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BLOCK DESIGNS AND STRONGLY REGULAR GRAPHS CONSTRUCTED FROM THE GROUP $U(3,4)$

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Abstract. We show a construction of the projective plane $PG(2,16)$ and the Hermitian unital $S(2,5,65)$ from the unitary group $U(3,4)$. Further, we construct two block designs, a 2-$(65,15,21)$ design and a 2-$(65,26,250)$ design, and two strongly regular graphs with parameters $(208,75,30,25)$ and $(416,100,36,20)$. These incidence structures are defined on the elements of the conjugacy classes of the maximal subgroups of $U(3,4)$. The group $U(3,4)$ acts transitively as an automorphism group of the so constructed designs and strongly regular graphs. The strongly regular graph with parameters $(416,100,36,20)$ has the full automorphism group of order 503193600, isomorphic to $G(2,4):Z_2$. Since the Janko group J_2 is a subgroup of $G(2,4)$, J_2 acts as an automorphism group of the constructed $SRG(416,100,36,20)$.

1. Introduction

An incidence structure is an ordered triple $\mathcal{D}=(\mathcal{P},\mathcal{B},\mathcal{I})$ where $\mathcal{P}$ and $\mathcal{B}$ are non-empty disjoint sets and $\mathcal{I}\subseteq\mathcal{P}\times\mathcal{B}$. The elements of the set $\mathcal{P}$ are called points, the elements of the set $\mathcal{B}$ are called blocks and $\mathcal{I}$ is called an incidence relation. The incidence matrix of an incidence structure is a $b\times v$ matrix $[m_{ij}]$, where b and v are the number of blocks and points respectively, such that $m_{ij}=1$ if the point P_j and block x_i are incident, and $m_{ij}=0$ otherwise.

An isomorphism from one incidence structure to another is a bijective mapping of points to points and blocks to blocks which preserves incidence.

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An isomorphism from an incidence structure $\mathcal{D}$ onto itself is called an automorphism of $\mathcal{D}$. The set of all automorphisms forms a group called the full automorphism group of $\mathcal{D}$ and is denoted by $Aut(\mathcal{D})$.

A t-(v,k,λ) design is a finite incidence structure $(\mathcal{P},\mathcal{B},\mathcal{I})$ satisfying the following requirements:

1. $|\mathcal{P}| = v$,
2. every element of $\mathcal{B}$ is incident with exactly k elements of $\mathcal{P}$,
3. every t elements of $\mathcal{P}$ are incident with exactly λ elements of $\mathcal{B}$.

A Steiner system $S(t,k,v)$ is a t-$(v,k,1)$ design. A 2-(v,k,λ) design is called a block design. A 2-(v,k,λ) design is called quasi-symmetric if the number of points in the intersection of any two blocks takes only two values. If $|\mathcal{P}| = |\mathcal{B}| = v$ and $2 \leq k \leq v-2$, then a 2-(v,k,λ) design is called a symmetric design. A symmetric 2-$(v,k,1)$ design is called a projective plane. A blocking set is a subset of the point set of a design that contains a point of every block, but that contains no complete block.

A semi-symmetric $(v,k,(\lambda))$ design is a finite incidence structure with v points and v blocks satisfying:

1. every point (block) is incident with exactly k blocks (points),
2. every pair of points (blocks) is incident with 0 or λ blocks (points).

Let $\mathcal{G} = (\mathcal{V},\mathcal{E},\mathcal{I})$ be a finite incidence structure. $\mathcal{G}$ is a graph if each element of $\mathcal{E}$ is incident with exactly two elements of $\mathcal{V}$. The elements of $\mathcal{V}$ are called vertices and the elements of $\mathcal{E}$ edges.

Two vertices u and v are called adjacent or neighbors if they are incident with the same edge. The number of neighbors of a vertex v is called the degree of v. If all the vertices of the graph $\mathcal{G}$ have the same degree k, then $\mathcal{G}$ is called k-regular.

Let $\mathcal{G}$ be a graph. Define a square $\{0,1\}$-matrix $A = (a_{uv})$ labelled with the vertices of $\mathcal{G}$ in such a way that $a_{uv} = 1$ if and only if the vertices u and v are adjacent. The matrix A is called the adjacency matrix of the graph $\mathcal{G}$.

An automorphism of a graph is any permutation of the vertices preserving adjacency. The set of all automorphisms forms the full automorphism group of the graph.

