Aversion to Price Risk and the Afternoon Effect

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Abstract

Many empirical studies of auctions show that prices of identical goods sold sequentially follow a declining path. Declining prices have been viewed as an anomaly, because the theoretical models of auctions predict that the price sequence should either be a martingale (with independent signals and no informational externalities), or a submartingale (with affiliated signals). This paper shows that declining prices, the afternoon effect, arise naturally when bidders are averse to price risk. A bidder is averse to price risk if he prefers to win an object at a certain price, rather than at a random price with the same expected value. When bidders have independent signals and there are no informational externalities, only the effect of aversion to price risk is present and the price sequence is a supermartingale. When there are informational externalities, even with independent signals, there is a countervailing, informational effect, which pushes prices to raise along the path of a sequential auction. This may help explaining the more complex price paths we observe in some auctions.

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"Theoretical work on auctions will almost certainly have to remove the assumption of risk neutral bidders if it is to explain the full range of interesting empirical results from real auctions." (Ashenfelter, 1989, p.31.)

1 Introduction

The classic theoretical models of sequential auctions of identical goods predict constant, or increasing, average prices across rounds. Weber (1983) and Milgrom and Weber (1982) showed that with independent private values the price sequence is a martingale (the expected value of P_{k+1}, the price in round $k+1$, conditional on P_k, the price in round k, is equal to P_k), while with affiliated values the price sequence is a submartingale (the expected value of P_{k+1} conditional on P_k is higher than P_k).

There is a substantial body of empirical evidence that is at odds with the classic theory: prices most frequently decline across rounds. The puzzle associated with this evidence has become known as the *afternoon effect* (because after a morning auction, often the second round takes place in the afternoon), or the *declining price anomaly*. Sequential auctions where prices have been shown to decline include wine (Ashenfelter, 1989, McAfee and Vincent, 1993), flowers (van den Berg et al., 2001), livestock (Buccola, 1982), gold jewelry (Chanel et al., 1996), china from shipwrecks (Ginsburgh and van Ours, 2007), stamps (Thiel and Petry, 1995), Picasso prints (Pesando and Shum, 1996), art (Beggs and Graddy, 1997), condominiums (Ashenfelter and Genesove, 1992), commercial real estate (Lusht, 1994). There is also experimental evidence of declining prices (Burns, 1985, and Keser and Olson, 1996). Ashenfelter and Graddy (2003) contains a general survey that focuses on art auctions.

In this paper, I propose a simple explanation of the afternoon effect: *aversion to price risk*. A bidder is averse to price risk if he prefers to win an object at a certain price, rather than at a random price with the same expected value.

The prediction that, with independent private values and risk neutral bidders, the price sequence is a martingale can be interpreted as a manifestation of the *law of one price*. Optimal bidder behavior would seem to require that the law of one price holds generally. After all, if the equilibrium price were known to be higher on average in a given round, shouldn't bidders lower their bids in that round?

It is easiest to explain the intuition for why aversion to price risk generates declining prices in the case of a two-round, second-price auction with private values and unit-demand bidders. The price in the last round will be determined by the second highest bid. The crucial observation is that in the first round each bidder chooses his optimal bid assuming that he will win and will be the price setter; that is, he assumes that his bid is tied with the highest bid of his opponents. This is because a small change in his bid only matters when the bidder wins and is the price setter. The fundamental implication of this observation is that in choosing his optimal bid, a bidder views the first-round price as certain (equal to his bid) and the second-round price as a random variable (equal to the second highest, second-round bid). Optimality requires that the bidder be indifferent between winning in the first or in the second round. Aversion to price risk then implies that the expected second-round price (conditional on the first-round price) must be lower than the first-round price. The difference is the risk premium that the bidder must receive to be

indifferent between winning at a random, rather than a certain, price.

The result that aversion to price risk generates a tendency for prices to decline, called an *aversion to price risk effect*, and its intuition, is very general. It holds for auctions with more than two rounds, for second-price, first-price, and English auctions. It also holds if there are informational externalities (i.e., if values are not purely private), and if bidders have multi-unit demand.

Aversion to price risk is different from risk aversion, which had been informally proposed by Ashenfelter (1989) as a possible explanation of declining prices. Models of risk aversion (e.g., Matthews, 1983, and McAfee and Vincent, 1993) assume that a bidder has a monetary value for the object, so that risk aversion is defined on the difference between the monetary value of the object and its price. Aversion to price risk, on the contrary, is defined on price alone. It implies separability of a bidder's payoff between utility from winning an object and utility from the bidder's monetary wealth (or disutility from paying the price).

McAfee and Vincent (1993) demonstrated that risk aversion is not a convincing explanation of the afternoon effect. They studied a two-round, private-value, second-price auction, and showed that prices decline only if bidders display increasing absolute risk aversion, which seems implausible. Under the more plausible assumption of decreasing absolute risk-aversion, an equilibrium in pure strategies does not exist and average prices need not decline.

To appreciate the difference between the concepts of risk aversion and aversion to price risk, it is useful to revisit the familiar result that with a single object for sale and risk aversion, the average price is higher in a first-price than in a second-price auction. It is simple to show that a similar result also holds when bidders are averse to price risk. The intuition, however, is different. In a first-price auction, the price is certain for the winning bidder; it equals his bid. In a second-price auction, on the contrary, the price is a random variable. On average the price must be lower in order to compensate the bidder for the price risk. (Appendix A shows that with aversion to price risk all standard auctions are payoff equivalent for the bidders.) Thus, in effect, in a first-price auction a bidder buys insurance against price variations. On the other hand, the intuition that is commonly given for why risk aversion raises the bid in a first-price auction is that, by bidding higher, a bidder buys insurance against the possibility of losing the auction (e.g., see Krishna, 2002, p.40).

There are other models in the literature that generate an afternoon effect. They include winners having the option to buy additional units (Black and de Meza, 1992), heterogeneity of objects (Engelbrecht-Wiggans, 1994, Bernhardt and Scoones, 1994, Gale and Hausch, 1994), ordering of the objects for sale by declining value (Beggs and Graddy, 1997), absentee bidders (Ginsburgh, 1998), an unknown number of objects for sale (Jeitschko, 1999), asymmetry among bidders (Gale and Stegeman, 2001), etc.

I view aversion to price risk as complementary to the other explanations given in the literature. The explanation based on aversion to price risk has the advantage of applying very generally, without requiring any additional modification of the classic model. This is an important advantage because, as the empirical evidence suggests, declining prices have been found to prevail even with no buyers' option to buy additional objects, with identical objects, etc.

While declining prices are much more common, increasing prices have also been documented in the empirical literature. For example, they were found for library books by Deltas and Kosmopolou (2001),

watches by Chanel et al. (1996), wool by Jones et al. (2004), and Israeli cable tv by Gandal (1997).

A second, general contribution of this paper is to show that affiliated types are not needed to explain increasing prices. Informational externalities alone, even with independent types, push prices to increase across rounds.[1] There are informational externalities (or interdependent types), if a bidder's payoff from winning an object directly depends on the types of the other bidders. To understand the intuition for the effect of informational externalities, consider a two-round, second-price auction with risk-neutral, unit-demand bidders. As I have already argued, a bidder must be indifferent between winning in the first and in the second round, conditional on the event that he wins the first round and he is the price setter. Thus, when bidders are risk neutral, the first-round price must be equal to the expected second-round price conditional on this event, which is lower than the expected second-round price conditional on the first-round price. This is because the first-round winner will generally bid higher and have a higher signal than the first-round price setter, and the value of an object is an increasing function of all bidders' signals.

When bidders are averse to price risk and there are informational externalities, the paper shows how to separate the aversion to price risk effect, which reduces prices from one round to the next, from the informational effect, which increases prices from round to round. The combined presence of the two effects may help explaining the more complex price paths, with prices increasing between some rounds and decreasing between others, that we sometimes observe in the data (e.g., see Jones et al., 2004).

Most of the paper studies the first-price and second-price sequential auctions with unit-demand bidders, but in Section 8 I show that the results extend to English auctions and, to some extent, to bidders with multi-unit demand.

The paper is organized as follows. Section 2 introduces the model. Section 3 presents the equilibria of the first-price and second-price auction. One delicate issue is what information is revealed from round to round. The natural candidate is to reveal the winning price in each round. However, the existence of an equilibrium with increasing bidding functions in a second-price auction is potentially problematic with this information policy, because the price setter in a round is a participant in the next round. I show that with no informational externalities an increasing equilibrium exists, but it does not if there are informational externalities. To study the second-price auction with informational externalities, I assume that only the winning bid is announced (Mezzetti et al., 2008, discuss this assumption in a model with affiliated values and risk neutral bidders). Section 4 presents the afternoon effect when there are no informational externalities. Section 5 looks at risk neutral bidders with informational externalities. Section 6 studies the general model with aversion to price risk and informational externalities. It defines and discusses the aversion to price risk effect and the informational effect. Section 7 presents a calibrated example that shows how the data from empirical studies can be reproduced for plausible values of an aversion to price risk and an informational externality parameter. Section 8 discusses extensions of the model and the robustness of the main results. In particular, it looks at English auctions and bidders with multi-unit demand. When studying sequential English auctions, it is important to formulate a tractable model that allows bidders not to reveal all the information in their hands during the first round. Section 9 concludes. Most of the proofs and additional

[1] Stochastic scale effects (Jeitschko and Wolfstetter, 1998) and uncertainty about the number of rounds (Feng and Chatterjee, 2005) may also generate increasing prices.

technical results are in the appendices.

2 The Model

There are K identical objects to be auctioned and N symmetric bidders, $N > K$. Each bidder has unit demand. Bidder i observes the realization x_i of a signal X_i, a random variable with support $[\underline{x}, \overline{x}]$. I assume that the signals are i.i.d. random variables with density f and distribution F. Let the random variable X_{-i} be the vector of signals of all bidders except i, with x_{-i} a realized value of X_{-i}. If i wins an object, the price he pays at auction, P, is a random variable which depends, through the bids, on his realized signal, or type, x_i and the types of all other bidders X_{-i}. Let p be a price realization. If a bidder does not win an object, he pays nothing.

Let q be the probability of winning an object, $G(p)$ a winning price distribution, and $L = (q, G(p))$ a compound lottery. Bidder i has von-Neumann Morgenstern preferences over lotteries, which only depend on the type profile $x \in X = X_i \times X_{-i}$ if i wins the object. Thus, we can normalize the utility of player i to zero when he does not win an object. Moreover, bidder i's preferences have an expected utility representation $U(x, q, G(p)) = \int qu(x, p)dG(p)$, where $u(x, p)$, the utility of the certain outcome of winning an object at price p, satisfies the following assumption.

Assumption A1. The function $u(x, p)$ is strictly decreasing in p and additively separable in x and p. For any distribution $G(p)$, let $E[P] = \int pdG(p)$. It is $u(x, E[P]) \geq \int u(x, p)dG(p)$.

If the inequality in *A1* is strict for all $G(p)$, we say that bidders are *averse to price risk.* If the inequality holds as an equality for all $G(p)$, then the model reduces to the classical case of risk neutral bidders. It is immediate from *A1* that there exist a function V and a convex function ℓ, such that we can write bidder i's utility when he wins an object at a price p and the type profile is $x = (x_i, x_{-i})$ as:

$$u(x, p) = V(x_i, x_{-i}) - \ell(p).$$

The first component, $V(x_i, x_{-i})$, is the utility bidder i receives from consuming one object when the type profile is x. The second component, $\ell(p)$, is the loss, or disutility, from a payment p to the seller. Think of $B(M-p)$ as the payoff, or benefit, from a money amount $M-p$, where M is the initial money endowment, then $-\ell(p) = B(M-p)$.

The realized valuation of bidder i, $V_i = V(x_i, x_{-i})$, depends on the value x_{-i} of the type of all other bidders. I will make the additional assumptions that $V(x_i, x_{-i})$ is positive, smooth, the same for all bidders i, symmetric in x_j, $j \neq i$, and increasing in all its arguments with $\frac{\partial V_i}{\partial x_i} \geq \frac{\partial V_i}{\partial x_j} \geq 0$. The latter assumption (which is commonly made when there are interdependent valuations, e.g., see Dasgupta Maskin, 2000) guarantees the allocational efficiency of the equilibrium.

It is useful to distinguish between the case of no *informational externalities*, when the valuation function of a bidder does not depend on the other bidders' types, $V(x_i, x_{-i}) = x_i$, and the case with informational externalities, in which V depends also on the signal realization x_{-i}. In the literature, the case of no informational externalities is referred to as *private values.*

Suppose that a bidder's value V_i for the object is the sum of two independent random variables W_i and Z, and that W_i has support $[0, 1]$, while Z has support $[100, 101]$. Suppose further that bidder i observes the value w_i taken by the random variable W_i, but not the value of Z. By defining the random variable $X_i = W_i + EZ$, we can think that each bidder observes the value of X_i, with support $[EZ, EZ + 1]$; clearly, there is a predominant common value component to a bidder's valuation, even though bidders have no private information about it. For this reason, I prefer to refer to $V_i = x_i$ as the case of no informational externalities, rather than private values. That bidders have no private information about the common value component of their valuations is consistent with the common practice of auction houses, such as Christie's and Sotheby's, to provide detailed expert estimates for each item at auction. When, on the other hand, there is private information about the common value component of the items for sale, then the appropriate model is one with informational externalities.

The loss function ℓ is the same for all bidders, strictly increasing with $\ell(0) = 0$, and convex, reflecting aversion to price risk on the part of bidders. Let $\phi = \ell^{-1}$ be the inverse of ℓ with respect to the realized price p; ϕ is strictly increasing and concave. An example of a loss function which will be used in Section 7 is the constant relative price-risk aversion function $\ell(p) = \frac{p^{1+r}}{1+r}$; another example is the constant absolute price-risk aversion function $\ell(p_i) = e^{\lambda p}$.

In the literature on risk aversion in auctions, it is commonly assumed that bidders have an equivalent monetary value for the object $v(x_i, x_{-i})$; risk aversion is captured by writing the payoff when winning at price p as $u\left(v(x_i, x_{-i}) - p\right)$, where u is a concave function. Most studies assume no informational externalities: $v(x_i, x_{-i}) = x_i$. As I pointed out in the introduction, McAfee and Vincent (1993) show that such a model only generates decreasing prices if bidders have increasing absolute risk aversion, and it is not very tractable (a pure strategy equilibrium does not exist with decreasing absolute risk aversion).

Bidders that are averse to price risk do not have an equivalent monetary value for the good on sale, but the good contributes additively to the utility of money. This is consistent, for example, with bidders viewing the good for sale as an asset that gives a stream of future payoffs. According to this interpretation, $V(x_i, x_{-i})$ is the discounted future payoff, while $\ell(p)$ is the current cost of acquiring the good at price p. This interpretation can also incorporate the possibility that a winner may resell the object in the future, so that the signal X_i includes information about future market value.

