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A generalization of paracompactness in terms of grills*

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Abstract. In this paper we initiate a generalized form of paracompactness via grills. We shall also define a grill-dependent regularity axiom on a topological space. From our results in such a generalized perspective, E. Michael's famous theorem on regular paracompactness and certain other results like "a T_2 paracompact space is regular" will follow as special cases.

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1. Introduction

In 1947, Choquet [3] introduced the concept of grills, which was subsequently found to be an extremely useful device, like filters and nets, for the investigations of many topological notions like compactifications, proximity spaces, different types of extension problems etc. (see [1, 2, 10] for details). In an earlier paper [7], we gave the formulation of a new topology on a given topological space, constructed from the existing topology of the space and a given grill. We also studied therein this topology in some detail specially under some special conditions imposed on the grill, in question.

Hence in this paper our aim is to introduce a kind of paracompactness-type of notion in a topological space by means of grills. Such an endeavour gives rise to a generalized version of paracompactness; a similar attempt towards such a generalization under the terminology "I-paracompactness" was undertaken by Hamlett et al. [5]. In the process we derive a type of regularity in terms of grills of some special form. All these ultimately facilitate us to achieve a general form of the well known Michael's theorem on regular paracompact spaces; the result that a T_2 paracompact space is regular is also obtained as a particular case of our generalized results.

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2. $\mathcal{G}$-paracompactness

We begin by recalling the definition of a grill, first proposed by Choquet [3].

Definition 1. *A collection $\mathcal{G}$ of nonempty subsets of a set X is called a grill if (i)$A \in \mathcal{G}$ and $A \subseteq B \subseteq X \Rightarrow B \in \mathcal{G}$, and (ii) $A \cup B \in \mathcal{G}(A, B \subseteq X) \Rightarrow A \in \mathcal{G}$ or $B \in \mathcal{G}$.*

In what follows, by a space X we shall mean a topological space (X, τ), and the notation $\mathcal{G}$ will stand for a grill on X. For any subset A of a space (X, τ) int$A$ and cl$A$ will, as usual, denote the interior and closure of A in (X, τ), respectively. We now define the proposed generalized form of paracompactness.

Definition 2. *Let $\mathcal{G}$ be a grill on a topological space (X, τ). Then the space X is said to be paracompact with respect to the grill $\mathcal{G}$ or simply $\mathcal{G}$-paracompact if every open cover $\mathcal{U}$ of X has a precise locally finite open refinement $\mathcal{U}^*$ (not necessarily a cover of X) such that $X \setminus \cup\, \mathcal{U}^* \notin \mathcal{G}$, where the statement "a cover $\mathcal{U} = \{U_\alpha : \alpha \in \Lambda\}$ has a precise refinement" means as usual, that there exists a collection $\mathcal{V} = \{V_\alpha : \alpha \in \Lambda\}$ of subsets of X such that $V_\alpha \subseteq U_\alpha$, for all $\alpha \in \Lambda$ (note that according to the terminology adopted in this paper, a refinement need not be a cover).*

Remark 1.

(a) Every paracompact space X is $\mathcal{G}$-paracompact, for every grill $\mathcal{G}$ on X; that the converse is false will be shown later (see Example 2). However, for the grill $\mathcal{G} = \mathcal{P}(X)\setminus\{\varnothing\}$, the concepts of paracompactness and $\mathcal{G}$-paracompactness coincide for any space X, where $\mathcal{P}(X)$ denotes the power set of X.

(b) If $\mathcal{G}_1$ and $\mathcal{G}_2$ are two grills on a space X with $\mathcal{G}_1 \subseteq \mathcal{G}_2$, then $\mathcal{G}_2$-paracompactness of $X \Rightarrow \mathcal{G}_1$-paracompactness of X. However, as will be seen in Example 2, under the stated condition that $\mathcal{G}_1 \subseteq \mathcal{G}_2$, $\mathcal{G}_1$-paracompactness may not imply $\mathcal{G}_2$-paracompactness. Moreover, it may so happen that a space X is $\mathcal{G}_1$-paracompact as well as $\mathcal{G}_2$-paracompact while the grills $\mathcal{G}_1$ and $\mathcal{G}_2$ are non-comparable; this is evident from the simple example as follows.

Example 1. *Let X be an uncountable set endowed with the co-countable topology τ. Consider the grills $\mathcal{G}_p = \{A \subseteq X : p \in A\}$ and $\mathcal{G}_q = \{A \subseteq X : q \in A\}$ on X for any two distinct points p and q of X. It is easy to see that (X, τ) is $\mathcal{G}_p$-paracompact as well as $\mathcal{G}_q$-paracompact, but $\mathcal{G}_p$ and $\mathcal{G}_q$ are clearly not comparable.*

In an earlier paper [7], we considered a new topology $\tau_\mathcal{G}$ associated with an arbitrary grill $\mathcal{G}$ on a topological space (X, τ). A brief description of this topology is as follows:

Let $\mathcal{G}$ be an arbitrary grill on a space (X, τ), and let $\Phi : \mathcal{P}(X) \to \mathcal{P}(X)$ (where $\mathcal{P}(X)$ denotes the power set of X) be an operator, given by $\Phi(A) = \{x \in X : A \cap U \in \mathcal{G}$ for all open sets U containing $x\}$. Then $\Psi : \mathcal{P}(X) \to \mathcal{P}(X)$, defined by $\Psi(A) = A \cup \Phi(A)$, is a Kuratowski closure operator, inducing a topology $\tau_\mathcal{G}$ (say) on X, strictly finer than τ, for which an open base is given by $\{U \setminus A : U \in \tau$ and $A \notin \mathcal{G}\}$.