Let $\mathcal{G}$ be a k-regular graph with n vertices. $\mathcal{G}$ is called a strongly regular graph with parameters (n,k,λ,μ) if any two adjacent vertices have λ common neighbors and any two non-adjacent vertices have μ common neighbors. A strongly regular graph with parameters (n,k,λ,μ) is usually denoted by $SRG(n,k,\lambda,\mu)$.

Let x and y $(x < y)$ be the two cardinalities of block intersections in a quasi-symmetric design $\mathcal{D}$. The block graph of the design $\mathcal{D}$ has as vertices the blocks of $\mathcal{D}$ and two vertices are adjacent if and only if they intersect in y points. The block graph of a quasi-symmetric 2-(v,k,λ) design is strongly regular. In a 2-$(v,k,1)$ design which is not a projective plane two blocks intersect in 0 or 1 points, therefore the block graph of this design is strongly regular (see [1]).

In this paper we consider structures constructed from the unitary group $U(3,4)$, the classical simple group of order 62400. The group $U(3,4)$ possesses 4 maximal subgroups (see [2]):

- $H_1 \cong (E_{16} : (Z_2 \times Z_2)) : Z_{15}$,
- $H_2 \cong A_5 \times Z_5$,
- $H_3 \cong (Z_5 \times Z_5) : S_3$,
- $H_4 \cong Z_{13} : Z_3$.

Generators of the group $U(3,4)$ and its maximal subgroups are available on the Internet:

`http://brauer.maths.qmul.ac.uk/Atlas/v3/clas/U34`.

The conjugacy classes of the subgroups $H_i,\ i = 1,2,3,4$, in $G \cong U(3,4)$ are denoted by $ccl_G(H_i),\ i = 1,2,3,4$. G is a simple group and $H_i,\ i = 1,2,3,4$, are maximal subgroups of G. Therefore,

$$N_G(H_i) = H_i \implies |ccl_G(H_i)| = |G : H_i|,\ i = 1,2,3,4.$$

For $i \in \{1,2,3,4\}$ we denote the elements of $ccl_G(H_i)$ by $H_i^{g_1}, H_i^{g_2}, \dots, H_i^{g_j}$, $j = |G : H_i|$.

2. Construction of $PG(2,16)$

Let G be a group isomorphic to the unitary group $U(3,4)$ and $H_1 \cong (E_{16} : (Z_2 \times Z_2)) : Z_{15}$, $H_2 \cong A_5 \times Z_5$ be maximal subgroups of G. Cardinality of the conjugacy class $ccl_G(H_1)$ is 65 and cardinality of the conjugacy class $ccl_G(H_2)$ is 208.

One can check, using GAP ([5]), that the intersection of any two elements, $H_1^{g_i} \in ccl_G(H_1),\ 1 \le i \le 65$, and $H_2^{g_j} \in ccl_G(H_2),\ 1 \le j \le 208$, is either $A_4 \times Z_5$ or Z_5. Further, for every $H_1^{g_i} \in ccl_G(H_1),\ 1 \le i \le 65$, the cardinality of the set $\{H_2^{g_j} \in ccl_G(H_2) \mid H_2^{g_j} \cap H_1^{g_i} \cong A_4 \times Z_5)\}$ is 5. Let us define the sets $S_i = \{H_1^{g_j} \in ccl_G(H_1) \mid H_2^{g_i} \cap H_1^{g_j} \cong A_4 \times Z_5\},\ 1 \le i \le 208$. For every $1 \le i,j \le 208$, $i \ne j$, the set $S_i \cap S_j$ has exactly one element.

That proves that the incidence structure $\mathcal{D}_1 = (\mathcal{P}_1, \mathcal{B}_1, \mathcal{I}_1)$ where $\mathcal{P}_1 = \{P_1^{(1)}, \dots, P_{65}^{(1)}\}$, $\mathcal{B}_1 = \{x_1^{(1)}, \dots, x_{208}^{(1)}\}$ and

$$(P_i^{(1)}, x_j^{(1)}) \in \mathcal{I}_1 \iff (H_1^{g_i} \cap H_2^{g_j} \cong A_4 \times Z_5)$$

is a Steiner system $S(2,5,65)$.