3 Sequential Auctions: Equilibrium

In this section I study the equilibrium bidding strategies in the sequential first-price and second-price auctions, in which one object is sold in each of K successive rounds. In round $k \leq K$ of a sequential auction, the bidding function of a remaining bidder depends not only on his type, but also on the common history of announced prices and bids from previous rounds. I will start by assuming that at the end of each round the bid of the winner is announced. In a first-price auction, this is equivalent to announcing the selling price, the standard practice in real auctions. I will look for symmetric equilibria in which the bid of a player is an increasing function of his type (everything else constant) in each round k. Thus, revealing the winning bid is equivalent to revealing the signal of the winning bidder.

By assuming that the winning bid is what is announced at the end of each round in a sequential second-price auction, I guarantee that the same information is revealed as in a sequential first-price auction format. On the other hand, if the bidding functions are increasing, revealing the price in a round of the sequential second-price auction amounts to revealing the type of the highest loser, a bidder who will be present in the next round. This has potentially quite different informational implications than revealing the type of the winner, which will not be present in future rounds (because of unit demand). I will show that in the case of no informational externalities the equilibrium bidding functions are the same irrespectively of whether the price or the winning bid are announced. I will also show, by way of an example, that with informational externalities an increasing equilibrium of the sequential second-price auction does not exist when the winning prices are announced.

The bidding functions of the sequential first-price and second-price auctions were first derived by Weber (1983) and Milgrom and Weber (1982) for the case of risk neutral bidders.[2] The next three theorems, whose proofs are in Appendix A, extend Milgrom and Weber's results to the case of aversion to price risk.

3.1 The Sequential First-Price Auction

In a sequential first-price auction, one object is sold in each of K rounds to the highest bidder at a price equal to the highest bid.

Define the random variable $Y_j^{(n)}$, an order statistic, as the j-th highest type of bidder out of n. Denote the distribution and density function of $Y_j^{(n)}$ as $F_j^{(n)}$ and $f_j^{(n)}$.

With a small abuse of notation, I will write bidder i's valuation as the following random variable:

$$V\left(X_i, Y_K^{(N-1)}, \cdots, Y_1^{(N-1)}\right) = E\left[V\left(X_i, X_{-i}\right) | X_i, Y_K^{(N-1)}, \cdots, Y_1^{(N-1)}\right].$$

Since I will look for an increasing equilibrium, and will assume that the winning prices are revealed, in round k of a sequential first-price auction the bid function will depend on the bidder's type x and on the types $y_1, ..., y_{k-1}$ of the winners in previous rounds.

Theorem 1 *Along the equilibrium path of the symmetric equilibrium of the sequential first-price auction with price (or winning bid) announcement, bidders follow the bidding functions*

$$\beta_k^{S1}(x; y_{k-1}, ..., y_1) = \phi\left(E\left[V\left(Y_K^{(N-1)}, Y_K^{(N-1)}, Y_{K-1}^{(N-1)}, ...\right) | Y_k^{(N-1)} \leq x \leq Y_{k-1}^{(N-1)} = y_{k-1}, ...\right]\right).$$

If there are no informational externalities, the bidding functions can be written as

$$\beta_k^{S1}(x; y_{k-1}, ..., y_1) = \beta_k^{S1}(x) = \phi\left(E\left[Y_K^{(N-1)} | Y_k^{(N-1)} \leq x \leq Y_{k-1}^{(N-1)}\right]\right).$$

In the formula for β_1^{S1} it should be understood that $Y_0^{(N-1)} = \overline{x}$, the top of the signal support. Observe that β_k^{S1} is an increasing function of all its arguments when there are informational externalities, while with no informational externalities β_k^{S1} does not depend on the types of the winners of previous rounds (equivalently, it does not depend on the price history).

[2] Milgrom and Weber (1982) worked with affiliated types. As they say in the foreword added to the published version, because of affiliation the proofs have to be considered in doubt; see Mezzetti et al. (2008), for some recent progress.

It is useful to view $\ell\left(\beta_k^{S1}(x;...)\right)$ as the *loss bid* associated with the *price bid* $\beta_k^{S1}(x;...)$. In the last round, a bidder of type x submits a loss bid equal to what he would value the object if he won and had the same signal realization as his highest opponent. Thus, for example, with no informational externalities his loss bid is equal to the highest expected value of his remaining opponents, conditional on their values being lower than his value. This is the same loss bid that the bidder would make in a single-sale, first-price auction. Bids in a round k before the last follow from the indifference condition that the loss bid in round k by a type x must be equal to the expected loss bid in round $k+1$ that a type x would make if he made a bid certain to lose in round k and discovered that round k winner has a type lower than x (see equation (14) in Appendix A).

3.2 The Sequential Second-Price Auction

In the sequential second-price auction, in each round one object is sold to the highest bidder at a price equal to the second highest bid. First, I will look at the case in which the winning bid is announced, as in the case of a sequential first-price auction. Then I will analyze the case in which the winning price is announced. In a sequential second-price auction, announcing the bids of winners is not equivalent to announcing the winning prices.

Theorem 2 *On the equilibrium path of the symmetric equilibrium of the sequential second-price auction with winning bid announcements, the bidding functions are*

$$\beta_k^{S2}(x;y_{k-1},...,y_1)=\phi\left(E\left[V(Y_K^{(N-1)},Y_K^{(N-1)},Y_{K-1}^{(N-1)},...)|Y_k^{(N-1)}=x\leq Y_{k-1}^{(N-1)}=y_{k-1},...\right]\right).$$

If there are no informational externalities, the bidding functions can be written as

$$\beta_k^{S2}(x;y_{k-1},...,y_1)=\beta_k^{S2}(x)=\phi\left(E\left[Y_K^{(N-1)}|Y_k^{(N-1)}=x\right]\right).$$

Given the history of the past winning types, in the last round a bidder of type x submits a loss bid equal to what he would value the object if he won and were pivotal. With no informational externalities, this loss bid reduces to the expected value of the object to the bidder. Bids in a round k before the last follow from the indifference condition that, conditional on being pivotal in round k, a bidder must be indifferent between winning in round k or in round $k+1$ (see equation (22) Appendix A).

Consider now the case in which the winning prices are announced after each bidding round. With no informational externalities, the equilibrium bidding functions in the sequential second-price auction are the same as when the winning bids are announced.

Theorem 3 *When there are no informational externalities, on the equilibrium path of the symmetric equilibrium of the sequential second-price auction with announcement of the winning prices, the bidding functions can be written as*

$$\beta_k^{S2p}(x;y_{k-1},...,y_2)=\phi\left(E\left[Y_K^{(N-1)}|Y_k^{(N-1)}=x\right]\right).$$

This theorem can be understood as follows. Without informational externalities, equilibrium bids do not depend on the past history of bids, no matter whether the winning bids, or the winning prices, are announced. In both cases, in the last round it is a dominant strategy to place a loss bid equal to the item's value. The bids in earlier rounds are then determined recursively, via the indifference condition (22). History does not matter because in each round k a bidder bids as if he were pivotal (i.e., as if $Y_k^{(N-1)} = x$).

When, on the other hand, there are informational externalities and the winning price is revealed in each round, an equilibrium of the sequential second-price auction with an increasing bidding function does not exist. The reason is simple: a bidder who, based on the history of prices, knows that he will lose in round k, but will almost certainly win in round $k+1$, has an incentive to deviate and make a very low bid in round k. By doing so he will avoid being the price setter in round k. The price setter in round k will be a bidder with a lower type, and hence in round $k+1$ all other bidders (including the future price setter) will have lower estimates of an object's conditional value and will make lower bids. As a result, the deviating bidder will profit by winning at a lower price in round $k+1$. This is made clear by the following simple example.

Example 1.

There are four bidders, three objects, and the common value of an object is $V = x_1 + x_2 + x_3 + x_4$. Without loss of generality, let $x_1 > x_2 > x_3 > x_4$ (bidders, of course, only know their own signals). Suppose there exists an increasing equilibrium. Then bidder 1 wins the first round and announcing the price reveals x_2, the signal of bidder 2. Suppose $x_3 = x_2 - \varepsilon$, with ε "arbitrarily small". At the beginning of the second round, bidder 3 knows that if he bids according to the equilibrium strategy, then with probability "arbitrarily close" to 1 he will be the price setter in round 2 and win an object in round 3. The price he will pay in round 3 is the bid of bidder 4. Since this is the last round, it is a weakly dominant strategy for bidder 4 to bid $b = \phi\left(E[X_1|X_1 \geq x_2] + x_2 + x_3 + x_4\right)$ (recall that x_2 and x_3 have been revealed by the price announcements, but x_1 has not). Now consider a deviation by bidder 3 in round 2; suppose he bids zero. Then the price setter in round 2 is bidder 4 and his signal is revealed. In round 3 bidder 4's weakly dominant bid is $\widehat{b} = \phi\left(E[X_1|X_1 \geq x_2] + x_2 + 2x_4\right)$, since bidder 4 assumes he is pivotal; that is, he assumes $x_3 = x_4$. After having deviated in round 2, in round 3 bidder 3's weakly dominant strategy is to use a loss bid equal to the conditional expected value of the object; that is, he will bid $\phi\left(E[X_1|X_1 \geq x_2] + x_2 + x_3 + x_4\right)$. It follows that by deviating bidder 3 will win in the third round and pay a price $\widehat{b}$ which is less than the price b he would pay if he followed the equilibrium strategy. Hence we have a contradiction; bidder 3 of type $x_3 = x_2 - \varepsilon$ has a profitable deviation in round 2 from the supposed increasing equilibrium.

4 The Afternoon Effect

In this section I show that declining prices (i.e., the afternoon effect) are a natural consequence of aversion to price risk, when there are no informational externalities.

Theorem 4 *When there are no informational externalities, the price sequences in a sequential first-price and in a sequential second-price auction are a supermartingale.*

Proof. The proofs for both cases follow the same logic. Note that if there are no informational externalities, then announcing either the winning bids or prices has no direct effect on the bidding functions. Consider first a sequential first-price auction. Suppose type x of bidder i wins auction $k < K$. Then it must be $Y_k^{(N-1)} \leq x \leq y_{k-1} < ... < y_1$. It follows that $P_k^{S1} = \beta_k^{S1}(x) = \phi\left(E\left[Y_K^{(N-1)}|Y_k^{(N-1)} \leq x \leq Y_{k-1}^{(N-1)}\right]\right)$ and

$$\begin{aligned}
E\left[P_{k+1}^{S1}|P_k^{S1}\right] &= E\left[P_{k+1}^{S1}|\beta_k^{S1}(x)\right] \\
&= E\left[\beta_{k+1}^{S1}(Y_k^{(N-1)})|Y_k^{(N-1)} \leq x = X_i \leq Y_{k-1}^{(N-1)} = y_{k-1}, ...\right] \\
&= E\left[\phi\left(\ell\left(\beta_{k+1}^{S1}(Y_k^{(N-1)})\right)\right)|Y_k^{(N-1)} \leq x = X_i \leq Y_{k-1}^{(N-1)} = y_{k-1}, ...\right] \\
&< \phi\left(E\left[\ell\left(\beta_{k+1}^{S1}(Y_k^{(N-1)})\right)|Y_k^{(N-1)} \leq x = X_i \leq Y_{k-1}^{(N-1)} = y_{k-1}, ...\right]\right) \\
&= \phi\left(E\left[Y_K^{(N-1)}|Y_k^{(N-1)} \leq x = X_i \leq Y_{k-1}^{(N-1)} = y_{k-1}, ...\right]\right) \\
&= \beta_k^{S1}(x) \\
&= P_k^{S1},
\end{aligned}$$

where the inequality follows from Jensen's inequality, given that ϕ is a concave function. This shows that the price sequence in a sequential first-price auction is a supermartingale.

Consider now a sequential second-price auction. Suppose in round k the winner is the bidder with signal $Y_k^{(N-1)}$, and bidder i of type x is the price setter; that is, $P_k^{S2} = \beta_k^{S2}(x; y_{k-1}, ..., y_1) = \beta_k^{S2}(x) = \phi\left(E\left[Y_K^{(N-1)}|Y_k^{(N-1)} = x\right]\right)$. In round $k+1$, bidder i of type x wins the auction, and the price setter is the bidder with the signal $Y_{k+1}^{(N-1)}$. It follows that

$$\begin{aligned}
E\left[P_{k+1}^{S2}|P_k^{S2}\right] &= E\left[P_{k+1}^{S2}|\beta_k^{S2}(x)\right] \\
&= E\left[\beta_{k+1}^{S2}(Y_{k+1}^{(N-1)})|Y_k^{(N-1)} = x\right] \\
&= E\left[\phi\left(\ell\left(\beta_{k+1}^{S2}(Y_{k+1}^{(N-1)})\right)\right)|Y_k^{(N-1)} = x\right] \\
&< \phi\left(E\left[\ell\left(\beta_{k+1}^{S2}(Y_{k+1}^{(N-1)})\right)|Y_k^{(N-1)} = x\right]\right) \\
&= \phi\left(E\left[Y_K^{(N-1)})|Y_k^{(N-1)} = x\right]\right) \\
&= \beta_k^{S2}(x) \\
&= P_k^{S2}.
\end{aligned}$$

Thus, the price sequence in a sequential second-price auction is also a supermartingale. ■

The intuition for the afternoon effect is essentially the same in a first-price and a second-price sequential auction. In each round before the last, conditional on having the highest remaining type and being the price setter, a bidder must be indifferent between winning now and winning in the next round.[3] But if a bidder is the price setter, then he knows the current price, while next round's price is random. Because of

[3] In a sequential first-price auction, if a bidder has the highest signal in round k and he bids according to the equilbrium bidding function, then he is automatically the price setter. On the other hand in a sequential second-price auction, conditioning on the highest-signal bidder *also* being the price setter amounts to requiring that his signal is tied with the signal of another bidder (i.e., the bidder is pivotal).

aversion to price risk, next round's expected price must then be lower than the price now. The difference is the risk premium the bidder must receive to be indifferent between the certain price now and the random price in the next round.