In [7] we derived certain interesting features of this topology along with some interactions of it with the given topology τ on X, especially when the grill satisfies a kind of 'suitability' condition, as stated below:

Definition 3. *The topology τ of a topological space (X, τ) is said to be suitable for a grill $\mathcal{G}$ on X if for any $A \subseteq X$, $A \setminus \Phi(A) \notin \mathcal{G}$, where $\Phi(A)$ is as defined above.*

Let us call a grill $\mathcal{G}$ on a space X a μ-grill if any arbitrary family $\{A_\alpha : \alpha \in \Lambda\}$ of subsets of X, $[\cup_\alpha A_\alpha \in \mathcal{G} \Rightarrow A_\alpha \in \mathcal{G}$ for at least one $\alpha \in \Lambda]$. There are many natural examples of μ-grills. For instance, for any non-empty subset A of a space X, the principal grill generated by A (see definition 5) is such a grill.

As to the sharing of $\mathcal{G}$-paracompactness between a topological space (X, τ) and its associated space $(X, \tau_{\mathcal{G}})$, we prove the next two results.

Theorem 1. *Let $\mathcal{G}$ be a μ-grill on a topological space (X, τ). Then $(X, \tau_{\mathcal{G}})$ is $\mathcal{G}$-paracompact if (X, τ) is so.*

Proof. Without any loss of generality, let us consider a cover $\mathcal{W}$ of X by basic open sets of $(X, \tau_{\mathcal{G}})$, given by $\mathcal{W} = \{W_\alpha : \alpha \in \Lambda\}$, where for each $\alpha \in \Lambda$, $W_\alpha = U_\alpha \setminus A_\alpha$ with $U_\alpha \in \tau$ and $A_\alpha \notin \mathcal{G}$. Then $\mathcal{U} = \{U_\alpha : \alpha \in \Lambda\}$ is a τ-open cover of X. By $\mathcal{G}$-paracompactness of (X, τ), $\mathcal{U}$ has a τ-locally finite τ-open precise refinement $\mathcal{V} = \{V_\alpha : \alpha \in \Lambda\}$ such that $X \setminus (\cup_{\alpha \in \Lambda} V_\alpha) \notin \mathcal{G}$. It suffices to show that $\mathcal{W}^* = \{V_\alpha \setminus A_\alpha : \alpha \in \Lambda\}$ is a precise $\tau_{\mathcal{G}}$-locally finite $\tau_{\mathcal{G}}$-open refinement of $\mathcal{W}$.

Clearly $\mathcal{W}^*$ is a $\tau_{\mathcal{G}}$-open precise refinement of $\mathcal{W}$. Also, since $\mathcal{V}$ is τ-locally finite and $\tau \subseteq \tau_{\mathcal{G}}$, $\mathcal{V}$ is $\tau_{\mathcal{G}}$-locally finite, and hence $\mathcal{W}^*$ is $\tau_{\mathcal{G}}$-locally finite. It thus remains to show that $X \setminus \cup_{\alpha \in \Lambda} (V_\alpha \setminus A_\alpha) \notin \mathcal{G}$. For this we see that
$X \setminus \cup_{\alpha \in \Lambda} (V_\alpha \setminus A_\alpha) = X \setminus [\cup_{\alpha \in \Lambda} (V_\alpha \cap A_\alpha^C)] = \cap_{\alpha \in \Lambda} [X \setminus (V_\alpha \cap (X \setminus A_\alpha))] = \cap_{\alpha \in \Lambda} [(X \setminus V_\alpha) \cup A_\alpha] = [\cap_{\alpha \in \Lambda} (X \setminus V_\alpha)] \cup [\cap_{\alpha \in \Lambda} A_\alpha] \cup [\cup_\Lambda ((\cap_{\alpha \in \Lambda_1} (X \setminus V_\alpha)) \cap (\cap_{\beta \in \Lambda_2} A_\beta))]$........(i), where in the last bracketed portion $\Lambda_1 \cup \Lambda_2 = \Lambda$ and $\Lambda_1 \cap \Lambda_2 = \varnothing$......(ii), and $\cup_\Lambda$ stands for the union over all possible partitions of Λ such that (ii) holds. Now, $\cap_{\alpha \in \Lambda} (X \setminus V_\alpha) = X \setminus (\cup \mathcal{V}) \notin \mathcal{G}$ and $\cap_{\alpha \in \Lambda} A_\alpha \notin \mathcal{G}$ (since $A_\alpha \notin \mathcal{G}$ for each α). Furthermore, for any partition $\{\Lambda_1, \Lambda_2\}$ of Λ with $\Lambda = \Lambda_1 \cup \Lambda_2$ and $\Lambda_1 \cap \Lambda_2 = \varnothing$, $[\cap_{\alpha \in \Lambda_1} (X \setminus V_\alpha)] \cap [\cap_{\beta \in \Lambda_2} A_\beta] \subseteq [\cap_{\beta \in \Lambda_2} A_\beta] \notin \mathcal{G}$. Thus $\cup_\Lambda [(\cap_{\alpha \in \Lambda_1} (X \setminus V_\alpha)) \cap (\cap_{\beta \in \Lambda_2} A_\beta)] \notin \mathcal{G}$(as $\mathcal{G}$ is a μ-grill). Hence from (i) it follows that $X \setminus \cup_{\alpha \in \Lambda} (V_\alpha \setminus A_\alpha) \notin \mathcal{G}$, and we are through. □