The intersection of any two different elements $H_2^{g_i}$ and $H_2^{g_j}$ of the set $ccl_G(H_2)$ is isomorphic to Z_5, $Z_2 \times Z_2$ or $Z_5 \times Z_5$. One can check that the incidence structure $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathcal{I})$ where $\mathcal{P} = \{P_1, \dots, P_{208}\}$, $\mathcal{B} = \{x_1, \dots, x_{208}\}$ and

$$(P_i, x_j) \in \mathcal{I} \iff (H_2^{g_i} \cap H_2^{g_j} \cong Z_5 \times Z_5)$$

is a semi-symmetric design with parameters $(208, 12, (1))$.

Let M_1 and M be the incidence matrices of $\mathcal{D}_1$ and $\mathcal{S}$ respectively, and I_{65} be the identity matrix of order 65. Then the matrix

$$P = \begin{bmatrix} I_{65} & M_1^T \\ M_1 & \mathrm{M} \end{bmatrix}$$

is the incidence matrix of a projective plane $PG(2,16)$, i.e., a symmetric $(273,17,1)$ design. Since the matrix P is symmetric, the projective plane admits a unitary polarity (for the definition see e.g. [3]). The absolute points and blocks are the conjugates of H_1, and the non-absolute points and blocks are the conjugates on H_2. The design $\mathcal{D}_1$ is the Hermitian unital in $PG(2,16)$ and it is resolvable (see [7]).

The group $U(3,4)$ acts transitively on the design $\mathcal{D}_1$ and the semi-symmetric design $\mathcal{S}$.

Using Nauty (see [4]) and GAP (see [5]), we have determined that the full automorphism group of $\mathcal{D}_1$ and $\mathcal{S}$ is a group of order 249600 isomorphic to $U(3,4):Z_4$. Note that $U(3,4):Z_4$ is the full automorphism group of $U(3,4)$.

The design $\mathcal{D}_1$ is a quasi-symmetric design with block intersections 1 and 0 and its block graph is a strongly regular graph with parameters $(208,75,30,25)$. Denote this graph by $\mathcal{G}_1$. It can be obtained directly from conjugates of H_2. The adjacency matrix of the graph $\mathcal{G}_1$ is the matrix $A_1 = (a_{ij}^{(1)})$ defined as follows:

$$a_{ij}^{(1)} = \begin{cases} 1, & \text{if } H_2^{g_i} \cap H_2^{g_j} \cong Z_2 \times Z_2, \\ 0, & \text{otherwise.} \end{cases}$$

The full automorphism group of $\mathcal{G}_1$ is isomorphic to $U(3,4):Z_4$. The group $U(3,4)$ acts transitively on the graph $\mathcal{G}_1$.

3. Construction of block designs 2-$(65,15,21)$ and 2-$(65,26,250)$

Let G be a group isomorphic to the unitary group $U(3,4)$ and $H_1 \cong (E_{16}:(Z_2 \times Z_2)):Z_{15}$, $H_3 \cong (Z_5 \times Z_5):S_3$ and $H_4 \cong Z_{13}:Z_3$ be maximal subgroups of G. Cardinality of the conjugacy class $ccl_G(H_1)$ is 65, cardinality of the conjugacy class $ccl_G(H_3)$ is 416, and cardinality of the conjugacy class $ccl_G(H_4)$ is 1600.

The intersection of any two elements, $H_1^{g_i} \in ccl_G(H_1)$, $1 \leq i \leq 65$, and $H_3^{g_j} \in ccl_G(H_3)$, $1 \leq j \leq 416$, is either Z_3 or Z_{10}.

One can check that the incidence structure $\mathcal{D}_2 = (\mathcal{P}_2, \mathcal{B}_2, \mathcal{I}_2)$ where $\mathcal{P}_2 = \{P_1^{(2)}, \ldots, P_{65}^{(2)}\}$, $\mathcal{B}_2 = \{x_1^{(2)}, \ldots, x_{416}^{(2)}\}$ and

$$(P_i^{(2)}, x_j^{(2)}) \in \mathcal{I}_2 \iff (H_1^{g_i} \cap H_3^{g_j} \cong Z_{10})$$

is a 2-$(65,15,21)$ design. Each block of the design $\mathcal{D}_2$ is a union of three disjoint blocks of the design $\mathcal{D}_1$ which form a triangle in the projective plane

$PG(2,16)$. A setwise stabilizer in $Aut(\mathcal{D}_1)$ of a union of three disjoint blocks of $\mathcal{D}_1$ which form a block of $\mathcal{D}_2$ is a group of order 600 isomorphic to $H_3 : Z_4$.

