Original scientific paper
Accepted 1.12.2008.

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Regular Polygons in the Taxicab Plane

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ABSTRACT

In this paper, we define taxicab regular polygons and determine which Euclidean regular polygons are also taxicab regular, and which are not. Finally, we investigate the existence or nonexistence of taxicab regular polygons.

Key words: taxicab distance, Euclidean distance, protractor geometry, regular polygon

MSC 2000: 51K05, 51K99.

Pravilni poligoni u taxicab ravnini

SAŽETAK

U radu definiramo pravilne taxicab poligone i određujemo koji su pravilni euklidski poligoni ujedno i pravilni taxicab poligoni, a koji nisu. Naposljetku, ispitujemo postojanje ili nepostojanje pravilnih taxicab poligona.

Ključne riječi: taxicab udaljenost, euklidska udaljenost, protractor geometrija, pravilni poligon

1 Introduction

A *metric geometry* consists of a set $\mathcal{P}$, whose elements are called *points*, together with a collection $\mathcal{L}$ of non-empty subsets of $\mathcal{P}$, called *lines*, and a distance function d, such that
1) every two distinct points in $\mathcal{P}$ lie on a unique line,
2) there exist three points in $\mathcal{P}$, which do not lie all on one line,
3) there exists a bijective function $f : l \to \mathbb{R}$ for all lines in $\mathcal{L}$ such that $|f(P) - f(Q)| = d(P,Q)$ for each pair of points P and Q on l.

A metric geometry defined above is denoted by $\{\mathcal{P},\mathcal{L},d\}$. However, if a metric geometry satisfies the plane separation axiom below, and it has an angle measure function m, then it is called *protractor geometry* and denoted by $\{\mathcal{P},\mathcal{L},d,m\}$.

4) For every l in $\mathcal{L}$, there are two subsets H_1 and H_2 of $\mathcal{P}$ (called *half planes* determined by l) such that
(i) $H_1 \cup H_2 = \mathcal{P} - l$ ($\mathcal{P}$ with l removed),
(ii) H_1 and H_2 are disjoint and each is convex,
(iii) If $A \in H_1$ and $B \in H_2$, then $[AB] \cap l \neq \varnothing$.

The taxicab metric was given by Minkowski [8] at the beginning of the last century. Later, taxicab plane geometry was introduced by Menger [6], and developed by Krause [5], using the taxicab metric $d_T(P,Q)$=$|x_1 - x_2|$+$|y_1 - y_2|$ instead of the well-known Euclidean metric $d_E(P,Q)$=$[(x_1-x_2)^2+(y_1-y_2)^2]^{1/2}$ for the distance between any two points $P = (x_1,y_1)$ and $Q = (x_2,y_2)$ in the Cartesian coordinate plane ($\mathbb{R}^2$). If L_E is the set of all lines, and m_E is the standard angle measure function of the Euclidean plane, then $\{\mathbb{R}^2,L_E,d_T,m_E\}$ is a model of protractor geometry, and it is called *taxicab plane* (see [3], [7]). The taxicab plane is one of the simple non-Euclidean geometries since it fails to satisfy the side-angle-side axiom, but it satisfies all the remaining twelve axioms of the Euclidean plane (see [5]). It is almost the same as the Euclidean plane $\{\mathbb{R}^2,L_E,d_E,m_E\}$ since the points are the same, the lines are the same, and the angles are measured in the same way. However, the distance functions are different. Since taxicab plane have distance function different from that in the Euclidean plane, it is interesting to study the taxicab analogues of topics that include the distance concept in the Euclidean plane. During the recent years, many such topics have been studied in the taxicab plane (see [10]). In this work, we study regular polygons in the taxicab plane.

