Differential sandwich theorems for certain subclasses of analytic functions

C. SELVARAJ* AND K. R. KARTHIKEYAN†

Abstract. *In this paper, we generalize the well known differential operator by using a generalized hypergeometric function. By applying this operator, we introduce a new class of non-Bazilević functions and derive subordination and superordination results for the class in the unit disk. Relevant connections of the results presented in this paper are also pointed out with various other known results.*

Key words: *analytic functions, differential operators, differential subordination, differential superordination*

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1. Introduction, definitions and preliminaries

Let $\mathcal{H}$ be the class of functions analytic in the open unit disc $\mathcal{U} = \{z : | z |< 1\}$. Let $\mathcal{H}(a, n)$ be the subclass of $\mathcal{H}$ consisting of functions of the form $f(z) = a + a_n z^n + a_{n+1}z^{n+1} + \dots$.
Let

$$\mathcal{A}_n = \{f \in \mathcal{H},\ f(z) = z + a_{n+1}z^{n+1} + a_{n+2}z^{n+2} + \dots\}$$

and let $\mathcal{A} = \mathcal{A}_1$. Let the functions f and g be analytic in $\mathcal{U}$. We say that the function f is subordinate to g if there exists a Schwarz function w, analytic in $\mathcal{U}$ with $w(0) = 0$ and $| w(z) |< 1$ such that $f(z) = g(w(z))$ for $z \in \mathcal{U}$. We denote it by $f(z) \prec g(z)$. In particular, if the function g is univalent in $\mathcal{U}$, the above subordination is equivalent to $f(0) = g(0)$ and $f(\mathcal{U}) \subset g(\mathcal{U})$. Let $p, h \in \mathcal{H}$ and let $\phi(r, s, t; z) : \mathbb{C}^3 \times \mathcal{U} \longrightarrow \mathbb{C}$. If p and $\phi(p(z), zp^{'}(z), z^2p^{''}(z); z)$ are univalent and if p satisfies the second order superordination

$$h(z) \prec \phi(p(z), zp^{'}(z), z^2p^{''}(z); z), \tag{1}$$

then p is a solution of the differential superordination (1). (If f is subordinate to F, then F is called to be superordinate to f) An analytic function q is called a

*Department of Mathematics, Presidency College, Chennai-600 005, Tamilnadu, India, e-mail: `pamc9439@yahoo.co.in`

†R. M. K. Engineering College, R. S. M. Nagar, Kavaraipettai-60 1206, Thiruvallur District, Gummidipoondi Taluk, Tamilnadu, India, e-mail: `kr_ karthikeyan1979@yahoo.com`

subordinant if $q \prec p$ for all p satisfying (1). A univalent subordinant $\hat{q}$ that satisfies $q \prec \hat{q}$ for all subordinants q of (1) is said to be the best subordinant. Recently Miller and Mocanu [7] obtained conditions on h, q and ϕ for which the following implication holds:

$$h(z) \prec \phi(p(z), zp^{'}(z), z^2p^{''}(z);\, z) \implies q(z) \prec p(z).$$

With the results of Miller and Mocanu [7], Bulboacă [3] investigated certain classes of first order differential superordinations as well as superordination-preserving integral operators [4]. Ali et al.[2] used the results obtained by Bulboacă [4] and gave the sufficient conditions for certain normalized analytic functions f to satisfy

$$q_1(z) \prec \frac{zf^{'}(z)}{f(z)} \prec q_2(z)$$

where q_1 and q_2 are given univalent functions in $\mathcal{U}$ with $q_1(0) = 1$ and $q_2(0) = 1$. Shanmugam et al. obtained sufficient conditions for a normalized analytic functions f to satisfy

$$q_1(z) \prec \frac{f(z)}{zf^{'}(z)} \prec q_2(z)$$

and

$$q_1(z) \prec \frac{z^2f^{'}(z)}{(f(z))^2} \prec q_2(z)$$

where q_1 and q_2 are given univalent functions in $\mathcal{U}$ with $q_1(0) = 1$ and $q_2(0) = 1$. The Hadamard product of two functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ and $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$ in $\mathcal{A}$ is given by $(f * g)(z) = z + \sum_{n=2}^{\infty} a_n\, b_n\, z^n$.

