7] I., KLUVANEK, G. KNOWLES, *Vector measures and control systems*, North-Holand, Amsterdam, 1975.

[8] D. R. LEWIS, *Integration with respect to vector measures*, Pac. J. Math. **33**(1970), 157–165.

[9] J. D. PRYCE, *A device of R. J. Whitley applied to pointwise compactness in spaces of continuous functions*, Proc. Lon. math. Soc. **23**(1971), 532–546.

[10] H. H. SCHAEFFER, *Topological Vector spaces*, Springer, New York, 1986.

[11] R. F. WHEELER, *Survey of Baire measures and strict topologies*, Expos. Math. **2**(1983), 97–190.