2 Taxicab Regular Polygons

As in the Euclidean plane, a *polygon* in the taxicab plane consists of three or more coplanar line segments; the line segments (*sides*) intersect only at endpoints; each endpoint (*vertex*) belongs to exactly two line segments; no two line segments with a common endpoint are collinear. If the number of sides of a polygon is n for $n \geqslant 3$ and $n \in \mathbb{N}$, then the polygon is called an *n-gon*. The following definitions for polygons in the taxicab plane are given by means of the taxicab lengths instead of the Euclidean lengths:

Definition 1 *A polygon in the plane is said to be taxicab equilateral if the taxicab lengths of its sides are equal.*

Definition 2 *A polygon in the plane is said to be taxicab equiangular if the measures of its interior angles are equal.*

Definition 3 *A polygon in the plane is said to be taxicab regular if it is both taxicab equilateral and equiangular.*

Definition 2 does not give a new equiangular concept because the taxicab and the Euclidean measure of an angle are the same. That is, every Euclidean equiangular polygon is also the taxicab equiangular, and vice versa. However, since the taxicab plane has a different distance function, Definition 1 and therefore Definition 3 are new concepts. In this study, we answer the following question: Which Euclidean regular polygons are also the taxicab regular, and which are not? Also we investigate the existence and nonexistence of taxicab regular polygons.

Proposition 1 *Let A, B, C and D be four points in the Cartesian plane such that $A \neq B$ and $d_E(A,B) = d_E(C,D)$, and let m_1 and m_2 denote the slopes of the lines AB and CD, respectively.*

(i) If $m_1 \neq 0 \neq m_2$, then $d_T(A,B) = d_T(C,D)$ iff $|m_1| = |m_2|$ or $|m_1 m_2| = 1$.

(ii) If $m_i = 0$ or $m_i \to \infty$, then $d_T(A,B) = d_T(C,D)$ iff $m_j = 0$ or $m_j \to \infty$, where $i, j \in \{1,2\}$ and $i \neq j$.

Proof. We know from [4] that for any two points P and Q in the Cartesian plane that do not lie on a vertical line, if m is the slope of the line PQ, then

$$d_E(P,Q) = \rho(m) d_T(P,Q) \tag{1}$$

where $\rho(m) = (1+m^2)^{1/2} \diagup (1+|m|)$. If P and Q lie on a vertical line, that is $m \to \infty$, then $d_E(P,Q) = d_T(P,Q)$.

(i) Let $m_1 \neq 0 \neq m_2$ and $d_E(A,B) = d_E(C,D)$; then by Equation (1), $\rho(m_1) d_T(A,B) = \rho(m_2) d_T(C,D)$. If $d_T(A,B) = d_T(C,D)$, then $\rho(m_1) = \rho(m_2)$, that is $(1+m_1^2)^{1/2} \diagup (1+|m_1|) = (1+m_2^2)^{1/2} \diagup (1+|m_2|)$. Simplifying the last equation, we get $(|m_1| - |m_2|)(|m_1 m_2| - 1) = 0$. Therefore $|m_1| = |m_2|$ or $|m_1 m_2| = 1$. If $|m_1| = |m_2|$ or $|m_1 m_2| = 1$, then $m_2 = m_1$, $m_2 = -m_1$, $m_2 = 1/m_1$ or $m_2 = -1/m_1$, and one can easily see by calculations that $\rho(m_1) = \rho(m_2)$. Therefore $d_T(A,B) = d_T(C,D)$.

(ii) Let $m_i = 0$ or $m_i \to \infty$. Then $\rho(m_i) = 1$. If $d_T(A,B) = d_T(C,D)$, then $\rho(m_j) = 1$. Thus, $m_j = 0$ or $m_j \to \infty$. If $m_j = 0$ or $m_j \to \infty$, then $\rho(m_j) = 1$, and therefore $d_T(A,B) = d_T(C,D)$. □

The following corollary follows directly from Proposition 1, and plays an important role in our arguments.

Corollary 2 *Let A, B and C be three non-collinear points in the Cartesian plane such that $d_E(A,B) = d_E(B,C)$. Then, $d_T(A,B) = d_T(B,C)$ iff the measure of the angle ABC is $\pi/2$ or A and C are symmetric about the line passing through B, and parallel to anyone of the lines $x = 0$, $y = 0$, $y = x$ and $y = -x$.*

Note that Proposition 1 and Corollary 2 indicate also Euclidean isometries of the plane that do not change the taxicab distance between any two points: The Euclidean isometries of the plane that do not change the taxicab distance between any two points are all translations, rotations of $\pi/2$ and $3\pi/2$ radians around a point, reflections about lines parallel to anyone of the lines $x = 0$, $y = 0$, $y = x$ and $y = -x$, and their compositions; there is no other bijections of $\mathbb{R}^2$ onto $\mathbb{R}^2$ which preserve the taxicab distance (see [9]).