Theorem 2. *Let $\mathcal{G}$ be a grill on a space (X, τ) such that $\tau \setminus \{\varnothing\} \subseteq \mathcal{G}$. If τ is suitable for $\mathcal{G}$ and $(X, \tau_{\mathcal{G}})$ is $\mathcal{G}$-paracompact, then (X, τ) is $\mathcal{G}$-paracompact.*

Proof. Let $\mathcal{U} = \{U_\alpha : \alpha \in \Lambda\}$ be a τ-open cover of X. Then $\mathcal{U}$ is a $\tau_{\mathcal{G}}$-open cover of X as well. Hence $\mathcal{U}$ has a $\tau_{\mathcal{G}}$-locally finite precise refinement $\{V_\alpha \setminus A_\alpha : \alpha \in \Lambda, V_\alpha \in \tau$ and $A_\alpha \notin \mathcal{G}\}$ such that $X \setminus \cup_{\alpha \in \Lambda} (V_\alpha \setminus A_\alpha) \notin \mathcal{G}$......(i)

We now show that $\mathcal{V} = \{V_\alpha : \alpha \in \Lambda\}$ is τ-locally finite. In fact, for each $x \in X$ there exists some $U \in \tau_{\mathcal{G}}$ such that $U \cap (V_\alpha \setminus A_\alpha) = \varnothing$ for all $\alpha \neq \alpha_1, \alpha_2, ...\alpha_n$ (say). But $U = V \setminus A$, where $V \in \tau$ and $A \notin \mathcal{G}$. Thus for any $\alpha \neq \alpha_1, \alpha_2, ...\alpha_n$, $(V \setminus A) \cap (V_\alpha \setminus A_\alpha) = \varnothing$, i.e., $(V \cap V_\alpha) \setminus (A \cup A_\alpha) = \varnothing$. Then either $(V \cap V_\alpha) = \varnothing$ or else $(\varnothing \neq)$ $(V \cap V_\alpha) \subseteq A \cup A_\alpha$. We claim that $V \cap V_\alpha = \varnothing$. For otherwise, $V \cap V_\alpha$ is a nonempty τ-open set $\Rightarrow V \cap V_\alpha \in \mathcal{G} \Rightarrow A \cup A_\alpha \in \mathcal{G}$, a contradiction. Thus $\mathcal{V}$ is τ-locally finite.

Again, $V_\alpha \setminus A_\alpha \subseteq U_\alpha$ and $V_\alpha \setminus A_\alpha \subseteq V_\alpha \Rightarrow V_\alpha \setminus A_\alpha \subseteq U_\alpha \cap V_\alpha \Rightarrow \cup_{\alpha \in \Lambda} (V_\alpha \setminus A_\alpha) \subseteq \cup_{\alpha \in \Lambda} (U_\alpha \cap V_\alpha) \Rightarrow X \setminus \cup_{\alpha \in \Lambda} (V_\alpha \setminus A_\alpha) \supseteq X \setminus \cup_{\alpha \in \Lambda} (U_\alpha \cap V_\alpha)$ and hence by (i), $X \setminus \cup_{\alpha \in \Lambda} (U_\alpha \cap V_\alpha) \notin \mathcal{G}$. Now, $\mathcal{W} = \{U_\alpha \cap V_\alpha : \alpha \in \Lambda\}$ is a τ-locally finite τ-open precise refinement of $\mathcal{U}$ such that $X \setminus (\cup \mathcal{W}) \notin \mathcal{G}$. Thus (X, τ) is $\mathcal{G}$-paracompact. □

From the last two theorems we have:

Corollary 1. *Let* (X, τ) *be a topological space and* $\mathcal{G}$ *a* μ*-grill on* X *such that* $\tau \setminus \{\varnothing\} \subseteq \mathcal{G}$ *and* τ *is suitable for* $\mathcal{G}$*. Then* (X, τ) *is* $\mathcal{G}$*-paracompact iff* $(X, \tau_{\mathcal{G}})$ *is* $\mathcal{G}$*-paracompact.*