To understand this intuition in more detail, consider round $k < K$ of the second-price auction. Suppose type x of bidder i has lost all preceding auctions. Suppose also that in round k bidder x considers raising his bid by a small amount ε above $\beta_k^{S2}(x)$. This will only make a difference if, after the deviation, he wins in round k, while he would have otherwise lost and won in round $k+1$. For this to happen, it must be that $Y_k^{(N-1)} \simeq x$; that is, we must be in the event in which bidder i with signal x is at the margin between winning and losing in round k (i.e., he must be pivotal, his signal must be tied with the signal of another bidder). Conditional on this event, the *marginal cost* of the deviation is the loss incurred in period k when bidding according to the deviation,

$$\ell(\beta_k^{S2}(x) + \varepsilon),$$

while the *marginal benefit* is the expected loss avoided in period $k+1$,

$$E\left[\ell(\beta_{k+1}^{S2}(Y_{k+1}^{(N-1)}))|Y_k^{(N-1)} \simeq X_i = x \leq Y_{k-1}^{(N-1)})\right].$$

Equating marginal cost and marginal benefit (and sending ε to zero) gives the following *indifference condition* (see equation (22) in Appendix A):

$$\ell(\beta_k^{S2}(x)) = E\left[\ell(\beta_{k+1}^{S2}(Y_{k+1}^{(N-1)}))|Y_k^{(N-1)} = X_i = x)\right].$$

The indifference condition says that the certain loss when winning in period k at a price $\beta_k^{S2}(x)$ must be equal to the expected loss when winning in period $k+1$. Since bidders are averse to price risk, it must then be the case that the expected price in round $k+1$ is less than the price $\beta_k^{S2}(x)$ in round k. For the marginal bidder to be indifferent between winning at a certain price now, or at an uncertain price in the next round, it must be the case that the next round's expected price (conditional on the current price) is lower than the current price. Hence prices must decline from one period to the next.

Now consider a sequential first-price auction. Suppose bidder i wins in round k if he bids as a type x; that is, suppose $Y_k^{(N-1)} \leq x \leq Y_{k-1}^{(N-1)}$. Bidder i can also always bid so low so as to lose in round k. With a losing bid, bidder i discovers the value of $Y_k^{(N-1)}$ (the signal of the winner when bidder i bids low). Bidder i can then win for sure in round $k+1$ by bidding $\beta_{k+1}^{S1}(Y_k^{(N-1)})$ (i.e., by bidding as if his type were $Y_k^{(N-1)}$). The indifference condition for sequential first-price auctions states that bidder i must be indifferent between winning in round k and in round $k+1$ (see equation (14) in Appendix A):

$$\ell(\beta_k^{S1}(x)) = E\left[\ell\left(\beta_{k+1}^{S1}(Y_k^{(N-1)})\right)|Y_k^{(N-1)} \leq x \leq Y_{k-1}^{(N-1)}\right].$$

The left hand side is the certain loss associated with the period k price; the right hand side is the expected loss associated with the random price in period $k+1$. Thus the price sequence must also be decreasing in a sequential first-price auction.

5 The Effect of Informational Externalities

In order to focus on the effect of informational externalities on the price sequence, it is best to look first at the standard risk neutral model (i.e., the case when ℓ is the identity function). I will show that when bidders are neutral with respect to price risk, and there are informational externalities, prices *increase* along the equilibrium path of a sequential auction with bid announcements. It is a bit of a surprise that this result, which applies to the standard model with risk neutral bidders and independent signals, does not seem to have appeared in previous literature. An increasing price sequence has only been derived in the model with affiliated signals by Milgrom and Weber (1982) (see also Mezzetti et al., 2008).

Theorem 5 *In a sequential first-price and in a sequential second-price auction with announcement of the winning bids, if bidders are risk neutral and there are informational externalities, then the price sequence is a submartingale.*

Proof. Recall that ℓ and ϕ coincide with the identity function. Consider first a sequential first-price auction. Suppose type x of bidder i wins auction $k < K$. It must be $Y_k^{(N-1)} \leq x < y_{k-1} < ... < y_1$, and $P_k^{S1} = \beta_k^{S1}(x; y_{k-1}, ..., y_1) = E\left[V\left(Y_K^{(N-1)}, Y_K^{(N-1)}, Y_{K-1}^{(N-1)}, ...\right)|Y_k^{(N-1)} \leq x \leq Y_{k-1}^{(N-1)} = y_{k-1}, ...\right]$. It follows that

$$\begin{aligned}
E\left[P_{k+1}^{S1}|P_k^{S1}\right] &= E\left[P_{k+1}^{S1}|\beta_k^{S1}(x; y_{k-1}, ..., y_1)\right] \\
&= E\left[\beta_{k+1}^{S1}(Y_k^{(N-1)}; x, y_{k-1}, .., y_1)|Y_k^{(N-1)} \leq x \leq Y_{k-1}^{(N-1)} = y_{k-1}, ..\right] \\
&\geq E\left[\beta_{k+1}^{S1}(Y_k^{(N-1)}; Y_k^{(N-1)}, y_{k-1}, .., y_1)|Y_k^{(N-1)} \leq x \leq Y_{k-1}^{(N-1)} = y_{k-1}, ..\right] \\
&= E\left[V\left(Y_K^{(N-1)}, Y_K^{(N-1)}, Y_{K-1}^{(N-1)}, ...\right)|Y_k^{(N-1)} \leq x \leq Y_{k-1}^{(N-1)} = y_{k-1}, ...\right] \\
&= \beta_k^{S1}(x; y_{k-1}, .., y_1) \\
&= P_k^{S1},
\end{aligned}$$

where the inequality follows from $\beta_{k+1}^{S1}(\cdot)$ being an increasing function of all its arguments and $Y_k^{(N-1)} \leq x$. Thus, the price sequence in a sequential first-price auction is a submartingale.

Now consider a sequential second-price auction. Suppose that in round k the winner is the bidder with signal $Y_k^{(N-1)}$, and bidder i of type x is the price setter; that is, $P_k^{S2} = \beta_k^{S2}(x; y_{k-1}, ..., y_1) = E\left[V(Y_K^{(N-1)}, Y_K^{(N-1)}, Y_{K-1}^{(N-1)}, ...)|Y_k^{(N-1)} = x \leq Y_{k-1}^{(N-1)} = y_{k-1}, ...\right]$. In round $k+1$, bidder i of type x wins the auction, and the price setter is the bidder with signal $Y_{k+1}^{(N-1)}$. It follows that

$$\begin{aligned}
E\left[P_{k+1}^{S2}|P_k^{S2}\right] &= E\left[P_{k+1}^{S2}|\beta_k^{S2}(x; y_{k-1}, .., y_1)\right] \\
&= E\left[\beta_{k+1}^{S2}(Y_{k+1}^{(N-1)}; Y_k^{(N-1)}, y_{k-1}, ..)|Y_{k+1}^{(N-1)} \leq X_1 = x \leq Y_k^{(N-1)}, ...\right] \\
&\geq E\left[\beta_{k+1}^{S2}(Y_{k+1}^{(N-1)}; Y_k^{(N-1)}, y_{k-1}, ..)|Y_{k+1}^{(N-1)} \leq X_1 = x = Y_k^{(N-1)}, ...\right] \\
&= E\left[V(Y_K^{(N-1)}, Y_K^{(N-1)}, Y_{K-1}^{(N-1)}, ...)|Y_k^{(N-1)} = X_1 = x \leq Y_{k-1}^{(N-1)} = y_{k-1}, ...\right] \\
&= \beta_k^{S2}(x; y_{k-1}, .., y_1) \\
&= P_k^{S2}.
\end{aligned}$$

Thus, the price sequence in a sequential second-price auction is also a submartingale. ■

The conventional wisdom that with independent signals and risk neutral bidders the price sequence is a martingale is only correct if there are no informational externality (i.e., in the common terminology, values are private). With informational externalities, prices tend to increase from one round to the next. To see why, consider a sequential second-price auction (the reasoning for a first-price auction is similar). As we argued in the previous section, a bidder must be indifferent between winning in the current round and winning in the next round, conditional on having the highest remaining signal and being the price setter. With risk neutral bidders, this implies that the current price must be equal to the expected price in the next round conditional on the event $\mathcal{E}$ that the current winner has the same signal as the price setter. With informational externalities, the next round expected price conditional on the event $\mathcal{E}$ is an underestimate of the next round expected price conditional on the current price. This is because the current winner will generally have a higher signal than the current price setter, and the value of an object directly depends in a positive way on the signals of all bidders.

It is useful to stress that the intuition behind the result in Milgrom and Weber (1982) that the price sequence is increasing when signals are affiliated random variables is similar to the intuition behind Theorem 5. With affiliated signals, the expected value of an object depends in a positive way on the signals of all bidders, and it is also the case that the current price is equal to an underestimate of the price in the next round (see Mezzetti et al., 2008, for a discussion).

6 Aversion to Price Risk and Informational Externalities: Effect Decomposition

While aversion to price risk pushes prices to decline over time, informational externalities introduce a tendency for prices to increase. If bidders are both averse to price risk and there are informational externalities, it is possible to decompose the two countervailing effects on the price sequence.

Given a bid loss ℓ^*, since $\ell\left(\phi(\ell^*)\right) = \ell^*$, we can think of $\phi(\ell^*)$ as the *implicit price* associated with ℓ^*; if a bidder pays a price $\phi(\ell^*)$, his loss is equal to ℓ^*.

Consider round $k+1$ of a sequential first-price auction, after the winners' signal history $x, y_{k-1}, ..., y_1$. (Note that the bidder with signal x is also the price setter in round k.) The round $k+1$ aversion to price risk effect is defined as the difference between the expected price and the implicit price associated with the expected loss in round $k+1$, conditional on the signal history $x, y_{k-1}, ..., y_1$:

$$\begin{aligned} A_{k+1}^{S1}(x, y_{k-1}, ..., y_1) = & E\left[\beta_{k+1}^{S1}(Y_k^{(N-1)}; x, y_{k-1}, ..)|Y_k^{(N-1)} \leq x \leq Y_{k-1}^{(N-1)} = y_{k-1}, ..\right] \\ & - \phi\left(E\left[\ell\left(\beta_{k+1}^{S1}(Y_k^{(N-1)}; x, y_{k-1}, ..)\right)|Y_k^{(N-1)} \leq x \leq Y_{k-1}^{(N-1)} = y_{k-1}, ..\right]\right) \end{aligned}$$

Since ϕ is a concave function, when bidders are averse to price risk the implicit price associated with the expected loss is higher than the expected price; the aversion to price risk effect is negative, $A_{k+1}^{S1}(\cdot) < 0$. If bidders are risk neutral, $A_{k+1}^{S1}(\cdot) = 0$.

The informational externality effect is defined as the increase in the implicit price in round $k+1$ due to the winning bidder in round k having signal $x \geq Y_k^{(N-1)}$, rather than the same signal $Y_k^{(N-1)}$ as the

winner in round $k+1$, conditional on the signal history $x, y_{k-1}, ..., y_1$:

$$I_{k+1}^{S1}(x, y_{k-1}, ..., y_1) = \phi\left(E\left[\ell\left(\beta_{k+1}^{S1}(Y_k^{(N-1)}; x, y_{k-1}, ..)\right)|Y_k^{(N-1)} \leq x \leq Y_{k-1}^{(N-1)} = y_{k-1}, ..\right]\right)$$
$$- \phi\left(E\left[\ell\left(\beta_{k+1}^{S1}(Y_k^{(N-1)}; Y_k^{(N-1)}, y_{k-1}, ..)\right)|Y_k^{(N-1)} \leq x \leq Y_{k-1}^{(N-1)} = y_{k-1}, ..\right]\right).$$

If there are informational externalities $I_{k+1}^{S1}(\cdot) > 0$, because β_{k+1}^{S1}, is an increasing function of all its variables; the informational externality effect is positive. If there are no informational externalities, bids do not depend on the signals of past winners and so the implicit price in round $k+1$ does not depend on the signal of the winning bidder in round k; hence $I_{k+1}^{S1}(\cdot) = 0$.

The aversion to price risk effect and the informational externality effect in a sequential second-price auction are defined as in the case of a first-price auction, except that conditioning now is on the event that in round k the winner is the bidder with signal $Y_k^{(N-1)}$, that bidder i of type x is the price setter, and that the previous rounds winners' signals are $y_{k-1}, ..., y_1$.

$$A_{k+1}^{S2}(x, y_{k-1}, ..., y_1) = E\left[\beta_{k+1}^{S2}(Y_{k+1}^{(N-1)}; Y_k^{(N-1)}, y_{k-1}, ..)|Y_{k+1}^{(N-1)} \leq X_1 = x \leq Y_k^{(N-1)}, ..\right]$$
$$- \phi\left(E\left[\ell\left(\beta_{k+1}^{S2}(Y_{k+1}^{(N-1)}; Y_k^{(N-1)}, y_{k-1}, ..)|\right)|Y_{k+1}^{(N-1)} \leq X_1 = x \leq Y_k^{(N-1)}, ..\right]\right).$$

$$I_{k+1}^{S2}(x, y_{k-1}, ..., y_1) = \phi\left(E\left[\ell\left(\beta_{k+1}^{S2}(Y_{k+1}^{(N-1)}; Y_k^{(N-1)}, y_{k-1}, ..)|\right)|Y_{k+1}^{(N-1)} \leq X_1 = x \leq Y_k^{(N-1)}, ..\right]\right)$$
$$- \phi\left(E\left[\ell\left(\beta_{k+1}^{S2}(Y_{k+1}^{(N-1)}; Y_k^{(N-1)}, y_{k-1}, ..)|\right)|Y_k^{(N-1)} = x\right]\right).$$

The next theorem, whose proof is in Appendix A, shows that in both sequential auctions the expected price in round $k+1$, conditional on the price in round k, is equal to the sum of the price in round k, the aversion to price risk effect, and the informational externality effect.

Theorem 6 *Suppose the price setter in round k has signal x and the history of winners' signals up to round $k-1$ is $y_{k-1}, ..., y_1$. In the sequential first-price auction ($j = 1$) and the sequential second-price auction ($j = 2$) with announcement of the winning bids we have:*

$$E\left[P_{k+1}^{Sj}|P_k^{Sj}\right] = P_k^{Sj} + A_{k=1}^{Sj}(x, y_{k-1}, ..., y_1) + I_{k=1}^{Sj}(x, y_{k-1}, ..., y_1).$$

7 A Calibrated Example

Can aversion to price risk explain the declining price sequences we observe in the data? What is the degree of aversion to price risk that is needed? May prices decline even with informational externalities? This section provides some answers to these questions. I introduce a simple parametric example, and show that its predictions match the data from a sample of empirical studies, for reasonable specifications of the parameters. The empirical reference points for the discussion in this section are the papers of Ashenfelter (1989) and McAfee and Vincent (1993) on sequential (mostly two-round) auctions of identical bottles of wine sold in equal lot sizes. Ashenfelter's (1989) data set included auctions between August 1985 and December 1987 in four different location (Christies's London and Chicago, Sotheby's London and

Buttersfield's San Francisco). McAfee and Vincent (1993) looked at auctions held at Christies's in Chicago in 1987. They both found evidence of declining prices; the price in the second auction was twice more likely to decrease than to increase. The average price ratio P_2/P_1 they found is displayed in Figure 1.