3 Euclidean Regular Polygons in Taxicab Plane

Since every Euclidean regular polygon is already taxicab equiangular, it is obvious that a Euclidean regular polygon is taxicab regular if and only if it is taxicab equilateral. So, to investigate the taxicab regularity of a Euclidean regular polygon, it is sufficient to determine whether it is taxicab equilateral or not. In doing so, we use following concepts:

Any Euclidean regular polygon can be inscribed in a circle and a circle can be circumscribed about any Euclidean regular polygon. A point is called the *center* of a Euclidean regular polygon if it is the center of the circle circumscribed about the polygon. A line l is called *axis of symmetry* (AOS) of a polygon if the polygon is symmetric about l, and in addition, if l passes through two distinct vertices of the polygon then l is called the *diagonal axis of symmetry* (DAOS) of the polygon. Clearly, every AOS of a Euclidean regular polygon passes through the center of the polygon.

Now, we are ready to investigate the taxicab regularity of Euclidean regular polygons.

Proposition 3 *No Euclidean regular triangle is taxicab regular.*

Proof. Since the Euclidean lengths of two consecutive sides are the same, and the angle between two consecutive sides is not a right angle, by Corollary 2, any two consecutive sides must be symmetric about a line parallel to anyone of the lines $x = 0$, $y = 0$, $y = x$ and $y = -x$, in order to have the same taxicab length. Suppose two consecutive sides are symmetric about a line parallel to anyone of the lines $x = 0$, $y = 0$, $y = x$ and $y = -x$. Figure 1 and Figure 2 show such Euclidean regular triangles. A simple calculation shows that none of the other two AOS's is parallel to anyone of the lines $x = 0$, $y = 0$, $y = x$ and $y = -x$. So, the triangles in Figure 1 and Figure 2 are not taxicab equilateral. Thus, no Euclidean regular triangle is taxicab regular. □

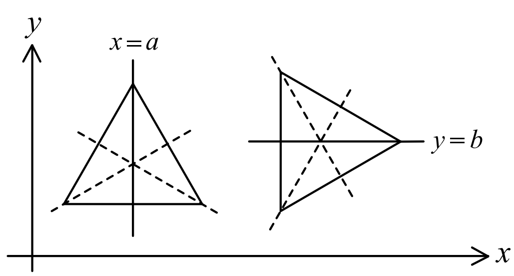

Figure 1

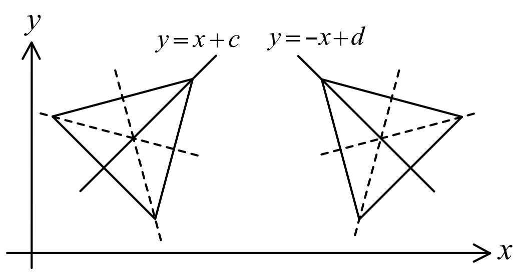

Figure 2

Corollary 4 *No Euclidean regular hexagon is taxicab regular.*

Proof. It is clear that every Euclidean regular hexagon is the union of six Euclidean regular triangles, and by Proposition 1 the taxicab lengths of the sides of one of the Euclidean regular triangles are the same as the taxicab lengths of corresponding parallel sides of the Euclidean regular hexagon as shown in Figure 3. Since no Euclidean regular triangle is taxicab equilateral, no Euclidean regular hexagon is taxicab equilateral, either. Thus, no Euclidean regular hexagon is taxicab regular. □

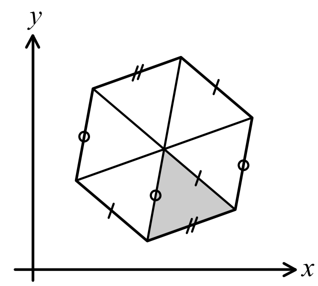

Figure 3

Proposition 5 *Every Euclidean regular quadrilateral (Euclidean square) is taxicab regular.*