For complex parameters $\alpha_1, \ldots, \alpha_q$ and

$$\beta_1, \ldots, \beta_s\ (\beta_j \in \mathbb{C} \setminus \mathbb{Z}_0^-;\, \mathbb{Z}_0^- = 0, -1, -2, \ldots;\, j = 1, \ldots, s),$$

we define the generalized hypergeometric function ${}_qF_s(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s; z)$ by

$${}_qF_s(\alpha_1, \alpha_2, \ldots, \alpha_q; \beta_1, \beta_2, \ldots, \beta_s; z) = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n\, \ldots\, (\alpha_q)_n}{(\beta_1)_n\, \ldots\, (\beta_s)_n} \frac{z^n}{n!}$$

$$(q \le s + 1;\, q, s \in \mathcal{N}_0 = \mathcal{N} \cup \{0\};\, z \in \mathcal{U}),$$

where $\mathcal{N}$ denotes the set of positive integers and $(x)_k$ is the Pochhammer symbol defined in terms of the Gamma function Γ by

$$(x)_k = \frac{\Gamma(x+k)}{\Gamma(x)} = \begin{cases} 1 & \text{if } k = 0 \\ x(x+1)(x+2)\ \ldots\ (x+k-1) & \text{if } k \in N = \{1, 2, , \ldots\}. \end{cases}$$

Corresponding to a function $\mathcal{G}_{q,\,s}(\alpha_1, \beta_1; z)$ defined by

$$\mathcal{G}_{q,\,s}(\alpha_1, \beta_1; z) := z\, {}_qF_s(\alpha_1, \alpha_2, \ldots, \alpha_q; \beta_1, \beta_2, \ldots, \beta_s; z) \tag{2}$$

We now define the following operator $D_\lambda^m(\alpha_1,\, \beta_1)f\, :\mathcal{U} \longrightarrow \mathcal{U}$ by

$$D_\lambda^0(\alpha_1,\, \beta_1)f(z) = f(z) * \mathcal{G}_{q,\,s}(\alpha_1,\, \beta_1;\, z)$$

$$D_\lambda^1(\alpha_1,\, \beta_1)f(z) = (1-\lambda)(f(z) * \mathcal{G}_{q,\,s}(\alpha_1,\, \beta_1;\, z)) + \lambda\, z(f(z) * \mathcal{G}_{q,\,s}(\alpha_1,\, \beta_1;\, z))' \quad (3)$$

$$D_\lambda^m(\alpha_1,\, \beta_1)f(z) = D_\lambda^1(D_\lambda^{m-1}(\alpha_1,\, \beta_1)f(z)) \quad (4)$$

If $f \in \mathcal{A}_1$, then from (3) and (4) we may easily deduce that

$$D_\lambda^m(\alpha_1,\, \beta_1)f(z) = z + \sum_{n=2}^{\infty} \left[1+(n-1)\lambda\right]^m \frac{(\alpha_1)_{n-1}\, \dots\, (\alpha_q)_{n-1}}{(\beta_1)_{n-1}\, \dots\, (\beta_s)_{n-1}}\, \frac{a_n z^n}{(n-1)!} \quad (5)$$

where $m \in N_0 = N \cup \{0\}$ and $\lambda \geq 0$. We remark that for a choice of the parameter $m = 0$ the operator $D_\lambda^0(\alpha_1,\, \beta_1)f(z)$ reduces to the well-known Dziok- Srivastava operator [5] and for $q = 2,\, s = 1;\, \alpha_1 = \beta_1,\, \alpha_2 = 1$ and $\lambda = 1$, we get the operator introduced by G. Ş. Sălăgean [9]. Also many (well known and new) integral and differential operators can be obtained by specializing the parameters.