Our principal concern of this paper, as already said in the introduction, is to exhibit how certain chosen kind of grills can affect interesting generalizations of the notions of paracompactness and the like. Such a study is scheduled to be taken up in the next section. For now, we show that a well known weaker form of paracompactness reduces to $\mathcal{G}$-paracompactness if the grill $\mathcal{G}$ can be chosen suitably. To that end we observe that

Result 1. *For any topological space* X, $\mathcal{G}_\delta = \{A \subseteq X : intcl\ A \neq \varnothing\}$ *is a grill on* X*.*

A well known weaker form of paracompactness is almost paracompactness (see [9]), the definition of which is recalled below:

Definition 4. *A topological space* (X, τ) *is said to be almost paracompact if every open cover* $\mathcal{U}$ *of* X *has a precise locally finite open refinement* $\mathcal{U}^*$ *such that* $X \setminus cl(\cup \mathcal{U}^*) = \varnothing$*.*

Theorem 3. *A topological space* (X, τ) *is almost paracompact iff* X *is* $\mathcal{G}_\delta$*-paracompact.*

Proof. Let $\mathcal{U}$ be an open cover of an almost paracompact space (X, τ). Then there exists a precise locally finite open refinement $\mathcal{U}^*$ of $\mathcal{U}$ such that $X \setminus \mathrm{cl}(\cup \mathcal{U}^*) = \varnothing$. We claim that $X \setminus (\cup\, \mathcal{U}^*) \notin \mathcal{G}$.
For otherwise, $X \setminus (\cup\, \mathcal{U}^*) \in \mathcal{G} \Rightarrow \mathrm{intcl}(X \setminus (\cup\, \mathcal{U}^*) \neq \varnothing \Rightarrow X \setminus \mathrm{clint}(\cup \mathcal{U}^*) \neq \varnothing$
$\Rightarrow X \setminus \mathrm{cl}(\cup \mathcal{U}^*) \neq \varnothing$, a contradiction. Thus (X, τ) is $\mathcal{G}_\delta$-paracompact.

We now prove a stronger converse that whenever $\mathcal{G}$ is any grill on X with $\tau \setminus \{\varnothing\} \subseteq \mathcal{G}$ (clearly $\mathcal{G}_\delta$ satisfies such a condition), then the almost paracompactness of (X, τ) is implied by the $\mathcal{G}$-paracompactness of X. We first observe that for such a grill $\mathcal{G}$, we have $\mathrm{int}A = \varnothing$ whenever $A (\subseteq X) \notin \mathcal{G}$ (as $\tau \setminus \{\varnothing\} \subseteq \mathcal{G}$). Now let $\mathcal{U}$ be an open cover of X. Then by the definition of $\mathcal{G}$-paracompactness there exists a precise locally finite open refinement $\mathcal{U}^*$ of $\mathcal{U}$ such that $X \setminus (\cup\, \mathcal{U}^*) \notin \mathcal{G}$. Thus $\mathrm{int}(X \setminus (\cup\, \mathcal{U}^*)) = \varnothing$, i.e., $X = \mathrm{cl}\ (\cup\, \mathcal{U}^*)$, proving (X, τ) to be almost paracompact. □

Example 2. *Let us consider the space* X *of Example 2.3 [6], given by* $X = \Re^+ \cup \{p\}$ *where* $\Re^+ = [0, \infty)$ *and* $p \notin \Re^+$ *equipped with the topology* τ *described as follows :* $\Re^+$ *has the usual topology and is an open subspace of* X*; and a basic neighbourhood of* $p \in X$ *is of the form* $O_n(p) = \{p\} \cup [\cup_{i=n}^{\infty} (2i, 2i+1)]$ *where* $n \in \omega$ *(= the first infinite cardinal). Then* (X, τ) *is almost paracompact but not paracompact (see [6] for details). Then by Theorem 3,* (X, τ) *is* $\mathcal{G}_\delta$*-paracompact without being paracompact. Incidentally we observe that for* $\mathcal{G} = \mathcal{P}(X) \setminus \{\varnothing\}$, X *is not* $\mathcal{G}$*-paracompact, although* $\mathcal{G}_\delta \subseteq \mathcal{G}$*.*

3. Principal grill $[A]$, $[A]$-regularity and $[A]$-paracompactness

We define now a kind of grills, associated with nonempty subsets of a set X.

Definition 5. *Let X be a nonempty set and $(\varnothing \neq)A \subseteq X$. Let us define $[A] = \{B \subseteq X : A \cap B \neq \varnothing\}$.*
It is easy to verify that $[A]$ is a grill on X. We shall call this grill the principal grill generated by A.

Remark 2. *In case of the principal grill generated by X, $[X]$-paracompactness reduces simply to paracompactness.*

In order to achieve a generalization of the famous Michael's theorem on regular paracompact spaces, we need to recall a kind of regularity axiom in terms of grills so as to serve our purpose.