	Mean Ratio P_2/P_1
Ashenfelter (1989): Christies's London	.9943
Ashenfelter (1989): Sotheby's London	.9875
Ashenfelter (1989): Christies's Chicago	.9884
Ashenfelter (1989): Butterfield's San Francisco	.9663
McAfee and Vincent (1993): Christies's Chicago	.9922

Figure 1: Price ratio in the data

I will make the following simplifying assumptions to the model. The value of an object to bidder i is $x_i + b\sum_{j\neq i} x_j$, with $b \in [0,1]$. If $b = 0$, there are no informational externalities. The random variables X_i are distributed on $[0,1]$ with distribution function $F(x) = x^a$, with $a > 0$. The loss function is:

$$\ell(p) = \frac{p^{1+r}}{1+r},$$

We can interpret $r = p\ell''/\ell'$ as a coefficient of relative price-risk aversion. The inverse of ℓ is

$$\phi(z) = (1+r)^{\frac{1}{1+r}}\, z^{\frac{1}{1+r}}.$$

I will restrict attention to the second-price auction and the case of two rounds, $K = 2$. In Appendix B, I compute the bidding functions, the expected price in round 2 conditional on the first-round price P_1, and the ratio of the conditional expected second-round price to the first-round price.

In all computations reported in this section, I will set $r = 2$, a commonly used value for relative risk aversion in computational macroeconomics (e.g., see Ljungqvist and Sargent, 2000); it implies that a bidder is willing to pay a price about 1% higher to avoid a 50-50 gamble of a 10% increase or a 10% decrease in price. The results do not seem overly sensitive to the value of r.

If we first postulate that in the auctions in question there were no informational externalities, that is $b = 0$, then the price ratio is (see (27)):

$$\frac{E\left[P_2|P_1\right]}{P_1} = \frac{(a(N-2))^{\frac{r}{1+r}}\left(a\left(N-2\right)+1\right)^{\frac{1}{1+r}}}{\left(a(N-2)+\frac{1}{1+r}\right)}. \tag{1}$$

We can use (1) to graph the price ratio $E\left[P_2|P_1\right]/P_1$ as a function of $A = a(N-2)$.

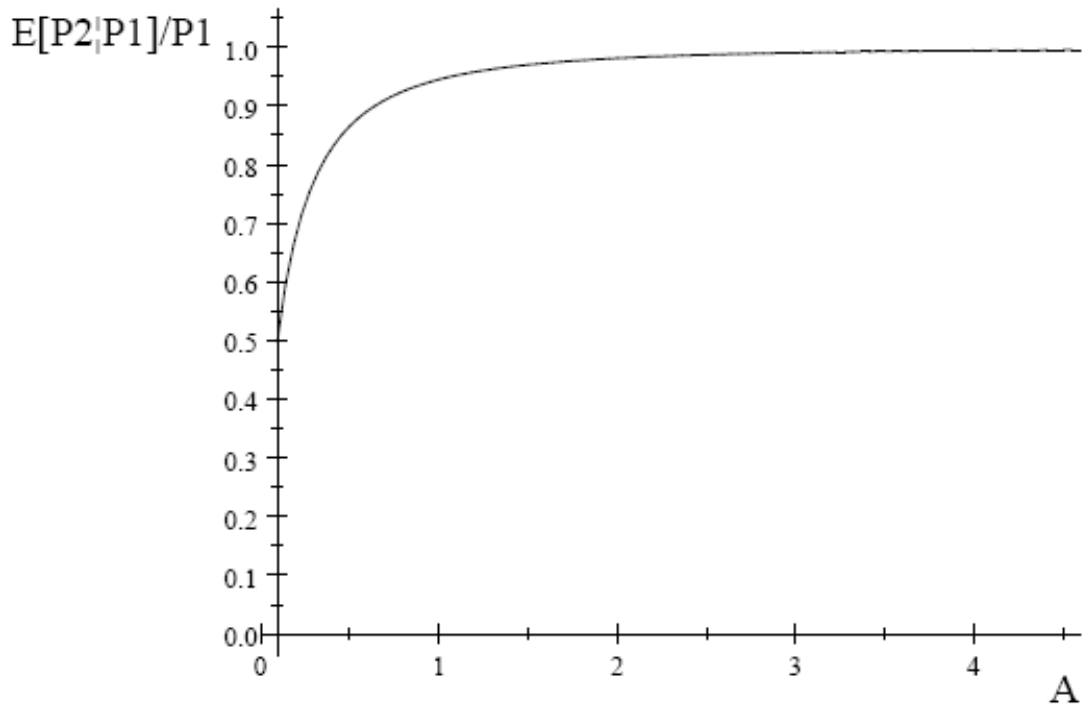

Figure 2: Price ratio with no informational externalities as a function of $A = a(N-2)$

The average price ratio in the data in Figure 1 ranges from 0.9663 to 0.9943, which correspond to values of A from 1.3903 to 3.9788. In most of the auctions considered the number of bidders was relatively small, typically well below 20. If we take $N = 10$, this gives values of a between 0.1738 and 0.4973 as those consistent with the data.

Suppose now that there are informational externalities, $b > 0$. In this case $\frac{E[P_2|P_1]}{P_1}$ depends on the type x of the first-round price setter. Appendix B reports the expected value of the price ratio $E\left[\frac{E[P_2|P_1]}{P_1}\right]$. With informational externalities, lower values of the parameter a are needed to match the data. Setting $r = 2$ as before, $N = 10$, and $a = 0.1$ yields the relationship between the price ratio and the informational externality parameter b shown in Figure 3. The range of the average price ratio in the data in Figure 1 corresponds to values of the informational externality parameter between 0.0827 and 0.1260. Prices may decline even if there are informational externalities. In fact, the presence of both aversion to price risk and informational externalities could help explaining why in some auctions prices decline and in other they increase; it could also help explaining why in some multiple round auctions prices decline between some

rounds and increase between other rounds (e.g. see Jones et al. 2004).

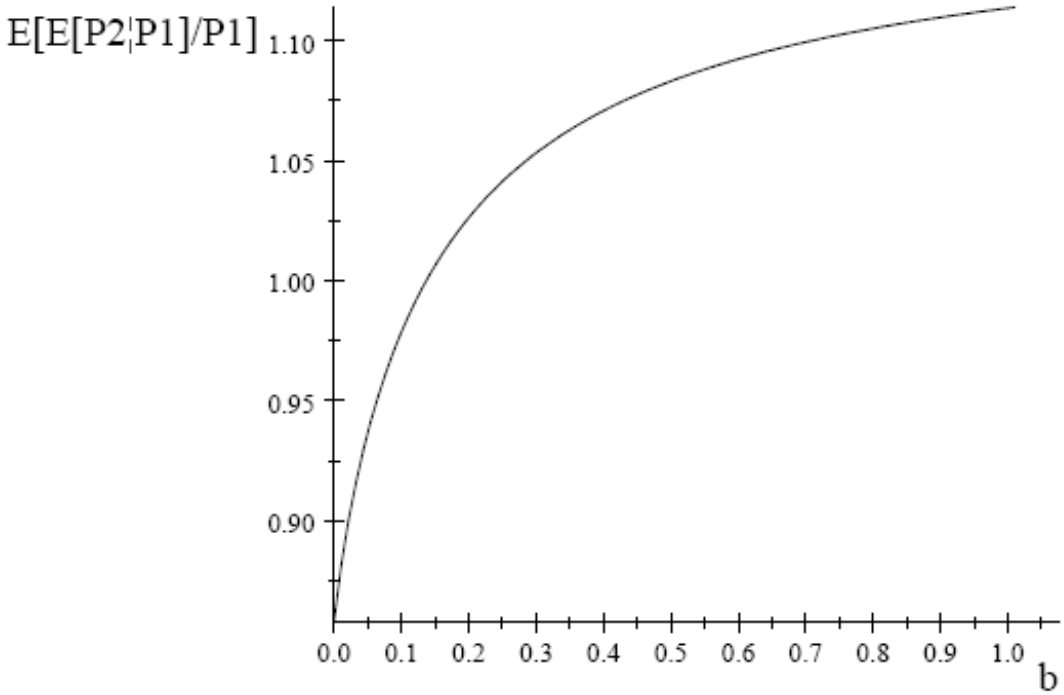

Figure 3: Price ratio as a function of the information externality parameter b

8 Extensions

In this section, I show that the aversion to price risk effect is present even when bidders demand multiple units, or the auction format is an oral ascending auction. While I do not prove it formally, if I allowed for informational externalities, the informational externality effect would also be present in these extensions.

8.1 Multi-Unit Demand

New issues and serious complications arise in models with multi-unit demand, which are generally not very tractable (e.g., see Milgrom, 2004). To keep the focus on the effect of aversion to price risk in a tractable model, in this subsection I extend Katzman's (1999) model of a two-round, second-price auction with no informational externalities.

I assume that each bidder extracts two values from the same distribution F. If x_h is the highest and x_l the lowest value extracted, then bidder i obtains a payoff of $x_h - \ell(p_k)$ if he only wins one object in round k at price p_k, and a payoff of $x_h + x_l - \ell(p_1) - \ell(p_2)$ if he wins two objects at prices p_1 and p_2. In Katzman (1999), ℓ is the identity function, and hence bidders are risk neutral.

The proof of the following theorem is in Appendix C.

Theorem 7 *On the equilibrium path of the two-round, second-price auction with multi-unit demand and no informational externalities, the bidding functions are*

$$\beta_2^{S2}\left(x_h, x_l | x_h > Y_1^{(2N-2)}\right) = \phi(x_l)$$

$$\beta_2^{S2}\left(x_h, x_l | x_h < Y_1^{(2N-2)}\right) = \phi(x_h)$$
$$\beta_1^{S2}(x_h, x_l) = \phi\left(E\left[Y_2^{(2N-2)} | Y_1^{(2N-2)} = x_h\right]\right).$$

The auction is efficient, the bidder or bidders with the two highest marginal valuations win the objects. In the second round, the loss bid is equal to the bidder's value for the object, which is equal to x_l if the bidder won in the first round and to x_h if he did not. In the first round, the loss bid is equal to the expected second highest value of a bidder's opponents, conditional on the highest value of the bidder's opponents being equal to the bidder's highest value. A bidder behaves as if he is only going to win an object, and his bid is selected so as to make him indifferent between winning in the first or in the second round.

Suppose that the first-round price setter is bidder i of type (x_h, x_l); the bidder with the highest value among bidder i's opponents won the first round. There are two possible events. The first event is that the first-round price setter, bidder i, wins the second round auction. In this case the expected second round price is the expected bid associated with the second highest value of bidder i's opponents; that is, $E\left[\phi(Y_2^{(2N-2)}) | Y_2^{(2N-2)} \leq x_h\right]$, which is equal to $E\left[\phi(Y_2^{(2N-2)}) | Y_1^{(2N-2)} = x_h\right]$, because of the independence of signals. This first event has probability $\Pr\{Y_2^{(2)} < x_h | Y_1^{(2)} > x_h\}$, the conditional probability that the first-round winner has a marginal value for the second object lower than x_h. The second event is that the first-round winner also wins the second auction, and hence the first-round price setter also sets the price in the second round; in this case the expected second round price is $\phi(x_h)$. This event happens with probability $\Pr\{Y_2^{(2)} > x_h | Y_1^{(2)} > x_h\}$. It follows that, conditional on bidder i of type (x_h, x_l) being the first-round price setter, the second round expected price is:

$$E\left[P_2 | P_1\right] = E\left[\phi\left(Y_2^{(2N-2)}\right) | Y_1^{(2N-2)} = x_h\right] \Pr\{Y_2^{(2)} < x_h | Y_1^{(2)} > x_h\} + \phi(x_h) \Pr\{Y_2^{(2)} > x_h | Y_1^{(2)} > x_h\}. \tag{2}$$

Define the aversion to price risk effect as:

$$\begin{aligned} A_2^{S2}(x_h) &= \left\{E\left[\phi\left(Y_2^{(2N-2)}\right) | Y_1^{(2N-2)} = x_h\right] - \phi\left(E\left[Y_2^{(2N-2)} | Y_1^{(2N-2)} = x_h\right]\right)\right\} \Pr\{Y_2^{(2)} < x_h | Y_1^{(2)} > x_h\} \\ &= \left\{E\left[\phi\left(Y_2^{(2N-2)}\right) | Y_1^{(2N-2)} = x_h\right] - \beta_1^{S2}(x_h, x_l)\right\} \Pr\{Y_2^{(2)} < x_h | Y_1^{(2)} > x_h\}. \end{aligned} \tag{3}$$

The aversion to price risk effect is the difference between the expected price and the implicit price associated with the expected loss in round 2, conditional on the first-round price setter winning the second round auction, multiplied by the probability that the first-round price-setter wins in the second round.

Define the multi-unit demand effect as:

$$M_2^{S2}(x_h) = \left[\phi(x_h) - \beta_1^{S2}(x_h, x_l)\right] \Pr\{Y_2^{(2)} > x_h | Y_1^{(2)} > x_h\}. \tag{4}$$

The multi-unit demand effect measures the effect on the price sequence of the same bidder winning both rounds. It is the difference between the second-round price and the first-round price, conditional on the first-round price-setter being the price setter also in the second round, multiplied by the probability that the first-round and second-round price setters are the same bidder, which implies that the same bidder wins both first and second round.

While the aversion to price risk effect is non-positive, the multi-unit demand effect is non-negative.[4] As the next theorem shows, whether the price sequence in a two-round second-price auction with no informational externalities is decreasing depends on whether the aversion to price risk effect dominates the multi-unit demand effect (the proof is in Appendix C).

Theorem 8 *Suppose that the price setter in the first round has signals* (x_h, x_l)*. Then the expected second round price in the two-round, second-price auction with multi-unit demand and no informational externalities is:*

$$E\left[P_2|P_1\right] = P_1 + A_2^{S2}(x_h) + M_2^{S2}(x_h)$$

Consider the example with $F(x) = x^a$ and $\ell(p) = p^{1+r}/(1+r)$ introduced in Section 7. Appendix C computes the bidding functions, the expected price in round 2 and the conditional price ratio. Letting $r = 2$ and $N = 10$, as in Section 7, Figure 4 graphs the expected price ratio as a function of a. The average price ratio in the data in Figure 1 ranges from 0.9663 to 0.9943. This corresponds to values of a ranging from 0.0728 to 0.1897. Aversion to price risk may generate declining prices even if bidders have multi-unit demand.