Now we introduce the following:
For $0 \leq \gamma \leq 1$, a function $f(z) \in \mathcal{A}$ is said to be in $\mathcal{N}_\lambda^m(\alpha_1, \beta_1;\, \gamma;\, \phi)$ if and only if it satisfies the condition

$$(D_\lambda^m(\alpha_1,\, \beta_1)f(z))' \left(\frac{z}{D_\lambda^m(\alpha_1,\, \beta_1)f(z)}\right)^{1+\gamma} \prec \phi(z), \quad \forall\, z \in \mathcal{U}. \quad (6)$$

For the choice $\phi(z) = \dfrac{1+z}{1-z}$, $m = 0, q = 2, s = 1\; \alpha_1 = \beta_1,\, \alpha_2 = 1$. The class $\mathcal{N}_\lambda^m(\alpha_1, \beta_1;\, \gamma;\, \phi)$ reduces to $\mathcal{N}(\gamma)\, (0 < \gamma < 1)$ introduced recently by Obradović [8]. He called this class a non-Bazilević type. Until now, this class was studied in a direction of finding necessary conditions over γ that imbedding this class into the class of univalent functions or into the class of starlike functions, which is still an open problem.

The purpose of this paper is to derive subordination and superordination results for the class $\mathcal{N}_\lambda^m(\alpha_1, \beta_1;\, \gamma;\, \phi)$. In order to prove our subordination and superordination results, we make use of the following known results.

Definition 1. *[7] Denote by Q the set of all functions f that are analytic and injective on $\overline{\mathcal{U}} - E(f)$, where*

$$E(f) \;=\; \{\zeta \in \partial\mathcal{U} :\, \lim_{z\to\zeta} f(z) = \infty\},$$

and are such that $f'(\zeta) \neq 0$ for $\zeta \in \partial\mathcal{U} - E(f)$.

Theorem 1. *[6]If $-1 \leq B < A \leq 1$, $\beta \in C \setminus \{0\}$ and the complex number η satisfies $Re(\eta) \geq -\beta(1-A)/(1-B)$, then the differential equation*

$$q(z) + \frac{zq'(z)}{\beta q(z) + \eta} = \frac{1+Az}{1+Bz}, \quad \forall\, z \in \mathcal{U}$$

has a univalent solution in $\mathcal{U}$ given by

$$q(z) = \begin{cases} \frac{z^{\beta+\eta}\, (1+Bz)^{\beta(A-B)/(B)}}{\beta \int_0^z t^{(\beta+\eta-1)}(1+Bt)^{\beta(A-B)/B}\, dt} - \frac{\eta}{\beta}, & if\, B \neq 0 \\ \frac{z^{\beta+\eta}\, \exp(\beta A z)}{\beta \int_0^z t^{(\beta+\eta-1)} \exp(\beta A z)\, dt} - \frac{\eta}{\beta}, & if\, B = 0. \end{cases} \quad (7)$$

If $p(z) = 1 + c_1 z + c_2 z^2 + \dots$ *is analytic in* $\mathcal{U}$ *and satisfies*

$$p(z) + \frac{zp'(z)}{\beta p(z) + \eta} \prec \frac{1 + Az}{1 + Bz} \quad \forall\, z \in \mathcal{U},$$

then

$$p(z) \prec q(z) \prec \frac{1 + Az}{1 + Bz} \quad \forall\, z \in \mathcal{U} \tag{8}$$

and $q(z)$ *is the best dominant of (8).*

Theorem 2. *[7] Let the function* q *be univalent in the open unit disc* $\mathcal{U}$ *and* θ *and* ϕ *be analytic in a domain* D *containing* $q(\mathcal{U})$ *with* $\phi(w) \neq 0$ *when* $w \in q(\mathcal{U})$*. set* $Q(z) = zq'(z)\phi(q(z))$, $h(z) = \theta(q(z)) + Q(z)$*. Suppose that*

1. Q *is starlike univalent in* $\mathcal{U}$*, and*
2. $Re\left(\frac{zh'(z)}{Q(z)}\right) > 0$ *for* $z \in \mathcal{U}$.