Definition 6 (see [8])**.** *Corresponding to given grill $\mathcal{G}$ on a topological space (X, τ), we define the space X to be $\mathcal{G}$-regular if for each closed subset F of X and each $x \in X \setminus F$, there exist disjoint open sets U and V such that $x \in U$ and $F \setminus V \notin \mathcal{G}$.*

Remark 3. *It follows from the above two definitions that the principal grill $[X]$ generated by X is, in fact, $\mathcal{P}(X) \setminus \{\varnothing\}$ (where $\mathcal{P}(X)$ stands, as usual, for the power set of X) and hence a space (X, τ) is $[X]$-regular iff (X, τ) is regular. Also, every regular space is $\mathcal{G}$-regular for any grill $\mathcal{G}$ on X, although not the converse is true as shown below.*

Example 3. *Let $X = \{a, b, c\}$, $\tau = \{\varnothing, X, \{a\}, \{b\}, \{a, b\}, \{a, c\}\}$. Then (X, τ) is a topological space. It is easy to verify that (X, τ) is not regular but is a $\mathcal{G}$-regular space, where $\mathcal{G}$ is the grill $\{\{a\}, \{b\}, \{a, b\}, \{a, c\}, \{b, c\}, X\}$ on X.*

Furthermore, we show by the following examples that $\mathcal{G}$-regularity of a space (X, τ) has nothing to do with the regularity of the topological space $(X, \tau_{\mathcal{G}})$.

Example 4. *Let $X = \{a, b, c\}$, $\tau = \{\varnothing, X, \{a\}, \{b\}, \{a, b\}\}$. Then (X, τ) is a topological space. Let $\mathcal{G} = \{\{a\}, \{a, b\}, \{a, c\}, X\}$. Then $\mathcal{G}$ is a grill on X. It is easy to verify that $(X, \tau_{\mathcal{G}})$ is not regular but (X, τ) is $\mathcal{G}$-regular.*

Example 5. *Let $X = \{a, b, c\}$, $\tau = \{\varnothing, X, \{a\}, \{b\}, \{a, b\}, \{a, c\}\}$. Then (X, τ) is a topological space. Let $\mathcal{G} = \{\{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, X\}$. Then $\mathcal{G}$ is a grill on X. It is easy to show that $(X, \tau_{\mathcal{G}})$ is regular but (X, τ) is not $\mathcal{G}$-regular.*

In case of a principal grill $[A]$ generated by a subset A of a space X, we obtain the following characterization for $[A]$-regularity.

Theorem 4. *Let A be any nonempty subset of a space (X, τ). Then (X, τ) is $[A]$-regular iff for each closed subset F of X and each $x \notin F$, there exist disjoint open sets U and V such that $x \in U$ and $F \cap A \subseteq V$.*

Proof. Let (X, τ) be $[A]$-regular and F a closed subset of X and $x \in X \setminus F$. Then there exist disjoint open sets U and V such that $x \in U$ and $F \setminus V \notin [A]$. Now, $F \setminus V \notin [A] \Rightarrow (F \setminus V) \cap A = \varnothing \Rightarrow F \cap A \cap (X \setminus V) = \varnothing \Rightarrow F \cap A \subseteq V$.
Conversely, let the given condition hold and let F be a closed subset of X with $x \in X \setminus F$. Then there exist disjoint open sets U and V such that $x \in U$ and $F \cap A \subseteq V$. Now, $F \cap A \subseteq V \Rightarrow F \cap A \cap (X \setminus V) = \varnothing \Rightarrow F \cap (X \setminus V) \notin [A] \Rightarrow (F \setminus V) \notin [A]$. □

As a particular case of the next theorem we obtain (see Corollary 3) the well known result that a paracompact T_2 space is regular.

Theorem 5. *Let $\mathcal{G}$ be a grill on a space (X, τ). If X is $\mathcal{G}$-paracompact and T_2, then X is $\mathcal{G}$-regular.*

Proof. Let F be a closed subset of $X$and $y \in X \setminus F$. Then the Hausdorffness of X implies that for each $x \in F$, there exist disjoint open sets G_x and H_x such that $y \in G_x$ and $x \in H_x$. Clearly $y \notin \mathrm{cl} H_x$. Then $\mathcal{U} = \{H_x : x \in F\} \cup \{X \setminus F\}$ is an open cover of X. Thus there exists a precise locally finite open refinement $\mathcal{U}^* = \{H'_x : x \in F\} \cup \{W\}$ such that $H'_x \subseteq H_x$ for each $x \in F$, $W \subseteq X \setminus F$ and $X \setminus (\cup\, \mathcal{U}^*) \notin \mathcal{G}$. Let $G = X \setminus \cup\{\mathrm{cl} H'_x : x \in F\}$ and $H = \cup\{H'_x : x \in F\}$. Then G and H are two nonempty disjoint open sets ($\mathcal{U}^*$, being locally finite, is closure preserving) such that $y \in G$, $F \setminus H \notin \mathcal{G}$ (as $X \setminus \cup\, \mathcal{U}^* \notin \mathcal{G}$), proving (X, τ) to be $\mathcal{G}$-regular. □