Proof. Since every side of the Euclidean square has the same Euclidean length and the angle between every two consecutive sides is a right angle (see Figure 4), by Corollary 2, every side has the same taxicab length. So, every Euclidean square is taxicab equilateral, and therefore is taxicab regular. □

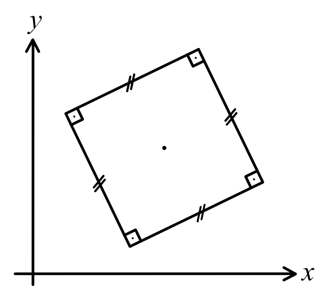

Figure 4

Proposition 6 *Every Euclidean regular octagon, one of whose DAOS's is parallel to anyone of the lines $x = 0$, $y = 0$, $y = x$ and $y = -x$, is taxicab regular.*

Proof. Let us consider the case $x = 0$. Clearly, every Euclidean regular octagon has four DAOS's, and if a DAOS of a Euclidean regular octagon is parallel to the line $x = 0$, then the other DAOS's are parallel to anyone of the lines $y = 0$, $y = x$ and $y = -x$ (see Figure 5). Since every two consecutive sides of such a Euclidean regular octagon are symmetric about a line parallel to anyone of the lines $x = 0$, $y = 0$, $y = x$ and $y = -x$, and every side has the same Euclidean length, by Corollary 2, these sides have the same taxicab length. So, a Euclidean regular octagon, one of whose DAOS's is parallel to the line $x = 0$ is taxicab equilateral, and therefore is taxicab regular. The other cases are similar. □

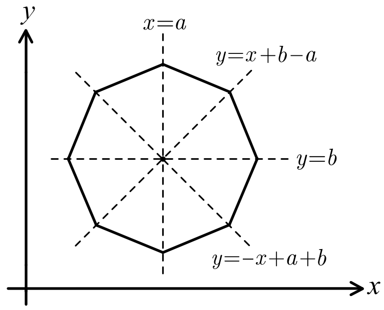

Figure 5

Theorem 7 *No Euclidean regular polygon, except the ones in Proposition 5 and Proposition 6, is taxicab regular.*

Proof. Let us classify Euclidean regular polygons as $(2n-1)$-gons and $2n$-gons for $n \geq 2$ ($n \in \mathbb{N}$), and investigate them separately:

(i) The Euclidean regular $(2n-1)$-gons: The case $n=2$ is proved in Proposition 3. Let $n>2$. It is clear that the number of AOS's of a Euclidean regular $(2n-1)$-gon is $2n-1$ (≥ 5), and each AOS of a Euclidean regular $(2n-1)$-gon passes through a vertex and the center of the polygon. Therefore, there exists at least one AOS which is not parallel to the lines $x=0$, $y=0$, $y=x$ and $y=-x$. Then, there are at least two consecutive sides symmetric about a line which is not parallel to the lines $x=0$, $y=0$, $y=x$ and $y=-x$. Also we know that the angle between two consecutive sides of Euclidean regular $(2n-1)$-gons is not a right angle. By Corollary 2, these consecutive sides do not have the same taxicab length. Thus, if $n>2$, then Euclidean regular $(2n-1)$-gons are not taxicab equilateral, and therefore are not taxicab regular. That is, no Euclidean regular $(2n-1)$-gon is taxicab regular.

(ii) The Euclidean regular $2n$-gons: The case $n=2$ is included in Proposition 5. The case $n=3$ is proved in Corollary 4. In order to exclude the case in Proposition 6, let us consider a Euclidean regular octagon, none of whose DAOS's is parallel to anyone of the lines $x=0$, $y=0$, $y=x$ and $y=-x$ for the case $n=4$. By Corollary 2, no two consecutive sides have the same taxicab length. Thus, such a Euclidean regular octagon is not taxicab equilateral, and therefore is not taxicab regular. Let $n>4$. Clearly, the number of the DAOS's of a Euclidean regular $2n$-gon is n. Therefore, there exists at least one DAOS which is not parallel to the lines $x=0$, $y=0$, $y=x$ and $y=-x$. Then, there are at least two consecutive sides symmetric about a line which is not parallel to the lines $x=0$, $y=0$, $y=x$ and $y=-x$. Also we know that the angle between two consecutive sides of Euclidean regular $2n$-gons is not a right angle for $n>4$. By Corollary 2, these consecutive sides do not have the same taxicab length. Thus, Euclidean regular $2n$-gons for $n>4$ are not taxicab equilateral, and therefore are not taxicab regular. The proof is completed. □