If

$$\theta(p(z)) + zp'(z)\phi(p(z)) \prec \theta(q(z)) + zq'(z)\phi(q(z)),$$

then $p(z) \prec q(z)$ *and* q *is the best dominant.*

Theorem 3. *[4] Let the function* q *be univalent in the open unit disc* $\mathcal{U}$ *and* ϑ *and* ϕ *be analytic in a domain* D *containing* $q(\mathcal{U})$*. Suppose that*

1. $Re\left(\frac{\vartheta'(q(z))}{\phi(q(z))}\right) > 0$ *for* $z \in \mathcal{U}$ *and*
2. $zq'(z)\phi(q(z))$ *is starlike univalent in* $\mathcal{U}$.

If $p \in \mathcal{H}[q(0), 1] \cap Q$*, with* $p(\mathcal{U}) \subseteq D$*, and* $\vartheta(p(z)) + zp'(z)\phi(p(z))$ *is univalent in* $\mathcal{U}$ *and*

$$\vartheta(q(z)) + zq'(z)\phi(q(z)) \prec \vartheta(p(z)) + zp'(z)\phi(p(z)),$$

then $q(z) \prec p(z)$ *and* q *is the best subordinant.*

2. Subordination and superordination for analytic functions

Theorem 4. *Let* δ *be a non-zero complex number and let the function* $q(z)$ *be analytic and univalent in* $\mathcal{U}$ *such that* $q(z) \neq 0, \forall\, z \in \mathcal{U}$*. Suppose that* $\frac{zq'(z)}{q(z)}$ *is starlike univalent in* $\mathcal{U}$*. Let*

$$Re\left\{\frac{1}{\delta}\, q(z) + 1 + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)}\right\} > 0 \tag{9}$$

and

$$\Psi_\lambda^m(\alpha_1, \beta_1, \delta, f)(z) := (D_\lambda^m(\alpha_1,\, \beta_1) f(z))' \left(\frac{z}{D_\lambda^m(\alpha_1,\, \beta_1) f(z)}\right)^{1+\gamma} \tag{10}$$

$$+\delta\left[\frac{z(D_\lambda^m(\alpha_1,\,\beta_1)f(z))''}{(D_\lambda^m(\alpha_1,\,\beta_1)f(z))'}+(1+\gamma)\left(1-\frac{z(D_\lambda^m(\alpha_1,\,\beta_1)f(z))'}{D_\lambda^m(\alpha_1,\,\beta_1)f(z)}\right)\right]$$

If q satisfies the following subordination:

$$\Psi_\lambda^m(\alpha_1,\beta_1,\delta,f)(z)\prec q(z)+\delta\frac{zq'(z)}{q(z)}$$

then for $0\le\gamma\le 1$,

$$(D_\lambda^m(\alpha_1,\,\beta_1)f(z))'\left(\frac{z}{D_\lambda^m(\alpha_1,\,\beta_1)f(z)}\right)^{1+\gamma}\prec q(z) \tag{11}$$

and q is the best dominant.

Proof. Let the function p be defined by

$$p(z):=(D_\lambda^m(\alpha_1,\,\beta_1)f(z))'\left(\frac{z}{D_\lambda^m(\alpha_1,\,\beta_1)f(z)}\right)^{1+\gamma}\qquad(z\in\mathcal{U};\;z\neq 0;\;f\in\mathcal{A})$$

By a straightforward computation, we have

$$\frac{zp'(z)}{p(z)}=\left[\frac{z(D_\lambda^m(\alpha_1,\,\beta_1)f(z))''}{(D_\lambda^m(\alpha_1,\,\beta_1)f(z))'}+(1+\gamma)\left(1-\frac{z(D_\lambda^m(\alpha_1,\,\beta_1)f(z))'}{D_\lambda^m(\alpha_1,\,\beta_1)f(z)}\right)\right].$$