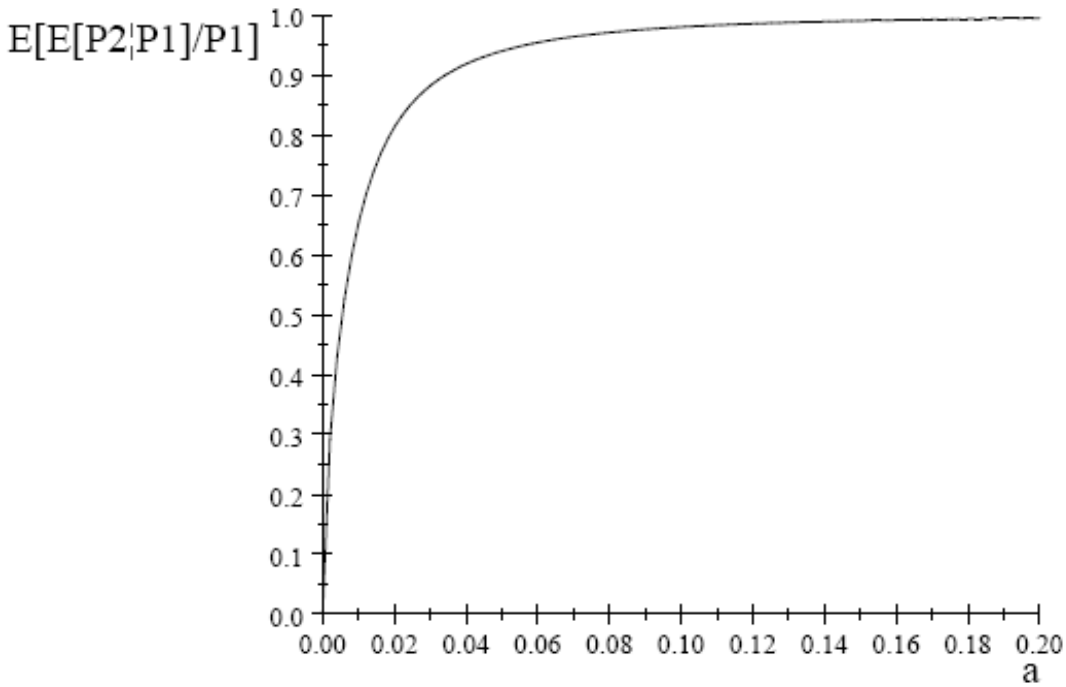

Figure 4: Price ratio as a function of a

8.2 The English Auction

Sequential auctions are often run using an English, or oral ascending, format. The fundamental difference between an English format and a sealed bid auction is that in the former information endogenously accrues to bidders in the course of each round of play. The main issue with analyzing sequential English auctions

[4] Katzman (1999) showed that with risk neutral bidders the price sequence is increasing.

is to find a formal model that is tractable and captures the information flow during the auction. The most commonly used model to study a static, single unit, English auction is the so-called Japanese version, in which a price clock moves continuously and bidders can only decide when to quit. Once a bidder quits, he cannot re-enter. When the second to last bidder quits, the clock stops and the last bidder standing wins at the current clock price. The main virtue of such an auction format is its simplicity. In the case of no informational externalities and risk neutral bidders, for example, it is a dominant strategy to quit when the price reaches one's value for the item. Using the Japanese format for sequential auctions is problematic. As Milgrom and Weber (1982) first pointed out, the equilibrium is the same as in the static English auction for multiple items. Consider for example the case of two objects, no informational externalities and risk neutral bidders. In the first round, it is a dominant strategy for all bidders to drop out at their value for the object if there are more than two bidders remaining, and to drop out immediately after the third to last bidder has dropped out. Thus, the price in the first round is equal to the third highest bidder's valuation. This is also the price that prevails in the second, and last, round. The price is the same in all rounds; in fact, a sequential Japanese auction is outcome equivalent to a static, multi-unit, Japanese auction. This is because the Japanese format forces all losing bidders to reveal their types during the first round. At the time the first round price is determined, all that is needed to determine prices in all rounds has been revealed. This is true independently of whether bidders are risk neutral or averse to price risk; aversion to price risk cannot lead to decreasing prices because bidders face no price risk!

Such a counterintuitive conclusion is a by-product of the extreme nature of the Japanese format. In practical ascending auctions, it is not the case that at the end of the first round all bidders in the room know the identity of all future winners and the types of all losing bidders. Bidders often stay silent at the beginning and only start bidding towards the end of a round. Some bidders stay silent throughout a round, and it is not at all clear what their values are. When bidders are allowed to decide how much information to reveal in the course of bidding, the analysis can be unwieldy. In this subsection, I will introduce a format of an ascending auction that is tractable, and allows bidders to hide their values during a round. The only point I want to make is that the afternoon effect is still present, as long as bidders in the early rounds are uncertain, at the time price is determined, about the valuation of some of their opponents (allowing for the probability of entry of new bidders in the second round would serve the same purpose).

I will assume that there are only two rounds and that there are no informational externalities. Bidders are averse to price risk and have unit demand. In each round of the auction, the auctioneer calls bidders to increase the current price by a fixed increment Δ (I will let Δ go to zero). If one or more bidders raise the price, then the bidder with the lowest label is selected as the new high bidder at the new price (the tie breaking rule is not important). If no bidder raises the price, then the auctioneer makes a second call, at the same price. If one or more bidders raises after the second call, then, as before, the bidder with the lowest label becomes the new high bidder. If no bidder raises after the second call, then the round ends and the current high bidder wins an object at the current price.

This sequential ascending auction format has an equilibrium in which only two bidders are initially active (say bidder 1 and bidder 2), in the first round. When one of the two active bidders does not raise the other active bidder's offer, then all other bidders enter the bidding (raise the current price) if it is

profitable for them to do so. Such an equilibrium has declining prices between rounds. This is because in the first round, when deciding to stop raising the price, the two initially active bidders are uncertain about the other bidders' valuations, and hence next round price. The uncertainty is due to all other bidders not being active in the early stages of the first round of the auction. The intuition for the occurrence of an afternoon effect is the same as in the case of a second-price auction. Bidders 1 and 2 are willing to pay a premium to insure themselves against a future uncertain price.

The proof of the following theorem is in Appendix C.

Theorem 9 *There is an equilibrium of the two-round, English auction, with no informational externalities, in which:* 1) *In the second round, each remaining bidder i answers the first call by raising the price if and only if he is not the current winner and the current price is below $\beta_2^E(x_i) = \phi(x_i)$.* 2) *In the first round, bidder $i = 1, 2$ answers the first call and raises bidder $j = 1, 2$, $j \neq i$, current winning price if and only if the current price is below $\beta_{1E}^E(x_i) = \phi\left(E\left[Y_2^{(N-1)}|Y_1^{(N-1)} = x_i\right]\right)$ and no other bidder has yet entered the bidding (i.e., ever raised the price). After a bidder different from 1 and 2 has entered the bidding for the first time, then bidder i raises the current winning price if and only if he is not the current winner and the current winning price is below $\beta_{1L}^E(x_i) = \phi(x_i)$.* 3) *In the first round, bidder $i \neq 1, 2$ does not enter the bidding as long as bidders 1 and 2 are raising each other prices. If one of bidders $1, 2$ does not answer the first-call to raise current price $p - \Delta$, then bidder i answer the second call and raises if and only if $p < \phi(x_i)$; after raising the price once, bidder i continues raising the price as long as he is not the current winner and the current winning price is below $\phi(x_i)$.*

The winners of the auction are the two highest valuation bidders. There are two possible outcomes. First, all bidders become active in the first round. In this case, the prices in the first and in the second round are the same and equal to $\phi(Y_3^{(N)})$; the types of all losing bidders will become known by the end of in the first round. This is the same outcome that would obtain in the equilibrium of the Japanese auction. The second possible outcome is that only bidder 1 and bidder 2 are active in the first round. This can happen only if they are the two highest valuation bidders. Without loss of generality, we can assume that bidder 1 has the highest valuation. Then the first round price is $P_1 = \phi\left(E\left[Y_2^{(N-1)}|Y_1^{(N-1)} = x_2\right]\right)$. The expected second round price, conditional on P_1, is $E\left[P_2|P_1\right] = E\left[\phi\left(Y_2^{(N-1)}\right)|Y_1^{(N-1)} = x_2\right]$; by Jensen's inequality, it is $E\left[P_2|P_1\right] < P_1$. Since the second outcome will occur with strictly positive probability, we have proved the following theorem.

Theorem 10 *The price sequence of the two-round English auction with no informational externalities is a supermartingale.*

9 Conclusions

The classic model of risk neutral bidders assumes additive separability in a bidder's preferences over objects and money. Aversion to price risk maintains additive separability, but postulates that a bidder prefers a certain price to an equivalent (on average) random price. Additive separability of preferences makes the

model very tractable. As I show in Lemma 11 in Appendix A, it implies that a bidder-payoff equivalence theorem holds. All auction mechanisms with the same allocation rule and which give the same payoff to the lowest bidder type are bidder-payoff equivalent.

Without additive separability, the effects due to aversion to price risk, aversion to quantity risk, and private information interact. The interaction between (price and quantity) risk aversion and private information in a single-sale model is the focus of Maskin and Riley (1984). McAfee and Vincent (1993) showed that sequential auctions without additive separability are not very tractable; equilibrium in their two-round, second-price auction with no informational externalities is in mixed strategies.

Aversion to price risk yields a simple explanation of declining prices in sequential auctions. In any given round, a small change in his own bid matters to a bidder only if it is, at the same time, the winning bid and the price setting bid. Thus, when his own bid matters, a bidder wins at a certain price, his own bid. Optimality requires that there is no first order effect on the bidder's payoff of a change in the bid. This implies that the bidder is indifferent between winning in the current round or in the next round. Since next round prince is random, in effect the bidder buys price insurance in the current round to protect himself against future price randomness.

The paper also uncovers an informational externality effect. When there is no aversion to price risk, but there are informational externalities, prices increase between rounds of sequential auctions. Again, the explanation is simple. The current price setting bidder must be indifferent between winning in the current or in the next round, assuming that he is also the current winner; that is, he must be indifferent between the current price (which he sets) and his estimate of the next round price. Since he assumes that he is also the current winner, the current price setter underestimates the signal of his highest opponent and true winner of the current round. Because of informational externalities, this amounts to underestimating next round price. Hence, on average next round price is higher than the current price.

Several empirical implications can be drawn from this paper. First, the more important a concern is price risk for bidders, the more we should expect prices to decline between rounds. Thus, for example, if there is a serious possibility that new bidders may enter in the next round, then price risk is more severe and we should expect prices to decline more.

Second, when informational externalities, or value interdependencies, are not very important, but bidders are averse to price risk, then prices are likely to decline. When value interdependencies are more important than price risk, then we should expect prices to increase between rounds. For example, if the auctioneer publishes all the information at his disposal (as the professional auction houses typically do), including value estimates of the objects for sale, then interdependencies are reduces and it is more likely that we see prices decline (as the data broadly suggests), rather than increase between rounds. If bidders are professionals, buying the goods for resale, and little information is provided about resale value by the auctioneer before the auction, then it is more likely that prices will increase between rounds.[5]

Third, when bidders have multi-unit demand, so, for example, if the same bidder wins multiple rounds,

[5] This seems broadly consistent with Deltas and Kosmopoulou (2005) study of an auction of rare library books, in which price estimates were not published and a lower bound on the number of professionals in the auction is estimated by the authors at about 25%.

then it is more likely that prices only decline moderately, or even increase, between rounds.[6]

Fourth, the less information about bidders' values transpires during a round, the more we should expect prices to decline. Thus, if each round is an oral ascending auction, the larger the number of bidders that remain silent during the initial rounds, the higher future price randomness, and hence the more likely are prices to decline between rounds.

More generally, the interaction between the aversion to price risk effect and the informational externality effect could help to explain the more complex price paths we sometimes observe in the data.

[6] This seems consistent with the findings in Jones et al. (2004).

Appendix A

This appendix contains the proofs of Theorems 1, 2, 3 and 6. First, it is it is useful to derive a lemma showing that bidders' payoffs are the same in every auction having the same outcome function and yielding the same payoff to the lowest type of bidder.

Suppose that k objects have already been sold to the k highest type bidders, $y_1, ..., y_k$; suppose also that the winners' types have been revealed. Consider a mechanism in which $\pi_{k+1}(x', y_{N-1}, ..., y_{k+1})$ is i's probability of winning one of the remaining objects and $p_{k+1}(x', y_{N-1}, ..., y_{k+1})$ is i's payment when he behaves as a type x'. Then, bidder i's expected payoff when his type is x, but he behaves as if his type were x' is

$$U_{k+1}(x'; x; y_k, ..., y_1) = \int_{\underline{x}}^{y_k} ... \int_{\underline{x}}^{y_{N-2}} \left[V(x, y_{N-1}, ..., y_1)\pi_i(x', y_{N-1}, ..., y_{k+1}) - \right.$$
$$\left. \ell(p_i(x'_i, y_{N-1}, ..., y_{k+1}))\right] f(y_{N-1}, ..., y_{k+1}|Y_k^{(N-1)} = y_k)dy_{N-1}...dy_{k+1},$$

where $f(y_{N-1}, ..., y_{k+1}|y_k)$ is the density of the order statistics $Y_{N-1}^{(N-1)}, ..., Y_{k+1}^{(N-1)}$ conditional on $Y_k^{(N-1)} = y_k$. (By independence, it is not necessary to condition on the order statistics $Y_h^{(N-1)}$ with $h < k$.)

Letting $U^*_{k+1}(x; y_k, ..., y_1) = U_{k+1}(x; x; y_k, ..., y_1)$ be the expected payoff in equilibrium of type x, and using a standard envelope argument yields

$$\frac{\partial U^*_{k+1}(x; y_k, ..., y_1)}{\partial x} \tag{5}$$
$$= \int_{\underline{x}}^{y_k} ... \int_{\underline{x}}^{y_{N-2}} \frac{\partial V(x, y_{N-1}, ..., y_1)}{\partial x} \pi_i(x, y_{N-1}, ..., y_{k+1}) f(y_{N-1}, ..., y_{k+1}|Y_k^{(N-1)} = y_k)dy_{N-1}...dy_{k+1}.$$

In particular, if the mechanism is efficient, as the sequential auctions studied in this paper, then (5) becomes

$$\frac{\partial U^*_{k+1}(x; y_k, .., y_1)}{\partial x} = E\left[\frac{\partial V(x, Y_{N-1}^{(N-1)}, .., Y_{k+1}^{(N-1)}, y_k, .., y_1)}{\partial x}\Big| x > Y_K^{(N-1)}\right] F_K(x|Y_k^{(N-1)} = y_k). \tag{6}$$

Equation (5), combined with $U^*_{k+1}(\underline{x}; ...) = \underline{u}$, yields the bidder-payoff equivalence lemma.