4 Existence of Taxicab Regular $2n$-gons

Now, we know which Euclidean regular polygons are taxicab regular, and which are not. Furthermore, we also know the existence of some taxicab regular polygons. However, we do not have general knowledge about the existence of taxicab regular polygons. In this section, we determine some of them. The following theorem shows that there exist taxicab regular $2n$-gons by means of taxicab circles. Recall that a taxicab circle with center A and radius r is the set of all points whose taxicab distance to A is r. This locus of points is a Euclidean square with center A, each side having slope ± 1, and each diagonal having length $2r$ (see [2]).

Theorem 8 *There exist two congruent taxicab regular* $2n$*-gons* ($n \geq 2$)*, having given any line segment as a side.*

Proof. Clearly, the measure of each interior angle of an equiangular $2n$-gon is $\pi(n-1)/n$ radians. Let us consider now any given line segment A_1A_2 in the taxicab plane. It is obvious that $(n-1)$ line segments A_iA_{i+1} ($2 \leq i \leq n$), having the same taxicab length $d_T(A_1,A_2)$, can be constructed such that the measure of the angle between every two consecutive segments is $\pi(n-1)/n$ radians, by using the taxicab circles with center A_i and radius $d_T(A_1,A_2)$, as in Figure 6. Also it is not difficult to see that $\measuredangle A_2A_1A_{n+1} + \measuredangle A_nA_{n+1}A_1 = \pi(n-1)/n$

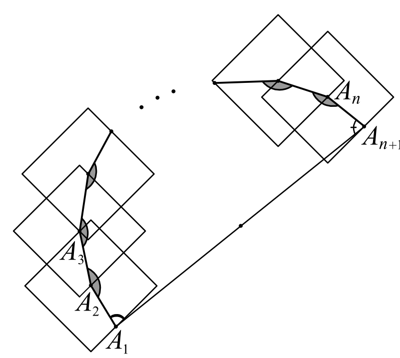

Figure 6

If we continue to construct line segments $A'_iA'_{i+1}$ which are symmetric to A_iA_{i+1} ($1 \leq i \leq n$) about the midpoint of A_1A_{n+1}, respectively, we get a $2n$-gon (see Figure 7). Since symmetry about a point (rotation of π radians around a point) preserves both taxicab lengths and angle measures, we have $d_T(A_i,A_{i+1}) = d_T(A'_i,A'_{i+1}) = d_T(A_1,A_2)$ ($1 \leq i \leq n$) and $\measuredangle A_i = \measuredangle A'_i = \pi(n-1)/n$ ($2 \leq i \leq n$). Also it is not difficult to see that $\measuredangle A_1 = \measuredangle A_{n+1} = \pi(n-1)/n$. Thus, the constructed $2n$-gon is taxicab regular. Furthermore, on the other side of the line A_1A_2, one can construct another taxicab regular $2n$-gon, having the same line segment A_1A_2 as a side, by using the same procedure (see Figure 8).

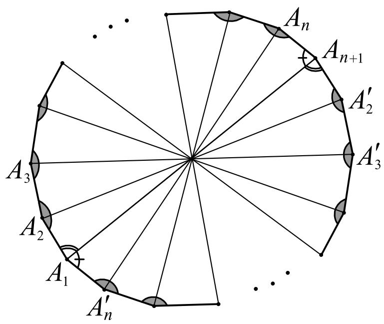

Figure 7

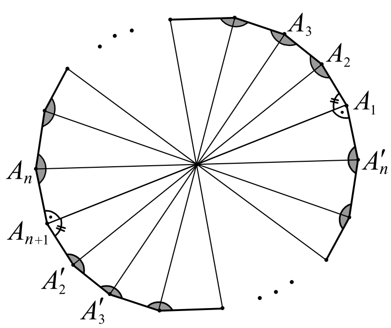

Figure 8

However, it is easy to see that these two taxicab regular $2n$-gons are symmetric about the midpoint of the line segment A_1A_2, and they are congruent. □

In every taxicab regular $2n$-gon, there are n line segments joining the corresponding vertices of the $2n$-gon ($A_iA'_i$, $1 \leq i \leq n$, for polygons in Figure 7 and Figure 8). We call each of these line segments an *axis* of the polygon. Clearly, axes of every taxicab regular $2n$-gon intersect at one and only one point.