By setting

$$\theta(w):=w\qquad\text{and}\qquad\phi(w):=\frac{\delta}{w},$$

it can be easily verified that θ is analytic in $\mathbb{C}$, ϕ is analytic in $\mathbb{C}\setminus\{0\}$ and that $\phi(w)\neq 0$ $(w\in\mathbb{C}\setminus\{0\})$. Also, by letting

$$Q(z)=zq'(z)\phi(q(z))=\delta\frac{zq'(z)}{q(z)}$$

and

$$h(z)=\theta(q(z))+Q(z)=q(z)+\delta\frac{zq'(z)}{q(z)}.$$

We find that $Q(z)$ is starlike univalent in $\mathcal{U}$ and that

$$Re\left(\frac{zh'(z)}{Q(z)}\right)=Re\left\{\frac{1}{\delta}\,q(z)+1+\frac{zq''(z)}{q'(z)}-\frac{zq'(z)}{q(z)}\right\}>0.$$

Assertion (11) of Theorem 4 now follows by an application of Theorem 2. □

For the choices $q(z)=\dfrac{1+Az}{1+Bz}$, $\;-1\le B<A\le 1$ and $q(z)=\left(\dfrac{1+z}{1-z}\right)^\mu$, $0<\mu\le 1$, in Theorem 4, we get the following results.

Corollary 1. *Let δ be a non-zero complex number and assume that (9) holds. If $f\in\mathcal{A}$ and*

$$\Psi_\lambda^m(\alpha_1,\beta_1,\delta,f)(z)\prec\frac{1+Az}{1+Bz}+\delta\frac{(A-B)z}{(1+Az)(1+Bz)}$$

where $\Psi_\lambda^m(\alpha_1, \beta_1, \delta, f)$ *is defined as in (10), then for* $0 \leq \gamma \leq 1$,

$$(D_\lambda^m(\alpha_1,\,\beta_1)f(z))^{'}\left(\frac{z}{D_\lambda^m(\alpha_1,\,\beta_1)f(z)}\right)^{1+\gamma} \prec \frac{1+Az}{1+Bz}$$

and $\dfrac{1+Az}{1+Bz}$ *is the best dominant.*

Corollary 2. *Let* δ *be a non-zero complex number and assume that (9) holds. If* $f \in \mathcal{A}$ *and*

$$\Psi_\lambda^m(\alpha_1, \beta_1, \delta, f)(z) \prec \left(\frac{1+z}{1-z}\right)^{\mu} + \frac{2\delta\mu\, z}{(1-z^2)},$$

where $\Psi_\lambda^m(\alpha_1, \beta_1, \delta, f)$ *is defined as in (10), then for* $0 \leq \gamma \leq 1$,

$$(D_\lambda^m(\alpha_1,\,\beta_1)f(z))^{'}\left(\frac{z}{D_\lambda^m(\alpha_1,\,\beta_1)f(z)}\right)^{1+\gamma} \prec \left(\frac{1+z}{1-z}\right)^{\mu}$$

and $\left(\dfrac{1+z}{1-z}\right)^{\mu}$ *is the best dominant.*

Theorem 5. *Let* $-1 \leq B < A \leq 1$, $0 \leq \gamma \leq 1$ *and* $\delta \in \mathbb{C}\backslash\mathbb{Z}_0^-$ ($\mathbb{Z}_0^- = \{0, -1, -2, \dots, \}$). *If*

$$(D_\lambda^m(\alpha_1,\,\beta_1)f(z))^{'}\left(\frac{z}{D_\lambda^m(\alpha_1,\,\beta_1)f(z)}\right)^{1+\gamma} \tag{12}$$

$$+\delta\left[\frac{z(D_\lambda^m(\alpha_1,\,\beta_1)f(z))^{''}}{(D_\lambda^m(\alpha_1,\,\beta_1)f(z))^{'}} + (1+\gamma)\left(1 - \frac{z(D_\lambda^m(\alpha_1,\,\beta_1)f(z))^{'}}{D_\lambda^m(\alpha_1,\,\beta_1)f(z)}\right)\right] \prec \frac{1+Az}{1+Bz},$$

then

$$(D_\lambda^m(\alpha_1,\,\beta_1)f(z))^{'}\left(\frac{z}{D_\lambda^m(\alpha_1,\,\beta_1)f(z)}\right)^{1+\gamma} \prec q(z) \prec \frac{1+Az}{1+Bz} \qquad \forall z \in \mathcal{U}, \tag{13}$$