Corollary 2. *Let A be a nonempty subset of a space (X, τ). If X is an $[A]$-paracompact Hausdorff space, then it is $[A]$-regular.*

If in the above corollary we put $A = X$, then by Remarks 2 and 3 we obtain:

Corollary 3. *A paracompact T_2 space is regular.*

Lemma 1. *For a nonempty subset A of a Hausdorff space (X, τ), let X be $[A]$-paracompact. Then for each $x \in X$ and each open set U containing x, there exists an open neighbourhood V of x such that cl $V \setminus U \subseteq X \setminus A$ i.e.,(cl $V \setminus U) \cap A = \varnothing$, and hence cl $V \cap A \subseteq U$.*

Proof. Let $x \in X$ and U be an open neighbourhood of x. Then $X \setminus U$ is a closed subset of X, not containing x. As (X, τ) is $[A]$-regular (by Corollary 2), we get by Theorem 4, two disjoint open sets G and V such that $x \in V$ and $(X \setminus U) \cap A \subseteq G$. Now, $G \cap$ cl $V = \varnothing \Rightarrow [(X \setminus U) \cap A] \cap$ cl $V = \varnothing \Rightarrow (X \setminus U) \cap A \cap$ cl $V = \varnothing \Rightarrow$ cl $V \cap (X \setminus U) \subseteq X \setminus A$, i.e., cl $V \setminus U \subseteq X \setminus A$ and hence the rest follows. □

Theorem 6. *Let (X, τ) be an $[A]$-paracompact, Hausdorff space for some nonempty subset A of X and $\mathcal{U} = \{U_\alpha : \alpha \in \Lambda\}$ be an open cover of X. Then there exists a precise locally finite open refinement $\{G_\alpha : \alpha \in \Lambda\}$ of $\mathcal{U}$ such that $A \subseteq \cup\{G_\alpha : \alpha \in \Lambda\}$ and cl $G_\alpha \cap A \subseteq U_\alpha \cap A$.*

Proof. Let $\mathcal{U} = \{U_\alpha : \alpha \in \Lambda\}$ be an open cover of X. Then by Lemma 1, for each $\alpha \in \Lambda$ and each $x \in U_\alpha$, there exists $V_{\alpha,x} \in \tau$ with $x \in V_{\alpha,x}$ such that cl $V_{\alpha,x} \cap A \subseteq U_\alpha$. Now, $\mathcal{V} = \{V_{\alpha,x} : x \in U_\alpha, \alpha \in \Lambda\}$ is an open cover of X. Hence by $[A]$-paracompactness of X, there exists a precise locally finite open refinement

$\mathcal{W} = \{W_{\alpha,x} : x \in U_\alpha, \alpha \in \Lambda\}$ of $\mathcal{V}$ such that $X \setminus (\cup\{W_{\alpha,x} : x \in U_\alpha, \alpha \in \Lambda\}) \notin [A]$, i.e., $A \subseteq \cup\{W_{\alpha,x} : x \in U_\alpha, \alpha \in \Lambda\}$. Now, for any $x \in U_\alpha$ and $\alpha \in \Lambda$, $W_{\alpha,x} \subseteq V_{\alpha,x}$ $\Rightarrow$ cl $W_{\alpha,x} \cap A \subseteq$ cl $V_{\alpha,x} \cap A \subseteq U_\alpha \cap A$. Let $G_\alpha = \cup_{x\in U_\alpha} W_{\alpha,x}$ for each $\alpha \in \Lambda$. Then clearly $\{G_\alpha : \alpha \in \Lambda\}$ is a precise locally finite open refinement of $\mathcal{U}$, and cl G_α =cl $(\cup_{x\in U_\alpha} W_{\alpha,x}) = \cup_{x\in U_\alpha} \mathrm{cl} W_{\alpha,x}$. So, (cl $G_\alpha) \cap A = \cup_{x\in U_\alpha}(\mathrm{cl}W_{\alpha,x} \cap A) \subseteq U_\alpha \cap A$. $\square$

We now state and prove our desired theorem as follows.

Theorem 7. *Let (X, τ) be a Hausdorff space and A a dense subset of X. Then the following statements are equivalent:*
(a) (X, τ) is $[A]$-paracompact.
(b) Each open cover of X has a precise locally finite refinement that covers A and consists of sets which are not necessarily closed or open.
(c) For each open cover $\mathcal{U} = \{U_\alpha : \alpha \in \Lambda\}$ of X, there exists a locally finite closed cover $\{F_\alpha : \alpha \in \Lambda\}$ of X such that $F_\alpha \cap A \subseteq U_\alpha$ for each $\alpha \in \Lambda$.

Proof. (a)$\Rightarrow$ (b): It is trivial.