Lemma 11 *Suppose $k = 0, ..., K-1$ objects have already been sold to the highest type bidders, and the winning types have been announced. Bidders' payoffs are the same in any mechanism $\langle \pi_{k+1}, p_{k+1}\rangle$ having the same outcome function π_{k+1} and yielding the same payoff to the lowest type. Equation (5) (equation (6) if the mechanism is efficient) and the boundary condition $U^*_{k+1}(\underline{x}; ...) = \underline{u}$ determine a bidder's payoff.*

We are now ready to prove Theorems 1-3

Proof of Theorem 1. Let $\beta_k^{S1}(x; y_{k-1}, ..., y_1)$ be round k equilibrium bidding function. Recall that, assuming that β_k^{S1} is increasing in x, on the equilibrium path the true types of the winning bidders are revealed. Suppose that if the winning bid in round k is higher than the highest equilibrium bid, then all bidders believes that the winning bidder's type is the same as the type of the previous round's winner; if the observed winning bid in round k is below the lowest equilibrium bid, then bidders believe that the winner's type is the lowest possible type.

Let $U_k^*(x; y_{k-1}, ...)$ be the expected payoff for a type x of bidder in the continuation equilibrium beginning in round k (i.e., the payoff conditional on having lost all previous auctions and on the history up to round k). In writing a bidder's payoff, I will use the function v_k, defined as follows:

$$v_k(x, y_k, ..., y_1) = E\left[V(X_i, X_{-i})|X_i = x, Y_k^{(N-1)} = y_k, ..., Y_1^{(N-1)} = y_1\right]. \tag{7}$$

Suppose that all the other bidders follow the equilibrium strategies, while bidder i is considering deviating in round k (only). First note that, given his beliefs, it is not profitable for bidder i to bid above the highest possible bid of the other bidders $\beta_k^{S1}(y_{k-1}; \cdot)$. Bidding below the lowest possible bid is equivalent to bidding the lowest bid; in both cases winning is a zero probability event. Hence if there is a profitable deviation, there is a profitable deviation with a bid in the range of possible bids. The payoff of bidder i of type x when he bids $b = \beta_k^{S1}(z; y_{k-1}, ...)$ (i.e., he bids like a type z) in round k is:

$$\begin{aligned} U_k(z; x; y_{k-1}, ...) = &\int_{\underline{x}}^{z} \left[v_k(x, y_k, ...) - \ell(\beta_k^{S1}(z; y_{k-1}, ...))\right] f_k^{(N-1)}\left(y_k|Y_{k-1}^{(N-1)} = y_{k-1}\right) dy_k \\ &+ \int_{z}^{\overline{x}} U_{k+1}^*(x; y_k, y_{k-1}, ..) f_k^{(N-1)}\left(y_k|Y_{k-1}^{(N-1)} = y_{k-1}\right) dy_k. \end{aligned} \tag{8}$$

Differentiating with respect to z yields the first order condition

$$\begin{aligned} &v_k(x, z, y_{k-1}, ..) f_k^{(N-1)}\left(z|Y_{k-1}^{(N-1)} = y_{k-1}\right) - \frac{d\left(\ell(\beta_k^{S1}(z; y_{k-1}, ...)) F_k^{(N-1)}\left(z|Y_{k-1}^{(N-1)} = y_{k-1}\right)\right)}{dz} \\ &- U_{k+1}^*(x; z, y_{k-1}, ..,) f_k^{(N-1)}\left(z|Y_{k-1}^{(N-1)} = y_{k-1}\right) = 0. \end{aligned} \tag{9}$$

Since on the equilibrium path it is $x \leq y_{k-1}$, and $z = x$ must be optimal, we obtain the following necessary condition for equilibrium:

$$\begin{aligned} &v_k(x, x, y_{k-1}, ..) f_k^{(N-1)}\left(x|Y_{k-1}^{(N-1)} = y_{k-1}\right) - \frac{d\left(\ell(\beta_k^{S1}(x; y_{k-1}, ..)) F_k^{(N-1)}\left(x|Y_{k-1}^{(N-1)} = y_{k-1}\right)\right)}{dx} \\ &- U_{k+1}^*(x; x, y_{k-1}, ..,) f_k^{(N-1)}\left(x|Y_{k-1}^{(N-1)} = y_{k-1}\right) = 0. \end{aligned} \tag{10}$$

Observe that if the signal of the winner in round $k < K$ is x, then in round $k+1$ bidder i with signal x wins with probability 1; hence, it is

$$U_{k+1}^*(x; x, y_{k-1}, ..,) = v_k(x, x, y_{k-1}, ...) - \ell(\beta_{k+1}^{S1}(x; x, y_{k-1}, ...)). \tag{11}$$

Since $U_{K+1}^*(x; x, y_{K-1}, ..,) = 0$, equation (11) also holds for $k = K$, provided we define

$$\beta_{K+1}^{S1}(x; x, y_{K-1}, ...) = \phi\left(v_K(x, x, y_{K-1}, ...)\right). \tag{12}$$

Using (11), equation (10) can be written as

$$\ell(\beta_{k+1}^{S1}(x; x, y_{k-1}, ...)) f_k^{(N-1)}\left(x|Y_{k-1}^{(N-1)} = y_{k-1}\right) - \frac{d\left(\ell(\beta_k^{S1}(x; \cdot)) F_k^{(N-1)}\left(x|Y_{k-1}^{(N-1)} = y_{k-1}\right)\right)}{dx} = 0. \tag{13}$$

Integrating (13) we obtain

$$\ell(\beta_k^{S1}(x;y_{k-1},..,y_1)) = \int_{\underline{x}}^{x} \ell\left(\beta_{k+1}^{S1}(\widetilde{x};\widetilde{x},y_{k-1},..,y_1)\right) \frac{f_k^{(N-1)}\left(\widetilde{x}|Y_{k-1}^{(N-1)}=y_{k-1}\right)}{F_k^{(N-1)}\left(x|Y_{k-1}^{(N-1)}=y_{k-1}\right)} d\widetilde{x} \tag{14}$$
$$= E\left[\ell\left(\beta_{k+1}^{S1}(Y_k^{(N-1)};Y_k^{(N-1)},y_{k-1},...)\right)|Y_k^{(N-1)} \leq x \leq Y_{k-1}^{(N-1)}=y_{k-1}\right].$$

By (12), for $k=K$, this yields

$$\ell(\beta_K^{S1}(x;y_{K-1},...)) = E\left[V(Y_K^{(N-1)},Y_K^{(N-1)},y_{K-1},...)|Y_K^{(N-1)} \leq x \leq Y_{K-1}^{(N-1)}=y_{K-1}\right].$$

Working backwards, (14) yields

$$\ell(\beta_k^{S1}(x;y_{k-1},...)) = E\left[V(Y_K^{(N-1)},Y_K^{(N-1)},Y_{K-1}^{(N-1)},...)|Y_k^{(N-1)} \leq x \leq Y_{k-1}^{(N-1)}=y_{k-1},...\right],$$

and hence on the equilibrium path the bidding function must satisfy

$$\beta_k^{S1}(x;y_{k-1},..,y_1) = \phi\left(E\left[V(Y_K^{(N-1)},Y_K^{(N-1)},Y_{K-1}^{(N-1)},..,Y_k^{(N-1)},y_{k-1},..)|Y_k^{(N-1)} \leq x \leq Y_{k-1}^{(N-1)}=y_{k-1}\right]\right). \tag{15}$$

Note from (15) that if values are private $V(Y_K^{(N-1)},Y_K^{(N-1)},Y_{K-1}^{(N-1)},...) = Y_K^{(N-1)}$, and β_k^{S1} is independent of $y_1,...,y_{k-1}$.

It remains to show that if all bidders follow the equilibrium bidding strategy in the rounds after k, and if in round k all other bidders follow the bidding strategy β_k^{S1} defined in (15), then it is also optimal for bidder i to follow it. Using (10) to replace the second term on the left hand side of equation (9) we obtain

$$\frac{\partial U_k}{\partial z} = \left[v_k(x,z,y_{k-1},...) - v_k(z,z,y_{k-1},...)\right] f_k^{(N-1)}\left(z|Y_{k-1}^{(N-1)}=y_{k-1}\right) \tag{16}$$
$$+ \left[U_{k+1}^*(z;z,y_{k-1},..,) - U_{k+1}^*(x;z,y_{k-1},..,)\right] f_k^{(N-1)}\left(z|Y_{k-1}^{(N-1)}=y_{k-1}\right).$$

Since v_k is increasing in x and $U_{K+1}^*=0$, for $k=K$ the sign of $\frac{\partial U_k}{\partial z}$ is the same as $x-z$; hence $z=x$ is optimal.

Now suppose $k<K$; take first the case $z \leq x$. Note that

$$U_{k+1}^*(x;z,y_{k-1},..,) = v_k(x,z,y_{k-1},...) - \ell(\beta_{k+1}^{S1}(z;z,y_{k-1},...)),$$

because in this case bidder i wins for sure in round k, and bids $\beta_{k+1}^{S1}(\min\{x,z\};z,y_{k-1},...) = \beta_{k+1}^{S1}(z;z,y_{k-1},...)$. It follows that $\frac{\partial U_k}{\partial z}=0$ for $z \leq x$ and bidder i has no incentive to bid less than the equilibrium strategy in round k.

Now suppose that $z>x$. By Lemma 11, equation (6), we have

$$\frac{\partial U_{k+1}^*(x;z,y_{k-1},..,y_1)}{\partial x} = E\left[\frac{\partial V(x,Y_{N-1}^{(N-1)},..,Y_{k+1}^{(N-1)},z,y_{k-1},..,y_1)}{\partial x}\Big|x > Y_K^{(N-1)}\right] F_K(x|Y_k^{(N-1)}=z)$$
$$< E\left[\frac{\partial V(x,Y_{N-1}^{(N-1)},..,Y_{k+1}^{(N-1)},z,y_{k-1},..,y_1)}{\partial x}\right]$$
$$= \frac{\partial v_k(x,z,y_{k-1},...)}{\partial x}. \tag{17}$$

Integrating between x and z, it follows that

$$U^*_{k+1}(z;z,y_{k-1},..,) - U^*_{k+1}(x;z,y_{k-1},..,) < v_k(z,z,y_{k-1},...) - v_k(x,z,y_{k-1},...),$$

and hence that $\frac{\partial U_k}{\partial z} < 0$ for $z > x$; bidder i has no incentive to bid more than the equilibrium strategy in round k. This concludes the proof of the theorem. ■

Proof of Theorem 2. Let $\beta^{S2}_k(x;y_{k-1},...,y_1)$ be round k equilibrium bidding function. Suppose that if the winning bid in round k is higher than the highest equilibrium bid, then all bidders believe that the winning bidder's type is the same as the type of the previous round's winner; if the observed winning bid in round k is below the lowest equilibrium bid, then bidders believe that the winner's type is the lowest possible type.

Let $U^*_k(x;y_{k-1},...)$ be the expected payoff for a type x of bidder at the beginning of round k. Suppose that all the other bidders follow the equilibrium strategies, while bidder i is considering deviating in round k. As for the case of a sequential first-price auction, if there is a profitable deviation, there is a profitable deviation with a bid in the range of possible bids. Recalling (7), the payoff of bidder i of type x when he bids $b = \beta^{S2}_k(z;y_{k-1},...)$ (i.e., he bids like a type z) in auction k can be written as:

$$\begin{aligned} U_k(z;x;y_{k-1},...) = &\int_{\underline{x}}^{z} \left[v_k(x,y_k,...) - \ell(\beta^{S2}_k(y_k;y_{k-1},...))\right] f^{(N-1)}_k\left(y_k|Y^{(N-1)}_{k-1} = y_{k-1}\right) dy_k \\ &+ \int_{z}^{\overline{x}} U^*_{k+1}(x;y_k,y_{k-1},..) f^{(N-1)}_k\left(y_k|Y^{(N-1)}_{k-1} = y_{k-1}\right) dy_k. \end{aligned}$$

Differentiating with respect to z yields the first order condition

$$\begin{aligned} &\left[v_k(x,z,y_{k-1},...) - \ell(\beta^{S2}_k(z;y_{k-1},...))\right] f^{(N-1)}_k\left(z|Y^{(N-1)}_{k-1} = y_{k-1}\right) \\ &- U^*_{k+1}(x;z,y_{k-1},..,) f^{(N-1)}_k\left(z|Y^{(N-1)}_{k-1} = y_{k-1}\right) = 0. \end{aligned} \tag{18}$$

Since on the equilibrium path $z = x$ must be optimal, the following is a necessary condition for equilibrium:

$$v_k(x,x,y_{k-1},...) - \ell(\beta^{S2}_k(x;y_{k-1},...)) - U^*_{k+1}(x;x,y_{k-1},..,) = 0. \tag{19}$$

If $k = K$, then $U^*_{k+1}(x;\cdot) = 0$, and (19) yields that on the equilibrium path the bidding function must satisfy

$$\begin{aligned} \beta^{S2}_K(x;y_{K-1},...,y_1) &= \phi\left(v_K(x,x,y_{K-1},...,y_1)\right) \\ &= \phi\left(E\left[V(Y^{(N-1)}_K, Y^{(N-1)}_K, Y^{(N-1)}_{K-1}...)|Y^{(N-1)}_K = x \le Y^{(N-1)}_{K-1} = y_{K-1},...\right]\right). \end{aligned} \tag{20}$$

If the signal of the winner in round $k < K$ is x, then in round $k+1$ bidder i with signal x wins with probability 1; hence, it is

$$U^*_{k+1}(x;x,y_{k-1},..,) = v_k(x,x,y_{k-1},...) - \int_{\underline{x}}^{x} \ell(\beta^{S2}_{k+1}(y_{k+1};x,y_{k-1},...)) f^{(N-1)}_{k+1}\left(y_{k+1}|Y^{(N-1)}_k = x\right). \tag{21}$$

Thus (19) can be written as

$$\begin{aligned} \ell(\beta^{S2}_k(x;y_{k-1},...)) &= \int_{\underline{x}}^{x} \ell(\beta^{S2}_{k+1}(y_{k+1};x,y_{k-1},...)) f^{(N-1)}_{k+1}\left(y_{k+1}|Y^{(N-1)}_k = x\right) \\ &= E\left[\ell(\beta^{S2}_{k+1}(Y^{(N-1)}_{k+1};Y^{(N-1)}_k,...))|Y^{(N-1)}_k = x \le Y^{(N-1)}_{k-1} = y_{k-1},...\right]. \end{aligned} \tag{22}$$

Recalling (20) and working backwards we obtain

$$\ell(\beta_k^{S2}(x; y_{k-1}, ..)) = E\left[V(Y_K^{(N-1)}, Y_K^{(N-1)}, Y_{K-1}^{(N-1)}, ...)|Y_k^{(N-1)} = x \leq Y_{k-1}^{(N-1)} = y_{k-1}, ...\right]$$

Thus, we have shown that on the equilibrium path the bidding function must satisfy

$$\beta_k^{S2}(x; y_{k-1}, .., y_1) = \phi\left(E\left[V(Y_K^{(N-1)}, Y_K^{(N-1)}, Y_{K-1}^{(N-1)}, ..)|Y_k^{(N-1)} = x \leq Y_{k-1}^{(N-1)} = y_{k-1}, ...\right]\right). \tag{23}$$

It only remains to show that if all bidders follow the equilibrium bidding strategy in the rounds after k, and if in round k all other bidders follow the bidding strategy β_k^{S2} defined in (23), then it is also optimal for bidder i to follow it. Using (19) to replace the second term on the left hand side of equation (18) we obtain

$$\begin{aligned}\frac{\partial U_k}{\partial z} &= [v_k(x, z, y_{k-1}, ...) - v_k(z, z, y_{k-1}, ...)]\, f_k^{(N-1)}\left(z|Y_{k-1}^{(N-1)} = y_{k-1}\right) \\ &\quad + \left[U_{k+1}^*(z; z, y_{k-1}, ..,) - U_{k+1}^*(x; z, y_{k-1}, ...)\right] f_k^{(N-1)}\left(z|Y_{k-1}^{(N-1)} = y_{k-1}\right).\end{aligned} \tag{24}$$

Consider $k = K$; since v_k is increasing in x and $U_{K+1}^* = 0$, the sign of $\frac{\partial U_K}{\partial z}$ is the same as $x - z$; hence $z = x$ is optimal.