where

$$q(z) = \begin{cases} \frac{\delta z^{1/\delta}\,(1+Bz)^{(A-B)/(\delta B)}}{\int_0^z t^{(1-\delta)/\delta}(1+Bt)^{(A-B)/\delta B}\,dt}, & if\, B \neq 0 \\ \frac{\delta z^{1/\delta}\, exp(Az/\delta)}{\int_0^z t^{(1-\delta)/\delta} exp(Az/\delta)\,dt}, & if\, B = 0, \end{cases} \tag{14}$$

and $q(z)$ *is the best dominant of (13).*

Proof. We define $p(z)$ by

$$p(z) = (D_\lambda^m(\alpha_1,\,\beta_1)f(z))^{'}\left(\frac{z}{D_\lambda^m(\alpha_1,\,\beta_1)f(z)}\right)^{1+\gamma} \qquad (z \in \mathcal{U};\; z \neq 0;\; f \in \mathcal{A})$$

We notice that $p(z) = 1 + c_1 z + c_2 z^2 + \dots$ is analytic in $\mathcal{U}$. By a straightforward computation, by using 12 we have

$$p(z) + \delta\frac{zp^{'}(z)}{p(z)} \prec \frac{1+Az}{1+Bz}, \quad \forall z \in \mathcal{U}.$$

Hence by using Theorem (1), we get

$$p(z) \prec q(z) \prec \frac{1+Az}{1+Bz}, \quad \forall z \in \mathcal{U}$$

where $q(z)$ is given by (14) for $\beta = \frac{1}{\delta}$ and $\eta = 0$ and is the best dominant. This completes the proof of Theorem 5. □

For $\gamma = 1$, $m = 0$, $q = 2, s = 1$, $\alpha_1 = \beta_1$ and $\alpha_2 = 1$, we have the following corollary:

Corollary 3. *Let* $-1 \leq B < A \leq 1$ *and* $\delta \in \mathbb{C}\backslash\mathbb{Z}_0^-$ $(\mathbb{Z}_0^- = \{0, -1, -2, \dots, \})$. *If*

$$\frac{z^2 f^{'}(z)}{\big(f(z)\big)^2} + \delta\left[\frac{zf^{''}(z)}{f^{'}(z)} + 2\left(1 - \frac{zf^{'}(z)}{f(z)}\right)\right] \prec \frac{1+Az}{1+Bz},$$

then

$$\frac{z^2 f^{'}(z)}{\big(f(z)\big)^2} \prec q(z) \prec \frac{1+Az}{1+Bz} \qquad \forall z \in \mathcal{U},$$

where $q(z)$ *is the best dominant and is given by (14).*

Next, by appealing to Theorem 3 of the preceding section, we prove the following:

Theorem 6. *Let* δ *be a non-zero complex number and let* q *be analytic and univalent in* $\mathcal{U}$ *such that* $q(z) \neq 0$ *and* $\dfrac{zq^{'}(z)}{q(z)}$ *starlike univalent in* $\mathcal{U}$. *Further, let us assume that*

$$Re\left[\frac{q(z)}{\delta}\right] > 0. \tag{15}$$

If $f \in \mathcal{A}$,

$$0 \neq (D_\lambda^m(\alpha_1, \beta_1)f(z))^{'}\left(\frac{z}{D_\lambda^m(\alpha_1, \beta_1)f(z)}\right)^{1+\gamma} \in \mathcal{H}\big[q(0), 1\big] \cap Q$$

and $\Psi_\lambda^m(\alpha_1, \beta_1, \delta, f)$ *is univalent in* $\mathcal{U}$, *then*

$$q(z) + \delta\frac{zq^{'}(z)}{q(z)} \prec \Psi_\lambda^m(\alpha_1, \beta_1, \delta, f)$$

implies

$$q(z) \prec (D_\lambda^m(\alpha_1, \beta_1)f(z))^{'}\left(\frac{z}{D_\lambda^m(\alpha_1, \beta_1)f(z)}\right)^{1+\gamma} \tag{16}$$

and q *is the best subordinant where* $\Psi_\lambda^m(\alpha_1, \beta_1, \delta, f)$ *is defined as in (10).*