(b)$\Rightarrow$ (c): Let $\{U_\alpha : \alpha \in \Lambda\}$ be an open cover of X. Then for any $x \in X$, there exists some $U_{\alpha(x)} \in \mathcal{U}$ such that $x \in U_{\alpha(x)}$. Then by Lemma 1, there exists some $H_x \in \tau$ with $x \in H_x$ such that cl $H_x \cap A \subseteq U_{\alpha(x)}$. Thus $\mathcal{H} = \{H_x : x \in X\}$ is an open cover of X, and hence by (b), there is a precise locally finite refinement $\{A_x : x \in X\}$ of $\mathcal{H}$ such that $A \subseteq \cup\{A_x : x \in X\}$. Since $\{A_x : x \in X\}$ is locally finite, so is $\{\mathrm{cl}\ A_x : x \in X\}$. Thus $\cup\{\mathrm{cl}A_x : x \in X\} =$ cl $[\cup\{A_x : x \in X\}] \supseteq$ cl $A = X$ (as A is dense in X)$\Rightarrow X = \cup\{\mathrm{cl}A_x : x \in X\}$. Now, $A_x \subseteq H_x \Rightarrow$ cl $A_x \subseteq \mathrm{cl}H_x \Rightarrow$ cl $A_x \cap A \subseteq \mathrm{cl}H_x \cap A \subseteq U_{\alpha(x)}$.
For each $\alpha \in \Lambda$, set $F_\alpha = \cup\{\mathrm{cl}A_x : \alpha = \alpha(x)\}$. Then F_α is closed for each $\alpha \in \Lambda$, as it is a union of locally finite closed sets. Thus $\{F_\alpha : \alpha \in \Lambda\}$ is locally finite and a cover of X. Finally, $F_\alpha \cap A = \cup\{\mathrm{cl}A_x : \alpha = \alpha(x)\} \cap A = \cup\{$ cl $A_x \cap A : \alpha = \alpha(x)\} \subseteq U_\alpha \cap A$, for each $\alpha \in \Lambda$.

(c)$\Rightarrow$ (a): Let $\mathcal{U} = \{U_\alpha : \alpha \in \Lambda\}$ be an open cover of X. In view of (c), let $\{F_\alpha : \alpha \in \Lambda\}$ be a locally finite closed cover of X such that $F_\alpha \cap A \subseteq U_\alpha$ for each $\alpha \in \Lambda$. For any $x \in X$, there exists $V_x \in \tau$ with $x \in V_x$ such that $V_x \cap F_\alpha \neq \varnothing$ for at most finitely many $\alpha \in \Lambda$. Now, $\mathcal{V} = \{V_x : x \in X\}$ is a cover of X. So there exists a locally finite closed cover $\{B_x : x \in X\}$ such that $B_x \cap A \subseteq V_x$, for all $x \in X$. Thus $\{B_x \cap A : x \in A\}$ is a cover of A.
Let us now consider $U(F_\alpha) = X \setminus \cup\{B_x : B_x \cap F_\alpha \cap A = \varnothing\}$. We first note that $U(F_\alpha)$ is open for each $\alpha \in \Lambda$. Now, $F_\alpha \cap A \subseteq U(F_\alpha)$. In fact, $y \in F_\alpha \cap A$ and $y \notin U(F_\alpha) \Rightarrow y \in F_\alpha \cap A$ and $y \in B_{y'}$ for some $y' \in X \Rightarrow B_{y'} \cap F_\alpha \cap A = \varnothing$. But $y \in F_\alpha \cap A$ and $y \in B_{y'} \Rightarrow y \in B_{y'} \cap F_\alpha \cap A$, a contradiction.

We shall now show that $\{U(F_\alpha) : \alpha \in \Lambda\}$ is locally finite. Indeed, each $x \in X$ has some open neighbourhood W intersecting finitely many B_x's, say $B_{x_1}, B_{x_2}, ...B_{x_n}$. Then W is contained in $\bigcup_{i=1}^{n} B_{x_i}$ (as $\{B_x : x \in X\}$ is a cover of X). Note that $B_x \cap U(F_\alpha) \neq \varnothing \Rightarrow B_x \cap F_\alpha \cap A \neq \varnothing$. Now each $B_x \cap A$ is contained in V_x, where V_x intersects at most finitely many F_α
$\Rightarrow B_x \cap A$ intersects at most finitely many F_α
$\Rightarrow$ each set B_x intersects at most finitely many $U(F_\alpha)$
$\Rightarrow W$ intersects at most finitely many $U(F_\alpha)$.

Thus $\{U(F_\alpha) : \alpha \in \Lambda\}$ is locally finite. Also, $\{U(F_\alpha) : \alpha \in \Lambda\}$ covers A, because $F_\alpha \cap A \subseteq U_\alpha$ and $\{F_\alpha \cap A : \alpha \in \Lambda\}$ is a cover of A.