Example. Using the procedure given in the proof of Theorem 8, one can easily construct taxicab regular $2n$-gons, having given any line segment as a side. To give examples, we construct a taxicab regular quadrilateral (taxicab square), a taxicab regular hexagon, and a taxicab regular octagon, having given line segment AB as a side, in Figure 9, 10 and 11:

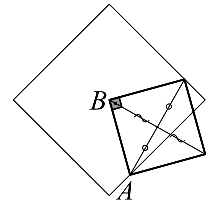

Figure 9

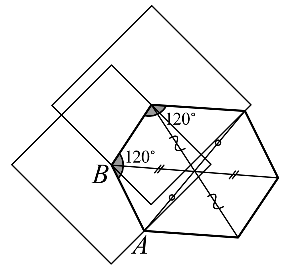

Figure 10

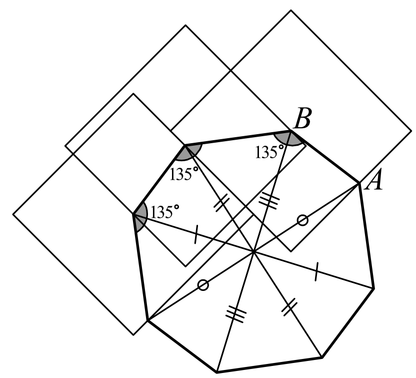

Figure 11

5 More About Taxicab Regular Polygons

By Proposition 5, we know that every Euclidean square is taxicab regular, that is, a taxicab square. By Proposition 11 below, we will see that every taxicab square is also Euclidean regular, that is, a Euclidean square. Thus, the Euclidean and the taxicab squares always have the same shape, and the only regular polygon having this property is square.

Proposition 9 *Let A, B, C and D be four points in the Cartesian plane such that $A \neq B$ and $d_T(A,B) = d_T(C,D)$, and let m_1 and m_2 denote the slopes of the lines AB and CD, respectively.*

(i) If $m_1 \neq 0 \neq m_2$, then $d_E(A,B) = d_E(C,D)$ iff $|m_1| = |m_2|$ or $|m_1m_2| = 1$.

(ii) If $m_i = 0$ or $m_i \to \infty$, then $d_E(A,B) = d_E(C,D)$ iff $m_j = 0$ or $m_j \to \infty$, where $i, j \in \{1,2\}$ and $i \neq j$.

Proof. The proof is similar to that of Proposition 1. □

The following corollary follows directly from Proposition 9:

Corollary 10 *Let A, B, and C be three non-collinear points in the Cartesian plane such that $d_T(A,B) = d_T(B,C)$. Then, $d_E(A,B) = d_E(B,C)$ iff the measure of the angle ABC is $\pi/2$ or A and C are symmetric about the line passing through B, and parallel to anyone of the lines $x = 0$, $y = 0$, $y = x$ and $y = -x$.*

Proposition 11 *Every taxicab square is Euclidean regular.*

Proof. Since every side of the taxicab square has the same taxicab length and the angle between every two consecutive sides is a right angle, by Corollary 10, every side has the same Euclidean length. So, every taxicab square is Euclidean equilateral, and therefore is Euclidean regular. □

We need a new notion to prove the next proposition: An equiangular polygon with an even number of vertices is called *equiangular semi-regular* if sides have the same Euclidean length alternately. There is always a Euclidean circle passing through all vertices of an equiangular semi-regular polygon (see [11]).