Proof. By setting

$$\vartheta(w) := w \qquad \text{and} \qquad \phi(w) := \frac{\delta}{w},$$

it can be easily verified that ϑ is analytic in $\mathbb{C}$, ϕ is analytic in $\mathbb{C} \setminus \{0\}$ and $\phi(w) \neq 0$ $(w \in \mathbb{C}\backslash\{0\})$. By the hypothesis of Theorem 6, $zq^{'}(z)\phi(q(z))$ is a starlike (univalent) function and

$$Re\frac{\vartheta^{'}(q(z))}{\phi(q(z))} = Re\left[\frac{q(z)}{\delta}\right] > 0.$$

Assertion (16) of Theorem 6 follows by an application of Theorem 3. □

Combining Theorems 4 and 2, we get the following sandwich theorem.

Theorem 7. *Let δ be a non-zero complex number and let q_1 and q_2 be univalent in $\mathcal{U}$ such that $q_1(z) \neq 0$ and $q_2(z) \neq 0$, $\forall z \in \mathcal{U}$ with $\dfrac{zq_1'(z)}{q_1(z)}$ and $\dfrac{zq_2'(z)}{q_2(z)}$ being starlike univalent. Suppose that q_1 satisfies (15) and q_2 satisfies (9). If $f \in \mathcal{A}$,*

$$(D_\lambda^m(\alpha_1,\, \beta_1)f(z))'\left(\frac{z}{D_\lambda^m(\alpha_1,\, \beta_1)f(z)}\right)^{1+\gamma} \in \mathcal{H}\big[q(0), 1\big] \cap Q, \quad \textit{and}$$

$\Psi_\lambda^m(\alpha_1, \beta_1, \delta, f)$ is univalent in $\mathcal{U}$, then

$$q_1(z) + \delta\frac{zq_1'(z)}{q_1(z)} \prec \Psi_\lambda^m(\alpha_1, \beta_1, \delta, f)(z) \prec q_2(z) + \delta\frac{zq_2'(z)}{q_2(z)}$$

implies

$$q_1(z) \prec (D_\lambda^m(\alpha_1,\, \beta_1)f(z))'\left(\frac{z}{D_\lambda^m(\alpha_1,\, \beta_1)f(z)}\right)^{1+\gamma} \prec q_2(z)$$

and q_1 and q_2 are the best subordinant and the dominant, respectively.

When $\gamma = 1$, $m = 0$, $q = 2, s = 1$, $\alpha_1 = \beta_1$ and $\alpha_2 = 1$, we have the following corollary

Corollary 4. *[10]Let δ be a non-zero complex number and let q_1 and q_2 be univalent in $\mathcal{U}$ such that $q_1(z) \neq 0$ and $q_2(z) \neq 0$, $(z \in \mathcal{U})$ with $\dfrac{zq_1'(z)}{q_1(z)}$ and $\dfrac{zq_2'(z)}{q_2(z)}$ being starlike univalent. Suppose that q_1 satisfies (15) and q_2 satisfies (9). If $f \in \mathcal{A}$,*

$$\frac{z^2 f'(z)}{\big\{f(z)\big\}^2} \in \mathcal{H}\big[q(0), 1\big] \cap Q,$$

and

$$\Psi_\lambda f(z) := \frac{z^2 f'(z)}{(f(z))^2} + \delta\left[\frac{zf''(z)}{f'(z)} + 2\left(1 - \frac{zf'(z)}{f(z)}\right)\right]$$

is univalent in $\mathcal{U}$, then

$$q_1(z) + \delta\frac{zq_1'(z)}{q_1(z)} \prec \Psi_\lambda^m(\alpha_1, \beta_1, \delta, f)(z) \prec q_2(z) + \delta\frac{zq_2'(z)}{q_2(z)}$$

implies

$$q_1(z) \prec \frac{z^2 f'(z)}{\big\{f(z)\big\}^2} \prec q_2(z)$$

and q_1 and q_2 are the best subordinant and the dominant, respectively.

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