Now suppose $k < K$; take first the case $z \leq x$. Note that

$$U_{k+1}^*(x; z, y_{k-1}, ..,) = v_k(x, z, y_{k-1}, ...) - \int_{\underline{x}}^{z} \ell(\beta_{k+1}^{S2}(y_{k+1}; z, y_{k-1}, ...)) f_{k+1}^{(N-1)}\left(y_{k+1}|Y_k^{(N-1)} = z\right),$$

because in this case bidder i wins for sure in round k. It follows that $\frac{\partial U_k}{\partial z} = 0$ for $z \leq x$ and bidder i has no incentive to bid less than the equilibrium strategy in round k.

Now take the case $z > x$. As shown in (17), by Lemma 11, equation (6), we have

$$\frac{\partial U_{k+1}^*(x; z, y_{k-1}, ..., y_1)}{\partial x} < \frac{\partial v_k(x, z, y_{k-1}, ...)}{\partial x}.$$

Integrating between x and z, it follows that

$$v_k(z, z, y_{k-1}, ...) - v_k(x, z, y_{k-1}, ...) > U_{k+1}^*(z; z, y_{k-1}, ..,) - U_{k+1}^*(x; z, y_{k-1}, ..,),$$

and hence that $\frac{\partial U_k}{\partial z} < 0$ for $z > x$; bidder i has no incentive to bid more than the equilibrium strategy in round k. This concludes the proof of the theorem. ■

Proof of Theorem 3. As in a static second-price auction, it is clear that in round K bidding according to the equilibrium strategy is a weakly dominant strategy; a bidder wins if and only if he obtains a positive payoff and the price he pays does not depend on his bid.

Now consider round $k < K$; suppose that all the other bidders follow their equilibrium strategies, as described in the theorem, while bidder i is considering deviating. Suppose first that bidder i of type x is the price setter in round $k-1$ and hence the bidder with the k-th highest signal (this implies that the k-th highest signal among his $N-1$ opponents is less than, or equal to, x). Note first that bidding as a type

$z > x$ yields the same payoff as bidding as a type x (he wins for sure). If he deviates in round k (only) and bids as if he were a type $z \leq x$, he either wins in round k, or in round $k+1$; he obtains a payoff

$$x - \int_{\underline{x}}^{z} E\left[Y_K^{(N-1)} | Y_k^{(N-1)} = y\right] f_k^{(N-1)}\left(y | Y_k^{(N-1)} \leq x\right) dy$$
$$- \int_{z}^{x} \int_{\underline{x}}^{y} E\left[Y_K^{(N-1)} | Y_{k+1}^{(N-1)} = t\right] f_{k+1}^{(N-1)}\left(t | Y_k^{(N-1)} = y\right) dt f_k^{(N-1)}\left(y | Y_k^{(N-1)} \leq x\right) dy.$$

Differentiating with respect to z yields

$$- E\left[Y_K^{(N-1)} | Y_k^{(N-1)} = z\right] f_k^{(N-1)}\left(z | Y_k^{(N-1)} \leq x\right)$$
$$+ \int_{\underline{x}}^{z} E\left[Y_K^{(N-1)} | Y_{k+1}^{(N-1)} = t\right] f_{k+1}^{(N-1)}\left(t | Y_k^{(N-1)} = z\right) dt f_k^{(N-1)}\left(z | Y_k^{(N-1)} \leq x\right),$$

which is equal to zero for all values of z. It follows that type x has no incentive to deviate in round k.

Now consider a type $x < y_k$, the price setter in round $k-1$. If in round k he bids as if he were a type $z \leq y_k$, then he loses and obtains the same (expected, future) payoff independently of his bid. It follows that he may as well bid as a type x; by equation (6), doing so gives him the (equilibrium) payoff

$$\left\{x - E\left[Y_K^{(N-1)} | Y_K^{(N-1)} < x, Y_k^{(N-1)} = y_k\right]\right\} \Pr\left[Y_K^{(N-1)} < x | Y_k^{(N-1)} = y_k\right], \tag{25}$$

where $\Pr\left[Y_K^{(N-1)} < x | Y_k^{(N-1)} = y_k\right]$ is the probability that $Y_K^{(N-1)} < x$ conditional on $Y_k^{(N-1)} = y_k$. If type x bids in round k as if he were a type $z = y_k$, then he ties with the round k winner and he may as well raise his bid and win for sure, or lower his bid and lose for sure. If he bids above the bid of the y_k type, so that he wins for sure, type x obtains a payoff

$$\begin{aligned}
&x - E\left[Y_K^{(N-1)} | Y_k^{(N-1)} = y_k\right] \\
&= \left\{x - E\left[Y_K^{(N-1)} | Y_K^{(N-1)} < x, Y_k^{(N-1)} = y_k\right]\right\} \Pr\left[Y_K^{(N-1)} < x | Y_k^{(N-1)} = y_k\right] \\
&+ \left\{x - E\left[Y_K^{(N-1)} | Y_K^{(N-1)} \geq x, Y_k^{(N-1)} = y_k\right]\right\} \Pr\left[Y_K^{(N-1)} \geq x | Y_k^{(N-1)} = y_k\right] \\
&< \left\{x - E\left[Y_K^{(N-1)} | Y_K^{(N-1)} < x, Y_k^{(N-1)} = y_k\right]\right\} \Pr\left[Y_K^{(N-1)} < x | Y_k^{(N-1)} = y_k\right].
\end{aligned}$$

It follows from (25) that it is not profitable for a bidder of type x to deviate and bid more than a type y_k. This concludes the proof of the theorem. ■

Proof of Theorem 6. Consider a sequential first-price auction, and suppose that the history of the

winners' signals up to round k is $x, y_{k-1}, ..., y_1$. Then we have:

$$
\begin{aligned}
&E\left[P_{k+1}^{S1}|P_k^{S1}\right] \\
&\quad = E\left[P_{k+1}^{S1}|\beta_k^{S1}(x; y_{k-1}, ..., y_1)\right] \\
&\quad = E\left[\beta_{k+1}^{S1}(Y_k^{(N-1)}; x, y_{k-1}, ..)|Y_k^{(N-1)} \leq x \leq Y_{k-1}^{(N-1)} = y_{k-1}, ..\right] \\
&\quad = \phi\left(E\left[\ell\left(\beta_{k+1}^{S1}(Y_k^{(N-1)}; x, y_{k-1}, ..)\right)|Y_k^{(N-1)} \leq x \leq Y_{k-1}^{(N-1)} = y_{k-1}, ..\right]\right) + A_{k=1}^{S1}(x, y_{k-1}, ..., y_1) \\
&\quad = \phi\left(E\left[\ell\left(\beta_{k+1}^{S1}(Y_k^{(N-1)}; Y_k^{(N-1)}, y_{k-1}, ..)\right)|Y_k^{(N-1)} \leq x \leq Y_{k-1}^{(N-1)} = y_{k-1}, ..\right]\right) \\
&\qquad\quad + A_{k=1}^{S1}(x, y_{k-1}, ..., y_1) + I_{k=1}^{S1}(x, y_{k-1}, ..., y_1) \\
&\quad = \phi\left(E\left[V\left(Y_K^{(N-1)}, Y_K^{(N-1)}, Y_{K-1}^{(N-1)}, ..\right)|Y_k^{(N-1)} \leq x \leq Y_{k-1}^{(N-1)} = y_{k-1}, ..\right]\right) \\
&\qquad\quad + A_{k=1}^{S1}(x, y_{k-1}, ..., y_1) + I_{k=1}^{S1}(x, y_{k-1}, ..., y_1) \\
&\quad = \beta_k^{S1}(x; y_{k-1}, .., y_1) + A_{k=1}^{S1}(x, y_{k-1}, ..., y_1) + I_{k=1}^{S1}(x, y_{k-1}, ..., y_1) \\
&\quad = P_k^{S1} + A_{k=1}^{S1}(x, y_{k-1}, ..., y_1) + I_{k=1}^{S1}(x, y_{k-1}, ..., y_1).
\end{aligned}
$$

Now consider a sequential second-price auction. Suppose that in round k the winner is the bidder with signal $Y_k^{(N-1)}$, and bidder i of type x is the price setter, while the history of the previous rounds winners' signals is $y_{k-1}, ..., y_1$. We have

$$
\begin{aligned}
&E\left[P_{k+1}^{S2}|P_k^{S2}\right] \\
&\quad = E\left[P_{k+1}^{S2}|\beta_k^{S2}(x; y_{k-1}, ..., y_1)\right] \\
&\quad = E\left[\beta_{k+1}^{S2}(Y_{k+1}^{(N-1)}; Y_k^{(N-1)}, y_{k-1}, ..)|Y_{k+1}^{(N-1)} \leq X_1 = x \leq Y_k^{(N-1)}, ..\right] \\
&\quad = \phi\left(E\left[\ell\left(\beta_{k+1}^{S2}(Y_{k+1}^{(N-1)}; Y_k^{(N-1)}, y_{k-1}, ..)|\right)|Y_{k+1}^{(N-1)} \leq X_1 = x \leq Y_k^{(N-1)}, ..\right]\right) + A_{k=1}^{S2}(x, y_{k-1}, ..., y_1) \\
&\quad = \phi\left(E\left[\ell\left(\beta_{k+1}^{S2}(Y_{k+1}^{(N-1)}; Y_k^{(N-1)}, y_{k-1}, ..)|\right)|Y_k^{(N-1)} = x\right]\right) \\
&\qquad\quad + A_{k=1}^{S2}(x, y_{k-1}, ..., y_1) + I_{k=1}^{S2}(x, y_{k-1}, ..., y_1) \\
&\quad = \phi\left(E\left[V(Y_K^{(N-1)}, Y_K^{(N-1)}, Y_{K-1}^{(N-1)}, ...)|Y_k^{(N-1)} = x \leq Y_{k-1}^{(N-1)} = y_{k-1}, ...\right]\right) \\
&\qquad\quad + A_{k=1}^{S2}(x, y_{k-1}, ..., y_1) + I_{k=1}^{S2}(x, y_{k-1}, ..., y_1) \\
&\quad = \beta_k^{S2}(x; y_{k-1}, .., y_1) + A_{k=1}^{S2}(x, y_{k-1}, ..., y_1) + I_{k=1}^{S2}(x, y_{k-1}, ..., y_1) \\
&\quad = P_k^{S2} + A_{k=1}^{S2}(x, y_{k-1}, ..., y_1) + I_{k=1}^{S2}(x, y_{k-1}, ..., y_1).
\end{aligned}
$$

■

Appendix B

In this appendix, I compute the bidding functions and price ratios for the example discussed in Section 7.

Recalling that

$$\phi(z) = (1+r)^{\frac{1}{1+r}}\, z^{\frac{1}{1+r}},$$

we can use Theorem 2 to calculate the bidding functions in the sequential second-price auction:

$$\beta_2^{S2}(x;y_1) = (1+r)^{\frac{1}{1+r}} \left(by_1 + \left(1 + b + (N-3)\frac{a}{a+1}b\right) x \right)^{\frac{1}{1+r}};$$

$$\beta_1^{S2}(x) = (1+r)^{\frac{1}{1+r}} \left(\frac{a\,(N-2)}{a\,(N-2)+1} x + b\left(x + (N-2)\,\frac{a}{a+1} x \right) \right)^{\frac{1}{1+r}}$$

The expected price in round 2, conditional on the first-round price P_1 is:

$$\begin{aligned}
&E\left[P_2|P_1 = \beta_1^{S2}(x)\right] \\
&= (1+r)^{\frac{1}{1+r}}\, E\left[\left(b\frac{a}{a+1}\frac{1-x^{a+1}}{1-x^a} + \left(1+b+(N-3)\frac{a}{a+1}b\right) Y_2^{(N-1)} \right)^{\frac{1}{1+r}} |Y_2^{(N-1)} \leq x \right] \\
&= (1+r)^{\frac{1}{1+r}} \int_0^x \left(b\frac{a}{a+1}\frac{1-x^{a+1}}{1-x^a} + \left(1+b+(N-3)\frac{a}{a+1}b\right) z \right)^{\frac{1}{1+r}} a(N-2)\frac{z^{a(N-2)-1}}{x^{a(N-2)}} dz \\
&= (1+r)^{\frac{1}{1+r}}\, x^{\frac{1}{1+r}} \int_0^1 \left(b\frac{a}{a+1}\frac{1-x^{a+1}}{x-x^{a+1}} + \left(1+b+(N-3)\frac{a}{a+1}b\right) z \right)^{\frac{1}{1+r}} a(N-2) z^{a(N-2)-1} dz
\end{aligned}$$

It follows that the ratio of the conditional expected second-round price to the first-round price is:

$$\frac{E\left[P_2|P_1\right]}{P_1} = \frac{\int_0^1 \left(b\frac{a}{a+1}\frac{1-x^{a+1}}{x-x^{a+1}} + \left(1+b+(N-3)\frac{a}{a+1}b\right) z \right)^{\frac{1}{1+r}} a(N-2) z^{a(N-2)-1} dz}{\left(\frac{a(N-2)}{a(N-2)+1} + b\left(1+(N-2)\,\frac{a}{a+1}\right)\right)^{\frac{1}{1+r}}}. \tag{26}$$

In the case of no informational externalities, that is $b = 0$, this becomes:

$$\frac{E\left[P_2|P_1\right]}{P_1} = \frac{\int_0^1 z^{\frac{1}{1+r}} a(N-2) z^{a(N-2)-1} dz}{\left(\frac{a(N-2)}{a(N-2)+1}\right)^{\frac{1}{1+r}}} = \frac{(a(N-2))^{\frac{r}{1+r}}\,(a\,(N-2)+1)^{\frac{1}{1+r}}}{\left(a(N-2)+\frac{1}{1+r}\right)}. \tag{27}$$

If there are informational externalities, $b > 0$, $\frac{E[P_2|P_1]}{P_1}$ depends on the signal x of the first-round price setter. Since x is the value of the second order statistic out of N draws, the expected value of the price ratio is

$$E\left[\frac{E\left[P_2|P_1\right]}{P_1}\right] =$$

$$\int_0^1 \frac{\int_0^1 \left(b\frac{a}{a+1}\frac{1-x^{a+1}}{x-x^{a+1}} + \left(1+\frac{b}{a+1}\left(1+(N-2)a\right)\right) z \right)^{\frac{1}{1+r}} a(N-2) z^{a(N-2)-1} dz}{\left(\frac{a(N-2)}{a(N-2)+1} + b\left(1+(N-2)\,\frac{a}{a+1}\right)\right)^{\frac{1}{1+r}}} N(N-1)a\left(1-x^a\right) x^{a(N-1)-1} dx$$

Appendix C

In this appendix I prove the theorems for the multi-unit and English auction presented in Section 8. I also compute the bidding functions, the expected price in round 2 and the conditional price ratio for a

multi-unit version of the example with $F(x) = x^a$ and $\ell(p) = p^{1+r}/(1+r)$ introduced in Section 7. I start with the multi-unit demand, sequential, second-price auction.