Proposition 12 *Every taxicab regular octagon, one of whose axes is parallel to anyone of the lines $x = 0$, $y = 0$, $y = x$ and $y = -x$, is Euclidean regular.*

Proof. In every taxicab regular octagon, sides have the same Euclidean length alternately since the measure of the angle between any two alternate sides is $\pi/2$ and sides have the same taxicab length, by Proposition 9 and Corollary 10. Therefore, every taxicab regular octagon is equiangular semi-regular. It is obvious that if any two consecutive sides of an equiangular semi-regular polygon have the same Euclidean length, then the polygon is Euclidean regular. Let us consider a taxicab regular octagon, $A_1A_2 \ldots A_8$, one of whose axes, let us say A_1A_5, is parallel to the line $y = 0$, for one case (see Figure 12).
Then there exist a Euclidean circle with diameter A_1A_5, passing through points $A_1, A_2, \ldots, A_8$, and there exist a taxicab circle with center A_1, passing through points A_2 and A_8. Since the Euclidean and the taxicab circles are both symmetric about the line A_1A_5, the intersection points of them, A_2 and A_8, are also symmetric about the same line. Then two consecutive sides A_1A_2 and A_1A_8 have the same Euclidean length by Corollary 10. Therefore, every taxicab regular octagon, one of whose axes is parallel to the line $y = 0$, is Euclidean regular. The other cases are similar. □

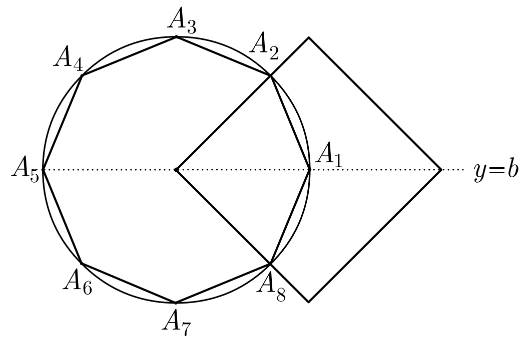

Figure 12

Theorem 13 *No taxicab regular polygon, except the ones in Proposition 11 and Proposition 12, is Euclidean regular.*

Proof. Assume that there exists a taxicab regular polygon, except the ones in Proposition 11 and Proposition 12, that is also Euclidean regular. Then there exists a Euclidean regular polygon, except the ones in Proposition 5 and Proposition 6, that is also taxicab regular. But this is in contradiction with Theorem 7. Therefore, no taxicab regular polygon, except the ones in Proposition 11 and Proposition 12, is Euclidean regular. □

6 On the Nonexistence of Taxicab (2*n*–1) - gons

The following theorem shows that there is no taxicab regular triangle:

Theorem 14 *There is no taxicab regular triangle.*

Proof. Every taxicab equiangular triangle is a Euclidean regular triangle. Since no Euclidean regular triangle is taxicab regular by Proposition 3, no taxicab equiangular triangle is taxicab regular. Therefore, there is no taxicab regular triangle. □

In addition to Theorem 14, we have seen that there is no taxicab regular 5-gon, 9-gon and 15-gon using a computer program called *Compass and Ruler* [12]. However, we could not reach any conclusion by reasoning about the existence or nonexistence of taxicab regular $(2n-1)$-gons for $n = 4$, $n = 6$, $n = 7$ and $n \geq 9$. Our conjecture is that there is no taxicab regular $(2n-1)$-gon since there is no center of symmetry of equiangular $(2n-1)$-gons. It seems interesting to study the open problem: Does there exist any taxicab regular $(2n-1)$-gon?

One can also consider the generalizations and variations of our problem. One of them is determining the regular polygons of the taxicab space. The taxicab distance between points $P = (x_1, y_1, z_1)$ and $Q = (x_2, y_2, z_2)$ in the

Cartesian coordinate space ($\mathbb{R}^3$) is defined by $d_T(P,Q) = |x_1 - x_2| + |y_1 - y_2| + |z_1 - z_2|$ (see [1]). Clearly, the concept of regular polygon can be defined similarly in the taxicab space; and if regular polygons are determined, then one can investigate regular polyhedra in the taxicab space. This is also interesting subject since there are only 5 types of regular polyhedra in the three dimensional Euclidean space. However, the results of this work cannot be generalized directly to the three dimensional taxicab space since the taxicab distance is not uniform in all directions.

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