Let now $\mathcal{U}^* = \{U(F_\alpha) \cap U_\alpha : \alpha \in \Lambda\}$. Then $\mathcal{U}^*$ is a precise locally finite open refinement of $\mathcal{U}$. Thus $F_\alpha \cap A \subseteq U_\alpha \cap U(F_\alpha)$, for all $\alpha \in \Lambda \Rightarrow A \subseteq \cup_{\alpha\in\Lambda}(F_\alpha \cap A) \subseteq \cup_{\alpha\in\Lambda}(U_\alpha \cap U(F_\alpha) \Rightarrow A \subseteq \cup\, \mathcal{U}^* \Rightarrow A \cap (X \setminus (\cup\, \mathcal{U}^*)) = \varnothing \Rightarrow X \setminus (\cup\, \mathcal{U}^*) \notin [A]$, proving (X, τ) to be $[A]$-paracompact. □

Taking $A = X$ in the above theorem and using Corollary 2, we arrive at the well known theorem of E. Michael:

Corollary 4. *In a regular space X, the following are equivalent:*
(a) X is paracompact.
(b) Every open cover of X has a locally finite refinement consisting of sets not necessarily open or closed.
(c) Each open cover of X has a closed locally finite refinement.

Finally, we show that $\mathcal{G}$-paracompactness is a topological invariant in the following sense.

Theorem 8. *Let $\mathcal{G}$ and $\mathcal{G}'$ be two grills respectively on two topological spaces (X, τ) and (Y, τ'). Let $f : (X, \tau) \to (Y, \tau')$ be a homeomorphism and $f(\mathcal{G}) \supseteq \mathcal{G}'$. If (X, τ) is $\mathcal{G}$-paracompact then (Y, τ') is $\mathcal{G}'$-paracompact [here the notation $f(\mathcal{G})$ stands for $\{f(G) : G \in \mathcal{G}\}$ which is clearly a grill in Y, as f is onto].*

Proof. Let $\{V_\alpha : \alpha \in \Lambda\}$ be an open cover of Y. Then by continuity and surjectiveness of f, $\{f^{-1}(V_\alpha) : \alpha \in \Lambda\}$ is an open cover of X. Hence by $\mathcal{G}$-paracompactness of (X, τ), there exists a locally finite precise open refinement $\{W_\alpha : \alpha \in \Lambda\}$ of $\{f^{-1}(V_\alpha) : \alpha \in \Lambda\}$ such that $X \setminus \cup_{\alpha\in\Lambda} W_\alpha \notin \mathcal{G}$. Since f is an open map, $\{f(W_\alpha) : \alpha \in \Lambda\}$ is an open precise refinement of $\{V_\alpha : \alpha \in \Lambda\}$ in (Y, τ'). We note that $\{f(W_\alpha) : \alpha \in \Lambda\}$ is locally finite as f is a homeomorphism. Now, as $X \setminus \cup_{\alpha\in\Lambda} W_\alpha \notin \mathcal{G}$, $Y \setminus \cup_{\alpha\in\Lambda} f(W_\alpha) = f(X \setminus \cup_{\alpha\in\Lambda} W_\alpha) \notin f(\mathcal{G})$ and hence $Y \setminus \cup_{\alpha\in\Lambda} f(W_\alpha) \notin \mathcal{G}'$. Thus (Y, τ') is $\mathcal{G}'$-paracompact. □

Corollary 5. *Let (X, τ) and (Y, τ') be two topological spaces, $(\varnothing \neq)A \subseteq X$, and $f : (X, \tau) \to (Y, \tau')$ a homeomorphism. If (X, τ) is $[A]$-paracompact then (Y, τ') is $[f(A)]$-paracompact.*

Proof. It is only sufficient to note that $[f(A)] = f([A])$ and the rest follows from Theorem 8. □

Taking $A = X$ in the above corollary and using Remark 2, we get

Corollary 6. *Let (X, τ) and (Y, τ') be two topological spaces, $f : (X, \tau) \to (Y, \tau')$ a homeomorphism. If (X, τ) is paracompact then (Y, τ') is paracompact.*

The next example shows that in Theorem 8 we cannot replace homeomorphism by a continuous, open surjection.

Example 6. *Let X be an uncountable discrete space and $Y = \{0, 1\}$ with Sierspinski's topology $\{Y, \varnothing, \{0\},\}$. Let $\mathcal{G} = \{(A \times \{1\}) : A$ is an infinite subset of $X\}$. It is easy to check that $\mathcal{G}$ is a grill on $X \times Y$. Now it is easy to see that $X \times Y$ is $\mathcal{G}$-paracompact (as $X \times Y$ is a paracompact space and using Remark 1 (a)). Let $p \notin X$ and consider the space $(X \cup \{p\}, \sigma)$ where $\sigma = \{U \subseteq X \cup \{p\} : p \in U\}$. Now consider the map $f : X \times Y \longrightarrow X \cup \{p\}$ defined by $f(x, 0) = p$ and $f(x, 1) = x$. It is easy to see that f is continuous, open and surjection and $\mathcal{G}' = f(\mathcal{G}) = \{A \subseteq X \cup \{p\} : A$ is infinite$\}$. Clearly, $X \cup \{p\}$ is not $\mathcal{G}'$-paracompact as the open cover $\{\{p, \alpha\} : \alpha \in X\}$ is an open cover of $X \cup \{p\}$ having no locally finite refinement whatsoever.*

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