Proof of Theorem 7. In the second round, it is a weakly dominant strategy to submit a loss bid equal to the object's value. Hence $\ell(\beta_2^{S2}(\cdot))$ equals x_l if the bidder won the first round, and x_h if the bidder lost.

Consider the first round. Suppose all other bidders bid according to the equilibrium bid functions in both periods. Note that the first-round bid function only depends on a bidder's high value. Hence, any profitable first-round deviation of bidder i of type x_h, x_l can be described as bidding as if his high value were z rather than x_h. The payoff from such a bid when $z \geq x_l$ is:

$$U_1(z; x_h, x_l) = \int_{\underline{x}}^{x_l} \left[x_h + x_l - \ell(\beta_1^{S2}(y_1, \cdot)) - y_1\right] f_1^{(2N-2)}(y_1)\, dy_1 + \int_{x_l}^{z} \left[x_h - \ell(\beta_1^{S2}(y_1, \cdot))\right] f_1^{(2N-2)}(y_1)\, dy_1$$
$$+ \int_z^{\overline{x}} \int_{\underline{x}}^{x_h} [x_h - y_2] f_2^{(2N-2)}\left(y_2 | Y_1^{(2N-2)} = y_1\right) dy_2 f_1^{(2N-2)}(y_1)\, dy_1.$$

Recall that $\ell\left(\beta_1^{S2}(z, \cdot)\right) = E\left[Y_2^{(2N-2)} | Y_1^{(2N-2)} = z\right]$. Hence, differentiating $U_1(z; x_h, x_l)$ with respect to z yields:

$$\begin{aligned}
& \left(x_h - \ell(\beta_1^{S2}(z, \cdot)) - \int_{\underline{x}}^{x_h} [x_h - y_2] f_2^{(2N-2)}\left(y_2 | Y_1^{(2N-2)} = z\right) dy_2\right) f_1^{(2N-2)}(z) \\
= & \left(\int_{\underline{x}}^{z} [x_h - y_2] f_2^{(2N-2)}\left(y_2 | Y_1^{(2N-2)} = z\right) dy_2 - \int_{\underline{x}}^{x_h} [x_h - y_2] f_2^{(2N-2)}\left(y_2 | Y_1^{(2N-2)} = z\right) dy_2\right) f_1^{(2N-2)}(z) \\
= & \left(\int_{x_h}^{z} [x_h - y_2] f_2^{(2N-2)}\left(y_2 | Y_1^{(2N-2)} = z\right) dy_2\right) f_1^{(2N-2)}(z),
\end{aligned}$$

which is negative if $z > x_h$ and zero if $z \leq x_h$, since $f_2^{(2N-2)}\left(y_2 | Y_1^{(2N-2)} = z\right) = 0$ for $y_2 > z$. Hence bidding as if $z = x_h$ (i.e., according to the equilibrium bidding function) is optimal (among all $z \geq x_l$). It remains to be shown that bidder i does not want to bid as if his high value is $z < x_l$.

Bidder i's payoff from bidding as if his high type is $z < x_l$ is:

$$\begin{aligned}
U_1(z; x_h, x_l) = & \int_{\underline{x}}^{z} \left[x_h + x_l - \ell(\beta_1^{S2}(y_1, \cdot)) - y_1\right] f_1^{(2N-2)}(y_1)\, dy_1 \\
& + \int_z^{\overline{x}} \int_{\underline{x}}^{x_h} [x_h - y_2] f_2^{(2N-2)}\left(y_2 | Y_1^{(2N-2)} = y_1\right) dy_2 f_1^{(2N-2)}(y_1)\, dy_1.
\end{aligned}$$

Differentiating with respect to z yields:

$$\left(x_h + x_l - \ell(\beta_1^{S2}(z, \cdot)) - z - \int_{\underline{x}}^{x_h} [x_h - y_2] f_2^{(2N-2)}\left(y_2 | Y_1^{(2N-2)} = z\right) dy_2\right) f_1^{(2N-2)}(z) = (x_l - z) f_1^{(2N-2)}(z),$$

which is positive. Hence bidding as if $z < x_l$ is never optimal. ■

Proof of Theorem 8. By (2), (3), and (4) we have that conditional on bidder i of type (x_h, x_l) being the first-round price setter, the second round expected price is:

$$\begin{aligned}
& E\left[P_2 | P_1\right] \\
& = \beta_1^{S2}(x_h, x_l) \Pr\{Y_2^{(2)} < x_h | Y_1^{(2)} > x_h\} + A_2^{S2}(x_h) + \beta_1^{S2}(x_h, x_l) \Pr\{Y_2^{(2)} > x_h | Y_1^{(2)} > x_h\} + M_2^{S2}(x_h) \\
& = \beta_1^{S2}(x_h, x_l) + A_2^{S2}(x_h) + M_2^{S2}(x_h) \\
& = P_1 + A_2^{S2}(x_h) + M_2^{S2}(x_h).
\end{aligned}$$

■

Now consider the example with $F(x) = x^a$ and $\ell(p) = p^{1+r}/(1+r)$. We have:

$$\begin{aligned}
\beta_2(x_h, x_l|x_h &> Y_1^{(2N-2)}) = (1+r)^{\frac{1}{1+r}} (x_l)^{\frac{1}{1+r}} \\
\beta_2(x_h, x_l|x_h &< Y_1^{(2N-2)}) = (1+r)^{\frac{1}{1+r}} (x_h)^{\frac{1}{1+r}}
\end{aligned}$$

$$\beta_1(x_h, x_l) = (1+r)^{\frac{1}{1+r}} \left(\frac{a\,(2N-3)}{a\,(2N-3)+1} x_h \right)^{\frac{1}{1+r}}$$

The expected price in the last round, round 2, conditional on the first-round price $P_1 = \beta_1(x_h, x_l)$ is:

$$E\left[\phi\left(Y_2^{(2N-2)}\right)|Y_1^{(2N-2)} = x_h\right] \Pr\{Y_2^{(2)} < x_h|Y_1^{(2)} > x_h\} + \phi(x_h) \Pr\{Y_2^{(2)} > x_h|Y_1^{(2)} > x_h\}$$

Note:

$$\Pr\{Y_2^{(2)} < x_h|Y_1^{(2)} > x_h\} = \int_{x_h}^{1} \int_0^{x_h} 2f(x)f(y) \frac{1}{1-F^2(x_h)} dxdy = \frac{2F(x_h)}{1+F(x_h)}$$

$$\begin{aligned}
&E\left[P_2|P_1 = \beta_1(x_h, x_l)\right] \\
&= (1+r)^{\frac{1}{1+r}} E\left[\left(Y_2^{(2N-2)}\right)^{\frac{1}{1+r}} |Y_2^{(2N-2)} \le x_h\right] \frac{2F(x_h)}{1+F(x_h)} + (1+r)^{\frac{1}{1+r}} (x_h)^{\frac{1}{1+r}} \frac{1-F(x_h)}{1+F(x_h)} \\
&= (1+r)^{\frac{1}{1+r}} \left\{ \int_0^{x_h} z^{\frac{1}{1+r}} a(2N-3) \frac{z^{a(2N-3)-1}}{x_h^{a(2N-3)}} dz \frac{2F(x_h)}{1+F(x_h)} + x_h^{\frac{1}{1+r}} \frac{1-F(x_h)}{1+F(x_h)} \right\} \\
&= (1+r)^{\frac{1}{1+r}} x_h^{\frac{1}{1+r}} \left\{ \int_0^{1} z^{\frac{1}{1+r}} a(2N-3) z^{a(2N-3)-1} dz \frac{2F(x_h)}{1+F(x_h)} + \frac{1-F(x_h)}{1+F(x_h)} \right\}
\end{aligned}$$

It follows that the ratio of the conditional expected second-round price to the first-round price is:

$$\begin{aligned}
\frac{E\left[P_2|P_1\right]}{P_1} &= \frac{\int_0^1 z^{\frac{1}{1+r}} a(2N-3) z^{a(2N-3)-1} dz \frac{2x_h^a}{1+x_h^a} + \frac{1-x_h^a}{1+x_h^a}}{\left(\frac{a(2N-3)}{a(2N-3)+1}\right)^{\frac{1}{1+r}}} \\
&= \left(\frac{a(2N-3)+1}{a(2N-3)}\right)^{\frac{1}{1+r}} \frac{(a(2N-3)) \frac{2x_h^a}{1+x_h^a} + \left(a(2N-3) + \frac{1}{1+r}\right) \frac{1-x_h^a}{1+x_h^a}}{\left(a(2N-3) + \frac{1}{1+r}\right)} \\
&= \left(\frac{a(2N-3)+1}{a(2N-3)}\right)^{\frac{1}{1+r}} \left(\frac{a(2N-3) + \frac{1}{1+r} \frac{1-x_h^a}{1+x_h^a}}{\left(a(2N-3) + \frac{1}{1+r}\right)} \right).
\end{aligned}$$

The expected value of the price ratio is

$$\begin{aligned}
&E\left[\frac{E\left[P_2|P_1\right]}{P_1}\right] \\
&= \frac{(a(2N-3)+1)^{\frac{1}{1+r}} \left(a(2N-3) + \frac{1}{1+r} \int_0^1 \frac{1-x^a}{1+x^a} N(N-1) 2ax^{2a-1} \left(1-x^{2a}\right) x^{2a(N-2)} dx\right)}{(a(2N-3))^{\frac{1}{1+r}} \left(a(2N-3) + \frac{1}{1+r}\right)}
\end{aligned}$$

Now consider the sequential English auction.

Proof of Theorem 9. In the second round, when bidder i is not the current winner and the current loss bid is below i's valuation of the object, it is a weakly dominant strategy for bidder i to raise the price. Thus, we only need to show that the first round strategies are optimal.

Consider bidder i, $i \neq 1, 2$. If all other bidders follow their equilibrium strategies, then in the first round bidder i cannot profitably deviate from his equilibrium strategy. On the equilibrium path, such a bidder only wins if his valuation x_i is one of the two highest, and when he wins (independently of the round) he pays a price equal to $\phi\left(Y_2^{(N-1)}\right)$ and makes a (positive) profit equal to $x_i - Y_2^{(N-1)}$. There is no deviation which would ever make bidder i either win at a lower price, or win and make a positive profit when he is not one of the two highest valuation bidders.

Now consider bidder $i = 1, 2$. Without loss of generality, let $i = 1$. An argument similar to the one in the previous paragraph shows that there is no profitable deviation for bidder 1 once one of the bidders $j \neq 1, 2$ has entered the bidding. It only remains to show that, before such an entry, bidder 1 of type x_1 will stop raising the price when it reaches the level $\beta_{1E}^E(x_1) = \phi\left(E\left[Y_2^{(N-1)} | Y_1^{(N-1)} = x_1\right]\right)$; that is, bidder 1 does not want to bid as if his type were $z \neq x_1$. Consider first the option of dropping out (temporarily) at a price $P = \beta_{1E}^E(z) < \beta_{1E}^E(x_1)$, and then resuming the bidding till the price reaches $\phi(x_1)$ if a bidder $j \neq 2$ enters. Let $U_1(x_1; x_1)$ be the payoff of type x_1 of bidder 1 if he follows his equilibrium strategy, raising the price above P, and let $U_1(z; x_1)$ be the payoff if he deviates and drops out (temporarily) at price P. The only difference in the payoffs from the two strategies is in the event that bidder 2's type x_2 is less than x_1, but higher than all other bidders' types. In such a case, bidder 1 wins in the first round at a price $\beta_{1E}^E(x_2)$ if he follows the equilibrium strategy, while he wins at a price $\phi(Y_1^{(N-2)})$ if he deviates. Thus,

$$\begin{aligned}
& U_1(x_1; x_1) - U_1(z; x_1) \\
= & \int_z^{x_1} \left[E\left[Y_1^{(N-2)} | Y_1^{(N-2)} \leq x_2\right] - \ell\left(\beta_{1E}^E(x_2)\right)\right] \frac{f(x_2)}{1 - F(z)} dx_2 \\
= & \int_z^{x_1} \left[E\left[Y_2^{(N-1)} | Y_1^{(N-1)} = x_2\right] - \ell\left(\beta_{1E}^E(x_2)\right)\right] \frac{f(x_2)}{1 - F(z)} dx_2 = 0,
\end{aligned}$$

and the deviation is not profitable.

Now consider the option of dropping out (temporarily) at a price $P = \beta_{1E}^E(z) > \beta_{1E}^E(x_1)$. The only difference in the payoffs from following the equilibrium strategy (dropping out at price $\beta_{1E}^E(x_1)$) and deviating is in the event that bidder 2's type x_2 is the highest of all bidders' types, including x_1, but it is less than z. In such a case, bidder 1 wins in the first round at a price $\beta_{1E}^E(x_2)$ if he deviates. He wins at a price $\phi(Y_1^{(N-2)})$ when $Y_1^{(N-2)} < x_1$ if he follows the equilibrium strategy. Since for $x_2 > x_1$ it is

$$\begin{aligned}
\left[x_1 - \ell\left(\beta_{1E}^E(x_2)\right)\right] F^{N-2}(x_2) &= \left[x_1 - E\left[Y_2^{(N-1)} | Y_1^{(N-1)} = x_2\right]\right] F^{N-2}(x_2) \\
&= \left[x_1 - E\left[Y_1^{(N-2)} | Y_1^{(N-2)} \leq x_2\right]\right] F^{N-2}(x_2) \\
&< \left[x_1 - E\left[Y_1^{(N-2)} | Y_1^{(N-2)} \leq x_1\right]\right] F^{N-2}(x_1),
\end{aligned}$$

the deviation is not profitable. ■

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