The intersection of any two elements, $H_1^{g_i} \in ccl_G(H_1)$, $1 \leq i \leq 65$, and $H_4^{g_j} \in ccl_G(H_4)$, $1 \leq j \leq 1600$, is either the trivial group or Z_3. The incidence structure $\mathcal{D}_3 = (\mathcal{P}_3, \mathcal{B}_3, \mathcal{I}_3)$, where $\mathcal{P}_3 = \{P_1^{(3)}, \ldots, P_{65}^{(3)}\}$, $\mathcal{B}_3 = \{x_1^{(3)}, \ldots, x_{1600}^{(3)}\}$ and

$$(P_i^{(3)}, x_j^{(3)}) \in \mathcal{I}_3 \iff (H_1^{g_i} \cap H_4^{g_j} \cong Z_3)$$

is a 2-$(65, 26, 250)$ design. Every block of $\mathcal{D}_3$ intersect 78 blocks of $\mathcal{D}_1$ in one point, 91 blocks in two points, and the remaining 39 blocks in four points. So, every block of $\mathcal{D}_3$ is a blocking set of the Hermitian unital $\mathcal{D}_1$.

The group $U(3,4)$ acts transitively on the designs $\mathcal{D}_2$ and $\mathcal{D}_3$. The full automorphism group of the designs $\mathcal{D}_2$ and $\mathcal{D}_3$ is isomorphic to $U(3,4) : Z_4$.

4. Construction of $SRG(416, 100, 36, 20)$

Let G be a group isomorphic to the unitary group $U(3,4)$ and $H_3 \cong (Z_5 \times Z_5) : S_3$ be a maximal subgroup of G. Cardinality of the conjugacy class $ccl_G(H_3)$ is 416.

The intersection of any two different elements $H_3^{g_i}$ and $H_3^{g_j}$ of the set $ccl_G(H_3)$ is isomorphic to Z_{10}, Z_2, S_3 or the trivial group.

The incidence structure $\mathcal{G}_2 = (\mathcal{V}_2, \mathcal{E}_2, \mathcal{I}_2)$, where $\mathcal{V}_2 = \{V_1^{(2)}, \ldots, V_{416}^{(2)}\}$ and vertices $V_i^{(2)}$ and $V_j^{(2)}$ are adjacent if and only if $H_3^{g_i} \cap H_3^{g_j} \cong S_3$, is a strongly regular graph with parameters $(416, 100, 36, 20)$.

The group $U(3,4)$ acts transitively on the graph $\mathcal{G}_2$. The full automorphism group of the graph $\mathcal{G}_2$ is a group of order 503193600 isomorphic to $G(2,4) : Z_2$. This is the full automorphism group of the exceptional group $G(2,4)$, which is the simple group of order 251596800. Since the Janko group J_2 is a subgroup of $G(2,4)$, J_2 acts as an automorphism group of the graph $\mathcal{G}_2$. The graph $\mathcal{G}_2$ was previously known. Namely, the Suzuki graph, a strongly regular graph with parameters $(1782, 416, 100, 96)$, is locally $\mathcal{G}_2$ (see [6]).

The graph $\mathcal{G}_2$ can be constructed from the design $\mathcal{D}_2$ in a similar way as $\mathcal{G}_1$ is constructed from $\mathcal{D}_1$. Any two blocks of $\mathcal{D}_2$ intersect in 2,3, or 5 points. The graph which has as its vertices the blocks of $\mathcal{D}_2$, two vertices being adjacent if and only if the corresponding blocks intersect in 3 points, is isomorphic to $\mathcal{G}_2$.

Strongly regular graphs described in this article, incidence matrices of the block designs and a semi-symmetric design, as well as generators of their full automorphism groups, are available at

`ftp://polifem.ffri.hr/matematika/vedrana/.`

In Table 1 we give the full automorphism groups of the constructed structures.

TABLE 1. Designs and strongly regular graphs.

Combinatorial structure	Order of the full automorphism group	Structure of the full automorphism group
$PG(2,16)$	34217164800	$P\Gamma L(3,16)$
$S(2,5,65)$	249600	$U(3,4):Z_4$
2-$(65,15,21)$ design	249600	$U(3,4):Z_4$
2-$(65,26,250)$ design	249600	$U(3,4):Z_4$
$(208,12,(1))$ design	249600	$U(3,4):Z_4$
$SRG(208,75,30,25)$	249600	$U(3,4):Z_4$
$SRG(416,100,36,20)$	503193600	$G(2,4):Z_2$

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